KÄHLER SPINORS

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/ /

Prof. Dr. Acar Işın
Director of the Institute
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to ***AYŞE*** *and* ***EMRAH*** *for their love*

SUMMARY

The Kähler equation which may be regarded as a square root of the Klein-Gordon equation is obtained and it is shown to be completely equivalent to four decoupled Dirac equations in Minkowski space-time even when minimal gauge interactions are introduced. In arbitrary space-times, however, there does not exist such an equivalence and the Kähler equation may be used to define a different notion of spinors. This notion is used to prove that the Susskind's formulation of lattice fermions is identical to the lattice transcription of the Kähler equation.

In the present work, self-consistent solutions to the Einstein-Kähler field equations in some cosmological space-times are examined. The isometries of the Robertson-Walker metric is used to construct an ansatz compatible with these isometries. Self-consistent solutions are then given to both the massive and massless equations in (i) Minkowski, (ii) de Sitter and (iii) Friedmann space-times. The special role of Minkowski space minimal left ideals and their isomorphism to Dirac spinors are emphasised.

ÖZET

Klein-Gordon denkleminin kare kökü gibi yorumlanabilen Kähler denklemi elde edilmiş ve Kähler denkleminin Minkowski uzay-zamanında, minimal etkileşmeler dahil edilse bile, birbirinden bağımsız dört Dirac denklemine eşdeğer olduğu ispatlanmıştır. Herhangi bir uzay zamanda böyle bir eşdeğerlik olmayıp Kähler denklemi yardımıyla farklı bir spinor kavramı geliştirilebilir. Bu yeni kavram kullanılarak, Susskind örgü fermion tanımının Kähler denkleminin örgü üzerine taşınmasından ibaret olduğu gösterilmiştir.

Bu çalışmada Einstein-Kähler alan denklemlerinin bazı kozmolojik uzay-zamanlardaki tutarlı çözümleri incelenmiştir. Robertson-Walker metriğinin isometrileri kullanılarak bu isometrilerle uyumlu bir ansatz oluşturulmuş ve sonra kütleli ve kütlesiz denklemlere (i) Minkowski, (ii) de Sitter ve (iii) Friedmann uzay zamanlarında tutarlı çözümler verilmiştir. Minkowski uzay-zamanda minimal sol ideallerinin özel rolü ve bu ideallerin Dirac spinorlerine eşdeğerliliği vurgulanmıştır.

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Miss **B. Öztunç** typed the manuscript.

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1. INTRODUCTION

Kähler introduced in 1962 a first order linear relativistic wave equation which iterates to the Klein-Gordon equation. He has furthermore shown that at least for flat space-time, Dirac equation satisfied by spinors can be re-interpreted in terms of a certain inhomogeneous differential form obeying this equation. The so called Kähler equation and its symmetries will be reviewed in detail in chapter 2. In arbitrary space-times Dirac equation and Kähler equation are known to be inequivalent. Graf (1978) used this inequivalence to propose an alternative spinor concept in arbitrary space-times.

The Kähler equation received little attention from physicists until Rabin (1982) and Becher and Joos (1982) took up the study of the lattice fermion problem. They showed that Susskind's (1977) formulation of fermions on a lattice is identical with lattice Kähler equation. The close analogy between the calculus of differential forms and lattice notions known from algebraic topology allows a straightforward transcription of Kähler equation onto a lattice.

In Minkowski space-time a Kähler form can be decomposed into four minimal ideals and each ideal can be identified with a decoupled Dirac spinor. Benn and Tucker (1982) argued that the observed four fundamental fermion generations might be a consequence of this property of Kähler field. In arbitrary space-times, however, the minimal ideals do not satisfy the corresponding Dirac equations. Banks et al. (1982) discussed the possible phenomenological applications of this failure of the above correspondence and noted that the gravitational interactions may alter the assigned internal fermion quantum numbers in four generation models. In chapter 4, we studied self-consistent solutions of the Einstein-Kähler system of equations in cosmologically relevant Minkowski, de Sitter and Friedmann space-times.

In the second chapter of the thesis we introduce the Kähler equation and study its formal properties. The correspondence between the Dirac equation and the Kähler equation in Minkowski space-time is established. Four pairwise orthogonal projectors which decompose any Kähler field into its minimal left ideals are given. The γ-matrix set corresponding to these projectors is explicitly constructed. Globalisation scheme of Graf is discussed under the heading of "Algebraic Spinors".

In the third chapter Kähler equation is transcribed onto a lattice by making use of the formal lattice-continuum correspondence. The equivalence of the Susskind fermions and lattice Kähler fermions is established.

In the fourth chapter we derive an ansatz for the Kähler field which respects the maximal symmetry of the Robertson-Walker metric. We then give self-consistent solutions to the Einstein-Kähler equations in both the massive and massless cases in Minkowski, de Sitter and Friedmann space-times. We show that in Minkowski space-time, the minimal left ideals can be identified with Dirac spinors. In the other cases minimal ideals do not satisfy the curved space Dirac equation.

A simple self-contained review of differential forms is given in an appendix in which the notational conventions are also set.

2. DIFFERENTIAL FORMS AS SPINORS

In this chapter we introduce the Kähler equation and study some of its formal aspects. Then, the equivalence of the Kähler equation and the Dirac equation in Minkowski space-time is established. Finally, spinors in arbitrary space-times are discussed.

2.1. Kähler Equation

The Klein-Gordon equation satisfied by a free relativistic tensor field Φ of mass m is

$$\Box\Phi=m^2\Phi \quad (2.1)$$

Here the covariant Laplace operator is given by

$$\Box=-d\delta-\delta d \quad (2.2)$$

where d is the exterior derivative and the co-derivative $\delta(=*d*)$ is with respect to the space-time metric of signature +2. It follows from Poincaré's lemma $d^2=0$ that equation (2.1) factorises according to

$$(d-\delta+m)(d-\delta-m)\Phi=0 \quad (2.3)$$

Kähler (1962) noted that the first order linear equation

$$(d-\delta-m)\Phi=0 \quad (2.4)$$

iterates to the Klein-Gordon equation (2.1) and thus may be regarded as a "square root", in the same sense as the Dirac equation is regarded as a "square root" of the Klein-Gordon equation. Then, he found a relation between the Kähler equation and the Dirac equation. We will make this relation explicit in the following section.

Here we first study some formal aspects of the equation (2.4). Before anything else it should be noted that, unless m vanishes, the operator $d-\delta$ should act on an inhomogeneous form Φ.

There are two trivial symmetries of the Kähler equation. The first symmetry is the covariance under general co-ordinate transformations which is made obvious by the exterior form notation we are using. The second trivial symmetry is the covariance under right v-multiplication by a constant form $U(U\varepsilon SU(4))$. That is, if Φ is a solution of Kähler equation then, Φ' given by

$$\Phi \rightarrow \Phi' = \Phi v U \qquad (2.5)$$

is also a solution of the Kähler equation. The non-trivial covariances of the Kähler equation are obtained by applying the Lie derivative with respect to a vector field on Φ. Lie derivative L_X commutes with Hodge * operation only if X is a Killing vector of the underlying space-time manifold (see Thirring, 1979). Therefore L_X commutes with Kähler operator $d-\delta$, when the Lie derivative is taken with respect to a Killing vector field. This implies, in turn, that if Φ is a solution of the Kähler equation, another solution is obtained by the transformation

$$\Phi \rightarrow \Phi' = L_X \Phi \qquad (2.6)$$

where X is a Killing vector.

There arise two more symmetries for the massless Kähler equation (m=0). First, the left Clifford multiplication of massless Kähler spinor by the constant 4-form $\varepsilon = *1$, which may be called the "chiral transformation" of differential forms, transforms Φ into another solution.

$$(d-\delta)(\varepsilon v \Phi) = -\varepsilon v\{(d-\delta)\Phi\} = 0 \qquad (2.7)$$

Next, the main automorphism A, defined by

$$A\omega=(-1)^{p}\omega \quad , \quad \omega\epsilon\Lambda^{p} \tag{2.8}$$

transforms solutions of the massless Kähler equation into solutions:

$$(d-\delta)(A\Phi)=-A(d-\delta)\Phi=0 \tag{2.9}$$

In general the symmetries of a physical system imply conserved currents and there is a standard way of obtaining these currents via Noether's construction. Nevertheless, it can be shown that (Benn and Tucker, 1983) if Φ and Ψ both satisfy the Kähler equation then the 3-form

$$j=S_3'\{A\Phi^{*}\Lambda*\Psi+\Psi\Lambda*A\Phi^{*}\} \tag{2.10}$$

is closed. Here $*$ as a superscript to the right of Φ denotes complex conjugation and S_3' projects the real part of the 3-form out of the expression inside the parantheses. By inserting ΦvU, $L_X\Phi$, $\epsilon v\Phi$ etc. instead of Ψ in the expression (2.10) the corresponding conserved currents may easily be obtained. If the space-time manifold does not admit isometries, then no conserved curents can be constructed for $L_X\Phi$.

On the other hand, if it is desired to derive the Kähler equation and the corresponding conserved currents from an action principle then we should vary the Lagrange 4-form

$$\mathcal{L}=S_4'\{A\Phi^{*}\Lambda*d\Phi-\frac{m}{2}A\Phi^{*}\Lambda*\Phi\} \tag{2.11}$$

As a side remark we note that, from this expression it is easy to minimally couple a U(1)-gauge potential A to the Kähler field introducing the U(1) covariant derivative

$D=d+ieA\wedge$ instead of d. The Lagrange 4-form

$$\mathcal{L}=S_4'\{A\Phi^*\wedge *D\Phi-\frac{m}{2}A\Phi^*\wedge *\Phi\} \qquad (2.12)$$

is invariant under the gauge transformations

$$\Phi \rightarrow e^{i\lambda e}\Phi$$
$$A \rightarrow A-d\lambda \qquad (2.13)$$

where λ is a function.

The stress energy-momentum 3-forms τ_a, are obtained from the Lagrange 4-form (2.11) as follows:

$$L_X\mathcal{L}=\tau_a\wedge L_X e^a \qquad (2.14)$$

where X generates arbitrary frame variations. For the action (2.11) we obtain

$$\tau_a=S_3'\{\Phi^*\wedge i_a*d\Phi-i_a d\Phi\wedge *A\Phi^*-\frac{m}{2}\Phi^*\wedge i_a*\Phi+\frac{m}{2}i_a\Phi\wedge *A\Phi^*\} \qquad (2.15)$$

where $i_a\equiv i_{X_a}$ and $\{X_a\}$ constitute a g-orthonormal frame. τ_a are the classical sources for the coupling of the system to Einsteinian gravitation.

2.2. Dirac Equation vs. Kähler Equation in Minkowski Space-Time

The Dirac equation satisfied by a four-component column matrix (spinor) ψ in Minkowski space-time is

$$(\gamma^a\nabla_a-m)\psi=0 \qquad (2.16)$$

where the gamma matrices (γ^a) are such that

$$\gamma^a\gamma^b+\gamma^b\gamma^a=2\eta^{ab}$$

This equation may as well be written in the form

$$(\gamma^a \nabla_a - m)\Psi = 0 \tag{2.17}$$

where Ψ is a 4x4 matrix whose first column is identical with ψ and all the rest of its entries are zero. As a matter of fact equation (2.17) makes sense for any matrix Ψ and it reduces to four decoupled Dirac equations, one for each column, in Minkowski space-time. This is achieved by the set of projectors

$$P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which acts from the right on equation (2.17). Thus, Dirac spinors may be regarded as minimal left ideals in the total matrix algebra $M_4(\mathbb{C})$. It is well-known that any 4x4 matrix Ψ can be expanded uniquely according to the formula

$$\Psi = S + V_a\gamma^a + \frac{1}{2!}F_{ab}\gamma^a\gamma^b + \frac{1}{3!}A_{abc}\gamma^a\gamma^b\gamma^c + \frac{1}{4!}P_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d \tag{2.18}$$

We associate with Ψ the differential form, called a Kähler spinor,

$$\Phi = S + V_a e^a + \frac{1}{2!}F_{ab}e^{ab} + \frac{1}{3!}A_{abc}e^{abc} + \frac{1}{4!}P_{abcd}e^{abcd} \tag{2.19}$$

It can be shown that the action of the Dirac operator $(\gamma^a\nabla_a - m)$ on Ψ is equivalent to the action of $(d-\delta-m)$ on Φ. In this sense a Kähler spinor is equivalent to 4 decoupled Dirac spinors in Minkowski space-time. As a side product, we also have proven above that the inhomogeneous form Φ carries 4 independent irreducible representations of the spin covering of the Lorentz group. We think it is remarkable that a set of irreducible tensor representations of the Lorentz group can be rearranged so as to form irreducible spinor representations.

We now wish to illustrate the converse of what we have shown. That is, if any inhomogeneous differential form Φ of type (2.22) is given, then it may be decomposed into 4 minimal left ideals all of which individually satisfy the Kähler equation in Minkowski space-time. This is achieved by, for instance, the following set of projectors which are of minimal rank and pairwise orthogonal;

$$P_1=\frac{1}{4}(1+e^{01})\vee(1+ie^{0123})$$

$$P_2=\frac{1}{4}(1+e^{01})\vee(1-ie^{0123})$$

$$P_3=\frac{1}{4}(1-e^{01})\vee(1+ie^{0123})$$

$$P_4=\frac{1}{4}(1-e^{01})\vee(1-ie^{0123}) \qquad (2.20)$$

Each minimal left ideal

$$\Phi_i=\Phi\vee P_i \qquad i=1,2,3,4.$$

satisfies the Kähler equation as a consequence of equation (2.5).

It is conventional to write the Dirac equation in terms of 4x4 complex γ matrices. In order to find the explicit γ matrix set corresponding to the above set of projectors, the isomorphism between the complex Clifford algebra $C_{1,3}(\mathbb{C})$ and the total matrix algebra $M_4(\mathbb{C})$ is exploited. It is always possible to construct a basis ε_{ij}, i,j=1,2,3,4, in the Clifford algebra $C_{1,3}(\mathbb{C})$ (Benn and Tucker, 1985) such that

$$\varepsilon_{ij}\vee\varepsilon_{jk}=\varepsilon_{ik} \qquad \text{(no sum on j)}$$

$$\varepsilon_{ij}\vee\varepsilon_{kl}=0 \qquad \text{for } j\neq k \qquad (2.21)$$

Then any element $\Phi\varepsilon C_{1,3}(\mathbb{C})$ can be written as

$$\Phi=\sum_{i,j}\Phi_{ij}\ \varepsilon_{ij} \qquad (2.22)$$

or conversely

$$\Phi_{ij}=4S_o(\Phi v\varepsilon_{ij}) \tag{2.23}$$

where S_o projects out the 0-form out of the parantheses. The Φ_{ij} can be regarded as elements of a 4x4 matrix Ψ. In particular if the orthonormal basis 1-forms e^a are considered, then the corresponding matrices are the Dirac γ-matrices; that is

$$\gamma^a_{ij}=4S_o(e^a v\varepsilon_{ij}) \tag{2.24}$$

The elements of the matrix basis ε_{ij} are explicitly given in Table 2.1.

Table 2.1 : The elements of matrix basis ε_{ij}.

ε_{ij}	1	2	3	4
1	P_1	e^2vP_2	$e^{02}vP_3$	e^0vP_4
2	e^2vP_1	P_2	$-e^0vP_3$	$-e^{02}vP_4$
3	$e^{02}vP_1$	e^0vP_2	P_3	e^2vP_4
4	e^0vP_1	$e^{02}vP_2$	e^2vP_3	P_4

By making use of (2.24) and Table 2.1, the γ-matrix set corresponding to the projector set (2.20) is obtained:

$$\gamma_0=\begin{bmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix}, \quad \gamma_1=\begin{bmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix}$$

$$\gamma_2=\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}, \quad \gamma_3=\begin{bmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix} \tag{2.25}$$

where σ_1, σ_2 and σ_3 are the standard Pauli matrices.

2.3. Algebraic Spinors

An algebraic spinor is usually defined as an element of a minimal left ideal in a Clifford algebra (Chevalley, 1954). Since Clifford algebra is semisimple, its finite dimensional irreducible representations are given by minimal left ideals. Any decomposition of a Clifford algebra into its minimal left ideals can be characterized by a set of v-idempotent projectors $\{P_i\}$, such that

$$\Sigma P_i = \mathbb{1}$$

$$P_i v P_j = \delta_{ij} P_i \qquad (2.26)$$

In this section we wish to give an appropriate globalization of an algebraic spinor to a spinor field over arbitrary Riemannian (or pseudo-Riemannian) manifolds. This may be achieved by defining spinor fields as cross sections of the spinor bundle whose fibers are representation modules of the complexified Clifford algebra. For an even dimensional manifold n=2r (or odd dimensional ones for which n=2r+1), the complexified algebra has 2^r dimensional modules as irreducible representations (Chevalley, 1954).

Graf (1978) proposed another way of globalization in which a spinor field corresponding to a projector P is considered as a cross section Ψ of the Clifford bundle such that $\Psi v P = \Psi$. Regarding the Clifford bundle embedded into a Kähler-Atiyah bundle, spinor field is then interpreted in a natural way as a differential form, in accordance with Kähler's previous work related with Minkowski space-time (1962). The main difference of the Graf's proposal is the use of vector bundles derived from the cotangent bundle, instead of principal bundles with structure groups homomorphic to the rotation groups

O(p,q). Graf's proposal gives rise to considerable technical simplifications.

Kähler equation (2.4) may not have spinorial solutions in the sense of Graf in arbitrary space-times. In order to decompose the Kähler spinor, which is a cross section of the Clifford-bundle, into its minimal left ideals, a complete set of projectors $\{P_i\}$ is first specified. These projectors satisfy the condition (2.26). The minimal left ideals are

$$\Phi_i = \Phi \vee P_i \qquad , \qquad i=1,2,3,4$$

These minimal ideals themselves do not satisfy the Kähler equation (2.4) in general. To see this we multiply the Kähler equation from the right by a projector P_i and carry it under the operator d-δ. After some manipulations we find

$$(d-\delta-m)\Phi_i = e^a \vee \Phi \vee \nabla_{X_a} P_i \tag{2.27}$$

where ∇_{X_a} is the covariant derivative with respect to the Christofell-Levi-Civita connection and e^a is a set of orthonormal basis 1-forms. The right hand side of (2.27) does not vanish necessarily. The existence of spinorial solutions to Kähler equation is therefore a topological property which is not shared by all manifolds. For this reason Graf suggested that any solution to the Kähler equation should be regarded as a spinor field corresponding to the trivial projector P=1. It is interesting to note that the projectors

$$P_+ = \frac{1}{2}(1+ie^{0123}) \quad , \quad P_- = \frac{1}{2}(1-ie^{0123}) \tag{2.28}$$

decompose any Kähler spinor Φ into two non-minimal left ideals $\Phi_\pm = \Phi \vee P_\pm$ of opposite parity and furthermore Φ_+ and Φ_- both satisfy the Kähler equation.

3. LATTICE FERMIONS

In conventional coloured quark-gluon model of strong interactions (QCD), the perturbative approach to quantisation fails due to the large value of the QCD coupling constant in the IR region. One is therefore forced to introduce Euclidean space-time lattice as an artifact in order to be able to discuss problems such as quark imprisonment in quantitative terms. The preliminary lattice calculations of the glue ball mass (Bhanot, 1981) and the hadron spectrum (Weingarten, 1982) as well as the discussions of the structure of the phase transitions of the quark-gluon matter on the lattice (Çelik, 1985) give promising results and justify such an approach. Formulation of lattice QCD, however, cannot be considered as yet satisfactory due to the several problems pertinent to the lattice transcription of the Dirac equation.

3.1. Problems With Lattice Fermions

Straightforward transcription of the Dirac equation to the lattice by replacing derivatives by nearest-neighbour difference operators leads to the spectrum degeneracy problem. In order to explain this problem in detail, we transcribe to the lattice the massless Dirac equation in 1+1 dimensions;

$$\partial_t \Psi = \alpha \partial_z \Psi \tag{3.1}$$

where Ψ is a two component spinor. α is the 2x2 matrix

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.2}$$

On one dimensional lattice with lattice spacing a, lattice version of equation (3.1) becomes

$$\partial_t \Psi = \frac{\alpha}{2a}[\Psi(n+1)-\Psi(n-1)] \tag{3.3}$$

where $\Psi(n)$ specifies the spinor at the n^{th} site. For plane wave solutions of the form $\Psi = \Psi_o e^{i(kna-\omega t)}$ we obtain the following dispersion relation

$$\omega^2 = \frac{1}{a^2}\sin^2 ka \tag{3.4}$$

with three solutions $k=0$, $+\pi/a$, $-\pi/a$ in the first Brillouin zone.

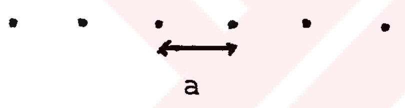


Figure 3.1a Space lattice

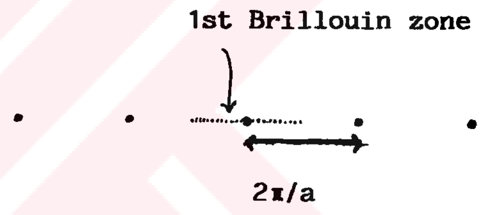


Figure 3.1b Momentum space lattice

This means that the continuum limit of equation (3.3) describes three fermions rather than just one. The equation (3.3) is invariant under the continuous, global chiral transformations:

$$\Psi \rightarrow \Psi' = e^{i\theta\gamma_5}\Psi \tag{3.5}$$

When the above discussion is extended to four dimensions, we see that four dimensional lattice Dirac equation describes 17 fermions in the continuum limit while preserving continuous chiral symmetry at m=0.

This difficulty was circumvented by Wilson (1974, 1975) by adding to the lattice Dirac equation a term which gives the unwanted fermions masses of the order of the inverse lattice spacing. In this way their masses become infinite in the continuum limit and they can be excluded from the low energy considerations. This extra form unfortunately destroys the chiral symmetry of the massless (m=0) Dirac equation. Another prescription due to Susskind (1977) is to reduce the number of degrees of freedom by using a single component field Φ on each lattice site. In this case although one obtains discrete chiral symmetry for m=0, in the continuum limit 4 fermions are described. A third transcription in a purely algebraic context was given by Becher (1981) and was later shown to be completely equivalent to that of Susskind (Dhar and Shankar, 1982). Since this latter formulation is related with Kähler formalism on the lattice, it will be discussed in the next section in detail.

Karsten and Smit (1981) and Rabin (1982) provided arguments on why it is impossible to find an undoubled lattice fermion formulation with chiral symmetry. Finally Nielsen and Ninomiya (1981a,b,c) and Karsten (1981) produced three different proofs of a no-go theorem which states that it is impossible to solve the doubling problem while preserving continuous chiral symmetry. The proof of this result depends on a set of physical assumptions. It is required that the interaction operator be local, translationally invariant over a finite number of lattice spacings and Hermitian. It may be possible to solve the doubling problem in a chirally invariant way by relaxing one or more of these assumptions (Drell et al., 1976; Jacobs, 1983).

3.2. Susskind Fermions

We write the Dirac equation in the form

$$\frac{d\Psi}{dt} = \dot{\Psi} = \vec{\alpha} \cdot \vec{\nabla}\Psi \tag{3.6}$$

where

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} , \quad \gamma_o = \begin{bmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{bmatrix} , \quad \gamma_5 = \begin{bmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{bmatrix} \tag{3.7}$$

σ 's are the familiar Pauli matrices and $1\!\!1$ is the 2x2 unit matrix. The number of degrees of freedom can be reduced by assigning to each lattice site a one componant field $\Phi(r)$, satisfying the canonical anticommutation relations

$$\{\Phi(n), \Phi(m)\} = \{\Phi^+(n), \Phi^+(m)\} = 0 \quad ,$$

$$\{\Phi^+(n), \Phi(m)\} = \delta_{nm} \tag{3.8}$$

Then the lattice which is three dimensional and of equal spacing is subdivided into four sublattices to accommodate the four components of the conventional Dirac field. The subdivision is accomplished by labeling the corners of the unit cell as shown in Figure 3.2a. The labeling is then carried periodically through the lattice. The planes x=0, y=0 and z=0 are illustrated in Figure 3.2b, Figure 3.2c and Figure 3.2d respectively. Now consider the Hamiltonian

$$H = \Sigma\Big(\frac{i}{2a}\,[\Phi^+(\vec{r})\,\Phi(\vec{r}+\hat{n}_z) - H.c]\,(-1)^{x+y}$$

$$+ \frac{i}{2a}\,[\Phi^+(\vec{r})\,\Phi(\vec{r}+\hat{n}_x) - H.c]$$

$$- \frac{1}{2a}\,[\Phi^+(\vec{r})\Phi(\vec{r}+\hat{n}_y) + H.c\,]\,(-1)^{x+y}\Big) \tag{3.9}$$

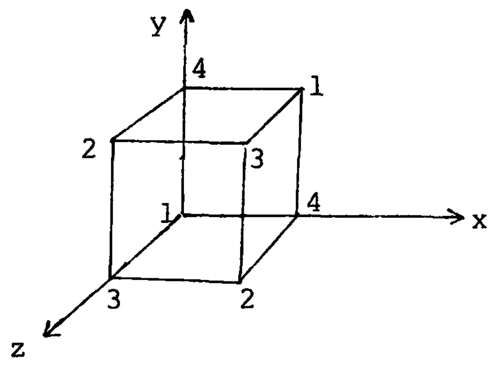


Figure 3.2a Labeling of lattice sites.

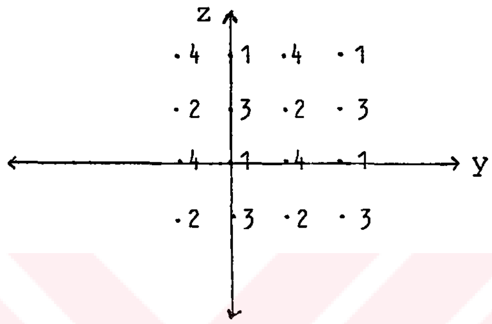


Figure 3.2b x=0 plane.

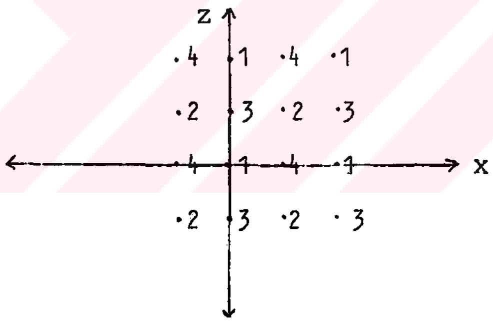


Figure 3.2c y=0 plane.

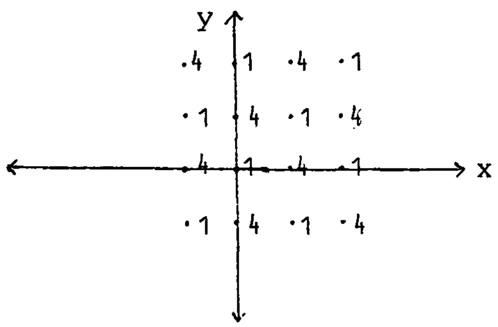


Figure 3.2d z=0 plane.

where a is the lattice spacing, $\hat{n}_x$, $\hat{n}_y$ and $\hat{n}_z$ are unit vectors, H.c stands for Hermitian conjugate and summation is over all the lattice points. Using the relations (3.8) and $\dot{\Phi}(\vec{r}) = (1/i)[\Phi(\vec{r}), H]$ we obtain the equation of motion

$$\dot{\Phi}(\vec{r}) = \frac{1}{2a}[\Phi(\vec{r}+\hat{n}_z) - \Phi(\vec{r}-\hat{n}_z)](-1)^{x+y}$$

$$+ \frac{1}{2a}[\Phi(\vec{r}+\hat{n}_x) - \Phi(\vec{r}-\hat{n}_x)]$$

$$+ \frac{i}{2a}[\Phi(\vec{r}+\hat{n}_y) - \Phi(\vec{r}-\hat{n}_y)](-1)^{x+y} \qquad (3.10)$$

Applying this to lattice points we can write it as

$$\Psi_1 = \Delta_z\Psi_3 + \Delta_x\Psi_4 + i\Delta_y\Psi_4$$

$$\Psi_2 = -\Delta_z\Psi_4 + \Delta_x\Psi_3 - i\Delta_y\Psi_3$$

$$\Psi_3 = \Delta_z\Psi_1 + \Delta_x\Psi_2 + i\Delta_y\Psi_2$$

$$\Psi_4 = -\Delta_z\Psi_2 + \Delta_x\Psi_1 - i\Delta_y\Psi_1 \qquad (3.11)$$

where $\Delta_x\Psi = (1/2a)[\Psi(\vec{r}+\hat{n}_x) - \Psi(\vec{r}-\hat{n}_x)]$ and Δy, Δz are given similarly. Equation (3.11) can be written in a more compact form as:

$$\dot{\Psi} = \vec{\alpha}\,.\,\vec{\Delta}\Psi \qquad (3.12)$$

In the continuum limit equation (3.12) becomes equivalent to the Dirac equation (3.6).

Let us subdivide the lattice into f sites for which y is even and g sites for which y is odd. The fields are relabeled accordingly:

$\Psi_i = f_i$ (y even)

$\Psi_i = g_i$ (y odd)

Next, we write equation (3.11) in Fourier transformed variables:

$$af=(\alpha_x \sin k_x a+\alpha_z \sin k_z a)f+(\alpha_y \sin k_y a)g$$

$$ag=(\alpha_x \sin k_x a+\alpha_z \sin k_z a)g+(\alpha_y \sin k_y a)f \qquad (3.13)$$

As $a \to 0$, the combinations $u=f+g$ and $\tilde{d}=f-g$ satisfy

$$\dot{u}=(\alpha_i k_i)u$$

$$\partial_t \tilde{d}=(\alpha_x k_x-\alpha_y k_y+\alpha_z k_z)\tilde{d} \qquad (3.14)$$

Let us introduce a new field variable d such that

$$d_1= \tilde{d}_2$$
$$d_2=-\tilde{d}_1$$
$$d_3=-\tilde{d}_4$$
$$d_4= \tilde{d}_3 \qquad (3.15)$$

Now u and d fields satisfy

$$\dot{u}=(\alpha_i k_i)u$$
$$\dot{d}=(\alpha_i k_i)d \qquad (3.16)$$

We therefore conclude that Susskind fermions on 3+1 dimensional lattice where time is continuous, describe two Dirac fermions in the continuum limit.

The symmetries of the Hamiltonian (3.9) include translations by even integers and rotations about any axis by angle π. These discrete lattice symmetries are promoted to continuous translational and rotational symmetries in the continuum limit as they do not mix the internal indices. Translations along the large diagonals as shown in Figure 3.3 are also symmetries of the Hamiltonian and mix the internal indices. These correspond to the discrete chiral symmetry which also becomes a continuous symmetry in the $a \to 0$ limit. Translation along a large diagonal induces the transformation

$$\begin{array}{l} \Psi_1 \to \Psi_3 \\ \Psi_2 \to \Psi_4 \\ \Psi_3 \to \Psi_1 \\ \Psi_4 \to \Psi_2 \end{array} \equiv \Psi \to \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \Psi = \gamma_5 \Psi \qquad (3.17)$$

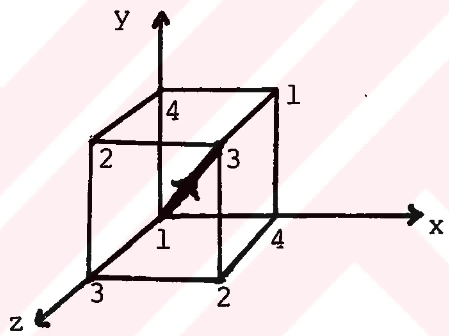


Figure 3.3 Translation along a large diagonal

In the above presentation of Susskind fermions we let the time variable be continuous. If the time is also discretized then the field variables and fermions appearing in the continuum limit double in nu·mber (Sharatchandra et al., 1981). To summarize, Susskind fermions show discrete chiral symmetry and describe four fermions in the continuum limit, in accordance with the no-go theorem of Nielsen and Ninomiya.

3.3. The Kähler Equation on the Lattice

Lattice Kähler equation was first studied by Rabin (1982) and independently by Becher and Joos (1982) in order to explain geometrically the connection between spectrum degeneracy and chiral symmetry. There is a formal correspondence between differential forms and their manipulations defined on the continuum and the quantities and operations defined on the lattice (Vaisman , 1973). Using this correspondence we write the Kähler equation on the lattice and show that Kähler equation is identical to the continuum limit of the Dirac equation satisfied by Susskind fermions.

For convenience, only in this section, we will make use of multiindex-notation and restrict ourselves to Euclidean hypercubic lattice. We write generically H,K,... for any ordered index set $\{\mu_1 \ldots\ldots \mu_p\}$. The derived index sets $H \cup K$, $H \cap K$, $H|K$ etc. are meant to be in their natural order. We furthermore introduce the following set of notations:

points $\quad {}^0C=(x,\emptyset)$

links $\quad {}^1C=(x,x+e_\mu)\equiv(x,\mu)$

plaquettes $\quad {}^2C=(x,x+e_{\mu 1},x+e_{\mu 2})\equiv(x,\mu_1\mu_2)$

.
.
.

p-cubes $\quad {}^pC=(x,x+e_{\mu 1},\ldots , x+e_{\mu p})\equiv(x,H) \qquad$ **(3.18)**

where e_μ is the unit vector in μ direction and H is the ordered set $\{\mu_1 \ldots\ldots \mu_p\}$. Weighted p-cubes are called p-chains. An inhomogeneous chain can be written as

$$C=\sum_{x,H}\alpha(x,H).(x,H) \qquad \textbf{(3.19)}$$

Elementary cochains $d^{x,H}$ dual to p-cells (x,H) are defined by the following relation

$$d^{x,H}.(x',H')=\delta^{x}_{x'}\ \delta^{H}_{H'} \tag{3.20}$$

The most general cochain is a weighted sum of elementary cochains and is given by

$$\check{\Phi}=\sum_{x,H} Q(x,H)d^{x,H} \tag{3.21}$$

It is a well known fact that when a manifold is approximated by a lattice, differential forms are mapped on cochains (Hocking and Young, 1961). The formula

$${}^{p}\check{\Phi}({}^{p}C_i)=\int_{{}^{p}C_i} {}^{p}\Phi \tag{3.22}$$

defines the mapping of a p-form to a p-cochain on the p dimensional lattice cell, ${}^{p}C$. The dual boundary operator $\check{\Delta}$ acting on co-chains is defined by

$$(\check{\Delta}\check{\Phi})(C)=\check{\Phi}(\partial C) \tag{3.23}$$

It follows from the Stokes' theorem that the definition given above transforms the exterior derivative d, into the dual boundary operator $\check{\Delta}$:

$$\int_{{}^{p}C} d({}^{p-1}\Phi)=\int_{\partial {}^{p}C} {}^{p-1}\Phi=\check{\Phi}(\partial {}^{p}C)=(\check{\Delta}\check{\Phi})({}^{p}C) \tag{3.24}$$

A list of lattice-continuum correspondences is given in Table 3.1. It allows straightforward transcription of the Kähler equation to the lattice:

$$(\check{\Delta}-\check{\nabla}-m)\check{\Phi}=0 \tag{3.25}$$

The action of $\check{\Delta}$ and $\check{\nabla}$ on $\check{\Phi}$ is given by

$$\check{\Delta}\check{\Phi}=\sum_{x,H}\Big(\sum_{\mu\epsilon H} \rho_{\{\mu\},H|\{\mu\}}\ [\Delta^{+}_{\mu} Q(x,H\,|\{\mu\})]\ d^{x,H}\Big)$$

and

$$\check{\nabla}\check{\Phi}=\sum_{x,H}\Big(\sum_{\mu\notin H}\rho_{\{\mu\},H}\,[\Delta_\mu^{-}Q(x,H\cup\{\mu\})]\,d^{x,H}\Big) \qquad (3.26)$$

Δ_μ^{+} and Δ_μ^{-} are the forward and backward difference operators respectively, and $\rho_{H,K}$ is a sign function:

$$\Delta_\mu^{+}Q(x,H)=\frac{1}{a}\,[\,Q(x+e_\mu,H)-Q(x,H)\,]$$

$$\Delta_\mu^{-}Q(x,H)=\frac{1}{a}\,[\,Q(x,H)-Q(x-e_\mu,H)\,]$$

$$\rho_{H,K}=\begin{cases}(-1)^{\nu}, & \text{where } \nu=\text{no. of pairs } (i,j)\in H\times K \text{ and } i>j\\ 0, & \text{if } H\cap K\neq\emptyset\\ +1, & \text{if } H=\emptyset \text{ or } K=\emptyset\end{cases} \qquad (3.27)$$

Table 3.1 Continuum-lattice correspondence

CONTINUUM	LATTICE
Weighted sums of points, curves, surfaces	Chains
Boundary, ∂	Boundary, ∂
Differential form Φ	Cochain $\check{\Phi}$
Exterior derivative $d, d^2=0$	Dual boundary op. $\check{\Delta}, \check{\Delta}^2=0$
Co-derivative $\delta, \delta^2=0$	Dual co-boundary op. $\check{\nabla}, \check{\nabla}^2=0$
Laplacian-$(d\delta+\delta d)$	Laplacian-$(\check{\Delta}\check{\nabla}+\check{\nabla}\check{\Delta})$
Exterior product $\wedge$	Exterior product $\wedge$
Interior product i_x	Interior product i_x
Clifford product $\vee$	Clifford product $\vee$

The exterior product, the interior product and the Clifford product need some comments. The exterior product of elementary cochains is defined by

$$d^{x,H}\wedge d^{y,K}=\begin{cases}\rho_{H,K}\ \delta^{x+e_H,y}\, d^{x,H\cup K} & \text{if } H\cap K=\emptyset\\ 0 & \text{otherwise}\end{cases} \tag{3.28}$$

where $e_H=\sum_{\mu\in H} e_\mu$. Because of the matching factor $\delta^{x+e_H,y}$, the exterior product is not local. Product rule with respect to the dual boundary operator holds:

$$\check{\Delta}(\check{\Phi}\wedge\check{\theta})=(\check{\Delta}\check{\Phi})\wedge\check{\theta}+(A\check{\Phi})\wedge(\check{\Delta}\check{\theta})$$

where

$$A^{p}\check{\Phi}=(-1)^{p}\,{}^{p}\check{\Phi} \tag{3.29}$$

The lattice correspondent of interior product is

$$i_{\tilde{e}^{\mu}}d^{x,H}=\begin{cases}\rho_{\{\mu\},H|\{\mu\}}\, d^{x,H|\{\mu\}} & \text{if } \{\mu\}\subset H\\ 0 & \text{otherwise}\end{cases} \tag{3.30}$$

Lattice interior product satisfies

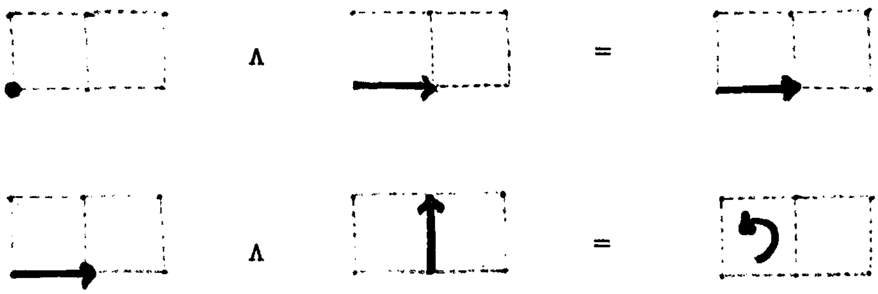
$$i_{\tilde{e}^{\mu}}(d^{x,H}\wedge d^{y,K})=(i_{\tilde{e}^{\mu}}d^{x,H})\wedge d^{y-e_\mu,K}+(Ad^{x,H})\wedge i_{\tilde{e}^{\mu}}d^{y,K} \tag{3.31}$$

Finally lattice Clifford product is defined by

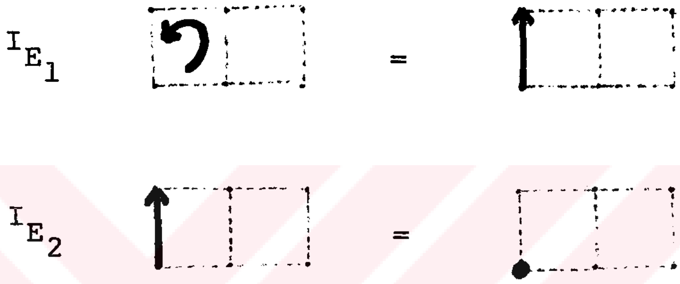
$$d^{x,H}\vee d^{y,K}=(-1)^{\binom{\lambda}{2}}(-1)^{\lambda(h-\lambda)}\,\rho_{\Lambda,H\Delta K}\rho_{H|\Lambda,K|\Lambda}\delta^{x+e_H,y}\,d^{x+e_\Lambda,H\Delta K} \tag{3.32}$$

Here $\Lambda=H\cap K$, $H\Delta K=H\cup K-\Lambda$ and λ and h are degrees of Λ and H respectively. Examples of lattice exterior product, interior product and Clifford product are shown in Figure 3.4. Chiral symmetry of the lattice Kähler equation can be formulated with the help of the constant 4 co-chain $\varepsilon=\sum_x d^{x,\{0123\}}$ which is the lattice analogue of dx^{0123}. The transformation $\check{\Phi}\to\varepsilon\vee\check{\Phi}$ is a symmetry of the massless lattice Kähler equation and it corresponds to discrete

Exterior product:



Interior product:



Clifford product:

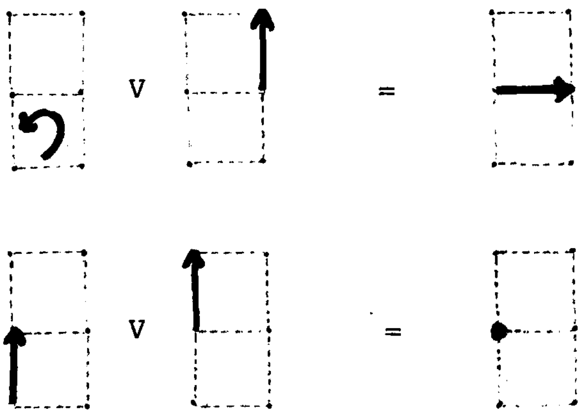


Figure 3.4. Examples of exterior, interior and Clifford product on lattice.

chiral symmetry. Explicitly,

$$(\check{\Delta}-\check{\nabla})(\varepsilon v\check{\Phi}) = -\varepsilon v(\check{\Delta}-\check{\nabla})\check{\Phi} = 0 \qquad (3.33)$$

In order to show that lattice Kähler equation iterates to the lattice Klein-Gordon equation we apply the conjugate lattice Kähler operator on expression (3.25):

$$(\check{\Delta}-\check{\nabla}+m)(\check{\Delta}-\check{\nabla}-m)\check{\Phi}$$

$$= (-\check{\Delta}\check{\nabla}-\check{\nabla}\check{\Delta}-m^2)\check{\Phi} = (\Delta^+_\mu\Delta^{-,\mu}-m^2)\check{\Phi} = 0 \qquad (3.34)$$

For the plane wave solutions of the form $Q(x,H)=U(p,H)e^{-ip_\mu x^\mu}$ (3.34) becomes

$$(\sum_\mu(\frac{2}{a}\sin\frac{p_\mu a}{2})^2-m^2)U(p,H)=0 \qquad (3.35)$$

This shows that there is no spectrum degeneracy in the first Brillouin zone, that is, the number of degrees of freedom on the lattice is the same as in the continuum.

The discussions given in the present and previous sections indicate that both the Kähler fermions and the Susskind fermions are 16 component objects which can be decomposed into 4 Dirac fields in the continuum limit. The massless lattice Kähler equation has a discrete chiral invariance just as the massless Dirac equation in Susskind's formulation. In fact the two formulations are seen to be identical after a simple relabelling of variables. Consider a hypercubic lattice and insert a new lattice site at the geometrical centre of each p-cell. Let the variable Q(x,H) associated with any p-cell be related with the site at the centre. These new sites form another lattice of one half the original spacing on which the fermion formulation is essentially that of Susskind. Mathematically, if $\sum_{x,H} Q(x,H)d^{x,H}$ satisfies the lattice Kähler equation then the Dirac spinor $\varphi(y)$ defined by

$$\psi_a(y) \equiv \psi_a(x + \frac{1}{2} e_H) \equiv \psi(x,H) = \sum_i \gamma^H_{ai} \, Q_i(x,H) \quad i=1,2,3,4 \quad \text{i fixed} \qquad (3.36)$$

satifies the lattice Dirac equation. Because of this equivalence we may say that the Kähler equation is the formal continuum limit of the Susskind formulation of Dirac's equation on the lattice. Although both formulations of lattice fermions are equivalent, the Kähler's approach is superior, firstly, because it is geometrically more intuitive and secondly, because continuum lattice correspondence makes many of the lattice definitions and manipulations more transparent. In particular, interactions and conserved currents may easily be constructed in the Kähler approach (Becher and Joos, 1982; Benn and Tucker, 1983).

4. FERMIONS IN COSMOLOGICAL SPACE-TIMES

We already noted that the Kähler equation in curved space-times may differ, in general, from the ordinary Dirac equation. According to the 4-generation fermion models based on the Kähler formalism (Benn and Tucker, 1982 ; Banks et al., 1982), the gravitational interactions can change the assigned internal quantum numbers of fermions (i.e. they change the ideal decomposition). Such effects should occur with amplitudes too small to be detected in present-day observations, however, they may have important consequences in the early phases of the evolution of the universe. Since we think that Kähler spinors may be more fundamental than the Dirac spinors, in the present chapter we investigate self-consistent solutions of the coupled Einstein-Kähler equations in some cosmological space-times.

4.1. Basic Ideas in Cosmology

The universe we observe appears homogeneous on the large scale. That is, it is the same to all observers wherever they are located. Most modern cosmological models also contain the assumption that the universe is isotropic, that is, to any observer it looks the same in all directions. As far as the present-day experiments are concerned, it seems unlikely on the grounds of probability alone, that our solar system occupies a special position in the universe. Hence, we conjecture isotropy everywhere in the universe and this implies its homogeneity. The assumption of large scale homogeneity and of large scale isotropy is called the **cosmological principle**. The status of this principle is that of a working hypothesis. The actual universe is irregular in detail, consisting of vast empty stretches that divide regions of concentrations of mass of many different shapes and sizes. In order to deal with this situation in a rational manner, we introduce an idealised view of the universe. We regard the universe to consist of a smoothed-out-pattern of a space-filling set of particles in motion, each of which is a potential center of mass of a galaxy, or of a cluster of galaxies. It is this structure that is assumed to be strictly homogeneous and isotropic.

In an evolving universe the notion of spatial homogeneity is not quite as simple a concept as it is in a static universe. Intuitively we think of it as meaning that all sufficiently large spatial samples of the universe are equivalent. But the question is "When?". Even the immediate neighbourhood of our solar system may be different from what it had been millions of years ago, let alone from other parts of the universe. In order to avoid this difficulty the following definition is used: Homogeneity means that the totality of observations any observer can make on the universe is identical to the totality of observations that any other observer can make on the universe. The most important corollary of such homogeneity is the existence of **cosmic time**, that is, of an absolute universe-wide sequence of moments. In fact, a homogeneous universe, if it is evolving, acts as the relevant synchronisation agent at each point of the universe.

The cosmological principle, taken together with the existence of a cosmic time, may be regarded as defining a group of transformations under which the large scale universe transforms into itself. The most general space-time metric that admits this group of transformations as its isometry group is the **Robertson-Walker metric:**

$$g= -dt^2 + R^2(t) \left[\frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{(1 + \frac{k}{4} r^2)^2} \right] \qquad (4.1)$$

in terms of isotropic co-ordinates (t, x^1, x^2, x^3) and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. t represents the cosmic time and R(t) is the expansion function of the 3-universe such that the ratio $H = \dot{R}/R$ is identified with Hubble's "constant". The curvature index k= 0 for flat 3-universe, while k= +1 for closed 3-universe and k= -1 for open 3-universe. Robertson-Walker metric (4.1) admits one time-like Killing vector

$$X = \frac{\partial}{\partial t} \tag{4.2}$$

and six space-like Killing vectors. These are the three rotation generators that account for the isotropy,

$$Y_i = \varepsilon_{ijk}\, x^j \frac{\partial}{\partial x^k} \quad , \qquad i = 1,2,3 \tag{4.3}$$

and the three generators of generalised translations that account for the homogeneity,

$$Y_{(i+3)} = \left(1 - \frac{k}{4} r^2\right) \frac{\partial}{\partial x^i} + \frac{1}{2} k\, x^i x^j \frac{\partial}{\partial x^j} \quad , \quad i = 1,2,3 \tag{4.4}$$

In all cosmological models we subject the Robertson-Walker metric to a set of field equations which then determine the expansion function R(t) and curvature index k. Here we will express the dynamics of our cosmological models by **Einstein's field equations**[1]

$$G_a = -K^2 \tau_a + \lambda {*e}_a \tag{4.5}$$

---

1 It should be noted, however, that the Robertson-Walker metric applies to all cosmological models, even to those outside the realm of relativistic theories of gravitation.

where $K^2 = 8\pi G$ is the universal gravitational coupling constant, λ is a cosmological constant and τ_a are the stress-energy-momentum 3-forms which describe the material contents of the universe. Since the texture of the material content of the actual universe is not amenable to an easy mathematical treatment, we generally adopt the theoretical device of regarding these contents as a uniformly distributed **cosmic dust** that has mass density $\rho(t)$. The effect of proper motion of the galaxies and the possible intergalactic presence of electromagnetic radiation, neutrinos, cosmic rays, quanta such as gravitons and gravitinos etc. are thought to cause pressure $p(t)$ and internal stresses. The latter were perhaps more important in the very early stages of the universe that was much denser. Nevertheless it is customary to describe the cosmic matter in homogeneous and isotropic models by the stress-energy-momentum 3-forms

$$\tau_a(\text{matter}) = \tau_{ab}(\text{matter})*e^b = \delta^o_a\,\rho*e^o + \delta^k_a p*e^k \qquad \textbf{(4.6)}$$

Substituting (4.1) and (4.6) into the field equations (4.5) we obtain the following two conditions:

$$-3\left(\frac{k+\dot{R}^2}{R^2}\right) + \lambda = -K^2\rho \qquad \textbf{(4.7a)}$$

$$2\frac{\ddot{R}}{R} + \left(\frac{k+\dot{R}^2}{R^2}\right) - \lambda = -K^2 p \qquad \textbf{(4.7b)}$$

Multiplying the right hand side of (4.7a) by R^3 and differentiating, we get $-3R^2\dot{R}$ times the left hand side of (4.7b), and thus

$$\frac{d}{dt}(\rho R^3) + 3p\dot{R}R^2 = 0 \qquad \textbf{(4.8)}$$

This is the relativistic equation of continuity.

The systematic study of the kinematics and dynamics of an expanding universe was first made by A. Friedmann. The so-called **Friedmann models** are concerned with dust universes that are isotropic and homogeneous. Then the pressure p= 0 and continuity equation (4.8) becomes

$$\frac{d}{dt}(\rho R^3)= 0$$

This condition may be interpreted to imply the constancy of the mass contained in a sphere of radius proportional to R. We set

$$K^2 \rho R^3 = 3C \tag{4.9}$$

and substitute in it (4.7a) to get

$$\dot{R}^2 = \frac{C}{R} + \frac{\lambda}{3} R^3 - k \tag{4.10}$$

This equation may be formally integrated by quadrature and the solutions are

$$t = \int \frac{dR}{\sqrt{\frac{C}{R} + \frac{\lambda}{3} R^2 - k}} \tag{4.11}$$

The well-known **Friedmann universe** is characterised by the choice of parameters $\lambda= 0$ and $k= 0$ so that $R(t)=R_o t^{2/3}$. The limit $t \to 0$ of the Friedmann universe corresponds to an essential singularity at which both the metric and the mass density cease to be well defined. This state of the universe is often characterised by the concept of the **big-bang**. That is, the universe is supposed to have a beginning at t= 0, when it was in a state of infinite mass density. Then it somehow exploded into existence.

There are various discussions on what the universe could have been like after the initial explosion. It is often argued that the early universe was largely made of high-intensity radiation rather than matter. The radiation dominated models can be best described by an ideal fluid that permeates the whole universe and is characterised by the equation of state $3p=\rho$. Then the continuity equation (4.8) implies $d/dt\ (\rho R^4)=0$. The radiation-dominated Friedmann universe corresponds to the choice of parameters $\lambda=0$, $k=0$ and $R(t)=R'_o t^{1/2}$. A comparison of these with the corresponding values for the matter dominated Friedmann universe shows that the expansion of the universe after the big-bang reduces the intensity of radiation much more rapidly than it reduces the mass density of matter, so that the present universe is matter-dominated as it should be. Nevertheless, a faint **cosmic background radiation** should also exist in the present epoch of the universe. This was indeed observed by A. Penzias and R. Wilson in 1965.

It may be interesting to note that the first relativistic cosmology ever considered was the **static Einstein universe.** In this case the universe is supposed to be empty and the expansion function $R(t)=R_o$. A positive definite cosmological constant $\lambda=3k/R_o^2$ is responsible for the curvature of the 3-universe. Einstein universe as a realistic model, however, had to be abandoned with the discovery of the expansion of the universe. Another empty universe model with zero density and pressure that might have a better chance of being realistic is the **de Sitter universe.** This model corresponds to the solution $\lambda>0$, $k=0$ and $R(t)=R_o\exp(\sqrt{\lambda/3}\,t)$. Due to its exponential expansion, the de Sitter universe respects the perfect cosmological principle; namely, that in addition to being spatially homogeneous and isotropic, the de Sitter universe is also temporally homogeneous. That is, it represents the same average physical aspects at all times. The fact that the de Sitter universe has no beginning may be philosophically more appealing.

4.2. Derivation of the Φ Ansatz

Using the isometries of the Robertson-Walker metric (4.1) we construct an ansatz for the Kähler field which respects these isometries. As stated in chapter 2, the Kähler field Φ will be considered as a section of a complex Clifford bundle over spacetime in which each fibre is given the structure of a Clifford algebra with product v and is related to the exterior product Λ in a canonical manner. Φ may be expressed in a local section by

$$\Phi = \sum_{p=0}^{4} \Phi^{(p)} \quad , \qquad \Phi^{p} \in \Lambda^{p}(M) \tag{4.12}$$

We next choose a local co-frame field for the Robertson-Walker metric (4.1) in terms of isotropic co-ordinates (x^1, x^2, x^3) and cosmic time t

$$e^{o} = dt$$

$$e^{j} = \frac{R(t)}{(1+\frac{k}{4}r^{2})} dx^{j} \quad , \qquad j = 1,2,3 \tag{4.13}$$

Our search for an ansatz for the Kähler field in this cosmological context will be motivated by imposing the maximally symmetric condition

$$L_{Y_i} \Phi = 0 \quad , \qquad i = 1,2,\ldots,6 \tag{4.14}$$

where L_{Y_i} denotes the Lie derivative with respect to the Killing vector Y_i[1]. Since Lie derivative is an order

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[1] It is known that an ideal decomposition of a Kähler spinor is not preserved under

preserving operation (4.14) can further be decomposed into

$$L_{Y_i}\, \Phi^{(p)} = 0 \qquad p = 0,1,..,4 \quad , \quad i = 1,2,...,6$$

Even though the space-time described by the Robertson-Walker metric (4.1) is not maximally symmetric; it contains a maximally symmetric subspace which is the 3 dimensional space. Making use of the general theorems for maximally symmetric spaces (see for example Weinberg, 1972), the solution to condition (4.14) is obtained:

$$\Phi = S(t)+V(t)e^{o}+A(t)e^{1}\Lambda e^{2}\Lambda e^{3}+P(t)e^{o}\Lambda e^{1}\Lambda e^{2}\Lambda e^{3} \qquad (4.15)$$

where S(t), V(t), A(t) and P(t) are complex functions of the cosmic time.

The Einstein-Kähler equations to be solved are

$$G_a = -K^2\tau_a(\Phi)-K^2\tau_a(\text{matter})+ \lambda{*}e_a \qquad (4.16)$$

$$(d-\delta-m)\,\Phi = 0 \qquad (4.17)$$

with the assumptions (4.13) and (4.15) the Einstein-Kähler equations give rise to the following system of coupled differential equations:

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the Lie derivative action. Nevertheless, it is possible to define a derivation that respects an ideal decomposition:

$$S_x \Phi = L_x\, \Phi + \frac{1}{4}\, \Phi\, V\, d\tilde{X} \;.$$

We would have imposed the maximally symmetric condition

$$S_{Y_i}\, \Phi = 0 \quad , \quad i = 1,2,...,6$$

if it didn't turn out here that its only possible solution is $\Phi = 0$.

$$2\ddot{R}/R+(k+\dot{R}^2)/R^2-\lambda = -p-\text{Re}\,\{V^*\dot{S}+P^*[\dot{A}+3(\dot{R}/R)A] - \frac{1}{2}m(|S|^2+|V|^2 +|P|^2+|A|^2)\}$$

$$-3(k+\dot{R}^2)/R^2+\lambda = -\rho-\text{Re}\,\{V^*\dot{S}-P^*[\dot{A}+3(\dot{R}/R)A] + \frac{1}{2}m(|S|^2-|V|^2 +|P|^2-|A|^2)\}$$

$$\dot{S}-mV=0$$

$$\dot{V}+3(\dot{R}/R)V+mS=0$$

$$\dot{A}+3(\dot{R}/R)A-mP=0$$

$$\dot{P}+mA=0 \qquad (4.18)$$

In the massless case ($m=0$) there is no coupling between the functions S,V,P and A, so that for any choice of the expansion function R(t), the last two sets of equations in (4.18) can be integrated immediately. It should also be noted that in this case Kähler stress-energy 3-forms vanish identically. On the other hand, in the massive case ($m\neq 0$), the first order equations satisfied by S and V may be combined to give the second order equation

$$\ddot{S}+3(\dot{R}/R)\dot{S}+m^2S=0 \qquad (4.19)$$

Once a solution for S is found, we determine V from the relation

$$V=m^{-1}\dot{S} \qquad (4.20)$$

The functions P and A are determined by similar equations, namely,

$$\ddot{P}+3(\dot{R}/R)\dot{P}+m^2P+0 \qquad (4.21)$$

and

$$A = -m^{-1}\dot{P} \tag{4.22}$$

In the massive case ($m \neq 0$), it proves convenient to simplify the first two equations in (4.18) by eliminating the functions V and A in favour of S and P:

$$2(\ddot{R}/R)+(k+\dot{R}^2)/R^2-\lambda = -p-(\frac{1}{2}m)(|\dot{S}|^2-|\dot{P}|^2)+(\frac{1}{2}m)(|S|^2-|P|^2)$$

$$-3(k+\dot{R}^2)/R^2+\lambda = -\rho-(\frac{1}{2}m)(|\dot{S}|^2-|\dot{P}|^2)-(\frac{1}{2}m)(|S|^2-|P|^2) \tag{4.23}$$

4.3. Solutions

In this section we state the consistent solutions of the Einstein-Kähler system of equations in some cosmological space-times (Dereli et al., 1984). We first consider the case for which $\rho = p = \lambda = 0$ and use the Minkowski metric specified by $R = R_o$ and $k = 0$. Then the massive equations (4.19)-(4.23) are solved by

$$S = a\sin(mt+\alpha) \quad , \quad P = a\,e^{i\beta}\sin(mt+\alpha) \tag{4.24}$$

where a and α are integration constants and β is an arbitrary phase difference. The expressions (4.24) yield

$$V = a\cos(mt+\alpha) \quad , \quad A = -a\,e^{i\beta}\cos(mt+\alpha) \tag{4.25}$$

In the massless case any choice of constants

$$S = S_o \quad , \quad V = V_o \quad , \quad A = A_o \quad , \quad P = P_o \tag{4.26}$$

is a solution to the equations (4.18) with m=0.

The same set of solutions (4.24)-(4.26) may also be obtained with the static Einstein metric instead of the

Minkowski metric. In this case the Universe is closed and has a constant scale factor, i.e. $R=R_o$ and $k=1$. The dynamics of the universe is driven by a cosmological term $\lambda=1/R_o^2$ together with a static uniform distribution of dust $\rho=2/R_o^2$, $p=0$.

Next we consider solutions for which the metric describes a de Sitter cosmology. In this case the expansion of the Universe is driven by a cosmological constant $\lambda > 0$. We set $\rho=p=0$ and consider $R=R_o \exp[(\lambda/3)^{1/2}t]$ and $k=0$. In this case (4.19) is nothing but the damped harmonic oscillator equation with the damping factor proportional to the Hubble parameter $\dot{R}/R$. The solutions to the massive ($m\neq 0$) equations (4.19)-(4.23) are given in terms of

$$\omega^2 = m^2 -(3/4)\lambda \quad . \tag{4.27}$$

We have for $\omega^2 > 0$,

$$S= \exp[-(3\lambda/4)^{1/2}t]\; a \sin(\omega t+\alpha)$$

$$P= \exp[-(3\lambda/4)^{1/2}t]\; a\, e^{i\beta} \sin(\omega t+\alpha)$$

$$V= (1-3\lambda/4m^2)^{1/2} \exp[-(3\lambda/4)^{1/2}t]\; a \cos(\omega t+\alpha) -(3\lambda/4m^2)^{1/2} \exp[-(3\lambda/4)^{1/2}t]\; a \sin(\omega t+\alpha)$$

$$A= -(1-3\lambda/4m^2)^{1/2} \exp[-(3\lambda/4)^{1/2}t]\; a\, e^{i\beta} \cos(\omega t+\alpha) +(3\lambda/4m^2)^{1/2} \exp[-(3\lambda/4)^{1/2}t]\; a\, e^{i\beta} \sin(\omega t+\alpha) \tag{4.28}$$

where a and α are integration constants and β is a phase difference. We cannot find oscillatory solutions when $\omega^2 < 0$. In this case

$$S= \exp[-(3\lambda/4)^{1/2}t]\; a \sinh(|\omega| t+\alpha)$$

$$P= \exp[-(3\lambda/4)^{1/2}t]\; a\, e^{i\beta} \sinh(|\omega| t+\alpha)$$

$$V= -(3\lambda/4m^2)^{1/2} \exp[-(3\lambda/4)^{1/2}t]\ a \sinh(|\omega|t+\alpha)$$
$$+(3\lambda/4m^2-1)^{1/2} \exp[-(3\lambda/4)^{1/2}t]\ a \cosh(|\omega|t+\alpha)$$

$$A= (3\lambda/4m^2)^{1/2} \exp[-(3\lambda/4)^{1/2}t]\ a\, e^{i\beta} \sinh(|\omega|t+\alpha)$$
$$-(3\lambda/4m^2-1) \exp[-(3\lambda/4)^{1/2}t]\ a\, e^{i\beta} \cosh(|\omega|t+\alpha) \qquad \textbf{(4.29)}$$

In the case of critical damping ($\omega^2=0$) we find

$$S= (a+bt) \exp[-(3\lambda/4)^{1/2}t]$$

$$P= (a+bt)\, e^{i\beta} \exp[-(3\lambda/4)^{1/2}t]$$

$$V= (b/m) \exp[-(3\lambda/4)^{1/2}t]$$
$$-(3\lambda/4m^2)^{1/2}(a+bt) \exp[-(3\lambda/4)^{1/2}t]$$

$$A= -(b/m)\, e^{i\beta} \exp[-(3\lambda/4)^{1/2}t]$$
$$+(3\lambda/4m^2)^{1/2}\, e^{i\beta}\, (a+bt) \exp[-(3\lambda/4)^{1/2}t] \qquad \textbf{(4.30)}$$

a and b are integration constants and β is a phase difference. The solutions to the massless Einstein-Kähler equations (4.18) with m=0 cannot be obtained from the solutions given above in the limit $m \to 0$. In fact m=0 solutions are

$$S=S_o$$

$$V=V_o \exp[-(3\lambda)^{1/2}t]$$

$$A=A_o \exp[-(3\lambda)^{1/2}t]$$

$$P=P_o \qquad \textbf{(4.31)}$$

Finally we consider solutions for which the metric describes a Friedmann cosmology. In this case the geometry is driven by a uniform distribution of matter. We have $\rho=\rho_o/t^2$ and $p=0$. The cosmological constant $\lambda=0$ and for

$k=0$, equation system (4.23) forces $R(t)=R_o t^{2/3}$ which is the expansion function of matter dominated Friedmann universe. The solution to the massive equations is found to be

$$S= a(mt)^{-1} \sin(mt+\alpha)$$

$$P= a\, e^{i\beta} (mt)^{-1} \sin(mt+\alpha)$$

$$V= -a(mt)^{-2} \sin(mt+\alpha)+a(mt)^{-1}\cos(mt+\alpha)$$

$$A= a\, e^{i\beta}(mt)^{-2}\sin(mt+\alpha)-a\, e^{i\beta}(mt)^{-1}\cos(mt+\alpha) \tag{4.32}$$

a and α are integration constants and β is a phase difference. For the massless ($m=0$) equations we find the solution

$$S= S_o$$

$$V= V_o t^{-2}$$

$$A= A_o t^{-2}$$

$$P= P_o \tag{4.33}$$

In chapter 2 we obtained the γ-matrix set (2-25) corresponding to the projector set (2-20). The Kähler field ansatz (4.10) in this matrix basis corresponds to the 4x4 matrix

$$\Psi = \begin{bmatrix} S-iP & 0 & 0 & V-iA \\ 0 & S+iP & V+iA & 0 \\ 0 & V-iA & S-iP & 0 \\ -V-iA & 0 & 0 & S-iP \end{bmatrix} \tag{4.34}$$

The minimal left ideals correspond to the column matrices

$$\psi_j^{(i)}= \Psi_{ji} \quad , \quad i= 1,2,3,4 \tag{4.35}$$

Maximally symmetric solutions to the Dirac equation in k=0 Robertson-Walker background with the set of γ-matrices (2.25) are found to be (Schrödinger 1940; Isham and Nelson 1974)

$$R^{-3/2}(t)\begin{bmatrix} a\ \sin(mt+\alpha) \\ b\ \sin(mt+\beta) \\ b\ \cos(mt+\beta) \\ -a\ \cos(mt+\alpha) \end{bmatrix} \qquad (4.36)$$

With an appropriate choice of the integration constants we see that (4.36) is exactly the same form as the matrices corresponding to the Minkowski space minimal ideals. On the other hand when we consider the de Sitter and Friedmann space-times the matrices corresponding to minimal ideals associated with those space-times fail to satisfy the corresponding Dirac equations.

5. CONCLUDING REMARKS

We introduced the Kähler equation and studied its symmetry properties in chapter 2. Then we showed that the Kähler equation can be completely reinterpreted in terms of the free Dirac equation. This reinterpretation remains true even when an abelian or a non-abelian gauge field is minimally coupled to the Kähler field. In arbitrary space-times, however, the Kähler equation and the Dirac equation are inequivalent in general. It is possible to give an alternative notion of a spinor based on the Kähler equation. In our opinion one of the most striking applications of the Kähler equation is found in the solution of the lattice fermion problem. In the third chapter we establish the connection between the Susskind reduction of the naive lattice Dirac equation and the lattice Kähler equation. We then show that the lattice Kähler formalism solves the fermion doubling problem in a chirally invariant manner.

Since we regard the Kähler equation as more fundamental than the Dirac equation, we examined in the fourth chapter simple classes of cosmological solutions to the coupled Einstein-Kähler system of equations. We have concentrated on space-time geometries described by a class of Robertson-Walker metrics and expressed the Kähler field in terms of a simple ansatz that respects the homogeneity and isotropy of these space-times. We constructed explicit solutions and examined their dependence on the mass parameter. We also investigated the structure of these solutions after projecting into left ideals generated by a set of projectors. This has enabled us to make a comparison between these solutions and the spinorial solutions to the Dirac equation in such cosmological backgrounds. This analysis may be expected to be of relevance in any model attempting to relate the spinor solutions of the Kähler equation to a cosmological context.

Appendix-A

DIFFERENTIAL FORMS

We present here a simple self-contained review of differential forms. We first discuss elementary concepts concerning tensors on manifolds. Then differential forms are introduced and some basic operations on them are defined. The definition of Kähler-Atiyah algebra follows a discussion of Clifford and Grassmann algebras in arbitrary differentiable manifolds. The corresponding algebras in the cotangent bundle over space-time are used in the text. A brief introduction to the theory of connections in principal bundles is also included. There are several approaches to the subject which are discussed in many text-books. This appendix depends heavily on Dereli's lecture notes (1982).

A-1. Elementary Concepts

We accept the meaning of differentiable manifold, chart and mapping between two manifolds to be intuitively clear. The classes of differentiability of all manifolds, objects and mappings are assumed to be C^{∞} unless stated otherwise.

Let M be a differentiable manifold of dimension n. A **curve** through a point P of M is a differentiable map γ from an open subset of R into M. The curve γ may be given in parametrised form by a function

$$f(t)=f(x^1(t),...,x^n(t)) \qquad \textbf{(A.1)}$$

where t usually stands for time. The **tangent** to the curve γ at point P is defined by the total derivative

$$\left.\frac{df}{dt}\right|_{t=0} \qquad \textbf{(A.2)}$$

The notion of the tangent to a curve is an intrinsic concept. Nevertheless, it is instructive to write the above definition in a coordinate chart,

$$\frac{df}{dt} = \sum_{i=1}^{n} \left(\frac{dx^i}{dt}\right) \frac{\partial f}{\partial x^i} \qquad \textbf{(A.3)}$$

It is seen from this expression that the quantities dx^i/dt act as components of an n-dimensional vector and $\partial / \partial x^i$ act as the basis vectors. The set of tangent vectors to all the curves passing through P forms a vector space and is called the **tangent space** of M at point P. It will be denoted $T_p(M)$. Elements of $T_p(M)$ are called **vectors**. In a coordinate chart (x^i), a basis of $T_p(M)$ is given by $\partial / \partial x^i$ which is called the natural basis or the coordinate basis. Of course any n linearly independent set of vectors $\{E_i\}$ in $T_p(M)$ can also be used as a basis. The basis vectors $\{\partial / \partial x^i\}$ and $\{E_i\}$ are, in general, related by GL(n,R) transformations. The transformations which preserve the lengths of basis vectors as well as the angles between them, form the orthogonal group SO(n). The vector space dual to $T_p(M)$ is known as the **cotangent space** of M at point P and is denoted by $T_p^*(M)$. The elements of $T_p^*(M)$ are linear functionals acting on vectors. They are called **co-vectors** or 1-forms. The coordinate basis $\{\partial / \partial x^i\}$ of $T_p(M)$ induces a unique basis for $T_p^*(M)$, denoted $\{dx^i\}$, through the duality relations

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j \qquad \textbf{(A.4)}$$

A **vector field** V on a manifold is a function which assigns to each $P \in M$ a vector V(P). On a coordinate chart

$$V(P) = V^i(P) \frac{\partial}{\partial x^i}(P) \qquad \textbf{(A.5)}$$

or in terms of $\{x^i\}$,

$$V=V^i(x)\frac{\partial}{\partial x^i} \tag{A.6}$$

where V^i's are called **contravariant** coordinate components. A co-vector field can similarly be defined in terms of $\{dx^i\}$ and **covariant** components ω_i:

$$\omega=\omega_i(x)dx^i \tag{A.7}$$

A tensor R of type (r,s) on M is a multilinear map

$$R:\underbrace{T^*(M)x\ldots xT^*(M)}_{r\ \text{times}}x\underbrace{T(M)x\ldots xT(M)}_{s\ \text{times}}\rightarrow R \tag{A.8}$$

If r=0, R is called a covariant tensor of rank s. Similarly if s=0 then R is called a contravariant tensor of rank r. When both r and s are non-zero, R is said to be a mixed tensor. The set of all tensors of type (r,s) on M is denoted by $T^{(r,s)}(T_p(M))$. A **tensor field** $T^{(r,s)}(M)$ of type (r,s) on a manifold M is an assignment of an element T of $T^{(r,s)}(T_p(M))$ to each point P of M. The concept of tensor fields is a natural generalization of vector and co-vector fields on a manifold. With the appropriate definitions of addition and multiplication by elements of R, the set of tensors of type (r,s) forms a vector space of dimension r+s.

The existence of a symmetric 2^{nd} rank covariant **metric tensor** g on M will be assumed. The coordinate components of g are

$$g_{ij}=g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) \tag{A.9}$$

We may find the inverse of the symmetric nxn matrix $(g)_{ij}$, provided det $(g_{ij})\neq 0$, and denote it by $(g)^{ij}$. That is, $g^{ij}g_{jk}=\delta^i_k$. Then the metric g and its inverse g^{-1} establish an isomorphism between the vectors and 1-forms. Namely; given the natural components V^i of a vector field V, it is possible to find a corresponding co-vector field $\tilde{V}$ whose natural components are

$$V_i = g_{ij} V^j \tag{A.10}$$

Conversely, given the components ω_i of a 1-form, it is possible to construct a vector with components

$$\omega^i = g^{ij} \omega_j \tag{A.11}$$

A set of n linearly independent basis vectors $\{E_i\}$ can be orthonormalized with respect to the given metric g. The set of dual co-vectors $\{e^i\}$ will be called the **orthonormal basis 1-forms.** Then, the metric tensor may also be written as

$$g = g_{ij} e^i \otimes e^j \tag{A.12}$$

where $\otimes$ denotes the symmetric tensor product.

An **exterior form** (or differential form) of degree p is a totally antisymmetric covariant tensor field of rank p. The set of all totally antisymmetric covariant tensors of rank p forms a $n!/\{p!(n-p)!\}$ dimensional space and is denoted $\Lambda^p(T(M))$ or simply Λ^p. By definition $\Lambda^o = R$ and $\Lambda^1 = T^*(M)$. We note that Λ^p contains only the zero tensor when $p > \dim \Lambda^1$.

A-2. Basic Operations on Exterior Forms

If $\omega \varepsilon \Lambda^p$ and $\nu \varepsilon \Lambda^q$ then the exterior product (or wedge product) of ω and ν is an element of Λ^{p+q}, denoted by and defined as the antisymmetric tensor product

$$\omega \Lambda \nu \equiv (\omega \otimes \nu)_A \tag{A.13}$$

where A stands for antisymmetrization.

Wedge product turns the 2^n dimensional vector space $\Lambda(M)=\sum_{r=0}^{n} \Lambda^r(T(M))$ into an associative algebra over R. This algebra is called the **exterior algebra** or **Grassmann algebra.** In a coordinate chart (x^i), a p-form ω may be expanded according to the formula

$$\omega=\frac{1}{p!}\,\omega_{i_1\ldots i_p}dx^{i_1}\Lambda\ldots\Lambda dx^{i_p} \tag{A.14}$$

Given a p-form ω and a q-form ν which has a similar expansion, it can be easily shown that

$$\omega\Lambda\nu=(-1)^{pq}\,\nu\Lambda\omega \tag{A.15}$$

A manifold M of dimension n is called to be **orientable** if it is possible to define an n-form ε which is not zero at any point of M. In a coordinate chart (x^i)

$$\varepsilon=dx^1\Lambda\ldots\Lambda dx^n=\frac{1}{p!}\,\varepsilon_{i_1\ldots i_n}dx^{i_1}\Lambda\ldots\Lambda dx^{i_n} \tag{A.16}$$

Here $\varepsilon_{i_1\ldots i_n}$ are the coordinate components of a totally antisymmetric covariant tensor of rank n and we choose $\varepsilon_{12\ldots n}=1$. For orientable manifolds, provided a symmetric covariant 2^{nd} rank metric tensor is defined on it, ε establishes a canonical isomorphism

$$*:\Lambda^p\rightarrow\Lambda^{n-p} \tag{A.17}$$

called the **Hodge map.** The Hodge map assigns the (n-p) form

$$*\omega=\frac{1}{p!(n-p)!}\,\varepsilon^{i_1\ldots i_p}{}_{i_{p+1}\ldots i_n}\,\omega_{i_1\ldots i_p}dx^{i_{p+1}}\Lambda\ldots\Lambda dx^{i_n} \tag{A.18}$$

to the p-form ω given by (A.14). It is easy to prove that

$$*o*=(-1)^{p(n-p)+(n-t)/2}\,\mathbb{1} \tag{A.19}$$

where $\mathbb{1}$ is the identity map and t is the signature of the metric.

By generalizing the product of a vector and a co-vector, a new operation called the **interior product**, can be defined. The interior product with respect to a vector $X \in T_p(M)$ is the linear map

$$i_X : \Lambda^p \rightarrow \Lambda^{p-1} \tag{A.20}$$

such that in a coordinate chart

$$i_X \omega = \frac{1}{(p-1)!} X^i \omega_{i_1 \ldots i_p} dx^{i_2} \Lambda \ldots \Lambda dx^{i_p} \tag{A.21}$$

where ω is a p-form. Interior product has the following properties:

$$i_X{}^2 = 0 \tag{A.22a}$$

$$i_X (\omega \Lambda \sigma) = i_X \omega \Lambda \sigma + (-1)^p \omega \Lambda i_X \sigma \tag{A.22b}$$

$$i_X (\omega + \sigma) = i_X \omega + i_X \sigma \tag{A.22c}$$

$$i_X f = 0 \tag{A.22d}$$

where $\omega \epsilon \Lambda^p$, $\sigma \epsilon \Lambda^q$ and $f \epsilon \Lambda^o$.

The differentiation of a function has a unique generalization into a derivative operation acting on exterior forms. It is known as the **exterior derivative** which is a linear map

$$d : \Lambda^p \rightarrow \Lambda^{p+1} \tag{A.23}$$

such that d takes a p-form ω given by the expression (A.14) into the (p+1) form

$$d\omega = \frac{1}{p!} \left(\frac{\partial \omega_{i_1 \ldots i_p}}{\partial x^i} dx^i \Lambda \right) dx^{i_1} \Lambda \ldots \Lambda dx^{i_p} \tag{A.24}$$

Exterior derivative operation satisfies the similar identities satisfied by i_X except the identity (A.22d) which is replaced by

$$d\omega=0 \quad , \quad \omega\varepsilon\Lambda^n \qquad \textbf{(A.25)}$$

where n is the dimension of the underlying manifold. Equipped with the operations d and i_X, we can define a linear operator L_X, called the **Lie derivative**,

$$L_X: \Lambda^p \to \Lambda^p$$

Note that L_X does not change the degree of a form. We define

$$L_X\omega=di_X\omega+i_Xd\omega \quad , \quad \omega\varepsilon\Lambda^p \qquad \textbf{(A.26)}$$

L_X commutes with d, and also satisfies the following identity

$$L_X(\omega\Lambda\nu)=L_X\omega\Lambda\nu+\omega\Lambda L_X\nu \quad , \omega\varepsilon\Lambda^p \ , \ \nu\varepsilon\Lambda^q \qquad \textbf{(A.27)}$$

The above definition of the Lie derivative applies to p-forms. Acting on vector fields, the Lie derivative is defined by the bracket action

$$L_XU=[X,U]=XU-UX \qquad X,U \ \varepsilon \ T_p(M) \qquad \textbf{(A.28)}$$

Lie bracket action is a derivation. Furthermore it satisfies the Jacobi identity

$$[[U,V],W]+[[W,U],V]+[[V,W],U]=0 \qquad \textbf{(A.29)}$$

Lie derivative of tensors of arbitrary rank with respect to a vector field X is defined recursively form the above rules.

Another degree preserving operator is the **Laplace-Beltrami** operator which generalizes the ordinary Laplacian to arbitrary metrics and manifolds:

$$\Box : \Lambda^p \rightarrow \Lambda^p \qquad \text{(A.30)}$$

defined by

$$-\Box = d{*}d{*}+{*}d{*}d \qquad \text{(A.31)}$$

A-3. Kähler-Atiyah Algebra

In the previous section we already noted that the space of differential forms over a manifold M is an algebra with respect to the antisymmetrized tensor product Λ. This algebra is called the Grassmann or exterior algebra over M and denoted by the pair $(\Lambda(M),\Lambda)$. We also know that the space of differential forms over M becomes an associative algebra with respect to a different type of multiplication called the **Clifford multiplication.** The Clifford multiplication of two inhomogeneous elements α and β of $\Lambda(M)$ is defined by

$$\alpha \vee \beta = \sum_p \frac{(-1)^{[p/2]}}{p!} g_{i_1 j_1} \cdots g_{i_p j_p} A^p(i_{E_{i_1}} \cdots i_{E_{i_p}} \alpha) \Lambda (i_{E_{j_1}} \cdots i_{E_{j_p}} \beta) \qquad \text{(A.32)}$$

where E_i is the basis for tangent space, $[p/2]$ means the integer part of the expression inside the brackets and A is an automorphism which sends a p form ω to $(-1)^p\omega$. We will denote the **Clifford algebra** by the pair $(\Lambda(M),\vee)$.

Before establishing the correspondence between the Clifford and Grassmann algebras, we provide the following definition from Porteous (1969): A subset S of an algebra A is said to **generate A as an algebra** if each element of A is expressible as a linear combination of a finite sequence of elements of A each of which is the product of a finite sequence of elements of S. We therefore see that the identity element $\mathbb{1}$ together with the orthonormal basis 1-forms $\{e^i\}$ generate both the exterior algebra and the Clifford algebra with the appropriate multiplication rules. The orthonormal basis 1-forms obey the anti-commutation relations

$$e^i \wedge e^j + e^j \wedge e^i = 0 \qquad \text{(A.33)}$$

and

$$e^i \vee e^j + e^j \vee e^i = 2g^{ij} \qquad \text{(A.34)}$$

with respect to the wedge product and the Clifford product, respectively. It follows from these rules that in the exterior algebra, square of any generating element is zero; whereas in the Clifford algebra, square of a generating element is either 1 or -1.

We will now define a new algebraic structure over the space of differential forms containing the Grassmann algebra and the Clifford algebra as substructures. For an exterior algebra we define a ∇ product on $\{e^i\}$ such that

$$e^i \nabla e^j = e^i \wedge e^j + I_{E_i} e^j \qquad \text{(A.35)}$$

Then, interchanging e^i and e^j and summing give

$$e^i \nabla e^j + e^j \nabla e^i = 2g^{ij} \qquad \text{(A.36)}$$

where $i_{E_i} e^j = g^{ij}$ has been used. This relation is the defining relation of a Clifford algebra. Since two Clifford algebras over the same vector space are isomorphic (universality of Clifford algebras), the ∇-generated algebra is the Clifford algebra $(\Lambda(M), \vee)$. Conversely for a Clifford algebra we define a Δ product on $\{e^i\}$ such that

$$e^i \Delta e^j = e^i \vee e^j - i_{E_i} e^j \qquad \text{(A.37)}$$

Repeating the same manipulations we obtain

$$e^i \Delta e^j + e^j \Delta e^i = 0 \qquad \text{(A.38)}$$

which is the defining relation of the exterior algebra.

On the space of differential forms not only can we impose the structure of an exterior algebra by means of $\wedge$ and i_X but also the structure of a Clifford algebra. Any of the two multiplications $\wedge$ and $\vee$ can be reduced to the other. Consequently we define the **Kähler-Atiyah algebra** as the quadruple $(\Lambda(M),\wedge,\vee,i_X)$ such that $e^i \vee e^j = e^i \wedge e^j + i_{E_i} e^j$ for any $e^i, e^j \in \Lambda^1$. Obviously both the Grassmann and Clifford algebras are substructures of the Kâhler-Atiyah algebra. The above presentation of Kähler-Atiyah algebra differs from Graf's definition as we do not distinguish between the exterior algebra and the Grassmann algebra in accordance with the definitions given by Porteous.

A-4. Fibre Bundles and Connections

A **fibre bundle** $E(M,F,\pi)$ is a manifold which is locally a direct product of a given manifold M, called the base manifold, and another manifold F called the fibre. $\pi : E \rightarrow M$ is the projection map which projects each fibre onto the point of M on which F is defined. If M is covered by a set of local coordinate neighbourhoods $\{U_i\}$, then the bundle E is topologically described over each neighbourhood U_i by the product $U_i x F$.

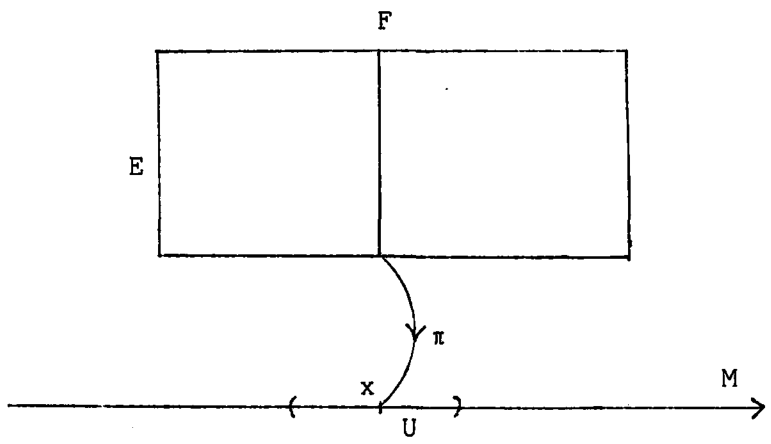

Figure A-1. Drawing picture of a fibre bundle.

To completely specify the bundle we must be given a set of transition functions $\{\Phi_{ij}\}$ which tell us how the fiber manifolds match up in the overlap between two neighbourhoods $U_i \cap U_j$, that is

$$\Phi_{ij} : F\big|_{U_i} \rightarrow F\big|_{U_j} \quad \text{in} \quad U_i \cap U_j \tag{A.39}$$

Although the local topology of the bundle is trivial, its global topology may be complicated due to relative twisting of neighbouring fibers. Global topology is determined by the transition functions and if all the transition functions are equal to unity then the bundle is called trivial. A **cross section** σ of a fiber bundle is a rule which assigns a preferred point on each fiber to each point x of the base manifold M such that $\pi^{-1} \circ \sigma = \mathbb{1}$.

A **vector bundle** is a fiber bundle whose fiber F is a linear vector space and whose transition functions belong to the general linear group of F. A **principal bundle** is a fiber bundle whose fiber is a Lie group G; the transition functions of a principal bundle belong to G and act on G by left (or right) multiplication.

Let us consider a vector bundle E, with a k dimensional real fiber $F=\mathbb{R}^k$ over an n-dimensional base space M; then k+n is called the **bundle dimension**. Transition functions of a vector bundle belong to $GL(k,\mathbb{R})$ and since $GL(k,\mathbb{R})$ preserves the usual operations of addition and scalar multiplication on a vector space, the fibers of E inherit the structure of a vector space. It is also possible to introduce a vector product on this vector space which turns the fibers into algebras. We can therefore think of a vector bundle as a family of vector spaces (or algebras) which are parametrised by the points of the base space M. If we replace $\mathbb{R}^k$ by $\mathbb{C}^k$ and $GL(k,\mathbb{R})$ by $GL(k,\mathbb{C})$ we obtain complex vector bundles.

We now briefly explain some vector bundles which have been mentioned in chapter 2. We let the **tangent bundle** $T(M)$ (the **cotangent bundle** $T^*(M)$) be the real vector bundle whose fiber at point $x \varepsilon M$ is given by the tangent space $T_x(M)$ (the cotangent space $T_x^*(M)$). A **frame** at $x \varepsilon M$ is a linear isomorphism

$$u:\mathbb{R}^n \to T_x(M)$$

Suppose $L_x(M)$ is the set of all frames at x then $L(M)$ denotes a **frame bundle.** By exterior algebra bundle, Clifford bundle and Kahler-Atiyah bundle we mean the bundles whose fibers are exterior algebras, Clifford algebras and Kahler-Atiyah algebras, respectively.

We now introduce connections on fibre bundles. Connections are of direct physical relevance and gives a rule for performing parellel translation of geometrical objects along some curve in base space. It also enables one to define a covariant derivative. Maxwell's theory of electromagnetism and Yang-Mills theories are essentially theories of connections on principal fibre bundles with a given gauge group as the fibre. Einstein's theory of gravitation deals with the Levi-Civita connection on the frame bundle over the space-time manifold. Since we are mainly interested in vector bundles, we define a **connection** in a vector bundle as a rule which assigns to each local trivialisation map $T_u:\pi^{-1}(U) \to F x U$ a $GL(k,\mathbb{R})$ valued 1-form ω_u on U. If T_v is another trivialisation and if g_{uv} is the transition function from T_u to T_v, then we require

$$\omega_v = g_{uv}^{-1} \omega_u g_{uv} + g_{uv}^{-1} dg_{uv} \qquad \textbf{(A.40)}$$

Connections may be defined in several different equivalent ways (Kobayashi and Nomizu, 1963): A connection on a vector bundle E, assigns to each $x \varepsilon U$ a subspace $H_x(M) C T_x(M)$

such that if $V_x(M) \subset T_x(M)$ is the set of all tangent vectors tangent to the fibre through x, then

$$T_x(M)=V_x(M) \oplus H_x(M) \qquad \textbf{(A.41)}$$

$V_x(M)$ is called the vertical subspace and $H_x(M)$ is called the horizontal subspace of $T_x(M)$. Any vector field Y on E may be uniquely decomposed into its vertical and horizontal components. Y is said to be horizontal if $Y_x \in H_x(M)$, $\forall x \in U$; and vertical if $Y_x \in V_x(M)$, $\forall x \in U$.

Another way of introducing the concept of a connection is through the use of covariant derivatives. Given a vector field X, and a section σ of a bundle E, we define an operator

$$\nabla_X : \sigma \to \nabla_X \sigma \qquad \textbf{(A.42)}$$

where

$$(\nabla_X \sigma)(x)=V(d\sigma(x))$$

$\nabla_X \sigma$ is called the **covariant derivative** of σ w.r.t. X and is a measure of how much σ fails to be parallel. σ is horizontal iff $\nabla_X \sigma=0$ for any X. Covariant derivative satisfies the following useful properties:

$$\nabla_{X+Y}\sigma=\nabla_X\sigma+\nabla_Y\sigma \qquad \textbf{(A.43a)}$$

$$\nabla_X(\sigma+\tau)=\nabla_X\sigma+\nabla_X\tau \qquad \textbf{(A.43b)}$$

$$\nabla_{fX}\sigma=f\nabla_X\sigma \qquad \textbf{(A.43c)}$$

$$\nabla_X(f\sigma)=f\nabla_X\sigma+(X.f)\sigma \qquad \textbf{(A.43d)}$$

where X and Y are two vector fields, σ and τ are two sections, f is a zero form.

The **covariant exterior derivative** associated with a connection 1-form ω in a vector bundle E is the linear operator:

$$D_\omega \equiv d+\omega\wedge \quad \text{(A.44)}$$

It should be noted that in general $D_\omega^2 \neq 0$, in contrast to $d^2=0$. We define the **curvature 2-form** of the connection ω

$$\Omega = D_\omega \omega \quad . \quad \text{(A.45)}$$

The **canonical form** (or solder form) on the frame bundle $L(M)$ is the R^n valued 1-form θ defined by

$$\theta(X)=u^{-1}(\pi(X)) \quad \text{(A.46)}$$

where X is a vector field, u is a frame and π is the projection map. Canonical form is horizontal and it is an intrinsic quantity associated with the frame bundle, that is, independent of the choice of a connection. The **torsion 2-form** of a linear connection ω

$$\Theta = D_\omega \theta \quad \text{(A.47)}$$

To summarise, let ω be a linear connection on $L(M)$ with curvature form Ω and torsion form Θ. Then we have the structure equations

$$\Omega = d\omega + \omega\wedge\omega \quad \text{(A.48a)}$$

and

$$\Theta = d\theta + \omega\wedge\theta \quad \text{(A.48b)}$$

The Bianchi identities

$$D_\omega \Omega = 0 \quad \text{(A.49a)}$$

and

$$D_\omega \Theta = \Omega\wedge\theta \quad \text{(A.49b)}$$

follow from the structure equations by taking the exterior derivative of both sides. By means of a pull back $\sigma^*: L(M) \to M$, the above structure equations can be equivalently formulated in terms of curvature and torsion 2-forms describing the geometry of M (see for example Benn et al., 1982). We then have

$$e^a = \sigma * \theta^a \tag{A.50a}$$

$$\omega^a{}_b = \sigma * \omega^a{}_b \tag{A.50b}$$

$$R^a{}_b = \sigma * \Omega^a{}_b \tag{A.50c}$$

$$T^a = \sigma * \Theta^a \tag{A.50d}$$

as the forms that enter the theories defined on M. The pull back σ^* commutes with d and the structure equations appear as the usual definitions of R and T on M. That is,

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \Lambda \omega^c{}_b \tag{A.51a}$$

$$T^a = de^a + \omega^a{}_b \Lambda e^b \tag{A.51b}$$

and the Bianchi identities follow from above:

$$D_\omega R^a{}_b = dR^a{}_b + \omega^a{}_c \Lambda R^c{}_b - R^a{}_c \Lambda \omega^c{}_b = 0 \tag{A.52a}$$

$$D_\omega T^a = dT^a + \omega^a{}_b \Lambda T^b = R^a{}_b \Lambda e^b \tag{A.52b}$$

The Einstein 3-forms on M are defined in terms of R

$$G_a \equiv \frac{1}{2} R^{bc} \Lambda *(e_a \Lambda e_b \Lambda e_c) = G_{ab} * e^b \tag{A.53}$$

In theories of gravitation Christofell-Levi-Civita spin connection 1-forms ω_{ab}, defined on the frame bundle L(M) and pulled back to space-time manifold M, are frequently used. Suppose the metric of the space-time is given by

$$g=\eta_{ab}e^{a}\otimes e^{b} \qquad \text{(A.54)}$$

then ω_{ab} is determined from the following conditions:

$$\omega_{ab}=-\omega_{ba} \qquad \forall\ a,b. \qquad \text{(A.55a)}$$

$$de^{a}+\omega^{a}{}_{b}\wedge e^{b}=0 \qquad \forall a. \qquad \text{(A.55b)}$$

The first condition is known as the metricity condition and follows from the covariant constancy of the metric. The second is the mathematical statement of the fact that the space-time manifold is Riemannian (torsionless).

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