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GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ALMOST PERFECT RINGS

by

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We have read the thesis entitled "**ALMOST PERFECT RINGS**" completed by **SİNEM BENLİ** under supervision of **ASSOC. PROF. DR. ENGİN MERMUT** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

Bazzoni and Salce have showed that all modules over a commutative domain R have a strongly flat cover if and only if every flat R-module is strongly flat and that holds if and only if R is almost perfect, that is, every proper quotient of R is a perfect ring. Facchini and Parolin defined a ring R to be right almost perfect if the quotient ring R/I is a right perfect ring for every nonzero proper two-sided ideal I of R. They proved that most of the properties of commutative almost perfect domains still hold in the noncommutative setting. In this thesis, we observe the relation between commutative almost perfect domains and C-rings of Renault, and then the relation between right almost perfect rings and C-$\mathcal{J}$-rings of Generalov. A ring R is said to be a left C-ring if for every left R-module M and for every essential proper submodule N of M, M/N has a simple submodule. For a set $\mathcal{J}$ of left ideals of a ring R, the ring R is said to be a left C-$\mathcal{J}$-ring if for any proper $\mathcal{J}$-dense left ideal I of R (that is, for every element r of R, the left ideal $(I : r)$ belongs to $\mathcal{J}$ and $(I : r)r$ not equal to 0), there exists an element r of R such that $(I : r)$ is a maximal left ideal. Facchini and Parolin have defined h-locality also for noncommutative rings. A ring R is said to be h-local if R/I is semilocal for every proper nonzero two-sided ideal I of R and every nonzero prime two-sided ideal of R is contained in a unique maximal two-sided ideal of R. We prove that for a prime ring R, R is right almost perfect if and only if it is h-local and a left C-$\mathcal{J}$-ring, where $\mathcal{J}$ is the Gabriel filter that consists of the left ideals I of R such that for every two-sided ideal J containing I properly, there exists an element r not contained in J with $(J : r)$ containing a nonzero two-sided ideal.

Keywords: Almost perfect rings, perfect rings, C-rings, C-$\mathcal{J}$-rings, torsion theory, hereditary torsion theory, Gabriel filter, h-local ring, $\mathcal{J}$-dense ideal, cotorsion, covers.

NEREDEYSE MÜKEMMEL HALKALAR

ÖZ

Bazzoni ve Salce değişmeli bir tamlık bölgesi üzerindeki her modülün bir güçlü düz örtüye sahip olması ile her düz R-modülün güçlü düz modül olmasının denk olduğunu göstermiştir ve bu durum ancak ve ancak R neredeyse mükememmel ise, yani R'nin bütün öz bölümleri mükemmel olan bir halka ise sağlanır. Facchini ve Parolin sağ neredeyse mükemmel halkaları, her sıfırdan farklı iki-taraflı öz I ideali için R/I bölüm halkası sağ mükemmel olan halkalar olarak tanımlamışlardır. Değişmeli neredeyse mükemmel tamlık bölgelerinin sahip olduğu özelliklerin çoğunun değişmeli olmayan uyarlamada hala geçerli olduğunu kanıtladılar. Bu tezde, değişmeli neredeyse mükemmel tamlık bölgeleri ile Renault tarafından tanımlanan C-halkalar arasındaki ilişkileri, ve daha sonra sağ neredeyse mükemmel halkalar ile Generalov tarafından tanımlanan sol C-$\mathcal{J}$-halkalar arasındaki ilişkileri gözlemledik. Eğer her sol R-modül M ve onun her büyük altmodülü N için M/N modülü basit bir altmodüle sahipse, R halkasına sol C-halka deriz. Bir $\mathcal{J}$ sol idealleri kümesi için, R halkasının sol C-$\mathcal{J}$-halka olması için gereken şart, R halkasının her J-yoğun sol ideali I (yani, R'nin her r elemanı için $(I : r)$ sol ideali $\mathcal{J}$'ye aittir ve $(I : r)r$ ifadesi 0'dan farkldır) için $(I : r)$ maksimal sol ideal olacak şekilde halkada bir r elemanı olmasıdır. Facchini ve Parolin değişmeli olmayan halkalar için h-lokal kavramını da tanımladı. R halkasının her sıfırdan farklı iki-taraflı öz I ideali için R/I bölüm halkası yarılokal ve R'nin her sıfırdan farklı iki-taraflı asal ideali R'nin tek bir iki-taraflı maksimal idealinde içeriliyorsa, R'ye h-lokal halka denir. Bir asal R halkasının sağ neredeyse mükemmel halka olması için gerek ve yeter şartın R nin h-lokal ve sol C-$\mathcal{J}$-halka olması olduğunu kanıtladık; burada $\mathcal{J}$ Gabriel filtresi öyle I sol ideallerinden oluşur ki I'yı öz içeren her iki-taraflı J ideali için halkanın öyle bir r elemanı vardır ki $(I : r)$ sol ideali sıfırdan farklı iki-taraflı bir ideal içerir.

Anahtar kelimeler : Neredeyse mükemmel halkalar, mükemmel halkalar, C-halkalar, C-$\mathcal{J}$-halkalar, burulma teorisi, kalıtsal burulma teorisi, Gabriel filtresi, h-lokal halka, $\mathcal{J}$-yoğun ideal, eş-burulma, örtüler.

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CHAPTER ONE
INTRODUCTION

Throughout the thesis, R will denote an associative ring with $1 \neq 0$ and all modules are unital right R-modules unless otherwise stated. Also, a *homomorphism* will be used to imply an R-module homomorphism unless otherwise stated. Whenever we say a *domain* we mean a ring without (left or right) zero-divisors that is not necessarily commutative, whereas we use *commutative domain* or *integral domain* to emphasize the commutativity. By an *ideal*, we always mean a two sided ideal.

In this thesis, first of all, we summarize and explain the motivation of the notion of right almost perfect rings from the beginning (see especially Section 1.3). We remind fundamentals of ring and module theory and we explain the concept of torsion theory (see Chapter 2). As a serious work, we investigate the work of Facchini and Parolin, namely, the extension of commutative almost perfect domains to noncommutative setting, and we explain it by giving much more details (see Chapter 3). Furthermore, we remind the notion of C-rings of Renault as well as $C_{\mathcal{J}}$-rings of Generalov. Finally, we pose their relation with commutative almost perfect domains and right almost perfect rings (see Chapter 4).

In this introductory chapter, we give a brief summary about the motivating ideas and necessary concepts for almost perfect rings. In the first section, we collect the main definitions and mention some new results we obtain in this thesis. For more details of the results, see Chapter 4. Section 1.2 is devoted to summarize the notion of covers and envelopes in relative homological algebra. In Section 1.3, we summarize the concept of cotorsion theory and explain the problem posed by Trlifaj which gave rise to the notion of commutative almost perfect domains. In the last section of this chapter, we list some properties of commutative almost perfect domains since they have generalizations to noncommutative setting by Facchini and Parolin.

1.1 Motivation and Main Results of This Thesis

The theory of covers and envelopes goes back to 1950's. Since its beginnings, the main concern about them has been showing their existence according to some classes of modules. In years, the existence of some types of covers and envelopes have been proved such as projective covers, pure-injective envelopes, torsion-free covers and flat covers. See Section 1.2 for the notions of covers and envelopes. After that, Trlifaj posed some open problems in the workshop 'Homological Methods in Module Theory', Cortona, 2000. One of them was the following: When is the class of strongly flat modules a cover class? Namely, over which commutative domains every module has a strongly flat cover? See Section 1.3 for the notion of strongly flat modules.

Salce and Bazzoni were motivated by this question and obtained the answer in 2002. They proved that all modules over a commutative domain R have a strongly flat cover if and only if every flat R-module is strongly flat, and also they showed that the commutative domains satisfying this property correspond to *commutative almost perfect domains* (see Theorem 1.3.4 and 1.3.5). A commutative ring R is said to be almost perfect if its nonzero proper quotients are perfect rings. Afterwards, this class of rings have been studied by Bazzoni, Salce, Zanardo and others. Several interesting properties and characterizations have been obtained for commutative almost perfect rings; for the properties that we are interested in see Section 1.4.

In this thesis, we investigate the notion of *right almost perfect rings*, which is a generalization of commutative almost perfect domains to the noncommutative case, introduced by Facchini & Parolin (2011): A ring R is said to be right almost perfect if the quotient ring R/I is a right perfect ring for every nonzero proper two-sided ideal I of R (for details and explanations, see Section 3.1). In their work, they also obtained a generalization to the noncommutative case for the theorem which gives equivalent conditions to the statement '*every torsion module over a commutative domain contains a simple submodule*' (see Theorem 1.4.4 for the commutative case and Theorem 3.3.4

for the noncommutative version). The important point was to define a suitable torsion theory for prime rings. Besides, they defined h-locality for noncommutative rings. A ring R is said to be h-local if R/I is semilocal for every proper nonzero two-sided ideal I of R and every nonzero prime two-sided ideal of R is contained in a unique maximal two-sided ideal of R. Moreover, they showed that most of the properties of commutative almost perfect domains still hold in the noncommutative setting (see Section 3.3). They gave some examples of right almost perfect rings (see Section 3.2).

After we study the structure of right almost perfect rings, we observe the common properties of C-rings of Renault and right almost perfect rings. The notion of C-rings was introduced by Renault in 1964 as follows: A ring R is said to be a left C-ring if for every left R-module M and for every essential proper submodule N of M, M/N has a simple submodule (see Section 4.1). At first, we obtain some one way implications for right almost perfect rings, such as for a prime local ring R, if R is a left C-ring, then it is right almost perfect (see Proposition 4.1.5 and Corollary 4.1.7). Also, for left bounded rings, we show an equivalence relation as follows: For a prime left bounded ring R, R is right almost perfect if and only if R is h-local and a left C-ring (see Proposition 4.1.9). Another situation that we obtain an equivalence is the commutative case, that is, for a commutative domain R, R is almost perfect if and only if R is h-local and R is a C-ring (see Corollary 4.1.11).

In an attempt to obtain an equivalence relation for the noncommutative case, we study $C_{\mathcal{J}}$-rings of Generalov. This class of rings forms a generalization of C-rings. $C_{\mathcal{J}}$-rings were introduced by Generalov in 1978 as follows: For a set $\mathcal{J}$ of left ideals of a ring R, the ring R is said to be a left $C_{\mathcal{J}}$-ring if for any proper $\mathcal{J}$-dense left ideal I of R, there exists an element r in R such that $(I : r)_l = \{x \in R : xr \in I\}$ is a maximal left ideal. Here, by a $\mathcal{J}$-dense left ideal I of the ring R, we mean a left ideal I of R such that for every $r \in R$, the left ideal $(I : r)_l = \{x \in R : xr \in I\}$ belongs to $\mathcal{J}$ and $(I : r)_l r \neq 0$ (see Section 4.2). As a main result, we prove that for a prime ring R, R is right almost perfect if and only if it is h-local and a left $C_{\mathcal{J}}$-ring, where $\mathcal{J}$ is the Gabriel topology (see Definition 2.2.20 for Gabriel topology) that consists of the left ideals I of R such that for every ideal J containing I properly, there exists an element r not contained in J such that $(J : r)_l$ contains a nonzero two-sided ideal (see Theorem 4.2.6 and Corollary 4.2.7).

1.2 Covers and Envelopes

In Ring and Module Theory, it has been always important to characterize rings via their modules. Since it is almost impossible to describe all modules over a given ring R, the idea is to approximate arbitrary modules by the modules from a particular class $\mathcal{X}$ of modules.

This procedure has been used to investigate injective envelopes by Eckmann & Schopf (1953), and they proved the existence of injective envelopes for modules over any ring R. Dually, Bass (1960) studied projective covers and introduced the notion of right perfect rings. Afterwards, many other varied notions of covers and envelopes were defined, such as pure-injective envelopes and torsion-free covers. The first problem that have been considered related to covers and envelopes are defining covers and envelopes in a general setting. Enochs (1981) first made a general definition of covers and envelopes by diagrams for a given class of modules. The same notion was also considered by Auslander & Buchweitz (1989). For a given class $\mathcal{X}$ of modules, determining whether every module has an $\mathcal{X}$-cover (or $\mathcal{X}$-envelope) or not has been another problem related to covers and envelopes.

Since the existence of projective covers is not so common and also it is believed that the duality between flat modules and injective modules is better than that between projective modules and injective modules, Enochs (1981) conjectured that over any ring R, every module has a flat cover. At first, they knew that the conjecture is true for right perfect rings since over right perfect rings every module has a projective cover and every flat module is projective.

Proposition 1.2.1. *(Xu, 1996, Proposition 1.3.1) Every right R-module over a right perfect ring R has a flat cover which is the same as its projective cover.*

In an attempt to enlarge the class of rings over which every module has a flat cover, Enochs (1963) initiated the study of torsion-free covers over integral domains. The fact that every module over an integral domain has a torsion-free cover provided the first class of rings, Prüfer domains (over such a domain, a module is flat if and only if it is torsion-free), over which every module has a flat cover. Afterwards, by Xu (1995), some improvements were obtained. The conjecture had been open for many years until Bican, El Bashir, & Enochs (2001) gave its proof in two different ways. We shall summarize the ideas of the way of Enochs in Section 1.3.

We shall state the general definitions of the notions of covers and envelopes, as well as collect some of their properties by following the books Xu (1996) and Enochs & Jenda (2000). For the unexplained terms and concepts of homological algebra, see for instance, Osborne (2000) or Bland (2011). All classes of modules are assumed to be closed under isomorphisms, under taking finite direct sums and direct summands.

Definition 1.2.2. Let $\mathcal{X}$ be a class of right R-modules. For an R-module M, an **$\mathcal{X}$-cover** is a module homomorphism $\varphi : X \to M$ with $X \in \mathcal{X}$ satisfying the following conditions:

(1) For every homomorphism $\varphi' : X' \to M$ with $X' \in \mathcal{X}$, there exists a homomorphism $f : X' \to X$ such that $\varphi f = \varphi'$, i.e., f completes the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & M \\ & \nwarrow^{f} & \uparrow{\scriptstyle \varphi'} \\ & & X' \end{array}$$

commutatively.

(2) For every endomorphism $f : X \to X$, if $\varphi f = \varphi$, then f must be an automorphism.

If the first condition holds (and perhaps not the second condition), $\varphi : X \to M$ is called an **$\mathcal{X}$-precover**.

Note that, an $\mathcal{X}$-cover need not be epic. Also, whenever it exists, it is unique up to isomorphism:

Proposition 1.2.3. *(Xu, 1996, Theorem 1.2.6) If $\varphi_i : X_i \to M$, $i = 1,2$, are two different $\mathcal{X}$-covers for a module M, then $X_1 \cong X_2$.*

Proof. By the definition of the notion of $\mathcal{X}$-covers, there exist module homomorphisms $f_1 : X_2 \to X_1$ and $f_2 : X_1 \to X_2$ such that $\varphi_1 f_1 = \varphi_2$ and $\varphi_2 f_2 = \varphi_1$. So, $\varphi_1 f_1 = \varphi_2$ implies that $(\varphi_2 f_2) f_1 = \varphi_2$, and $\varphi_2 f_2 = \varphi_1$ implies that $(\varphi_1 f_1) f_2 = \varphi_1$. Then, $f_2 f_1$ and $f_2 f_1$ are automorphisms by the hypothesis (2) in the definition. Hence, $X_1 \cong X_2$. □

In addition, a class $\mathcal{X}$ of modules over any ring R is said to be a **cover class**, if every R-module has an $\mathcal{X}$-cover.

Definition 1.2.4. Let $\mathcal{X}$ be a class of right R-modules. For an R-module M, an **$\mathcal{X}$-envelope** of M is a homomorphism $\varphi : M \to X$ such that the following hold:

(1) For every $\varphi' : M \to X'$ with $X' \in \mathcal{X}$, there exists a homomorphism $f : X \to X'$ such that $\varphi' = f\varphi$.

(2) If f is an endomorphism of X with $\varphi = f\varphi$, then f must be an automorphism.

Similarly, if the first one holds (and perhaps not the second), $\varphi : M \to X$ is called an **$\mathcal{X}$-preenvelope**, and also envelopes, if exist, are unique up to isomorphism.

By specializing the class of modules, all the existing covers and envelopes can be obtained. As an example of consistency, we shall show that projective covers and $\mathcal{P}$-covers coincide, where $\mathcal{P}$ is the class of all projective right R-modules for a ring R. Recall that, if P is a projective module and $\varphi : P \to M$ is an epimorphism, then φ is called a **projective cover** in case $\mathrm{Ker}\varphi$ is a small submodule of P. A submodule K of an R-module M is said to be **small** in M if for every submodule $L \subseteq M$, $K + L = M$ implies $L = M$.

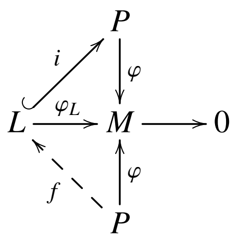
Theorem 1.2.5. *(Xu, 1996, Theorem 1.2.12) Let $\mathcal{P}$ be the class of all projective right R-modules for a ring R. For a right R-module M and a homomorphism $\varphi : P \to M$, with $P \in \mathcal{P}$, the following are equivalent:*

(1) $\varphi : P \to M$ is a $\mathcal{P}$-cover,

(2) $\varphi : P \to M$ is a projective cover.

Proof. (1) $\Rightarrow$ (2): First we see that φ is epic. By using the fact that any right R-module is an image of a projective module, we have a projective module $P' \in \mathcal{P}$ such that

$\varphi^{'} : P^{'} \to M$ is epic. Since φ is a $\mathcal{P}$-cover, there exists a homomorphism $f : P^{'} \to P$ such that $\varphi f = \varphi^{'}$. This implies that φ is also an epimorphism. To see that $\mathrm{Ker}\varphi$ is small in P, let $K = \mathrm{Ker}\varphi$ and $K + L = P$ for a submodule $L \subseteq P$. We claim that $L = P$. Since the restriction $\varphi_L : L \to M$ is epic, we have the following commutative diagram:



Since φ is a $\mathcal{P}$-cover, if must be an automorphism. Then $\mathrm{Im}(if) = P$, and so $P \subseteq L$.

(2) ⇒ (1): Clearly $\varphi : P \to M$ is a $\mathcal{P}$-precover. Now suppose that $f : P \to P$ is an endomorphism with $\varphi f = \varphi$. Since $P = \mathrm{Ker}\varphi + f(P)$ and $\mathrm{Ker}\varphi$ is small in P, we have $f(P) = P$. Consider the short exact sequence

$$0 \longrightarrow \mathrm{Ker} f \hookrightarrow P \xrightarrow{f} P \longrightarrow 0 .$$

Since it is splitting, there exists a homomorphism $g : P \to P$ such that $fg = 1_P$. Then g is monic and $P = \mathrm{Ker} f + \mathrm{Im} g$. But $\varphi = \varphi f$, hence $\mathrm{Ker} f \subseteq \mathrm{Ker}\varphi$ is small in P. It requires that $\mathrm{Im} g = P$. Therefore, g is an isomorphism, and so is $g^{-1} = f$. □

1.3 Strongly Flat Covers Over Commutative Domains

In this section, we will summarize the work done by Bazzoni & Salce (2002). For undefined notions and more explanation, see Enochs & Jenda (2000) and Trlifaj (2000). We shall start by presenting the concept of cotorsion theory. Note that we use $\mathrm{Ext}(M,N)$ to indicate $\mathrm{Ext}^1_R(M,N)$, that is, the class of all the equivalence classes of short exact sequences starting with the R-module N and ending with the R-module M.

For a given class $\mathcal{X}$ of right R-modules, let

$$^{\perp}\mathcal{X} = \{F \in \mathcal{M}od\text{-}R : \mathrm{Ext}(F,X) = 0 \ \text{ for every } \ X \in \mathcal{X}\}$$

and

$$\mathcal{X}^{\perp} = \{G \in \mathcal{M}od\text{-}R : \mathrm{Ext}(X,G) = 0 \ \text{ for every } \ X \in \mathcal{X}\}.$$

These classes are called **orthogonal classes** of $\mathcal{X}$. Note that, for any class $\mathcal{X}$ of modules, we have $\mathcal{X} \subseteq {}^{\perp}(\mathcal{X}^{\perp})$ and $\mathcal{X} \subseteq ({}^{\perp}\mathcal{X})^{\perp}$. Also, if $\mathcal{X}_1 \subseteq \mathcal{X}_2$, then ${}^{\perp}\mathcal{X}_2 \subseteq {}^{\perp}\mathcal{X}_1$ and $\mathcal{X}_2^{\perp} \subseteq \mathcal{X}_1^{\perp}$. By using these relations, we have ${}^{\perp}\mathcal{X} = {}^{\perp}(({}^{\perp}\mathcal{X})^{\perp})$ and $\mathcal{X}^{\perp} = ({}^{\perp}(\mathcal{X}^{\perp}))^{\perp}$ for every class $\mathcal{X}$ of modules.

Definition 1.3.1. A pair $(\mathcal{A},\mathcal{B})$ of classes of R-modules is said to be a **cotorsion pair** or **cotorsion theory** if $\mathcal{A}^{\perp} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$.

For a given cotorsion pair $(\mathcal{A},\mathcal{B})$, a class $\mathcal{D}$ is said to **generate** the cotorsion pair $(\mathcal{A},\mathcal{B})$ if ${}^{\perp}\mathcal{D} = \mathcal{A}$ (and so $\mathcal{D} \subseteq \mathcal{B}$), whereas a class $\mathcal{G}$ is said to **cogenerate** the cotorsion pair $(\mathcal{A},\mathcal{B})$ if $\mathcal{G}^{\perp} = \mathcal{B}$ (and so $\mathcal{G} \subseteq \mathcal{A}$).

As examples for cotorsion pairs, $(\mathcal{M}od\text{-}R,\mathcal{I})$, $(\mathcal{P},\mathcal{M}od\text{-}R)$ can be given where $\mathcal{I}$ and $\mathcal{P}$ denote the classes of injective and projective R-modules, respectively. We remark that if $(\mathcal{A},\mathcal{B})$ is a cotorsion pair, then $\mathcal{A}$ and $\mathcal{B}$ are both closed under extensions and summands, also $\mathcal{A}$ contains all projective modules whereas $\mathcal{B}$ contains all injective modules. Moreover, $\mathcal{A}$ is closed under arbitrary direct sums, and $\mathcal{B}$ is closed under arbitrary direct products.

A cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to have **enough projectives** if for every module M, there is an exact sequence $0 \longrightarrow B \longrightarrow A \longrightarrow M \longrightarrow 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Besides, $(\mathcal{A},\mathcal{B})$ is said to have **enough injectives** if for every module M, there exists an exact sequence $0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Moreover, if a cotorsion pair has enough injectives and enough projectives, it is called **complete**. However, we have that a cotorsion pair $(\mathcal{A},\mathcal{B})$ has enough projectives if and only if it has enough injectives. Also, it is easy to see that having such an exact sequence $0 \longrightarrow B \longrightarrow A \longrightarrow M \longrightarrow 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ implies that $A \to M$ is an $\mathcal{A}$-precover of M. Similarly, if the cotorsion pair $(\mathcal{A},\mathcal{B})$ has enough injectives, then every module M has a $\mathcal{B}$-preenvelope.

Definition 1.3.2. An R-module C is said to be **cotorsion** (in the sense of Enochs) if

$\text{Ext}(F,C) = 0$ for every flat R-module F.

The pair $(\mathcal{F},\mathcal{C})$ where $\mathcal{F}$ is the class of all flat R-modules and $\mathcal{C}$ is the class of all cotorsion modules forms a cotorsion pair, and it is called *flat cotorsion pair*. If we turn back to the proof of *flat cover conjecture* given by Enochs, the sketch of the proof is as follows: Eklof & Trlifaj (2001) proved that every cotorsion pair which is cogenerated by a set of modules has enough projectives and enough injectives. Bican, El Bashir, & Enochs (2001) proved that the flat cotorsion pair is cogenerated by a set of modules. Hence, every module M over any ring R has a flat precover. It had been also proved by Enochs (1981) that if a class $\mathcal{X}$ of modules is closed under direct limits and every module admits an $\mathcal{X}$-precover, then $\mathcal{X}$-covers exist. Since the class of flat modules over any ring R is closed under direct limits, we conclude that the class of flat modules is a cover class.

Definition 1.3.3. Let R be a commutative domain with the field of fractions Q. An R-module C is said to be **weakly cotorsion** if $\text{Ext}(Q,C) = 0$. It is also called *Matlis cotorsion* or *cotorsion in the sense of Matlis*. An R-module M is said to be **strongly flat** if $\text{Ext}(M,C) = 0$ for every weakly cotorsion module C.

Hence, over any integral domain R, the pair $(\mathcal{SF},\mathcal{WC})$ is a cotorsion pair where $\mathcal{SF}$ and $\mathcal{WC}$ denotes the classes of all strongly flat modules and all weakly cotorsion modules, respectively. Furthermore, we can partially order the cotorsion pairs in the following way: $(\mathcal{A},\mathcal{B}) \geq (\mathcal{A}',\mathcal{B}')$ means the class $\mathcal{B}$ contains the class $\mathcal{B}'$, or equivalently, the class $\mathcal{A}'$ contains the class $\mathcal{A}$. Since the field of fraction Q of a commutative domain R is a flat R-module, every cotorsion module is weakly cotorsion, and so every strongly flat module is flat. Thus, we have the subsequent relation for the cotorsion pairs we mention up to present:

$$(\mathcal{P},\mathcal{M}od\text{-}R) \geq (\mathcal{SF},\mathcal{WC}) \geq (\mathcal{F},\mathcal{C}) \geq (\mathcal{M}od\text{-}R,\mathcal{I}) \text{ if } R \text{ is a commutative domain.}$$

Note that all these cotorsion pairs are complete.

Bazzoni & Salce (2002) have been motivated by the question posed by Trlifaj

(2000). Trlifaj (2000) asked the following question: when is the class of strongly flat modules over a commutative domain R a cover class? Actually, this is a part of the following more general question: For a class $\mathcal{X}$ of modules, is the property of being closed under direct limits necessary to be a cover class? Because the property of being closed under direct limits imply for a complete cotorsion pair to be a cover class. The general question is still open.

Over a commutative domain R, the class $\mathcal{SF}$ is not closed under direct limits in general, but its closure under direct limits is the class of flat modules. Note that over Dedekind domains $\mathcal{SF} = \mathcal{F}$, hence $\mathcal{SF}$-covers exist for these domains. Bazzoni & Salce (2002) proved the following main result.

Theorem 1.3.4. *(Bazzoni & Salce, 2002, Theorem 2.10) Let R be a commutative domain. Every module admits an $\mathcal{SF}$-cover if and only if the class of strongly flat modules coincides with the class of flat modules. Thus, in particular, $\mathcal{SF}$ is a cover class if and only if it is closed under direct limits.*

Moreover, Bazzoni & Salce (2002) also characterized the commutative domains for which $\mathcal{SF} = \mathcal{F}$, similar to the characterization of perfect rings by Bass (1960).

Theorem 1.3.5. *(Bazzoni & Salce, 2002, Theorem 4.5) The following are equivalent for a commutative domain R:*

(1) R satisfies $\mathcal{SF} = \mathcal{F}$,

(2) R is h-local and every localization R_P of R at a maximal ideal P is almost perfect,

(3) R is almost perfect, i.e., R/I is perfect for every nonzero proper ideal I of R,

(4) $\mathcal{SF}$ is a cover class,

(5) $\mathcal{SF}$ is closed under direct limits,

(6) $\mathcal{WC} = \mathcal{C}$.

1.4 Commutative Almost Perfect Domains

In this section, we aim to collect some of the important results about commutative almost perfect domains given by Bazzoni & Salce (2003). The reason why we are interested in these properties of a commutative almost perfect domain is that they have generalizations to noncommutative case given by Facchini & Parolin (2011). Moreover, the properties of a commutative almost perfect domain we state here will be necessary while we show their relation with C-rings in Section 4.1.

To begin with, let us recall the notion of a torsion module over an integral domain. In the case R is an integral domain, we call an element m of a module M_R **torsion element** if there is a nonzero element $r \in R$ such that $mr = 0$. The set of all torsion elements of a module M_R, denoted by $t(M)$, is a submodule of M. The submodule $t(M)$ is called the **torsion submodule of** M. Also an R-module M is said to be a **torsion module** if $t(M) = M$, and it is said to be a **torsion-free module** if $t(M) = 0$. Note that the quotient module $M/t(M)$ is torsion-free for every R-module M.

Now, remember the characterization of commutative perfect rings. Although Bass (1960) defined the notion of a perfect ring for arbitrary rings, it gives us a bit more advantages when we consider that the ring R is commutative. For the definitions of the terms used in the following theorems and propositions, see Section 2.1 and especially Subsection 2.1.2. Besides, it will be beneficial to note in advance that the notion of a right perfect ring and a right T-nilpotent ideal coincides with their left counterparts since the ring R is commutative.

Theorem 1.4.1. *(Bazzoni & Salce, 2003, Theorem 1.1) If R is a commutative ring, then the following are equivalent:*

(1) R is a perfect ring, i.e., R is semilocal and $\mathrm{Jac}(R)$ *is T-nilpotent,*

(2) R satisfies the DCC *on principal ideals,*

(3) R is a finite direct product of local rings with T-nilpotent maximal ideals,

(4) R is semilocal and every localization of R at a maximal ideal is a perfect ring,

(5) R is semilocal and semiartinian.

Furthermore, R is a perfect domain if and only if it is a field.

Definition 1.4.2. A commutative ring R is called **almost perfect** if for every nonzero proper ideal I of R, the quotient ring R/I is a perfect ring.

The next proposition enables to restrict the study of commutative almost perfect rings to the commutative domain case.

Proposition 1.4.3. *(Bazzoni & Salce, 2003, Propposition 1.3) Let R be a commutative almost perfect ring. If R is not a domain, then R is a perfect ring.*

We shall state a theorem, used in below Theorem 1.4.5, from the book Enochs & Jenda (2000), which has an extension to noncommutative case by Facchini & Parolin (2011) (see Proposition 3.3.4).

Theorem 1.4.4. *(Enochs & Jenda (2000, Theorem 4.4.1) and Bazzoni & Salce (2003, Theorem 2.2)) The following are equivalent for an integral domain R with the field of fractions Q:*

(1) Every nonzero torsion module contains a simple submodule,

(2) Every torsion module over R is semiartinian,

(3) For every nonzero proper ideal I of R, R/I contains a simple submodule,

(4) For every nonzero proper R-submodule N of Q, Q/N has a simple submodule,

(5) Q/R is semiartinian,

(6) A module M is injective if and only if $\mathrm{Ext}(S, M) = 0$ *for every simple module S,*

(7) If $\{\varphi_i : T_i \to M_i\}_{i\in I}$ is a family of torsion-free covers, then their product $\prod_{i\in I} \varphi_i : \prod_{i\in I} T_i \to \prod_{i\in I} M_i$ is again a torsion-free cover.

Recall that a commutative domain R is said to be h**-local** if R is of *finite character*, that is, each nonzero proper ideal I of R is contained in at most finitely many maximal ideals of R and every nonzero prime ideal of R is contained in only one maximal ideal of R.

Theorem 1.4.5. *(Bazzoni & Salce, 2003, Theorem 2.3) The following are equivalent for a commutative domain R.*

(1) R is almost perfect,

(2) R is h-local and R satisfies one of the equivalent conditions of Theorem 1.4.4.

Corollary 1.4.6. *(Bazzoni & Salce, 2003, Corollary 2.4) If R is a commutative local domain and Q is its field of fractions, then the following are equivalent:*

(1) R is almost perfect,

(2) Q/R is semiartinian,

(3) Every nonzero torsion module is semiartinian.

CHAPTER TWO
PRELIMINARIES

This chapter is prepared with the aim of collecting the definitions and results frequently used throughout the thesis. In the first section of this chapter, we concern about results related to ring and module theory. In the second section, using the book Stenström (1975), we summarize torsion theory which was defined by Gabriel (1962) and Maranda (1964).

2.1 Ring and Module Theory

In this section, our objective is to give a brief information about some notions of ring and module theory. We do not deal with every term in modules and rings. Actually, we accept the fundamentals of module theory and ring theory. The definitions and results we remind here will be used in the next chapters commonly. We shall usually state the definitions and results for right modules which have obvious left versions. For further and deeper results and detailed proofs, see for example Anderson & Fuller (1992), Lam (2001), Farb & Dennis (1993) and Bland (2011).

A right R-module M is said to be a **semisimple module** if it is a direct sum of simple submodules, or equivalently, if every submodule of M is a direct summand.

Definition 2.1.1. A ring R is called **right semisimple** if any of the following equivalent conditions hold:

(1) R is semisimple as a right R-module,

(2) Every right R-module is semisimple,

(3) Every short exact sequence of right R-modules splits.

Also, left semisimple rings can be defined similarly. Note that, for a right semisimple ring R, since the right R-module R is finitely generated and from Definition 2.1.1-(1), it follows that R_R satisfies both ACC and DCC. Thus, a right semisimple ring R is both right noetherian and right artinian. Moreover, the

Wedderburn & Artin Theorem which presents the structure of semisimple rings gives us that a ring is right semisimple if and only if it is left semisimple. Therefore, we drop the term left or right.

A ring R is called **simple** if it has no nontrivial two-sided ideals. It is clear that saying the ring is simple does not imply that it is simple as a module over itself whereas any ring which is simple as a module over itself is a simple ring. Besides, we know that simple modules are semisimple, but it is not true that every simple ring is semisimple. To make it correct, we need one more condition as we can see in the following theorem.

Theorem 2.1.2. *(Farb & Dennis, 1993, Theorem 1.15) Let R be a ring. Then the following are equivalent:*

(1) R is a simple right artinian ring,

(2) R is isomorphic to a matrix ring over a division ring,

(3) R is semisimple as a right R-module and all simple modules over R are isomorphic,

(4) R is homogeneous semisimple as a right R-module,

(5) R is right artinian and has a faithful simple module.

Because of the left-right symmetry in (2), they are all equivalent to the left-handed versions of (1), (3), (4) and (5). From now on, for semisimple rings we use the term *semisimple artinian* to emphasize the above equivalences.

We endure by giving a few definitions of some types of ideals. The **Jacobson radical** of a ring R, denoted by $\mathrm{Jac}(R)$, is defined to be the intersection of all the maximal right ideals of R. However, it coincides with the intersection of all the maximal left ideals, and also we shall remark that it is a two-sided ideal (see Lam (2001, Lemma 4.1 and Corollary 4.2)). A one-sided (or two-sided) ideal I of R is said to be **nil** if I consists of nilpotent elements while it is said to be **nilpotent** if $I^n = 0$ for some positive integer n. Clearly, every nilpotent ideal is nil; the converse holds for right noetherian rings:

Proposition 2.1.3. *(Anderson & Fuller, 1992, Theorem 15.22) If R is a right noetherian ring, then every nil one-sided ideal of R is nilpotent.*

Theorem 2.1.4. *(Anderson & Fuller, 1992, Theorem 15.20) A ring R is left artinian if and only if R is left noetherian,* $\mathrm{Jac}(R)$ *is nilpotent, and* $R/\mathrm{Jac}(R)$ *is semisimple artinian.*

Theorem 2.1.5. *(Anderson & Fuller, 1992, Corollary 15.23) Let R be a left noetherian ring. If* $R/\mathrm{Jac}(R)$ *is semisimple artinian and if* $\mathrm{Jac}(R)$ *is nil, then R is left artinian.*

Proposition 2.1.6. *(Lam, 2001, Lemma 4.11) If a right (resp., left) ideal* $I \subseteq R$ *is nil, then* $I \subseteq \mathrm{Jac}(R)$.

Proposition 2.1.7. *(Lam, 2001, Proposition 4.6) If I is an ideal of R lying in* $\mathrm{Jac}(R)$*, then* $\mathrm{Jac}(R/I) = \mathrm{Jac}(R)/I$.

The Jacobson radical of a ring R turns out to be a useful tool when determining whether the ring is semisimple artinian as in the following way:

Theorem 2.1.8. *(Lam, 2001, Theorem 4.14) A ring R is semisimple artinian if and only if R is right artinian and* $\mathrm{Jac}(R) = 0$.

In the next definition, we can find an important class of rings which generalizes one-sided artinian rings: A ring R is said to be **semiprimary** if $\mathrm{Jac}(R)$ is nilpotent and $R/\mathrm{Jac}\,R$ is semisimple artinian.

Another important class of rings is primitive rings. They are a generalization of simple rings. Before we state its definition, we shall mention some module theoretic notions. The annihilator of an element m of a R-module M is the set $\{r \in R : mr = 0\}$ which has a right ideal structure. The **annihilator** of a right R-module M, which forms a two-sided ideal, is defined as follows: $\mathrm{ann}(M) = \{r \in R : mr = 0$ for every $m \in M\}$. We say a right R-module M is **faithful** if its annihilator is zero. A ring R is called a **right primitive ring** if it has a faithful simple right R-module. We call an ideal I of R a **right primitive ideal** if it is the annihilator of a simple right R-module. Left primitive rings and left primitive ideals are defined similarly. In spite of the name right and left, primitive ideals are always two-sided. In addition, the quotient of a ring R by a right primitive ideal I becomes a right primitive ring.

Now, we remind the socle and the radical of a right R-module. For a right R-module M, its **socle**, denoted by $\mathrm{Soc}(M)$, is defined to be the sum of all simple submodules of M. Dually, the **radical** of M, denoted by $\mathrm{Rad}(M)$, is the intersection of all maximal submodules in M. Note that $\mathrm{Rad}(M) = M$ if and only if M has no maximal submodule.

Another class of modules that we shall use are semiartinian modules. A right R-module M is called **semiartinian** if every nonzero homomorphic image of M, i.e., every nonzero quotient module of M, has a nonzero socle, that is, has a simple submodule. Equivalently, a module M is semiartinian if and only if it is an essential extension of its socle.

Definition 2.1.9. A ring R is said to be **right semiartinian ring** if it is semiartinian as a right R-module over itself, that is, $\mathrm{Soc}(R/I)$ is nonzero for every proper right ideal I of R. Equivalently, a ring R is right semiartinian if and only if every right R-module is semiartinian.

There is a correspoding notion of a left semiartinian ring. Also, every artinian module is of course semiartinian, as well as right artinian rings are right semiartinian. For more details, see for instance, the books Stenström (1975, Ch. VIII, §2) and Dung, Huynh, Smith, & Wisbauer (1994, p. 26–28).

2.1.1 Prime and Semiprime Rings

Prime rings, which generalize domains, will play an important role in the characterization of almost perfect rings. Because of this reason, we summarize definitions and some necessary properties about prime rings.

Definition 2.1.10. An ideal P in a ring R is said to be a **prime ideal** if $P \neq R$ and, for all ideals I and J of R, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. Also, an ideal P of a ring R is called **semiprime** if for every ideal I of R, $I^2 \subseteq P$ implies that $I \subseteq P$.

In fact, for the preceding definitions, we have frequently used equivalent conditions which can be seen for example in Lam (2001, Proposition 10.2 and Proposition 10.9).

Definition 2.1.11. A ring R is called a **prime** (resp., **semiprime**) ring if the zero ideal 0 is a prime (resp., semiprime) ideal.

Lemma 2.1.12. *(Lam, 2001, p. 158) Let R be a ring and I a two-sided ideal in R. Then, the quotient ring R/I is a prime ring if and only if I is a prime ideal in R.*

The subsequent theorem relates semiprime rings to semisimple rings.

Theorem 2.1.13. *(Lam, 2001, Theorem 10.24) For any ring R, the following three statements are equivalent:*

(1) R is semisimple artinian,

(2) R is semiprime and right artinian,

(3) R is semiprime and satisfies DCC *on principal right ideals.*

2.1.2 Perfect and Semiperfect Rings

First of all, it will be advantageous to give a brief summary about the notions of local and semilocal rings. In commutative algebra, *local rings* are defined to be rings as those which have a unique maximal ideal. Its generalization to arbitrary rings is defined as follows: a ring R is said to be **local** if the quotient ring $R/\operatorname{Jac}(R)$ is a division ring, or equivalently, if R has a unique maximal right ideal. As a matter of fact, there are a few more conditions that are equivalent to this definition. For them, see for example Anderson & Fuller (1992, Proposition 15.15). It is worthwhile to note that, in contrast to commutative case, if a ring R is local, then it has a unique maximal ideal, while having a unique maximal ideal is not sufficient to be a local ring. As an example, simple rings have a unique maximal ideal (the zero ideal) but need not be local.

A ring R is said to be **semilocal** if the quotient ring $R/\operatorname{Jac}(R)$ is semisimple artinian.

Proposition 2.1.14. *(Lam, 2001, Proposition 20.2) For a ring R, consider the following two conditions:*

(1) *R is semilocal,*

(2) *R has finitely many maximal ideals.*

In general, we have (2) $\Rightarrow$ (1). *The converse holds if* $R/\operatorname{Jac}(R)$ *is commutative.*

Note the following well-known fact which we use in Chapter 3; for completeness, we shall also give its proof.

Proposition 2.1.15. *If R is a semilocal ring, then R has no infinite orthogonal set of idempotents.*

Proof. Suppose for the contrary that R has an orthogonal set of nonzero idempotents, say $\{e_i : i \in I\}$, where I is an infinite index set and $e_i \neq e_j$ for all $i \neq j$ in I. Then $\{e_i + \operatorname{Jac}(R) : i \in I\}$ is an infinite orthogonal set of idempotents (not necessarily nonzero) in the quotient ring $R/\operatorname{Jac}(R)$. The submodule $U = \bigoplus_{i\in I}(e_i + \operatorname{Jac}(R))(R/\operatorname{Jac}(R)) \subseteq R/\operatorname{Jac}(R)$ is a direct summand of the right $R/\operatorname{Jac}(R)$-module $R/\operatorname{Jac}(R)$ since $R/\operatorname{Jac}(R)$ is semisimple artinian. Hence, $R/\operatorname{Jac}(R) = U \oplus V$ for some submodule $V \subseteq R/\operatorname{Jac}(R)$, and it must be a finite sum of nonzero submodules of $R/\operatorname{Jac}(R)$ since it is finitely generated. So, $e_i + \operatorname{Jac}(R) = 0 + \operatorname{Jac}(R)$, that is, $e_i \in Jac(R)$, for all but finitely many $i \in I$. It follows that $1 - e_i$ is a unit in R, and $(1 - e_i)R = R$ for all but finitely many $i \in I$. Then, $R = e_iR \oplus (1 - e_i)R$ implies $e_iR = 0$ and so $e_i = 0$ for all but finitely many $i \in I$ which is a contradiction with the infinity assumption on the set $\{e_i : i \in I\}$. □

Perfect rings were introduced by Bass (1960) in the discussion of dualizing the injective envelopes. This class of rings result in a homological characterization for modules, namely: Right perfect rings are exactly the rings whose right modules have projective covers, and they are precisely the rings whose right flat modules are projective. Due to these reasons, it became a remarkable class of rings. We shall start with the definition of semiperfect rings.

Definition 2.1.16. A ring R is called **semiperfect** if R is semilocal and idempotents of $R/\operatorname{Jac}(R)$ can be lifted to R.

There is no distinction between being a right or left semiperfect ring. Moreover, it contains the class of local rings, whereas it is contained in the class of semilocal rings (see Lam (2001, Chapter 8))

Theorem 2.1.17. *(Lam, 2001, Theorem 23.11) Let R be a commutative ring. The ring R is semiperfect if and only if it is a finite direct product of local rings.*

Before we give the definition of a right perfect ring, we need the notion of a right T-nilpotent ideal:

Definition 2.1.18. A one-sided ideal J of a ring R is called **right T-nilpotent** (resp., **left T-nilpotent**) if for any sequence $(a_k)_{k=1}^{\infty}$ of elements in J, there exists an integer $n \geq 1$ such that $a_n \dots a_2 a_1 = 0$ (resp., $a_1 a_2 \dots a_n = 0$).

Clearly, for a one-sided ideal I of a ring R, we have:

$$I \text{ is nilpotent} \Rightarrow I \text{ is right (left) } T\text{-nilpotent} \Rightarrow I \text{ is nil.}$$

Definition 2.1.19. A ring R is called a **right perfect ring** (resp., **left perfect ring**) if R is semilocal and the Jacobson radical Jac(R) of R is right T-nilpotent (resp., left T-nilpotent).

It can be easily seen that the class of right perfect rings is a generalization of semiprimary rings, in particular, one-sided artinian rings. Now we state the famous theorem called Bass' Theorem P (see Bass (1960)).

Theorem 2.1.20. *(Anderson & Fuller, 1992, Theorem 28.4) For any ring R, the following conditions are equivalent:*

(1) R is right perfect,

(2) R is semilocal and R is right max , i.e., every nonzero right R-module contains a maximal submodule,

(3) Every (right) R-module has a projective cover,

(4) Every flat (right) R-module is projective,

(5) R satisfies the DCC *on principal left ideals,*

(6) R does not contain an infinite orthogonal set of nonzero idempotents and R is left semiartinian, i.e., any nonzero left R-module N contains a simple submodule.

Remark 2.1.21. Right perfect rings do not have to be left perfect. For instance, see the example given by H. Bass in (Lam, 2001, p. 345).

We shall end this section with the following theorem which can be considered as an analogue of Theorem 2.1.17.

Theorem 2.1.22. *(Lam, 2001, Theorem 23.24) Let R be a commutative ring. R is perfect if and only if it is a finite direct product of local rings each of which has a T-nilpotent maximal ideal.*

2.2 Torsion Theory

In commutative algebra, the *field of fractions* of an integral domain, or more generally the *total ring of fractions* of a commutative ring and *localization* in commutative rings are well-known tools. Nevertheless, extending this idea to noncommutative case was not that simple. It was first investigated by Ore (1931) for the case R is a domain, and then Asano (1939) considered the existence of a *classical ring of fractions* (or a *total ring of fractions*) with respect to a multiplicative set of regular elements for an arbitrary ring R. The conditions that will allow us to construct a *general ring of fractions* with respect to a multiplicative closed set that may contain zero divisors was studied by Elizarov (1960). For all the discussion above, we recommend for example Goodearl & Warfield Jr (2004) or Stenström (1975).

In particular, to each ring of fractions of a ring R, there is an associated notion of torsion for R-modules. In this section, we are interested in this most general concept of torsion theories instead of the notion of torsion related to various multiplicative closed sets in a ring R. Our objective is to state some definitions and results relevant to torsion theory which will be used in Chapters 3 and 4. For the details of proofs and more about the theory in this section, see Stenström (1975, Chapter VI).

The concept of torsion theory for abelian categories has been introduced by Dickson (1966) formally even though the concept is in the work of Gabriel (1962) and Maranda (1964) earlier.

Before we give the necessary definitions and important results of torsion theory, we will mention the basic concepts of category theory in a concise manner since it is the natural language to use in torsion theory. On the ground that it is a wide theory, one can need more definitions or details than we collect here. For this reason, see for instance, Stenström (1975) or Osborne (2000).

Definition 2.2.1. A **category** $\mathcal{C}$ consists of a class of objects, $\mathrm{Obj}\,\mathcal{C}$, and morphism sets $\mathrm{Mor}_{\mathcal{C}}(A,B)$ for every $A,B \in \mathrm{Obj}\,\mathcal{C}$ (an element f of $\mathrm{Mor}_{\mathcal{C}}(A,B)$ is denoted by $f : A \mapsto B$) with a composition law for $\mathrm{Mor}_{\mathcal{C}}(B,C) \times \mathrm{Mor}_{\mathcal{C}}(A,B) \to \mathrm{Mor}_{\mathcal{C}}(A,C)$ denoted by $(g,f) \mapsto gf$ that satisfies the followings:

(1) Composition is associative, that is, if $f \in \mathrm{Mor}(C,D)$, $g \in \mathrm{Mor}(B,C)$ and $h \in \mathrm{Mor}(A,B)$, then $(fg)h = f(gh)$.

(2) Each $\mathrm{Mor}(A,A)$ contains a distinguished element 1_A and each 1_A is an identity, that is, if $f \in \mathrm{Mor}(A,B)$, then $f = f1_A = 1_B f$.

If $\mathcal{B}$ and $\mathcal{C}$ are categories, then $\mathcal{B}$ is a **subcategory** of $\mathcal{C}$ in the case (*i*) $\mathrm{Obj}\,\mathcal{B}$ is a subclass of $\mathrm{Obj}\,\mathcal{C}$, (*ii*) $\mathrm{Mor}_{\mathcal{B}}(A,B)$ is a subset of $\mathrm{Mor}_{\mathcal{C}}(A,B)$ for all $A,B \in \mathrm{Obj}\,\mathcal{B}$, and (*iii*) the composition in $\mathcal{B}$ is the same as in $\mathcal{C}$. Also, a category is said to be **small** if the class of objects actually is a set.

Definition 2.2.2. A (covariant) **functor** $\mathbf{F}{:}\mathcal{K} \to \mathcal{M}$, where $\mathcal{K},\mathcal{M}$ are categories, is a function which assigns each object A of $\mathrm{Obj}\,\mathcal{K}$ to the object $\mathbf{F}(A)$ of $\mathrm{Obj}\,\mathcal{M}$ as well as each morphism $f : A \to B$ in $\mathrm{Mor}_{\mathcal{K}}(A,B)$ to the morphsim $\mathbf{F}(f) : \mathbf{F}(A) \to \mathbf{F}(B)$ in $\mathrm{Mor}_{\mathcal{M}}(\mathbf{F}(A),\mathbf{F}(B))$ such that $\mathbf{F}(gf) = \mathbf{F}(g)\mathbf{F}(f)$ for all morphisms f,g whenever the composite is defined and $\mathbf{F}(1_A) = 1_{\mathbf{F}(A)}$ for all $A \in \mathrm{Obj}\,\mathcal{K}$.

The **identity functor** $\mathbf{I} : \mathcal{K} \to \mathcal{K}$ assigns every object A to itself and every morphism $f : A \to B$ again to itself.

As an important remark, we shall stress that for a ring R, the module categories $R\text{-}\mathcal{M}od$ and $\mathcal{M}od\text{-}R$ are abelian categories which are locally small, complete and cocomplete (see Stenström (1975, p. 87–89, 99)).

Now, we return to our main concept of torsion theory. Gabriel (1962) and Maranda (1964) describes this notion in three equivalent ways:

(i) by the class of torsion modules,

(ii) by the right ideals that serve as annihilators of torsion elements,

(iii) by the functor assigning to each module its torsion submodule.

Even though the notion of preradicals defined for locally small, complete and cocomplete abelian categories in Stenström (1975), for our purposes, we prefer to state all definitions for the category $\mathcal{M}od\text{-}R$ of right R-modules. Similar definitions are given for the category $R\text{-}\mathcal{M}od$ of left R-modules also.

Definition 2.2.3. A **preradical** $r : \mathcal{M}od\text{-}R \to \mathcal{M}od\text{-}R$ is defined to be a functor such that it assigns each module M to its submodule $r(M)$ and each R-homomorphism $f : M \to N$ induces a homomorphism $r(M) \to r(N)$ via restricting f to the submodule $r(M)$. Note that this just means $f(r(M)) \subseteq r(N)$.

So a preradical is a subfunctor of the identity functor. Provided that we consider the class of all preradicals of $\mathcal{M}od\text{-}R$, it forms a complete lattice (see Stenström (1975, p. 63–64) for the notions *lattice* and *complete lattice*).

Suppose that r_1 and r_2 are preradicals. Then, one can define a preradical r_1r_2 by $r_1r_2(M) = r_1(r_2(M))$. Also, we may define a preradical $(r_1 : r_2)$, so that $(r_1{:}r_2)(M)/r_1(M) = r_2(M/r_1(M))$. A preradical is said to be **idempotent** if $rr = r$ and is said to be **radical** if $(r : r) = r$, that is, if $r(M/r(M)) = (r : r)(M)/r(M) = r(M)/r(M) = 0$.

We can associate two classes of modules to each preradical r :

$$\mathcal{T}_r = \{M \in \mathcal{M}od\text{-}R : r(M) = M\}$$

and

$$\mathcal{F}_r = \{M \in \mathcal{M}od\text{-}R : r(M) = 0\}.$$

Proposition 2.2.4. *(Stenström, 1975, Proposition 1.2) For a preradical r, $\mathcal{T}_r$ is closed under quotient modules and direct sums, and $\mathcal{F}_r$ is closed under submodules and direct products.*

Definition 2.2.5. A class C of modules is called a **pretorsion class** if it is closed under quotient modules and direct sums while it is called a **pretorsion-free class** if it is closed under submodules and direct products.

Thus, the preceding proposition gives that the classes $\mathcal{T}_r$ and $\mathcal{F}_r$ associated to a preradical r are pretorsion and pretorsion-free, respectively.

Conversely, if we take a pretorsion class C of right R-modules, there is a corresponding idempotent preradical t defined for all modules M by

$$t(M) = \sum_{N \subseteq M, N \in C} N,$$

where the summation is over all submodules N of M such that $N \in C$. Hence, every module M has a largest submodule $t(M)$ in C.

The above argument gives more precisely the following one-to-one correspondence.

Proposition 2.2.6. *(Stenström, 1975, Proposition 1.4) There is a bijective correspondence between idempotent preradicals of $\mathcal{M}od\text{-}R$ and pretorsion classes of modules of $\mathcal{M}od\text{-}R$.*

A pretorsion class is said to be **hereditary** if it is closed under submodules.

The next proposition enables to connect the left exactness of a preradical and the hereditary property.

Proposition 2.2.7. *(Stenström, 1975, Proposition 1.7) The following are equivalent for a preradical r:*

(1) r is a left exact functor.

(2) If $N \subseteq M$, then $r(N) = r(M) \cap N$.

(3) r is idempotent and $\mathcal{T}_r$ is closed under submodules.

Corollary 2.2.8. *(Stenström, 1975, Corollary 1.8) There is a bijective correspondence between left exact preradicals and hereditary pretorsion classes.*

Definition 2.2.9. A **torsion theory** for $\mathcal{M}od\text{-}R$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of right R-modules such that

(1) $\operatorname{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

(2) If $\operatorname{Hom}(C, F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$.

(3) If $\operatorname{Hom}(T, C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

Therefore, for a torsion theory $(\mathcal{T}, \mathcal{F})$ for $\mathcal{M}od\text{-}R$, we have that:

$$C \in \mathcal{T} \Leftrightarrow \operatorname{Hom}(C, F) = 0 \text{ for all } F \in \mathcal{F}$$

and

$$C \in \mathcal{F} \Leftrightarrow \operatorname{Hom}(T, C) = 0 \text{ for all } T \in \mathcal{T}.$$

The class $\mathcal{T}$ is called a **torsion class** and its elements are called **torsion modules**, whereas $\mathcal{F}$ is said to be a **torsion-free class** and its elements are called **torsion-free modules**.

A torsion theory generated by a given class $\mathcal{C}$ of right R-modules is obtained as follows:

$$\mathcal{F} = \{F \in \mathcal{M}od\text{-}R : \operatorname{Hom}(C, F) = 0 \text{ for all } C \in \mathcal{C}\}$$

and

$$\mathcal{T} = \{F \in \mathcal{M}od\text{-}R : \text{Hom}(T, F) = 0 \text{ for all } F \in \mathcal{F}\}.$$

In this case, $\mathcal{T}$ is the smallest torsion class containing C.

Proposition 2.2.10. *(Stenström, 1975, Proposition 2.1) The following properties of a class $\mathcal{T}$ of modules are equivalent:*

(1) $\mathcal{T}$ is a torsion class for some torsion theory,

(2) $\mathcal{T}$ is closed under quotient modules, direct sums and extensions.

Accordingly, if $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then in particular, $\mathcal{T}$ is a pretorsion class, and we can associate an idempotent preradical t by Proposition 2.2.6. In fact, a module M is torsion-free if and only if $t(M) = 0$. Furthermore, t is actually a radical. On the other hand, for a given idempotent radical t of $\mathcal{M}od\text{-}R$, the pretorsion class $\mathcal{T}_t$ we associate to t will be a torsion class. By these explanations, we state the following proposition:

Proposition 2.2.11. *(Stenström, 1975, Proposition 2.3) There is a bijective correspondence between torsion theories and idempotent radicals.*

Proposition 2.2.12. *(Stenström, 1975, Proposition 2.5) If C is a class of modules closed under quotient modules, then the torsion theory generated by C consists of all modules T such that each nonzero quotient module of T has a nonzero submodule in C.*

A torsion theory $(\mathcal{T}, \mathcal{F})$ is said to be **hereditary** if $\mathcal{T}$ is hereditary, i.e., $\mathcal{T}$ is closed under submodules.

By recalling Proposition 2.2.7, and combining Corollary 2.2.8 and Proposition 2.2.11, we obtain the next proposition:

Proposition 2.2.13. *Stenström (1975, Proposition 3.1) There is a bijective correspondence between hereditary torsion theories and left exact radicals.*

At this point, we shall continue a few more propositions which will be useful in Section 3.3.

Proposition 2.2.14. *(Stenström, 1975, Proposition 3.2) A torsion theory* $(\mathcal{T},\mathcal{F})$ *is hereditary if and only if* $\mathcal{F}$ *is closed under injective envelopes.*

Proposition 2.2.15. *(Stenström, 1975, Proposition 3.3) The torsion class* $\mathcal{T}$ *generated by a given class* C *of modules closed under submodules and quotient modules is hereditary.*

Proposition 2.2.16. *(Stenström, 1975, Proposition 3.6) A hereditary torsion theory is generated by the family of those cyclic modules* R/I *which are torsion modules, where* I *runs through right ideals of* R*.*

As a consequence, a hereditary torsion theory is uniquely determined by the family of right ideals I of R such that R/I is a torsion module; such a family of right ideals is a family of neighborhoods of 0 for a certain topology on R. Due to this cause, it may be suitable to remind some topological structures. A **topological ring** is a ring with a topology that makes it a topological group under addition and makes the multiplication $R \times R \to R$ a continuous map. The topology of R is characterized by the filter $\mathcal{S}$ of neighborhoods of 0 satisfying the following conditions (for the notion of *filter*, see Stenström (1975, p. 64)):

$(N1)$ For every $U \in \mathcal{S}$, there exists $V \in \mathcal{S}$ such that $V + V \subseteq$ U.

$(N2)$ If $U \in \mathcal{S}$, then $-U \in \mathcal{S}$.

$(N3)$ For every $r \in R$ and $U \in \mathcal{S}$, there exists $V \in \mathcal{S}$ such that $rV \subseteq U$ and $Vr \subseteq U$.

$(N4)$ For every $U \in \mathcal{S}$, there exists $V \in \mathcal{S}$ such that $VV \subseteq U$.

A **topological right *R*-module** can be defined in a similar manner (see Stenström (1975, p. 144)).

We pay attention to the topologies defined by ideals or submodules.

Definition 2.2.17. A topological ring R is **right linearly topological** if there is a basis of neighborhoods of 0 consisting of right ideals.

Recall that for a right ideal I and element r of a ring R, we may define the right ideal $(I : r)_r = \{x \in R : rx \in I\}$. Similarly, for a left ideal I, we have the left ideal $(I : r)_l = \{x \in R : xr \in I\}$.

If R is a right linearly topological ring, then the set $\mathcal{J}$ of all open right ideals satisfies:

$(T1)$ If $I \in \mathcal{J}$ and $I \subseteq J$, then $J \in \mathcal{J}$.

$(T2)$ If $I, J \in \mathcal{J}$, then $I \cap J \in \mathcal{J}$.

$(T3)$ If $I \in \mathcal{J}$ and $r \in R$, then $(I : r)_r = \{x \in R : rx \in I\} \in \mathcal{J}$.

Conversely, if $\mathcal{J}$ is a set of right ideals of R satisfying $(T1), (T2), (T3)$, then there is a unique right linear topology on R with $\mathcal{J}$ as a basis of neighborhoods of 0.

Likewise, a **linearly topological right R-module** M over a right linearly topological ring R with $\mathcal{J}$ as the set of all open right ideals can be defined (see Stenström (1975, p.144)). In particular, for any right R-module M, there is a strongest linear topology on M whose open submodules are given by

$$\mathcal{J}(M) = \{L \subseteq M : (L : m)_r \in \mathcal{J} \text{ for all } m \in M\}.$$

This topology is called **$\mathcal{J}$-topology** on M. Besides, a module M_R is a linearly topological module under its discrete topology, that is, the $\mathcal{J}$-topology of M is discrete if and only if the annihilator ideals

$$\text{ann}(m) = \{r \in R : mr = 0\} = (0 : m)_r \in \mathcal{J} \text{ for every } m \in M.$$

Also, M is said to be **$\mathcal{J}$-discrete** if the $\mathcal{J}$-topology of M is discrete.

Lemma 2.2.18. *(Stenström, 1975, Lemma 4.1) The class of $\mathcal{J}$-discrete modules is a hereditary pretorsion class.*

As a consequence of the above lemma, we can associate a left exact preradical t to the class of $\mathcal{J}$-discrete modules by Corollary 2.2.8.

Proposition 2.2.19 (Proposition 4.2)**.** *(Stenström, 1975) There is a bijective correspondence between:*

(1) Right linear topologies on R,

(2) Hereditary pretorsion classes of R-modules,

(3) Left exact preradicals of $\mathcal{M}$od-R.

It is benefical to stress that the hereditary pretorsion class we associate to a right linear topology $\mathcal{J}$ is

$$\{M_R : \text{ann}(m) \in \mathcal{J} \text{ for every } m \in M\},$$

that is, the class of $\mathcal{J}$-discrete modules, whereas the corresponding linear topology for a hereditary pretorsion class C is the following set of right ideals of R:

$$\{I_R \subseteq R_R : (R/I)_R \in C\}.$$

Adding the below new axiom ($T4$) to the list ($T1$), ($T2$), ($T3$), we obtain the definition of a (right) Gabriel topology:

Definition 2.2.20. A (right) **Gabriel topology** on a ring R is a family $\mathcal{J}$ of right ideals of R satisfying the following axioms $(T1) - (T4)$:

$(T1)$ If $I \in \mathcal{J}$ and $I \subseteq J$, then $J \in \mathcal{J}$.

$(T2)$ If $I, J \in \mathcal{J}$, then $I \cap J \in \mathcal{J}$.

$(T3)$ If $I \in \mathcal{J}$ and $r \in R$, then $(I : r)_r = \{x \in R : rx \in I\} \in \mathcal{J}$.

$(T4)$ If I is a right ideal of R and there exists $J \in \mathcal{J}$ such that $(I : j)_r = \{r \in R : jr \in I\} \in \mathcal{J}$ for every $j \in J$, then $I \in \mathcal{J}$.

Theorem 2.2.21. *(Stenström, 1975, Theorem 5.1) There is a bijective correspondence between:*

(1) Right Gabriel topologies on R,

(2) Hereditary torsion theories for R,

(3) Left exact radicals of $\mathcal{M}od$-R.

By the preceding theorem, we obtain that for a Gabriel topology $\mathcal{J}$, the corresponding hereditary torsion class consists of all modules which are discrete in their $\mathcal{J}$-topology. These modules are called $\mathcal{J}$-torsion modules.

The following serve as a useful tool.

Lemma 2.2.22. *(Stenström, 1975, Lemma 5.2) If $\mathcal{J}$ is a non-empty set of right ideals of R satisfying $(T3)$ and $(T4)$, then it also satisfies $(T1)$ and $(T2)$.*

Now, let us consider the class Top(R) of the topologies on R. We say τ_1 is weaker than τ_2, or equivalently, τ_2 is stronger than τ_1, if $\tau_1 \subseteq \tau_2$. Then, Top(R) forms a complete lattice. Moreover, every intersection of Gabriel topologies is again a Gabriel topology, and so there is a closure operator $\mathbf{J}$ on Top(R) which to each topology τ associates the weakest Gabriel topology $\mathbf{J}(\tau)$ stronger than τ.

Proposition 2.2.23. *(Stenström, 1975, Proposition 5.4) If τ is a topology, then the Gabriel topology $\mathbf{J}(\tau)$ is equal to the set*

$$\{I_R \subseteq R_R : \ \textit{for every} \ \ J \supseteq I, \ \ J \neq R, \ \ \textit{there exists} \ \ r \notin J \ \ \textit{such that} \ \ (J:r)_r \in \tau\}.$$

We shall note as an important remark that if τ is a topology, then the corresponding hereditary torsion class for the Gabriel topology $\mathbf{J}(\tau)$ is the class of modules generated by the class of τ-discrete modules.

CHAPTER THREE
ALMOST PERFECT RINGS

In this chapter, the notion of *a right almost perfect ring* will be introduced and some properties of this class of rings investigated by Facchini & Parolin (2011) will be explained in detail.

3.1 An Overview of Almost Perfect Rings

As we mentioned before, we call a ring R **right almost perfect** if R/I is a right perfect ring for every proper nonzero two-sided ideal I of R. One can define **left almost perfect rings** similarly. This class of rings generalizes right perfect rings, in other saying, right perfect rings are right almost perfect since quotient rings of right perfect rings are right perfect by Lam (2001, Corollary 24.19) or Anderson & Fuller (1992, Corollary 28.7).

To begin with, we can say that the notion of almost perfect rings is not left-right symmetric, in other words, there are left almost perfect rings that are not right almost perfect as the following example shows:

Example 3.1.1. (Facchini & Parolin, 2011, §3 Example (4)) Let k be a field and k_ω be the k-algebra of all matrices consisting of countably many rows and columns with entries in k such that each row has only finitely many nonzero entries. If we consider the set N of all strictly lower triangular matrices in k_ω with only finitely many nonzero entries, then $R = k + N$ forms a subalgebra of k_ω. In this case, $E_{ij} \in N$ if and only if $i > j$ where E_{ij} denotes the matrix units. Thus, the Jacobson radical $\mathrm{Jac}(R)$ of R is N. It is shown that R is left perfect but not right perfect in Bass (1960, p. 476). As a result, it is a left almost perfect ring. Now we claim that it is not right almost perfect. In order to show this, we take into account the principal ideal I of R generated by $E_{2,1}$. It can be seen that I is the vector space generated by all E_{i1} with $i \geq 2$. Then $I \subseteq \mathrm{Jac}(R)$ and so $\mathrm{Jac}(R/I) = \mathrm{Jac}(R)/I = N/I$. But, $\mathrm{Jac}(R/I) = N/I$ is not a right T-nilpotent ideal: For the sequence $\{E_{k,k-1}\}_{k=3}^{\infty} = \{\ldots, E_{5,4}, E_{4,3}, E_{3,2}\}$ of elements in N, the products $E_{n,n-1}E_{n-1,n-2} \cdots E_{4,3}E_{3,2} = E_{n,2}$ are not in I for every positive integer n.

So the quotient ring R/I is not right perfect which implies that R is not right almost perfect.

As a further information, there is no relation between almost perfect rings and semiperfect rings:

Example 3.1.2. (Facchini & Parolin, 2011, §3, Example (5)) The ring $\mathbb{Z}$ of integers is almost perfect inasmuch as every quotient ring of $\mathbb{Z}$ is artinian but it is not semiperfect, the reason is that the ring $\mathbb{Z}/\operatorname{Jac}(\mathbb{Z}) = \mathbb{Z}$ is not semisimple artinian, and so $\mathbb{Z}$ is not semilocal.

Example 3.1.3. (Facchini & Parolin, 2011, §3, Example (5)) A commutative valuation domain of Krull dimension ≥ 2 is semiperfect, but is not almost perfect.

Recall that if R is a commutative almost perfect ring which is not a domain, then R must be a perfect ring (see Proposition 1.4.3). As an analogue of this theorem, Facchini & Parolin (2011) shows that for non-prime rings, right almost perfect rings and right perfect rings coincide:

Theorem 3.1.4. *(Facchini & Parolin, 2011, Theorem 3.1) If R is right almost perfect and not a prime ring, then R is right perfect.*

Proof. With the purpose of showing that R is right perfect, the proof of the theorem separates into two cases depending on having nilpotent two-sided ideals or not.

First case: Suppose that R has a nonzero nilpotent two-sided ideal, that is, there exists a nonzero two-sided ideal J of R such that $J^n = 0$, where n is the smallest positive integer with this property. Therefore, with the hypothesis that R is not prime, it follows that there exists a nonzero two-sided ideal K with $K^2 = 0$. In particular, K is nilpotent, hence nil. By Proposition 2.1.6, we have $K \subseteq \operatorname{Jac}(R)$, and so $\operatorname{Jac}(R)$ is nonzero. Because R is right almost perfect, the quotient ring $R/\operatorname{Jac}(R)$ is right perfect, and by Theorem 2.1.20, $R/\operatorname{Jac}(R)$ is semisimple artinian. This gives us that R is semilocal. Now we can conclude that R has no infinite orthogonal set of idempotents by

Proposition 2.1.15. In order to obtain that R is right perfect by using Theorem 2.1.20-(6), we claim that R is left semiartinian, i.e., every nonzero left R-module contains a simple submodule. Take a nonzero left R-module M. Here, we have two observations. Firstly, if $KM = 0$, that is, $K \subseteq \text{ann}(M)$, then M is a left R/K-module. But R/K is right perfect since R is right almost perfect. Therefore R/K is left semiartinian which means every left R/K-module has a simple submodule. Consequently, M has a simple R/K-submodule which is also a simple R-module. On the other hand, if $KM \neq 0$, then KM is a nonzero left R/K-module. On the ground that R/K is right perfect, and so left semiartinian, the module KM has a simple R/K-submodule which is also a simple R-module. Thus, M has a simple R-submodule as required. Subsequently, R is left semiartinian.

Second case: This time we assume that R has no nonzero nilpotent two-sided ideal. Since R is not prime, there exist two-sided nonzero ideals I and J of R with $IJ = 0$. Then $(I \cap J)^2 \subseteq IJ = 0$ implies that $I \cap J = 0$ because R has no nonzero nilpotent two sided ideal. Our first claim is that R contains no infinite orthogonal set of idempotents. Suppose for the contrary that R contains an infinite orthogonal set E of idempotents in R. Let $E_I = \{e + I : e \in E\}$. Then by Theorem 2.1.20-(6),E_I must be finite due to the fact that R/I is right perfect. Therefore, there is a partition of E into finitely many subsets $E_1, E_2, \ldots, E_n$ with the property that for every $e, f \in E$; $e - f \in I$ if and only if e and f belong to same block partition E_i of the partition. But E is infinite, hence one of the blocks is infinite, say E_t. Thus E_t is an infinite orthogonal set of idempotents of R. If we take the set $E_{t,J} = \{e + J : e \in E_t\}$, it will be an orthogonal set of idempotents of R/J. As in the above argument, the set $E_{t,J}$ must be finite. Similarly, there is a partition of E_t into finitely many subsets $E_1^{'}, \ldots, E_m^{'}$ with the property that for every $e, f \in E_t$; $e - f \in J$ if and only if e and f belong to same block partition $E_j^{'}$ of the partition of E_t. Since E_t is infinite, one of these blocks must be infinite, say $E_l^{'}$. However, for every $e, f \in E_l^{'}$, we have $e - f \in J$ because e and f belong to the same block $E_l^{'}$. Also, $e - f \in I$ because both e and f are in E_t. Subsequently, $e - f \in I \cap J = 0$, that is, $e = f$ for every $e, f \in E_l^{'}$. It means that $E_l^{'}$ has just one element. This gives a contradiction with our assumption on $E_l^{'}$. Now we pose our second claim: R is left semiartinian. For the proof of this claim, let M be a left R-module. If $IM = 0$, then M has a simple R/I-submodule, hence a simple R-submodule. In the case $IM \neq 0$, IM has a module structure over the right perfect and so left semiartinian ring R/J. Thus, IM has a simple R/J-submodule, hence a simple R-module. As a result, M has a simple R-submodule in any case. □

By the above theorem, we understand that to investigate the structure of right almost perfect rings, we should only focus on the prime case, because otherwise the ring R will be a right perfect ring. For this reason, we state some properties about prime rings which will be useful for the following theorems in this chapter.

Lemma 3.1.5. *(Facchini & Parolin, 2011, Lemma 4.1) Every prime right perfect ring is a simple artinian ring.*

Proof. Since R is a right perfect ring, it satisfies the DCC on principal left ideals by Theorem 2.1.20. In view of Theorem 2.1.13 and by considering the fact that every prime ring is semiprime, we can conclude that R is semisimple artinian. Now we need to see that semisimple prime rings are simple. Suppose for the contrary that the ring R is not simple, i.e., R contains a nonzero two-sided ideal I properly. Since R is semisimple artinian, I_R is a direct summand of R_R, that is, $R_R = I_R \oplus B_R$ for some nonzero right ideal B of R. But R is prime, hence $BI \subseteq B \cap I = 0$ gives us $I = 0$ or $B = 0$ which contradicts with our assumption. Therefore R is simple artinian. □

Corollary 3.1.6. *(Facchini & Parolin, 2011, Corollary 4.2) A nonzero two-sided ideal of a right almost perfect ring R is a maximal ideal if and only if it is a prime ideal if and only if it is a right primitive ideal.*

Proof. We already know that every maximal ideal is prime. For the second implication, suppose that I is a nonzero prime ideal of R. Then R/I is right perfect. Besides, by Lemma 2.1.12, R/I is a prime ring. Consequently, by the above lemma, R/I is a simple artinian ring. Theorem 2.1.2 gives us that R/I is semisimple artinian and all simple modules over the ring R/I are isomorphic. For this unique simple right R/I-module, say M, $\mathrm{ann}_{R/I}(M)$ must be equal to zero as R/I is a simple ring. So $\mathrm{ann}_R(M) = I$. As a result, I is the right annihilator of the unique simple right R-module M which means I is a right primitive ideal. In an effort to see the last implication, i.e., to see that I is maximal whenever it is right primitive, we consider the ring R/I. Since R/I is a right perfect ring with a faithful simple right R/I-module, we have $\mathrm{Jac}(R/I) = \bigcap \mathrm{ann}(S) = 0$, where S ranges over all the simple right R/I-modules. It gives us that $(R/I)/(\mathrm{Jac}(R/I)) \cong R/I$ is semisimple artinian. It follows that R/I is simple artinian, otherwise $R/I = B_1 \oplus B_2 \cdots \oplus B_n$ for $n > 1$ where B_i's are homogeneous semisimple components of R/I. But it requires that $B_iB_j = 0$ and $B_j \subseteq \mathrm{ann}(B_i)$ for every $i \neq j$. It contradicts with R/I having a faithful simple right R/I-module. Thus n must be equal to 1. Therefore I is maximal. □

Now, we can give our attention to the notion of h-locality. Matlis (1964) gives the definition of this notion in the study of commutative domains whose torsion modules admit primary decompositions and this class of domains generalize Dedekind domains. Actually, the notion of h-locality had been studied first by Jaffard (1952) and then the h-local domain property was studied by many others for their own purposes; see Fontana, Houston, & Lucas (2012, §2.1) or Fuchs & Salce (2001, Chapter IV, §3).

Definition 3.1.7. A commutative domain R is said to be ***h*-local** if

(1) R is of *finite character*, that is, each nonzero proper ideal I of R is contained in at most finitely many maximal ideals of R.

(2) Every nonzero prime ideal of R is contained in only one maximal ideal of R.

It is worth to explain that in Definition 3.1.7 the condition (1) means that the quotient ring R/I is semilocal for every nonzero proper ideal I of R. Indeed, for a nonzero ideal $I \subseteq R$, there are at most finitely many maximal ideals $M_1, \ldots, M_n$ of R that contains I if and only if the ring R/I has finitely many maximal ideals $M_1/I, \ldots M_n/I$ if and only if R/I is semilocal by the commutative part of Proposition 2.1.14. Likewise, the condition (2) says that the ring R/P is local for every nonzero prime ideal P of R. The reason is that, for a prime ideal P of R, we know that there exists a unique maximal ideal M of R that contains P, that is, M/P is the unique maximal ideal in R/P if and only if R/P is local since R is commutative.

Facchini & Parolin (2011) extend this notion to general (noncommutative) rings in the following way:

Definition 3.1.8. A ring R is said to be ***h*-local** if

(1) The ring R/I is semilocal for every proper nonzero two-sided ideal I of R.

(2) Every nonzero prime two-sided ideal of R is contained in a unique maximal two-sided ideal of R.

Moreover, we can easily see that local rings are h-local as follows. Suppose that R is a local ring. Then R has a unique maximal ideal, namely $\text{Jac}(R)$, and every two-sided proper ideal I of R is contained in $\text{Jac}(R)$. Then $(R/I)/\text{Jac}(R/I) = (R/I)/(\text{Jac}(R)/I) \cong R/\text{Jac}(R)$ by Proposition 2.1.7, and $R/\text{Jac}(R)$ is semisimple artinian. Hence R/I is semilocal. Also, for every prime ideal P of R, we have $P \subseteq \text{Jac}(R)$.

We immediately deduce the following corollary:

Corollary 3.1.9. *(Facchini & Parolin, 2011, Corollary 4.3) Every right almost perfect ring is h-local.*

Proof. It follows from Theorem 2.1.20 that R/I is semilocal for every nonzero prime ideal I and from Corollary 3.1.6 that each prime ideal is contained in a unique maximal ideal P. □

We conclude this section by collecting the discussion appearing up to present. In general, for an arbitrary ring R, one has two exclusive cases according to the ideal 0 is maximal or not. The case that the zero ideal is maximal corresponds to simple rings. If R is not simple, that is, if 0 is not a maximal ideal in R, again we can separate into cases whether R is semilocal or not. On the ground of these reasons, Facchini & Parolin (2011) indicate that right almost perfect rings belong to exactly one of the following three classes and give the conditions for a ring R to belong to these classes:

Theorem 3.1.10. *(Facchini & Parolin, 2011, p. 201) Right almost perfect rings belong to exactly one of the following three classes.*

(1) First class: Simple rings.

(2) Second class: Semilocal non-simple right almost perfect rings. A ring R belongs to this second class if and only if R/P is simple artinian for every nonzero prime ideal $P \subseteq R$ and every nonzero non-faithful left R-module is semiartinian.

(3) Third class: Non-simple, non-semilocal right almost perfect rings. A ring R belongs to this third class, that is, R is not simple, not semilocal, and right almost perfect if and only if $\mathrm{Jac}(R) = 0$*, every nonzero element of R belongs to only finitely many maximal ideals of R, the ring R/P is simple artinian for every nonzero prime ideal ideal P of R and every nonzero non-faithful left R-module is semiartinian.*

3.2 Examples of Almost Perfect Rings

In this section, we would like to collect a few more examples of *right or left almost perfect rings* in order to make the notion more concrete and understandable. This section is based on Facchini & Parolin (2011, §3 and §5).

Example 3.2.1. *Simple rings*. Since simple rings have no proper nonzero two sided ideals, they are trivially both right and left almost perfect.

Example 3.2.2. *Nearly simple chain rings*. In the first place, let us consider the intersection of all nonzero two-sided ideals of a ring R and call it A. Then the ideal A is either the zero ideal or the least nonzero two-sided ideal of R. Now, suppose that this second case holds, that is, R has a least nonzero two-sided ideal, A. In this situation we have the following observation: R is right almost perfect if and only if the quotient ring R/A is right perfect. It can be seen straightforwardly by observing that for every nonzero two-sided ideal I of R, A is contained in I and $R/I \cong (R/A)/(I/A)$. As an example, take into account the case where R has exactly three two-sided ideals: 0, R and I. Then R is right almost perfect if and only if the simple ring R/I is right perfect. However, for simple rings we have that it is right perfect if and only if it is simple artinian since semisimplicity implies being artinian. As a consequence, we obtain that R is right almost perfect if and only if R/I is simple artinian, where R is a ring with exactly three ideals. Particularly, such a ring R is right almost perfect if and only if it is left almost perfect. An example of these rings with three ideals is given by the nearly simple chain rings.

Definition 3.2.3. A **nearly simple chain ring** R is a noncommutative right and left chain ring (i.e., right ideals and left ideals are linearly ordered), with exactly three two-sided ideals: 0, R and Jac(R).

On the ground that chain rings are local, $R/\operatorname{Jac}(R)$ is a division ring, hence $R/\operatorname{Jac}(R)$ is semisimple which means artinian, and so both left and right perfect. Therefore, nearly simple chain rings are right and left almost perfect. Note that nearly simple chain rings belong to the second class in Theorem 3.1.10.

Example 3.2.4. *Von Neumann regular rings that are right V-rings but not left V-rings.* For a field k and an infinite-dimensional vector space V_k, if we take into account the endomorphism ring $\operatorname{End}(V_k)$ and its two-sided ideal S consisting of endomorphisms of finite rank, then we obtain that the k-subalgebra $R = k + S$ which has just three two-sided ideals: 0, R and S. Also, R is prime and clearly right and left almost perfect. Moreover, the Jacobson radical Jac(R) is 0. Otherwise, $\operatorname{Jac}(R) = S$ which is impossible due to the fact that we have $1 - E_{11}$ is not invertible (where $E_{11} \in S$), and so $E_{11} \notin \operatorname{Jac}(R)$. This fact brings about that R is not semilocal. Consequently, this class of rings constitutes an example for the third class of right almost perfect rings that we explained in Theorem 3.1.10. Further, R is not right perfect and is not left perfect.

3.3 Main Results by Facchini

After we met the notion of (noncommutative) *right almost perfect rings*, we turned our attention to the properties of commutative almost perfect domains (see Section 1.4) that still hold in the noncommutative setting; we present them in this section from Facchini & Parolin (2011).

In the work of extending the characterizations and results to the noncommutative case, at first, we need the notion of *torsion module* over a noncommutative ring. We know that for each ring of fractions of a ring R, there is a notion of torsion for R-modules which corresponds to the ring of fractions. However, the example of a nearly simple chain domain R (hence an example of right and left almost perfect domain) which is also an Ore domain given by Puninski (2001) indicates that the conditions of Theorem 1.4.4 do not hold if we consider the torsion modules as those in which every element is annihilated by a nonzero element of R and Q is the classical ring of fractions of R, which is a division ring. In this torsion theory, we obtain a nonzero torsion module whose socle is zero (for the example and explanations, see Facchini & Parolin (2011, p. 195–196)). For these reasons, Facchini & Parolin (2011) uses a different torsion theory as explained below.

Firstly, let us remind the following well-known notions:

Definition 3.3.1. A submodule N of a module M is said to be **essential** in M if for every submodule $L \subseteq M$, $N \cap L = 0$ implies $L = 0$.

Definition 3.3.2. Let I be a *one-sided* ideal of a ring R. The **core** of I, denoted by $\mathrm{core}(I)$, is the largest *two-sided* ideal of R contained in I. Observe that $\mathrm{core}(I)$ is the annihilator in R of the right R-module $(R/I)_R$ for a right ideal I of R. Similarly, this holds for left ideals of R.

Lemma 3.3.3. *(Facchini & Parolin, 2011, Lemma 4.4) In a prime ring R, every right ideal with a nonzero core is essential in the right R-module R.*

Proof. Assume that I is a right ideal with $\mathrm{core}(I) \neq 0$. For a nonzero right ideal J of R, we know that $J\,\mathrm{core}(I) \neq 0$ since R is prime. The inclusions $J\,\mathrm{core}(I) \subseteq \mathrm{core}(I) \cap J \subseteq I \cap J$ show that the ideals I and J have nonzero intersection. Thus, I is essential in R_R. □

On any *prime* ring R, we can define a *natural topology*, which is right and left linearly topological, by taking the set $\mathcal{B}$ of all nonzero two-sided ideals of R as a basis of neighborhoods of zero. The set $\mathcal{D}$ of left ideals of R that are open in this topology satisfies the conditions $(T1), (T2)$ and $(T3)$ that we state in Section 2.2. We have

$$\mathcal{D} = \{{}_RI \subseteq_R R : \text{ there exists } K \in \mathcal{B} \text{ such that } K \subseteq I\} = \{{}_RI \subseteq_R R : \mathrm{core}(I) \neq 0\},$$

and so $(T1)$ is trivial. For $K_1 \subseteq I$ and $K_2 \subseteq J$ where $I, J \in \mathcal{D}$, it comes that $K_1K_2 \subseteq I \cap J$, and so $(T2)$ holds. $(T3)$ can be seen as follows: For $I \in \mathcal{D}$ and $r \in R$, clearly

$$K \subseteq (I : r)_l = \{x \in R : xr \in I\} \in \mathcal{D},$$

where K is some ideal in $\mathcal{B}$ chosen such that $K \subseteq I$ by the definition of $\mathcal{D}$. As Facchini & Parolin (2011, p. 197) point out, $(T4)$ does not hold for this topology in general, and so this left linear topology does not always form a left Gabriel topology.

For the next step, consider the class $\mathcal{P}$ of all left R-modules whose elements are annihilated by an element of $\mathcal{B}$, that is, the class of $\mathcal{D}$-discrete modules,

$$\mathcal{P} = \{{}_RM : \quad \text{for every} \;\; m \in M, \text{there exisits} \;\; K \in \mathcal{B} \;\; \text{such that} \;\; Km = 0 \;\; \}.$$

It is clearly closed under quotient modules, i.e., under homomorphic images and also under submodules. In order to see that it is closed under direct sums, take a collection $\{M_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}$. Let $m \in \bigoplus_{\lambda\in\Lambda} M_\lambda$. Then $m = m_1 + \cdots + m_n$ for some $m_i \in M_{\lambda_i}$ and $\lambda_i \in \Lambda$ for $i = 1, 2, \ldots, n$. If we take $K = K_1 \cdots K_n$, where each K_i belongs to $\mathcal{B}$ and $K_i m_i = 0$ for $i = 1, 2, \ldots, n$, then since R is prime, K is nonzero. Also, $K \subseteq K_i$ since each of $K_1, \ldots, K_n$ are two-sided ideals. So, we have $Km_i \subseteq K_i m_i = 0$ for each $i = 1, \ldots, n$, and so $Km = 0$. Thus, $\mathcal{P}$ is a hereditary pretorsion class. Therefore, by Proposition 2.2.12, the torsion theory $\mathcal{T}$ generated by $\mathcal{P}$ consists of all left R-modules T such that each nonzero quotient module of T has a nonzero submodule contained in $\mathcal{P}$. Namely,

$$\mathcal{T} = \{T : T/N \;\; \text{has a nonzero submodule contained in} \;\; \mathcal{P} \;\; \text{for every} \;\; N \subsetneq T\}.$$

Also, $\mathcal{T}$ is hereditary by Proposition 2.2.15, and $\mathcal{T}$ is the smallest torsion class that contains $\mathcal{P}$. On the other hand, the torsion theory $\mathcal{T}$ generated by $\mathcal{P}$ corresponds to the left Gabriel topology $\mathbf{J}(\mathcal{D})$ by the last paragraph of Page 30, and we have the following by Proposition 2.2.23:

$$\mathbf{J}(\mathcal{D}) = \{{}_RI \subseteq_R R : \text{for every} \;\; J \supseteq I, \;\; J \neq R, \;\; \text{there exist} \;\; r \notin J \;\; \text{such that} \;\; \text{core}(J : r)_l \neq 0\}.$$

Here, by the arguments in Section 2.2, $\mathbf{J}(\mathcal{D})$ is the weakest Gabriel topology stronger than $\mathcal{D}$, and so $\mathcal{B} \subseteq \mathcal{D} \subseteq \mathbf{J}(\mathcal{D})$. In addition, in this torsion theory, the module ${}_RR$ is torsion-free. Suppose for the contrary that it is not. Then, $t({}_RR) \neq 0$ for the left exact radical t corresponding to this torsion class $\mathcal{T}$. It requires that there exists a nonzero left ideal I of R contained in $\mathcal{T}$. So for a nonzero $r \in I$, the cyclic submodule Rr is torsion since $\mathcal{T}$ is hereditary. Then Rr contains a nonzero submodule K in $\mathcal{P}$, that is, for all $x \in K \subseteq Rr$, there exists $I \in \mathcal{B}$ such that $Ix = 0$, which is a contradiction since R is prime.

In the rest of this and the next chapter, whenever we say a torsion module over a prime ring, we mean a module in $\mathcal{T}$. We follow the proofs of Facchini & Parolin (2011) by giving detailed explanations.

The following proposition is a noncommutative analogue of Theorem 1.4.4.

Proposition 3.3.4. *(Facchini & Parolin, 2011, Proposition 4.5) The following conditions are equivalent for a prime ring R:*

(1) Every nonzero torsion left R-module contains a simple submodule,

(2) Every torsion left R-module is semiartinian,

(3) R/I is a left semiartinian ring for every nonzero proper two-sided ideal I of R,

(4) For every proper left ideal L with a nonzero core, the cyclic left R-module R/L contains a simple submodule.

Proof. (1) $\Rightarrow$ (2): Let M be a torsion left R-module. By Proposition 2.2.10, $\mathcal{T}$ is closed under quotient modules, i.e., homomorphic images. So every nonzero homomorphic image of M is torsion. The condition (1) says that every homomorphic image contains a simple submodule.

(2) $\Longrightarrow$ (3): Assume that I is a nonzero proper two-sided ideal of R. We claim that R/I is a left semiartinian ring, that is, every nonzero left R/I-module has a simple submodule. In order to show this, let M be a nonzero left R/I-module. We can view M as a left R-module whose elements are annihilated by I, which is an element of $\mathcal{B}$.

Therefore ${}_RM \in \mathcal{P} \subseteq \mathcal{T}$, hence M is a nonzero torsion left R-module, and so semiartinian by (2). It gives us that every left R/I-module is semiartinian.

(3) $\Longrightarrow$ (4): Suppose that L is a left ideal of R with $\operatorname{core}(L) \neq 0$. (3) gives that the quotient ring $R/\operatorname{core}(L)$ is a left semiartinian ring, hence we obtain the left $R/\operatorname{core}(L)$-module R/L is semiartinian. But it is also semiartinian as a left R-module which means that it contains a simple R-submodule.

(4) $\Longrightarrow$ (1) : Let M be a nonzero torsion left R-module, i.e., $M \in \mathcal{T}$. If M is simple, then it is trivial, so assume that M is not a simple module. Then there exists a nonzero submodule N of M such that $N \in \mathcal{P}$ by Proposition 2.2.12. It means that for every $n \in N$, there exist $I \in \mathcal{B}$ such that $In = 0$. Now, set $L = \operatorname{ann}(n)$. So L is a proper left ideal of R containing I. In particular, L has a nonzero core. Therefore, by (4), the cyclic left R-module $R/L \cong Rn$ contains a simple submodule which means ${}_RM$ contains a simple submodule. □

The next theorem gives the noncommutative analogue of Theorem 1.4.5.

Theorem 3.3.5. *(Facchini & Parolin, 2011, Theorem 4.6) For a prime ring R, the following statements are equivalent:*

(1) The ring R is right almost perfect,

(2) The ring R is h-local and satisfies one of the equivalent conditions of Proposition 3.3.4.

Proof. (1) $\Rightarrow$ (2). Suppose that R is a right almost perfect ring. By Corollary 3.1.9, R is h-local. For the remaining part, we will show that it satisfies the third condition of Proposition 3.3.4. Let I be a nonzero proper two-sided ideal of R. Then the ring R/I is right perfect, hence left semiartinian by Theorem 2.1.20.

(2) $\Rightarrow$ (1). Let R be a h-local ring, and assume that every torsion left R-module is semiartinian. Every left R-module M with a nonzero annihilator is torsion. Because $\operatorname{ann}(M) \in \mathcal{B}$, so clearly $M \in \mathcal{P}$ which gives $M \in \mathcal{T}$. Let I be a nonzero proper two-sided ideal of R. We claim that R/I is right perfect. On the ground that every nonzero

left R/I-module is a left R-module annihilated by the nonzero proper ideal I, every left R/I-module is semiartinian. Thus, the ring R/I is left semiartinian. Besides, R/I is semilocal as R is h-local. By Proposition 2.1.15, R/I contains no infinite set of orthogonal idempotents. We conclude that R/I is right perfect by Theorem 2.1.20-(6). Hence R is right almost perfect. □

The following corollary is a noncommutative analogue of Corollary 1.4.6.

Corollary 3.3.6. *(Facchini & Parolin, 2011, Corollary 4.7) Let R be a prime local ring. Then, R is right almost perfect if and only if R satisfies one of the equivalent conditions of Proposition 3.3.4.*

Proof. First part by Theorem 3.3.5. Converse part is obtained by the fact that local rings are h-local and again by Theorem 3.3.5. □

We shall state another characterization of right almost perfect rings by Facchini & Parolin (2011).

Proposition 3.3.7. *(Facchini & Parolin, 2011, Proposition 4.8) A ring R is right almost perfect if and only if it satisfies the following conditions:*

(1) R is h-local,

(2) Every nonzero right R-module M_R with $\mathrm{Rad}(M) = M$ *is faithful.*

Proof. (⇒): The condition (1) is satisfied by Corollary 3.1.9. In order to see (2), let M be a nonzero right R-module with $\mathrm{Rad}(M_R) = M_R$. Suppose for the contrary that it is not faithful, i.e., the two-sided ideal $I = \mathrm{ann}(M)$ is nonzero. Also, since the quotient ring R/I is a right perfect ring by the hypothesis, it is also a right max ring by Theorem 2.1.20-(2). But it means that every nonzero right R/I-module has a maximal submodule, and so $\mathrm{Rad}(M_{R/I}) \neq M_{R/I}$. It contradicts with $\mathrm{Rad}(M_R) = M_R$, so $I := \mathrm{ann}(M)$ must be zero.

(⇐): Suppose that (1) and (2) hold. Let I be a nonzero proper two-sided ideal of R. Since R is h-local, the factor ring R/I is semilocal. Now, we claim that R/I is right max. For the proof of the claim, let M be a nonzero right R/I-module. Then M, viewed as a right R-module, is not faithful because $I \subseteq \mathrm{ann}(M_R)$. So, by the condition (2), $\mathrm{Rad}(M_R) \neq M_R$, that is, M_R has a maximal submodule, as well $M_{R/I}$. Therefore, R/I is right perfect by Theorem 2.1.20-(2). □

Facchini & Parolin (2011) also consider the *left noetherian* case for prime rings and obtain stronger results for right almost perfect rings. Before we state this stronger characterization, we shall explain the advantages of being left noetherian for a ring R.

Lemma 3.3.8. *(Facchini & Parolin, 2011, p. 199) Let R be a left noetherian ring. Then, R is right perfect if and only if R is left artinian.*

Proof. If R is right perfect, then R is semilocal and $\mathrm{Jac}(R)$ is right T-nilpotent. By Proposition 2.1.3, $\mathrm{Jac}(R)$ is also nilpotent. Then, by Theorem 2.1.4, we conclude that R is left artinian. Conversely, if R is left artinian, it is also semilocal. Also, $\mathrm{Jac}(R)$ is nilpotent, hence right T-nilpotent. □

In the case R is a left noetherian prime ring, the set $\mathcal{B}$ of nonzero two-sided ideals of R becomes a basis for a left Gabriel topology, so that the left Gabriel topology $\mathbf{J}(\mathcal{D})$ turns out to be the linear topology $\mathcal{D}$, and $\mathcal{P} = \mathcal{T}$.

We are ready to state the following characterization for left noetherian rings.

Theorem 3.3.9. *(Facchini & Parolin, 2011, Proposition 5.1) The following conditions are equivalent for a left noetherian prime ring R:*

(1) Every nonzero torsion left R-module contains a simple submodule,

(2) Every torsion left R-module is semiartinian,

(3) The ring R/I is a left semiartinian ring for every nonzero proper two-sided ideal I of R,

(4) The ring R/I is left artinian for every nonzero proper two-sided ideal I of R,

(5) R is right almost perfect.

Proof. (1) ⇔ (2) ⇔ (3) has been shown in Proposition 3.3.4.

(3) ⇒ (4): Since R is left noetherian, the left Gabriel topology corresponding to semiartinian modules consists of the left ideals A of R with R/A of finite length. If I is two-sided and R/I is left semiartinian, then $1+I$ is annihilated by I, hence I belongs to the left Gabriel topology corresponding to the semiartinian modules, and so R/I is a left R-module of finite length. Therefore, R/I is left artinian.

(4) ⇒ (5): By Lemma 3.3.8, R/I is right perfect for every nonzero proper two-sided ideal I of R.

(5) ⇒ (1): Let M be a nonzero torsion left R-module. We claim that ${}_RM$ contains a simple submodule. Take a nonzero element m of M. Since $M \in \mathcal{T} = \mathcal{P}$, every element of M is annihilated by an element of $\mathcal{B}$. So, there exists $I \in \mathcal{B}$ such that $Im = 0$. Thus, $I \subseteq \mathrm{ann}(Rm)$. Moreover, the ring R/I is left semiartinian since it is right perfect. Then the module ${}_{R/I}(Rm)$ contains a simple R/I-submodule, and so a simple R-submodule. □

CHAPTER FOUR
ALMOST PERFECT RINGS AND $C_{\mathcal{J}}$-RINGS

In this chapter, our main object is to state and prove the relations between almost perfect rings and C-rings of Renault (1964), and as well as $C_{\mathcal{J}}$-rings of Generalov (1978). In an attempt to declare the relations between these classes of rings, we divided the investigation into two cases. First, we state the results related with C-rings and commutative C-domains. Secondly, we show the relation concerning about $C_{\mathcal{J}}$-rings. Before anything else, we shall start the sections by presenting the concepts of C-rings and of $C_{\mathcal{J}}$-rings.

4.1 C-rings of Renault

Definition 4.1.1. (Renault, 1964) A ring R is said to be **right C-ring**, if for every right R-module M and for every essential proper submodule N of M, $\mathrm{Soc}(M/N) \neq 0$, that is, the quotient module M/N has a simple submodule.

Proposition 4.1.2. *(Renault, 1964, Proposition 1.2) A ring R is right C-ring if and only if for every essential right ideal I of R,* $\mathrm{Soc}(R/I) \neq 0$.

A module M_R is said to be **singular** if its every element is singular, that is, for every $m \in M$, $\mathrm{ann}(m)$ is essential in R_R. Also, we have the following proposition that will provide an equivalence for the notion of C-ring.

Proposition 4.1.3. *(Lam, 1999, p. 269) Let N be a submodule of the R-module M, where M is a free R-module. Then, the quotient module M/N is singular if and only if N is essential in M.*

Since R is a free right module over itself and every cyclic R-module is isomorphic to the right R-module R/I for some right ideal I of R, right C-rings are also defined to be as those rings such that every cyclic singular right R-module has a nonzero socle.

A submodule N of a right R-module M is said to be **neat** if any simple module S is projective relative to the projection $M \to M/N$. Besides, a submodule N of M is said to be **closed** in M if it has no essential extension in M, or equivalently, if there exists a submodule N' of M such that N is a **complement** of N' in M, that is, N is maximal with respect to the property $N \cap N' = 0$. For every module M and submodule N of M, if N is a closed submodule of M, then N is a neat submodule of M. The converse does not hold and indeed that characterizes C-rings. That is, C-rings are characterized as the rings over which every neat submodule is a closed submodule. Similarly, left C-rings are defined. For all of the above argument, see Generalov (1978), Mermut (2004, §3.3) and Clark, Lomp, Vanaja, & Wisbauer (2006, §10).

A module M is called **max-injective** if for every maximal right ideal I of R, every homomorphism $I \to M$ can be extended to a homomorphism $R \to M$. A module M is max-injective if and only if $\mathrm{Ext}(S, M) = 0$ for every simple module S (for details see for example Özdemir (2011, §4.2)). Smith (1981, Lemma 4) proved that R is a right C-ring if and only if every max-injective right R-module is injective (without using the C-ring terminology).

By combining all of these results, we obtain a list of characterizations for right C-rings as follows:

Proposition 4.1.4. *(Hatipoğlu, 2014, Proposition 3.3) The following are equivalent for a ring R and if one of the equivalent conditions is satisfied, then the ring R is said to be a right C-ring:*

(1) For every right R-module M and for every essential proper submodule N of M, $\mathrm{Soc}(M/N) \neq 0$*,*

(2) For every essential right ideal I of R, $\mathrm{Soc}(R/I) \neq 0$*,*

(3) Every singular module is semiartinian,

(4) For every cyclic singular module M, $\mathrm{Soc}(M) \neq 0$*,*

(5) For every right R-module M, all neat submodules of M are closed,

(6) Every max-injective right R-module is injective.

Clearly right semiartinian rings are right C-rings. Another example of right C-rings are left perfect rings since left perfect rings are right semiartinian. Commutative noetherian rings whose nonzero prime ideals are maximal are C-rings (see Mermut (2004, Proposition 3.3.6)). In particular, commutative artinian rings and Dedekind domains are such C-rings.

Recall that in a prime ring R, every right ideal with a nonzero core is essential in the right R-module R_R (see Lemma 3.3.3).

Proposition 4.1.5. *If R is a prime left C-ring, then it satisfies the last condition of Proposition 3.3.4, that is, for every proper left ideal I of R with a nonzero core, the cyclic left R-module R/I contains a simple submodule.*

Proof. Let I be a proper left ideal of R with $\text{core}(I) \neq 0$. By Lemma 3.3.3, I is essential in the left R-module R. Since R is a left C-ring, the left R-module R/I contains a simple submodule. So, R satisfies condition (4) in Proposition 3.3.4. □

Corollary 4.1.6. *For a prime ring R, if R is h-local and a left C-ring, then it is a right almost perfect ring.*

Proof. Since R is a left C-ring, it satisfies one of the equivalent conditions in Proposition 3.3.4 by Proposition 4.1.5. Thus, R is right almost perfect by Theorem 3.3.5. □

Corollary 4.1.7. *For a prime local ring R, if R is a left C-ring, then it is right almost perfect.*

Corollary 4.1.8. *For a left noetherian prime ring R, if R is a left C-ring, then it is right almost perfect.*

Proof. If R is a left C-ring, then every nonzero torsion left R-module (in the sense of Facchini) contains a simple submodule by Proposition 4.1.5. Hence, R is right almost perfect by Theorem 3.3.9. □

Recall that a ring R is said to be **right bounded** if every essential right ideal of R contains an ideal which is essential as a right ideal. Left bounded rings are defined similarly. Note that a prime ring R is right bounded if and only if every essential right ideal I of R contains a nonzero ideal, that is, $\operatorname{core}(I) \neq 0$ (see Goodearl & Warfield Jr (2004, p. 156)).

Proposition 4.1.9. *For a prime left bounded ring R, R is right almost perfect if and only if R is h-local and a left C-ring.*

Proof. Since R is a prime left bounded ring, we have $\operatorname{core}(I) \neq 0$ for every essential left ideal I of R. Thus, R/I contains a simple submodule for every essential left ideal I of R since R is right almost perfect. This means that R is a left C-ring. Converse part holds by Corollary 4.1.6. □

Another situation that we obtain equivalence is the commutative case. For commutative domains, we have:

Proposition 4.1.10. *(Mermut, 2004, Proposition 3.3.9) A commutative domain R is a C-ring if and only if every nonzero torsion module has a simple submodule.*

Corollary 4.1.11. *For a commutative domain R, R is almost perfect if and only if R is h-local and R is a C-ring.*

Proof. Straightforward by Proposition 4.1.10 and Theorem 1.4.5. □

Corollary 4.1.12. *For a local commutative domain R, R is almost perfect if and only if R is a C-ring.*

4.2 $C_{\mathcal{J}}$-rings of Generalov

Definition 4.2.1. (Generalov, 1978) Let $\mathcal{J}$ be a set of right ideals of a ring R. A submodule N of a module M is said to be **$\mathcal{J}$-pure** if for any commutative diagram of the form

$$\begin{array}{ccc} I & \xrightarrow{\ i\ } & R \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow{\ j\ } & M \end{array}$$

where $I \in \mathcal{J}$, i and j are injections, and f and g are arbitrary homomorphisms, there exists a homomorphism $h : R \to N$ such that $hi = f$.

Definition 4.2.2. Let $\mathcal{J}$ be a set of right ideals of a ring R. A submodule N of a module M is called **$\mathcal{J}$-dense** if for every $m \in M$ the right ideal $(N : m)_r = \{r \in R : mr \in N\}$ belongs to $\mathcal{J}$ and $m(N : m)_r \neq 0$. In this case, the module M is said to be an **essential $\mathcal{J}$-extension** of N.

Definition 4.2.3. A submodule N of a module M is called a **weakly $\mathcal{J}$-pure** submodule if it does not have any proper essential $\mathcal{J}$-extension in M, that is, if N is $\mathcal{J}$-dense in $K \subseteq M$, then $N = K$.

Let us observe the following conditions that we may want the set $\mathcal{J}$ of right ideals of a ring R to satisfy.

$(F1)$ If $I \in \mathcal{J}$ and the right ideal J contains I, then $J \in \mathcal{J}$.

$(F2)$ If $I \in \mathcal{J}$ and $r \in R$, then $(I : r)_r \in \mathcal{J}$.

$(F3)$ If a right ideal I is contained in $J \in \mathcal{J}$ and $(I : r)_r \in \mathcal{J}$ for every $r \in J$, then $I \in \mathcal{J}$.

If the set $\mathcal{J}$ of right ideals of R satisfies all three conditions, then it is said to be a **radical filter**.

Lemma 4.2.4. *A set $\mathcal{J}$ of right ideals is a Gabriel topology if and only if it is a radical filter.*

Proof. $(\Rightarrow)$ part is clear. Conversely, suppose that $\mathcal{J}$ is a radical filter. $(T3)$ comes from $(F2)$ easily. For $(T4)$, suppose that I is a right ideal and there exists a right ideal $J \in \mathcal{J}$ such that $(I : r)_r \in \mathcal{J}$ for every $r \in J$. We aim to obtain $I \in \mathcal{J}$. Since $I \cap J$ is a right ideal that is contained in J and $(I \cap J : r)_r = (I : r)_r \in \mathcal{J}$ for every $r \in J$, we obtain by $(F3)$ that the right ideal $I \cap J$ belongs to $\mathcal{J}$. Also, $(F1)$ implies that $I \in \mathcal{J}$. Thus, it is a Gabriel topology by Lemma 2.2.22. □

From now on, let us suppose that $\mathcal{J}$ satisfies $(F1)$ unless otherwise stated.

Definition 4.2.5. Let $\mathcal{J}$ be a set of right ideals of a ring R. The ring R is said to be a **right $C_{\mathcal{J}}$-ring** if for any proper $\mathcal{J}$-dense right ideal I of R, there exists an element

$r \in R$ such that $(I : r)_r$ is a right maximal ideal. Left $C_{\mathcal{J}}$-rings for a set $\mathcal{J}$ of left ideals of R are defined similarly.

$C_{\mathcal{J}}$-rings generalizes the concept of C-rings. If $\mathcal{J}$ is the set of all right ideals of R, then being a right $C_{\mathcal{J}}$-ring means just being a right C-ring.

We can state our main theorem:

Theorem 4.2.6. *Let R be a prime ring and $\mathcal{J} = \mathbf{J}(\mathcal{D})$ the left Gabriel topology in the sense of Facchini. Then, R is a left $C_{\mathcal{J}}$-ring if and only if it satisfies the following condition which is one of the equivalent conditions of Proposition 3.3.4 for prime rings: For every proper left ideal I of R with a nonzero core, the cyclic left R-module R/I contains a simple submodule.*

Proof. Suppose that R is a left $C_{\mathcal{J}}$-ring. We will show that the condition (4) of Proposition 3.3.4 holds. Let I be a proper left ideal of R with $\mathrm{core}(I) \neq 0$. Since the core of I is nonzero, I is essential in the left R-module $_RR$ by Lemma 3.3.3, which means $(I : r)_l r \neq 0$ for every $0 \neq r \in R$. Also, $\mathrm{core}(I)$ is a two-sided ideal, so it is contained in $\mathcal{B} \subseteq \mathcal{D} \subseteq \mathbf{J}(\mathcal{D})$. By $(T2)$, $I \in \mathbf{J}(\mathcal{D})$. Moreover, for every nonzero $r \in R$, $\mathrm{core}(I) \subseteq (I : r)_l$, and so the left ideal $(I : r)_l$ is in $\mathbf{J}(\mathcal{D})$. So, I is $\mathcal{J}$-dense in $_RR$. Due to the fact that R is a left $C_{\mathcal{J}}$ ring, there exists $r \in R$ such that $(I : r)_l$ is a maximal left ideal, that is, $R/(I : r)_l \cong R(r + I)$ is a simple left R-module in the left R-module R/I.

For the converse part, assume (4) of Proposition 3.3.4 holds. Let I be a $\mathcal{J}$-dense left ideal of R, i.e., for every $r \in R$ and for the element $r + I \in R/I$, $(I : r)_l = (0 : r + I)_l = \mathrm{ann}(r + I) \in \mathcal{J}$ and $(I : r)_l r \neq 0$. Recall that the torsion class corresponding to the Gabriel topology $\mathcal{J} = \mathbf{J}(\mathcal{D})$ consists of left R-modules M such that $\mathrm{ann}(m) \in \mathbf{J}(\mathcal{D})$ for every $m \in M$. Thus, the left R-module R/I is torsion. By Proposition 2.2.12, R/I contains a nonzero submodule $N = U/I$ in $\mathcal{P}$. Then, for a nonzero $x = u + I \in N$, there exists $K \in \mathcal{B}$ such that $Kx = 0$. But, $K \subseteq (I : u)$, that is, $K \subseteq \mathrm{ann}(x)$. It follows that $\mathrm{core}(\mathrm{ann}(x)) \neq 0$. By assumption $R/\mathrm{ann}(x)$ contains a simple submodule, and then $R/\mathrm{ann}(x) \cong Rx \subseteq R/I$ implies that R/I also contains a simple submodule. □

We end the section with immediate corollaries of Proposition 4.2.6.

Corollary 4.2.7. *For a prime ring R, R is right almost perfect if and only if R is h-local and R is a left $C_{\mathcal{J}}$-ring, where $\mathcal{J} = \boldsymbol{J}(\mathcal{D})$ is the left Gabriel topology in the sense of Facchini.*

Corollary 4.2.8. *For a local prime ring R, R is right almost perfect if and only if R is a left $C_{\mathcal{J}}$-ring, where $\mathcal{J} = \boldsymbol{J}(\mathcal{D})$ is the left Gabriel topology in the sense of Facchini.*

CHAPTER FIVE
CONCLUSION

In this thesis, we summarized and explained the motivation of the notion of right almost perfect rings from the beginning. We investigated the work of Facchini and Parolin and explained it by giving much more details. Furthermore, we remind the notion of C-rings of Renault and $C_{\mathcal{J}}$-rings of Generalov. Finally, we pose their relation with commutative almost perfect domains and right almost perfect rings.

In the study of right almost perfect rings, there is still some characterizations and properties of commutative almost perfect domains that we do not have noncommutative analogues such as in Bazzoni & Salce (2002). The motivation to investigate commutative almost perfect domains was to consider strongly flat covers. So, as a major object, it can be dealt with the problem of generalizing the following theorem proved by Bazzoni and Salce to noncommutative setting: 'For a commutative domain R; every R-module has a strongly flat cover if and only if every flat R-module is strongly flat if and only if R is almost perfect.' Namely, the next step may be to find a class of rings so that we can define the notion of strongly flat modules coinciding with its commutative counterpart. For this reason, it is necessary to introduce an analogue of the field fractions Q of a commutative domain for an arbitrary prime ring. So, the question is: for which class of prime rings R, its maximal quotient ring becomes a flat R-module and can we define the notion of strongly flat module over it? Facchini suggests for a prime ring R that the candidate for Q is the ring of quotients $\varinjlim \mathrm{Hom}({}_R I, {}_R R)$, where the direct limit is over ${}_R I \in \mathbf{J}(\mathcal{D})$, of the prime ring R with respect to the Gabriel topology $\mathbf{J}(\mathcal{D})$. The first attempt for this may be to understand the case for semiprime right Goldie rings. Because in this case, its maximal ring of quotients will be a flat module over the ring.

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APPENDICES

NOTATIONS

R	an associative ring with unit unless otherwise stated
M	a right R-module unless otherwise stated
$M \oplus N$	the direct sum of modules M and N
$\mathcal{M}od\text{-}R$	the category of right R-modules
$R\text{-}\mathcal{M}od$	the category of left R-modules
$\mathrm{Obj}\,C$	the class of objects for a category C
$\mathrm{Mor}(A,B)$	the set of all morphisms from an object A to a object B in a category
$\mathrm{Ext}(M,N)$	All the equivalence classes of short exact sequences starting with the R-module N and ending with the R-module M
$\mathbb{Z}$	the ring of integers
$\cong$	isomorphic
Ker	the kernel of the map f
Im	the image of the map f
$\mathrm{ann}(M)$	the annihilator of the R-module M
$\mathrm{ann}(m)$	the annihilator of an element m of the R-module M
$\mathrm{Soc}(M)$	the socle of the R-module M
$\mathrm{Rad}(M)$	the radical of the R-module M
ACC	ascending chain condition
DCC	descending chain condition
$\mathrm{Jac}(R)$	the Jacobson radical of the ring R
$\mathrm{core}(I)$	the core of the one-sided ideal I
$\mathbf{J}(\tau)$	the Gabriel topology for the topology τ

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