TURKISH REPUBLIC
ERCIYES UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

SCOTT TOPOLOGY

MUHAMMAD QASIM

SUPERVISOR
Prof. Dr. Mehmet BARAN

MSc. Thesis

December 2014
KAYSERİ

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I declare that all informations in this work were obtained in accordance with academic and ethical rules. All results and material that not been at the essence of this work are also transferred and expressed by giving reference as required by these rules and behavior.

MUHAMMAD QASIM

The MSc thesis entitled **"SCOTT TOPOLOGY"** has been prepared in accordance with Erciyes University Graduate School of Natural and Applied Sciences Institute Thesis Preparation and Writing Guide.

Student
MUHAMMAD QASIM

Supervisor
Prof. Dr. Mehmet BARAN

Chairman of the Department of Mathematics
Prof. Dr. İlhan ÖZTÜRK

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Juror : Prof. Dr. Mehmet BARAN

Juror : Assoc. Prof. Dr. Ali DELİCEOĞLU

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Director of the Institute

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SCOTT TOPOLOGY

MUHAMMAD QASIM
Erciyes University, Graduate School of Natural and Applied Sciences
MSc. Thesis, December 2014
Supervisor: Prof. Dr. Mehmet BARAN

ABSTRACT

The topology introduced on a directed completely partial order (or on any complete lattice) was first formulated for the lattice $L = \mathcal{O}(X)$ of open sets of a topological space in 1970 by B. J. Day and G. M. Kelly. But it's credit goes to Dana Scott for defining this topology in all generality and for demonstrating its usefulness in his article on " Continuous lattice". The name Scott topology was first used by Isbell in 1975, and the name was used in the Seminar on Continuity in Semilattices (SCS) for several year.

This dissertation are mainly consists of three chapters.

In first chapter, fundamental notions, some theorems and several examples of partial orders, directed completely partial orders, some topological and categorical concepts which will be used in other chapters has been given.

In the second chapter, Scott-open set and Scott closed sets has been defined, some important properties of Scott topology, approximation relation and basic properties of this relation, Scott open set through this relation has been investigated. Moreover, the Scott-continuous function and relation between topologically continuous function and Scott continuous function has been studied.

Finally, in the last chapter, DCPO and CPO categories has been defined, some important special objects of DCPO has been studied. Moreover, cartesian closed property of DCPO has been proved, sober space and spatial lattice has been defined. In addition to relation with Scott topological space and sober space has been investigated.

Keywords: DCPO , Scott open set, Scott topology, Scott continuous function, cartesian closed, sober space.

SCOTT TOPOLOJİ

MUHAMMAD QASIM
Erciyes Üniversitesi, Fen Bilimleri Enstitüsü
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ÖZET

Yönlendirilmiş tam kısmı sıralı küme (veya herhangi bir tam latis) üzerinde tanımlanmış topoloji ilk kez 1970'te B.J. Day ve G.M. Kelly tarafından topolojik uzayının açık kümesinin $L = \mathcal{O}(X)$ latisi için formülize edildi. Yalnız, bu topolojinin tüm genellemeleriyle tanımlanması ve kullanılabilirliğin gösterilmesi Dana Scott'ın "continuous lattice" adlı makalesine aittir. Scott topoloji ismi ilk kez 1975'te Isbell tarafından kullanılmıştır ve bu isim birkaç yıl SCS'te kullanılmıştır.

Bu tez üç bölümden oluşturmaktadır.

İlk bölümde, diğer bölümlerde kullanılacak kısmi sıralama kümeler, tam yönlendirilmiş kısmi sıralı kümeler, bazı topolojik ve kategoriksel kavramlar hakkındaki temel tanımlar, bazı teorem ve çeşitli örnekler verilmiştir.

İkinci bölümde, Scott açık kümeler ve Scott kapalı kümeler tanımlanmıştır ve Scott topolojisinin bazı temel özellikleri, yaklaşım bağıntı ve bu bağıntının temel özellikleri incelenmiştir. İlaveten, Scott sürekli fonksiyonu ve topolojik sürekli fonksiyon ile Scott sürekli fonksiyonu arasındaki ilişki araştırılmıştır.

Son olarak, DCPO ve CPO kategorileri tanımlanmıştır, DCPO 'nın özel objeleri araştırılmıştır. İlaveten, DCPO 'nın Kartezyen kapalı özelliği ispatlanmıştır, sober uzayı and spatial latis tanımlanmıştır. Ayrıca, Scott topolojik uzayı ve sober uzayı arasındaki ilişki de incelenmiştir.

Anahtar Kelimeler: DCPO, Scott açık küme, Scott topoloji, Scott sürekli fonksiyon, Kartezyen kapalı, sober uzayı.

TABLE OF CONTENT

SCOTT TOPOLOGY

LIST OF FIGURES

INTRODUCTION

Topology has proved to be an important tool for certain aspects of theoretical computer science. Conversely, the problems that arise in the computational setting have provided new and interesting stimuli for topology. These problems also have increased the interaction between topology and related areas of mathematics such as order theory and topological algebra [1].

Domain theory traces its history back to the need to define mathematical models of programming languages. The impetus was the introduction of a variety of high-level programming languages and the increasing complexity of their design and use in the 1960's. This led to an acknowledged need for models for programming languages that would support precise reasoning about program behavior. Such models were required both to give an unambiguous definition of a given programming language [2].

The field of Denotational Semantics was introduced by Christopher Strachey at Oxford University in the mid-sixties to meet this need. Strachey and Dana Scott [3] provided denotations for language constructs using higher order functions in some mathematical universe. The techniques developed in denotational semantics were successful for procedural languages, functional languages, and later parallel languages. The initial problem was the lack of a theory for producing mathematical models that met all the requirements:

(a) Modelling recursion required functions to have fixed points.

(b) Modelling functional languages required a cartesian closed category so that the set of functions between objects was itself an object of the category.

(c) Modelling more complicated languages required solutions to recursive definitions of the universes themselves (e.g., $U \cong (U \to U) + (U \times U) + B$) [2].

In 1969, Dana Scott [4] discovered a theory that could provide a rigorous mathematical foundation for denotational semantics. This theory, called Domain Theory, has evolved to become not only an important tool for applications in computer science, but also an exciting field of ongoing research in pure mathematics. Domains carry several intrinsic topologies: the most fundamental is the Scott topology which is crucial to the theory. The others- the Lawson topology and the μ- topology also play important roles in the theory and in the applications of domain theory to computer science and to other areas [2].

The topology introduced on a directed completely partial order (or on any complete lattice) was first formulated for the lattice $L = \mathcal{O}(X)$ of open sets of a topological space in 1970 by B. J. Day and G. M. Kelly [5] . But it's credit goes to Dana Scott for defining this topology in all generality and for demonstrating its usefulness in his article on " Continuous lattice" [6]. The name Scott topology was first used by Isbell [7] in 1975, and the name was used in the Seminar on Continuity in Semilattices (SCS) for several year.

In mathematics, there is an ample supply of categorical dualities between certain categories of topological spaces and categories of partially ordered sets. Today, these dualities are usually collected under the label Stone duality. These concepts are named in honor of Marshall Stone. Sober space and spatial lattice are the key factor in the Stone duality. Every Sober space is Scott topology. But what about its converse? In December 1978, Peter T. Johnstone [8] discovered a counterexample that answers this question in the negative.

Categories are algebraic structures with many complementary natures, e.g., geometric, logical, computational, combinatorial, just as groups are many-faceted algebraic structures. In 1945 Eilenberg and MacLane introduced a category in a purely auxiliary fashion, as preparation for what they called functors and natural transformations. Eilenberg and MacLane [9] later wrote that their goal was to understand natural transformations; in order to do that, functors had to be defined, which required categories.

category theory simplifies the communication among the people working in different fields by creating a new language which is economical regarding new ideas and their expression and it also provides a new meaning to the old problems by raising the different theorems and structures independent from each other. Category theory has applicative roots especially in theoretical Computer Science, cohomology theory, DNA and RNA codes of Molecular Biology and logics [10].

In this thesis, we survey the topological and categorical concepts for the Scott topology, and showed cartesian closed property of Scott topology-which is of fundamental importance in domain theory, and showed relation between Scott topological space and sober space.

CHAPTER 1

BASIC DEFINITIONS

The relationship between topology and order theory has plenty effects on Computer Science. This relationship happens over a specific order, called a *partial order* and together with a set is called POSET. In this chapter, we will give some basic definitions which lead us to Scott Topology.

1.1. Partial Order Sets

Definition 1.1.1. Let$D$ be any non-empty set. Then $\leqslant$ is called *partial order* relation, if for every $a, b, c \in D$.

(i) (Reflexivity): a $\leqslant$ a

(ii) (Anti-Symmetry): a $\leqslant$ b $\wedge$ b $\leqslant$ a $\Rightarrow$ a $=$ b

(iii) (Transitivity): a $\leqslant$ b $\wedge$ b $\leqslant$ c $\Rightarrow$ a $\leqslant$ c

The set D together with a *partial order* $\leqslant$ is called a *partially ordered set* (POSET).

Example 1.1.1. Let X be a non-empty set. The set $P(X)$ of all subsets of X by $(\subseteq)$ relation forms a poset. Let $\forall A, B, C \in P(X)$

(i) Since every set is the subset of itself. Then $A \subseteq A$.

(ii) If $A \subseteq B$ and $B \subseteq A$, then by the definition of equality, we have $A = B$.

(iii) If $A \subseteq B$ and $B \subseteq C$, then by the definition of subset, we have $A \subseteq C$.

Example 1.1.2. The set $\mathbb{N}$ of natural numbers forms a poset by $\leqslant$ order on $\mathbb{R}$. It is easy to see that reflexivity, anti-symmetry and transitivity are satisfied.

Example 1.1.3. The set $\{2, 3, 4, 6, 8\}$ under divisibility relation forms a poset with a diagram as:

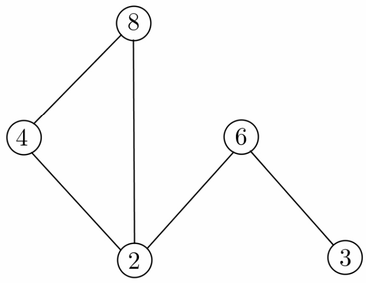


Figure 1.1. Hasse diagram of {2,3,4,6,8} ordered by divisibility

Example 1.1.4. If $X = \{1, 2, 3\}$, then the poset
$P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$ under $\subseteq$ relation represented by the diagram as below:

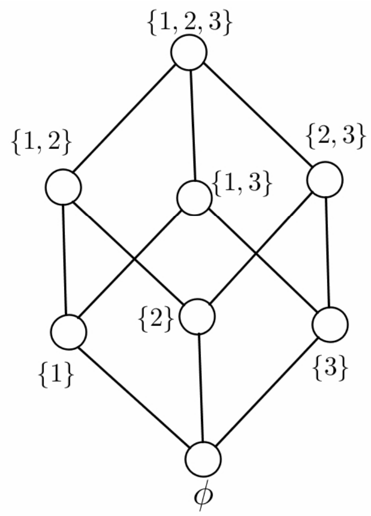


Figure 1.2. Hasse diagram of set of three elements ordered by inclusion

Definition 1.1.2. Let D be a non-empty set. Then $\leqslant$ is called *discrete order* relation, if for any $x, y \in D$
$x \leqslant y$ if and only if $x = y$.

Definition 1.1.3. Let D be a partially ordered set. An element $a \in D$ is called *maximal* if whenever $a \leqslant x$ then $x = a$. Similarly an element $a \in D$ is called *minimal* if whenever $x \leqslant a$ then $x = a$. If there is an element $\top \in D$ such that $\forall x \in D$, $x \leqslant \top$, then $\top$ is called *maximum*(or top) element, denoted by maxD. On other hand; if there is an element $\perp \in D$ such that $\forall x \in D$, $\perp \leqslant x$, then $\perp$ is called *minimum*(or bottom) element, denoted by minD.

Maximal and minimal element of any given set may be more than one. But maximum and minimum element of any given set is unique.

Definition 1.1.4. Let B be a subset of a poset D. An element $u \in D$ is an *upper bound* of B if $\forall x \in B$, $x \leqslant u$. If the set of all upper bounds for B has a smallest element,that element is called *least upper bound*(or *supremum*) of B-denoted by sup$B$ (or $\bigvee B$). sup$B$ may or may not belong to B. If it does, then it is the largest element of B.

An element $l \in D$ is a *lower bound* of B if $\forall x \in B$, $l \leqslant x$. If the set of all lower bound for B has a largest element, then that element is called *greatest lower bound*(or *infimum*) of B-denoted by inf$A$ (or $\bigwedge B$). Similarly inf$B$ may or may not belong to B. If it does, then it is the smallest element of B. If B has both an upper bound and a lower bound, then B is called *bounded*.

Example 1.1.5. Let $(\mathbb{R}, \leqslant)$ be a partial order set and $A = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$.
There is no maxmum element of A.
$supA = \sqrt{2}$

Example 1.1.6. Let $X = \{a, b, c, d, e, f\}$ be a set ordered by following diagram:

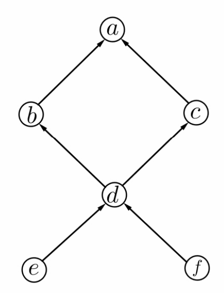


Figure 1.3. Hasse diagram of set of six-elements ordered by direction of arrows

Here $(x, y) \in R$ iff $x = y$ or one can go from x to y in upward direction. Let $C = \{b, c, d\}$.

$minC = \{d\}$

There is no maximum element of C.

$supC = \{a\}$

$infC = \{d\}$

$minimalC = \{d\}$

$maximalC = \{b, c\}$

Definition 1.1.5. Let $(D, \leqslant)$ be a poset. If every finite subset of D has both supremum and infimum, then D is called a *lattice*.

Similarly, D is called a *complete lattice*, if all subsets of D have both supremum and infimum [11].

Example 1.1.7. Let $D = \mathbb{Z}^+$ be the set of positive integers and $\leqslant$ be the order. Then D is a lattice.

Example 1.1.8. Let X be a non-empty set and $\subseteq$ be the order. Then $(P(X), \subseteq)$ is a complete lattice.

Definition 1.1.6. Let $(D, \leqslant)$ be a poset. A subset U of D is a *down set*(or *lower set*) if, whenever $x \in U$ and $z \leqslant x$ then we have $z \in U$. Similarly a subset V of a poset D is an *up set*(or *upper set*) if, whenever $x \in V$ and $x \leqslant y$, we have $y \in V$. For any $x \in D$, down set $\downarrow x = \{y \in D : y \leqslant x\}$; and the up set $\uparrow x = \{y \in D : x \leqslant y\}$. For any set $A \subseteq D$, we define the down set $\downarrow A = \{y \in D : \exists x \in A, y \leqslant x\}$ and the up set $\uparrow A = \{y \in D : \exists x \in A, x \leqslant y\}$ [12].

Example 1.1.9. Let $\mathbb{R}$ be the set of real numbers under $\leqslant$ order. Let $C, D \subseteq \mathbb{R}$ be subsets of $\mathbb{R}$ such that $C = [100, \infty)$ and $D = (-\infty, 50]$, then C is an up set and D is a down set.

1.2. Directed Complete Posets(DCPO)

Definition 1.2.1. Given a partial order $(D, \leqslant)$, a non-empty subset $\triangle \subseteq D$ is called *Directed* if,for all $x, y \in \triangle$, there is a $z \in \triangle$ such that $x \leqslant z$ and $y \leqslant z$ [13].

we will write $\triangle \subseteq_{dir} D$ if $\triangle$ is a directed subset of D.

Example 1.2.1. The natural number $\mathbb{N}$ under the order $\leqslant$, then relation is a partial order $(\mathbb{N}, \leqslant)$. Let $\triangle$ be any subset of $\mathbb{N}$ and consider $x, y \in \triangle$, then it is quite easily seen that

$k = max(x, y) \in \triangle \ni x \leqslant z \wedge x \leqslant z$. Hence $\triangle \subseteq_{dir} \mathbb{N}$. Similarly $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are directed sets under the usual order.

Proposition 1.2.1. Let D be a poset. A non-empty chain in D is directed [14].

Proof: Let D be a poset and $\triangle$ be a non-empty chain in D. let $u, v \in \triangle$. Since in the chain, each two elements are comparable, then $u \leqslant v$ or $v \leqslant u$. If $u \leqslant v$, then $u \leqslant v$ and $v \leqslant v$. Similarly, if $v \leqslant u$, we have $v \leqslant u$ and $u \leqslant u$. Thus $\triangle$ is directed. ■

Proposition 1.2.2. In a finite poset D, a subset has top element $''\top''$ if and only if it is directed [14].

Proof: $(\Rightarrow)$ Let D be a finite poset and $\triangle \subseteq D$ be a non-empty subset with a top element $\top_\triangle$. Then for any $u \in \triangle$,$u \leqslant \top_\triangle$. Consequently, $\forall u, v \in \triangle$, take $w = \top_\triangle \in \triangle$ so that $u \leqslant w$ and $v \leqslant w$. Thus $\triangle$ is directed.

$(\Leftarrow)$ Let $\triangle \subseteq D$ be a directed subset. Then, $\triangle \neq \emptyset$. Since D is finite, so is $\triangle$. Let $\triangle = \{u_1, u_2, ..., u_n\}$. Now for any $u_i, u_j \in \triangle$,$\exists u_k \in \triangle$ such that $u_i \leqslant u_k$ and $u_j \leqslant u_k$. Also for any $u_m \in \triangle$,$\exists u_w \in \triangle$ such that $u_k \leqslant u_w$ and $u_m \leqslant u_w$. Thus, by the transitivity of $'' \leqslant''$ and the directness of $\triangle$, $u_w = \{u_i, u_j, u_k, u_m, u_w\}$. Continuing in this fashion our process must come to an end since $\triangle$ is finite. That is, there must be an element $u \in \triangle$ such that $u = max\{u_1, u_2, ..., u_n\}$. Hence, $\triangle$ has a top element. ■

Definition 1.2.2. **(i)** A partial order $(D, \leqslant)$ is called a *directed complete partial order*(DCPO) if each $\triangle \subseteq_{dir} D$ has a supremum, denoted by $\bigvee \triangle$.

(ii) A complete partially order set (CPO) is a DCPO with a least element, denoted by $\perp$ [13].

Example 1.2.2. Every finite poset is a dcpo.

Example 1.2.3. Let $\mathbb{N}_n$ be the set of the first n natural numbers. Then $(\mathbb{N}_n, \leqslant)$ is a dcpo. Moreover, since $\forall k \in \mathbb{N}_n$, $0 \leqslant k$, $(\mathbb{N}_n, \leqslant)$ is a cpo with a least element 0.

Example 1.2.4. The set of real numbers $\mathbb{R}$, the set of rational numbers $\mathbb{Q}$, the set of natural numbers $\mathbb{N}$ and the set of integers $\mathbb{Z}$ fail to be dcpos under $(\leqslant)$ order, because all these sets are directed subsets of themselves and no one has a supremum.

Example 1.2.5. Let X be any non-empty set. Define $X_\perp = X \cup \{\perp\}$ where $\perp \notin X$, and for $x, y \in X_\perp$, define $x \leqslant y$ if and only if $x = \perp$ or $x = y$. Then $(X_\perp, \leqslant)$ is a cpo.

Definition 1.2.3. Let $(D, \leqslant)$ and $(D^{'}, \sqsubseteq)$ be partial order set. A function $f : D \to D^{'}$ is called *monotonic*(or *order preserving*) if, for all $x, y \in D$, if $x \leqslant y$, then $f(x) \sqsubseteq f(y)$.

Example 1.2.6. Let $f : \mathbb{N} \to \mathbb{N}$ be the function $f(n) = n + 5$. Then f is monotonic, given by the partial order $\leqslant$ on $\mathbb{N}$.

Example 1.2.7. Let $\mathbb{N}_n$ be the set of first n natural numbers and $g : \mathbb{N}_n \to \mathbb{N}_{n+1}$ be the function $g(n) = n + 1$. Then g is monotonic, given the partial order $\leqslant$.

Definition 1.2.4. Let $(D, \leqslant)$ be a dcpo set. An element a of D is called *compact* if, for any directed subset $\triangle$ of D, $a \leqslant \bigvee \triangle$ implies that $\exists u \in \triangle$ such that $a \leqslant u$ [13].

The set of all compact elements of D is denoted by K_D. That is,
$K_D = \{d \in D | \text{d is compact}\}$. The set K_D is called the *base* of D.

Lemma 1.2.1. Whenever it exists, the supremum of any finite set of compact elements is compact [15].

Proof: Let D be a dcpo and let $A = \{a_i\}_{i=1}^{n}$ be a finite set of compact elements in D. Suppose that $SupA = b \in D$. By the definition of the supremum, we have $a_i \leqslant b$, $\forall a_i \in A$. Now, let $\triangle$ be any directed subset of D such that $b \leqslant \bigvee \triangle$. So, we have $a_i \leqslant b \leqslant \bigvee \triangle$, $\forall a_i \in A$. Since a_i is compact for all i, there exists $u_i \in \triangle$ such that $a_i \leqslant u_i$ for all $i = 1, 2, 3, ..., n$. Let $u = max\{u_1, u_2, ..., u_n\}$. Then, u exists in $\triangle$. Thus $a_i \leqslant u$,$\forall a_i$ in A. Hence u is a upper bound of A. Since b is the least upper bound of A, then $b \leqslant u$ and so b is compact. ■

Proposition 1.2.3. Let $(D, \leqslant)$ be a dcpo set. If each directed subset of D contains its supremum, then $K_D = D$.

Proof: Clearly $K_D \subseteq D$. Now, let $a \in D$ and let $\triangle$ be a directed subset of D such that $a \leqslant \bigvee \triangle$. Since $\bigvee \triangle \in \triangle$ (by hypothesis), then take $u = \bigvee \triangle \in \triangle$ and so $a \leqslant u$. Therefore, a is compact. Thus, $D \subseteq K_D$ and consequently $K_D = D$. ■

Example 1.2.8. For each finite subset A of $\mathbb{N}$, $K_A = A$.

Similarly, for any finite subset B of $\mathbb{Z}$, $K_B = B$

Definition 1.2.5. Let $(D, \leqslant)$ be a dcpo. Then D is said to be *algebraic* if, for every $x \in D$, the set $\downarrow_K x = \{a \in K_D | a \leqslant x\}$ is directed and $x = \bigvee \downarrow_K x$ [13].

Example 1.2.9. Let $D = \{1, 2, 3, ..., n\}$, where $n \in \mathbb{N}$, by the $\leqslant$ order . Then D is an algebraic dcpo.

Proof: Clearly, D is a finite poset and hence is a dcpo. Also, $K_D = D$ (since D is finite). Therefore, for any $x \in D$, $\downarrow_K x = \{a \in K_D | a \leqslant x\} = \{a \in D | a \leqslant x\} = \downarrow x$ is directed and $\bigvee \downarrow_K x = \bigvee \downarrow x = x$. Hence, D is an algebraic dcpo. ■

Example 1.2.10. Let $B = [2, 3]$ be subset of $\mathbb{R}$ under $(\leqslant)$. Then, B is a dcpo. Since 2 is the bottom element in B, then for any directed subset U of B with $2 \leqslant \bigvee U$, there is $u \in U$ such that $2 \leqslant u$. Thus $2 \in K_B$.

Now, for any $x \in B$ with $x \neq 2$, we have $U = (z, x)$, $z \in B$ is directed with $\bigvee U = x$ but U contains no element u such that $x \leqslant u$. So $x \notin K_B$, Hence, $K_B = \{2\}$ and $\forall x \in B$, $\downarrow_K x = \{2\}$. Thus B is not algebraic.

Example 1.2.11. Let $D = (-\infty, 0]$ be the subset of $\mathbb{R}$ ordered by $(\leqslant)$ relation. D is a dcpo. If $y \in D$ and for $x \leqslant y$, let U be the interval of real numbers (x, y), which is a directed subset of D with $y = \bigvee U$. So, $y \leqslant \bigvee U = y$ and there is no $u \in U$ such that $y \leqslant u$. Therefore, y is not compact and consequently, $K_D = \emptyset$. Hence D is not algebraic.

Example 1.2.12. If D is a finite algebraic dcpo, then each element in D is compact. That is; $K_D = D$.

Proof: Let D be a finite algebraic dcpo and let $x \in D$. Then $\downarrow_K x$ is a finite directed subset of compact elements with x as its join. Then x is compact. ■

Definition 1.2.6. Let D be a poset and $\emptyset \neq I \subseteq D$. Then I is said to be an *ideal*, if it satisfies the following conditions:

(i) I is a down set.

(ii) I is a directed set [16].

Example 1.2.13. Let $V = \mathbb{Z}^-$ be the set of negative integers, ordered as given below:

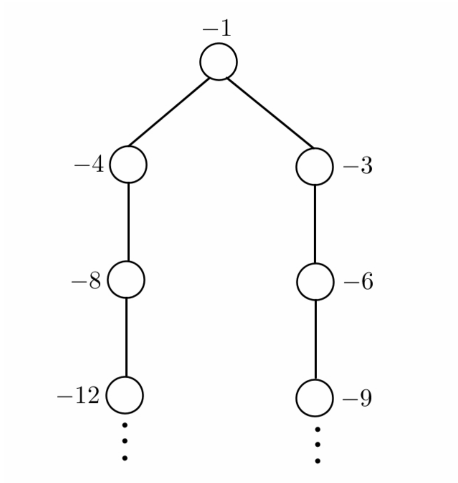


Figure 1.4. Negative integers of multiples of 3 and multiples of 4 ordered by comparing with -1

$-1 \geqslant -4 \geqslant -8 \geqslant -12 \geqslant -16 \geqslant ...$ and $-1 \geqslant -3 \geqslant -6 \geqslant -9 \geqslant -12 \geqslant ...$ and for each $x \in \{-4, -8, -12, ...\}$ multiples of 4 and $y \in \{-3, -6, -9, ...\}$ multiples of 3, x and y are incomparable.
Clearly, V is an ideal.

Now, we will move to dual concept of " ideal ", named as " filter".

Definition 1.2.7. Let D be poset and $\emptyset \neq U \subseteq D$. U is said to be *filtered* if $\forall x, y \in U$, there exists $z \in U$ such that $z \leqslant x$ and $z \leqslant y$ [16].

Example 1.2.14. Any subset of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ is a filtered set ordered by ($\leqslant$) relation. So we can openly say that every chain is filtered set.

Lemma 1.2.2. If D is any poset with a bottom element $\bot$, then any subset of D containing $\bot$ is a filtered set.

Proof: Let D be a poset with the bottom element $\bot$ and let U be any subset of D such that $\bot \in U$. Then for any $x, y \in U$, $\bot \leqslant x$ and $\bot \leqslant y$. Hence, U is a filtered set. ■

Example 1.2.15. Every cpo is a filtered set.

Definition 1.2.8. Let D be a poset and let $F \neq \emptyset$ and $F \subseteq D$. Then F is said to be a *filter* if F is a filtered upper set, that is
(i) $\forall x, y \in F$, there exists $z \in F$ such that $z \leqslant x$ and $z \leqslant y$.

(ii) $\forall x \in F, \forall y \in D$, $x \leqslant y$ implies $y \in F$. Similarly, we can define a filter on a lattice. Let $(L, \leqslant)$ be a lattice and let $F \neq \emptyset$ and $F \subseteq L$. Then F is called *filter on lattice* if
(i) $\forall x \in F, \forall y \in D$, $x \leqslant y$ implies $y \in F$.
(ii) $\forall x, y \in F, x \wedge y \in F$ [17].

Example 1.2.16. Let $U = \mathbb{Z}^+$ be the set of positive integers, ordered as given below:

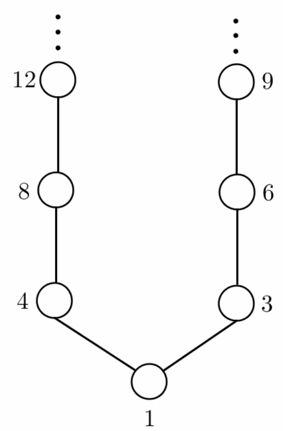


Figure 1.5. Positive integers of multiples of 3 and multiples of 4 ordered by comparing with 1

$1 \leqslant 4 \leqslant 8 \leqslant 12 \leqslant 16 \leqslant ...$ and $1 \leqslant 3 \leqslant 6 \leqslant 9 \leqslant 12 \leqslant ...$ and for each $x \in \{4, 8, 12, ...\}$ and $y \in \{3, 6, 9, ...\}$, x and y are incomparable.
Clearly U is a filter.

1.3. Topological Concepts

Topology is an area of mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing or gluing. This includes such properties as connectedness, continuity and boundary. Topology developed as a field of study out of geometry and set theory, through analysis of such concepts as space, dimension, and transformation. Such ideas go back to Leibniz, who in the 17th century envisioned the geometria situs (Greek-Latin for "geometry of place") and analysis situs (Greek-Latin for "picking apart of place"). The term topology was introduced by Johann Benedict Listing in the 19th century [18], although it was not until the first decades of the 20th century that the idea of a topological space was developed. By the middle of the 20th century, topology had become a major branch of mathematics. In 1914, Felix Hausdorff coined the term "topological space" and gave the definition for what is now called a Hausdorff space [19]. Currently, a topological space is a slight generalization of Hausdorff spaces, given in 1922 by Kazimierz Kuratowski [20].

Definition 1.3.1. Let $X \neq \emptyset$ be a set. Then a *topology* on X is a subset τ of $P(X)$ satisfying the following conditions:

(i) X and $\emptyset$ belong to τ.

(ii) If $U_1, U_2, U_3, ..., U_n \in \tau$, where $n \in \mathbb{N}$, then $\bigcap\limits_{k=1}^{n} U_k \in \tau$.

(iii) If $\{U_i : i \in I\}$ is an indexed family of sets, each of which belong to τ, then $\bigcup\limits_{i \in I} U_i \in \tau$.

We will call the elements of a topology on any set X, *open* subsets of X.

Definition 1.3.2. If X is a topological space and $A \subseteq X$, we say A is *closed* if $A^c = X - A$ is open.

Example 1.3.1. Let $X \neq \emptyset$ and let $\tau = \{\emptyset, X\}$. Then (X, τ) is a topological space named as indiscrete topological space.

Example 1.3.2. Let X be any set and let $\tau = P(X)$. Then (X, τ) is a topological space named as discrete topological space.

Definition 1.3.3. Suppose that τ, τ' be two topologies on a given set X. We say that τ is *coarser* than τ', or τ' is *finer* than τ if $\tau \subseteq \tau'$.

Example 1.3.3. The left ray topology on $\mathbb{R}$ is coarser than the usual topology, since each set of the form $(-\infty, a)$ which is open in the left ray topology, is also open in usual topology while the set $B = (10, 12)$ belongs to the usual topology but not to the left ray topology.

Theorem 1.3.1. If $\mathfrak{F}$ is the collection of closed sets in a topological space (X, τ), then

(i) X and $\emptyset$ both belong to $\mathfrak{F}$

(ii) any arbitrary intersection of members of $\mathfrak{F}$ belongs to $\mathfrak{F}$

(iii) any finite union of members of $\mathfrak{F}$ belongs to $\mathfrak{F}$

Proof: It follows easily from [21]. ■

Definition 1.3.4. Let (X, τ) be a topological space and A is a subset of X, then the *closure* of A is denoted by $\overline{A}$ or $Cl(A)$ is the intersection of all closed sets containing A or all closed super set of A. i.e. the smallest closed set containing A.
On the other hand it can also be as let (X, τ) be a topological space and let A be any subset of X. A point $x \in X$ is said to be *adherent* to A if each neighborhood of x contains a point of A (which may be x itself). The set of all points of X adherent to A is called *closure* (or *adherence*) of A and is denoted by $\overline{A}$. In symbols
$\overline{A} = \{x \in X : \forall U_x \in \tau, U_x \cap A \neq \emptyset\}$

Theorem 1.3.2. A subset A of a space X is closed if and only if $\overline{A} = A$.

Example 1.3.4. Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $A = \{b, d\}$ be a subset of X.
Open sets are $\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X$
Closed sets are $\emptyset, \{b, c, d\}, \{a, d\}, \{d\}, X$
Closed sets containing A are $X, \{b, c, d\}$
$\overline{A} = X \cap \{b, c, d\} = \{b, c, d\}$.

Example 1.3.5. Let X be an infinitive set with
$\tau_{cofinite} = \{U \subseteq X | U^c \text{ is finite}\} \cup \{\emptyset\}$ and $A \subseteq X$.

$$\overline{A} = \begin{cases} \text{A} & \text{if } A \text{ is finite} \\ \text{X} & \text{if } A \text{ is infinite} \end{cases}$$

Theorem 1.3.3. Let A and B be the subsets of the space X. Then

(i) $\overline{\emptyset} = \emptyset$

(ii) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$

(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(iv) $\overline{\overline{A}} = \overline{A}$

Proof: It follows from [21]. ■

Definition 1.3.5. Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in X$ is an *interior point* of A if there exists an open set U containing x such that $U \subseteq A$. The set of interior of A is called the interior of A and is denoted by $Int(A)$ or A^o.

A point $x \in X$ is an *exterior point* of A if there exists an open set U containing x such that $U \cap A = \emptyset$. The set of exterior points of A is called the exterior of A and is denoted by $Ext(A)$ or $(X - A)^o$.

A point of $x \in X$ is a *boundary point* of A if every open set in X containing x contains at least one point of A and at least one point of $X - A$. The set of boundary points of A is called the boundary of A and is denoted by $Bd(A)$ or $\partial(A)$.

In other words $\partial(A) = \overline{A} - A^o$.

Example 1.3.6. Let $X = \{a, b, c, d, e\}$ with the topology

$\tau = \{\emptyset, \{a\}, \{a, c, d\}, \{b, c, d, e\}, \{c, d\}, X\}$ and $A = \{b, c, d\}$ be the subset of X.

$Int(A) = \{c, d\}$

$Ext(A) = \{a\}$

$\partial(A) = \{b, e\}$

Definition 1.3.6. Let X be a topological space, $x \in X$ and $A \subseteq X$. Then x is a *accumulation point* of A if every open set containing x contains at least one point of A different from x. For any set A in the space X, the set of all accumulation points of A is called the *derived* set of A. The derived set of A is denoted by A'.

In other words, $x \in A' \Leftrightarrow \forall U \in \tau$ such that $x \in U$, $(U - \{x\}) \cap A \neq \emptyset$

Example 1.3.7. Let X be any infinitive set with topology

$\tau_{cofinite} = \{U \subseteq X | U^c \text{ is finite}\} \cup \{\emptyset\}$ and $A \subseteq X$. Then

$$A' = \begin{cases} \emptyset & \text{if } A \text{ is finite} \\ \text{X} & \text{if } A \text{ is infinite} \end{cases}$$

Theorem 1.3.4. Let A be the subset of the topological space (X, τ); let A' be the accumulation point of A. Then

$\overline{A} = A \cup A'$

Proof: It follows from [22]. ■

Definition 1.3.7. Let X be a topological space. Then $A \subseteq X$ is *dense* in X if $\overline{A} = X$.

Definition 1.3.8. Let (X, τ) be a topological space. A *base* for τ is a collection $\mathfrak{B}$ of subsets of X such that :

(i) each member of $\mathfrak{B}$ is also a member of τ.

(ii) if $U \in \tau$ and $U \neq \emptyset$, then U is the union of sets belonging to $\mathfrak{B}$

Since $\mathfrak{B} \subseteq \tau$ and if $U \neq \emptyset$, then $U \in \tau$ if and only if U is the union of members of $\mathfrak{B}$. Therefore, a base for τ completely determines τ by arbitrary unions of members of $\mathfrak{B}$.

Theorem 1.3.5. $\mathfrak{B}$ is a base for a topology on X if and only if

(i) $X = \bigcup_{B \in \mathfrak{B}} B$

(ii) whenever $B_1, B_2 \in \mathfrak{B}$ with $x \in B_1 \cap B_2$ there exists $B_3 \in \mathfrak{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Proof: It follows from [21]. ■

Definition 1.3.9. Let $\mathfrak{B}$ be the base for the set X, then we define the *topology generated* by $\mathfrak{B}$ as follows: A subset U of X is open if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B$ and $B \subseteq U$.

Example 1.3.8. Let $\mathfrak{Q} = \{(q_1, q_2) \mid q_1, q_2 \in \mathbb{Q}, q_1 < q_2\}$. Then $\mathfrak{Q}$ is a basis for $\mathbb{R}$. Indeed, let $s \in \mathbb{Z} \subseteq \mathbb{R}$. Then the open interval $(s-1, s+1) \in \mathfrak{Q}$ contains s. Let $(q_1, q_2), (p_1, p_2) \in \mathfrak{Q}$ such that both contain s. If $(q_1, q_2) \subseteq (p_1, p_2)$, then (q_1, q_2) serves as the set we need. A similar argument holds if $(p_1, p_2) \subseteq (q_1, q_2)$, Assume that $(q_1, q_2) \nsubseteq (p_1, p_2)$, and without loss of generality, suppose that $q_2 < p_2$. Then $s \in (p_1, q_2) \subseteq (q_1, q_2) \cap (p_1, p_2)$.

Example 1.3.9. Let $\mathfrak{Q} = \{(q_1, q_2) \mid q_1, q_2 \in \mathbb{Q}, q_1 < q_2\}$, we have that $\mathfrak{Q}' = \{\cup Q' \mid Q' \subseteq \mathfrak{Q}\}$ is the topology generated by $\mathfrak{Q}$, Since $\mathfrak{Q}$ is basis for $\mathbb{R}$ and union of collection of all basis gives us topology. So $(\mathbb{R}, \mathfrak{Q}')$ is a topological space.

Definition 1.3.10. Let (X, τ) be a topological space and $U \subseteq X$, $x \in U$, then *neighborhood* of x is a set U which contains an open set V containing x, i.e., $x \in V \subseteq U$ where $V \in \tau$. If U is an open set then U is said to be *open neighborhood*.

Proposition 1.3.1. Let (X, τ) be a topological space and $U \subseteq X$. Then U is open if and only if U is neighborhood of each elements in U.

Proof: It follows from [23]. ■

Definition 1.3.11. Let (X, τ) be a topological space and let $\emptyset \neq A \subseteq X$, the collection $\tau_A = \{U \cap A : U \in \tau\}$ forms a topology on A is called *subspace topology* of A. This topological space is denoted by (A, τ_A).

Example 1.3.10. Let $\mathbb{R}$ be real numbers with $\mathcal{U}$ standard topology and $\mathbb{Z} \subseteq \mathbb{R}$. Then $\tau_{\mathbb{Z}} = \{U \cap \mathbb{Z} : U \in \mathcal{U}\} = P(\mathbb{Z})$.

Definition 1.3.12. Let (X, τ) and (Y, τ') be two topological spaces. A function $f : X \longrightarrow Y$ is called *continuous* if, $\forall U \in \tau'$, $f^{-1}(U) \in \tau$.

Example 1.3.11. Let $\mathbb{R}$ denote the set of real number with standard topology, and let $\mathbb{R}_l$ denote the set of real numbers with lower limit topology. Let $f : \mathbb{R} \longrightarrow \mathbb{R}_l$ be the identity function ; $f(x) = x, \forall x \in \mathbb{R}$. Then f is not continuous; the inverse image of $[a, b)$ of $\mathbb{R}_l$ equals itself, which is not open in $\mathbb{R}$. But the identity function $g : \mathbb{R}_l \longrightarrow \mathbb{R}$ is continuous, since the inverse image of (a, b) of $\mathbb{R}$ is itself, which is open in $\mathbb{R}_l$.

Definition 1.3.13. Let (X, τ) be a topological space. If it satisfies the following conditions, then (X, τ) is called T_0-*space*.

For all $x, y \in X$ with $x \neq y$, there is either an open set containing x but not y or an open set containing y not x, i.e., there exist open sets U and V such that $x \in U$, $y \notin U$ or $y \in V$, $x \notin V$.

Example 1.3.12. Let X be a non-empty set with $\tau_{cofinite} = \{U \subseteq X | U^c \text{ is finite}\} \cup \{\emptyset\}$ topology. Then it is a T_0-space. Indeed, for each distinct pair $x, y \in X$, $\{y\}^c$ is open for x and does not contain y.

Definition 1.3.14. Let (X, τ) be a topological space. If it satisfies the following conditions, then (X, τ) is called T_1-*space*.

For all $x, y \in X$ with $x \neq y$, there is either an open set containing x but not y and an open set containing y not x, i.e., there exist open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Example 1.3.13. Let $\mathbb{R}$ be real numbers with standard topology $\mathcal{U}$. Then, $(\mathbb{R},\mathcal{U})$ is T_1-space.

Definition 1.3.15. Let (X, τ) be a topological space.

If for all $x, y \in X$ with $x \neq y$, there exists open sets U contains x and V contains y such that $U \cap V = \emptyset$, then (X, τ) is called T_2-*space* or *Hausdorff space*.

Example 1.3.14. The discrete space $(X, P(X))$ is T_2-space. Indeed, $\forall x \neq y$, there exists an open set $x \in \{x\} = U$ and $y \in \{y\} = V$ such that $U \cap V = \{x\} \cap \{y\} = \emptyset$.

Theorem 1.3.6. Every T_1-space is T_0-space and every T_2-space is T_1-space.

Proof: It follows from their definitions. ■

Definition 1.3.16. Let (X, τ) be a topological space, $A \subseteq X$ and let $\mathcal{G} = \{U_i | i \in I\}$ be a family of subsets of X. If $A \subseteq \bigcup_{i \in I} U_i$, then $\mathcal{G}$ family of subsets of A is called *cover* of A. If I is finite, then $\mathcal{G} = \{U_i | i \in I\}$ is called a *finite cover* of A. If each $U_i, i \in I$ is open in X and $\mathcal{G} = \{U_i | i \in I\}$ is called *open cover* of A.

Definition 1.3.17. Let $\mathcal{G} = \{U_i | i \in I\}$ be a cover of $A \subseteq X$. Then the family $\mathcal{G}' = \{U_{ik} | i_k \in J \subseteq I\}$ is a *subcover* of $\mathcal{G} = \{U_i | i \in I\}$ of A if $\mathcal{G}' = \{U_{ik} | i_k \in J \subseteq I\}$ covers A.

Example 1.3.15. Let $\mathbb{R}$ be a real numbers with standard topology and let $\forall n \in \mathbb{N}, U_n = (-n, n)$ and $V_n = (-2n, 2n)$. Then $\mathcal{G} = \{U_n | n \in \mathbb{N}\}$ and $\mathcal{H} = \{V_n | n \in \mathbb{N}\}$ are open covers of $\mathbb{R}$ and $\mathcal{H}$ is subcover of $\mathcal{G}$.

Definition 1.3.18. Let (X, τ) be a topological space. If each open cover of X has a finite subcover then (X, τ) is called *compact space*.

Example 1.3.16. Let $\mathbb{R}$ be set of real numbers with standard topology $\mathcal{U}$. Then $(\mathbb{R},\mathcal{U})$ is not compact. Indeed, it has no finite subcover which covers $\mathbb{R}$.

Example 1.3.17. Let X be an infinite set and let
$\tau_{cofinite} = \{U \subseteq X | U^c \text{ is finite}\} \cup \{\emptyset\}$ be a topology. Then $(X, \tau_{cofinite})$ is compact.

Proof: Let $\mathfrak{G} = \{U_i | i \in I, U_i \in \tau_{cofinite}\}$ be the open cover of X, i.e., $X = \bigcup_{i \in I} U_i$. Let $U_{i0} \in \tau$, i.e., ${U_{i0}}^c$ is finite. ${U_{i0}}^c = \{a_1, a_2, a_3, ..., a_n\}$.

$$X = {U_{i0}}^c \cup U_{i0} = U_{i0} \cup \{a_1, a_2, a_3, ..., a_n\} = \bigcup_{i \in I} U_i.$$

$$a_1 \in \bigcup_{i \in I} U_i \Rightarrow \exists i_1 \in I \ni a_1 \in U_{i1} \Rightarrow \{a_1\} \subseteq U_{i1}$$

$$a_2 \in \bigcup_{i \in I} U_i \Rightarrow \exists i_2 \in I \ni a_2 \in U_{i2} \Rightarrow \{a_2\} \subseteq U_{i2}$$

$$\vdots$$

$$a_n \in \bigcup_{i \in I} U_i \Rightarrow \exists i_n \in I \ni a_n \in U_{in} \Rightarrow \{a_n\} \subseteq U_{in}$$

$$\Rightarrow \{a_1\} \cup \{a_2\} \cup ... \cup \{a_n\} \subseteq \{a_1, a_2, a_3, ..., a_n\} \subseteq U_{i1} \cup U_{i2} \cup ... \cup U_{in}$$

$$\Rightarrow X = {U_{i0}}^c \cup U_{i0} \subseteq U_{i0} \cup U_{i1} \cup U_{i2} \cup ... \cup U_{in}$$

$\Rightarrow X = \bigcup_{k=0}^{n} U_{ik}$. Hence $(X, \tau_{cofinite})$ is compact. ■

1.4. Categorical Concepts

Categories are algebraic structures with many complementary natures, e.g., geometric, logical, computational, combinatorial, just as groups are many-faceted algebraic structures. In 1945 Eilenberg and MacLane introduced a category in a purely auxiliary fashion, as preparation for what they called functors and natural transformations. Eilenberg and MacLane [9] later wrote that their goal was to understand natural transformations; in order to do that, functors had to be defined, which required categories.

The reason why we are studying category theory is that it simplifies the communication among the people working in different fields by creating a new language which is economical regarding new ideas and their expression and it also provides a new meaning to the old problems by raising the different theorems and structures independent from each other. Category theory has applicative roots especially in theoretical Computer Science, cohomology theory, DNA and RNA codes of Molecular Biology and logics [10].

Definition 1.4.1. A *category* is a quadruple $\mathcal{E} = (\mathcal{O}, hom, id, \circ)$ consisting of,

(1) a class $\mathcal{O}$ whose members are called *$\mathcal{E}$-objects* denoted by $Ob(\mathcal{E})$,

(2) for each pair (A, B) of $\mathcal{O}$-objects, a set $hom(A, B)$ whose members are called *$\mathcal{E}$-morphisms* from A to B denoted by $Mor(\mathcal{E})$,

(3) for each $\mathcal{E}$-objects A, a morphisms

$$A \xrightarrow{id_A} A$$

called $\mathcal{E}$-*identity* on A,

(4) a composition law associating with each $\mathcal{E}$-morphism

$$A \xrightarrow{f} B$$

and each $\mathcal{E}$-morphism

$$B \xrightarrow{g} C$$

an $\mathcal{E}$- morphism

$$A \xrightarrow{g \circ f} C$$

called *composite* of f and g, subject to the following conditions:

(i) Associative Property: for each morphisms

$$A \xrightarrow{f} B$$

,

$$B \xrightarrow{g} C$$

and

$$C \xrightarrow{h} D$$

, the equation $h \circ (g \circ f) = (h \circ g) \circ f$ holds.

(ii) Identity Property: for each $\mathcal{E}$-morphisms

$$A \xrightarrow{f} B$$

the equation $id_B \circ f = f$ and $f \circ id_A = f$ holds [24].

Example 1.4.1. The category **Set** whose object class is the class of all sets; $hom(A, B) = \{f | f : A \longrightarrow B \text{ function}\}$ is the set of all functions from A to B, id_A is the identity function on A, and $\circ$ is the usual composition of functions.

Example 1.4.2. The category **Top** whose object class is the class of all topological space, morphisms are all continuous function between topological spaces, $id_{(X,\tau)}$ is identity morphism on (X,τ) and $\circ$ is the usual composition of topological spaces.

Example 1.4.3. The category **Grp** whose objects are groups, morphisms are all homomorphisms between groups, $1_{(G,\star)}$ is identity morphisms on $(G,\star)$, and $\circ$ is the usual composition of groups.

Example 1.4.4. The category **POSET** whose objects are partial ordered set; $hom((D,\leqslant),(D',\leqslant')) = \{f|f:(D,\leqslant)\longrightarrow(D',\leqslant')\ni\forall x,y\in D, x\leqslant y\Rightarrow f(x)\leqslant' f(y)\}$ morphisms are all order preserving between partial ordered sets, $1_{(D,\leqslant)}$ is identity morphisms on $(D,\leqslant)$, and $\circ$ is the usual composition of partial ordered sets.

Definition 1.4.2. (1) A category $\mathcal{C}$ is said to be a *subcategory* of a category $\mathcal{E}$ provided that the following conditions are satisfied:

(i) $Ob(\mathcal{C})\subseteq Ob(\mathcal{E})$

(ii) for each $C,C'\in Ob(\mathcal{C})$, $hom_{\mathcal{C}}(C,C')\subseteq hom_{\mathcal{E}}(C,C')$

(iii) for each $\mathcal{C}$-objects C, the $\mathcal{E}$-identity on C is the $\mathcal{C}$-identity on C,

(iv) the composition law in $\mathcal{C}$ is the restriction of the composition law in $\mathcal{E}$ to the morphisms of $\mathcal{C}$.

(2) $\mathcal{C}$ is called a *full subcategory* of $\mathcal{E}$ if, in addition to the above, for each $C,C'\in Ob(\mathcal{C}), hom_{\mathcal{C}}(C,C') = hom_{\mathcal{E}}(C,C')$ [24].

Example 1.4.5. For any category $\mathcal{C}$, the empty category and $\mathcal{C}$ itself are full subcategories of $\mathcal{C}$.

Example 1.4.6. **Haus** the class of all Hausdorff spaces specifies the full subcategory of **Top**. Indeed, $Ob(\mathcal{C})\subseteq Ob(\mathcal{E})$ and morphisms are same in $\mathcal{C}$ and $\mathcal{E}$.

Definition 1.4.3. Let $\mathcal{E}$ be a category and let $f:B\longrightarrow C$ be the morphism in $\mathcal{E}$. If for each $g,h:A\longrightarrow B$ morphisms in $\mathcal{E}$, $f\circ g = f\circ h$ implies $g=h$, then f is called *monomorphism* [24].

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \xrightarrow{f} C$$

Example 1.4.7. Let $\mathbf{Set}$ and $A, B \in Ob(\mathbf{Set})$. $f : A \longrightarrow B$ is monomorphism $\Leftrightarrow f$ is $1:1$

Example 1.4.8. Let $\mathbf{Top}$ and $(A, \tau), (B, \sigma) \in Ob(Top)$. $f : (A, \tau) \longrightarrow (B, \sigma)$ is monomorphism $\Leftrightarrow f$ is $1:1$ and continuous.

Definition 1.4.4. Let $\mathcal{E}$ be a category and let $f : A \longrightarrow B$ be the morphism in $\mathcal{E}$. If for each $g, h : B \longrightarrow C$ morphisms in $\mathcal{E}$, $g \circ f = h \circ f$ implies $g = h$, then f is called *epimorphism* [24].

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$$

Example 1.4.9. Let $\mathbf{Set}$. $f : A \longrightarrow B$ is epimorphism $\Leftrightarrow f$ is onto.

Example 1.4.10. Let $\mathbf{Top}$. $f : (A, \tau) \longrightarrow (B, \sigma)$ is epimorphism $\Leftrightarrow f$ is onto and continuous.

Definition 1.4.5. A morphism $f : A \longrightarrow B$ in a category $\mathcal{E}$ is called an *isomorphism* provided that there exists a morphism $g : B \longrightarrow A$ with $g \circ f = id_A$ and $f \circ g = id_B$ and it is denoted by $A \cong B$ [24].

Example 1.4.11. Let $\mathbf{Set}$. $f : A \longrightarrow B$ is isomorphism $\Leftrightarrow f$ is $1:1$ and onto.

Example 1.4.12. Let $\mathbf{Top}$. $f : (A, \tau) \longrightarrow (B, \sigma)$ is isomorphism $\Leftrightarrow f$ is homeomorphism.

Example 1.4.13. Let $\mathbf{Grp}$. $f : (A, \diamond) \longrightarrow (B, \star)$ is isomorphism $\Leftrightarrow f$ is group isomorphism.

Example 1.4.14. Every identity id_A is an isomorphism and $id_A^{-1} = id_A$.

Definition 1.4.6. Let $\mathcal{E}$ be a category and let i be any object of $\mathcal{E}$. If for each A object of $\mathcal{E}$ category $\mathcal{E}(i, A) = \{f \mid f : i \longrightarrow A\}$ there exist exactly one morphism from i to A, then i is said to be *initial object* [24].

Example 1.4.15. In $\mathbf{Set}$ category empty set is an initial object and in $\mathbf{Top}$ category, $i = (\emptyset, \{\emptyset\})$ is an initial object. Similarly in $\mathbf{Grp}$ category, $i = (G = \{e\}, \circ)$ is an initial object.

Definition 1.4.7. Let $\mathcal{E}$ be a category and let T be any object of $\mathcal{E}$. If for each A object of $\mathcal{E}$ category $\mathcal{E}(A,T) = \{f \mid f : A \longrightarrow T\}$ there exist exactly one morphism from A to T, then T is said to be *Terminal object* [24].

Example 1.4.16. In $\mathbf{Set}$ category, $T = \{x\}$ is a terminal object and, in $\mathbf{Top}$ category, $T = (\{x\}, \{\emptyset, \{x\}\})$ is a terminal object. Similarly, in $\mathbf{Grp}$ category, $T = (G = \{e\}, \circ)$ is a terminal object.

Definition 1.4.8. An object Z of $\mathcal{E}$ category is called *zero object* provided that it is both an initial and terminal object.

Example 1.4.17. $\mathbf{Set}$, $\mathbf{Top}$ and $\mathbf{POSET}$ don't have zero objects, but $\mathbf{Grp}$ has zero object.

Definition 1.4.9. A morphism

$$A \xrightarrow{f} B$$

of $\mathcal{E}$ category is called a *section* provided that there exists a morphism

$$B \xrightarrow{g} A$$

of $\mathcal{E}$ category such that $g \circ f = id_A$ [24].

Example 1.4.18. If T is a terminal object, then every morphism with domain T is a section.

Example 1.4.19. In $\mathbf{Vec}$, the sections are exactly the injective linear transformation.

Definition 1.4.10. A morphism

$$A \xrightarrow{f} B$$

of $\mathcal{E}$ category is called a *retraction* provided that there exists a morphism

$$B \xrightarrow{g} A$$

of $\mathcal{E}$ category such that $f \circ g = id_B$ [24].

Example 1.4.20. The retractions in $\mathbf{Set}$ are precisely surjective functions.

Example 1.4.21. In $\mathcal{E} = \mathbf{Vec}$, the sections are exactly the surjective linear transformation.

Definition 1.4.11. Let $\mathcal{E}$ be a category and I be any set and $\{A_i\}_{i\in I}$ be the class of object of $\mathcal{E}$ category, and let $P, X \in Ob(\mathcal{E})$ and let $p_i : P \longrightarrow \{A_i\}_{i\in I}$ be morphism in $\mathcal{E}$. If for each given morphism $f_i : X \longrightarrow \{A_i\}_{i\in I}$ there exists a unique morphism $\varphi\colon X \longrightarrow P$ such that $p_i \circ \varphi = f_i$, then $(P, \{p_i\}_{i\in I})$ is called *product* of $\{A_i\}_{i\in I}$ objects [24].

$$\begin{array}{ccc} P & \xrightarrow{p_i} & A_i \\ \varphi \uparrow & \nearrow_{f_i} & \\ X & & \end{array}$$

Example 1.4.22. In the category of $\mathbf{Set}$, given two sets A_1 and A_2, the projections from the cartesian product $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$ form a product $\pi_i : A_1 \times A_2 \to \{A_i\}_{i=1,2}$. Indeed, for a given $f_i : A \longrightarrow \{A_i\}_{i=1,2}$ morphism there is a unique $f : A \longrightarrow A_1 \times A_2$ with $f_i = \pi_i \circ f$, namely $f(a) = (f_1(a), f_2(a))$

Example 1.4.23. In the categories $\mathbf{Vec}$, $\mathbf{Grp}$ the "direct product", and in $\mathbf{Top}$ the " topological products", considered as sources via the projections, are products.

Definition 1.4.12. Let $\mathcal{E}$ be a category and I be any set and $\{A_i\}_{i\in I}$ be the class of object of $\mathcal{E}$ category, and let $P, X \in Ob(\mathcal{E})$ and let $q_i : \{A_i\}_{i\in I} \longrightarrow Q$ be morphism in $\mathcal{E}$. If for each given morphism $f_i : \{A_i\}_{i\in I} \longrightarrow X$ there exists a unique morphism $\varphi\colon Q \longrightarrow X$ such that $\varphi \circ q_i = f_i$, then $(P, \{p_i\}_{i\in I})$ is called *coproduct* of $\{A_i\}_{i\in I}$ objects [24].

$$\begin{array}{ccc} Q & \xleftarrow{q_i} & A_i \\ \varphi \downarrow & \swarrow_{f_i} & \\ X & & \end{array}$$

Example 1.4.24. In $\mathbf{Top}$, coproducts are called "topological sums" and can be constructed as for sets by supplying the disjoint union with the final topology.

In category theory it is the morphisms, rather than objects, that have primary role. Now, we take a more global view point and consider categories themselves as structured objects. The " morphisms" between them but preserve their structure are called *functors*.

Definition 1.4.13. Let $\mathcal{E}$ and $\mathcal{C}$ be two categories. A *functor* F from $\mathcal{E}$ to $\mathcal{C}$ is a function that assigns, each $A \in Ob(\mathcal{E})$ objects to $F(A) \in Ob(\mathcal{C})$, and to each $\mathcal{E}$-morphism

$$A \xrightarrow{f} A'$$

a $\mathcal{C}$-morphism

$$F(A) \xrightarrow{F(f)} F(A')$$

, in such a way that

(1) F preserves composition; i.e., $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined, and

(2) F preserves identity morphisms: i.e., $F(id_A) = id_{F(A)}$ for each $A \in Ob(\mathcal{E})$ [24].

Example 1.4.25. $U : \mathbf{Top} \longrightarrow \mathbf{Set}$ is given by $U(X, \tau) = X$ and $f : (X, \tau) \to (Y, \sigma)$ is continuous function, $U(f) = f$ is a functor. Indeed, let $(X, \tau), (Y, \sigma), (Z, \gamma)$ be objects of $\mathbf{Top}$ and let

$$(X, \tau) \xrightarrow{f} (Y, \sigma) \xrightarrow{g} (Z, \gamma)$$

be morphism in $\mathbf{Top}$. Since

$$X \xrightarrow{U(f)} Y \xrightarrow{U(g)} Z$$

is morphism in $\mathbf{Set}$, by taking image of $f \circ g$ in $\mathbf{Set}$ category, it is clear to see that $U(f \circ g) = U(g) \circ U(f)$. Similarly, if

$$(X, \tau) \xrightarrow{id_{(X,\tau)}} (X, \tau)$$

is a morphism in $\mathbf{Top}$, then

$$X \xrightarrow{U(id_{(X,\tau)})} X$$

will be a morphism in $\mathbf{Set}$ and by taking image $U(id_{(X,\tau)}) = id_{U(X,\tau)}$.

Example 1.4.26. $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ is given by $U(G, \circ) = G$, $f : (G, \circ) \to (H, \star)$ group homomorphism, $U(f) = f$ is a functor.

Definition 1.4.14. Let $F : \mathcal{E} \longrightarrow \mathcal{C}$ be a functor.

(1) F is called *faithful* provided that all that hom-set restrictions $F : hom_{\mathcal{E}}(A, A') \longrightarrow hom_{\mathcal{C}}(B, B')$ are injective: i.e., $\forall A, A' \in Ob(\mathcal{E})$ and for each $f, g : A \longrightarrow A'$ morphisms, $F(f) = F(g)$ implies $f = g$.

(2) F is called *full* provided that all hom-set restrictions are surjective: i.e., $\forall A, A' \in Ob(\mathcal{E})$ and for each $f : F(A) \longrightarrow F(A')$ morphism, there exists at least $g : A \longrightarrow A'$ morphism such that $F(g) = f$.

(3) F is called *amnestic* provided that an $\mathcal{E}$-isomorphism f is an identity whenever $F(f)$ is an identity: i.e., for any $f : A \longrightarrow A$ morphism, if $F(f) = Id = 1_{F(A)}$ and f is isomorphism, then $f = 1_A$.

(4) F is called *concrete* if F is both faithful and amnestic [24].

Example 1.4.27. $U : \mathbf{Top} \longrightarrow \mathbf{Set}$ is faithful and amnestic, so it is a concrete but not full. Because continuity of morphisms in $\mathbf{Top}$ category may not preserve. But $D : \mathbf{Set} \longrightarrow \mathbf{Top}$ is given by $D(A) = (A, P(A))$ discrete topological space, is concrete (i.e. faithful, amnestic) and full functor.

Example 1.4.28. $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ is concrete (faithful and amnestic) and but not full functor.

Definition 1.4.15. Let $F, G : \mathcal{E} \longrightarrow \mathcal{C}$ be functors. A *natural transformation* η from F to G is a function that assigns to each $A \in Ob(\mathcal{E})$, a $\mathcal{C}$-morphism $\eta_A : F(A) \longrightarrow G(A)$ in such a way that the following naturality condition holds: for each $\mathcal{E}$-morphism

$$A \xrightarrow{f} B$$

the square

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\eta_B]{} & G(B) \end{array}$$

commutes [24].

Example 1.4.29. Let $U : \mathbf{Top} \longrightarrow \mathbf{Set}$ be the forgetful functor and let $D : \mathbf{Set} \longrightarrow \mathbf{Top}$ be "discrete functor " defined by for a $A \in Ob(\mathbf{Set})$ and $D(A) \in Ob(\mathbf{Top})$, then $D(A) = (A, P(A))$. Thus, $\eta : I \longrightarrow UD$ is a natural transformation.

CHAPTER 2

SCOTT TOPOLOGY

Continuous lattice, a type of complete lattice (a partially ordered set in which all subsets have both a supremum (join) and an infimum (meet)), first studied under this name by D. Scott [4], examples of which occur in many areas of algebra, analysis and topology. Continuous lattices are usually defined in terms of an auxiliary relation, the way-below relation, which is definable in any complete lattice. There are two intrinsic topologies which are of importance in the study of continuous lattices, Scott topology and Lawson topology. The Scott topology was defined by D.S. Scott [6] in 1972 in all generality and demonstrated its usefulness in his article on " Continuous lattice". The name Scott topology was first used by Isbell [7] in 1975.

In this chapter, we will start with the definition of Scott open and Scott closed sets, Scott-continuous function and will discuss some important properties of Scott topology and later on, we will define approximation relation and will discuss some basic properties of this relation, continuous poset, Scott bases and Scott topology through this particular relation.

2.1. Scott Open Sets

Definition 2.1.1. Let $(D, \leqslant)$ be a dcpo and $U \subseteq D$. Then U is called *Scott open* if the following two conditions hold:

(i) U is an up set(or upper set), i.e., if $x \in U$ and $x \leqslant y$, then $y \in U$.

(ii) $\bigvee \triangle \in U$ implies $\triangle \cap U \neq \emptyset$ for all directed sets $\triangle \subseteq_{dir} D$ [25].

Proposition 2.1.1. Let $(D, \leqslant)$ be a dcpo and $\sigma(D) = \{U \subseteq D | U$is Scott open $\}$ be the

Scott topology. Then, $(D, \sigma(D))$ is a topological space.

Proof:

(i) Clearly D itself a Scott open set, and $\emptyset$ is vacuously Scott open set.

(ii) If $\{U_i : i \in I\}$ is an indexed family of sets, each U_i belong to $\sigma(D)$, then we need to proof that $\bigcup_{i \in I} U_i \in \sigma(D)$.
(a) In order to show $\bigcup_{i \in I} U_i$ is an upper set. Let $x \in \bigcup_{i \in I} U_i$ implies for some $i \in I$, $x \in U_i$ and $x \leqslant y$ implies $y \in U_i$. So for some $i \in I$, $y \in \bigcup_{i \in I} U_i$. Hence $\bigcup_{i \in I} U_i$ is upper set.
(b) Let $\triangle \subseteq_{dir} D$ and let $\bigvee \triangle \in \bigcup_{i \in I} U_i$. then we need to show that $\bigcup_{i \in I} U_i \cap \triangle$ is non-empty. Since $U_i \subseteq \bigcup_{i \in I} U_i \Rightarrow \underbrace{U_i \cap \triangle}_{\neq \emptyset} \subseteq \bigcup_{i \in I} U_i \cap \triangle$. Hence $\bigcup_{i \in I} U_i \cap \triangle \neq \emptyset$.

(iii) If $U, V \in \sigma(D)$, then we should prove $U \cap V \in \sigma(D)$.
(a) in order to show upper property of $U \cap V$, let $x \in U \cap V$ and let $x \leqslant y$. Since $x \in U \cap V \Rightarrow x \in U$ and $x \in V$, from our assumption and upper property of U and V, we have $x \leqslant y$, $y \in U$ and $y \in V \Rightarrow y \in U \cap V$. Hence $U \cap V$ is an upper set.
(b) Let $\triangle \subseteq_{dir} D$ and let $\bigvee \triangle \in U \cap V$. We need to prove that $(U \cap V) \cap \triangle$ is non-empty. Since U and V is upper sets of $\sigma(D)$, then $\exists x \in U \cap \triangle$ and $\exists y \in V \cap \triangle$. If $x = y$, then $U \cap V \cap \triangle$ is non-empty. If $x \neq y$, since $\triangle$ is directed, then $\exists z \in \triangle$ such that $x \leqslant z$ and $y \leqslant z$. Since $x \in U$, $y \in V$, U and V are upper sets, so $z \in U$ and $z \in V \Rightarrow z \in U \cap V$. Hence $(U \cap V) \cap \triangle \neq \emptyset$.

Thus, $\sigma(D)$ is a *Scott topology* over D. ■

Definition 2.1.2. A subset $F \subseteq D$ is called *Scott closed* if its complement is Scott open, i.e., F^c is an upper set and for any $\triangle \subseteq_{dir} D$ directed set that has a supremum $\bigvee \triangle$ with $\bigvee \triangle \in F^c$, then $\triangle \cap F^c \neq \emptyset$.

Proposition 2.1.2. A subset F of a poset D is Scott closed if it holds the following properties:

(i) F is a down set.

(ii) if $\triangle \subseteq_{dir} D$ contained in F and $\bigvee \triangle$ exists, then $\bigvee \triangle \in F$.

Proof: Since F is Scott closed subset if F^c is Scott open, i.e., F^c is an upper set, so F

is a down set. Similarly for any $\triangle \subseteq_{dir} D$, with $\bigvee \triangle \in F^c$, then $F^c \cap \triangle \neq \emptyset$ and so $\triangle \nsubseteq F$. The proof follows easily by contrapositive of this statement. ■

Example 2.1.1. Let $D = [5, 15] \subseteq \mathbb{R}$ and $\leqslant$ be an usual order on D. It is obvious that D is dcpo. D is a Scott open with respect to $\leqslant$ order. Indeed, let $U = (7, 15] \subseteq D$. Clearly U is an upper set and let $\triangle \subseteq_{dir} D$ be directed set such that $\bigvee \triangle \in U$. Therefore, $7 < \bigvee \triangle \leqslant 15$. So, $\triangle \nsubseteq [5, 7)$. Because if $\triangle \subseteq [5, 7)$, then $\bigvee \triangle \notin U$ which is not possible. Thus $\triangle \cap U \neq \emptyset$. Hence, U is Scott open.

Example 2.1.2. The right ray topology on $\mathbb{R}$ is the Scott topology. Indeed, let $\tau_{(a,\infty)} = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$. Since $\mathbb{R}$ is an upper set and for all $\triangle \subseteq_{dir} D = \mathbb{R}$ with $\bigvee \triangle \in \mathbb{R}$, $\mathbb{R} \cap \triangle \neq \emptyset$. Now, let $a \in \mathbb{R}$, then $U = (a, \infty) \in \tau_{(a,\infty)}$ is an upper set. Let $\triangle \subseteq_{dir} D = \mathbb{R}$ with $\bigvee \triangle$ exists such that $\bigvee \triangle \in U$. Thus, $a < \bigvee \triangle < \infty$ and hence, a is not an upper bound of $\triangle$, therefore there exists an $k \in \triangle$ such that $a < k \leqslant \bigvee \triangle$. So, $k \in U$ and $U \cap \triangle \neq \emptyset$. Hence, U is Scott open as desired.

Example 2.1.3. If D is a discrete poset, then every subset is Scott open.

Example 2.1.4. $U = [a, \infty) \subseteq \mathbb{R}$ is not Scott open under $\leqslant$ order. Indeed, let $\triangle = [b, a) \subseteq \mathbb{R}$, where $b < a$. Then $\triangle$ is directed set and $\bigvee \triangle = \{a\} \in U$. Moreover, $\triangle \cap U = \emptyset$. Thus, U is not Scott open.

Example 2.1.5. The set of real numbers $\mathbb{R}$ under usual order $\leqslant$ is Scott closed. Indeed, clearly $\mathbb{R}$ is a down set. If $\triangle \subseteq \mathbb{R}$ is a directed subset and $\bigvee \triangle$ exists,then $\bigvee \triangle \in \mathbb{R}$. Therefore, $\mathbb{R}$ is Scott closed.

Proposition 2.1.3. Let D be a dcpo. Then, the set
$U_x = D - \downarrow x = \{z \in D : z \nleqslant x\}$ is a Scott open set [13].

Proof: Let $z \in U_x$ and let $y \in D$ such that $z \leqslant y$. Assume that $y \notin U_x$. Then, $y \leqslant x$ and hence by transitivity, $z \leqslant x$. So, $z \notin U_x$ which contradicts with our assumption. Hence, $y \in U_x \Rightarrow U_x$ is an upper set.

Now, let $\triangle \subseteq_{dir} D$ be directed set such that $\bigvee \triangle$ exists and $\bigvee \triangle \in U_x$. Since $\bigvee \triangle \in U_x$, then $\bigvee \triangle \nleqslant x$. Suppose that for any $k \in \triangle$, $k \notin U_x$ and consequently $k \leqslant x, \forall k \in \triangle$. Thus, x is an upper bound of $\triangle$ implies $\bigvee \triangle \leqslant x$ which contradicts with our supposition. Thus, $\triangle \cap U_x \neq \emptyset$. Hence U_x is a Scott open set as desired. ■

Proposition 2.1.4. Let D be a dcpo and let K_D be denote the set of all compact elements in D. Then, for any $d \in K_D, \uparrow d = \{x : d \leqslant x\}$ is a Scott open [15].

Proof: Since $\uparrow d = \{x : d \leqslant x\}$, it is obvious that $\uparrow d$ is an upper set. Let $\triangle \subseteq_{dir} D$ be directed set with supremum $\bigvee \triangle$ such that $\bigvee \triangle \in \uparrow d$, we have $d \leqslant \bigvee \triangle$. Since d is compact, then there exists an $k \in \triangle$ such that $d \leqslant k$. So, $k \in \uparrow d \cap \triangle$. Thus, $\uparrow d$ is a Scott open set. ■

Proposition 2.1.5. In a finite poset, every ideal is Scott closed [14].

Proof: Let D be a finite poset and let I be an ideal subset of D. Since D is finite, it is dcpo. Since I is an ideal, then $I = \downarrow I$, i.e., I is a down set.

Now, let $\triangle \subseteq_{dir} I$ be directed set such that $\bigvee \triangle$ exists. Again by finiteness of D, $\bigvee \triangle \in \triangle$ and hence $\bigvee \triangle \in I$. Thus, I is Scott closed as desired. ■

Proposition 2.1.6. Let D be an algebraic dcpo. Then,the family
$\uparrow K_D = \{\uparrow d : d \leqslant x\}$ forms a base for $\sigma(D)$ Scott topology on D [13].

Proof: In order to show that $\uparrow K_D$ is a base for Scott topology, we should satisfy the following conditions.
(i) For any $x \in D$, there exists a compact element $d \in K_D$ such that $d \leqslant x$, that is $x \in \uparrow d$. So, $D \subseteq \bigcup_{d \in K_D} \uparrow d$ implies $D = \bigcup_{d \in K_D} \uparrow d$
(ii) Let $x \in \uparrow d \cap \uparrow d'$, for any $d, d' \in K_D$. So, $d, d' \in \downarrow_K x$. Since, $\downarrow_K x$ is a directed, then there exists $d'' \in \downarrow_K x$ such that $d \leqslant d''$ and $d' \leqslant d''$. By theorem 1.3.5., $\uparrow K_D$ is a base for $\sigma(D)$. ■

Definition 2.1.3. Let (X, τ) be a topological space and define a binary relation $\leqslant'$ on X by $x \leqslant' y$ if and only if $\forall U \in \tau, x \in U$ implies $y \in U$
Then $\leqslant'$ is called *specialization order* on X.

Proposition 2.1.7. A specialization order $(\leqslant')$ on (X, τ) is always preorder [26].

Proof: In order to show preordered property of specialization order, we should satisfy reflexivity and transitivity.
(i) (reflexivity):Let $x \in X$. Since $\forall U \in \tau, x \in U$. Hence, by definition of $\leqslant'$, $x \leqslant' x$.

(ii) (transitivity): Let $x, y, z \in X$, $x \leqslant' y$ and $y \leqslant' z$. We need to show that $x \leqslant' z$. Since $x \leqslant' y, \forall U \in \tau, x \in U \Rightarrow y \in U$ and
$y \leqslant' z, \forall V \in \tau, y \in V \Rightarrow z \in V$. So, if $x \in U$ then $z \in U$ and consequently $x \leqslant' z$. ■

Proposition 2.1.8. Let (X, τ) be a topological space.
(i) (X, τ) is T_0 if and only if the specialization order on X is a partial order.
(ii) (X, τ) is T_1 if and only if the specialization order on X is a discrete order [26].

Proof: $(i)(\Leftarrow)$ Let $(\leqslant')$ be the specialization order on X. By Proposition 2.1.7. $\leqslant'$ is preorder. Suppose that $x, y \in X$ and since $\leqslant'$ is antisymmetric relation on X. If $x \neq y$ either $x \nleqslant' y$ or $y \nleqslant' x$. This means that either there is an open set U such that $x \in U$ and $y \notin U$ or there is an open set V such that $x \notin V$ and $y \in V$. Hence, (X, τ) is T_0 space.
$(\Rightarrow)$ Let (X, τ) be T_0 space and let $(\leqslant')$ be the specialization order on X. By Proposition 2.1.7. $\leqslant'$ is preorder. So, we just need to proof $(\leqslant')$ is antisymmetric. Let $x, y \in X$ with $x \neq y$ and since (X, τ) is T_0 space, there exists an open set U such that $x \in U$ and $y \notin U$ or there exists an open set V such that $y \in V$ and $x \notin V$. This implies $x \nleqslant' y$ or $y \nleqslant' x$. So, $(\leqslant')$ is antisymmetric. Hence, $(\leqslant')$ is a partial order on X.

$(ii)(\Rightarrow)$ Suppose that (X, τ) is T_1 space. Then for each distinct points $x, y \in X$ there exist open sets U and V such that $x \in U$ and $y \notin U$, and $x \notin V$ and $y \in V$, so that x and y are not related by $(\leqslant')$. Thus the order $(\leqslant')$ is the discrete order.
$(\Leftarrow)$ Let the specialization order $(\leqslant')$ be a discrete order and $x, y \in X$ with $x \neq y$. Since $\leqslant'$ is discrete, $x \nleqslant' y$ and $y \nleqslant' x$. It follows that there exist open sets U and V such that $x \in U$ and $y \notin U$, and $x \notin V$ and $y \in V$. This implies (X, τ) is T_1 space. ■

Lemma 2.1.1. Let $(D, \leqslant)$ be a dcpo and let $(D, \sigma(D))$ be Scott topological space. Then, The specialization order $\leqslant'$ on $(D, \sigma(D))$ is a partial order $\leqslant$ [26].

Proof: We need to show $\leqslant' = \leqslant$. Let $(x, y) \in \leqslant$ and $U \in \sigma(D)$. Suppose $x \in U$. Since U is an upper set under $\leqslant$, and $x \leqslant y$, so $y \in U$. That is, $(x, y) \in \leqslant'$. Thus, $\leqslant \subseteq \leqslant'$.

Now, let $(x', y') \in \leqslant'$. Since $U_{y'} = \{a \in D : a \nleqslant y'\}$ is a Scott open and hence in $\sigma(D)$. Suppose $x' \nleqslant y'$. Then $x' \in U_{y'}$, and since $(x', y') \in \leqslant'$, it follows that $y' \in U_{y'}$, i.e. $y' \nleqslant y'$ which contradicts with our assumption. Thus, $x' \leqslant y'$, and so $\leqslant' \subseteq \leqslant$. ■

Lemma 2.1.2. Let $(D, \leqslant)$ be a dcpo. Then, Scott topological space $(D, \sigma(D))$ is T_0 space.

Proof: It follows from Proposition 2.1.8.(i) and Lemma 2.1.1. ■

Note that the Scott topological space $(D, \sigma(D))$ is not T_1 space, since for any distinct points $x, y \in D$, if $x < y$, then any upper set contains x must contain y. Thus, $(D, \sigma(D))$ is a non-Hausdorff space.

Definition 2.1.4. An *Alexandroff space* (or Alexandroff-discrete space) is a topological space in which the intersection of any family of open sets is open.

This space was first studied in 1937 by P.Alexandroff [27] under the name of Diskrete R$\ddot{a}$ume (discrete space). The name is not valid now, since discrete space is a space where all subsets are open.

Given a partially ordered set $(D, \leqslant)$, T_0-*Alexandroff space* τ on D can be defined by choosing the open sets to be upper set, i.e.,
$\tau = \{U \subseteq D | \forall x, y \in D, x \in U \wedge x \leqslant y \Rightarrow y \in U\}$. The induced topology on D-denoted by $(D, \tau(\leqslant))$ is a T_0-Alexandroff space [12].
In another words, If $(D, \tau(\leqslant))$ is a T_0-Alexandroff space, then a subset A of D is open if and only if $A = \uparrow A$, i.e., A is an upper set. For each $x \in D$, $\uparrow x$ or $V(x) = \bigcap_{U \in \tau, x \in U} U$ will denote the minimal basic neighborhood base of x.

Proposition 2.1.9. Let X be a non-empty set and let (X, τ) be a topological space and let $\leqslant$ be an order on X. Then, the following are equivalent to each other.
(i) (X, τ) is Alexandroff space.
(ii) $(X, \tau(\leqslant))$ is a T_0-Alexandroff space [14].

Proof: $(i) \Rightarrow (ii)$ Assume that arbitrary intersection of open sets is open, i.e., $\forall U_i \in \tau, i \in I, \bigcap_{i \in I} U_i \in \tau$. Let $\mathcal{B}_x = \{U_i \in \tau : x \in U_i\}$ be a neighborhood of $x \in X$. Then, $V(x) = \bigcap_{U_i \in \tau, x \in U_i} U_i$ is open. So, $V(x) \subseteq U_i$ for all $U_i \in \mathcal{B}_x$. Thus, $\mathcal{B'}_x = \{V(x)\}$ is the minimal basic neighborhood base of x.
$(ii) \Rightarrow (i)$ Let for each $x \in X$, x be a minimal basic neighborhood base, i.e., $\forall x \in X, \exists \mathcal{B}_x = \{V(x)\}$ be a minimal basic neighborhood base. Let $\{U_i : i \in I\}$ be the

collection all open sets in X and let $k \in \bigcap_{i\in I} U_i$. Then $\forall i \in I, k \in U_i$. So, $\forall i \in I, k \in V(k) \subseteq U_i$. Therefore, $V(k) \subseteq \bigcap_{i\in I} U_i$. Thus, $k \in V(k) \subseteq \bigcap_{i\in I} U_i$. Hence, arbitrary intersection of open sets is open. ■

Lemma 2.1.3. Every Scott open set is a Alexandroff open. On any poset D, the Scott topology is coarser than Alexandroff topology.

Proof: Since every Scott open set has upper set property, so every Scott open is Alexandroff open set.

Let $\mathcal{B} = \{\uparrow x : x \in D\}$ be a basis of Alexandroff space and $\uparrow x$ be an element of $\mathcal{B}$. Since, $\uparrow x$ is in $\uparrow K_D = \{\uparrow d : d \leqslant x\}$ which is a basis for Scott topology. Thus, $\mathcal{B} \subseteq \uparrow K_D$. Therefore, $\sigma(D) \subseteq \tau(D)$. Hence, $\sigma(D)$ Scott topology is coarser than $\tau(D)$ Alexandroff topology. ■

Example 2.1.6. Consider the set $\mathbb{R}$ of real numbers under the order $\leqslant$. Let $\mathcal{B} = \{[x, \infty) : x \in \mathbb{R}\}$. Since $\mathcal{B}$ has upper property, then $\mathcal{B}$ is a base for T_0-Alexandroff topology on $\mathbb{R}$. Now, let $\sigma(\mathbb{R}) = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ denote the right ray topology on $\mathbb{R}$. Then, $\sigma(\mathbb{R})$ is the Scott topology on $\mathbb{R}$. Since $(a, \infty) = \bigcup_{n\in\mathbb{N}} [a + 1/2n, \infty) \in \tau(\mathbb{R})$, where $\tau(\mathbb{R})$ stands for T_0-Alexandroff topology, then $\sigma(\mathbb{R}) \subseteq \tau(\mathbb{R})$.

Proposition 2.1.10. On a finite poset D, the T_0-Alexandroff topology agrees with Scott topology, i.e., $\tau(D) = \sigma(D)$ [14].

Proof: Since every upper set in Scott topology is also upper set in T_0-Alexandroff topology, i.e., $\sigma(D) \subseteq \tau(D)$. So, we just need to show that $\tau(D) \subseteq \sigma(D)$. Let U be any Alexandroff open set and let $\triangle$ be any directed subset of D such that $\bigvee \triangle \in U$. Since $\triangle$ is directed and D is finite. Thus $\triangle$ has a top element such that $\top = \bigvee \triangle \in \triangle$. So, $U \cap \triangle \neq \emptyset$. Hence, U is a Scott open set and $\tau(D) \subseteq \sigma(D)$. We have $\tau(D) = \sigma(D)$. ■

Example 2.1.7. Let $D = \{1, 2, 3, ..., n\}$, where $n \in \mathbb{N}$, by the $\leqslant$ order . Then D is an algebraic dcpo and $K_D = D$. So, $\mathcal{B} = \{\uparrow d : d \in K_D\} = \{\uparrow x : x \in D\}$ is a base for Scott topology and for T_0-Alexandroff topology as well.

Proposition 2.1.11. Let D be a dcpo and $(D, \sigma(D))$ be Scott topological space. Then, $\forall x \in D, \downarrow x = \overline{\{x\}}$, where $\overline{\{x\}}$ is closure with respect to $\sigma(D)$ [25].

Proof: Since $\downarrow x$ is a down set and hence a closed set containing x. So, $\overline{\{x\}} \subseteq\downarrow x$. Conversely, let $y \in\downarrow x$, so, $y \leqslant x$. We need to show that $y \in \overline{\{x\}}$. Let $y \in U, \forall U \in \sigma(D)$. Since $y \leqslant x$ and $U \in \sigma(D)$, $x \in U$. Thus, $U \cap \{x\} = \{x\} \neq \emptyset$. Hence $y \in \overline{\{x\}}$. Hence, $\downarrow x \subseteq \overline{\{x\}}$.

Thus, $\downarrow x = \overline{\{x\}}$ as desired. ■

Example 2.1.8. Let $D = \{a, b, c, d\}$ be a set ordered as in the following diagram. xRy if and only if $x = y$ or one can from x to y in upward direction.

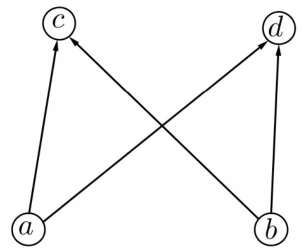


Figure 2.1. Hasse diagram of set of four-elements ordered by direction of arrows

Since D is finite. Thus, $\tau(D) = \sigma(D)$. So, in order to find Scott topology on D, we just need to find upper set of D. Thus, $\sigma(D) = \{\emptyset, D, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$. So, the closed sets are $D, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}$. Since, $\downarrow a = \{a\} and \downarrow d = \{d\}$ are down sets, so their closures will be equal to themselves.

$\overline{\{a\}} = \{a\}$

$\overline{\{b\}} = \{a, b, d\}$

$\overline{\{c\}} = \{a, c, d\}$

$\overline{\{d\}} = \{d\}$

Now, let us check the hereditary property of Scott topology.

Proposition 2.1.12. A subspace of Scott topology is a Scott subspace [14].

Proof: Let $(D, \sigma(D))$ be a Scott topological space and let A be any subset of D. Then, A has subspace topology $\sigma_A = \{A \cap U : U \in \sigma(D)\}$. Let $B \in \sigma_A$. So, there exists $U \in \sigma(D)$ such that $B = A \cap U$, and let $x \in B$ and $y \in A$ such that $x \leqslant y$. Since,

$x \in U \in \sigma(D)$, then $y \in U$ and hence, $y \in A \cap U = B$. Thus, B is an upper set w.r.t. to A.

Now, let for any $\triangle \subseteq_{dir} A$ be directed set such that $\bigvee \triangle$ exists and $\bigvee \triangle \in B$. Then, $\triangle \cap A = \triangle \neq \emptyset$. Since U is a Scott open, then $\triangle \cap U \neq \emptyset$. Therefore, $\triangle \cap B = \triangle \cap (A \cap U) = (\triangle \cap A) \cap U = \triangle \cap U \neq \emptyset$. Hence B is a Scott open w.r.t. to A. ■

Proposition 2.1.13. The finite product of Scott topological spaces is a Scott topological space.

Proof: Let $(D_1, \sigma(D_1)), (D_2, \sigma(D_2)), ..., (D_n, \sigma(D_n))$ be Scott topological spaces, $D = D_1 \times D_2 \times ... \times D_n$ and τ be product topology on D. We need to show that τ is a Scott topology on D. Let $x = (x_1, x_2, ..., x_n) \in U$ and let $x = (x_1, x_2, ..., x_n) \leqslant y = (y_1, y_2, ..., y_n)$. Since

$x_1 \leqslant y_1$ and U_1 is a Scott open implies $y_1 \in U_1$. Similarly,

$x_2 \leqslant y_2$ and U_2 is a Scott open implies $y_2 \in U_2$.

$\vdots$

$x_n \leqslant y_n$ and U_n is a Scott open implies $y_n \in U_n$. Therefore,

$y = (y_1, y_2, ..., y_n) \in U = U_1 \times U_2 \times ... \times U_n$. Thus, U is an upper set.

Now, let $\triangle = \triangle_1 \times \triangle_2 \times ... \times \triangle_n \subseteq_{dir} D = D_1 \times D_2 \times ... \times D_n$ with $\bigvee \triangle \in U$. Since

U_1 is Scott open and $\triangle_1 \subseteq_{dir} D_1$, then $\triangle_1 \cap D_1 \neq \emptyset$. Similarly

U_2 is Scott open and $\triangle_2 \subseteq_{dir} D_2$, then $\triangle_2 \cap D_2 \neq \emptyset$.

$\vdots$

U_n is Scott open and $\triangle_n \subseteq_{dir} D_n$, then $\triangle_n \cap D_n \neq \emptyset$. Therefore,

$\triangle \cap U = (\triangle_1 \times \triangle_2 \times ... \times \triangle_n) \cap (U_1 \times U_2 \times ... \times U_n)$

$= \underbrace{(\triangle_1 \cap U_1)}_{\neq \emptyset} \times \underbrace{(\triangle_2 \cap U_2)}_{\neq \emptyset} \times ... \times \underbrace{(\triangle_n \cap U_n)}_{\neq \emptyset}$ implies $\triangle \cap U \neq \emptyset$. Hence, U is a Scott open set. Thus, τ is a Scott topology. ■

2.2. The Approximation Relation

Definition 2.2.1. Let $(D, \leqslant)$ be a poset. Then, for any $x, y \in D$

x *approximate* y if and only if for all $\triangle \subseteq_{dir} D$ with $\bigvee \triangle$, $y \leqslant \bigvee \triangle$ implies that there exists $k \in \triangle$ such that $x \leqslant k$ [28].

In other words, x *approximate* y if and only if every directed set with join (supremum) above y has a member above x [29].

x approximate y is denoted by $x \ll y$. $\Uparrow x = \{a \in D : x \ll a\}$ is called *way-above* set. Similarly $\Downarrow x = \{a \in D : a \ll x\}$ is called *way-below* set.

Example 2.2.1. Let $D = [5, 8]$, then $(D, \leqslant)$ is a dcpo. Clearly $5 \ll 6$, since for any directed subset $\triangle$ of D with $6 \leqslant \bigvee \triangle$, there exists $k \in \triangle$ such that $5 \leqslant k$. Similarly $5 \ll 7$ and $5 \ll 8$. Actually,$\forall x \in D$, $5 \ll x$.

Proposition 2.2.1. Let $(D, \leqslant)$ be a poset. The following are true.
(i) $x \ll y \Rightarrow x \leqslant y$. But its converse is not always true.
(ii) $z \leqslant x \ll y \leqslant w \Rightarrow z \ll w$.
(iii) If $\perp$ is the least element, then $\forall x \in D$, $\perp \ll x$ [29].

Proof: (i) Suppose that $x \ll y$. Let $\{y\}$ be a directed subset of D with $\bigvee\{y\} = y$ and $y \leqslant \bigvee\{y\} = y$. Since $x \ll y$, then there exists $k \in \{y\}$ such that $x \leqslant k$. Thus, $x \leqslant k = y$.

But its converse is not always true. Let $D = [5, 6] \cup \{3\}$ and ordered over D is defined as follows: the elements of $[5, 6]$ are ordered by $\leqslant$. For 3 and any $x \in [5, 6)$, x and 3 are not comparable. But for $x = 6$, $3 \leqslant 6$. Then D is clearly a dcpo. Let $\triangle = [5, 6)$, then $\triangle$ is a directed subset of D with $\bigvee \triangle = 6$. Clearly $6 \leqslant \bigvee \triangle = 6$. But, for any $x \in \triangle$, $3 \nleqslant x$. Hence, 3 not approximate 6.

(ii) Assume that $z \leqslant x \ll y \leqslant w$. Let $\triangle \subseteq_{dir} D$ with supremum $\bigvee \triangle$ such that $w \leqslant \bigvee \triangle$. Since $y \leqslant w$, then $y \leqslant \bigvee \triangle$. Now, $x \ll y$ and $y \leqslant \bigvee \triangle$, then $\exists k \in \triangle$ such that $x \leqslant k$. Thus, $z \leqslant x \Rightarrow z \leqslant k$.

(iii) Let $x \in D$ and let $\triangle \subseteq_{dir} D$ with supremum $\bigvee \triangle$ such that $x \leqslant \bigvee \triangle$. Since $\triangle$ is directed, then $\triangle \neq \emptyset$. So, there exists $k \in \triangle$ such that $\perp \leqslant k$, because $\perp \leqslant k, \forall k \in D$. Thus, $\perp \ll x$,$\forall x \in D$. ■

The approximation relation is not always have reflexive property. Let us give an example to hold our statement.

Example 2.2.2. Let $D = [5, 6] \cup \{3\}$ and let ordered over D be defined as follows: the elements of $[5, 6]$ are ordered by $\leqslant$. For 3 and any $x \in [5, 6)$, x and 3 are not comparable. But for $x = 6$, $3 \leqslant 6$. Then D is clearly a dcpo. Let $\triangle = [5, 6)$, then $\triangle$ is a directed subset of D with $\bigvee \triangle = 6$ and $3 \leqslant \bigvee \triangle = 6$. Since $\forall k \in \triangle$, 3 and k are not comparable, then there is no $k \in \triangle$ such that $3 \leqslant k$. Thus, 3 doesnot approximate 3.

Proposition 2.2.2. Let D be a dcpo. then an element $d \in K$ is compact if and only if $d \ll d$ [14].

Proof: $(\Rightarrow)$ Let D be a dcpo and let $d \in K$ be a compact element and let $\triangle$ be a directed subset with supremum $\bigvee \triangle$ such that $d \leqslant \bigvee \triangle$. Then by the definition of the compact element, there exists $k \in \triangle$ such that $d \leqslant k$. Thus, $d \ll d$.
$(\Leftarrow)$ Let $d \ll d$ and let $\triangle_{dir} \subseteq D$ with supremum $\bigvee \triangle$, such that $d \leqslant \bigvee \triangle$. Since d approximate d, then there exists $k \in \triangle$ such that $d \leqslant k$ and this implies d is a compact element. ■

Definition 2.2.2. (i) A poset D is said to be *continuous* if for every $x \in D$, there exists a directed set $\triangle_x \subseteq \Downarrow x$ such that $x = \bigvee \triangle_x$.
In another words, if $\Downarrow x$ is directed and $x = \bigvee \Downarrow x$, then D is said to be *continuous*.
(ii) A dcpo which is continuous as a poset is called a *domain*.
(iii) A domain which is a complete lattice is called a *continuous lattice* [25].

Example 2.2.3. The set of all real numbers $\mathbb{R}$ is continuous under the order " $\leqslant$ " and hence $\mathbb{R}$ is a domain.

Proof: Let $y \in \mathbb{R}$ be any arbitrary fixed point and let $z \in \mathbb{R}$.
(i) Suppose $z < y$. Then, let $\triangle$ be any directed subset with supremum $\bigvee \triangle$ such that $y \leqslant \bigvee \triangle$. Then, $z < \bigvee \triangle$ and hence z is not upper bound of $\triangle$ in $\mathbb{R}$ under the usual order " $\leqslant$ ". Therefore, there exists $k \in \triangle$ such that $z \leqslant k$ and hence $z \ll y$.
(ii) Assume that $z = y$. Then, z doesnot approximate y. If we take $\triangle = (-\infty, y)$ which is directed subset of $\mathbb{R}$, then $\bigvee \triangle = y$. So, $y \leqslant \bigvee \triangle$ but there is no $k \in \triangle$ such that $z = y \leqslant k$.
(iii) Suppose that $y > z$. i.e. $y \nleq z$. So, y doesnot approximate z.
Thus, from above $(i), (ii), (iii)$ cases, we have $\Downarrow x = (-\infty, y)$, $\forall y \in \mathbb{R}$ which is a directed subset of $\mathbb{R}$ with supremum y. Hence, $\mathbb{R}$ is continuous with respect to " $\leqslant$ " order.
Thus, $\mathbb{R}$ is a domain under usual order " $\leqslant$ ". ■

Proposition 2.2.3. Every algebraic dcpo is domain [14].

Proof: Let D be an algebraic dcpo and let $x \in D$. Since, from the definition of algebraic dcpo, we have $\downarrow_K x = \{d \in K_D | d \leqslant x\}$ is directed and $x = \bigvee \downarrow_K x$. So, we just need to prove that $\downarrow_K x \subseteq \Downarrow x$. Let $d \in \downarrow_K x$. Then d is compact and $d \leqslant x$. Since, $d \leqslant d \ll d \leqslant x$ implies $d \ll x$. Thus, $d \in \Downarrow x$. Hence, D is continuous. Thus D an algebraic dcpo is a domain. ■

Lemma 2.2.1. Let D be a poset. Then, for any $x, y \in D$, $y \in int(\uparrow x)$ implies $x \ll y$, where $int(\uparrow x)$ denotes the interior of $\uparrow x$ in Scott topology [14].

Proof: Let $\triangle$ be a directed subset of D such that $\bigvee \triangle$ exists and let $y \leqslant \bigvee \triangle$. Since, $y \in int(\uparrow x)$, $y \leqslant \bigvee \triangle$ and $int(\uparrow x)$ is an up set, then $\bigvee \triangle \in int(\uparrow x)$. Since $int(\uparrow x)$ is Scott open, there exists $k \in \triangle$ such that $k \in int(\uparrow x) \subseteq \uparrow x$. So, $k \in \uparrow x$ and $x \leqslant k$. Hence, $x \ll y$. ■

Proposition 2.2.4. If D is a continuous poset, then the approximating relation $\ll$ has the interpolation property:
$x \ll z \Rightarrow \exists y \in D$ such that $x \ll y \ll z$ [16].

Proof: Let $C = \{u \in D : \exists y \in D$ such that $u \ll y \ll z\}$. It can be easily deduced from the directedness and the approximating property of $\Downarrow x$ for every $x \in D$ that C is directed and has z as its supremum. Thus, it follows from $x \ll z$ that there is some $u \in C$ such that $x \leqslant u$. By the construction of C, there is some $y \in D$ such that $x \leqslant u \ll y \ll z$ as desired. ■

Definition 2.2.3. Let D be a poset and let F be a filter in D. F is called *open filter* if it is Scott open set [16].

Proposition 2.2.5. Let D be a continuous dcpo and let $x, y \in D$. Then, the following are equivalent.
(i) $x \ll y$
(ii) There exists an open filter F with $y \in F \subseteq \uparrow x$
(iii) $y \in int(\uparrow x)$, where $int(\uparrow x)$ denotes interior of $\uparrow x$ in Scott topology [16].

Proof: $(i) \Rightarrow (ii)$ Since $x \ll y$ and D is continuous, then there exists x_1 such that $x \ll x_1 \ll y$. Similarly there exists x_2 such that $x \ll x_2 \lll\ll x_1 \ll y$, continuing this process we can get $\{x_n\}_{n\in\mathbb{N}}$ satisfying $x \ll x_{n+1} \ll x_n \ll y$. Let $F = \bigcup(\uparrow x_n)$, where $n \in \mathbb{N}$. We need to show that F is Scott open set. Clearly F is an upper set. Let $\triangle$ be any directed subset of D such that $\bigvee \triangle \in F$. Then, for some $n \in \mathbb{N}$, $x_n \leqslant \bigvee \triangle$. Since $x_{n+1} \ll x_n$, then by the definition of $\ll$, $x_{n+1} \leqslant k$, where $k \in \triangle$ and thus $k \in \uparrow x_{n+1} \subseteq F$ implies $k \in F$. Thus, $\triangle \cap F \neq \emptyset$. Hence, F is a Scott open set. Since, for each $n \in \mathbb{N}$, $x \ll x_{n+1} \ll x_n \ll y$ implies $x \leqslant x_{n+1} \leqslant x_n \leqslant y$. Thus, F is a filtered set and $y \in F \subseteq \uparrow x$.
$(ii) \Rightarrow (iii)$ Since F is a Scott open. Thus, $y \in int(\uparrow x)$.
$(iii) \Rightarrow (i)$. The proof follows by applying Lemma 2.2.1. ■

Proposition 2.2.6. Let D be a continuous dcpo and let $x \in D$ and Let $(D, \sigma(D))$ be Scott topological space and let A be any subset of D. Then,
(i) the set $\Uparrow x = \{y \in D : x \ll y\}$ is Scott open.
(ii) $int(A) = \bigcup\{\Uparrow x : \Uparrow x \subseteq A\}$, where $int(A)$ denotes interior of A with respect to $\sigma(D)$ [16].

Proof: (i) Since D is a continuous poset, then, we can say that $y \in \Uparrow x$ if and only if $y \in int(\uparrow x)$, where $int(\uparrow x)$ denotes the interior of $\uparrow x$ with respect to Scott topology. Therefore, $\Uparrow x = int(\uparrow x)$. Hence, $\Uparrow x$ is Scott open.
(ii) The proof follows directly from (i). ■

2.3. Scott Continuous Function

Definition 2.3.1. Let $(D, \leqslant)$ and $(D', \leqslant')$ be two dcpos. A function $f : (D, \leqslant) \longrightarrow (D', \leqslant')$ is said to be Scott continuous function if for any directed $\triangle \subseteq_{dir} D$, f is monotonic and $f(\bigvee \triangle) = \bigvee f(\triangle)$ [25].

Proposition 2.3.1. Let $(D, \leqslant)$ and $(D', \leqslant')$ be two dcpos, and let $f : (D, \leqslant) \rightarrow (D', \leqslant')$ be a function, and let $\triangle \subseteq_{dir} D$ and $f(\triangle)$ be directed subset of D'.
Then, if $f(\bigvee \triangle) = \bigvee f(\triangle)$ then f is monotonic [26].

Proof: Let $x, y \in D$ and let $x \leqslant y$. Then, $\triangle = \{x, y\}$ is directed and hence, by assumption, $f(\triangle) = \{f(x), f(y)\}$ is directed. Furthermore, again by our assumption, $f(y) = f(\bigvee \triangle) = \bigvee f(\triangle)$, that is $f(x) \leqslant' f(y)$. Hence, f is monotonic. ■

Proposition 2.3.2. Let $(D, \leqslant)$ and $(D', \leqslant')$ be two dcpos, and let $f : (D, \leqslant) \to (D', \leqslant')$ be a function and $\triangle \subseteq_{dir} D$ be directed subset of D.
Then $f(\triangle)$ is directed and $\bigvee f(\triangle) \leqslant' f(\bigvee \triangle)$ [26].

Proof: We first need to show that $f(\triangle)$ is directed. Let $x, y \in D$. Since $\triangle$ is directed subset of D, there exists $z \in D$ such that $x \leqslant z$ and $y \leqslant z$. Hence, by assumption f is monotonic, $f(x) \leqslant' f(z)$ and $f(y) \leqslant' f(z)$, where $f(x), f(y), f(z) \in D'$. Thus, $f(\triangle)$ is directed.
Now, let $\triangle \subseteq_{dir} D$ be directed set and let $k = \bigvee \triangle$, where $k \in \triangle$. We want to show that $\bigvee f(\triangle) \leqslant' f(k)$. So, for any $t \in \triangle$, since $\triangle$ is directed, we have $t \leqslant k$ so that $f(t) \leqslant' f(k)$, since f is monotonic. This shows $\bigvee f(\triangle) \leqslant' f(k) = f(\bigvee \triangle)$ as desired.
■

Example 2.3.1. Consider $(\mathbb{N}_n, \leqslant)$ and $(\mathbb{N}_{n+1}, \leqslant)$ be dcpos and let $g : \mathbb{N}_n \to \mathbb{N}_{n+1}$ be the function $g(n) = n + 1$. We want to show that g is Scott continuous. g is monotonic. So, let $\triangle \subseteq_{dir} \mathbb{N}_n$. Then $\bigvee \triangle$ is simply the largest natural number in $\triangle$, call it δ. Since $k \leqslant \delta, \forall k \in \triangle$, it follows from the monotonicity of g that $g(k) \leqslant g(\delta), \forall k \in \triangle$. That is, $g(\delta)$ is the largest natural number in $g(\triangle)$, hence $\bigvee g(\triangle) = g(\delta) = g(\bigvee \triangle)$. Since $\triangle$ is arbitrary, g is Scott continuous.

Proposition 2.3.3. Let $(D, \leqslant)$ be a dcpo and let $id_D : (D, \leqslant) \to (D, \leqslant)$ be the identity function. Then, id_D is a Scott continuous.

Proof: Let $x, y \in D$ and let $x \leqslant y$, clearly $id_D(x) = x \leqslant y = id_D(y)$. Now, let $\triangle \subseteq_{dir} D$. Then, we have $id_D(\bigvee \triangle) = \bigvee \triangle$ and $\bigvee id_D(\triangle) = \bigvee \triangle$.
Thus, $id_D(\bigvee \triangle) = \bigvee id_D(\triangle)$. ■

Proposition 2.3.4. Let $(D, \leqslant)$, $(D', \leqslant')$ and $(D'', \leqslant'')$ be dcpos, and let $f : (D, \leqslant) \to (D', \leqslant')$ and $g : (D', \leqslant') \to (D'', \leqslant'')$ be Scott continuous functions. Then, $g \circ f : (D, \leqslant) \to (D'', \leqslant'')$ is Scott continuous function.

Proof: Let $x, y \in D$ and let $x \leqslant y$. Then $f(x) \leqslant' f(y)$, and so $g(f(x)) \leqslant'' g(f(y))$. Thus, $g \circ f$ is monotonic. Now, let $\triangle \subseteq_{dir} D$. We need to show that

$g(f(\bigvee \triangle)) = \bigvee g(f(\triangle))$. Since f is Scott continuous, $g(f(\bigvee \triangle)) = g(\bigvee f(\triangle))$ and since g is continuous, $g(\bigvee f(\triangle)) = \bigvee g(f(\triangle))$. Thus, $g(f(\bigvee \triangle)) = \bigvee g(f(\triangle))$. ■

Proposition 2.3.5. Let $(D, \leqslant)$ and $(D', \leqslant')$ be a dcpos, and let $(D, \sigma(D))$ and $(D', \sigma(D'))$ be Scott topological spaces. Then,
$f : (D, \sigma(D)) \longrightarrow (D', \sigma(D'))$ is topologically continuous if and only if $f : (D, \leqslant) \longrightarrow (D', \leqslant')$ is Scott continuous [26].

Proof: ($\Rightarrow$) Let f be topologically continuous and $x, y \in D$ with $x \leqslant y$. We wish to show that $f(x) \leqslant' f(y)$, or $f(x) \in \downarrow f(y)$. Assume the contrary, consider $U = D' - \downarrow f(y) = \{f(x) \in D' | f(x) \nleq' f(y)\}$. Then, $f(x) \in U$ and U is Scott open, hence $x \in f^{-1}(U)$ is also Scott open. Since $x \leqslant y$ and $f^{-1}(U)$ is upper set, $y \in f^{-1}(U)$, which implies $f(y) \in U = D' - \downarrow f(y)$, a contradiction. Therefore, $f(x) \leqslant' f(y)$. Thus, f is monotonic. Now, let $\triangle \subseteq_{dir} D$ be directed set and let $k = \bigvee \triangle$, where $k \in \triangle$. We want to show that $f(k) = \bigvee f(\triangle)$. First, for any $t \in \triangle$, since $\triangle$ is directed, we have $t \leqslant k$ so that $f(t) \leqslant' f(k)$, since f is monotonic. This shows $\bigvee f(\triangle) \leqslant' f(k)$.
Now, suppose that r is any upper bound of $f(\triangle)$. We want to show that $f(k) \leqslant' r$, or $f(k) \in \downarrow r$. Assume contrary, then $f(k)$ lies in $U = D' - \downarrow r = \{f(k) \in D' | f(k) \nleq' r\}$, a Scott open, so $\bigvee \triangle = k \in f^{-1}(U)$, $f^{-1}(U)$ is also Scott open, which implies some $t \in \triangle$ with $t \in f^{-1}(U)$, or $f(t) \in U$. This means $f(t) \nleq' r$, a contradiction. Therefore, $f(k) \leqslant' r$ implies $f(k) \leqslant' \bigvee f(\triangle)$. Thus, f is a Scott continuous function.

($\Leftarrow$) Let f be Scott continuous function and $U \in \sigma(D')$. We want to show that $f^{-1}(U) \in \sigma(D)$. In other words, we need to prove that $f^{-1}(U)$ is an upper set and for any $\triangle \subseteq_{dir} D$ directed set with $\bigvee \triangle \in f^{-1}(U)$, $f^{-1}(U) \cap \triangle \neq \emptyset$. Let $y \in f^{-1}(U)$ with $y \leqslant x$, which implies $f(y) \leqslant' f(x)$, f is a monotonic. Since $f(y) \in U$ and U is Scott open, $f(x) \in U$, so $x \in f^{-1}(U)$. Therefore, $f^{-1}(U)$ is an upper set.
Now, suppose that $\triangle \subseteq_{dir} D$ is directed set with $\bigvee \triangle \in f^{-1}(U)$. Since f is Scott continuous function, $\bigvee f(\triangle) = f(\bigvee \triangle) \in U$ and since $f(\triangle)$ is directed, there is $y \in f(\triangle) \cap U$, which means there is $x \in D$ such that $f(x) = y$ and $x \in \triangle \cap f^{-1}(U)$. This shows $\triangle \cap f^{-1}(U) \neq \emptyset$. Thus, $f^{-1}(U) \in \sigma(D)$. Hence, f is topologically continuous. ■

Lemma 2.3.1. Let $(D, \leqslant)$ be an algebraic dcpo.

(i) For $x, y \in D$, $x \leqslant y$ if and only if $\downarrow_K x \subseteq \downarrow_K y$.

(ii) If $\triangle \subseteq_{dir} D$ then $\downarrow_K \bigvee \triangle = \bigcup\{\downarrow_K x : x \in \triangle\}$ [26].

Proof: (i) $(\Rightarrow)$ Let $x, y \in D$ and $x \leqslant y$, and let $d \in \downarrow_K x = \{d \in K_D : d \leqslant x\}$. Since $x \leqslant y$ and $d \leqslant x$ implies $d \leqslant y$. Therefore, $d \in \downarrow_K$. Thus, $\downarrow_K x \subseteq \downarrow_K y$.

$(\Leftarrow)$ Let $\downarrow_K x \subseteq \downarrow_K y$. Since $(D, \leqslant)$ is algebraic, by taking supremum $\bigvee \downarrow_K x \leqslant \bigvee \downarrow_K y$ implies $x \leqslant y$.

(ii) Let $\triangle \subseteq_{dir} D$ be directed set. If $d \in \downarrow_K \bigvee \triangle$, then $d \in K_D$ and $d \leqslant \bigvee \triangle$. Hence there is $x \in \triangle$ such that $d \leqslant x$, this is $d \in \downarrow_K x$. Therefore, $\downarrow_K \bigvee \triangle \subseteq \bigcup\{\downarrow_K x : x \in \triangle\}$

Now, let $d \in \downarrow_K x$. Therefore, $d \leqslant x$, for some $x \in \triangle$. Since $(D, \leqslant)$ is algebraic, $d \leqslant \bigvee \triangle$ that implies $\bigcup\{\downarrow_K x : x \in \triangle\} \subseteq \downarrow_K \bigvee \triangle$. Thus, $\downarrow_K \bigvee \triangle = \bigcup\{\downarrow_K x : x \in \triangle\}$. ■

Proposition 2.3.6. Let $(D, \leqslant)$ and $(D', \leqslant')$ be a dcpos and suppose $(D, \leqslant)$ is algebraic. Then a function $f : (D, \leqslant) \longrightarrow (D', \leqslant')$ is Scott continuous if and only if for each $x \in D$, $f(x) = \bigvee\{f(a) : a \in \downarrow_K x\}$ [26].

Proof: $(\Rightarrow)$ Let f be a Scott continuous function and let $x \in D$. Since D is algebraic, $\downarrow_K x$ is directed set and $x = \bigvee \downarrow_K x$. By continuity it follows that $f(x) = (\bigvee \downarrow_K x) = \bigvee f(\downarrow_K x) = \bigvee\{f(a) : a \in \downarrow_K x\}$.

$(\Leftarrow)$ Suppose $f(x) = \bigvee\{f(a) : a \in \downarrow_K x\}$ for each $x \in D$. Suppose $x \leqslant y$ in D. Then, $\downarrow_K x \subseteq \downarrow_K y$. Thus, for each $a \in \downarrow_K x$, $f(a) \leqslant' \bigvee\{f(b) : b \in \downarrow_K y\} = f(y)$ and hence $f(x) = \{f(a) : a \in \downarrow_K x\} \leqslant' f(y)$, f is monotonic.

Now, we need to prove that $f(\bigvee \triangle) = \bigvee f(\triangle)$ for any $\triangle \subseteq_{dir} D$. Since f is monotonic, then we just need to show $f(\bigvee \triangle) \leqslant' \bigvee f(\triangle)$. For any $\triangle \subseteq_{dir} D$, $\downarrow_K \bigvee \triangle = \bigcup\{\downarrow_K x : x \in \triangle\}$ and by assumption $f(\bigvee \triangle) = \bigvee\{f(a) : a \in \downarrow_K \bigvee \triangle\}$. Let $a \in \downarrow_K \bigvee \triangle$ and choose $x \in \triangle$ such that $a \in \downarrow_K x$. Then, $f(a) \leqslant' f(x) \leqslant' \bigvee\{f(y) : y \in \triangle\} = \bigvee f(\triangle)$. It follows that $f(\bigvee \triangle) = \bigvee\{f(a) : a \in \downarrow_K \bigvee \triangle\} \leqslant' \bigvee f(\triangle)$. Therefore, $f(\bigvee \triangle) = \bigvee f(\triangle)$. Thus, f is Scott continuous. ■

We obtain the following important result.

Proposition 2.3.7. Let $(D, \leqslant)$ and $(D', \leqslant')$ be a dcpos and suppose $(D, \leqslant)$ is algebraic, and let $K_D = \{d \in D | d \text{ is compact }\}$. Then each monotonic function $f : K_D \longrightarrow D'$ has a unique Scott continuous extension $\overline{f} : D \longrightarrow D'$ [26].

Proof: Assume that $f : K_D \longrightarrow D'$ is monotonic and define $\overline{f} : D \longrightarrow D'$ by $\overline{f}(x) = \bigvee\{f(a) : a \in \downarrow_K x\}$. The function $\overline{f}$ is defined on all of D since, for $x \in D$, $\downarrow_K x$ is directed and hence $\{f(a) : a \in \downarrow_K x\}$ is directed by monotonicity. The Scott continuity and uniqueness follow immediately from proposition 2.3.6. ■

CHAPTER 3

CATEGORICAL PROPERTIES OF SCOTT TOPOLOGY

3.1. Category of DCPO and CPO

Definition 3.1.1. The category $\mathbf{DCPO}$ whose object class is the class of all directed completely partial orders;
$hom((D,\leqslant),(D',\leqslant')) = \{f | f : (D,\leqslant) \longrightarrow (D',\leqslant') \text{ is Scott continuous}\}$ morphisms all Scott continuous functions between directed completely partial orders. Indeed,

$$(D,\leqslant) \xrightarrow{f} (D',\leqslant') \xrightarrow{g} (D'',\leqslant'')$$

First, we need to prove that $g \circ f$ is Scott continuous. Let $x, y \in D$ and let $x \leqslant y$. Then $f(x) \leqslant' f(y)$, and so $g(f(x)) \leqslant'' g(f(y))$. Thus, $g \circ f$ is monotonic. Now, let $\triangle \subseteq_{dir} D$. We need to show that $g(f(\bigvee \triangle)) = \bigvee g(f(\triangle))$. Since f is Scott continuous, $g(f(\bigvee \triangle)) = g(\bigvee f(\triangle))$ and since g is continuous, $g(\bigvee f(\triangle)) = \bigvee g(f(\triangle))$. Therefore, $g(f(\bigvee \triangle)) = \bigvee g(f(\triangle))$. Thus, $g \circ f : (D,\leqslant) \rightarrow (D'',\leqslant'')$ is Scott continuous function.

(i) $\mathbf{Identity\ Property}$:

Let $id_{(D,\leqslant)}$ be an identity function.

$$(D,\leqslant) \xrightarrow{id_{(D,\leqslant)}} (D,\leqslant)$$

and let $x, y \in D$ and let $x \leqslant y$, clearly $id_D(x) = x \leqslant y = id_D(y)$. Now, Let $\triangle \subseteq_{dir} D$. Then, we have $id_D(\bigvee \triangle) = \bigvee \triangle$ and $\bigvee id_D(\triangle) = \bigvee \triangle$. Thus, $id_D(\bigvee \triangle) = \bigvee id_D(\triangle)$. So identity morphism is Scott continuous.

$$(D,\leqslant) \xrightarrow{I_D} (D,\leqslant) \xrightarrow{f} (D',\leqslant')$$

Since $\forall x \in D$, $f \circ I_x(x) = f(x)$. so $f \circ I_x = f$. Similarly

$$(D, \leqslant) \xrightarrow{f} (D', \leqslant') \xrightarrow{Id_{(D',\leqslant')}} (D', \leqslant')$$

$I_{(D',\leqslant')} \circ f = f$. Note that $I_{(D',\leqslant')} \circ f$ and $f \circ I_{(D,\leqslant)}$ are Scott continuous.

(ii) **Associative Property** :

Let $(D, \leqslant), (D', \leqslant'), (D'', \leqslant''), (D''', \leqslant''')$ be objects of $\mathbf{DCPO}$ and

$$(D, \leqslant) \xrightarrow{f} (D', \leqslant') \xrightarrow{g} (D'', \leqslant'') \xrightarrow{h} (D''', \leqslant''')$$

be Scott continuous function. Since $\forall x \in D$,
$h \circ (g \circ f)(x) = h \circ (g(f(x))) = h(g(f(x)))$
$(h \circ g) \circ f(x) = (h \circ g)(f(x)) = h(g(f(x)))$. Thus,
$h \circ (g \circ f) = (h \circ g) \circ f$ and $h \circ (g \circ f)$ is Scott continuous.

Hence, $\mathbf{DCPO}$ is a category with directed completely partial orders as objects and Scott continuous functions between directed completely partial orders as morphisms.

Definition 3.1.2. The category $\mathbf{CPO}$ whose object class is the class of all completely partial orders and morphisms are same like in $\mathbf{DCPO}$;
$hom((D, \leqslant), (D', \leqslant')) = \{f | f : (D, \leqslant) \longrightarrow (D', \leqslant')\text{is Scott continuous}\}$ functions between completely partial orders.

Proposition 3.1.1. The category $\mathbf{CPO}$ is the full subcategory of $\mathbf{DCPO}$.

Proof: Since $Ob(\mathbf{CPO}) \subseteq Ob(\mathbf{DCPO})$ and for each $(D, \leqslant), (D', \leqslant') \in Ob(\mathbf{CPO})$, $hom_{\mathbf{CPO}}(D, D') = hom_{\mathbf{DCPO}}(D, D')$, $\mathbf{CPO}$ is the full subcategory of $\mathbf{DCPO}$. ∎

Proposition 3.1.2. The category $\mathbf{DCPO}$ has a terminal object.

Proof: Let $\mathbf{DCPO}$ be a category of directed completely partial orders and, let $T = (\{\perp\}, \leqslant_\perp)$ be directed completely partial order space. Then, for each $(D, \leqslant)$ object of $\mathbf{DCPO}$ $hom_{\mathbf{DCPO}}(D, T) = \{f \mid f : (D, \leqslant) \longrightarrow T = (\{\perp\}, \leqslant_\perp)\}$ is one point-set namely, constant function.

Thus, $T = (\{\perp\}, \leqslant_\perp)$ is a terminal object of $\mathbf{DCPO}$. ∎

Proposition 3.1.3. $Scott : \mathbf{DCPO} \to \mathbf{Top}$ is given by for each $(D, \leqslant)$ directed completely partial order set, $Scott(D, \leqslant) = (D, \sigma(D))$ Scott topological space and for each $f : (D, \leqslant) \to (D', \leqslant')$ Scott continuous function $Scott(f) = f : (D, \sigma(D)) \to (D', \sigma(D'))$ continuous is faithfull and full functor [30].

Proof: First, we need to show that $Scott$ is a functor. Let $(D, \leqslant), (D', \leqslant'), (D'', \leqslant'')$ be objects of $\mathbf{DCPO}$ and let

$$(D, \leqslant) \xrightarrow{f} (D', \leqslant') \xrightarrow{g} (D'', \leqslant'')$$

be Scott continuous function. Then

$$(D, \sigma(D)) \xrightarrow{Scott(f)} (D', \sigma(D')) \xrightarrow{Scott(g)} (D'', \sigma(D''))$$

is continuous, it is clear to see that $Scott(g \circ f) = g \circ f = Scott(g) \circ Scott(f)$. Since

$$(D, \leqslant) \xrightarrow{id_{(D,\leqslant)}} (D, \leqslant)$$

is Scott continuous,

$$(D, \sigma(D)) \xrightarrow{Scott(id_{(D,\leqslant)})} (D, \sigma(D))$$

is continuous and $Scott(id_{(D,\leqslant)}) = id_{Scott(D,\leqslant)}$. Thus, $Scott : \mathbf{DCPO} \to \mathbf{Top}$ is a functor.

Now, we show that $Scott$ is full functor. Let $\forall (D, \leqslant), (D', \leqslant') \in Ob(\mathbf{DCPO})$ and $f : Scott((D, \leqslant)) = (D, \sigma(D)) \longrightarrow Scott((D', \leqslant')) = (D', \sigma(D'))$ be continuous. We need to check that, there exists a $g : (D \leqslant) \longrightarrow (D', \leqslant')$ Scott continuous function such that $Scott(g) = f$. By proposition 2.3.5.

Hence, $Scott : \mathbf{DCPO} \to \mathbf{Top}$ is full functor.

Let $(D, \leqslant)$ and $(D', \leqslant')$ be directed partial order sets and $f, g : (D, \leqslant) \longrightarrow (D', \leqslant')$ be Scott continuous function with $Scott(f) = Scott(g)$. We need to show that $f = g$. Since $Scott(f) = f : (D, \sigma(D)) \to (D', \sigma(D'))$ and $Scott(g) = g : (D, \sigma(D)) \to (D', \sigma(D'))$ are continuous. Thus, $f = g$.

Hence, $Scott : \mathbf{DCPO} \to \mathbf{Top}$ is faithful functor. ■

3.2. Cartesian Products

Definition 3.2.1. Let $(D, \leqslant)$ and $(D', \leqslant')$ be two dcpos. Then the *cartesian product* of D and D', denoted by $(D \times D', \leqslant_\times)$, where $\leqslant_\times$ is defined by $\forall x, y \in D$, $x', y' \in D'$
$(x, x') \leqslant_\times (y, y')$ if and only if $x \leqslant y$ and $x' \leqslant' y'$ [26].
Note that $\leqslant_\times$ is a partial ordering on $D \times D'$.

Lemma 3.2.1. Let $(D, \leqslant)$ and $(D', \leqslant')$ be two dcpos. If $\triangle \subseteq D \times D'$ is directed set then $\pi_i(\triangle)$ is directed for $i = 1, 2$ and $\bigvee \triangle = (\bigvee \pi_1(\triangle), \bigvee \pi_2(\triangle))$, where π_i are projection functions, $i = 1, 2$ [31].

Proof: Suppose $\triangle \subseteq D \times D'$ is directed. Each π_i is monotonic and hence $\pi_i(\triangle)$ is directed. Let $a = \bigvee \pi_1(\triangle) \in D$ and $b = \bigvee \pi_2(\triangle)$. We must show that $\bigvee \triangle = (a, b)$. Clearly (a, b) is an upper bound for $\triangle$. Let $(x, y) \in \triangle$, $\pi_1(x, y) = x \leqslant a$ and $\pi_2(x, y) = y \leqslant' b$. It follows that $(x, y) \leqslant_\times (a, b)$ and consequently (a, b) is an upper bound for $\triangle$. Now, we need to show that (a, b) is a least upper bound for $\triangle$. Suppose (c, d) is an upper bound for $\triangle$ with $(x, y) \in \triangle$ and $(x, y) \leqslant_\times (c, d)$. So, in particular $x \leqslant c$. Thus, c is an upper bound for $\pi_1(\triangle)$ and hence $a \leqslant c$. Similarly, $b \leqslant' d$ and hence $(a, b) \leqslant_\times (c, d)$, i.e., (a, b) is a least upper bound for $\triangle$. ■

Definition 3.2.2. Let $(D, \leqslant)$, $(D', \leqslant')$ and $(D'', \leqslant'')$ be dcpos. A function $f : D \times D' \to D''$ is called *Scott continuous in its first argument*, if for each $b \in D'$, the function $x \mapsto f(x, b)$ from D into D'' is Scott continuous. Similarly, $f : D \times D' \to D''$ is *Scott continuous in its second argument*, if for each $a \in D$, the function $y \mapsto f(a, y)$ from D' into D'' is Scott continuous [26].

Proposition 3.2.1. Let $(D, \leqslant)$, $(D', \leqslant')$ and $(D'', \leqslant'')$ be dcpos. A function
$f : D \times D' \to D''$ is Scott continuous if and only if f is Scott continuous in each argument [26].

Proof: $(\Rightarrow)$ Let $f : D \times D' \to D''$ be Scott continuous. Fix $b \in D'$ and define $g : D \to D''$ by $g(x) = f(x, b)$. The function g is clearly monotonic, since if $x \leqslant y$ in D then $(x, b) \leqslant_\times (y, b)$ in $D \times D'$ and hence $f(x, b) \leqslant'' f(y, b)$. Let $\triangle \subseteq_{dir} D$ be directed set, then $g(\bigvee \triangle) = f(\bigvee \triangle, b) = f(\bigvee(\triangle \times \{b\})) = \bigvee f(\triangle \times \{b\}) = \bigvee g(\triangle)$. Thus,

g is Scott continuous. By the same argument, for $a \in D$, $g : D' \to D \times D'$ given by $g(x) = f(a, x)$ is Scott continuous.

($\Leftarrow$) Let $f : (D \times D', \leqslant_\times) \to (D'', \leqslant'')$ be Scott continuous in each of its arguments. First of all, f is monotonic. For suppose $(x, y) \leqslant_\times (z, w)$. Then $f(x, y) \leqslant'' f(z, y) \leqslant'' f(z, w)$ by monotonicity in each argument.

Now, let $\triangle \subseteq_{dir} D \times D'$ be directed set and let $a = \bigvee \pi_1(\triangle)$ and $b = \bigvee \pi_2(\triangle)$, so that $\bigvee \triangle = (a, b)$ by lemma 3.2.1. Then $f(\bigvee \triangle) = f(a, b) = f(a, \bigvee \pi_2(\triangle)) = \bigvee_{y \in \pi_2(\triangle)} f(a, y)$, where last equality follows from the continuity of f in the second argument. Similarly, we obtain for fixed $y \in \pi_2(\triangle)$, $f(a, y) = f(\bigvee \pi_1(\triangle), y) = \bigvee_{x \in \pi_1(\triangle} f(x, y)$. It follows that $f(\bigvee \triangle) = \bigvee_{y \in \pi_2(\triangle)} \bigvee_{x \in \pi_1(\triangle)} f(x, y)$. Let $x \in \pi_1(\triangle)$ and $y \in \pi_2(\triangle)$ and choose z and w such that $(x, z) \in \triangle$ and $(w, y) \in \triangle$. Since $\triangle$ is directed there is $(u, v) \in \triangle$ such that $(x, z) \leqslant_\times (u, v)$ and $(w, y) \leqslant_\times (u, v)$. But then $(x, y) \leqslant_\times (u, v)$ and hence f is monotonic, $f(x, y) \leqslant'' f(u, v) \leqslant'' \bigvee f(\triangle)$. Thus, $f(\bigvee \triangle) = \bigvee_{y \in \pi_2(\triangle)} \bigvee_{x \in \pi_1(\triangle)} f(x, y) \leqslant'' \bigvee f(\triangle)$ and hence f is Scott continuous. ■

Proposition 3.2.2. (Products in **DCPO**). Let $(D, \leqslant)$ and $(D', \leqslant')$ be two dcpos. Then, $(D \times D', \leqslant_\times)$ is a product of D and D' [31].

Proof: We first need to show that $(D \times D', \leqslant_\times)$ is a dcpo. Let $\triangle \subseteq_{dir} D \times D'$ be directed set. Define $\triangle_D = \{x | (x, x') \in \triangle\}$ and $\triangle_{D'} = \{x' | (x, x') \in \triangle\}$. Then $(\bigvee \triangle_D, \bigvee \triangle_{D'})$ is the supremum of $\triangle$. Let (z, z') be an upper bound for $\triangle$. Then, z is an upper bound for $\triangle_D$ and z' is an upper bound for $\triangle_{D'}$. Since $\bigvee \triangle_D$ is a supremum for $\triangle_D$, it follows that $\bigvee \triangle_D \leqslant z$. Similarly, $\bigvee \triangle_{D'} \leqslant' z'$. Thus, $(\bigvee \triangle_D, \bigvee \triangle_{D'}) \leqslant_\times (z, z')$.

Now, we want to show that π_1 and π_2 are Scott continuous. Let $(x, x') \leqslant_\times (y, y')$. Then, $\pi_1(x, x') = x \leqslant y = \pi_1(y, y')$. Thus, π_1 is monotonic. Let $\triangle \subseteq_{dir} D \times D'$ be directed set. Then, $\pi_1(\bigvee \triangle) = \bigvee \triangle_D$. Moreover, since $\pi_1(\triangle) = \triangle_D$, it follows that $\bigvee \pi_1(\triangle) = \bigvee \triangle_D$ and consequently $\pi_1(\bigvee \triangle) = \bigvee \pi_1(\triangle)$. Thus, π_1 is Scott continuous. Similarly, π_2 is also Scott continuous.

Finally, let $(D'', \leqslant'')$ be a dcpo with Scott continuous functions $f : D'' \to D$ and $g : D'' \to D'$. We need to show that there exist a unique function $f \times g : D'' \to D \times D'$ such that $\pi_1 \circ (f \times g) = f$ and $\pi_2 \circ (f \times g)$. First, we want to show that $f \times g$ is Scott continuous.

Let $x \leqslant'' y$. Since f and g is Scott continuous, it follows that $f(x) \leqslant f(y)$ and $g(x) \leqslant' g(y)$. Thus, $(f \times g)(x) = (f(x), g(x)) \leqslant_\times (f(y), g(y)) = (f \times g)(y)$. Hence, $f \times g$ is monotonic. Let $\triangle \subseteq_{dir} D''$ be directed subset of D''. Since $f \times g$ is monotonic, we have $\bigvee(f \times g)(\triangle) \leqslant_\times (f \times g)(\bigvee \triangle)$. Since f and g is Scott continuous, we have the following.

$\bigvee\{f(x)|x \in \triangle\} = f(\bigvee \triangle)$, $\bigvee\{g(x)|x \in \triangle\} = g(\bigvee \triangle)$.

Let $\bigvee(f \times g)(\triangle) = (z, z')$. Then, z is an upper bound for $\bigvee\{f(x)|x \in \triangle\}$ and z' is an upper bound for $\bigvee\{g(x)|x \in \triangle\}$. Thus, $\bigvee\{f(x)|x \in \triangle\} \leqslant z$ and $\bigvee\{g(x)|x \in \triangle\} \leqslant' z'$. Hence, $(f \times g)(\bigvee \triangle) \leqslant_\times \bigvee(f \times g)(\triangle)$. Extensionality ensures the uniqueness of $f \times g$. ■

Definition 3.2.3. Let $(D, \leqslant)$ and $(D', \leqslant')$ be two dcpos. Then, the *function space* of D into D', denoted by D'^D, is the following partially ordered set. For $f, g \in D'^D$

$f \leqslant_{D'^D} g$ if and only if $f(x) \leqslant' g(x)$ for each $x \in D$.

The *evaluation function* $e_{D,D'} : D \times D'^D \to D'$ is defined by $e_{D,D'}(x, f) = f(x)$ [26].

Theorem 3.2.1. Let $(D, \leqslant)$ and $(D', \leqslant')$ be two dcpos. Then, $(D'^D, \leqslant_{D'^D})$ is a dcpo. Furthermore, if $\triangle \subseteq_{dir} D'^D$ is directed then $\bigvee \triangle$ is the function defined by $(\bigvee \triangle)(x) = \bigvee\{f(x) : f \in \triangle\}$ [26].

Proof: Let $\triangle \subseteq_{dir} D'^D$ be a directed set and let $\triangle_x = \{f(x) : f \in \triangle\}$ for each $x \in D$. Then, $\triangle_x \neq \emptyset$ since $\triangle \neq \emptyset$. Suppose $f(x), f(y) \in \triangle_x$. Since $\triangle$ is directed there is $h \in \triangle$ such that $f \leqslant h$ and $g \leqslant h$. But this implies that $f(x) \leqslant' h(x)$ and $g(x) \leqslant' h(x)$ and hence $\triangle_x$ is a directed set in D'. Thus, $\bigvee \triangle_x$ exists in D' for each $x \in D$. Define $k : D \to D'$ by $k(x) = \bigvee \triangle_x$. We want to show that k is Scott continuous.

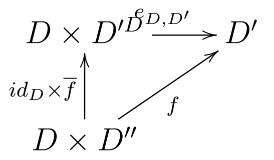
First suppose $x \leqslant y$ in D. Then, for $f \in \triangle$, $f(x) \leqslant' f(y) \leqslant' \bigvee \triangle_y = k(y)$. So, $k(x) = \bigvee \triangle_x = \bigvee\{f(x) : f \in \triangle\} \leqslant' k(y)$. Thus, k is monotonic.

Now, let $\triangle' \subseteq_{dir} D$ be directed subset of D. Then, $k(\bigvee \triangle') = \bigvee_{f\in\triangle} f(\bigvee \triangle') = \bigvee_{f\in\triangle} \bigvee_{x\in\triangle'} f(x) = \bigvee_{x\in\triangle'} \bigvee_{f\in\triangle} f(x) = \bigvee_{x\in\triangle'} k(x) = \bigvee k(\triangle')$. Thus, k is Scott continuous. ■

Proposition 3.2.3. Let $(D, \leqslant)$, $(D', \leqslant')$. $(D'', \leqslant'')$ and $(D'^D, \leqslant_{D'^D})$ be dcpos, and let $f : D \times D'' \to D'$ be a Scott continuous function and let $x \in D$. Then,

$\overline{f} : D'' \to D'^D$ and $e_{D,D'} : D \times D'^D \to D'$, defined by $\overline{f}(y)(x) = f(x, y)$ and

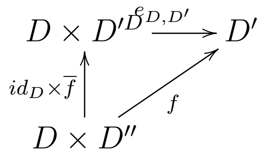
$e_{D,D'}(x, f) = f(x)$, respectively are Scott continuous functions and $\overline{f} : D'' \to D'^D$ is unique and following diagram is commutative [31].



Proof: First, we need to show that $\overline{f}$ is Scott continuous. Let $u \leqslant'' v$ in D''. Then, for each $x \in D$, $\overline{f}(u)(x) = f(x, u) \leqslant' f(x, v) = \overline{f}(v)(x)$, by monotonicity of f, showing that $\overline{f}(u) \leqslant_{D'^D} \overline{f}(v)$. Thus, $\overline{f}$ is monotonic. Let $\triangle \subseteq_{dir} D''$ be directed set. Then, for each $x \in D$, $\overline{f}(\bigvee \triangle)(x) = f(x, \bigvee \triangle) = f(\bigvee(\{x\} \times \triangle)) = \bigvee f(\{x\} \times \triangle) = \bigvee\{f(x, u) : u \in \triangle\} = \bigvee\{\overline{f}(u)(x) : u \in \triangle\} = (\bigvee \overline{f}(\triangle))(x)$.
Thus, $\overline{f}(\bigvee \triangle) = \bigvee \overline{f}(\triangle)$. So, $\overline{f}$ is Scott continuous.
Since $\overline{f}$ and id_D are Scott continuous, it follows that $id_D \times \overline{f}$ is Scott continuous.

Now, we need to show that $e_{D,D'}$ is Scott continuous. By proposition 3.2.1. it suffices to show that $e_{D,D'}$ is Scott continuous in its each argument. This is trivially true of the first argument, since the fixed f in the second argument is Scott continuous. Now fix $x \in D$ and let $h : D'^D \to D'$ such that $h(f) = e_{D,D'}(x, f) = f(x)$. Clearly, h is monotonic. Let $\triangle \subseteq_{dir} D'^D$ be directed set. Then, $h(\bigvee \triangle) = (\bigvee \triangle)(x) = \bigvee\{f(x) : f \in \triangle\} = \bigvee\{h(f) : f \in \triangle\} = \bigvee h(\triangle)$. Thus, h is Scott continuous and hence, $e_{D,D'}$ is Scott continuous.

To see $f = e_{D,D'} \circ (id_D \times \overline{f})$, let $(x, x'') \in D \times D''$. Then $(id_D \times \overline{f})(x, x'') = (x, \overline{f}(x''))$, and $e_{D,D'}(x, \overline{f}(x'')) = \overline{f}(x'')(x) = f(x, x'')$. Thus, the diagram

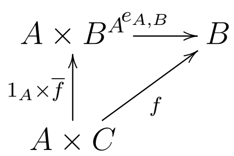


commutes.

Finally, to show the uniqueness of $\overline{f} : D'' \to D'^D$, let $f' : D'' \to D'^D$ be Scott continuous function such that the diagram is commutative. We need to show that $\overline{f} = f'$. For each $x'' \in D''$, $f'(x'') \in D'^D$, i.e., for each $x \in D$, $f'(x'')(x) \in D'$. Then,
$f = (e_{D,D'} \circ (id_D \times f')(x, x'')) = e_{D,D'}(x, f'(x'')) = f'(x'')(x) = f(x, x'') = \overline{f}(x'')(x)$

and consequently, $f'(x'')(x) = \overline{f}(x'')(x)$. Thus, for each $x \in D$ and $x'' \in D''$, $f' = \overline{f}$. Hence, $\overline{f}$ is unique. ∎

Definition 3.2.4. A category $\mathcal{C}$ is called *cartesian closed* provided that the following conditions are satisfied:
(i) For each $A, B \in Ob(\mathcal{C})$ there exists a product $A \times B$ in $\mathcal{C}$, i.e., for any $A, B \in Ob(\mathcal{C})$, there exists an object $A \times B$ in $\mathcal{C}$ and some morphisms $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ such that for $C \in Ob(\mathcal{C})$ and morphisms $f : C \to A$, $g : C \to B$ there is a unique morphism $f \times g : C \to A \times B$ such that $\pi_1 \circ (f \times g) = f$ and $\pi_2 \circ (f \times g) = g$.
(ii) $\forall A \in Ob(\mathcal{C})$ the following holds:
$\forall B \in Ob(\mathcal{C})$, there exists some $\mathcal{C}$-objects B^A (called power object) and some $\mathcal{C}$-morphism $e_{A,B} : A \times B^A \to B$ (called evaluation morphism) such that for each $C \in Ob(\mathcal{C})$ and each $\mathcal{C}$-morphism $f : A \times C \to B$, there exists a unique $\mathcal{C}$-morphism $\overline{f} : C \to B^A$ such that the diagram



commutes [32].

Proposition 3.2.4. The category $\mathbf{DCPO}$ is cartesian closed.

Proof: It follows from Proposition 3.2.2. and Proposition 3.2.3. ∎

3.3. Sober Space

Definition 3.3.1. Let $(D, \leqslant)$ be a complete lattice. D is called a *frame* if it satisfies the infinite distributivity law, that is $\forall x, y_i \in D, i \in I$
$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i)$ [17].

Example 3.3.1. Let X be a non-empty set and $\subseteq$ be the order. Then $(P(X), \subseteq)$ is a complete lattice and it satisfies infinite distributivity law, thus $(P(X), \subseteq)$ is a frame.

Definition 3.3.2. Let $(D, \leqslant)$ and $(D', \leqslant')$ be frame and let $f : (D, \leqslant) \to (D', \leqslant')$. f is called *frame-morphism* if it preserves arbitrary joins (supremum) and finite meets (infimum), that is, for every $A \subseteq D$, $f(\bigvee A) = \bigvee f(A)$ and for every finite $B \subseteq D$, $f(\bigwedge B) = \bigwedge f(B)$ [17].

Definition 3.3.3. $\mathbf{Frm}$ is a category whose objects are frames and morphisms are frame-morphisms. Similarly, $\mathbf{CLat}$ is a category whose objects are complete lattices and morphisms are frame-morphisms between complete lattices.

An element of a topological space X is naturally equipped with following three pieces of information. We can associate with it its filter F_x of open neighborhoods, the complement of its closure and a map from $\mathbf{1}$ (one-element topological space) to X.

Definition 3.3.4. Let $(D, \leqslant)$ be a lattice and let F be a filter on D. A filter $F \subseteq D$ is called *prime* if $\bigvee M \in F$ implies $F \cap M \neq \emptyset$ for all finite $M \subseteq D$. Similarly,
A filter $F \subseteq D$ is called *completely prime filter* if $\bigvee M \in F$ implies $F \cap M \neq \emptyset$ for any arbitrary $M \subseteq D$ [11].
In other words, the filter F is completely prime filter if and only if, for every family $(u_i)_{i \in I}$ of elements of complete lattices D whose least upper bound $\bigvee_{i \in I} u_i$ is in F, u_i is already in F for some $i \in I$ [33].

Definition 3.3.5. Let D be a complete lattice. The points of D are completely prime filter of D [11].

Lemma 3.3.1. Every completely prime filter is Scott open.

Proof: It follows from Definition 1.2.8. and Definition 3.3.5. ■

Definition 3.3.6. Let $(D, \leqslant)$ be a complete lattice. The collection $pt(D)$ of all points is turned into a topological space by declaring all those subsets of $pt(D)$ to be open those are in the form of
$O_x = \{F \in pt(D) | x \in F\}$, $x \in D$ [11].

Proposition 3.3.1. Let $(D, \leqslant)$ be a complete lattice. The collection of all sets O_u, where $u \in D$, forms a topology on $pt(D)$.

Proof: (i) Clearly $pt(D)$ itself is open and $\emptyset$ is vacuously open.

(ii) Consider finite intersection, for every completely prime filter F, $F \in \bigcap_{i=1}^{n} O_{x_i}$ implies $x_1, x_2, ..., x_n \in F$. So, $\bigwedge\{x_1, x_2, ..., x_n\} \in F$, since F is a filter. Conversely, if $\bigwedge\{x_1, x_2, ..., x_n\}$ is in F, then all larger elements, in particular $x_1, x_2, ..., x_n$ are in F, since F is upper closed. Thus, $\bigcap_{i=1}^{n} O_{x_i} = O_{\bigwedge\{x_1, x_2, ..., x_n\}}$.

(iii) Consider unions, for every completely prime filter F, $F \in \bigcup_{i\in I} O_{x_i}$ implies for some $i \in I$, $x_i \in F$. If so, then $\bigvee_{i\in I} x_i \in F$, since F is upper closed.
Conversely, if $\bigvee_{i\in I} x_i \in F$, then some $x_i \in F$, because F is completely prime filter. Thus, $\bigcup_{i\in I} O_{x_i} = O_{\bigvee_{i\in I} x_i}$.

So, similar to a topological space (in which an element x belongs to an open set O if $x \in O$), in a complete lattice a point F belongs to an open set O_x if $x \in F$. ■

Proposition 3.3.2. $pt : \mathbf{CLat} \to \mathbf{Top}$ defined on $D \in Ob(\mathbf{CLat})$ to $pt(D)$ a topological space of all points, and on frame morphism $g : D \to D'$ by letting
$pt(g) : pt(D') \to pt(D)$ map every completely filter F of D' to $g^{-1}(F)$ [33].

Proof: We first have to check that $pt(g)(F)$ is a complete prime filter whenever F is a completely prime filter. This follows from the fact that g is a frame morphism, i.e., $pt(g)(F)$ is completely prime filter because, if $\bigvee_{i\in I} x_i \in pt(g)(F) = g^{-1}(F)$, then $g(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} g(x_i) \in F$, so some $g(x_i) \in F$ implies $x_i \in g^{-1}(F)$.
Secondly, we must show that $pt(g)$ is continuous. Let O_x be an open set in $pt(D)$. Then,
$pt(g)^{-1}(O_x) = \{F \in pt(D') | g^{-1}(F) \in O_x\} = \{F \in pt(D') | x \in g^{-1}(F)\}$
$= \{F \in pt(D') | g(x) \in F\} = O_{g(x)}$.
pt preserves identities and composition which can be easily seen. ■

Proposition 3.3.3. $\Omega : \mathbf{Top} \to \mathbf{CLat}$ defined on $(X, \tau) \in Ob(\mathbf{Top})$ to $(\tau, \subseteq)$ a lattice with $\subseteq$ order, and on continuous map $(X, \tau) \to (Y, \sigma)$ by letting
$\Omega(f) : \Omega(Y, \sigma) = (\sigma, \subseteq) \to \Omega(X, \tau) = (\tau, \subseteq)$ map every open set U of σ to $f^{-1}(U)$ [11].

Proof: Let $(X, \tau) \in Ob(\mathbf{Top})$. We may think τ as an ordered set where the order relation is set inclusion $\subseteq$. So, $\Omega((X, \tau)) = (\tau, \subseteq) \in Ob(\mathbf{Clat})$ because infinite joins (supremum) exists.
Let $f : (X, \tau) \to (Y, \sigma)$ be continuous function, i.e., $\forall U \in \sigma$, $f^{-1}(U) \in \tau$. We need to show that $\Omega(f) : \Omega((Y, \sigma)) = (\sigma, \subseteq) \to \Omega((X, \tau)) = (\tau, \subseteq)$ is frame morphism. Let A be any subset of σ. Then, $\Omega(f)(\bigcup\limits_{i\in I} A_i) = f^{-1}(\bigcup\limits_{i\in I} A_i) = \bigcup\limits_{i\in I} f^{-1}(A_i) = \bigcup\limits_{i\in I} \Omega(f)(A_i)$.
Similarly, let B be a finite subset of σ. Then, $\Omega(f)(\bigcap\limits_{i\in I} B_i) = f^{-1}(\bigcap\limits_{i\in I} B_i) = \bigcap\limits_{i\in I} f^{-1}(B_i)$
$= \bigcap\limits_{i\in I} \Omega(f)(B_i)$. Thus, $\Omega(f)$ is a frame morphism.
Ω preserves identities and composition which can be easily seen. ■

Proposition 3.3.4. Let $\Omega : \mathbf{Top} \rightarrow \mathbf{CLat}$ and $pt : \mathbf{CLat} \rightarrow \mathbf{Top}$ be functors. Then, $\eta : I \rightarrow pt \circ \Omega$ and $\varepsilon : I \rightarrow \Omega \circ pt$ are natural transformations [11].

Proof: Let (X, τ) be a $Ob(\mathbf{Top})$. (X, τ) can be mapped into the space of points of its open set lattice, i.e., $x \in X$ map to completely prime filter F_x of its open neighborhood. This assignment, which we denote by $\eta_{(X,\tau)} : (X, \tau) \rightarrow pt(\Omega((X, \tau)))$ is continuous and open onto its image. Let $U \in \tau$. Then we get by simply unwinding the definitions: $F_x \in O_U \Leftrightarrow U \in F_x \Leftrightarrow x \in U$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be continuous. It also commutes with continuous function: $pt(\Omega(f))(\eta_{(X,\tau)}(x)) = \Omega(f)^{-1}(F_x) = F_{f(x)} = \eta_{(Y,\sigma)} \circ f(x)$. Thus, $\Omega : \mathbf{Top} \rightarrow \mathbf{CLat}$ is a natural transformation.

The same procedure holds for complete lattice. Let $\varepsilon_D : D \rightarrow \Omega(pt(D))$ be the map which assigns O_x to $x \in D$. It is a frame morphism by Proposition 3.3.1. Let $h : (D, \leqslant) \rightarrow (D', \leqslant')$ be morphism. It also commutes with frame morphism: $\Omega(pt(h))(\varepsilon_{(D,\leqslant)}(x)) = pt(h)^{-1}(O_x) = O_{h(x)} = \varepsilon_{(D',\leqslant')} \circ h(x)$. Thus, $\varepsilon : I \rightarrow \Omega \circ pt$ is a natural transformation. ■

Theorem 3.3.1. The functors $\Omega : \mathbf{Top} \rightarrow \mathbf{CLat}$ and $pt : \mathbf{CLat} \rightarrow \mathbf{Top}$ are dual adjoints of each other. The units are η and ε [11].

Proof: Since $\varepsilon : I \rightarrow \Omega \circ pt$ and $\eta : I \rightarrow pt \circ \Omega$ are natural transformation. We just need to show that

$$\Omega((X, \tau)) \xrightarrow{\varepsilon_{\Omega((X,\tau))}} \Omega(pt(\Omega((X, \tau)))) \xrightarrow{\Omega(\eta_{(X,\tau)})} \Omega((X, \tau))$$

$\Rightarrow \Omega(\eta_{(X,\tau)})(\varepsilon_{\Omega((X,\tau))}) = id$ and similarly,

$$pt((D, \leqslant)) \xrightarrow{\eta_{pt((D,\leqslant))}} pt(\Omega(pt((D, \leqslant)))) \xrightarrow{pt(\varepsilon_{(D,\leqslant)})} pt((D, \leqslant))$$

$\Rightarrow pt(\varepsilon_{(D,\leqslant)})(\eta_{pt((D,\leqslant))}) = id$.

Let $U \in \tau$. $\Omega(\eta_{(X,\tau)})(\varepsilon_{\Omega((X,\tau))})(U) = \eta^{-1}_{(X,\tau)}(O_U) = \{x \in X | \eta_{(X,\tau)}(x) \in O_U\}$

$= \{x \in X | F_x \in O_U\} = \{x \in X | U \in F_x\} = \{x \in X | x \in U\} = U = id(U)$.

Similarly, other equality can be proved with similar fashion. Thus, $\Omega : \mathbf{Top} \rightarrow \mathbf{CLat}$ and $pt : \mathbf{CLat} \rightarrow \mathbf{Top}$ are dual adjoints of each other. ■

Definition 3.3.7. Let X be a topological space and let $F \subseteq X$ be a closed subset of X. F is called *irreducible* if it can not be written as the union of two smaller closed subsets; that is, whenever F_1 and F_2 are closed sets with $F = F_1 \cup F_2$, then $F = F_1$ or $F = F_2$ [33].

Proposition 3.3.5. Let X be a topological space. Then, $\eta_X : X \to pt(\Omega(X))$ is injective if and only if X satisfies the T_0 separation axiom. It is surjective if and only if every irreducible closed set is the closure of an element of X.

Proof: It follows from [11]. ■

Definition 3.3.8. A topological space X is called *sober* if η_X is bijective [11].

Definition 3.3.9. A complete lattice D is called *spatial* if ε_D is bijective [11].

Theorem 3.3.2. For any complete lattice D the topological space $pt(D)$ is sober. For any topological space X the lattice $\Omega(X)$ is spatial.

Proof: It follows easily from [33]. ■

Theorem 3.3.3. Any Hausdorff space X is sober space; Any sober space is T_0.

Proof: Let X be a topological space and let F be an irreducible subset of X. We need to prove that F is one point subset. Since F is an irreducible, so it has at least one point. If we show that F has two points, then this will lead us to a contradiction. Let $x, y \in F$ be two distinct points. By the definition of hausdorffness, there exist open subsets U, V of X such that $U \ni x$, $V \ni y$ and $U \cap V = \emptyset$. Then, the sets $F\backslash U$ and $F\backslash V$ are both closed subsets and their union is F, i.e., $F = (F\backslash U) \cup (F\backslash V)$ and they are both proper subsets of F, since $x \notin F\backslash U$ and $y \notin F\backslash V$. Thus, we have written F as a union of two proper closed subsets, contradicting the assumption of irreducibility. Hence, F is one point subset of X and, closure of F is equal to itself. Thus, X is a sober space. By Proposition 3.3.5. and Definition 3.3.8, every sober space is T_0.

■

Proposition 3.3.6. The specialization order of any sober space (X, τ_s) forms a dcpo, whose Scott topology contains τ_s [13].

Proof: Let $\leqslant_s$ be a specialization order on X and $\triangle$ be directed under the specialization order $\leqslant_s$. We show that $\overline{\triangle}$ is irreducible. Let $\overline{\triangle} = F_1 \cup F_2$, and let F_1 and F_2 two closed subset. suppose $\triangle \nsubseteq F_1$ and $\triangle \nsubseteq F_2$. Let $x \in \triangle\backslash F_1$ and $y \in \triangle\backslash F_2$, and let $z \geqslant_s x, y$

in $\triangle$. $z \in \triangle$, since $\triangle$ is directed and $\triangle \subseteq F_1 \cup F_2$, we have, say $z \in F_1$. Then $x \in F_1$, since F_1 is down set: contradiction. Thus, $\triangle \subseteq F_1$. Hence $\overline{\triangle} = F_1$ or $\overline{\triangle} = F_2$, i.e. $\overline{\triangle}$ is irreducible. So, $\overline{\triangle} = \overline{\{y\}}$ for some y.

Now, we need to show that y is an upper bound. Let $k \in \triangle$ and let U be an open set in τ_s. Suppose $y \notin U$. Then $\overline{\triangle} = \overline{\{y\}} \subseteq X \backslash U$, and $k \notin U$. Hence, $k \leqslant_s y$. We claim that if $\triangle \cap U = \emptyset$, then $y \notin U$. Indeed, $\triangle \cap U = \emptyset$ implies $\overline{\triangle} \cap U = \emptyset$ and $y \notin U$.

Now, we need to prove that y is a least upper bound. Let z be an upper bound of $\triangle$, and suppose $z \notin U$. Then, $\triangle \cap U = \emptyset$ by the definition of $\leqslant_s$, and $y \notin U$ follows by claim.

Now, we will show that any open set U is Scott open. we now know that $y = \bigvee \triangle$ and by the claim, $\bigvee \triangle = y \in U$, then $\triangle \cap U \neq \emptyset$. Thus, U is Scott open. ■

This naturally leads us to ask about opposite direction, i.e., Is every Scott topology on dcpo always sober? In 1978, P.T. Johnstone [8] discovered a counterexample that answers this question in the negative.

Proposition 3.3.7. Let $X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$, consider $\mathbb{N} \cup \{\infty\}$ with the order: $n \leqslant n'$ if and only if $n \leqslant n'$ in $\mathbb{N}$ or $n' = \infty$. Consider the following partial order on X.
$(m, n) \leqslant (m', n')$ if and only if ($m = m'$ and $n \leqslant n'$) or ($n' = \infty$ and $n \leqslant m'$) [13].

Proof: First, we need to check that $(X, \leqslant)$ is dcpo. We claim that any element (m, ∞) is maximal. Let $(m, \infty) \leqslant (m', n')$: if $m = m'$ and $\infty \leqslant n'$, then $\infty = n'$, while other alternative ($n' = \infty$ and $\infty \leqslant m'$) cannot arises because m' ranges over $\mathbb{N}$. In particular, there is no maximum element, since the elements (m, ∞) are comparable only when they are equal.

Let $\triangle$ be directed. If it contains some (m, ∞), then it has a maximum. Otherwise let $(m', n'), (m'', n'')$ be two elements of $\triangle$: a common upper bound in $\triangle$ can only be in the form of (m''', n'''), with $m''' = m' = m''$. Hence $\triangle = \{m\} \times \triangle'$ for some m and some $\triangle' \subseteq_{dir} \mathbb{N}$. It is then obvious that $\triangle$ has a least upper bound.

Next, we observe that a non-empty Scott open contains all elements (p, ∞), for p sufficiently large. Indeed, if $(m, n) \in U$, where U is Scott open, then $p \geqslant n$ implies $(m, n) \leqslant (p, \infty)$. In particular, any finite intersection of non-empty Scott open sets is not empty. In other words, X is irreducible. If X were a sober space then we should have $X = \downarrow x$ for some x, but we have seen that X has no maximum. ■

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CURRICULUM VITAE

PERSONAL INFORMATION

Name, Surname	Muhammad QASIM
Nationality	Pakistani
Date of Birth and Place	28 October 1990, Chakwal
Tel	+90 554 268 91 49
email	qasim99956@gmail.com
Address	ERU, Faculty of Science, Departement of Mathematics, Kayseri

EDUCATION

Degree	**Institute**	**Grad. Date**
Undergraduate	Erciyes University, Faculty of Science	2013
Higher Secondary School	PakTurk International School and Colleges, Islamabad	2008
Secondary School	Govt. High School, Balkassar	2006

LANGUAGES

Turkish, English, Urdu, Punjabi