DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON THE SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS

by
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İZMİR

ON THE SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS

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We have read the thesis entitled **"ON THE SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS"** completed by **SETENAY AKDUMAN** under supervision of **ASSOC. PROF. DR. SEDEF KARAKILIÇ** and **PROF. DR. ALEXANDER PANKOV** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

The time independent Schrödinger operator is one of the fundamental operator in quantum physics.

In this thesis, firstly, we obtain asymptotic formulas of arbitrary order for the eigenvalues of the multidimensional Schrödinger operator with a matrix potential and the Neumann boundary condition, when the corresponding eigenvalue of the unperturbed operator is near the diffraction plane.

Secondly, we introduce a detailed analysis of the spectral properties of Schrödinger operators with non-regular potentials on infinite metric graphs such as a characterization of the bottom of essential spectrum, the discreteness of the negative part of the spectrum and of the whole spectrum, exponential decay of eigenfuctions. Here we suppose that the potential is locally integrable and its negative part is bounded in certain integral sense.

Keywords: Schrödinger operator, matrix potential, Neumann condition, perturbation, asymptotic formulas, metric graph, spectrum, eigenspace of Schrödinger operators, exponential decay.

SCHRÖDINGER OPERATÖRLERİNİN SPEKTRAL ÖZELLİKLERİ ÜZERİNE

ÖZ

Zamandan bağımsız Schrödinger operatörü kuantum fiziğinin temel operatörlerinden biridir.

Bu tezde ilk olarak, Neumann sınır koşulları ile tanımlanan matris potansiyelli çok boyutlu Schrödinger operatörünün özdeğerleri için keyfi dereceden asimptotik formüller elde edilmiştir. Bu kısımda, özdeğerlerin kırınım düzlemine yakın olduğu varsayılmıştır.

İkinci olarak ise, sonsuz metrik grafikleri üzerinde tanımlı düzenli olmayan potansiyele sahip Schrödinger operatörünün; esaslı spektrumunun alt sınırının karakterizasyonu, spektrumunun negatif kısmının ve tüm spektrumunun diskritliği, özvektörlerin üstel azalması gibi spektral özelliklerinin detaylı bir analizi yapılmıştır. Bu kısımda, potansiyelin lokal olarak integrallenebilir olduğu ve potansiyelin negatif kısmının integral anlamında sınırlı olduğu varsayılmıştır.

Anahtar kelimeler: Schrödinger operatörü, matris potensiyeli, Neumann koşulu, pertürbasyon, asimptotik formüller, metrik grafiği, spektrum, Schrödinger operatörünün özuzayı, üstel azalma.

CONTENTS

CHAPTER ONE
INTRODUCTION

The spectral theory of operators in a finite dimensional space first appeared in the study of frequencies of small vibrations of mechanical systems. If the vibrations of a string is in consideration, then an eigenvalue problem for a differential operator arises. For instance, for an inhomogeneous string, it is necessary to consider the general Sturm-Liouville problem with variable coefficients. Finally, study of vibrations of a membrane or a three dimensional elastic body leads to the eigenvalue problems for multidimensional differential operators.

One of the richest source of the spectral theory is the quantum physics and most of the theory is dedicated to the Schrödinger operator $L(V)$ defined by the differential expression

$$L(V)u(x) = (-\boldsymbol{\Delta} + V(x))u(x)$$

which is a fundamental operator of quantum physics. The Schrödinger operator can be considered as the energy operator of one or several particles depending on the form of the potential $V(x)$. According to the fundamental principles of quantum physics, the possible values of the energy of a particle belong to the spectrum of the Schrödinger operator and eigenfunctions describe the state of the particle.

This thesis includes two independent studies on the spectral theory of Schrödinger operators.

The first study, which is the subject of Chapter Two, is on the Schrödinger operator whose potential is a real-valued, symmetric matrix V. In the sequel, we denote this operator by $L(V)$. More precisely, $L(V)$ is defined by

$$L(u(x)) = (-\boldsymbol{\Delta} + V(x))u(x) \tag{1.1}$$

and the Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial F} = 0, \tag{1.2}$$

in $L_2^m(F)$ where F is the d-dimensional rectangle $F = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_d]$, $a_1, a_2, \ldots, a_d$ are arbitrary real numbers, ∂F is the boundary of F, $m \geqslant 2, d \geqslant 2$, $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal of the boundary of F, $\mathbf{\Delta}$ is a diagonal $m \times m$ matrix whose diagonal elements are the scalar Laplace operators $\Delta = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \cdots + \frac{\partial^2}{\partial {x_d}^2}$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, V is a real-valued symmetric matrix $V(x) = (v_{ij}(x)), i, j = 1, 2, \ldots, m, v_{ij}(x) \in L_2(F)$, that is, $V^T(x) = V(x)$.

We denote the eigenvalue and eigenfunction pairs of $L(V)$ by Λ_N and ψ_N, respectively.

In Chapter Two, we obtain asymptotic formulas for the eigenvalue Λ_N of $L(V)$ when the corresponding eigenvalue of the unperturbed operator $L(0)$, which is defined by (1.1) when $V(x) = 0$ and the boundary condition (1.2), is roughly speaking, near diffraction plane.

In the second study, which covers Chapters Three, we consider Schrödinger operators with non-regular potentials on infinite metric graphs. The potentials are supposed to be locally integrable with negative part bounded in certain integral sense. Defining a self-adjoint Schrödinger operator, we start with a second-order symmetric differential operator

$$L_0 u = -\frac{d^2 u}{dx^2} + V(x)u$$

on the domain that consists of sufficiently smooth compactly supported functions satisfying the Kirchhoff conditions at the vertices of a metric graph Γ.

In Chapter Three, first we show that the Friedrichs extension, L, of L_0 is the only self-adjoint extention of L_0. Our next result is a characterization of the bottom of essential spectrum. In the rest of Chapter Three, we begin with a sufficient condition for the discreteness of the negative part of spectrum. Then we obtain a necessary and sufficient condition for the discreteness of whole spectrum. Finally, we show that, under natural assumptions, eigenfunctions corresponding to isolated eigenvalues of finite multiplicity decay at infinity exponentially fast.

Now, in Section 1.1 and Section 1.2, we give the literature surveys, fundamental definitions and facts related with our first and second studies, respectively.

1.1 Introduction to the Perturbation Theory of the Schrödinger Operator with a Matrix Potential

In this section, we give a brief discussion of what is known from the literature and what is presented in this thesis about the perturbation theory of the multidimensional Schrödinger operator with a matrix potential.

As the eigenvalue problem of the operator $L(V)$ defined by (1.1) and (1.2), most of the problems related with spectral theory fail to be explicitly soluble, they need a qualitative and asymptotic study.

In this direction, perturbation theory which was created by Rayleigh and Schrödinger is an important tool in the spectral theory of linear differential operators. The main problem is to seek an approximate solution of the eigenvalue problem for a linear operator slightly different from a simplier one for which the problem is completely solved. More precisely, for the Schrödinger operator $L(V)$, it is essential to know how the eigenvalues of the unperturbed operator $L(0)$ is affected under perturbation.

The most significant progress has been achieved in one dimensional case. The crucial property in analysis of the problem in one dimensional case is that the distance between the consecutive eigenvalues (which occurs in the denominator of the perturbation series) becomes larger and larger so that the perturbation theory can be applied to obtain the asymptotic formulas for sufficiently large eigenvalues.

For physical applications, it is important to have a perturbation theory of the Schrödinger operator in many dimensional cases because of the fact that the Hilbert space for N particles in $\mathbb{R}^d$ is $L_2(\mathbb{R}^{N\cdot d})$. However, in many dimensional case, (even in two or three dimensions), the problem is considerably difficult. In this case, to construct a perturbation theory turns out to be rather difficult, because of the denseness of the eigenvalues of the free operator which are situated very close to each other in a high energy region. Therefore, when perturbation disturbs them, they strongly influence each other. This presents considerable difficulties as the arbitrarily small differences become small divisors in an asymptotic expansion, in particular, "the small denominators problem". Thus, to describe the perturbation of one of the eigenvalues, we must also study all the other surrounding eigenvalues.

In order to overcome this difficulty, for the first time in papers (Veliev, 1987, 2006, 2007, 2015), the eigenvalues of the unperturbed operator $L(0)$ is divided into two groups: Non-resonance and resonance ones. In these papers, various asymptotic formulas were obtained for the perturbations of each group.

Now to give the precise definitions of these groups, we first introduce the following notations:

The eigenvalues and the eigenfunctions of the unperturbed operator $L(0)$:

The eigenvalues of the operator $L(0)$ which is defined by (1.1) when $V(x) = 0$ and the boundary condition (1.2) are $|\gamma|^2$ and the corresponding eigenspaces are

$$E_\gamma = \text{span}\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \ldots, \Phi_{\gamma,m}(x)\},$$

where $\gamma \in \frac{\Gamma^{+0}}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \ldots, \frac{n_d\pi}{a_d}) : n_k \in \mathbb{Z}^+ \cup \{0\}, k = 1, 2, \ldots, d\}$,
$\Phi_{\gamma,j}(x) = (0, \ldots, 0, u_\gamma(x), 0, \ldots, 0)$, $j = 1, 2, \ldots, m$,
$u_\gamma(x) = \cos\frac{n_1\pi}{a_1}x_1 \cos\frac{n_2\pi}{a_2}x_2 \cdots \cos\frac{n_d\pi}{a_d}x_d$, $u_0(x) = 1$ when $\gamma = (0, 0, \ldots, 0)$. The non-zero component $u_\gamma(x)$ is in the j-th component.

It can be easily calculated that the norm of $u_\gamma(x)$, $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^d) \in \frac{\Gamma+0}{2}$ in $L_2(F)$ is $\sqrt{\frac{\mu(F)}{|A_\gamma|}}$, where $\mu(F)$ is the measure of the d-dimensional rectangle F, $|A_\gamma|$ is the number of vectors in

$$A_\gamma = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \frac{\Gamma}{2} : |\alpha_k| = |\gamma^k|,\ k = 1, 2, \ldots, d \right\},$$

$$\frac{\Gamma}{2} = \left\{ \left(\frac{n_1 \pi}{a_1}, \frac{n_2 \pi}{a_2}, \ldots, \frac{n_d \pi}{a_d} \right) : n_k \in \mathbb{Z}, k = 1, 2, \ldots, d \right\}.$$

Equivalently,

$$\|u_\gamma(x)\| = \sqrt{\frac{a_1 a_2 \cdots a_d}{2^{d-s}}},$$

where s, $(0 \leq s \leq d)$, is the number of components γ^k of $(\gamma^1, \gamma^2, \ldots, \gamma^d)$ such that $\gamma^k = 0$.

Since $\{u_\gamma(x)\}_{\gamma \in \frac{\Gamma+0}{2}}$ is a complete system in $L_2(F)$, for any $q(x)$ in $L_2(F)$ we have

$$q(x) = \sum_{\gamma \in \frac{\Gamma+0}{2}} \frac{|A_\gamma|}{\mu(F)} (q, u_\gamma) u_\gamma(x), \tag{1.3}$$

where $(\cdot, \cdot)$ is the inner product in $L_2(F)$.

In this thesis, it is convenient to use the equivalent decomposition (see Karakılıç, Atılgan et al. (2005))

$$q(x) = \sum_{\gamma \in \frac{\Gamma}{2}} q_\gamma u_\gamma(x), \tag{1.4}$$

where $q_\gamma = \frac{1}{\mu(F)} (q(x), u_\gamma(x))$ for the sake of simplicity. That is, the decomposition (1.3) and (1.4) are equivalent for any $d \geq 1$.

Since $v_{ij}(x) \in L_2(F)$ by (1.4), it has the following decomposition

$$v_{ij}(x) = \sum_{\gamma \in \frac{\Gamma}{2}} v_{ij\gamma} u_\gamma(x) \tag{1.5}$$

for $i, j = 1, 2, \ldots, m$ where $v_{ij\gamma} = \dfrac{(v_{ij}, u_\gamma)}{\mu(F)}$.

Throughout Chapter Two, which is devoted to our first study, we have the following assumption:

Assumption on the potential $V(x)$:

We assume that the Fourier coefficients $v_{ij\gamma}$ of $v_{ij}(x)$ satisfy

$$\sum_{\gamma\in\frac{\Gamma}{2}} |v_{ij\gamma}|^2(1+|\gamma|^{2l}) < \infty \tag{1.6}$$

for each $i,j=1,2,\ldots,m,\quad l > \frac{(d+20)(d-1)}{2}+d+3$ which implies

$$v_{ij}(x) = \sum_{\gamma\in\Gamma^{+0}(\rho^\alpha)} v_{ij\gamma}u_\gamma(x) + O(\rho^{-p\alpha}), \tag{1.7}$$

where $\Gamma^{+0}(\rho^\alpha)=\{\gamma\in\frac{\Gamma}{2}: 0\le|\gamma|<\rho^\alpha\}$, $p=l-d$, $\alpha<\frac{1}{d+20}$, ρ is a large parameter and $O(\rho^{-p\alpha})$ is a function in $L_2(F)$ with norm of order $\rho^{-p\alpha}$.

Indeed, we have

$$\begin{aligned}
&\left\|\sum_{\gamma\in\frac{\Gamma}{2}\setminus\Gamma^{+0}(\rho^\alpha)} v_{ij\gamma}u_\gamma\right\|^2 = \left\|\sum_{|\gamma|>\rho^\alpha} v_{ij\gamma}u_\gamma\right\|^2 = \sum_{|\gamma|>\rho^\alpha}|v_{ij\gamma}|^2\|u_\gamma\|^2\\
&=\sum_{|\gamma|>\rho^\alpha}|v_{ij\gamma}|^2\,\frac{a_1a_2\cdots a_d}{2^{d-s}} \le a_1a_2\cdots a_d\sum_{|\gamma|>\rho^\alpha}|v_{ij\gamma}|^2 = a_1a_2\cdots a_d\sum_{|\gamma|>\rho^\alpha}\left[\frac{|v_{ij\gamma}||\gamma|^l}{|\gamma|^l}\right]^2\\
&\le a_1a_2\cdots a_d\left[\sum_{|\gamma|>\rho^\alpha}\frac{|v_{ij\gamma}||\gamma|^l}{|\gamma|^l}\right]^2 \le a_1a_2\cdots a_d\left(\sum_{|\gamma|>\rho^\alpha}|v_{ij\gamma}|^2|\gamma|^{2l}\right)\left(\sum_{|\gamma|>\rho^\alpha}\frac{1}{|\gamma|^{2l}}\right).
\end{aligned}$$

The first sum in the last expression is convergent by (1.6). The second sum is big-oh of $\rho^{-p\alpha}$ by using the integral test. Thus, (1.7) holds .

Moreover, by (1.6), we have

$$M_{ij}\equiv\sum_{\gamma\in\frac{\Gamma}{2}}|v_{ij\gamma}|<\infty,\ \text{ for all }\ i,j=1,2,\ldots,m. \tag{1.8}$$

Actually,

$$\sum_{\gamma\in\frac{\Gamma}{2}}|v_{ij\gamma}| = \sum_{\gamma\in\frac{\Gamma}{2}}\frac{|v_{ij\gamma}||\gamma|^l}{|\gamma|^l} < \left(\sum_{\gamma\in\frac{\Gamma}{2}}|v_{ij\gamma}|^2|\gamma|^{2l}\right)^{\frac{1}{2}}\left(\sum_{\gamma\in\frac{\Gamma}{2}}\frac{1}{|\gamma|^{2l}}\right)^{\frac{1}{2}} < \infty.$$

As noted in Hald & McLaughlin (1996), $q(x)$ satisfies (1.6) if $q(x) \in W_2^l(F)$, for sufficiently large l and support of the gradient of q is in the interior of F. Also if $q(x) \in W_2^l(F)$ is a periodic function with respect to a lattice $\Omega = \{(m_1a_1, m_2a_2, \ldots, m_da_d) : m_k \in \mathbb{Z}, k = 1, 2, \ldots, d\}$ then it also satisfies condition (1.6).

Resonance and Non-Resonance Domains: *(For more detailed analysis of these domains, see Veliev (2015))*

Definition 1.1.1. *Let ρ be a large parameter, $\alpha < \frac{1}{d+20}$, $\alpha_k = 3^k\alpha$, $k = 1, 2, \ldots, d-1$ and*

$$V_b(\rho^{\alpha_1}) \equiv \left\{x \in \mathbb{R}^d : ||x|^2 - |x+b|^2| < \rho^{\alpha_1}\right\},$$

$$E_1(\rho^{\alpha_1}, p) \equiv \bigcup_{b\in\Gamma(p\rho^\alpha)} V_b(\rho^{\alpha_1}),$$

$$U(\rho^{\alpha_1}, p) \equiv \mathbb{R}^d \setminus E_1(\rho^{\alpha_1}, p),$$

$$E_k(\rho^{\alpha_k}, p) = \bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_k\in\Gamma(p\rho^\alpha)}\left(\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})\right),$$

where $\Gamma(p\rho^\alpha) \equiv \left\{b \in \frac{\Gamma}{2} : 0 < |b| < p\rho^\alpha\right\}$ and the intersection $\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ in E_k is taken over $\gamma_1, \gamma_2, \ldots, \gamma_k$ which are linearly independent vectors and the length of γ_i is not greater than the length of the other vector in $\Gamma\cap\gamma_i\mathbb{R}$. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain*, and the eigenvalue $|\gamma|^2$ is called a* non-resonance eigenvalue *if $\gamma \in U(\rho^{\alpha_1}, p)$. The domains $V_b(\rho^{\alpha_1})$, for $b \in \Gamma(p\rho^\alpha)$ are called* resonance domains *and the eigenvalue $|\gamma|^2$ is a* resonance eigenvalue *if $\gamma \in V_b(\rho^{\alpha_1})$.*

The domain $V_b(\rho^{\alpha_1}) \setminus E_2$, i.e., the part of resonance domain $V_b(\rho^{\alpha_1})$, which does not contain the intersections of two resonance domains is called a *single resonance domain*.

As noted in Veliev (2006), Veliev (2007) and Veliev (2015), the single resonance domain $V_b(\rho^{\alpha_1}) \setminus E_2$ has asymptotically full measure on $V_b(\rho^{\alpha_1})$, that is, if

$$2\alpha_2 - \alpha_1 + (d+3)\alpha < 1 \text{ and } \alpha_2 > 2\alpha_1 \tag{1.9}$$

hold, then

$$\frac{\mu\left(\left(V_b(\rho^{\alpha_1}) \setminus E_2\right) \cap B(\rho)\right)}{\mu\left(V_b(\rho^{\alpha_1}) \cap B(\rho)\right)} \to 1, \text{ as } \rho \to \infty,$$

where $B(\rho) = \left\{x \in \mathbb{R}^d : |x| \leqslant \rho\right\}$, for a large parameter $\rho \gg 1$ and $E_2 = \bigcup\limits_{\gamma_1, \gamma_2 \in \Gamma(p\rho^{\alpha})} (V_{\gamma_1}(\rho^{\alpha_2}) \cap V_{\gamma_2}(\rho^{\alpha_2}))$. Since $\alpha < \frac{1}{d+20}$, the conditions in (1.9) hold.

How the eigenvalues $|\gamma|^2$ of the unperturbed operator $L(0)$ is affected under perturbation is an important problem. We study this problem by using energy as a large parameter, in other words when $|\gamma| \sim \rho$, that is, there exist positive constants c_1, c_2 such that $c_1\rho < |\gamma| < c_2\rho$, c_1, c_2 do not depend on ρ and ρ is a big parameter.

For the scalar case, $m = 1$, non-resonance and resonance domains were first introduced in Veliev (1987) and more recently in Veliev (2015). Corresponding to each group he obtained various asymptotic formulas for the eigenvalues of the periodic Schrödinger operator with quasiperiodic boundary conditions in an arbitrary dimension $d \geq 2$. In Feldman et al. (1991), Friedlander (1990), Karpeshina (1996) and Karpeshina (2002), the authors obtained asymptotic formulas for quasiperiodic boundary conditions in two and three dimensions. Hald & McLaughlin (1996) obtained asymptotic formulas for Dirichlet boundary condition in two dimensions. In Atılgan et al. (2002), Karakılıç, Atılgan et al. (2005) and Karakılıç, Veliev et al. (2005), the authors obtained asymptotic formulas for the eigenvalues of the Schrödinger operator with Dirichlet and Neumann boundary conditions in an arbitrary dimension.

For the matrix case, asymptotic formulas for the eigenvalues of the Schrödinger operator with quasiperiodic boundary conditions are obtained in Karpeshina (2002). In Coşkan & Karakılıç (2011), in an arbitrary dimension, the asymptotic formulas of arbitrary order for the eigenvalue of the operator $L(V)$ which corresponds to the non-resonance eigenvalue $|\gamma|^2$ of $L(0)$ are obtained.

In Chapter Two, we obtain the high energy asymptotics of arbitrary order in an arbitrary dimension $(d \geq 2)$ for the eigenvalue of $L(V)$ corresponding to resonance eigenvalue $|\gamma|^2$ when γ belongs to the single resonance domain, that is, $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, where δ is from $\{e_1, e_2, \ldots, e_d\}$ and $e_1 = \Big(\frac{\pi}{a_1}, 0, \ldots, 0\Big)$, $e_2 = \Big(0, \frac{\pi}{a_2}, \ldots, 0\Big), \ldots, e_d = \Big(0, 0, \ldots, \frac{\pi}{a_d}\Big)$.

1.2 Introduction to Schrödinger Operators on Infinite Metric Graphs

This section has a survey nature about quantum graphs and is devoted to the essential ingredients of the proofs which are presented in Chapter Three.

The name "quantum graph" refers to a graph considered as a one-dimensional simplicial complex and equipped with a differential operator ("Hamiltonian"). Such objects naturally arise as simplified models in mathematics, physics, chemistry and engineering, when one considers wave propagation through a quasi-one-dimensional system that looks like a thin neighborhood of a graph. The works on quantum graph theory and their applications have brought together tools and intuition coming from graph theory, combinatorics, mathematical physics, PDEs, and spectral theory.

In paper Kuchment (2004), one can find some basic notions and results concerning quantum graphs and their spectra. In Kuchment (2005), a continuation of Kuchment (2004), some combinatorial spectral results are considered too. While combinatorial spectral graph theory has been initiated quite long time ago, the theory of quantum graphs is not developed well enough. A contemporary presentation of the subject in a monograph form is given in Berkolaiko & Kuchment (2013).

Now, we introduce the main players of the quantum graph theory: metric graphs and differential operators on them.

Metric Graphs:

Let $\Gamma = (E, V)$ be an undirected graph with the set of edges E and the set of vertices V. Multiple edges and loops are allowed. At the same time we assume that the graph is connected in the sense that any two vertices are terminal vertices of at least one path of edges. Recall that the *degree* d_v of a vertex $v \in V$ is the number of edges emanating from v. We assume that all the vertices of the graph Γ have finite and positive degrees. For any vertex $v \in V$ we denote by E_v the set of edges adjacent to v.

The graph Γ is said to be a *metric graph* if each edge e is identified with an $[0, l_e]$ of the real line. The endpoints of the interval correspond to the vertices of the edge. The induced coordinate on the edge e is denoted by x_e though we often skip the index e in this notation. The distance $d(x, y)$ between two points x and y in Γ is defined as the length of a shortest path that connects these points. Since the graph is connected, the distance is well defined. Fixing an arbitrary vertex $o \in V$, we set

$$d(x) = d(o, x)\,.$$

In addition, there is a natural measure, dx, on Γ which coincides with the Lebesgue measure on each edge. In particular, integration over Γ makes sense.

In Chapter Three, we accept the following assumptions:

(i) The sets of edges and vertices are countably infinite;

(ii) There exist two positive constants $\underline{l}$ and $\bar{l}$ such that

$$\underline{l} \le l_e \le \bar{l}$$

for all $e \in E$.

Assumption (i) means that the graph Γ is a non-compact metric space. Considering metric graphs, many authors allow infinite edges with the vertex of degree 1 at infinity. However, such an edge can be replaced by an infinite chain of edges each of which has a fixed length. Therefore, this case reduces to the case when Assumption (ii) is satisfied. This assumption is imposed for a convenience only.

Schrödinger Operators on Infinite Metric Graphs:

Differential equations on metric graphs (networks) is a relatively new area of mathematical research though the first publication in which equations of such type appear is paper Kirchhoff (1847). Last decades demonstrate a great progress in this area as well as in its applications. Various aspects of differential equations and operators on metric graphs are reflected in monographs Berkolaiko & Kuchment (2013), Blank et al. (2008), Mehmeti et al. (2001), Pokornyi et al. (2005) and survey articles Von Below & Mugnolo (2013), Kuchment (2002, 2004, 2005), Pokornyi & Pryadiev (2004) (see also references therein). A substantial part of this activity, known under the name "Quantum Graphs", is dealing with the spectral theory of Schrödinger and other quantum mechanical operators on metric graphs (see Berkolaiko & Kuchment (2013), Blank et al. (2008), Kuchment (2004, 2005) and references therein).

In Chapter Three, to define a self-adjoint Schrödinger operator, we start with a second-order symmetric differential operator

$$L_0 u = -\frac{d^2 u}{dx^2} + V(x)u$$

on the domain that consists of sufficiently smooth compactly supported functions satisfying the Kirchhoff conditions at the vertices of a metric graph Γ.

Assumptions on the potentials:

In Chapter Three, the potentials are supposed to be locally integrable with negative part bounded in certain integral sense (see, assumptions (V1) and (V2)). In the case of operators on real line, these assumptions turn into the assumption that the potential is of local Kato class, while its negative part is of Kato class (see, e.g., Cycon et al. (2009), Simon (1982)). Under our assumptions, L_0 is a symmetric, bounded below operator in the space $L^2(\Gamma)$.

The Friedrichs extension, L, of L_0 is the object of Chapter Three, where first we show that L is the only self-adjoint extension of L_0. In other words, the operator L_0 is essentially self-adjoint. Actually, we introduce the maximal extension, $\tilde{L}$, of L_0 and prove that it coincides with the operator L. To achieve this aim, we are using an appropriate variation of the method of Agmon (1985) based on an estimate of L^2-norm of a function u in terms of L^2-norm of $\tilde{L}u$. Then we obtain a characterization of the bottom of essential spectrum in terms of the Rayleigh quotient similar to the well-known Persson's theorem for Schrödinger operators on $\mathbb{R}^d$ (see, e.g., Cycon et al. (2009)). Being of interest in its own, this is a key tool in our next result devoted to the discreteness of spectrum. We begin with a sufficient condition for the discreteness of the negative part of spectrum which is similar to an old result Birman (1959) (see also Glazman (1965)) for Schrödinger operators on real line. Then we obtain a necessary and sufficient condition for the discreteness of whole spectrum. This is a generalization of a classical result by Molchanov (1953) (see also Glazman (1965)) on one-dimensional Schrödinger operators. The condition we provide is similar to that in the Molchanov's result and means that the potential growths to infinity at infinity in an integral sense. Recently, another result of Molchanov's type is published in Kovaleva & Popov (2015). In that paper it is assumed that the potential is continuous and bounded below. Under this assumption those authors prove the following sufficient condition: if the potential tends to infinity at infinity, then the spectrum is discrete. In this sense that result is weaker than one obtained in Chapter Three. On the other hand, Kovaleva & Popov (2015) deals with slightly more general vertex conditions allowing δ-interaction at vertices. However, we would like to point out that our approach extends to this case without any difficulty.

While the theory of Schrödinger operators on the Euclidean space is currently well-developed, the theory of quantum graphs, i.e., Schrödinger type operators on metric graphs, is relatively new, and many important problems in this area are still open. Most of results obtained so far concern the case when the potential is sufficiently regular. However, as it is well-known the potential represents external force field which often has singularities. Due to this fact, in Chapter Three, we study Schrödinger operators with locally integrable potentials on infinite metric graphs. Such potentials form a sufficiently wide class and allow many important singularities.

A well-known fact in the theory of Schrödinger operators on the Euclidean space is that, under natural assumptions, eigenfunctions corresponding to isolated eigenvalues of finite multiplicity decay at infinity exponentially fast (see, e.g., Simon (1982)). From the point of view of quantum mechanics this means that bound states of a quantum system are well-localized in space. A natural conjecture is that a similar statement holds true on metric graphs.

In the theory of Schrödinger operators on $\mathbb{R}^d$ there are several ways to study exponential decay of eigenfunctions. Commonly known of them are discussed in Agmon (2014), Agmon (1985), Bardos & Merigot (1977), Hislop & Sigal (1996), Reed & Simon (1975), Simon (1982). All these methods are not appropriate in the case of quantum graphs because they heavily relay on either smooth structure, or linear one, of the underlying Euclidean space. Also we mention a geometric method suggested in Agmon (2014), Agmon (1985). However, this method applies only in the case of eigenvalues below the essential spectrum. We intend to employ another approach which is typically used to obtain estimates of Green's functions. In case of eigenfunctions it is used in Kurbatov (2012).

In the rest of Chapter Three, we prove that, under natural assumptions, eigenfunctions corresponding to isolated eigenvalues of finite multiplicity decay at infinity exponentially fast. Our approach to obtain exponential decay of eigenfunctions on metric graphs is an adaptation of that in Kurbatov (2012). The key point is to study the resolvent of “twisted” operators defined in terms of certain function η on the graph. This will be done by means of analytic functional calculus. The function η has to be chosen so that the twisted operator is well-defined and twisting preserves the Kirchhoff vertex conditions. Therefore, η must be sufficiently regular. In particular, η has to be smooth at interior points of the edges and satisfy the Kirchhoff conditions at the vertices. On the other hand, the function η has to allow us to control the distance function on the graph. The best choice would be to use a function η which coincides with the distance function. However, this is not possible because the distance function is only continuous and does not satisfy the vertex condition (3.3). Actually, we use as η an appropriate regularization of the distance function.

CHAPTER TWO
ASYMPTOTIC FORMULAS FOR THE SINGLE RESONANCE EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH A MATRIX POTENTIAL

Schrödinger operators are inexhaustible as mathematics itself. Indeed, the method of approach to the spectral theory of these operators may differ according to the properties of its potential as well as its domain which is described by the boundary conditions.

By this reason, as in papers Veliev (1987), Veliev (2006), Veliev (2007), Veliev (2015), Karakılıç, Atılgan et al. (2005) and Karakılıç, Veliev et al. (2005), Atılgan et al. (2002), Coşkan & Karakılıç (2011), in this chapter, we study the operator $L(V)$. The crucial difference between this chapter and the study Coşkan & Karakılıç (2011) is that here we obtained the asymptotic formulas for the resonance eigenvalues, $|\gamma|^2$ when γ belongs to the single resonance domain, that is, $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, where δ is from $\{e_1, e_2, \ldots, e_d\}$ and $e_1 = \left(\frac{\pi}{a_1}, 0, \ldots, 0\right)$, $e_2 = \left(0, \frac{\pi}{a_2}, \ldots, 0\right)$, $e_d = \left(0, \ldots, \frac{\pi}{a_d}\right)$, $d \geq 2$, while in Coşkan & Karakılıç (2011) the authors obtained asymptotic formulas for the non-resonance eigenvalues.

To obtain asymptotic formulas for eigenvalues of $L(V)$ when $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, we use a similar approach as in Veliev (2015) and Karakılıç, Veliev et al. (2005). In this chapter, there are some additional technicalities.

2.1 The Operator $L(P(s))$

Now let $H_\delta = \{x \in \mathbb{R}^d : x \cdot \delta = 0\}$ be the hyperplane which is orthogonal to δ. Then we define the following sets:

$$\Omega_\delta = \{\omega \in \Omega : w \cdot \delta = 0\} = \Omega \cap H_\delta,$$

$$\Gamma_\delta = \{\gamma \in \frac{\Gamma}{2} : \gamma \cdot \delta = 0\} = \frac{\Gamma}{2} \cap H_\delta.$$

Here "$\cdot$" denotes the inner product in $\mathbb{R}^d$. Clearly, for all $\gamma \in \frac{\Gamma}{2}$, we have the following decomposition

$$\gamma = j\delta + \beta, \beta = (\beta^1, \ldots, \beta^d) \in \Gamma_\delta, j \in \mathbb{Z}. \tag{2.1}$$

Note that; if $\gamma = j\delta + \beta \in V_\delta(\rho^{\alpha_1}) \backslash E_2$, then

$$|j| < r_1, \quad r_1 = \rho^{\alpha_1}|\delta|^{-2} + 1, \ \ |\beta^k| > \frac{1}{3}\rho^{\alpha_1}, \ \forall k : e_k \neq \delta. \tag{2.2}$$

We write the decomposition (1.3) of $v_{ij}(x)$ as

$$v_{ij}(x) = \sum_{\gamma' \in \frac{\Gamma}{2}} v_{ij\gamma'} u_{\gamma'}(x) = p_{ij}(s) + \sum_{\gamma \in \frac{\Gamma}{2} \backslash \delta\mathbb{R}} v_{ij\gamma} u_\gamma(x), \tag{2.3}$$

where

$$p_{ij}(s) = \sum_{n \in \mathbb{Z}} p_{ijn} \cos ns, \ \ p_{ijn} = v_{ij(n\delta)}, \ \ s = x \cdot \delta, \ i, j = 1, 2, \ldots, m. \tag{2.4}$$

In order to obtain the asymptotic formulas for the single resonance eigenvalues $|\gamma|^2$ ($\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$), we consider the operator $L(V)$ as the perturbation of $L(P(s))$ where $L(P(s))$ is defined by the differential expression

$$Lu = -\boldsymbol{\Delta} u + P(s)u \tag{2.5}$$

and the Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial F} = 0,$$

$$P(s) = (p_{ij}(s)), \ i, j = 1, 2, \ldots, m. \tag{2.6}$$

It can be easily verified by the method of separation of variables that the eigenvalues and the corresponding eigenfunctions of $L(P(s))$, indexed by the pairs $(j, \beta) \in \mathbb{Z} \times \Gamma_\delta$, are $\lambda_{j,\beta} = \lambda_j + |\beta|^2$ and $\chi_{j,\beta}(x) = u_\beta(x) \cdot \varphi_j(s) = (u_\beta(x)\varphi_{j1}, u_\beta(x)\varphi_{j2}, \ldots, u_\beta(x)\varphi_{jm})$, respectively, where $\beta \in \Gamma_\delta$, λ_j is the eigenvalue and $\varphi_j(s) = (\varphi_{j,1}(s), \varphi_{j,2}(s), \ldots, \varphi_{j,m}(s))$ is the corresponding eigenfunction of the operator $T(P(s))$ defined by the differential expression

$$T(P(s))Y = -\left|\frac{\pi}{a_i}\right|^2 Y'' + P(s)Y \tag{2.7}$$

and the boundary condition

$$Y'(0) = Y'(\pi) = 0. \tag{2.8}$$

The eigenvalues of the operator $T(0)$, defined by (2.7) when $P(s) = 0$ and the boundary condition (2.8), are $|n\delta|^2 = |\frac{n\pi}{a_i}|^2$ with the corresponding eigenspace $E_n = span\,\{C_{n,1}(s), C_{n,2}(s), \ldots, C_{n,m}(s)\}$, where $C_{n,i}(s) = (0, \ldots, \cos ns, \ldots, 0)$, the non-zero component $\cos ns$ stands in the ith place and $n \in \mathbb{Z}^+ \cup \{0\}$. It is well known that (for example, see Naimark et al. (1967)) the eigenvalue λ_j of $T(P(s))$ satisfying $|\lambda_j - |j\delta|^2| < \sup P(s)$, satisfies the following relation

$$\lambda_j = |j\delta|^2 + O\left(\frac{1}{|j\delta|}\right). \tag{2.9}$$

By the above equation, the eigenvalue $|\gamma|^2 = |\beta|^2 + |j\delta|^2$ of $L(0)$ corresponds to the eigenvalue $|\beta|^2 + \lambda_j$ of $L(P(s))$.

Note that $L_2^m(F)$ is the space of all vector valued functions $f(x) = (f_1(x), \ldots, f_m(x))$ whose components $f_i(x)$, $i = 1, \ldots, m$, $m \geq 2$, are in $L_2(F)$. We denote the inner product in $L_2^m(F)$ by $\langle \cdot, \cdot \rangle$ which is defined by using the inner product $(\cdot, \cdot)$ in $L_2(F)$ as follows:

$$\begin{aligned} f(x) = (f_1(x), \ldots, f_m(x)),\ g(x) = (g_1(x), \ldots, g_m(x)) \in L_2^m(F) \\ \Rightarrow\ \langle f, g \rangle = (f_1, g_1) + \cdots + (f_m, g_m), \end{aligned} \tag{2.10}$$

for $x \in \mathbb{R}^d$, $d \geq 2$. Also for any $f \in L_2^m[0, \pi]$, since $\{C_{n,i}\}_{n\in\mathbb{Z}^+\cup\{0\},\, i=1,2,\ldots,m}$ is a complete system, by (2.10) we have the decomposition

$$f(s)=\sum_{n\in\mathbb{Z}^{+}\cup\{0\}}\sum_{i=1}^{m}\frac{2}{\pi}\left\langle f(s),C_{n,i}(s)\right\rangle C_{n,i}(s)$$
$$=\left(\sum_{n\in\mathbb{Z}^{+}\cup\{0\}}\frac{2}{\pi}\left(f_{1}(s),\cos ns\right)\cos ns,\ldots,\sum_{n\in\mathbb{Z}^{+}\cup\{0\}}\frac{2}{\pi}\left(f_{m}(s),\cos ns\right)\cos ns\right). \tag{2.11}$$

On the other hand, by equivalence of the decompositions (1.3) and (1.4) ($q(x)=q(s)\in L_2^m[0,\pi]$, when $d=1$), it is convenient to use the decomposition

$$f(s)=\sum_{n\in\mathbb{Z}}\sum_{i=1}^{m}\frac{1}{\pi}\left\langle f(s),C_{n,i}(s)\right\rangle C_{n,i}(s).$$

In the sequel, for the sake of simplicity, we use the brief notation $\left\langle f(s),C_{n,i}(s)\right\rangle$ instead of $\frac{1}{\pi}\left\langle f(s),C_{n,i}(s)\right\rangle$, since the constants which do not depend on ρ are inessential in our calculations.

The system of eigenfunctions $\{\chi_{j,\beta}\}_{j,\beta}$ is complete in $L_2^m(F)$. Indeed; suppose that there exists a non-zero function $f(x)\in L_2^m(F)$ which is orthogonal to each $\chi_{j,\beta},\ j\in\mathbb{Z}$, $\beta\in\Gamma_\delta$. Since $C_{n,i},\ i=1,2,\ldots,m$ can be decomposed by φ_j, by (2.1), and the definition of $\chi_{j,\beta}$, the function $\phi_{i,\gamma}=u_\beta(x)\cdot C_{n,i}$, $i=1,2,\ldots,m$ can be decomposed by the system $\{\chi_{j,\beta}\}_{j\in\mathbb{Z},\beta\in\Gamma_\delta}$. Thus, the assumption $\left\langle \chi_{j,\beta}(x),f(x)\right\rangle=0$ for $j\in\mathbb{Z}$, $\beta\in\Gamma_\delta$ implies that $\left\langle f(x)\ ,\ \phi_{i,\gamma}\right\rangle=0$, $\forall\gamma\in\frac{\Gamma}{2}$ and $i=1,2,\ldots,m$, which contradicts to the fact that $\{\phi_{i,\gamma}(x)\}_{\gamma\in\frac{\Gamma}{2}\ ,\ i=1,\ldots,m}$ is a basis for $L_2^m(F)$.

To prove the asymptotic formulas, we use the binding formula

$$\left(\Lambda_N-\lambda_{j,\beta}\right)\left\langle \psi_N\ ,\ \chi_{j,\beta}\right\rangle=\left\langle \psi_N,\left(V-P\right)\chi_{j,\beta}\right\rangle, \tag{2.12}$$

for the eigenvalue, eigenfunction pairs Λ_N, $\psi_N(x)$ and $\lambda_{j,\beta}$, $\chi_{j,\beta}$ of the operators $L(V)$ and $L(P(s))$, respectively. The formula (2.12) can be obtained by multiplying the equation $L(V)\psi_N(x)=\Lambda_N\psi_N(x)$ by $\chi_{j,\beta}$ and using the facts that $L(P(s))$ is self-adjoint and $L(P(s))\chi_{j,\beta}=\lambda_{j,\beta}\ \chi_{j,\beta}$.

Now our aim is to decompose $(V-P)\,\chi_{j,\beta}$ with respect to the basis $\{\chi_{j',\beta'}\}_{j'\in\mathbb{Z},\beta'\in\Gamma_\delta}$. By (2.3) and (1.7), we have

$$v_{ij}(x)-p_{ij}(s)=\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d_{ij}(\beta_1,n_1)\cos n_1 s\ u_{\beta_1}(x)+O(\rho^{-p\alpha}), \tag{2.13}$$

where

$$\Gamma'(\rho^\alpha)=\{(\beta_1,n_1):\beta_1\in\Gamma_\delta\backslash\{0\},n_1\in\mathbb{Z},n_1\delta+\beta_1\in\Gamma(\rho^\alpha)\}$$

and

$$d_{ij}(\beta_1,n_1)=\frac{1}{\mu(F)}\int\limits_F v_{ij}(x)\cos n_1 s\ u_{\beta_1}(x)dx.$$

For $(\beta_1,n_1)\in\Gamma'(p\rho^\alpha)$, we have $|n_1\delta+\beta_1|<p\rho^\alpha$ and since β_1 is orthogonal to δ,

$$|\beta_1|<p\rho^\alpha,\ \ |n_1|<p\rho^\alpha\ \ |n_1|<\frac{1}{2}r_1 \tag{2.14}$$

(see (2.2)).

Clearly (see equation (22) in Karakılıç et al. (2005)), we have, for all $i,j=1,2,\ldots,m$,

$$\begin{aligned}\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} & d_{ij}\left(\beta_1,n_1\right)\left(\cos n_1 s\right)u_{\beta_1}(x)u_\beta(x)\\ &=\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d_{ij}\left(\beta_1,n_1\right)\left(\cos n_1 s\right)u_{\beta_1+\beta}(x)\end{aligned} \tag{2.15}$$

for all $\beta\in\Gamma_\delta$ satisfying $\left|\beta^k\right|>\frac{1}{3}\rho^{\alpha_1}$, $\forall k:e_k\neq\delta$.

By using the definition of $\chi_{j,\beta}$, $P(s)$, the decompositions (2.13) and (2.15), we have

$$\begin{aligned}&(V-P)\,\chi_{j,\beta}=\\ &\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)}\sum_{k=1}^{m}\left(d_{1k}\left(\beta_1,n_1\right)\left(\cos n_1 s\right)\varphi_{j,k}(s)u_{\beta+\beta_1},\right.\\ &\qquad\left.\ldots,d_{mk}\left(\beta_1,n_1\right)\left(\cos n_1 s\right)\varphi_{j,k}(s)u_{\beta+\beta_1}\right)+O\left(\rho^{-p\alpha}\right).\end{aligned} \tag{2.16}$$

Now we consider the following decompositions:

$$\varphi_{j,k}(s) = \sum_{n\in\mathbb{Z}} (\varphi_{j,k}, \cos ns)\ \cos ns, \tag{2.17}$$

$$\begin{aligned} \cos n_1 s\ \varphi_{j,k}(s) &= \sum_{n\in\mathbb{Z}} (\varphi_{j,k}, \cos ns)\ .\ \cos n_1 s . \cos ns \\ &= \sum_{n\in\mathbb{Z}} (\varphi_{j,k}, \cos ns)\ .\frac{1}{2}[\cos(n_1+n)s + \cos(n_1-n)s] \\ &= \sum_{n\in\mathbb{Z}} (\varphi_{j,k}, \cos ns)\ . \cos(n_1+n)s \end{aligned} \tag{2.18}$$

for each $j \in \mathbb{Z}$, $k = 1, 2, \ldots, m$.

On the other hand; the decomposition of $\varphi_j(s) = (\varphi_{j,1}(s), \ldots, \varphi_{j,m}(s))$ with respect to the basis $\{C_{n,i}(s) = (0, 0, \ldots, \cos ns, 0, \ldots, 0)\}_{n\in\mathbb{Z}, i=1,2,\ldots,m}$ is given by

$$\begin{aligned} \varphi_j(s) &= (\varphi_{j,1}, \varphi_{j,2}, \ldots, \varphi_{j,m}) \\ &= \sum_{n\in\mathbb{Z}} \sum_{i=1}^{m} \langle\ \varphi_j(s), C_{n,i}(s)\ \rangle C_{n,i}(s) \\ &= \left(\sum_{n\in\mathbb{Z}} \langle\ \varphi_j(s), C_{n,1}(s)\ \rangle \cos ns, \ldots, \sum_{n\in\mathbb{Z}} \langle\ \varphi_j(s), C_{n,m}(s)\ \rangle \cos ns \right). \end{aligned} \tag{2.19}$$

Thus, (2.17), (2.18) and (2.19), gives

$$\begin{aligned} \varphi_{j,k}(s) &= \sum_{n\in\mathbb{Z}} \langle\ \varphi_j(s), C_{n,k}(s)\ \rangle\ \cos ns \\ \cos n_1 s\ \varphi_{j,k}(s) &= \sum_{n\in\mathbb{Z}} \langle\ \varphi_j(s), C_{n,k}(s)\ \rangle\ \cos(n+n_1)s. \end{aligned} \tag{2.20}$$

Lemma 2.1.1. *Let r be a number no less than r_1 $(r \geq r_1)$ and j, n be integers satisfying $|j| + 1 < r$, $|n| \geq 2r$. Then*

$$\langle\ \varphi_j(s)\ ,\ C_{n,i}(s)\ \rangle = O\left(\rho^{-(l-1)\alpha}\right)\ , \forall i = 1, 2, \ldots, m \tag{2.21}$$

and

$$\varphi_j(s) = \sum_{|n|<2r} \sum_{i=1}^{m} \langle\ \varphi_j(s)\ ,\ C_{n,i}(s)\ \rangle\ C_{n,i}(s) + O\left(\rho^{-(l-2)\alpha}\right). \tag{2.22}$$

Proof. We use the following binding formula for $T(0)$ and $T(P(s))$

$$\left(\lambda_j - |n\delta|^2\right)\langle \varphi_j(s), C_{n,k}(s)\rangle = \langle \varphi_j(s), P(s)C_{n,k}(s)\rangle \tag{2.23}$$

and the obvious decomposition, which can be obtained by definition of $P(s)$ and (1.7),

$$\begin{aligned} P(s)C_{n,k}(s) &= \left(\sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{1kn_1}\cos n_1 s\cos ns, \ldots, \sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{mkn_1}\cos n_1 s\cos ns\right) \\ &\qquad + O\left(|n\delta|^{-(l-1)}\right) \\ &= \left(\sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{1kn_1}\cos(n-n_1)s, \ldots, \sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{mkn_1}\cos(n-n_1)s\right) \\ &\qquad + O\left(|n\delta|^{-(l-1)}\right) \\ &= \sum_{t=1}^{m}\sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{tkn_1}C_{n-n_1,k}(s) + O\left(|n\delta|^{-(l-1)}\right). \end{aligned} \tag{2.24}$$

Putting above equation (2.24) into (2.23), we get

$$\begin{aligned} &\left(\lambda_j - |n\delta|^2\right)\langle \varphi_j(s), C_{n,k}(s)\rangle \\ &= \langle \varphi_j(s), \sum_{t_1=1}^{m}\sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{t_1kn_1}C_{n-n_1,k}\rangle + O\left(|n\delta|^{-(l-1)}\right) \\ &= \sum_{t_1=1}^{m}\sum_{|n_1\delta|<\frac{|n\delta|}{2l}} p_{t_1kn_1}\langle \varphi_j(s), C_{n-n_1,k}(s)\rangle + O\left(|n\delta|^{-(l-1)}\right). \end{aligned} \tag{2.25}$$

By assumption $|n| \geq 2r$ and $|j|+1 < r$, thus if $|n_1\delta| < \frac{|n\delta|}{2l}$ then $||(n-n_1)\delta|^2-|j|| > \frac{|n|}{5}$ which together with (2.9) imply $|\lambda_j - |(n-n_1)\delta|^2| > c|n\delta|$. So that in (2.23) if we substitute $(n-n_1)\delta$ instead of $n\delta$, we get

$$\langle \varphi_j(s), C_{n-n_1,k}(s)\rangle = \frac{\langle \varphi_j(s), P(s)C_{n-n_1,k}\rangle}{\lambda_j - |(n-n_1)\delta|^2}. \tag{2.26}$$

Now using (2.26) in (2.25), we get

$$\left(\lambda_j - |n\delta|^2\right)\langle \varphi_j(s), C_{n,k}(s)\rangle = \sum_{t_1=1}^{m} \sum_{|n_1\delta|<\frac{|n\delta|}{2l}} \frac{p_{t_1 k n_1}\langle \varphi_j(s), P(s)\, C_{n-n_1,k}(s)\rangle}{(\lambda_j - |(n-n_1)\delta|^2)} + O\left(|n\delta|^{-(l-1)}\right).$$

Again putting (2.24) into the last equation, we obtain

$$\begin{aligned}
&\left(\lambda_j - |n\delta|^2\right)\langle \varphi_j(s), C_{n,k}(s)\rangle \\
&= \sum_{t_1=1}^{m} \sum_{|n_1\delta|<\frac{|n\delta|}{2l}} \frac{p_{t_1 k n_1}\langle \varphi_j(s), \sum\limits_{t_2=1}^{m} \sum\limits_{|n_2\delta|<\frac{|n\delta|}{2l}} p_{t_2 k n_2}\, C_{n-n_1-n_2,k}(s)\rangle}{(\lambda_j - |(n-n_1)\delta|^2)} + O\left(|n\delta|^{-(l-1)}\right) \\
&= \sum_{t_1,t_2=1}^{m} \sum_{\substack{|n_1\delta|<\frac{|n\delta|}{2l} \\ |n_2\delta|<\frac{|n\delta|}{2l}}} \frac{p_{t_1 k n_1} p_{t_2 k n_2}\langle \varphi_j(s), C_{n-n_1-n_2,k}(s)\rangle + O\left(|n\delta|^{-(l-1)}\right)}{(\lambda_j - |(n-n_1)\delta|^2)}. \qquad (2.27)
\end{aligned}$$

In this way, iterating $p_1 = [\frac{l}{2}]$ times and dividing both sides of the obtained equation by $\lambda_j - |n\delta|^2$, we have

$$\langle \varphi_j(s) C_{n,k}(s)\rangle = \sum_{t_1,t_2,\ldots,t_{p_1}=1}^{m} \sum_{\substack{|n_1\delta|<\frac{|n\delta|}{2l} \\ |n_2\delta|<\frac{|n\delta|}{2l} \\ \vdots \\ |n_{p_1}\delta|<\frac{|n\delta|}{2l}}} \frac{p_{t_1 k n_1} p_{t_2 k n_2} \cdots p_{t_{p_1} k n_{p_1}} \langle \varphi_j, C_{n-n_1-\cdots-n_{p_1},k}\rangle}{\prod\limits_{s=0}^{p_1-1} (\lambda_j - |(n-n_1-\cdots-n_s)\delta|^2)} + O(|n\delta|^{-(l-1)}), \qquad (2.28)$$

where the integers $n, n_1, \ldots, n_{p_1}$ satisfy the conditions

$$|n_s| < \frac{|n|}{2l}, \quad s = 1, \ldots, p_1, \quad |j| + 1 < \frac{|n|}{2}.$$

These conditions and the assumptions $|n| > 2r$, $|j| + 1 < r$ imply that

$$||n - n_1 - \cdots - n_s| - |j|| > \frac{|n|}{5}, \quad s = 0, 1, 2, \ldots, p_1.$$

This together with (2.9), give

$$\frac{1}{|\lambda_j - |(n-n_1-\cdots-n_s)\delta|^2|} = \frac{1}{\left||j\delta|^2 + O\left(\frac{1}{|j\delta|}\right) - |\left(n-n_1-\cdots-n_s\right)\delta|^2\right|}$$
$$= O\left(|n\delta|^{-2}\right) \tag{2.29}$$

for $s = 0, \ldots, p_1 - 1$. Hence by (2.28), (2.29) and (1.8), we have

$$\langle \varphi_j(s), C_{n,k}(s) \rangle = O\left(|n\delta|^{-(l-1)}\right).$$

Since $|n\delta| \geq 2r \geq r_1 > 2\rho^{\alpha}$, $O\left(|n\delta|^{-(l-1)}\right) = O(\rho^{-(l-1)\alpha})$ from which we get the proof of (2.21).

To prove (2.22), we write the Fourier series of $\varphi_j(s)$ with respect to the basis $\{C_{n,1}(s), \ldots, C_{n,m}(s)\}_{n\in\mathbb{Z}}$ as follows:

$$\begin{aligned}\varphi_j(s) &= \sum_{n\in\mathbb{Z}} \langle \varphi_j(s), C_{n,k}(s)\rangle C_{n,k}(s) \\ &= \sum_{|n|<2r} \langle \varphi_j(s), C_{n,k}(s)\rangle C_{n,k}(s) + \sum_{|n|\geqslant 2r} \langle \varphi_j(s), C_{n,k}(s)\rangle C_{n,k}(s).\end{aligned}$$

From which together with (2.21), we get (2.22). $\square$

Using the first relation (2.21) in Lemma 2.1.1 and (2.20), we also have

$$\cos n_1 s\ \varphi_{j,k}(s) = \sum_{|n|<2r} \langle\ \varphi_j(s)\ ,\ C_{n,k}(s)\ \rangle\ \cos(n+n_1)s + O\left(\rho^{-(l-2)\alpha}\right). \tag{2.30}$$

Putting this last relation (2.30) into (2.16), we get

$$\begin{aligned}&(V-P)\chi_{j,\beta} \\ &= \sum_{(\beta_1,n_1)\in\Gamma'(\rho^{\alpha})} \sum_{|n|<2r} \sum_{k=1}^{m} (d_{1k}\left(\beta_1,n_1\right) \langle\ \varphi_j(s)\ ,\ C_{n,k}(s)\ \rangle\ \cos(n+n_1)s\ u_{\beta+\beta_1}, \ldots, \\ &d_{mk}\left(\beta_1,n_1\right) \langle\ \varphi_j(s)\ ,\ C_{n,k}(s)\ \rangle\ \cos(n+n_1)s\ u_{\beta+\beta_1}) + O\left(\rho^{-p\alpha}\right)\end{aligned} \tag{2.31}$$

(note that $p=(l-d), d\geq 2 \Rightarrow \frac{1}{\rho(l-2)}<\frac{1}{\rho^{p\alpha}}$. Hence $O\left(\rho^{-p\alpha}\right)+O\left(\rho^{-(l-2)\alpha}\right)=O\left(\rho^{-p\alpha}\right)$).

Now, in order to decompose $(V-P)\chi_{j,\beta}$ with respect to $\left\{\chi_{j+j_1',\beta_1'}\right\}$ we consider the inner product $\langle (V-P)\chi_{j,\beta}, \chi_{j+j_1',\beta_1'}\rangle$, that is, by the definition of $\chi_{j+j_1',\beta_1'}$ and (2.31), the inner products $(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \varphi_{j+j_1',t}(s)\ u_{\beta_1'})$, $t=1,2,\ldots,m$. Using the decomposition (2.20), instead of j, we substitute $j+j_1'$ to get

$$\begin{aligned}
&\left(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \varphi_{j+j_1',t}(s)\ u_{\beta_1'}\right)\\
&\qquad=\left(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \sum_{n'\in\mathbb{Z}}\langle\ \varphi_{j+j_1'}(s)\ ,\ C_{n',t}(s)\ \rangle\ \cos n's\ u_{\beta_1'}\right)\\
&\qquad=\sum_{n'\in\mathbb{Z}}\langle\ \overline{\varphi_{j+j_1'}(s)\ ,\ C_{n',t}(s)}\ \rangle\left(\cos(n+n_1)s\ u_{\beta+\beta_1}, \cos n's\ u_{\beta_1'}\right).
\end{aligned}$$

Note that if $\beta_1'\neq\beta+\beta_1$ or $n'\neq n+n_1$ then $(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \cos n's\ u_{\beta_1'})=0$. Thus,

$$\begin{aligned}
&\left(\cos(n+n_1)s\ u_{\beta+\beta_1}\ ,\ \varphi_{j+j_1',t}(s)\ u_{\beta_1'}\right)\\
&\qquad=\begin{cases} 0 & ,\ \text{if}\quad \beta_1'\neq\beta+\beta_1\quad\text{or}\quad n'\neq n+n_1\\ \langle\ \overline{\varphi_{j+j_1'}(s)\ ,\ C_{n+n_1,t}(s)}\ \rangle & ,\ \text{otherwise.}\end{cases}
\end{aligned}$$

Using the last equality and (2.31), we get

$$\begin{aligned}
&(V-P)\chi_{j,\beta}\\
&=\sum_{\substack{j_1'\in\mathbb{Z}\\(\beta_1,n_1)\in\Gamma'(\rho^\alpha)}}\left(\sum_{|n|<2r}\sum_{k=1}^{m}\sum_{i=1}^{m}d_{ik}\left(\beta_1,n_1\right)\langle\varphi_j,C_{n,k}\rangle\langle\overline{\varphi_{j+j_1'},C_{n+n_1,i}}\rangle\right)\chi_{j+j_1',\beta+\beta_1}\\
&\qquad\qquad+O(\rho^{-p\alpha}).\qquad(2.32)
\end{aligned}$$

Lemma 2.1.2. *Let r be a number no less than r_1 $(r \geq r_1)$, j, n and $n_1$be integers satisfying $|n| < 2r$, $|n_1| < \frac{1}{2}r_1$ and $|j| + 1 < r$, then*

$$\sum_{\substack{j_1 \in \mathbb{Z} \\ |j_1| \geqslant 6r}} \langle \varphi_{j+j_1}, C_{n,i} \rangle = O\left(\rho^{-(l-2)\alpha}\right), \forall i = 1, 2, \ldots, m.$$

Proof. By the binding formula (2.23) for $T(0)$ and $T(P(s))$ we have

$$\left(\lambda_{j+j_1} - |(n+n_1)\delta|^2\right) \langle \varphi_{j+j_1}, C_{n+n_1,k} \rangle = \langle \varphi_{j+j_1}, P(s) C_{n+n_1,k} \rangle. \tag{2.33}$$

If $|j_1| \geq 6r$ then the assumptions of this lemma imply $||j + j_1| - |n + n_1|| > \frac{r}{2}$. Thus, using (2.33) and the fact that $\lambda_{j+j_1} = |(j + j_1)\delta|^2 + O\left(\frac{1}{|(j+j_1)\delta|}\right)$, we get

$$|\sum_{j_1 \geqslant 6r} \langle \varphi_{j+j_1}, C_{n+n_1,k} \rangle| = |\sum_{j_1 \geqslant 6r} \frac{\langle \varphi_{j+j_1}, P(s) C_{n+n_1,k} \rangle}{\lambda_{j+j_1} - |(n+n_1)\delta|^2}|.$$

Using the decomposition of $p_{tk}(s) = \left(\sum_{|l_1\delta| < |r\delta|} v_{tk,l_1\delta} \cos l_1 s\right) + O(|r\delta|^{-(l-1)})$ and iterating the obtained formula $p_1 = [\frac{l}{2}]$ times as in the proof of Lemma 2.1.1, we get

$$\begin{aligned} &|\sum_{|j_1| \geqslant 6r} \langle \varphi_{j+j_1}, C_{n+n_1,k} \rangle| \\ &= |\sum_{j_1 \geqslant 6r} \sum_{\substack{|l_1\delta| < |r\delta| \\ |l_2\delta| < |r\delta| \\ \vdots \\ |l_{p_1}\delta| < |r\delta|}} \sum_{t_1, t_2, \ldots, t_{p_1} = 1}^{m} \frac{v_{t_1 k, l_1 \delta} \cdots v_{t_{p_1} k, l_{p_1} \delta} \langle \varphi_{j'}, C_{n+n_1-l_1-\cdots-l_{p_1},k} \rangle}{\prod_{s=0}^{p_1-1} |\lambda_{j+j_1} - (n+n_1-l_1-\cdots-l_s)\delta|^2}. \end{aligned} \tag{2.34}$$

Since $|n| < 2r$ and $|n_1| < \frac{1}{2}r_1 < \frac{1}{2}r$, $|n+n_1| < \frac{5r}{2}$. Also, $|n+n_1-l_1-\cdots-l_s| < 3r$ and $\frac{1}{|\lambda_{j+j_1} - |(n+n_1-l_1-\cdots-l_s)\delta|^2|} = O\left(|r|^{-2}\right)$. Substituting this result into (2.34) and using (1.8), we get the proof. $\square$

By Lemma 2.1.2, the equation (2.32) becomes;

$$(V-P)\chi_{j,\beta} = O(\rho^{-p\alpha}) +$$
$$\sum_{\substack{|j_1'|<6r \\ (\beta_1,n_1)\in\Gamma'(\rho^\alpha)}} \left(\sum_{|n|<2r} \sum_{k=1}^{m} \sum_{i=1}^{m} d_{ik}(\beta_1,n_1) \langle \varphi_j, C_{n,k}\rangle \overline{\langle \varphi_{j+j_1'}, C_{n+n_1,i}\rangle} \right) \chi_{j+j_1',\beta+\beta_1}$$
$$= O(\rho^{-p\alpha}) +$$
$$\sum_{\substack{|j_1|<6r \\ (\beta_1,n_1)\in\Gamma'(\rho^\alpha)}} \left(\sum_{|n|<2r} \sum_{k=1}^{m} \sum_{i=1}^{m} d_{ik}(\beta_1,n_1) \langle \varphi_j, C_{n,k}\rangle \overline{\langle \varphi_{j+j_1}, C_{n+n_1,i}\rangle} \right) \chi_{j+j_1,\beta+\beta_1},$$

that is,

$$(V-P)\chi_{j,\beta} = \sum_{(\beta_1,j_1)\in \boldsymbol{Q}(\rho^\alpha,6r)} S(j,\beta,j+j_1,\beta+\beta_1)\,\chi_{j+j_1,\beta+\beta_1} + O(\rho^{-p\alpha}), \quad (2.35)$$

for every j satisfying $|j|+1<r$, where

$$\boldsymbol{Q}(\rho^\alpha,6r) = \{(\beta,j) : |j\delta| < 6r\ ,\ 0<|\beta|<\rho^\alpha\},$$

and

$$S(j,\beta,j+j_1,\beta+\beta_1)$$
$$= \sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^\alpha)} \left(\sum_{|n|<2r} \sum_{k=1}^{m} \sum_{i=1}^{m} d_{ik}(\beta_1,n_1) \langle \varphi_j, C_{n,k}\rangle \overline{\langle \varphi_{j+j_1}, C_{n+n_1,i}\rangle} \right).$$

We need to prove that

$$\sum_{(\beta_1,j_1)\in \boldsymbol{Q}(\rho^\alpha,6r)} |S(j,\beta,j+j_1,\beta+\beta_1)| < c_3. \quad (2.36)$$

By the definition of $S\left(j,\beta,j+j_1,\beta+\beta_1\right)$, $d_{ik}\left(\beta_1,n_1\right)$ and (1.8), we have

$$\begin{aligned}
&\sum_{(\beta_1,j_1)\in\boldsymbol{Q}(\rho^{\alpha},6r)}\left|S\left(j,\beta_1,j',\beta+\beta_1\right)\right| \\
&\leq \sum_{n_1:(\beta_1,n_1)\in\Gamma'(\rho^{\alpha})}\sum_{i,k=1}^{m}\left|d_{ik}(\beta_1,n_1)\right|\sum_{|n|<2r}\left|\langle\varphi_j,C_{n,k}\rangle\right|\sum_{|j_1|<6r}\left|\overline{\langle\varphi_{j+j_1},C_{n+n_1,i}\rangle}\right| \\
&\leq c_4\sum_{|n|<2r}\left|\langle\varphi_j,C_{n,k}\rangle\right|\sum_{|j_1|<6r}\left|\langle\varphi_{j+j_1},C_{n+n_1,i}\rangle\right|. \qquad (2.37)
\end{aligned}$$

Now we prove that

$$\sum_{n\in\mathbb{Z}}\left|\langle\varphi_j,C_{n,k}\rangle\right|<c_5 \quad \text{and} \quad \sum_{j_1\in\mathbb{Z}}\left|\langle\varphi_{j+j_1},C_{n+n_1,i}\rangle\right|<c_6. \qquad (2.38)$$

For this, let

$$A=\left\{n\in\mathbb{Z}\ \mid\ |n\delta|^2\in[\lambda_{j-1},\lambda_{j+1}]\ \right\}$$

and

$$B=\left\{j_1\in\mathbb{Z}\ \mid\ \lambda_{j+j_1}\in\left[\ |(n+n_1)\delta|^2-1,\ \ |(n+n_1)\delta\ |^2+1\ \right]\ \right\},$$

then it follows from (2.9) that the number of elements in the sets A and B are less than c_7. So if we isolate the terms with $n\in A$ and $j_1\in B$ in the first and second summations of inequalities in (2.38), respectively, applying (2.23) to the other terms then using the facts

$$\sum_{n\notin A}\frac{1}{|\lambda_j-|n\delta|^2|}<c_8 \quad \text{and} \quad \sum_{j_1\notin\beta}\frac{1}{|\lambda_{j+j_1}-|(n+n_1)\delta|^2|}<c_9,$$

we get (2.38), hence by (2.37), (2.36) is proved.

2.2 The Iterability Condition

The expressions (2.35) and (2.12) together imply that

$$\begin{aligned}&(\Lambda_N - \lambda_{j',\beta'}) \langle \psi_N \ , \ \chi_{j',\beta'} \rangle \\ &= \sum_{(\beta_1, j_1) \in \boldsymbol{Q}(\rho^\alpha, 6r)} S\left(j', \beta', j' + j_1, \beta' + \beta_1\right) \langle \psi_N \ , \ \chi_{j'+j_1, \beta'+\beta_1} \rangle + O(\rho^{-p\alpha}). \end{aligned} \quad (2.39)$$

If the condition (iterability condition for the triple (N, j', β'))

$$|\Lambda_N - \lambda_{j',\beta'}| > c_{10} \quad (2.40)$$

holds then the formula (2.39) can be written in the following form

$$\langle \psi_N \ , \ \chi_{j',\beta'} \rangle = \sum_{(\beta_1, j_1) \in \boldsymbol{Q}(\rho^\alpha, 6r)} \frac{S\left(j', \beta', j' + j_1, \beta' + \beta_1\right) \langle \psi_N \ , \ \chi_{j'+j_1, \beta'+\beta_1} \rangle}{\Lambda_N - \lambda_{j',\beta'}} + O(\rho^{-p\alpha}). \quad (2.41)$$

Using (2.39) and (2.41), we are going to find Λ_N which is close to $\lambda_{j,\beta}$, where $|j|+1 < r_1$. For this, first in (2.39) instead of j', β', taking j, β, hence instead of r taking r_1, we get

$$\begin{aligned}&(\Lambda_N - \lambda_{j,\beta}) \langle \psi_N \ , \ \chi_{j,\beta} \rangle \\ &= \sum_{(\beta_1, j_1) \in \boldsymbol{Q}(\rho^\alpha, 6r_1)} S\left(j, \beta, j + j_1, \beta + \beta_1\right) \langle \psi_N \ , \ \chi_{j+j_1, \beta+\beta_1} \rangle + O(\rho^{-p\alpha}). \end{aligned} \quad (2.42)$$

To iterate it by using (2.41) for $j' = j + j_1$ and $\beta' = \beta + \beta_1$, we will prove that there is a number N such that

$$|\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}| > \frac{1}{2}\rho^{\alpha_2}, \quad (2.43)$$

where $|j + j_1| < 7r_1 \equiv r_2$, since $\lambda_{j,\beta}$ and $|j_1| < 6r_1$. Then $(j + j_1, \beta + \beta_1)$ satisfies (2.40). This means that, in formula (2.39), the pair (j', β') can be replaced by the pair

$(j+j_1,\beta+\beta_1)$. Then, (2.39) instead of r taking r_2, we get

$$\langle\psi_N\ ,\ \chi_{j+j_1,\beta+\beta_1}\rangle = O(\rho^{-p\alpha})+ \sum_{(\beta_2,j_2)\in\boldsymbol{Q}(\rho^\alpha,6r_2)} \frac{S\left(j+j_1,\beta+\beta_1,j+j_1+j_2,\beta+\beta_1+\beta_2\right)\langle\psi_N\ ,\ \chi_{j+j_1+j_2,\beta+\beta_1+\beta_2}\rangle}{\Lambda_N-\lambda_{j+j_1,\beta+\beta_1}}.$$

Putting the above formula into (2.42), we obtain

$$(\Lambda_N-\lambda_{j,\beta})\,c(N,j,\beta) = O(\rho^{-p\alpha})+ \sum_{\substack{(\beta_1,j_1)\in\boldsymbol{Q}(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in\boldsymbol{Q}(\rho^\alpha,6r_2)}} \frac{S\left(j,\beta,j^1,\beta^1\right)S\left(j^1,\beta^1,j^2,\beta^2\right)c(N,j^2,\beta^2)}{\Lambda_N-\lambda_{j^1,\beta^1}}, \tag{2.44}$$

where $c(N,j,\beta)=\langle\psi_N,\chi_{j,\beta}\rangle$, $j^k=j+j_1+j_2+\cdots+j_k$ and $\beta^k=\beta+\beta_1+\beta_2+\cdots+\beta_k$. Thus, we are going to find a number N such that $c(N,j,\beta)$ is not too small and the condition (2.43) is satisfied.

Lemma 2.2.1. *(a) Suppose $g_1(x),g_2(x),\ldots,g_{p_2}(x)\in L_2^m(F)$ where $p_2=[\frac{d}{2\alpha_2}]+1$. Then for every eigenvalue $\lambda_{j,\beta}$ of the operator $L(P(s))$, there exists an eigenvalue Λ_N of $L(V)$ satisfying*

(i) $|\Lambda_N-\lambda_{j,\beta}|<2M$, where $M=\|V\|$,

(ii) $|c(N,j,\beta)|>\rho^{-q\alpha}$, where $q\alpha=[\frac{d}{2\alpha}+2]\alpha$,

(iii) $|c(N,j,\beta)|^2>\frac{1}{2p_2}\sum_{i=1}^{p_2}\left|\langle\psi_N,\frac{g_i}{\|g_i\|}\rangle\right|^2>\frac{1}{2p_2}\left|\langle\psi_N,\frac{g_i}{\|g_i\|}\rangle\right|^2$, $\forall i=1,2,\ldots,p_2$.

(b) Let $\gamma=\beta+j\delta\in V_\delta'(\rho^{\alpha_1})$ and $(\beta_1,j_1)\in\boldsymbol{Q}(\rho^\alpha,6r_1),(\beta_k,j_k)\in\boldsymbol{Q}(\rho^\alpha,6r_k)$, where $r_k=7r_{k-1}$ for $k=2,3,\ldots,p$. Then for $k=1,2,3,\ldots,p_1$, we have

$$|\lambda_{j,\beta}-\lambda_{j^k,\beta^k}|>\frac{3}{5}\rho^{\alpha_2},\quad \forall\beta^k\neq\beta. \tag{2.45}$$

Proof. (a) Let A,B,C be the set of indexes N satisfying (i), (ii), (iii), respectively. Using the binding formula (2.12) for $L(V)$ and $L(P(s))$ and the Bessel's inequality, we get

$$\sum_{N\notin A}|c(N,j,\beta)|^2=\sum_{N\notin A}\left|\frac{(\psi_N,(V-P)\chi_{j,\beta})}{\Lambda_N-\lambda_{j,\beta}}\right|^2$$
$$\leq\frac{1}{4M^2}\|(V-P)\chi_{j,\beta}\|^2\leq\frac{1}{4}.$$

Hence by Parseval's relation, we obtain

$$\sum_{N\in A}|c(N,j,\beta)|^2>\frac{3}{4}.$$

Using the fact that the number of indexes N in A is less than $\rho^{d\alpha}$ and by the relation $N\notin B\Rightarrow|c(N,j,\beta)|\leq\rho^{-q\alpha}$, we have

$$\sum_{N\in A\backslash B}|c(N,j,\beta)|^2<\rho^{d\alpha}\rho^{-q\alpha}<\rho^{-\alpha},$$

since $\alpha<\frac{1}{d+20}$. On the other hand by the relation $A=(A\setminus B)\cup(A\cap B)$ and the above inequalities, we get

$$\frac{3}{4}<\sum_{N\in A}|c(N,j,\beta)|^2=\sum_{N\in A\backslash B}|c(N,j,\beta)|^2+\sum_{N\in A\cap B}|c(N,j,\beta)|^2,$$

which implies

$$\sum_{N\in A\cap B}|c(N,j,\beta)|^2>\frac{3}{4}-\rho^{-\alpha}>\frac{1}{2}. \tag{2.46}$$

Now, suppose that $A\cap B\cap C=\emptyset$, i.e., for all $N\in A\cap B$, the condition (iii) does not hold. Then by (2.46) and Bessel's inequality, we have

$$\frac{1}{2}<\sum_{N\in A\cap B}|c(N,j,\beta)|^2\leq\sum_{N\in A\cap B}\frac{1}{2p_2}\sum_{i=1}^{p_2}\left|\left\langle\psi_N,\frac{g_i}{\|g_i\|}\right\rangle\right|^2$$
$$=\frac{1}{2p_2}\sum_{i=1}^{p_2}\sum_{N\in A\cap B}\left|\left\langle\psi_N,\frac{g_i}{\|g_i\|}\right\rangle\right|^2$$
$$<\frac{1}{2p_2}\sum_{i=1}^{p_2}\left\|\frac{g_i}{\|g_i\|}\right\|^2=\frac{1}{2},$$

which is a contradiction.

(b) The definition of $\lambda_{j,\beta}$ gives

$$\begin{aligned}|\lambda_{j,\beta}-\lambda_{j^k,\beta^k}| &= ||\beta|^2+\lambda_j-|\beta+\beta_1+\cdots+\beta_k|^2-\lambda_{j^k}| \\ &\geq |||\beta|^2-|\beta+\beta_1+\cdots+\beta_k|^2|-|\lambda_j-\lambda_{j^k}||. \qquad (2.47)\end{aligned}$$

The condition of the lemma $(\beta_1, j_1) \in \boldsymbol{Q}(\rho^\alpha, 6r_1), (\beta_k, j_k) \in \boldsymbol{Q}(\rho^\alpha, 6r_k)$ and the relation $\beta + j\delta \in V_\delta(\rho^{\alpha_1}) \setminus E_2$ together with $|j\delta| < c_{11}\rho^{\alpha_1}$ (see (2.2)) and $|j_i\delta| < c_{12}\rho^{\alpha_1}$ (see (2.14)) imply that

$$\begin{aligned}\rho^{\alpha_2} &< ||\beta|^2+|j\delta|^2-|\beta^k|^2-|j^k\delta|^2| \\ &< ||\beta|^2-|\beta^k|^2|+c_{12}\rho^{\alpha_1}, \quad \beta_1+...+\beta_k\neq 0,\end{aligned}$$

since $\beta, \beta_1, ..., \beta_k$ are orthogonal to δ. That is, we have

$$||\beta|^2-|\beta_k|^2|>c_{13}\rho^{\alpha_2}.$$

This last inequality together with (2.47) and the asymptotic formula (2.9) give

$$|\lambda_{j,\beta}-\lambda_{j^k,\beta^k}|>c_{14}\rho^{\alpha_2}. \qquad \square$$

2.3 Asymptotic Formulas

Now we consider the following function

$$g_i(x)=\sum_{\substack{(\beta_1,j_1)\in\boldsymbol{Q}(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in\boldsymbol{Q}(\rho^\alpha,6r_2)}}\frac{S(j,\beta,j^1,\beta^1)\,S(j^1,\beta^1,j^2,\beta^2)\,\chi_{j^2,\beta^2}}{(\lambda_{j,\beta}-\lambda_{j^1,\beta^1})^i}, \quad 1\leq i\leq p_2. \qquad (2.48)$$

Since $\{\chi_{j^2,\beta^2}(x)\}$ is a total system and $\beta_1 \neq 0$ by (2.36) and (2.45), we have

$$\sum_{(j',\beta')} |\langle g_i(x), \chi_{j',\beta'}\rangle|^2 = \sum_{\substack{(\beta_1,j_1)\in \boldsymbol{Q}(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in \boldsymbol{Q}(\rho^\alpha,6r_2)}} \frac{|S\left(j,\beta,j^1,\beta^1\right)S\left(j^1,\beta^1,j^2,\beta^2\right)|^2}{|\left(\lambda_{j,\beta}-\lambda_{j^1,\beta^1}\right)^i|^2} \leq c_{15}\,\rho^{-2i\alpha_2}, i.e.,$$

$$g_i(x) \in L_2^m(F) \quad and \quad \|g_i(x)\| = O(\rho^{-i\alpha_2}), \quad \forall i = 1,2,\ldots,p_2. \tag{2.49}$$

Theorem 2.3.1. *For every eigenvalue $\lambda_{j,\beta}$ of the operator $L(P(s))$ with $\beta + j\delta \in V'_\delta(\rho^{\alpha_1})$, there exists an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = \lambda_{j,\beta} + O\left(\rho^{-\alpha_2}\right). \tag{2.50}$$

Proof. By Lemma 2.2.1, for the chosen $g_i(x), i = 1,2,\ldots,p_2$ in (2.48), there exists a number N, satisfying (i), (ii), (iii). Since $(\beta_1, j_1) \in \boldsymbol{Q}(\rho^\alpha, 6r_1)$, by part (b) of Lemma 2.2.1, we have

$$|\lambda_{j,\beta} - \lambda_{j^1,\beta^1}| > c_{16}\rho^{\alpha_2}.$$

The above inequality together with (i) imply

$$|\Lambda_N - \lambda_{j^1,\beta^1}| > c_{17}\rho^{\alpha_2}.$$

Using the following well known decomposition

$$\frac{1}{[\Lambda_N - \lambda_{j^1,\beta^1}]} = \sum_{i=1}^{p_2} \frac{[\Lambda_N - \lambda_{j,\beta}]^{i-1}}{[\lambda_{j,\beta} - \lambda_{j^1,\beta^1}]^i} + O\left(\rho^{-(p_2+1)\alpha_2}\right),$$

and (2.48), we see that the formula (2.44) can be written as

$$\begin{aligned}
&(\Lambda_N - \lambda_{j,\beta})\, c(N,j,\beta)\\
&= O(\rho^{-p\alpha}) + \sum_{\substack{(\beta_1,j_1)\in \boldsymbol{Q}(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in \boldsymbol{Q}(\rho^\alpha,6r_2)}} \frac{S\left(j,\beta,j^1,\beta^1\right) S\left(j^1,\beta^1,j^2,\beta^2\right) \langle \psi_N, \chi_{j^2,\beta^2}\rangle}{\Lambda_N - \lambda_{j^1,\beta^1}}\\
&= \sum_{i=1}^{p_2} \left[(\Lambda_N - \lambda_{j,\beta})^{i-1} \left\langle \psi_N, \frac{g_i}{\|g_i\|} \right\rangle \right] \|g_i\| + O\left(\rho^{-(p_2+1)\alpha_2}\right).
\end{aligned}$$

Now dividing both sides of the last equation by $c(N,j,\beta)$ and using (ii), (iii) we have

$$\begin{aligned}
&|\Lambda_N-\lambda_{j,\beta}|\le O\left(\rho^{-(p_2+1)\alpha_2+q\alpha}\right)+\frac{\left|\left\langle\psi_N,\frac{g_1}{\|g_1\|}\right\rangle\right|}{|c(N,j,\beta)|}\|g_1\|\\
&+\frac{|\Lambda_N-\lambda_{j,\beta}|\left|\left\langle\psi_N,\frac{g_2}{\|g_2\|}\right\rangle\right|}{|c(N,j,\beta)|}\|g_2\|+\cdots+\frac{|\Lambda_N-\lambda_{j,\beta}|^{(p_2-1)}\left|\left\langle\psi_N,\frac{g_{p_2}}{\|g_{p_2}\|}\right\rangle\right|}{|c(N,j,\beta)|}\|g_{p_2}\|\\
&\le(2p_2)^{\frac{1}{2}}\left(\|g_1\|+2M\|g_2\|+\cdots+(2M)^{p_2-1}\|g_{p_2}\|\right)+O\left(\rho^{-(p_2+1)\alpha_2+q\alpha}\right).
\end{aligned}$$

Hence by (2.49), we obtain

$$\Lambda_N=\lambda_{j,\beta}+O\left(\rho^{-\alpha_2}\right),$$

since $(p_2+1)\alpha_2-q\alpha>\alpha_2$. Theorem is proved. □

It follows from (2.45) and (2.50) that the triples (N,j^k,β^k) for $k=1,2,\ldots,p_1$, satisfy the iterability condition (2.40). By (2.41) instead of j',β' and r taking j^2,β^2 and r_3, we have

$$c(N,j^2,\beta^2)=\sum_{(\beta_3,j_3)\in\boldsymbol{Q}(\rho^\alpha,6r_3)}\frac{S(j^2,\beta^2,j^3,\beta^3)(\psi_N,\chi_{j^3,\beta^3})}{\Lambda_N-\lambda_{j^2,\beta^2}}+O(\rho^{-p\alpha}).\tag{2.51}$$

To obtain the other terms of the asymptotic formula of Λ_N, we iterate the formula (2.44). Now we isolate the terms with multiplicand $c(N,j,\beta)$ in the right hand side of (2.44).

$$\begin{aligned}
&(\Lambda_N-\lambda_{j,\beta})c(N,j,\beta)=O(\rho^{-p\alpha})\\
&+\sum_{\substack{(\beta_1,j_1)\in\boldsymbol{Q}(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in\boldsymbol{Q}(\rho^\alpha,6r_2)\\(j+j_1+j_2,\beta+\beta_1+\beta_2)=(j,\beta)}}\frac{S\left(j,\beta,j^1,\beta^1\right)S\left(j^1,\beta^1,j,\beta\right)}{\Lambda_N-\lambda_{j^1,\beta^1}}c(N,j,\beta)\\
&+\sum_{\substack{(\beta_1,j_1)\in\boldsymbol{Q}(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in\boldsymbol{Q}(\rho^\alpha,6r_2)\\(j+j_1+j_2,\beta+\beta_1+\beta_2)\neq(j,\beta)}}\frac{S\left(j,\beta,j^1,\beta^1\right)S\left(j^1,\beta^1,j^2,\beta^2\right)}{\Lambda_N-\lambda_{j^1,\beta^1}}c(N,j^2,\beta^2).
\end{aligned}\tag{2.52}$$

Substituting the equation (2.51) into the second sum of the equation (2.52), we get

$$
\begin{aligned}
&(\Lambda_N - \lambda_{j,\beta})c(N,j,\beta) \\
&= \sum_{\substack{(\beta_1,j_1)\in \boldsymbol{Q}(\rho^\alpha,6r_1)\\ (\beta_2,j_2)\in \boldsymbol{Q}(\rho^\alpha,6r_2)\\ (j^2,\beta^2)=(j,\beta)}} \frac{S\left(j,\beta,j^1,\beta^1\right)S\left(j^1,\beta^1,j,\beta\right)}{\Lambda_N - \lambda_{j^1,\beta^1}} c(N,j,\beta) \\
&+ \sum_{\substack{(\beta_1,j_1)\in \boldsymbol{Q}(\rho^\alpha,6r_1)\\ (\beta_2,j_2)\in \boldsymbol{Q}(\rho^\alpha,6r_2)\\ (j^2,\beta^2)\neq(j,\beta)\\ (\beta_3,j_3)\in \boldsymbol{Q}(\rho^\alpha,6r_3)}} \frac{S\left(j,\beta,j^1,\beta^1\right)S\left(j^1,\beta^1,j^2,\beta^2\right)S\left(j^2,\beta^2,j^3,\beta^3\right)}{(\Lambda_N - \lambda_{j^1,\beta^1})(\Lambda_N - \lambda_{j^2,\beta^2})} c(N,j^3,\beta^3) \\
&+ O(\rho^{-p\alpha}).
\end{aligned} \tag{2.53}
$$

Again isolating terms $c(N,j,\beta)$ in the last sum of the equation (2.53), we obtain

$$
\begin{aligned}
&(\Lambda_N - \lambda_{j,\beta})c(N,j,\beta) \\
&= [\sum_{\substack{(\beta_1,j_1)\in \boldsymbol{Q}(\rho^\alpha,6r_1)\\ (\beta_2,j_2)\in \boldsymbol{Q}(\rho^\alpha,6r_2)\\ (j^2,\beta^2)=(j,\beta)}} \frac{S\left(j,\beta,j^1,\beta^1\right)S\left(j^1,\beta^1,j,\beta\right)}{\Lambda_N - \lambda_{j^1,\beta^1}} \\
&+ \sum_{\substack{(\beta_1,j_1)\in \boldsymbol{Q}(\rho^\alpha,6r_1)\\ (\beta_2,j_2)\boldsymbol{Q}(\rho^\alpha,6r_2)\\ (\beta_3,j_3)\in \boldsymbol{Q}(\rho^\alpha,6r_3)\\ (j^2,\beta^2)\neq(j,\beta)\\ (j^3,\beta^3)=(j,\beta)}} \frac{S\left(j,\beta,j^1,\beta^1\right)S\left(j^1,\beta^1,j^2,\beta^2\right)S\left(j^2,\beta^2,j,\beta\right)}{(\Lambda_N - \lambda_{j^1,\beta^1})(\Lambda_N - \lambda_{j^2,\beta^2})}] c(N,j,\beta) \\
&+ \sum_{\substack{(\beta_1,j_1)\in \boldsymbol{Q}(\rho^\alpha,6r_1)\\ (\beta_2,j_2)\in \boldsymbol{Q}(\rho^\alpha,6r_2)\\ (\beta_3,j_3)\in \boldsymbol{Q}(\rho^\alpha,6r_3)\\ (j^2,\beta^2)\neq(j,\beta)\\ (j^3,\beta^3)\neq(j,\beta)}} \frac{S\left(j,\beta,j^1,\beta^1\right)S\left(j^1,\beta^1,j^2,\beta^2\right)S\left(j^2,\beta^2,j^3,\beta^3\right)}{(\Lambda_N - \lambda_{j^1,\beta^1})(\Lambda_N - \lambda_{j^2,\beta^2})} c(N,j^3,\beta^3) \\
&+ O(\rho^{-p\alpha}).
\end{aligned} \tag{2.54}
$$

In this way, iterating $2p$ times, we get

$$
(\Lambda_N - \lambda_{j,\beta})c(N,j,\beta) = \left[\sum_{k=1}^{2p} \tilde{S}_k\right] c(N,j,\beta) + \tilde{R}_{2p} + O(\rho^{-p\alpha}), \tag{2.55}
$$

where

$$\tilde{S}_k(\Lambda_N, \lambda_{j,\beta}) = \sum_{\substack{(\beta_1, j_1)\in \boldsymbol{Q}(\rho^\alpha, 6r_1) \\ (\beta_{k+1}, j_{k+1})\in \boldsymbol{Q}(\rho^\alpha, 6r_{k+1}) \\ (j^{k+1}, \beta^{k+1}) = (j,\beta) \\ (j^s,\beta^s)\neq (j,\beta),\ s=2,\ldots,k}} \left(\prod_{i=1}^{k} \frac{S\left(j^{i-1}, \beta^{i-1}, j^i, \beta^i\right)}{(\Lambda_N - \lambda_{j^i,\beta^i})} \right) S\left(j^k, \beta^k, j, \beta\right) \tag{2.56}$$

and

$$\tilde{R}_k = \sum_{\substack{(\beta_1, j_1)\in \boldsymbol{Q}(\rho^\alpha, 6r_1) \\ (\beta_{k+1}, j_{k+1})\in \boldsymbol{Q}(\rho^\alpha, 6r_{k+1}) \\ (j^s,\beta^s)\neq (j,\beta),\ s=2,\ldots,k+1}} \left(\prod_{i=1}^{k} \frac{S\left(j^{i-1}, \beta^{i-1}, j^i, \beta^i\right)}{(\Lambda_N - \lambda_{j^i,\beta^i})} \right) S\left(j^k, \beta^k, j^{k+1}, \beta^{k+1}\right) c(N, j^{k+1}, \beta^{k+1}). \tag{2.57}$$

Now we estimate $\tilde{S}_k$ and $\tilde{R}_k$. For this, we consider the terms which appear in the denominators of (2.56) and (2.57). By the conditions under the summations in (2.56) and (2.57), we have $j_1+j_2+\ldots+j_i \neq 0$ or $\beta_1+\beta_2+\ldots+\ldots\beta_i \neq 0$, for $i = 2, 3, \ldots, k$.

If $\beta_1 + \beta_2 + \ldots + \ldots \beta_i \neq 0$, then by (2.45) and (2.50), we have

$$|\Lambda_N - \lambda_{j^i,\beta^i}| > \frac{1}{2}\rho^{\alpha_2}. \tag{2.58}$$

If $\beta_1 + \beta_2 + \ldots + \ldots \beta_i = 0$, i.e., $j_1 + j_2 + \ldots + j_i \neq 0$, then by a well-known theorem

$$|\lambda_{j,\beta} - \lambda_{j^i,\beta^i}| = |\lambda_j - \lambda_{j^i}| > c_{18},$$

hence by (2.50), we obtain

$$|\Lambda_N - \lambda_{j^i,\beta^i}| > \frac{1}{2}c_{19}. \tag{2.59}$$

Since $\beta_k \neq 0$ for all $k \leq 2p$, the relation $\beta_1 + \beta_2 + \cdots + \beta_i = 0$ implies $\beta_1 + \beta_2 + \cdots + \beta_{i\pm 1} \neq 0$. Therefore the number of multiplicands $\Lambda_N - \lambda_{j^i,\beta^i}$ in (2.57) satisfying

(2.58) is no less then p. Thus, by (2.36), (2.58) and (2.59), we get

$$\tilde{S}_1 = O(\rho^{-\alpha_2}), \quad \tilde{R}_{2p} = O(\rho^{-p\alpha_2}). \tag{2.60}$$

Theorem 2.3.2. *(a) For every eigenvalue $\lambda_{j,\beta}$ of $L(P(s))$ such that $\beta + j\delta \in V'_\delta(\rho^{\alpha_1})$, there exists an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = \lambda_{j,\beta} + E_{k-1} + O(\rho^{-k\alpha_2}), \tag{2.61}$$

where $E_0 = 0$, $E_s = \sum_{k=1}^{2p} \tilde{S}_k(E_{s-1} + \lambda_{j,\beta}, \lambda_{j,\beta})$, $s = 1, 2, \ldots$.

(b) If

$$|\Lambda_N - \lambda_{j,\beta}| < c_{20} \tag{2.62}$$

and

$$|c(N, j, \beta)| > \rho^{-q\alpha} \tag{2.63}$$

hold then Λ_N satisfies (2.61).

Proof. By Lemma 2.2.1 (a) - (b), there exists N satisfying the conditions (2.62) and (2.63) in part (b). Hence it suffices to prove part (b). By (2.45) and (2.62), the triples (N, j^k, β^k) satisfy the iterability condition in (2.40). Hence we can use (2.55) and (2.60). Now we prove the theorem by induction:

For $k = 1$, to prove (2.61), we divide both sides of the equation (2.55) by $c(N, j, \beta)$ and use the estimations (2.60).

Suppose that (2.61) holds for $k = s$, i.e.,

$$\Lambda_N = \lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}). \tag{2.64}$$

To prove that (2.61) is true for $k = s+1$, in (2.55) we substitute the expression (2.64) for Λ_N into $\sum_{k=1}^{2p} \tilde{S}_k(\Lambda_N, \lambda_{j,\beta})$, then we get

$$\begin{aligned}&(\Lambda_N - \lambda_{j,\beta})c(N,j,\beta)\\&= \left[\sum_{k=1}^{2p} \tilde{S}_k\left(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}\right)\right] c(N,j,\beta) + \tilde{R}_{2p} + O(\rho^{-p\alpha}),\end{aligned} \tag{2.65}$$

dividing both sides of the last equality by $c(N,j,\beta)$ and using Lemma 2.2.1(ii), we obtain

$$\Lambda_N = \lambda_{j,\beta} + \sum_{k=1}^{2p} \tilde{S}_k\left(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}\right) + O(\rho^{-(p-q)\alpha}). \tag{2.66}$$

Now we add and subtract the term $\sum_{k=1}^{2p} \tilde{S}_k\left(E_{s-1} + \lambda_{j,\beta}, \lambda_{j,\beta}\right)$ in (2.66), then we have

$$\begin{aligned}&\Lambda_N = \lambda_{j,\beta} + E_s + O(\rho^{-(p-q)\alpha})\\&+\left[\sum_{k=1}^{2p} \tilde{S}_k\left(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}\right) - \sum_{k=1}^{2p} \tilde{S}_k\left(E_{s-1} + \lambda_{j,\beta}, \lambda_{j,\beta}\right)\right].\end{aligned} \tag{2.67}$$

Now, we first prove that $E_j = O(\rho^{-\alpha_2})$ by induction. $E_0 = 0$. Suppose that $E_{j-1} = O(\rho^{-\alpha_2})$, then $a = \lambda_{j,\beta} + E_{j-1}$ satisfies (2.58) and (2.59). Hence we get

$$\tilde{S}_1(a, \lambda_{j,\beta}) = O(\rho^{-\alpha_2}) \Rightarrow E_j = O(\rho^{-\alpha_2}). \tag{2.68}$$

To prove the theorem, we need to show that the expression in the square brackets in (2.67) is equal to $O(\rho^{-(s+1)\alpha_2})$. This can be easily checked by (2.68) and the obvious relation

$$\frac{1}{\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}) - \lambda_{j^k,\beta^k}} - \frac{1}{\lambda_{j,\beta} + E_{s-1} + \lambda_{j^k,\beta^k}} = O(\rho^{-(s+1)\alpha_2}), \tag{2.69}$$

for $\beta^k \neq \beta$. The theorem is proved. □

CHAPTER THREE
THE SPECTRAL THEORY OF SCHRÖDINGER OPERATORS ON INFINITE METRIC GRAPHS

This chapter is organized as follows. In Section 3.1, we give description of main functional spaces on metric graphs. Section 3.2 contains the definition and main properties of Schrödinger operators with locally integrable potentials on metric graphs. In Section 3.3, first, we prove essential self-adjointness of the Hamiltonian. Then we obtain a description of the bottom of essential spectrum. In Section 3.4, we prove theorems on the discreteness of the negative part of the spectrum and of the whole spectrum that extend some classical results for one dimensional Schrödinger operators. Finally, in Section 3.5, we show under natural assumptions that eigenfunctions corresponding to isolated eigenvalues of finite multiplicity decay at infinity exponentially fast.

3.1 Main Functional Spaces on Metric Graphs

Let us introduce basic functional spaces on metric graphs (we employ the standard notations of functional spaces on intervals of real line). We denote by $L^2(\Gamma)$ the space of all complex valued functions which are square integrable on Γ with respect to the measure dx. More explicitly, this space consists of all measurable functions f such that $f|_e \in L^2(e)$ for all $e \in E$ and

$$\|f\|^2 = \|f\|^2_{L^2} = \sum_{e\in E} \|f\|^2_{L^2(e)} < \infty \,.$$

The Sobolev space $H^1(\Gamma)$ consists of all *continuous* complex valued functions f on Γ such that $f|_e \in H^1(e)$ for all edges $e \in E$ and

$$\|f\|^2_{H^1} = \sum_{e\in E} \|f\|^2_{H^1(e)} < \infty \,.$$

We also need the standard space $L^1_{loc}(\Gamma)$ with respect to the measure dx. It consists of all functions which are absolutely integrable on every edge. Finally, we introduce the space $BS(\Gamma)$ of Stepanov bounded functions (known also as uniform L^1 space see Simon (1982)). It consists of all functions $f \in L^1_{loc}(\Gamma)$ such that

$$\|f\|_{BS} = \sup_{e \in E} \|f\|_{L^1(e)} < \infty \,.$$

We need the following simple lemma.

Lemma 3.1.1. *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\int_\Gamma |f(x)||u(x)|^2 dx \le \|f\|_{BS} \left(\varepsilon \|u'\|^2 + C_\varepsilon \|u\|^2 \right) ,$$

whenever $f \in BS(\Gamma)$ and $u \in H^1(\Gamma)$.

Proof. Without loss of generality we may suppose that u is a real valued function. Since $u \in H^1(\Gamma)$, then it is continuous and, hence, there exists $y_e \in e$ such that

$$u^2(y_e) = \frac{1}{l_e} \int_e u^2(x) dx \,.$$

On the other hand, for any $\varepsilon > 0$

$$\left| \frac{d}{dx} u^2(x) \right| = |2u'(x)u(x)| \le \varepsilon |u'(x)|^2 + \varepsilon^{-1} u^2(x) \,,$$

which yields

$$\int_e \left| \frac{d}{dx} u^2(x) \right| \le \varepsilon \|u'\|^2_{L^2(e)} + \varepsilon^{-1} \|u\|^2_{L^2(e)} \,.$$

Then we have

$$\begin{aligned} u^2(y) - u^2(y_e) = \int_{y_e}^{y} \frac{d}{dx} u^2(x) dx &\le \int_e \left| \frac{d}{dx} u^2(x) \right| dx \\ &\le \varepsilon \|u'\|^2_{L^2(e)} + \varepsilon^{-1} \|u\|^2_{L^2(e)}, \end{aligned}$$

and, by the definition of y_e,

$$u^2(y) \leq \varepsilon \|u'\|^2_{L^2(e)} + (\varepsilon^{-1} + l_e^{-1})\|u\|^2_{L^2(e)} \,.$$

Hence,

$$\begin{aligned}\sum_{e\in E}\int_e |f(y)|u^2(y)dy \leq & \left(\sup_{e\in E}\int_e |f(y)|dy\right)\sum_{e\in E}\|u^2\|_{L^\infty(e)} \\ \leq & \|f\|_{BS}\left(\varepsilon\sum_{e\in E}\|u'\|^2_{L^2(e)} + C_\varepsilon \sum_{e\in E}\|u\|^2_{L^2(e)}\right) \\ = & \|f\|_{BS}\left(\varepsilon\|u'\|^2 + C_\varepsilon\|u\|^2\right) \,,\end{aligned}$$

where $C_\varepsilon = \varepsilon^{-1} + \underline{l}^{-1}$. $\square$

Let us introduce more functional spaces on metric graphs. The space $L^\infty(\Gamma)$ consists of essentially bounded functions with the standard ess sup-norm.

Also we need some spaces of compactly supported functions and local Sobolev spaces. The space $H^1_{comp}(\Gamma)$ consists of all compactly supported functions from $H^1(\Gamma)$. This is a locally convex linear topological space. The space $H^1_{loc}(\Gamma)$ consists of all continuous functions u such that $u \in H^1(e)$ for all edge e. This is also a locally convex space. The negative Sobolev space $H^{-1}(\Gamma)$ is the space of all continuous anti-linear functionals on $H^1_{comp}(\Gamma)$, *i.e.*, the anti-dual space to $H^1_{comp}(\Gamma)$. The duality pairing, as well as the inner product in L^2, is denoted by $(.,.)$.

Certainly,

$$H^1_{comp}(\Gamma) \subset H^1(\Gamma) \subset L^2(\Gamma) \subset H^{-1}(\Gamma) \subset H^{-1}_{loc}(\Gamma)$$

(continuous and dense embeddings). In what follows we also need the continuous embedding

$$H^1(\Gamma) \subset L^\infty(\Gamma) \,.$$

Indeed, by (Brezis, 2010, Theorem 8.8), there exists a constant $K > 0$ independent of

$e \in E$ and such that $|v(x)| \leq K\|v\|_{H^1(e)}$ for all $v \in H^1(e)$. This implies immediately that

$$|v(x)| \leq K\|v\|_{H^1(\Gamma)}, \quad x \in \Gamma \tag{3.1}$$

for all $v \in H^1(\Gamma)$.

Throughout this chapter we shall use the following sequence of cut-off functions defined on Γ (see, Kuchment (2004)). Fix a vertex $o \in \Gamma$. For any integer $n > 0$, let $\Gamma_n \subset \Gamma$ be the union of all edges e such that both endpoints of e are at a distance at most n from o. This is an exhausting sequence of compact sets. Let us fix a C^2 function $\phi(x)$ on $[0, \underline{l}/4]$ such that it is equal to 1 in a small neighborhood of 0, equal to 0 in a small neighborhood of $\underline{l}/4$ and $0 \leq \phi \leq 1$. We define the cut-off function φ_n on Γ as follows. It is equal to 1 on Γ_n and to 0 on all edges which do not have vertices in Γ_n. Now let e be an edge with only one vertex v in Γ_n. Identifying e with the interval $[0, l_e]$, without loss of generality we may suppose that the vertex v corresponds to the endpoint 0. Then we define φ_n to be equal to ϕ on $[0, \underline{l}/4]$ and 0 on the remaining part of $[0, l_e]$. It is easily seen that $0 \leq \varphi_n \leq 1$, and there exists a constant $C > 0$ independent of n such that $|\varphi_n'| \leq C$ and $|\varphi_n''| \leq C$ on all edges. Notice that in Section 3.3 we shall use cut-off functions with a specific choice of the function ϕ. Namely, suppose that $0 < \varepsilon_0 < \varepsilon_1 < \underline{l}/4$. Making use of a partition of unity, we can choose the function ϕ that satisfies the following additional properties: $\phi(x) = 1$ for $x \in [0, \varepsilon_0]$ and $\phi(x) = 1 - (x - \varepsilon_0)^6$ for $x \in (\varepsilon_0, \varepsilon_1]$. Then a straightforward verification shows that the function $(1 - \phi^2(x))^{1/2}$ is of class C^2.

3.2 Schrödinger Operators

Let $V(x)$ be a real function on Γ. We consider the Schrödinger operator associated to the differential expression

$$\mathcal{L} = -\frac{d^2}{dx^2} + V(x)$$

together with certain conditions at the vertices of Γ. Throughout this chapter we accept the following assumptions

$(V1)$ *The function V is locally integrable on* $\Gamma : V(x) \in L^1_{loc}(\Gamma)$;

$(V2)$ $V_- \in BS(\Gamma) : \sup\limits_{e\in E} \int\limits_e V_-(x) < \infty$.

Here and thereafter we use the following notation $a_+ = \max\{a, 0\}$ and $a_- = -\min\{a, 0\}$.

Let $\mathcal{D}_0(\Gamma)$ be the space of all compactly supported function φ on Γ such that $\varphi|_e \in C^2(e)$ for all edges $e \in E$,

$$\varphi \text{ is continuous at all vertices of } \Gamma \tag{3.2}$$

and

$$\sum_{e\in E_v} \frac{d\varphi}{dn_e}(v) = 0 \tag{3.3}$$

for all vertices $v \in V$, where $\frac{d}{dn_e}$ stands for the outward derivatives at the endpoints of the edge e. Condition (3.3) is the so-called Kirchhoff vertex condition. In the case when the degree $d_v = 1$ this is the standard Neumann boundary condition (equally well it can be replaced by the Dirichlet condition). Making use of the cut-off functions φ_n and standard approximation techniques on finite intervals it is not difficult to see that the space $\mathcal{D}_0(\Gamma)$ is dense in the space $H^1(\Gamma)$ and, hence, in $L^2(\Gamma)$.

First we introduce the operator L_0 in $L^2(\Gamma)$ with the domain $D(L_0) = \mathcal{D}_0(\Gamma)$ defined by

$$(L_0\varphi)(x) = (\mathcal{L}\varphi)(x)$$

on every edge $e \in E$. It can be easily shown that L_0 is a symmetric operator. Actually, the standard integration by parts over all edges shows that $(L_0u, v) = (u, L_0v)$ for each $u, v \in \mathcal{D}_0(\Gamma)$.

Lemma 3.1.1 implies that for every $\theta > 0$ there exists a constant $C_\theta > 0$, depending on $\|V_-\|_{BS}$, such that

$$\int_\Gamma V_-(x)|u(x)|^2 dx \le \theta\|u'\|^2 + C_\theta\|u\|^2 \tag{3.4}$$

for all $u \in H^1(\Gamma)$.

Let $q_0(u) = (L_0 u, u)$, $u \in \mathcal{D}_0(\Gamma)$, be the quadratic form associated to the operator L_0. Making use of inequality (3.4) with θ small enough, we obtain that there exist constants $\alpha_0 > 0$ and $\lambda_0 > 0$ such that

$$\begin{aligned}
q_0(u) =& \int_\Gamma \left(|u'(x)|^2 + V(x)|u(x)|^2\right) dx \\
\ge& \int_\Gamma \left(|u'(x)|^2 + \left(|V(x)| - 2V_-(x)\right)|u(x)|^2\right) dx \\
=& \int_\Gamma \left(|u'(x)|^2 + |V(x)||u(x)|^2 - 2V_-(x)|u(x)|^2\right) dx \\
\ge& \int_\Gamma \left(|u'(x)|^2 + |V(x)||u(x)|^2\right) dx - 2\theta\|u'\|^2 - 2C_\theta\|u\|^2 \\
=&(1-2\theta)\int_\Gamma |u'(x)|^2 dx + \int_\Gamma |V(x)||u(x)|^2 dx - 2C_\theta\|u\|^2 \\
>&(1-2\theta)\int_\Gamma \left(|u'(x)|^2 + |V(x)||u(x)|^2\right) dx - 2C_\theta\|u\|^2 \\
=&\alpha_0 \int_\Gamma \left(|u'(x)|^2 + |V(x)||u(x)|^2\right) dx - \lambda_0\|u\|^2_{L^2(\Gamma)}
\end{aligned} \tag{3.5}$$

for all $u \in \mathcal{D}_0(\Gamma)$. Here we choose $\alpha_0 = 1 - 2\theta$ and $\lambda_0 = 2C_\theta$ with θ small enough.

In particular, q_0 is bounded below as well as the operator L_0. Hence, q_0 is a closable quadratic form, by the Friedrichs extension theorem. Its closure is denoted by q. Obviously,

$$q_0(u) \le \int_\Gamma \left(|u'(x)|^2 + |V(x)||u(x)|^2\right) dx \tag{3.6}$$

for all $u \in \mathcal{D}_0(\Gamma)$. Together with (3.5), this implies that the domain $D(q)$ of q consists of all $u \in H^1(\Gamma)$ such that

$$\int_\Gamma \left(|u'(x)|^2 + |V(x)||u(x)|^2\right) dx < \infty\,.$$

Furthermore, inequalities (3.5) and (3.6) hold for the form q and all $u \in D(q)$. Being equipped with the norm

$$(q(u) + (\lambda_0 + 1)\|u\|^2)^{1/2},$$

the space $D(q)$ is a Hilbert space continuously embedded into $H^1(\Gamma)$.

Since the form q is closed and bounded below, it generates the self-adjoint operator L, the so-called Friedrichs extension of L_0, which is bounded below (see, *e.g.*, Blank et al. (2008) and Reed & Simon (1975)). The operator L is defined as follows. A function $u \in D(q)$ belongs to the domain $D(L)$ of L and $Lu = f \ni L^2(\Gamma)$ if and only if

$$\int_\Gamma (u'\overline{\varphi'} + Vu\overline{\varphi})\,dx = \int_\Gamma f\overline{\varphi}\,dx \tag{3.7}$$

for all $\varphi \in D(q)$. Testing (3.7) on smooth functions with compact support in any open edge, we see that on each edge the function u satisfies the equation $\mathcal{L}u = f$ in the weak sense. Hence, the derivative u' is an absolutely continuous function on each edge. Choosing a test function φ such that its support belongs to a sufficiently small neighborhood of some vertex v, while on a smaller neighborhood $\varphi = 1$, we obtain, after integration by parts, that u satisfies vertex condition (3.3).

3.3 Essential Self-Adjointness and the Bottom of Essential Spectrum

In this section, first, we show that the Friedrichs extension, L, of L_0 is the only self-adjoint extension of L_0. In other words, the operator L_0 is essentially self-adjoint.

For this aim, let us consider the maximal operator $\tilde{L} = L_0^*$ associated to the differential expression $\mathcal{L}$ and vertex conditions (3.2) and (3.3). The domain $D(\tilde{L})$ consists of all functions $u \in L^2(\Gamma)$ such that

(i) *u and u' are absolutely continuous functions on each edge, and, hence, $u'' \in L^1_{loc}(\Gamma)$;*

(ii) *u satisfies vertex conditions (3.2) and (3.3);*

(iii) $\mathcal{L}u \in L^2(\Gamma)$.

The operator $\tilde{L}$ is defined by $\tilde{L}u = \mathcal{L}u$ for all $u \in D(\tilde{L})$.

Obviously, $L \subset \tilde{L}$. Actually, we have the following

Theorem 3.3.1. *Under assumptions* $(V1)$ *and* $(V2)$*,* $L = \tilde{L}$*.*

Proof. Replacing $V(x)$ by $V(x) + \lambda$ with sufficiently large λ, we may suppose that $L_0 \geq I$ in the sense that

$$(L_0\varphi, \varphi) = \int_\Gamma (|\varphi'|^2 + V(x)|\varphi|^2)dx \geq \|\varphi\|^2 \tag{3.8}$$

for all $\varphi \in D(L_0)$.

The key point of the proof is the following inequality. Suppose that $u \in D(\tilde{L})$ and $f = \tilde{L}u$. Then

$$\|u\| \leq \|f\| \,. \tag{3.9}$$

Notice that it is enough to prove inequality (3.9) for real valued functions only.

By the definition of $\tilde{L}$, $Vu \in L^1_{loc}(\Gamma)$ and

$$\int_\Gamma (u'\varphi' + Vu\varphi)dx = \int_\Gamma f\varphi dx \tag{3.10}$$

for all $\varphi \in \mathcal{D}_0(\Gamma)$. Let $\psi \in \mathcal{D}_0(\Gamma)$. Elementary differentiation shows that $\psi^2 u \in D(\tilde{L})$ and has a compact support. A standard approximation argument in one dimension shows that there exists a sequence $u_k \in \mathcal{D}_0$ such that $u_k \to u$ and $u_k' \to u'$ locally uniformly on Γ. Then $\psi^2 u_k \in \mathcal{D}_0(\Gamma)$. Taking $\varphi = \psi^2 u_k$ in (3.10) and passing to the limit, we obtain that

$$\int_\Gamma \left(u'(u\psi^2)' + Vu^2\psi^2 \right) dx = \int_\Gamma fu\psi^2 dx \,. \tag{3.11}$$

By the identity

$$u'(u\psi^2)' = ((u\psi)')^2 - u^2(\psi')^2 \,,$$

equation (3.11) becomes

$$\int_\Gamma \left(((u\psi)')^2 + Vu^2\psi^2 - u^2(\psi')^2\right) dx = \int_\Gamma fu\psi^2 dx\,. \tag{3.12}$$

Inequality (3.8) and a density argument imply that

$$\int_\Gamma \left(((u\psi)')^2 + V(u\psi)^2\right) dx \geq \int_\Gamma (u\psi)^2 dx\,. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain that

$$\int_\Gamma \left((u\psi)^2 - |\psi'|^2u^2\right) dx \leq \int_\Gamma fu\psi^2 dx\,. \tag{3.14}$$

Taking as ψ the cut-off function φ_n constructed in Section 3.1, we obtain

$$\begin{aligned}\int_\Gamma (u\varphi_n)^2 dx &\leq \int_\Gamma fu\varphi_n^2 dx + \int_\Gamma u^2|\varphi_n'|^2 dx\\ &\leq \left(\int_\Gamma (u\varphi_n)^2 dx\right)^{1/2} \cdot \left(\int_\Gamma (f\varphi_n)^2 dx\right)^{1/2} + \int_\Gamma u^2|\varphi_n'|^2 dx\,.\end{aligned}$$

Since $0 \leq \varphi_n \leq 1$ and $\varphi_n = 1$ on Γ_n, the last inequality and the inequality $2ab \leq a^2+b^2$ imply that

$$\int_{\Gamma_n} u^2 dx \leq \int_\Gamma (u\varphi_n)^2 dx \leq \int_\Gamma f^2 dx + 2\int_\Gamma u^2|\varphi_n'|^2 dx\,.$$

Since φ_n' is bounded uniformly with respect to n, $\varphi_n' = 0$ on Γ_n and $u \in L^2(\Gamma)$, we have that

$$\int_\Gamma u^2|\varphi_n'|^2 dx \leq C\int_{\Gamma\setminus\Gamma_n} u^2 dx \to 0 \text{ as } n \to \infty\,,$$

and inequality (3.9) follows.

To complete the proof it is enough to show that $D(\tilde{L}) \subset D(L)$. Since $L \geq I$, then it possesses the bounded inverse operator L^{-1}. Let $u \in D(\tilde{L})$ and $v = u - L^{-1}\tilde{L}u$. Then $v \in D(\tilde{L})$ and $\tilde{L}v = 0$ because $L \subset \tilde{L}$. However, inequality (3.9) implies that the operator $\tilde{L}$ has zero kernel and, hence, $v = 0$. As consequence, $v = 0$ and $u \in D(L)$. This completes the proof. □

Theorem 3.3.1 shows that the operator L is the only self-adjoint extension of L_0. Hence, the operator L_0 is essentially self-adjoint, and L is the closure of L_0. In particular, $\mathcal{D}_0(\Gamma)$ is dense in $D(L)$ with respect to the graph norm.

Our next aim is to obtain a characterization of the bottom of essential spectrum.

For any compact subset K of Γ we denote by $\mathcal{D}_0(\Gamma \setminus K)$ the set of all functions $\varphi \in \mathcal{D}_0(\Gamma)$ such that $\operatorname{supp}\varphi \subset \Gamma \setminus K$. We denote by $\sigma(L)$ and $\sigma_{ess}(L)$ the spectrum and the essential spectrum of L.

It is well known that the bottom of the spectrum of a self-adjoint bounded below operator coincides with the infimum of its Raylaigh quotient. Due to the density of $\mathcal{D}_0(\Gamma)$ in $D(L)$ with respect to the graph norm, this implies immediately that $\inf \sigma(L) = \Lambda(L)$, where

$$\Lambda(L) = \inf\left\{ \frac{(L\varphi, \varphi)}{\|\varphi\|^2} : \varphi \in \mathcal{D}_0(\Gamma), \varphi \neq 0 \right\}.$$

Theorem 3.3.2. *Under assumptions $(V1)$ and $(V2)$*

$$\inf \sigma_{ess}(L) = \sup_{\Gamma_n \Subset \Gamma} \inf\left\{ \frac{(L\varphi, \varphi)}{\|\varphi\|^2} : \varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n), \varphi \neq 0 \right\}, \tag{3.15}$$

where the supremum is taken over the sequence of compact sets Γ_n defined in Section 3.1.

To prove Theorem 3.3.2 we need the following lemma.

Lemma 3.3.3. *Suppose that, for some $\lambda \in \mathbb{R}$,*

$$(L\varphi, \varphi) \geq \lambda \|\varphi\|^2 \tag{3.16}$$

for all $\varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n)$. Then there exists a non-negative function $W \in \mathcal{D}_0(\Gamma)$ such that

$$(L_W\varphi, \varphi) \geq \lambda \|\varphi\|^2 \tag{3.17}$$

for all $\varphi \in \mathcal{D}_0(\Gamma)$, where $L_W = L + W$.

Proof. Let φ_n be the sequence of cut-off functions with special choice of the function ϕ (see the end of Section 3.1). Then $\varphi_n \in \mathcal{D}_0$, and the function $\psi_n(x) = (1-\varphi_n^2(x))^{1/2}$ is C^2-smooth on each edge and satisfies conditions (3.2) and (3.3). Furthermore, it is not difficult to verify the following identity.

$$\varphi'' = \varphi_n(\varphi_n\varphi)'' + \psi_n(\psi_n\varphi)'' + [(\varphi_n')^2 + (\psi_n')^2]\varphi\,. \tag{3.18}$$

Let $\varphi \in \mathcal{D}_0(\Gamma)$. Using (3.18), we obtain that

$$\begin{aligned}(L\varphi,\varphi) = (L(\varphi_n\varphi),\varphi_n\varphi) + (L(\psi_n\varphi),\psi_n\varphi)\\ - \int_\Gamma [(\varphi_n')^2 + (\psi_n')^2]|\varphi|^2 dx\,.\end{aligned} \tag{3.19}$$

Since supp$\psi_n \subset \Gamma \setminus \Gamma_n$, it follows from (3.16) that

$$(L(\psi_n\varphi),\psi_n\varphi) \geq \int_\Gamma \psi_n^2|\varphi|^2 dx\,. \tag{3.20}$$

By the definition of $\Lambda(L)$,

$$(L(\varphi_n\varphi),\varphi_n\varphi) \geq \Lambda(L)\int_\Gamma \varphi_n^2|\varphi|^2 dx\,. \tag{3.21}$$

Combining (3.19), (3.20) and (3.21), we obtain that

$$\begin{aligned}(L\varphi,\varphi) &\geq \int_\Gamma (\psi_n^2 + \Lambda(L)\varphi_n^2 - (\varphi_n')^2 - (\psi_n')^2)|\varphi|^2 dx\\ &= \int_\Gamma \alpha(x)|\varphi|^2 dx\end{aligned} \tag{3.22}$$

for all $\varphi \in \mathcal{D}_0(\Gamma)$, where

$$\alpha(x) = \psi_n^2 + \Lambda(L)\varphi_n^2 - (\varphi_n')^2 - (\psi_n')^2$$

is continuous function such that $\alpha(x) = 1$ outside a compact set. Then $\alpha(x) - 1$ has compact support. Let $\kappa := \sup(1-\alpha(x))$ and set

$$W(x) = \alpha(x) + \kappa \cdot \varphi_{n'}(x)\,,$$

where $n' > n$ is large enough. With this choice of W, (3.22) yields (3.17), and the proof is complete. □

Proof of Theorem 3.3.2. Fix a number λ such that

$$\lambda < \sup_{\Gamma_n \Subset \Gamma} \inf\left\{\frac{(L\varphi,\varphi)}{\|\varphi\|^2} : \varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n), \varphi \neq 0\right\}.$$

Hence, there exists $n \in \mathbb{N}$ such that

$$(L\varphi,\varphi) \geq \lambda\|\varphi\|^2$$

for all $\varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n)$. By Lemma 3.3.3, there exists a non-negative function $W \in \mathcal{D}_0(\Gamma)$ such that

$$((L_W)\varphi,\varphi) \geq \lambda\|\varphi\|^2$$

for all $\varphi \in \mathcal{D}_0(\Gamma)$, which implies that

$$\Lambda(L_W) = \inf \sigma(L_W) \geq \lambda\,. \tag{3.23}$$

Since W is a compactly supported function, the multiplication operator by W is compact. Hence, by Weyl's theorem, $\sigma_{ess}(L) = \sigma_{ess}(L_W)$. As consequence,

$$\inf \sigma_{ess}(L) \geq \inf \sigma(L_W) \geq \lambda\,,$$

and we conclude that

$$\inf \sigma_{ess}(L) \geq \sup_{\Gamma_n \Subset \Gamma} \inf\left\{\frac{(L_0\varphi,\varphi)}{\|\varphi\|^2} : \varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n), \varphi \neq 0\right\}. \tag{3.24}$$

To prove the reverse inequality let μ be any positive number such that

$\mu < \inf \sigma_{ess}(L)$. Let E_μ be the spectral projector of L that corresponds to the part of the spectrum below μ. Then E_μ is a finite rank projection so that

$$E_\mu = \sum_{i=1}^{N} (\cdot, \phi_i)\phi_i \, ,$$

where $\phi_i \in D(L)$ are orthonormal eigenfunctions of L with eigenvalues below μ. Hence, for any function $\varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n)$

$$\begin{aligned} \|E_\mu \varphi\| &\leq \sum_{i=1}^{N} |(\varphi, \phi_i)| \cdot \|\phi_i\|^2 \\ &\leq \sum_{i=1}^{N} \left(\int_{\Gamma \setminus \Gamma_n} |\phi_i|^2 dx \right)^{\frac{1}{2}} \|\phi_i\| \cdot \|\varphi\| \, . \end{aligned}$$

Therefore, for any $\varepsilon \in (0, 1)$, there exists an $n = n(\varepsilon) \in \mathbb{N}$ such that

$$\|E(\mu)\varphi\| \leqslant \varepsilon \cdot \|\varphi\| \tag{3.25}$$

for all $\varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n)$.

Now we have

$$\begin{aligned} (L\varphi, \varphi) &= \|L^{\frac{1}{2}}\varphi\|^2 \\ &= \|L^{\frac{1}{2}}(I - E_\mu)\varphi\|^2 + \|L^{\frac{1}{2}} E_\mu \varphi\|^2 \\ &\geq \|L^{\frac{1}{2}}(I - E_\mu)\varphi\|^2 = (L(I - E_\mu)\varphi, \varphi) \\ &\geq \mu \|(I - E_\mu)\varphi\|^2 \\ &\geq \mu \|\varphi - E_\mu \varphi\|^2 \\ &\geq \mu (\|\varphi\| - \|E_\mu \varphi\|)^2. \end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26), we obtain that

$$(L\varphi, \varphi) \geq \mu \cdot (1 - \varepsilon)^2 \cdot \|\varphi\|^2$$

for any $\varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n)$, which implies that

$$\sup_{\Gamma_n \Subset \Gamma} \inf \left\{ \frac{(L_0\varphi, \varphi)}{\|\varphi\|^2} : \varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n), \varphi \neq 0 \right\} \geq \mu \cdot (1-\varepsilon)^2 .$$

Letting $\varepsilon \to 0$, we conclude that

$$\sup_{\Gamma_n \Subset \Gamma} \inf \left\{ \frac{(L_0\varphi, \varphi)}{\|\varphi\|^2} : \varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n), \varphi \neq 0 \right\} \geq \inf \sigma_{ess}(L) .$$

Together with (3.24), this proves (3.15).

The proof is complete. $\square$

3.4 Discreteness of Spectrum

We begin with a sufficient condition for the discreteness of the negative part of spectrum.

Theorem 3.4.1. *In addition to Assumptions $(V1)$ and $(V2)$, suppose that*

$$\int_e V_-(x)dx \to 0$$

in the sense that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\int_e V_-(x)dx < \varepsilon$$

for all $e \in E$ such that $e \subset \Gamma \setminus \Gamma_n$. Then the negative part of spectrum, $\sigma(L) \cap (-\infty, 0)$, is discrete.

Proof. We need to prove that there is no negative part of the essential spectrum. By Theorem 3.3.2, it is enough to show that for every $\varepsilon \in (0,1)$ there exists $n \in \mathbb{N}$ such that

$$\int_\Gamma (|u'(x)|^2 - V_-(x)|u(x)|^2)dx \geq -\varepsilon \int_\Gamma |u(x)|^2 dx \tag{3.27}$$

for all $u \in \mathcal{D}_0(\Gamma \setminus \Gamma_n)$.

Let $\eta > 0$. Then there exists $n(\eta) \in \mathbb{Z}$ such that

$$\int_e V_-(x_e)dx_e < \eta$$

for all $e \in E$ such that $e \subset \Gamma \setminus \Gamma_{n(\eta)}$. For any $u \in \mathcal{D}_0(\Gamma \setminus \Gamma_{n(\eta)})$ there exists $x_{0,e} \in e$ such that

$$u^2(x_{0,e}) = \frac{1}{l_e} \int_e u^2(x)dx\,.$$

Then, for $x \in e$,

$$|u^2(x) - u^2(x_{0,e})| = 2\left|\int_{x_{0,e}}^x u'udx\right| \leq \int_e (u'^2 + u^2)dx\,.$$

From this, we have

$$\begin{aligned}
\int_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} V_-u^2(x)dx &= \int_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} V_-(u^2(x) - u^2(x_{0,e}))dx \\
&\quad + \int_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} V_-u^2(x_{0,e})dx \\
&\leq \eta \int_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} (u'^2(x) + u^2(x))dx + \frac{\eta}{l_e}\int_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} u^2(x)dx \\
&\leq \eta \int_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} (u'^2(x) + u^2(x))dx + \frac{\eta}{\underline{l}}\int_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} u^2(x)dx \\
&\leq \eta\left(1 + \frac{1}{\underline{l}}\right)\int_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} (u'^2(x) + u^2(x))dx\,.
\end{aligned}$$

Summing up over $e \subset \Gamma \setminus \Gamma_{n(\eta)}$, we obtain that

$$\sum_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} \int_e V_-u^2(x)dx \leq \eta\left(1 + \frac{1}{\underline{l}}\right) \sum_{e\subset\Gamma\setminus\Gamma_{n(\eta)}} \int_e (u'^2(x) + u^2(x))dx\,.$$

Taking η sufficiently small so that $\varepsilon = \eta(1 + \underline{l}^{-1}) < 1$, we obtain that

$$\int_\Gamma (u'^2(x) - V_-u^2(x))dx \geq -\varepsilon \int_\Gamma u^2(x)dx\,,$$

for all $u \in \mathcal{D}_0(\Gamma \setminus \Gamma_n)$ with $n = n(\eta)$. The proof is complete. $\square$

Now we prove a criterion for the discreteness of the whole spectrum. We employ the following terminology. An *interval* in Γ is a subinterval of any edge.

Theorem 3.4.2. *Under Assumptions* $(V1)$ *and* $(V2)$*,* $\sigma(L)$ *is discrete if and only if for every* $\alpha \in (0, \underline{l})$

$$\int_S V(x)dx \to \infty \tag{3.28}$$

as the interval S *of length* α *escapes to infinity (this means that for every* $M > 0$*, there exists* $n \in \mathbb{N}$ *such that*

$$\int_S V(x)dx \geq M$$

for all intervals S *of length* α *with the property that* $S \subset \Gamma \setminus \Gamma_n$ *) .*

Proof. (a) *Sufficiency*. Suppose that (3.28) holds. Without loss of generality, we assume that $L \geq I$. To prove that the spectrum of L is discrete it is sufficient to show that the negative part of $\sigma(L - \mu I)$ is discrete for all $\mu > 0$. By Theorem 3.3.2, to do this we have to prove the existence of $n \in \mathbb{N}$ such that

$$\int_\Gamma (|u'|^2 + V|u|^2)dx \geq \mu \int_\Gamma |u|^2 dx \tag{3.29}$$

for all $u \in \mathcal{D}_0(\Gamma \setminus \Gamma_n)$.

Now, given $\alpha \in (0, \underline{l})$, we choose $n(\alpha)$ such that for every interval $S \subset \Gamma \setminus \Gamma_{n(\alpha)}$ of length α

$$\int_S V(x)dx \geq 1\,. \tag{3.30}$$

For an edge e, we introduce the number

$$N(e) = \begin{cases} \dfrac{l_e}{\alpha} & \text{if } \frac{l_e}{\alpha} \text{ is an integer} \\ \left[\dfrac{l_e}{\alpha}\right] + 1 & \text{otherwise}\,. \end{cases}$$

Then we cover the edge e by intervals S_k^e, $k = 1, \dots, N(e)$, of length α in such a way that at most two of the intervals overlap.

Due to the continuity of u, in each interval S_k^e we can choose a point x_k^e such that

$$u^2(x^k) = \frac{\int_{S_k^e} V(x)u^2(x)dx}{\int_{S_k^e} V(x)dx}\,. \tag{3.31}$$

Estimating $|u^2(x) - u^2(x_k^e)|$ as in the proof of Theorem 3.4.1, we obtain the inequality

$$\begin{aligned}\int_{S_k^e} u^2(x)dx &= \int_{S_k^e} u^2(x_k^e)dx + \int_{S_k^e} (u^2(x) - u^2(x_k^e))dx \\ &\le \alpha\,\frac{\int_{S_k^e} V(x)u^2(x)dx}{\int_{S_k^e} V(x)dx} + \alpha \int_{S_k^e} (u'^2 + u^2)dx\,.\end{aligned}$$

Hence, by (3.30)

$$\int_{S_k^e} u^2(x)dx \le \alpha \int_{S_k^e} V(x)u^2(x)dx + \alpha \int_{S_k^e} (u'^2 + u^2)dx\,. \tag{3.32}$$

Summing up inequalities (3.32) over all $e \in \Gamma \setminus \Gamma_{n(\eta)}$ and all k, we obtain that

$$\int_\Gamma u^2(x)dx \le \alpha \int_\Gamma V(x)u^2(x)dx + \alpha \int_\Gamma (u'^2 + u^2)\,dx$$

for all $u \in \mathcal{D}_0(\Gamma \setminus \Gamma_{n(\eta)})$.

Since we suppose that $L \ge I$, then $q(\cdot)^{1/2}$ is an equivalent norm on the form domain $D(q)$. Since the embedding $D(q) \subset H^1(\Gamma)$ is continuous, $\|u\|_{H^1}^2 \le Cq(u)$. Therefore,

$$\int_{\Gamma\setminus\Gamma_{n(\eta)}} u^2(x)dx \le (1+C)\alpha \int_{\Gamma\setminus\Gamma_{n(\eta)}} (u'^2 + V(x)u^2)dx\,.$$

Taking $\alpha = ((1+C)\mu)^{-1}$, we obtain (3.29).

(b) *Necessity*. Assume that the spectrum $\sigma(L)$ is discrete, but for some $\alpha \in (0, \underline{l})$ there exist $\rho > 0$ and a sequence of intervals S_k of length α escaping to infinity and such that

$$\int_{S_k} V(x)dx \le \rho\,. \tag{3.33}$$

Obviously, we may assume that each edge contains at most one such interval and

$S_k \subset \Gamma \setminus \Gamma_k$. Now, we choose a sequence of $\psi_k \in \mathcal{D}_0(\Gamma)$ such that $\operatorname{supp} \psi_k \subset S_k$, $0 \le \psi_k \le 1$, $|\psi_k'| \le C$ for some $C > 0$, and $\psi_k = 1$ on some subinterval of length δ in S_k. Then $\psi_k \in \mathcal{D}_0(\Gamma \setminus \Gamma_k)$ and

$$\begin{aligned}\int_{S_k} (\psi_k'^2(x) + V(x)\psi_k^2(x))dx &\le \int_{S_k} \psi_k'^2(x)dx + \int_{S_k} V(x)\psi_k^2(x)dx \\ &\le \alpha C^2 + \int_{S_k} V(x)dx \\ &\le \alpha C^2 + \rho\,. \end{aligned} \tag{3.34}$$

On the other hand,

$$\int_{S_k} \psi_k^2(x)dx \ge \delta\,.$$

Hence,

$$\int_{\Gamma} (\psi_k'^2(x) + V(x)\psi_k^2(x))dx \le \frac{\alpha C^2 + \rho}{\delta} \int_{\Gamma} \psi_k^2(x)dx\,.$$

By Theorem 3.3.2, it follows that for

$$\sigma(L - \mu I) \cap (-\infty, 0)$$

contains points of essential spectrum whenever

$$\mu > \frac{\alpha C^2 + \rho}{\delta}\,.$$

This contradiction proves the required. □

3.5 Exponential Decay of the Eigenfunctions

Now before giving our result on exponential decay of eigenfunctions, we begin with some properties of Schrödinger operators with locally integrable potentials on metric graphs obtained in Section 3.2.

Under Assumption $(V1)$, the differential expression

$$\mathcal{L} = -\frac{d^2}{dx^2} + V(x)$$

generates an operator $\mathcal{L} : H^1_{loc}(\Gamma) \to H^{-1}_{loc}(\Gamma)$ defined as follows

$$(\mathcal{L}u, \upsilon) = \int_\Gamma (u'(x)\upsilon'(x) + V(x)u(x)\upsilon(x))\,dx, \quad \forall \upsilon \in H^1_{comp}(\Gamma).$$

In Section 3.2, we have started with a second order symmetric differential operator

$$L_0 u = -\frac{d^2 u}{dx^2} + V(x)u \tag{3.35}$$

on the domain $\mathcal{D}_0(\Gamma)$ that consists of all compactly supported function u on Γ such that $u|_e \in C^2(e)$ for all edges $e \in E$, satisfies vertex conditions (3.2) and (3.3). Under our assumptions, L_0 is a symmetric, bounded below operator in the space $L^2(\Gamma)$.

Let $q_0(u) = (L_0 u, u)$, $u \in \mathcal{D}_0(\Gamma)$, be the quadratic form associated to the symmetric operator L_0. Under Assumptions $(V1)$ and $(V2)$, the form q_0 is bounded below and, hence, closable. Furthermore, the closure q of q_0 generates a self-adjoint, semi-bounded below extension L of L_0. The operator L is the only self-adjoint extension of L_0. In other words, the operator L_0 is essentially self-adjoint. The form domain $D(q)$ is given by

$$D(q) = H_+ = \left\{ u \in H^1(\Gamma) : \int_\Gamma |V(x)||u(x)|^2 dx < \infty \right\}.$$

This is a Hilbert space continuously and densely embedded into $L^2(\Gamma)$. Its dual space is denoted by H_-. Obviously, $L^2(\Gamma) \subset H_- \subset H^{-1}_{loc}$ (continuously and densely). The operator L extends to a bounded linear operator $\hat{L} : H_+ \to H_-$ (see (Reed & Simon, 1980, Section VIII.6)). Actually, $\hat{L}$ is the restriction of $\mathcal{L}$ to H_+, while L is the restriction of $\mathcal{L}$ to the domain $D(L)$ of L. The domain $D(L)$ consists of all functions $u \in L^2(\Gamma)$ such that u and u' are absolutely continuous on each edge of Γ, satisfy vertex conditions (3.2) and (3.3), and $\mathcal{L}u \in L^2(\Gamma)$. Equipping $D(L)$ with the graph norm, we often regard L as a bounded linear operator from $D(L)$ into $L^2(\Gamma)$.

To obtain exponential decay of the eigenfunctions, we begin with the following lemma.

Lemma 3.5.1. *There exists a function η such that it is continuous on Γ, $\eta|_e \in C^2(e)$ on each edge e, with η' and η'' bounded on Γ, η satisfies the Kirchhoff vertex conditions, and*

$$d(x) - c_0 \leq \eta(x) \leq d(x) + c_0\,, \quad x \in \Gamma\,, \tag{3.36}$$

with $c_0 > 0$ independent of x.

Proof. We construct the function η so that at any vertex $\eta = d$. Consider any edge e that connects two vertices v and w, and identify it with the interval $[-m, m]$, where $2m = l_e$, so that the endpoint $-m$ corresponds to the vertex v. Let $a = d(v)$ and $b = d(w)$. Then on this edge

$$d(x) = \min[a + x + m, b + m - x]\,.$$

We define η to be the cubic polynomial such that

$$\eta(-m) = a, \quad \eta'(-m) = 0,$$
$$\eta(m) = b, \quad \eta'(m) = 0\,,$$

Then an elementary calculation shows that η is of the form

$$\eta(x) = \alpha x^3 + \beta x + \gamma\,,$$

where α, β and γ are given by

$$\alpha = \frac{a-b}{4m^3}\,,$$

$$\beta = -\frac{3(a-b)}{4m}$$

and

$$\gamma = \frac{a+b}{2}\,.$$

Now let

$$\theta(x) = \frac{b-a}{2m}x + \frac{a+b}{2}\,.$$

The maximum value of $d(x)$ is $d(x_0) = (a+b)/2 + m$, where $x_0 = (b-a)/2$. It is easily seen that $\theta(x) \leq d(x)$, while the maximum value of $d(x) - \theta(x)$ on $[-m, m]$ is $d(x_0) - \theta(x_0)$. A straightforward calculation shows that

$$d(x_0) - \theta(x_0) = m - \frac{(b-a)^2}{4m} \leq \frac{\bar{l}}{2}\,.$$

Making use of the elementary calculus, we see that the function $\theta(x) - \eta(x)$ attains extreme values

$$\pm\frac{(b-a)}{6\sqrt{3}}$$

at the points $\pm m/\sqrt{3}$. Thus, on $[-m, m]$

$$|\theta(x) - \eta(x)| \leq \frac{|b-a|}{6\sqrt{3}}\,.$$

Since obviously $|b-a| \leq \bar{l}$, we obtain (3.36) with

$$c_0 = \frac{\bar{l}}{2} + \frac{\bar{l}}{6\sqrt{3}}\,.$$

The proof is complete. □

The relation (3.36) shows that exponential estimates in terms of the distance function are equivalent to exponential estimates in terms of the function η. This is a serious reduction to obtain exponential decay. Because now "twisted"operator $\mathcal{L}_\epsilon$ right below is well-defined.

Let $\epsilon \in \mathbb{R}$. For $u \in L^2_{loc}(\Gamma)$, we set

$$(\Phi_\epsilon u)(x) = e^{\epsilon\eta(x)}u(x)\,.$$

It is easily seen that $\Phi_\epsilon u \in L^2_{loc}(\Gamma)$, and Φ_ϵ is a linear continuous operator in $L^2_{loc}(\Gamma)$.

Moreover,

$$\Phi_\epsilon H^1_{loc}(\Gamma) \subset H^1_{loc}(\Gamma)$$

and

$$\Phi_\epsilon H^1_{comp}(\Gamma) \subset H^1_{comp}(\Gamma)\,,$$

and Φ_ϵ is a continuous linear operator in these spaces. This operator extends to a linear continuous operator

$$\Phi_\epsilon : H^{-1}_{loc}(\Gamma) \to H^{-1}_{loc}(\Gamma)$$

by the formula

$$(\Phi_\epsilon u, v) = (u, \Phi_\epsilon v), \quad \forall v \in H^1_{comp}(\Gamma)$$

for any $u \in H^{-1}_{loc}(\Gamma)$. This is indeed an extension of Φ_ϵ previously defined on $L^2_{loc}(\Gamma)$.

Now we define twisted operator

$$\mathcal{L}_\epsilon : H^1_{loc}(\Gamma) \to H^{-1}_{loc}(\Gamma)$$

by

$$\mathcal{L}_\epsilon u = \Phi_\epsilon \mathcal{L} \Phi_{-\epsilon} u, \quad u \in H^1_{loc}(\Gamma)\,.$$

An explicit expression for this operator is the following

$$\mathcal{L}_\epsilon = \mathcal{L} + \epsilon \mathcal{B}_\epsilon,$$

where

$$\mathcal{B}_\epsilon = 2\eta'(x)\frac{d}{dx} + \eta''(x) - \epsilon \eta'^2(x).$$

The operator $\mathcal{B}_\epsilon$ maps continuously $H^1(\Gamma)$ into $L^2(\Gamma)$.

Lemma 3.5.2. *The restriction of $\mathcal{L}_\epsilon$ to $D(L)$ defines a closed linear operator L_ϵ in $L^2(\Gamma)$, with the domain $D(L_\epsilon) = D(L)$ and non-empty resolvent set, provided $|\epsilon| \leq \epsilon_0$ for some $\epsilon_0 > 0$.*

Proof. Without loss of generality, we may assume that the operator L is positive definite. Hence, as a bounded operator from $D(L)$ into $L^2(\Gamma)$, L is invertible. In addition, we suppose that $|\epsilon| \leq 1$. Since $\eta', \eta'' \in L^\infty(\Gamma)$, the restriction of $\mathcal{B}_\epsilon$ to the space $H^1(\Gamma)$ is a (uniformly with respect to ϵ) bounded linear operator $B_\epsilon : H^1(\Gamma) \to L^2(\Gamma)$. Since the embedding $D(L) \subset H^1(\Gamma)$ is continuous and $D(L)$ is equipped with the graph norm, then

$$\|\epsilon B_\epsilon u\| \leq C|\epsilon|\|Lu\|\,, \quad \forall u \in D(L)\,.$$

Hence, the operator $L_\epsilon : D(L) \to L^2(\Gamma)$ has a bounded inverse if $|\epsilon|$ is sufficiently small. As consequence, the operator L_ϵ , with the domain $D(L_\epsilon) = D(L)$ is a closed operator in $L^2(\Gamma)$. □

Remark 3.5.3. *Notice that L_ϵ as a bounded operator from $D(L)$ into $L^2(\Gamma)$ depends continuously on ϵ. Then so is the resolvent. This implies that L_ϵ as a closed operator in $L^2(\Gamma)$ is continuous with respect to ϵ in a neighborhood of $\epsilon = 0$ in the sense of Kato's generalized convergence Kato (1966).*

Let $L^2_\epsilon(\Gamma)$ be the image of $L^2(\Gamma)$ under the transformation Φ_ϵ, *i.e.*,

$$L^2_\epsilon(\Gamma) = \Phi_\epsilon L^2(\Gamma)\,.$$

This is a Hilbert space with the norm induced from $L^2(\Gamma)$

$$\|\Phi_\epsilon u\|_\epsilon = \|u\|\,, \quad u \in L^2_\epsilon(\Gamma)\,.$$

Assuming that $|\epsilon|$ is small enough, we introduce the operator

$$L_{(\epsilon)} = \Phi_{-\epsilon} L_\epsilon \Phi_\epsilon\,.$$

This is a closed operator in $L^2_{-\epsilon}(\Gamma)$ with the domain

$$D(L_{(\epsilon)}) = \Phi_{-\epsilon} D(L)\,.$$

By the definition of $L_{(\epsilon)}$, this operator is unitary equivalent to L_ϵ. Notice that, due to the properties of the function η, functions in $D(L_{(\epsilon)})$ satisfy vertex conditions (3.2) and (3.3). Actually, $L_{(\epsilon)}$ is the restriction of $\mathcal{L}$ to the domain $D(L_{(\epsilon)})$.

Theorem 3.5.4. *Under Assumptions* $(V1)$ *and* $(V2)$*, let* λ_0 *be an isolated eigenvalue of finite multiplicity for the operator* L*. Then there exist* $\epsilon > 0$ *and* $C > 0$ *such that for any normalized eigenfunction* $u \in L^2(\Gamma)$ *with the eigenvalue* λ_0

$$|u(x)| \leq Ce^{-\epsilon d(x)}\,. \tag{3.37}$$

Proof. First we note that, by Lemma 3.5.1, it is enough to prove the estimates

$$|u(x)| \leq Ce^{-\epsilon\eta(x)} \tag{3.38}$$

instead of (3.37).

Since λ_0 is an isolated eigenvalue, there is a sufficiently small closed disc centered at λ_0 and such that it contains the only point λ_0 of the spectrum of L. Denote by γ the boundary of this disc, with counterclockwise orientation. Then the image of the Riesz projector

$$P_0 = \frac{1}{2\pi i}\int_\gamma (\lambda I - L)^{-1} d\lambda$$

is the eigenspace E_0 of the operator L, with the eigenvalue λ_0, and the multiplicity of λ_0 is dim$E_0 = k < \infty$.

By Lemma 3.5.2 and Remark 3.5.3, in a small neighborhood of $\epsilon = 0$ the operator L_ϵ depends continuously in ϵ with respect to Kato's generalized convergence. Hence, by (Kato, 1966, Theorem 3.16 of Ch. 4), for all ϵ in a neighborhood of $\epsilon = 0$, the circle γ does not intersect the spectrum of L_ϵ, the Riesz projector

$$P_\epsilon = \frac{1}{2\pi i}\int_\gamma (\lambda I - L_\epsilon)^{-1} d\lambda$$

as a bounded operator in $L^2(\Gamma)$ depends continuously on ϵ in this neighborhood, and $\dim E_\epsilon = k < \infty$ is independent of ϵ, where E_ϵ is the image of P_ϵ.

Now we turn to the operator $L_{(\epsilon)}$ which is unitary equivalent to L_ϵ and, as consequence, has the same spectrum. The Riesz projector associated to the part of spectrum surrounded by γ is unitary equivalent to P_ϵ because

$$\begin{aligned} P_{(\epsilon)} = \Phi_{-\epsilon} P_\epsilon \Phi_\epsilon &= \frac{1}{2\pi i} \int\limits_\gamma \Phi_{-\epsilon}(\lambda I - L_\epsilon)^{-1} \Phi_\epsilon d\lambda \\ &= \frac{1}{2\pi i} \int\limits_\gamma (\Phi_{-\epsilon}(\lambda I - L_\epsilon)\Phi_\epsilon)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int\limits_\gamma (\lambda I - L_{(\epsilon)})^{-1} d\lambda. \end{aligned}$$

Hence, the image $E_{(\epsilon)}$ of $P_{(\epsilon)}$ is isomorphic to E_ϵ and, therefore, $\dim E_{(\epsilon)} = k$.

Let $\epsilon > 0$. Then

$$L^2_{-\epsilon}(\Gamma) \subset L^2(\Gamma)\,,$$

$$D(L_{(\epsilon)}) \subset D(L)$$

and the operator $L_{(\epsilon)}$ is the restriction of the operator L. Hence, the resolvent $(\lambda I - L_{(\epsilon)})^{-1}$ of $L_{(\epsilon)}$ is the restriction of the resolvent $(\lambda I - L)^{-1}$ of L to the space $L^2_{-\epsilon}(\Gamma)$. This implies immediately that the projector $P_{(\epsilon)}$ is the restriction of the projector $P_{(0)} = P_0$. Therefore, $E_{(\epsilon)}$ is a subspace of E_0. Since the dimensions of these two spaces are equal to k, we have that $E_0 = E_{(\epsilon)}$. This means that the eigenspace E_0 is, in fact, a subspace of $L^2_{-\epsilon}(\Gamma)$.

Now let $u \in E_0$ be an eigenfunction with $\|u\| = 1$. Then $u = \Phi_{-\epsilon} v$ for some $v \in L^2(\Gamma)$. Since Φ_ϵ induces an isomorphism between E_0 and E_ϵ, and $\|u\| = 1$, then $\|v\|$ is bounded by a constant independent of u. By the definition of L_ϵ, the function $v \in D(L)$ satisfies

$$L_\epsilon v = \lambda_0 v\,.$$

Recall that the resolvent of L_ϵ acts as a bounded operator from $L^2(\Gamma)$ into $D(L)$ and, hence, into $H^1(\Gamma)$. Now the last equation implies that $\|v\|_{H^1}$ and, hence, $\|v\|_{L^\infty}$ are bounded by a constant independent of u, that is, there exists a constant $K > 0$ such that

$$|v(x)| \leq K\|v\|_{H^1(\Gamma)},\ x \in \Gamma.$$

As a consequence,

$$|u(x)| = |e^{-\epsilon\eta(x)}v(x)| \leq Ce^{-\epsilon\eta(x)}\ ,$$

where the constant $C > 0$ depends only on λ_0 and ϵ. The proof is complete. $\square$

CHAPTER FOUR
CONCLUSION

One of the richest source of the spectral theory is the quantum physics and most of the theory is dedicated to the Schrödinger operator $L(V)$ defined by the differential expression

$$L(V)u(x) = (-\boldsymbol{\Delta} + V(x))u(x)$$

which is a fundamental operator of quantum physics. The Schrödinger operator can be considered as the energy operator of one or several particles depending on the form of the potential $V(x)$. According to the fundamental principles of quantum physics, the possible values of the energy of a particle belong to the spectrum of the Schrödinger operator and eigenfunctions describe the state of the particle.

This thesis includes two independent studies on the spectral theory of Schrödinger operators.

The first study is on the Schrödinger operator defined by (1.1)-(1.2), whose potential is a real-valued, symmetric matrix $V = (v_{ij}(x))$, $i, j = 1, 2, \ldots, m$. We denote this operator by $L(V)$.

We denote the eigenvalue and eigenfunction pairs of $L(V)$ by Λ_N and ψ_N, respectively.

The eigenvalues of the unperturbed operator $L(0)$ which is defined by (1.1) when $V(x) = 0$ and the boundary condition (1.2) are $|\gamma|^2$ and the corresponding eigenspaces are

$$E_\gamma = \text{span}\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \ldots, \Phi_{\gamma,m}(x)\},$$

where $\gamma \in \frac{\Gamma^{+0}}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \ldots, \frac{n_d\pi}{a_d}) : n_k \in \mathbb{Z}^+ \cup \{0\}, k = 1, 2, \ldots, d\}$,
$\Phi_{\gamma,j}(x) = (0, \ldots, 0, u_\gamma(x), 0, \ldots, 0)$, $j = 1, 2, \ldots, m$,
$u_\gamma(x) = \cos\frac{n_1\pi}{a_1}x_1 \cos\frac{n_2\pi}{a_2}x_2 \cdots \cos\frac{n_d\pi}{a_d}x_d$, $u_0(x) = 1$ when $\gamma = (0, 0, \ldots, 0)$. The non-zero component $u_\gamma(x)$ is in the j-th component.

We assume that the Fourier coefficients $v_{ij\gamma}$ of $v_{ij}(x)$ satisfy

$$\sum_{\gamma\in\frac{\Gamma}{2}} |v_{ij\gamma}|^2(1+|\gamma|^{2l}) < \infty$$

for each $i,j = 1,2,\ldots,m, \quad l > \frac{(d+20)(d-1)}{2} + d + 3.$

In Chapter Two, we obtain asymptotic formulas for the eigenvalue Λ_N of $L(V)$ when the corresponding eigenvalue of the unperturbed operator $L(0)$, is roughly speaking, near diffraction plane.

As the eigenvalue problem of the operator $L(V)$ defined by (1.1) and (1.2), most of the problems related with spectral theory fail to be explicitly soluble, they need a qualitative and asymptotic study.

The most significant progress has been achieved in one dimensional case. The crucial property in analysis of the problem in one dimensional case is that the distance between the consecutive eigenvalues (which occurs in the denominator of the perturbation series) becomes larger and larger so that the perturbation theory can be applied to obtain the asymptotic formulas for sufficiently large eigenvalues.

However, in many dimensional case, (even in two or three dimensions), the problem is considerably difficult. In this case, to construct a perturbation theory turns out to be rather difficult, because of the denseness of the eigenvalues of the free operator which are situated very close to each other in a high energy region. Therefore, when perturbation disturbs them, they strongly influence each other. This presents considerable difficulties as the arbitrarily small differences become small divisors in an asymptotic expansion, in particular, “the small denominators problem”. Thus, to describe the perturbation of one of the eigenvalues, we must also study all the other surrounding eigenvalues.

In order to overcome this difficulty, for the first time in papers (Veliev, 1987, 2006, 2007, 2015), the eigenvalues of the unperturbed operator $L(0)$ is divided into two groups: Non-resonance and resonance ones (see Definition 1.1.1).

In Chapter Two, we obtain the high energy asymptotics of arbitrary order in an arbitrary dimension $(d \geq 2)$ for the eigenvalue of $L(V)$ corresponding to resonance eigenvalue $|\gamma|^2$ when γ belongs to the single resonance domain, that is, $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, where δ is from $\{e_1, e_2, \ldots, e_d\}$ and $e_1 = \left(\frac{\pi}{a_1}, 0, \ldots, 0\right)$, $e_2 = \left(0, \frac{\pi}{a_2}, \ldots, 0\right), \ldots, e_d = \left(0, 0, \ldots, \frac{\pi}{a_d}\right)$.

In order to obtain the asymptotic formulas for the single resonance eigenvalues $|\gamma|^2$ $(\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2)$, we consider the operator $L(V)$ as the perturbation of $L(P(s))$ where $L(P(s))$ is defined by the differential expression

$$Lu = -\boldsymbol{\Delta} u + P(s)u$$

and the Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial F} = 0,$$

$$P(s) = (p_{ij}(s)), \; i, j = 1, 2, \ldots, m,$$

$$p_{ij}(s) = \sum_{n \in \mathbb{Z}} p_{ijn} \cos ns, \;\; p_{ijn} = v_{ij(n\delta)}, \;\; s = x \cdot \delta, \; i, j = 1, 2, \ldots, m.$$

It can be easily verified by the method of separation of variables that the eigenvalues and the corresponding eigenfunctions of $L(P(s))$, indexed by the pairs $(j, \beta) \in \mathbb{Z} \times \Gamma_\delta$, are $\lambda_{j,\beta} = \lambda_j + |\beta|^2$ and $\chi_{j,\beta}(x) = u_\beta(x) \cdot \varphi_j(s) = (u_\beta(x)\varphi_{j1}, u_\beta(x)\varphi_{j2}, \ldots, u_\beta(x)\varphi_{jm})$, respectively, where $\beta \in \Gamma_\delta$, λ_j is the eigenvalue and $\varphi_j(s) = (\varphi_{j,1}(s), \varphi_{j,2}(s), \ldots, \varphi_{j,m}(s))$ is the corresponding eigenfunction of the operator $T(P(s))$ defined by the differential expression

$$T(P(s))Y = -\left|\frac{\pi}{a_i}\right|^2 Y'' + P(s)Y \tag{4.1}$$

and the boundary condition

$$Y'(0) = Y'(\pi) = 0. \tag{4.2}$$

The eigenvalues of the operator $T(0)$, defined by (4.1) when $P(s) = 0$ and the boundary condition (4.2), are $|n\delta|^2 = |\frac{n\pi}{a_i}|^2$ with the corresponding eigenspace $E_n = span\{C_{n,1}(s), C_{n,2}(s), \ldots, C_{n,m}(s)\}$, where $C_{n,i}(s) = (0, \ldots, \cos ns, \ldots, 0)$, the non-zero component $\cos ns$ stands in the ith place and $n \in \mathbb{Z}^+ \cup \{0\}$. It is well known that (for example, see Naimark et al. (1967)) the eigenvalue λ_j of $T(P(s))$ satisfying $|\lambda_j - |j\delta|^2| < \sup P(s)$, satisfies the following relation

$$\lambda_j = |j\delta|^2 + O\left(\frac{1}{|j\delta|}\right).$$

By the above equation, the eigenvalue $|\gamma|^2 = |\beta|^2 + |j\delta|^2$ of $L(0)$ corresponds to the eigenvalue $|\beta|^2 + \lambda_j$ of $L(P(s))$.

As a result, we proved the following theorems

- ***Theorem 2.3.1*** *For every eigenvalue $\lambda_{j,\beta}$ of the operator $L(P(s))$ with $\beta + j\delta \in V'_\delta(\rho^{\alpha_1})$, there exists an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = \lambda_{j,\beta} + O\left(\rho^{-\alpha_2}\right).$$

- ***Theorem 2.3.2***

  (a) *For every eigenvalue $\lambda_{j,\beta}$ of $L(P(s))$ such that $\beta + j\delta \in V'_\delta(\rho^{\alpha_1})$, there exists an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = \lambda_{j,\beta} + E_{k-1} + O(\rho^{-k\alpha_2}), \tag{4.3}$$

  where $E_0 = 0$, $E_s = \sum_{k=1}^{2p} \tilde{S}_k(E_{s-1} + \lambda_{j,\beta}, \lambda_{j,\beta})$, $\quad s = 1, 2, \ldots.$

  (b) *If*

$$|\Lambda_N - \lambda_{j,\beta}| < c_{20}$$

and

$$|c(N, j, \beta)| > \rho^{-q\alpha}$$

hold then Λ_N *satisfies* (4.3).

For the operator $L(V)$ defined by (1.1), (1.2) we may suggest the following open problem:

For $m \geq 1$ (both scalar and matrix cases) one may study the asymptotic behaviour of the eigenfunctions of $L(V)$ for both non-resonance and resonance domains.

In the second study, we consider Schrödinger operators with non-regular potentials on infinite metric graphs.

Let $\Gamma = (E, V)$ be an undirected graph with the set of edges E and the set of vertices V. The graph Γ is said to be a *metric graph* if each edge e is identified with an $[0, l_e]$ of the real line.

The distance $d(x, y)$ between two points x and y in Γ is defined as the length of a shortest path that connects these points. Since the graph is connected, the distance is well defined.

For the second study, in Chapter Three, we assumed that

(i) The sets of edges and vertices are countably infinite;

(ii) There exist two positive constants $\underline{l}$ *and* $\bar{l}$ *such that*

$$\underline{l} \leq l_e \leq \bar{l}$$

for all $e \in E$.

Differential equations on metric graphs (networks) is a relatively new area of mathematical research though the first publication in which equations of such type appear is paper Kirchhoff (1847).

In Chapter Three, to define a self-adjoint Schrödinger operator, we start with a second-order symmetric differential operator

$$L_0 u = -\frac{d^2 u}{dx^2} + V(x)u$$

on the domain that consists of sufficiently smooth compactly supported functions satisfying the Kirchhoff conditions at the vertices of a metric graph Γ. The potentials are supposed to be locally integrable with negative part bounded in certain integral sense (see, assumptions (V1) and (V2)). In the case of operators on real line, these assumptions turn into the assumption that the potential is of local Kato class, while its negative part is of Kato class (see, e.g., Cycon et al. (2009), Simon (1982)). Under our assumptions, L_0 is a symmetric, bounded below operator in the space $L^2(\Gamma)$.

While the theory of Schrödinger operators on the Euclidean space is currently well-developed, the theory of quantum graphs, i.e., Schrödinger type operators on metric graphs, is relatively new, and many important problems in this area are still open. Most of results obtained so far concern the case when the potential is sufficiently regular. However, as it is well-known the potential represents external force field which often has singularities. Due to this fact, in Chapter Three, we study Schrödinger operators with locally integrable potentials on infinite metric graphs. Such potentials form a sufficiently wide class and allow many important singularities.

First, we show that the Friedrichs extension, L, of L_0 is the only self-adjoint extension of L_0. In other words, the operator L_0 is essentially self-adjoint. For this aim, let us consider the maximal operator $\tilde{L} = L_0^*$ associated to the differential expression $\mathcal{L}$ and vertex conditions (3.2) and (3.3). The domain $D(\tilde{L})$ consists of all functions $u \in L^2(\Gamma)$ such that

(i) u *and* u' *are absolutely continuous functions on each edge, and, hence,* $u'' \in L^1_{loc}(\Gamma)$*;*

(ii) u *satisfies vertex conditions (3.2) and (3.3);*

(iii) $\mathcal{L}u \in L^2(\Gamma)$.

The operator $\tilde{L}$ is defined by $\tilde{L}u = \mathcal{L}u$ for all $u \in D(\tilde{L})$.

Obviously, $L \subset \tilde{L}$. Actually, we have the following theorem

- ***Theorem 3.3.1*** *Under assumptions* $(V1)$ *and* $(V2)$, $L = \tilde{L}$.

Next we obtained a characterization of the bottom of essential spectrum:

- ***Theorem 3.3.2*** *Under assumptions* $(V1)$ *and* $(V2)$

$$\inf \sigma_{ess}(L) = \sup_{\Gamma_n \Subset \Gamma} \quad \inf \left\{ \frac{(L\varphi, \varphi)}{\|\varphi\|^2} : \varphi \in \mathcal{D}_0(\Gamma \setminus \Gamma_n), \varphi \neq 0 \right\},$$

where the supremum is taken over the sequence of compact sets Γ_n *defined in Section 3.1.*

For the discreteness of spectrum, we begin with a sufficient condition for the discreteness of the negative part of spectrum.

- ***Theorem 3.4.1*** *In addition to Assumptions* $(V1)$ *and* $(V2)$*, suppose that*

$$\int_e V_-(x)dx \to 0$$

in the sense that for every $\varepsilon > 0$ *there exists* $n \in \mathbb{N}$ *such that*

$$\int_e V_-(x)dx < \varepsilon$$

for all $e \in E$ such that $e \subset \Gamma \setminus \Gamma_n$. Then the negative part of spectrum, $\sigma(L) \cap (-\infty, 0)$, is discrete.

Then we proved a criterion for the discreteness of the whole spectrum.

- ***Theorem 3.4.2*** *Under Assumptions $(V1)$ and $(V2)$, $\sigma(L)$ is discrete if and only if for every $\alpha \in (0, \underline{l})$*

$$\int_S V(x)dx \to \infty$$

  as the interval S of length α escapes to infinity (this means that for every $M > 0$, there exists $n \in \mathbb{N}$ such that

$$\int_S V(x)dx \geq M$$

  for all intervals S of length α with the property that $S \subset \Gamma \setminus \Gamma_n$) .

The last result of our second study is on the exponential decay of eigenfunctions:

- ***Theorem 3.5.4*** *Under Assumptions $(V1)$ and $(V2)$, let λ_0 be an isolated eigenvalue of finite multiplicity for the operator L. Then there exist $\epsilon > 0$ and $C > 0$ such that for any normalized eigenfunction $u \in L^2(\Gamma)$ with the eigenvalue λ_0*

$$|u(x)| \leq Ce^{-\epsilon d(x)}\,.$$

In the second study, we consider Schrödinger operators with non-regular potentials on infinite metric graphs. Our above results can be extended to the case when Kirchhoff vertex conditions are replaced by general self-adjoint vertex conditions. This is a relatively new problem for us.

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