

**UNIVERSAL MODULES OF DIFFERENTIAL
OPERATORS**

**DİFERANSİYEL OPERATÖRLERİN EVRENSEL
MODÜLLERİ**

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ABSTRACT

UNIVERSAL MODULES OF DIFFERENTIAL OPERATORS

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This thesis is concerned with universal differential operator modules of order n . Let R be a commutative k -algebra where k is an algebraically closed field of characteristic zero. Suppose that $J_n(R)$ is the universal module of differential operators of order n with the universal differential operator Δ_n and $\Omega_n(R)$ is the universal module of derivations of order n with the universal operator δ_n . Firstly, we obtain the following result:

Let m and n be positive integers such that $m < n$. We have the following short exact sequence of R -modules:

$$0 \rightarrow \ker\theta \rightarrow \Omega_n(R) \xrightarrow{\theta} \Omega_m(R) \rightarrow 0.$$

Moreover, $\ker\theta$ is generated by the set

$$\{\delta_n(r_0 \dots r_m) + \sum_{\substack{T \neq \emptyset \\ T \subseteq \{0, \dots, m\}}} (-1)^{|T|} r_T \delta_n(r_{T'})\}$$

where $r_i \in R$ for $i = 0, \dots, m$; T' is the complement of the set T in the set $\{0, \dots, m\}$ and

$$r_T = \prod_{\substack{k \in T \\ T \subseteq \{0, \dots, m\}}} r_k.$$

Next, we consider the map

$$J_n(R) \xrightarrow{\alpha} J_{n-1}(\Omega_1(R))$$

and obtain some results on $\ker\alpha$ and $\operatorname{coker}\alpha$ where R is a domain of dimension one or two.

Then we focus on the behavior of the Betti series of the universal module of derivations. Firstly, we showed that the Betti series of $\Omega_2(R_m)$ is rational under some conditions where R is the coordinate ring of an affine irreducible curve represented by $\frac{k[x_1, x_2, \dots, x_s]}{(f)}$ and m is a maximal ideal of R . Next, we generalize this result for the universal module of n th order derivations and we proved the following theorem:

Let $k[x_1, x_2, \dots, x_s]$ be a polynomial algebra and m be a maximal ideal of $k[x_1, x_2, \dots, x_s]$ containing an irreducible element f . Let

$$d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) \in m\Omega_n(k[x_1, x_2, \dots, x_s])$$

for $0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_s \leq n - 1$. Assume that $R = \frac{k[x_1, x_2, \dots, x_s]}{(f)}$ is not a regular ring at $\bar{m} = m/(f)$. Then $B(\Omega_n(R_{\bar{m}}), t)$ is a rational function.

Furthermore, we showed that under some conditions the Betti series of

$$\Omega_n((k[U \times A_k^t])_{\bar{m}})$$

is a rational function where $k[U \times A_k^t]$ is the coordinate ring of the product of U and A_k^t , $\bar{m} = m/(f)$ and m is a maximal ideal of $k[x_1, \dots, x_s, y_1, \dots, y_t]$ containing the irreducible element f .

Key words: Differential Operator, Universal Module, Minimal Resolution, Betti Series

ÖZET

DİFERANSİYEL OPERATÖRLERİN EVRENSEL MODÜLLERİ

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R ve k birimli ve değişmeli halkalar olmak üzere R bir k -cebiri olsun. F bir R -modül olmak üzere $D \in Hom_k(R, F)$ dönüşümüne k üzerinde n . mertebeden türev operatörü denir eğer R 'den aldığımız keyfi $n + 1$ tane eleman $\{x_0, \dots, x_n\}$ için aşağıdaki koşul sağlanırsa:

$$D(x_0 \dots x_n) = \sum_{s=1}^n (-1)^{s+1} \sum_{i_1 < \dots < i_s} x_{i_1} \dots x_{i_s} D(x_0 \dots \widehat{x_{i_1}} \dots \widehat{x_{i_s}} \dots x_n).$$

Burada $\widehat{x_i} = 1$ olarak alınacaktır [1].

Yukarıdaki tanım göz önüne alınırsa, 1. mertebeden türev operatörünün R 'den F 'ye bilinen türev olduğu kolaylıkla görülebilir.

q . mertebeden türevler için evrensel modül inşa etme fikri [2, Nakai]'ye kadar uzanır. Nakai, bu çalışmasında sadece 1. mertebeden türev operatörleri için evrensel modülü tanımlamış ve bunların varlığını ispatlamıştır. Yüksek mertebeden türevler için evrensel modülün, $\Omega_k^q(R)$, varlığı ilk defa Osborn [3] tarafından ispatlanmıştır. Nakai [1] ve Osborn [3] tarafından elde edilen özellikleri göz önüne alırsak, $\Omega_k^q(R)$ aşağıdaki özellikleri sağlar:

- (i) q . mertebeden bir türev operatörü (kanonik) $\delta_q : R \rightarrow \Omega_k^q(R)$ vardır,
- (ii) $\Omega_k^q(R)$, R -modül olarak $\{\delta_q(r) : r \in R\}$ kümesi tarafından üretilir,

(iii) F herhangi bir R -modül ve $D : R \rightarrow F$, q . mertebeden herhangi bir türev operatörü olmak üzere, tek bir R -modül homomorfizması

$$\alpha : \Omega_k^q(R) \rightarrow F$$

vardır ve $\alpha \delta_q = D$ sağlanır.

Evrensel diferansiyel operatör modülleri bir halkanın cebirsel yapısını anlamak için kullanılan en etkili araçlardan biridir. Böylelikle, cebirlerle ilgili problemler modül teoriye aktarılmış olunur. Örneğin, aşağıdaki sonuç yardımıyla regüler halkaları karakterize edebiliriz [5, Theo. 15.2.9]:

A karakteristiği sıfır olan bir cisim üzerinde afin tamlık bölgesi ve B , A 'nın bir maksimal idealindeki lokalizasyonu olmak üzere I , B 'nin maksimal ideali olsun. Bu durumda aşağıda verilen ifadeler denktir:

- (i) $\{b_1, \dots, b_n\}$ kümesi I 'nin minimal üreteç kümesi olmak üzere $\Omega_k^1(B)$, B üzerinde rankı n olan bir serbest modüldür ve tabanı $\{db_1, \dots, db_n\}$ kümesi olur.
- (ii) $\Omega_k^1(B)$, B üzerinde bir serbest modüldür .
- (iii) B regülerdir.

Dolayısıyla, yukarıdaki ifadenin bir sonucu olarak söyleyebiliriz ki, A 'nın regüler olması için gerek ve yeter koşul $\Omega_k^1(A)$ 'nın projektif olmasıdır. Buna ek olarak, literatürde regüler halkaların karakterizasyonu ile ilgili Nakai ve Zariski-Lipman tarafından ortaya atılan ve hala açık olan iki önemli problem vardır. Nakai'nin ortaya attığı problem Mount ve Villamayor tarafından [6] aşağıdaki şekilde ifade edilmiştir:

Nakai Sanısı: R karakteristiği sıfır olan bir cisim üzerinde afin bir halka olsun. $Der_k(R)$ ile R üzerinde tanımlı yüksek mertebeden türevlerin cebirini, $der_k(R)$ ile $Der_k(R)$ 'nin 1. mertebeden türevlerle üretilen alt cebirini gösterelim. Bu durumda, $der_k(R) = Der_k(R)$ olmasıyla R 'nin regüler olması denktir.

Zariski-Lipman Sanısı: $Der_k(R)$ serbest R -modül ise R regülerdir.

Bazı özel durumlarda, bu ifadeler ispatlanmıştır. Bu problemlerden yola çıkarak sorulabilecek en doğal sorulardan bir tanesi, bu iki iddianın arasında bir bağlantı olup olmadığıdır. Bu soru, 1978 yılında Becker [8] tarafından cevaplanmıştır. Becker, Nakai'nin sanısının Zariski-Lipman'nın sanısını gerektirdiğini ispatlamıştır.

1996 yılında, Erdoğan [9] tarafından n . mertebeden evrensel diferansiyel operatör modüllerinin projektif boyutları ile ilgili önemli sonuçlar elde edilmiştir:

Teorem S bir afin tamlık bölgesi olmak üzere $S = k[x_1, \dots, x_s]/(f)$ biçiminde temsil edilsin. Bu durumda $pdJ_n(S) \leq 1$ sağlanır.

1999 yılında, Çimen ve Erdoğan tarafından [10] n . mertebeden evrensel diferansiyel operatör modüllerinin projektif boyutları ile ilgili aşağıdaki teorem ispatlanmıştır:

Teorem U indirgenmiş bir hiperyüzey ve A_k^t bir afin t -uzayı olsun. $k[U \times A_k^t]$, U ve A_k^t 'nin çarpımının koordinat halkası olmak üzere

$$pdJ_n(k[U \times A_k^t]) \leq 1$$

sağlanır.

Bu tezin ilk kısmında, diferansiyel operatörlerin evrensel modüllerinin tarihsel gelişimi incelenerek bu alanda elde edilen önemli sonuçlara yer verildi. İkinci kısmın amacı ise diferansiyel operatörler ve bunların evrensel modülleriyle ilgili ilerideki çalışmalarımıza temel olacak teoriyi oluşturmaktır. Bu kısımda öncelikle n . mertebeden diferansiyel operatörün ve n . mertebeden türev operatörünün tanımı yapılarak, bunlar için evrensel modüllerin varlığı ve tekliği ispatlandı. Daha sonra bölüm halkaları, lokal halkalar ve regüler halkalar gibi özel durumlarda evrensel modüllerin özellikleri incelendi. Üçüncü kısımda ise, evrensel modüllerin projektif boyutları ile ilgili bazı sonuçlara örneklerle birlikte yer verildi. Ayrıca, aşağıdaki sonuç elde edildi:

Teorem R bir k -cebir olmak üzere m ve n , $m < n$ olacak şekilde pozitif tam sayılar olsun. δ_n ve δ_m , sırasıyla R 'nin n . ve m . mertebeden evrensel türev operatörleri olsunlar. Bu durumda elimizde R -modüllerin

$$0 \rightarrow \zeta ek\theta \rightarrow \Omega_n(R) \xrightarrow{\theta} \Omega_m(R) \rightarrow 0$$

tam dizisi vardır. Ayrıca, $i = 0, \dots, m$ için $r_i \in R$ ve T' , T 'nin $\{0, \dots, m\}$ kümesi içindeki tümleyeni olmak üzere

$$r_T = \prod_{\substack{k \in T \\ T \subseteq \{0, \dots, m\}}} r_k$$

verilsin. Bu durumda, $\zeta ek\theta$

$$\{\delta_n(r_0 \dots r_m) + \sum_{\substack{T \neq \emptyset \\ T \subseteq \{0, \dots, m\}}} (-1)^{|T|} r_T \delta_n(r_{T'})\}$$

kümesi tarafından üretilir.

Buna ek olarak, R bir tamlık bölgesi olmak üzere R 'nin boyutunun bir veya iki olduğu durumlarda

$$J_n(R) \xrightarrow{\alpha} J_{n-1}(\Omega_1(R))$$

dönüşümünün çekirdeği ve eşçekirdeği (cokernel) ile ilgili bazı sonuçlar elde edildi. (R, m) bir lokal halka olsun. $\Omega_n(R)$ 'nin Betti serisi, $n \geq 1$ olmak üzere

$$B(\Omega_n(R), t) = \sum_{i \geq 0} \text{boy}_{R/m} \text{Ext}^i(\Omega_n(R), \frac{R}{m}) t^i$$

olarak tanımlanır.

Dördüncü bölümde, 2. mertebeden evrensel türev modülünün Betti serisinin rasyonelliği incelendi ve elde edilen bu sonuçlar n . mertebeden evrensel türev modülüne genellenerek aşağıdaki teoremler ispatlandı:

Teorem $k[x_1, x_2, \dots, x_s]$, k üzerinde bir polinomlar cebiri ve m , $k[x_1, x_2, \dots, x_s]$ 'nin indirgenemez f elemanını içeren bir maksimal ideali olsun. $0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_s \leq n-1$ olmak üzere

$$d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) \in m \Omega_n(k[x_1, x_2, \dots, x_s])$$

sağlandığını kabul edelim ve $R = \frac{k[x_1, x_2, \dots, x_s]}{(f)}$, $\bar{m} = m/(f)$ 'de regüler olmasın.

Bu durumda, $B(\Omega_n(R_{\bar{m}}), t)$ rasyoneldir.

A_k^t bir afin t -uzayı olsun. $k[A_k^t]$, A_k^t 'nin koordinat halkasını göstermek üzere $k[y_1, \dots, y_t]$ biçimindedir. U indirgenmiş bir hiperyüzey olsun. Bu durumda $k[U]$, $k[x_1, \dots, x_s]/(f)$ biçimindedir.

Teorem $R = k[x_1, \dots, x_s, y_1, \dots, y_t]$ bir polinomlar cebiri olmak üzere m , R 'nin indirgenemez f elemanını içeren bir maksimal ideali verilsin. Ayrıca, $0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_s + \beta_1 + \dots + \beta_t \leq n-1$ olmak üzere

$$d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} y_1^{\beta_1} y_2^{\beta_2} \dots y_t^{\beta_t} f) \in m \Omega_n(k[x_1, x_2, \dots, x_s, y_1, \dots, y_t])$$

sağlansın. Diğer taraftan, $k[U \times A_k^t]$ 'nin $\bar{m} = m/(f)$ 'de regüler olmadığını kabul edelim. Bu durumda, $\Omega_n((k[U \times A_k^t])_{\bar{m}})$ evrensel modülünün Betti serisi rasyoneldir.

Anahtar Kelimeler: Diferansiyel Operatör, Evrensel Modül, Minimal Çözülüş, Betti Serisi

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1 INTRODUCTION

1.1 Historical Background of Universal Modules

Definition 1.1.1 [1] *Let R and k be commutative rings with identity and let R be a k -algebra. An n th order derivation D of R into an R -module F over k is an element of $\text{Hom}_k(R, F)$ such that for any set of $n + 1$ elements $\{x_0, \dots, x_n\}$ of R we have the following identity:*

$$D(x_0 \dots x_n) = \sum_{s=1}^n (-1)^{s+1} \sum_{i_1 < \dots < i_s} x_{i_1} \dots x_{i_s} D(x_0 \dots \widehat{x_{i_1}} \dots \widehat{x_{i_s}} \dots x_n)$$

where the hat over x_i 's means that it is missed.

By using the above definition, it can be easily seen that a first order derivation is just the ordinary derivation of R into an R -module F .

The idea of constructing a universal object, $\Omega_k^q(R)$, for q^{th} order derivations goes as far back as [2, Nakai]. In this work, he constructed a universal object for just 1st order derivations and proved some functorial properties of $\Omega_k^1(R)$. Universal module for high order derivations was defined by Osborn [3], in 1967. In this paper, a more general version of derivations was introduced, φ -derivations, where A and B are k -algebras and $\varphi : A \rightarrow B$ is an algebra homomorphism. Note that a φ -derivation is a derivation of the given order where φ is the identity homomorphism on A . Later developments on high order derivations and their universal modules have been proved by Heyneman and Sweedler [4], in 1969.

By using the results proved in [1] and [3], a universal object for q^{th} order derivations, $\Omega_k^q(R)$, is an R -module satisfying the following properties:

- (i) There exists a canonical q^{th} order derivation $\delta_q : R \rightarrow \Omega_k^q(R)$,
- (ii) $\Omega_k^q(R)$ is generated as an R -module by $\{\delta_q(r) : r \in R\}$,
- (iii) Given any R -module F together with q^{th} order derivation $D : R \rightarrow F$, there exists a unique R -module homomorphism $\alpha : \Omega_k^q(R) \rightarrow F$ such that $\alpha \delta_q = D$.

In 1970, Nakai [1] gave some fundamental computations on high order derivations, introduced the module of high order differentials and proved some important functorial properties of it.

In [1, prop. 2], it is proved that:

If R is a polynomial algebra $k[x_\lambda : \lambda \in \Lambda]$ over k with indeterminates $\{x_\lambda : \lambda \in \Lambda\}$, then the universal module of derivations of order n is a free R -module.

Universal differential operator module is a powerful tool in understanding the algebraic structure of a ring. So, by this way we are able to reduce questions about algebras to module theory. For example, there is a well-known result which helps to characterize regular rings [5, Theo. 15.2.9]:

Theorem 1.1.2 *Let A be an affine domain over a field of characteristic zero and let B be the localization of A at some maximal ideal. Assume that I is the maximal ideal of B . Then the followings are equivalent:*

- (i) $\Omega_k^1(B)$ is free of rank n over B with a basis db_1, \dots, db_n where b_1, \dots, b_n is a minimal generating set for I .
- (ii) $\Omega_k^1(B)$ is free over B .
- (iii) B is regular.

Hence, as a corollary we have:

A is regular if and only if $\Omega_k^1(A)$ is projective.

Moreover, there are two important conjectures on characterizing regular rings. Nakai's Conjecture is stated in [6] as follows:

Nakai's Conjecture: Assume that R is an affine ring of an algebraic variety defined over a field k of characteristic zero. Denote by $Der_k(R)$, the algebra of high order derivations of R into itself, and denote by $der_k(R)$, the subalgebra of $Der_k(R)$ which is generated by the first order derivations of R into itself.

Is the condition $Der_k(R) = der_k(R)$ equivalent to the regularity of R ?

Second conjecture is given by Lipman as follows:

Zariski-Lipman Conjecture: If $Der_k(R)$ is free then R is regular.

It is proved that both Nakai's and Zariski-Lipman's conjectures are true for some important cases. Of course, it is natural and interesting to ask whether the conjectures given above have a relation. This question is answered by Becker [8], in 1978. It is proved that if the Nakai's conjecture is true, then so is Zariski-Lipman's.

Later work on universal module of differential operators has been done by Erdoğan [9], in 1996. The result in this paper involves a study of the projective dimension of universal modules of differential operators of order n and it is proved that:

Theorem 1.1.3 *Let S be an affine domain represented by $S = k[x_1, \dots, x_s]/(f)$. Then the projective dimension of $J_n(S)$ is less than or equal to 1.*

Another result on projective dimension has been given by Çimen and Erdoğan [10], in 1999. In this paper, it is proved that

Theorem 1.1.4 *Let U be a reduced hypersurface and A_k^t be an affine t -space. Suppose that $k[U \times A_k^t]$ is the coordinate ring of the product of U and A_k^t . Then the projective dimension of $J_n(k[U \times A_k^t])$ is at most one.*

Further results on identifying the projective dimensions of $\Omega_n(R)$, are proved in 2006 by Olgun and Erdoğan [11] where R is an affine algebra represented by $k[x_1, \dots, x_n]/(f)$. Moreover, in this paper, the generators of the kernel of the map

$$\Omega_n(R) \rightarrow \Omega_1(R)$$

are determined where R is an affine algebra.

In 2003, Erdoğan [12] proved the following:

Theorem 1.1.5 *Let R be an affine regular algebra. Then*

$$0 \rightarrow \Omega_n(R) \xrightarrow{\varphi} J_p(\Omega_n(R)) \rightarrow \text{coker} \varphi \rightarrow 0$$

is an exact sequence of R -modules where $\Omega_n(R)$ denotes the universal module of derivations of order n and $J_p(R)$ denotes the universal module of differential operators of order p .

Another interesting exact sequence constructed in [13] by Erdoğan is the following

$$\Omega_2(R) \rightarrow J_1(\Omega_1(R)) \rightarrow \wedge^2(\Omega_1(R)) \rightarrow 0$$

where $\wedge^2 \Omega_1(R)$ denotes the second exterior power of $\Omega_1(R)$. Additionally, in 1996, Hart [14] proved that the above map is also injective.

Besides, in [12, Theorem 7], it is showed that the regularity of an affine algebra R is equivalent to the projectivity of $\wedge^2 \Omega_1(R)$.

In 2005, Olgun and Erdoğan [15] examined the structure of the universal module over the tensor product algebra $R \otimes S$ and proved

$$0 \rightarrow \frac{N + K\Omega_n(R \otimes_k S)}{K\Omega_n(R \otimes_k S)} \rightarrow \frac{\Omega_n(R \otimes_k S)}{K\Omega_n(R \otimes_k S)} \rightarrow \Omega_n(R/I \otimes_k S/J) \rightarrow 0$$

is an exact sequence of $(\frac{R \otimes_k S}{K})$ -modules where I is an ideal of R , J is an ideal of S , K is given by $I \otimes S + R \otimes J$ and N is a submodule of $\Omega_n(R \otimes_k S)$ generated by the elements of the form $\{\delta_n(x) : x \in K\}$. Moreover, they investigated the homological dimension of $\Omega_n(R \otimes_k S)$.

Before concluding this introductory section on the history of differential operators, it might be interesting if we give the following theorem which gives the relations between differential operators and geometry.

Theorem 1.1.6 [2, Corollary 1] *Let P be a point of an algebraic set V . Then under some suitable conditions the necessary and sufficient condition for P to be a simple point of V is that $\Omega_1(R)$ is a free R -module where R is the local ring corresponding the point P of V .*

The purpose of this thesis is to further study the universal modules of differential operators of order n . The thesis proceeds as follows:

The aim of chapter 2 is to develop the theory of differential operators and their universal modules. Firstly, we give the definition of differential operators of order n and high order derivations of order n . Next, we construct the universal modules for both and prove their existence and uniqueness. And, we end this section by examining some properties of the universal modules for some particular cases, such as factor rings, local rings and regular rings.

Chapter 3 includes some well-known results on projective dimension of the universal module of differential operators of order n . Next, we give some examples on computing the projective dimensions of the universal modules. Furthermore, in this section we obtain the following result:

Theorem 1.1.7 *Let R be a k -algebra and m, n be positive integers such that $m < n$. Suppose that δ_n and δ_m denote the universal operators of R of order n and m , respectively. Then we have the following short exact sequence of R -modules:*

$$0 \rightarrow \ker \theta \rightarrow \Omega_n(R) \xrightarrow{\theta} \Omega_m(R) \rightarrow 0.$$

Moreover, $\ker\theta$ is generated by the set

$$\{\delta_n(r_0\dots r_m) + \sum_{\substack{T \neq \emptyset \\ T \subseteq \{0, \dots, m\}}} (-1)^{|T|} r_T \delta_n(r_{T'})\}$$

where $r_i \in R$ for $i = 0, \dots, m$; T' is the complement of T in the set $\{0, \dots, m\}$ and

$$r_T = \prod_{\substack{k \in T \\ T \subseteq \{0, \dots, m\}}} r_k.$$

Note that this result is indeed a generalization of the result proved in the paper [11, Olgun and Erdoğan]. Then we give some examples to discuss the result more closely. Next, we consider the map

$$J_n(R) \xrightarrow{\alpha} J_{n-1}(\Omega_1(R))$$

and obtain some results on $\ker\alpha$ and $\operatorname{coker}\alpha$ where R is a domain of dimension one or two.

Now, let us recall the definition of the Betti series:

Definition 1.1.8 *Let (R, m) be a local ring. The Betti series of $\Omega_n(R)$ is defined to be the series*

$$B(\Omega_n(R), t) = \sum_{i \geq 0} \dim_{R/m} \operatorname{Ext}^i(\Omega_n(R), \frac{R}{m}) t^i \text{ for all } n \geq 1.$$

In chapter 4, we present our contribution which includes a study on the behavior of the Betti series of the universal modules. Firstly, we discuss the rationality of the Betti series of $\Omega_2(R_m)$ where R is a coordinate ring of an affine irreducible curve represented by $\frac{k[x_1, x_2, \dots, x_s]}{(f)}$ and m is a maximal ideal of R . Next, we generalize these results for the universal modules of differential operators of order n . We obtain the following theorem:

Theorem 1.1.9 *Let $k[x_1, x_2, \dots, x_s]$ be a polynomial algebra and m be a maximal ideal of $k[x_1, x_2, \dots, x_s]$ containing an irreducible element f . Let*

$$d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) \in m \Omega_n(k[x_1, x_2, \dots, x_s])$$

for $0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_s \leq n - 1$. Assume that $R = \frac{k[x_1, x_2, \dots, x_s]}{(f)}$ is not a regular ring at $\bar{m} = m/(f)$. Then $B(\Omega_n(R_{\bar{m}}), t)$ is a rational function.

Let U be a reduced hypersurface and A_k^t be an affine t -space. Additionally, we showed that under some conditions the Betti Series of

$$\Omega_n((k[U \times A_k^t])_{\bar{m}})$$

is a rational function where $k[U \times A_k^t]$ is the coordinate ring of the product of U and A_k^t , m is a maximal ideal of $k[x_1, \dots, x_s, y_1, \dots, y_t]$ containing the irreducible element f and $\bar{m} = m/(f)$.

2 UNIVERSAL MODULES

This chapter summarizes the elementary theory of differential operators and their universal modules. In the subsection 1, we give the definition of differential operators of order n and then we construct a universal object $J_n(R)$ which is unique up to isomorphism. Next, we define high order derivations and their universal modules, $\Omega_n(R)$. Then, inevitably, we give the relation between $J_n(R)$ and $\Omega_n(R)$. Subsection 4 concerns the universal modules of local rings. In the subsection 5, we give some examples which illustrate the theory and next, we concentrate on universal modules of factor rings. So, we are able to compute the universal modules where it is of the form R/I . Then we give the relation between universal modules and vector spaces. In the subsection 8, we examine the universal modules of field extensions. And, we close this section by proving some important results on universal modules of regular algebras. Note that the definitions, results and examples in this chapter come from [1, Nakai], [3, Osborn], [4, Heyneman and Sweedler], [13, Erdogan], [16, Poulton] and [17, Sweedler].

2.1 Modules of Differential Operators

Throughout our work, unless the contrary is stated explicitly, by a ring, we mean a commutative ring with identity. Let k be an algebraically closed field of characteristic zero, R be a k -algebra and let M and N be R -modules. $Hom_k(M, N)$ denotes the set of all k -linear maps from M to N . With the following operations $Hom_k(M, N)$ becomes an R - R bimodule:

$$rf : m \mapsto rf(m)$$

$$fr : m \mapsto f(rm)$$

where $f \in Hom_k(M, N)$, $m \in M$ and $r \in R$. The commutator of f and r is denoted by $[f, r]$ and defined as:

$$[f, r] := fr - rf.$$

Moreover, we know that $[f, r] \in Hom_k(M, N)$.

Definition 2.1.1 *The differential operator module of order n from M to N is denoted by $D_R^n(M, N)$ and is defined recursively:*

Firstly, we set

$$D_R^0(M, N) := \text{Hom}_R(M, N).$$

Assume that $D_R^{n-1}(M, N)$ has been defined. Then

$$D_R^n(M, N) := \{f \in \text{Hom}_k(M, N) : [f, r] \in D_R^{n-1}(M, N), \forall r \in R\}.$$

Let us define $D_R^n(M, N) = 0$, where n is a negative integer.

Definition 2.1.2 *The space of k -linear differential operators from M to N is defined as:*

$$D_R(M, N) := \bigcup_{n \geq 0} D_R^n(M, N).$$

Proposition 2.1.3 *$D_R^n(M, N)$ is an R -submodule of $\text{Hom}_k(M, N)$.*

Proof. The proof proceeds by induction on n . Firstly, let $n = 0$. Then by definition, we know

$$D_R^0(M, N) = \text{Hom}_R(M, N)$$

which is an R -module. Assume that the proposition is true for $n - 1$, that is, assume that $D_R^{n-1}(M, N)$ is an R -module. We want to prove it for n . Let $f, g \in D_R^n(M, N)$ and $r, s \in R$. So, we have

$$[f + g, r] = [f, r] + [g, r].$$

By the definition of differential operators, $[f, r]$ and $[g, r]$ belong to $D_R^{n-1}(M, N)$ and by using the induction assumption, we obtain

$$[f + g, r] \in D_R^{n-1}(M, N)$$

for all $r \in R$. Hence, $f + g \in D_R^n(M, N)$.

On the other hand, by using the commutativity of R , we have

$$[sf, r] = s[f, r].$$

Since $D_R^{n-1}(M, N)$ is an R -module, we get

$$[sf, r] \in D_R^{n-1}(M, N)$$

for all $r \in R$. Thus, $sf \in D_R^n(M, N)$. ■

Proposition 2.1.4 *For every integer n , we have*

$$D_R^n(M, N) \subseteq D_R^{n+1}(M, N).$$

Proof. The proof follows by induction on n . For the case $n = 0$, we have

$$f \in D_R^0(M, N) = \text{Hom}_R(M, N)$$

and hence, we obtain

$$[f, r] = 0 \in \text{Hom}_R(M, N).$$

So, $f \in D_R^1(M, N)$. Now, assume that the assertion is true for $n - 1$, in other words, we have

$$D_R^{n-1}(M, N) \subseteq D_R^n(M, N).$$

Let f be an element of $D_R^n(M, N)$. Then by the assumption,

$$[f, r] \in D_R^{n-1}(M, N) \subseteq D_R^n(M, N)$$

for all $r \in R$. Therefore, $f \in D_R^{n+1}(M, N)$. ■

Observe that by the propositions (2.1.3) and (2.1.4), we filter $\text{Hom}_k(M, N)$ by increasing submodules $D_R^n(M, N)$.

Proposition 2.1.5 *Let M, N and K be R -modules. Let $f \in D_R^n(M, N)$ and $g \in D_R^m(N, K)$. Then*

$$gf \in D_R^{m+n}(M, K).$$

In particular, if $u \in \text{Hom}_R(M, N)$ and $v \in \text{Hom}_R(N, K)$, then

$$v \circ f \in D_R^n(M, K) \text{ and } g \circ u \in D_R^m(M, K).$$

Proof. We prove it by induction on $m + n$. For the first case, let $m = n = 0$. The assertion is clear since if $f \in \text{Hom}_R(M, N)$ and $g \in \text{Hom}_R(N, K)$, then

$$gf \in \text{Hom}_R(M, K).$$

Now assume that the expression is true for the integers less than $m + n$. Let $f \in D_R^n(M, N)$ and $g \in D_R^m(N, K)$. Then we have

$$[gf, r] = g[f, r] + [g, r]f$$

for all $r \in R$. On the other hand, $g \in D_R^m(N, K)$ and $[f, r] \in D_R^{n-1}(M, N)$ and by the induction hypothesis, we obtain

$$g[f, r] \in D_R^{m+n-1}(M, K).$$

Similarly, $[g, r]f \in D_R^{m+n-1}(M, K)$. So,

$$[gf, r] = g[f, r] + [g, r]f \in D_R^{m+n-1}(M, K)$$

for all $r \in R$ and this means $gf \in D_R^{m+n}(M, K)$ as required. ■

Corollary 2.1.6 $D_R(M, M) = D_R(M)$ is a k -subalgebra of $\text{End}_k(M)$.

Definition 2.1.7 $D_R(M)$ is called the ring of differential operators of M .

Example 2.1.8 Let R be the polynomial algebra $R = k[x, y, z]$. Then

$$D_R^0(R) \cong R,$$

$$D_R^1(R) = \langle 1, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle,$$

$$D_R^2(R) = \langle 1, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}, \frac{\partial^2}{\partial xy}, \frac{\partial^2}{\partial xz}, \frac{\partial^2}{\partial yz} \rangle,$$

More generally, we have

$$D_R^n(R) = \langle \{1, \frac{\partial^t}{\partial x^i \partial y^j \partial z^k} : 1 \leq t = i + j + k \leq n\} \rangle.$$

Proposition 2.1.9 Let R and S be commutative k -algebras, let M, N be R -modules and let M', N' be S -modules. If $f \in D_R^r(M, N)$ and $g \in D_S^t(M', N')$, then

$$f \otimes g \in D_{R \otimes S}^{r+t}(M \otimes M', N \otimes N').$$

Proof. The proof proceeds by induction on $r + t$. Let $r + t = 0$. If $f \in \text{Hom}_R(M, N)$ and $g \in \text{Hom}_S(M', N')$, then we know

$$f \otimes g \in \text{Hom}_{R \otimes S}(M \otimes M', N \otimes N')$$

with the following definition

$$(f \otimes g)(m \otimes m') := f(m) \otimes g(m').$$

Suppose that the result is true for all values less than $r + t$.

Claim. We have the following equality:

$$[f \otimes g, u \otimes v] = [f, u] \otimes (\tilde{v} \circ g) + (f \circ \tilde{u}) \otimes [g, v]$$

where $u \in R$, $v \in S$ and $\tilde{u} \in \text{Hom}_R(M, M)$ and $\tilde{v} \in \text{Hom}_R(N', N')$ such that

$$\tilde{u} : m \mapsto um,$$

$$\tilde{v} : n' \mapsto vn'.$$

Proof of Claim. If we apply $a \otimes b$ both sides, then we get

$$\begin{aligned} [f \otimes g, u \otimes v](a \otimes b) &= [(f \otimes g)(u \otimes v) - (u \otimes v)(f \otimes g)](a \otimes b) \\ &= f(ua) \otimes g(vb) - uf(a) \otimes vg(b) \end{aligned}$$

and

$$\begin{aligned} ([f, u] \otimes (\tilde{v} \circ g) + (f \circ \tilde{u}) \otimes [g, v])(a \otimes b) &= [f, u](a) \otimes (\tilde{v} \circ g)(b) + (f \circ \tilde{u})(a) \otimes [g, v](b) \\ &= (f(ua) - uf(a)) \otimes vg(b) \\ &\quad + f(ua) \otimes (g(vb) - vg(b)). \end{aligned}$$

So, we can conclude that they are equal. Then by using the induction hypothesis, we get

$$[f \otimes g, u \otimes v] \in D_{R \otimes S}^{r+t-1}(M \otimes M', N \otimes N')$$

for all $(u \otimes v) \in R \otimes S$. Therefore, $f \otimes g \in D_{R \otimes S}^{r+t}(M \otimes M', N \otimes N')$. ■

2.2 Universal Modules of Differential Operators

Let R be a k -algebra and let $r_i, r_j, s_i, s_j \in R$. Then $R \otimes_k R$ becomes a k -algebra with the given operation

$$\left(\sum_i r_i \otimes s_i\right) \cdot \left(\sum_j r_j \otimes s_j\right) = \sum_{i,j} r_i r_j \otimes s_i s_j.$$

Further, $\text{Hom}_k(M, N)$ is endowed an $R \otimes_k R$ -module structure with

$$(r \otimes_k s)f : m \mapsto rf(sm)$$

where $r, s \in R$, $f \in \text{Hom}_k(M, N)$ and $m \in M$.

Let us define the multiplication map,

$$\begin{aligned} \theta : R \otimes_k R &\longrightarrow R \\ \sum_{i=1}^n a_i \otimes b_i &\mapsto \sum_{i=1}^n a_i b_i. \end{aligned}$$

By this map, we have

$$0 \longrightarrow \ker\theta \longrightarrow R \otimes_k R \xrightarrow{\theta} R \longrightarrow 0$$

exact sequence of R -modules. For notational simplicity, we denote $\ker\theta = I$.

Lemma 2.2.1 *I is an ideal of $R \otimes_k R$ and generated by the set*

$$\{1 \otimes r - r \otimes 1 : r \in R\}.$$

Proof. It is easy to see that the elements of the form

$$\{1 \otimes r - r \otimes 1 : r \in R\}$$

belong to I . Conversely, let

$$\alpha = \sum_i r_i \otimes s_i \in I.$$

By the definition of the map, we have

$$\sum_i r_i s_i = 0.$$

Therefore, we obtain

$$\alpha = \sum_i r_i \otimes s_i = \sum_i r_i \otimes s_i - \left(\sum_i r_i s_i\right) \otimes 1 = \sum_i (r_i \otimes 1)(1 \otimes s_i - s_i \otimes 1)$$

as desired. ■

Proposition 2.2.2 *Let M and N be R -modules and let $f \in \text{Hom}_k(M, N)$. Then*

$$[f, r] = (1 \otimes r - r \otimes 1)f$$

for all $r \in R$.

Proof. By using $R - R$ bimodule and $R \otimes_k R$ -module structures of $\text{Hom}_k(M, N)$, we have

$$\begin{aligned} [f, r](m) &= (fr - rf)(m) \\ &= f(rm) - rf(m) \\ &= [(1 \otimes r)f](m) - [(r \otimes 1)f](m) \\ &= (1 \otimes r - r \otimes 1)f(m). \end{aligned}$$

This means that, $[f, r] = (1 \otimes r - r \otimes 1)f$. ■

Since I is an ideal of $R \otimes_k R$, I^{n+1} is an ideal of $R \otimes_k R$ for all $n \geq 1$ and it is generated by the elements of the form

$$\prod_{i=0}^n (1 \otimes r_i - r_i \otimes 1)$$

where $r_0, r_1, \dots, r_n \in R$. Moreover, we have the following equality

$$\prod_{i=0}^n (1 \otimes r_i - r_i \otimes 1) = \sum_{T \subseteq \{0, \dots, n\}} (-1)^{|T|} r_T \otimes r_{T'}$$

where T is any subset of $\{0, \dots, n\}$; T' is the complement of T in $\{0, \dots, n\}$; $|T|$ denotes the number of elements of T ;

$$r_T = \prod_{k \in T} r_k \text{ and } r_\emptyset = 1.$$

Proposition 2.2.3 *Let M and N be R -modules and let $f \in \text{Hom}_k(M, N)$. Then f is a differential operator of order n if and only if $I^{n+1}f = 0$.*

Proof. We prove it by induction on n . Let $r \in R$. For $n = 0$, by considering the definition in (2.1.1), we obtain

$$\begin{aligned} f \in D_R^0(M, N) &\Leftrightarrow [f, r] = 0 \text{ for all } r \in R, \\ &\Leftrightarrow (1 \otimes r - r \otimes 1)f = 0 \text{ for all } r \in R, \\ &\Leftrightarrow If = 0. \end{aligned}$$

Let us assume that the assertion is true for n . We shall prove it for $n + 1$.

By using the induction hypothesis, we have

$$\begin{aligned}
f \in D_R^{n+1}(M, N) &\Leftrightarrow [f, r] \in D_R^n(M, N) \text{ for all } r \in R, \\
&\Leftrightarrow (1 \otimes r - r \otimes 1)f \in D_R^n(M, N) \text{ for all } r \in R, \\
&\Leftrightarrow If \in D_R^n(M, N) \text{ for all } r \in R, \\
&\Leftrightarrow I^{n+2}f = 0.
\end{aligned}$$

Hence, we get the required result. ■

Corollary 2.2.4 *Let M and N be R -modules and let $f \in D_R^n(M, N)$. Then*

$$f(r_0 \dots r_n m) = \sum_{\substack{T \subseteq \{0, 1, \dots, n\} \\ |T| \geq 1}} (-1)^{|T|+1} r_T f(r_{T'} m) \quad (1)$$

where $r_0, r_1, \dots, r_n \in R$ and $m \in M$.

Proof. Let $f \in D_R^n(M, N)$. Then by using the proposition (2.2.3), we have $I^{n+1}f = 0$.

Therefore, we get

$$\begin{aligned}
0 &= [(1 \otimes r_0 - r_0 \otimes 1)(1 \otimes r_1 - r_1 \otimes 1) \dots (1 \otimes r_n - r_n \otimes 1)f](m) \\
&= \left[\sum_{T \subseteq \{0, 1, \dots, n\}} (-1)^{|T|} (r_T \otimes r_{T'}) f \right](m) \\
&= \sum_{T \subseteq \{0, 1, \dots, n\}} (-1)^{|T|} r_T f(r_{T'} m).
\end{aligned}$$

So, this ensures that $f(r_0 \dots r_n m) = \sum_{\substack{T \subseteq \{0, 1, \dots, n\} \\ |T| \geq 1}} (-1)^{|T|+1} r_T f(r_{T'} m)$. ■

Remark 2.2.5 *Let $f \in D_R^0(M, N) = \text{Hom}_R(M, N)$. Then*

$$[f, r_0](m) = f(r_0 m) - r_0 f(m) = 0$$

for any $r_0 \in R$ and $m \in M$. Therefore, we get $[f, r_0] = 0$.

Remark 2.2.6 *Let $f \in D_R^1(M, N)$. By considering the equation given in (1), we obtain*

$$[f, r_0, r_1](m) = f(r_0 r_1 m) - r_0 f(r_1 m) - r_1 f(r_0 m) + r_0 r_1 f(m) = 0$$

for any $r_0, r_1 \in R$ and $m \in M$. Hence, we get $[f, r_0, r_1] = 0$.

More generally, let $f \in D_R^n(M, N)$. Then by (1), we have

$$[f, r_0, r_1, \dots, r_n] = 0$$

for any $r_0, r_1, \dots, r_n \in R$.

Definition 2.2.7 Let M and N be R -modules and let

$$\Delta_n : M \longrightarrow N$$

be a differential operator of order n . If for any R -module K and for any differential operator

$$d : M \longrightarrow K$$

of order n , there exists a unique R -module homomorphism

$$\alpha : N \longrightarrow K$$

which makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{d} & K \\ \Delta_n \downarrow & & \downarrow id \\ N & \xrightarrow{\alpha} & K \end{array}$$

commutative, then

$$\Delta_n : M \longrightarrow N$$

is said to be the universal differential operator of order n . And N is called the universal differential operator module of order n .

Let M be an R -module and consider the tensor product $R \otimes_k M$. $R \otimes_k M$ is an $R \otimes R$ -module with

$$(r \otimes s)(r' \otimes m) = (rr' \otimes sm)$$

where $r, s, r' \in R$ and $m \in M$.

Note that since I^{n+1} is an ideal of $R \otimes_k R$ for $n \geq 1$, we can define the quotient module

$$R \otimes_k M / I^{n+1}(R \otimes_k M).$$

Definition 2.2.8 Let R be a k -algebra and let M be an R -module. The quotient module

$$R \otimes_k M / I^{n+1}(R \otimes_k M)$$

is called the universal differential operator module of order n of M and denoted by $J_n(M)$. Moreover, the universal differential operator Δ_n is defined as the composite of the following maps

$$\begin{aligned} \Delta_n : M &\rightarrow R \otimes_k M \rightarrow J_n(M) \\ m &\mapsto 1 \otimes m \mapsto 1 \otimes m + I^{n+1}(R \otimes_k M). \end{aligned}$$

Proposition 2.2.9 $\Delta_n : M \rightarrow J_n(M)$ is a differential operator of order n .

Proof. It is easy to see that Δ_n is k -linear. Further, by the definition of Δ_n we see that $I^{n+1}\Delta_n = 0$. By using the proposition (2.2.3), we get the result. ■

Next, we prove the existence and uniqueness of the universal module of differential operators.

Proposition 2.2.10 Let M be an R -module. Then the map

$$\Delta_n : M \rightarrow J_n(M)$$

is the universal differential operator of order n of M .

Proof. Let K be an R -module and let

$$f : M \longrightarrow K$$

be a differential operator of order n . Our aim is to show that there exists a unique R -module homomorphism

$$\alpha : J_n(M) \longrightarrow K$$

such that $\alpha\Delta_n = f$. Let us define the map

$$\begin{aligned} F : R \otimes_k M &\rightarrow K \\ r \otimes m &\mapsto rf(m). \end{aligned}$$

Then we have $Fi = f$ where

$$i : M \longrightarrow R \otimes M$$

is given by $i(m) = 1 \otimes m$. Since f is a differential operator of order n , by using the proposition (2.2.3), we see $I^{n+1}f = 0$.

Claim. We have $F(I^{n+1}(R \otimes_k M)) = 0$.

Proof of Claim. Let $r, r_0, r_1, \dots, r_n \in R$ and $m \in M$. Then considering the equality given in (1) and the fact that f is a differential operator of order n enables us the following:

$$\begin{aligned} F\left(\prod_{i=0}^n (1 \otimes r_i - r_i \otimes 1)(r \otimes m)\right) &= F\left(\sum_{T \subseteq \{0,1,\dots,n\}} (-1)^{|T|} (r_T \otimes r_{T'})(r \otimes m)\right) \\ &= F\left(\sum_{T \subseteq \{0,1,\dots,n\}} (-1)^{|T|} (r_T r \otimes r_{T'} m)\right) \\ &= \sum_{T \subseteq \{0,1,\dots,n\}} (-1)^{|T|} r_T r f(r_{T'} m) \\ &= r\left(\sum_{T \subseteq \{0,1,\dots,n\}} (-1)^{|T|} r_T f(r_{T'} m)\right) = 0. \end{aligned}$$

Therefore, we obtain the uniquely induced map

$$\begin{aligned} \bar{F} : R \otimes_k M / I^{n+1}(R \otimes_k M) &\longrightarrow K \\ r \otimes m + I^{n+1}(R \otimes_k M) &\longrightarrow r f(m) \end{aligned}$$

such that $\bar{F}p = F$ where p is the natural homomorphism

$$p : R \otimes_k M \longrightarrow R \otimes_k M / I^{n+1}(R \otimes_k M).$$

So, we see that

$$\bar{F}pi = Fi = f.$$

Thus, the map $pi = \Delta_n$ is the universal differential operator of order n . ■

Proposition 2.2.11 *Let M be an R -module and let Δ'_n and $J'_n(M)$ be another universal differential operator and universal differential operator module of M , respectively. Then there exists a unique R -module isomorphism*

$$\alpha : J_n(M) \longrightarrow J'_n(M)$$

such that $\Delta'_n = \alpha \Delta_n$.

Proof. Since we know that

$$\Delta'_n : M \longrightarrow J'_n(M)$$

satisfies the universal property, we obtain the following commutative diagrams:

$$\begin{array}{ccc} M & \xrightarrow{\Delta'_n} & J'_n(M) \\ \Delta_n \downarrow & & \downarrow id \\ J_n(M) & \xrightarrow{\alpha} & J'_n(M) \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{\Delta_n} & J_n(M) \\ \Delta'_n \downarrow & & \downarrow id \\ J'_n(M) & \xrightarrow{\beta} & J_n(M) \end{array}$$

such that

$$\alpha \circ \Delta_n = \Delta'_n \quad \text{and} \quad \beta \circ \Delta'_n = \Delta_n.$$

Hence, we get

$$\beta\alpha\Delta_n(m) = \Delta_n(m)$$

and

$$\alpha\beta\Delta'_n(m) = \Delta'_n(m)$$

for each $m \in M$. On the other hand, the identity maps $1_{J_n(M)}$ and $1_{J'_n(M)}$ satisfy

$$1_{J_n(M)}\Delta_n(m) = \Delta_n(m) \quad \text{and} \quad 1_{J'_n(M)}\Delta'_n(m) = \Delta'_n(m)$$

for each $m \in M$. So, by the uniqueness we see $\alpha\beta = 1_{J'_n(M)}$ and $\beta\alpha = 1_{J_n(M)}$.

Then we conclude that α is an isomorphism, as desired. ■

Proposition 2.2.12 *Let M and N be R -modules. Then the map*

$$\psi : \text{Hom}_R(J_n(M), N) \longrightarrow D_R^n(M, N), \quad \alpha \mapsto \alpha\Delta_n$$

is an R -module isomorphism.

Proof. Let $D \in D_R^n(M, N)$. Since $J_n(M)$ satisfies the universal property, there exists an R -linear map

$$\alpha : J_n(M) \longrightarrow N$$

such that $\alpha\Delta_n = D$. Then $\psi(\alpha) = \alpha\Delta_n = D$. Hence, ψ is surjective.

Let $\alpha \in \text{Hom}_R(J_n(M), N)$ and let $\psi(\alpha) = 0$. By the definition of the map ψ , we obtain $\alpha\Delta_n(m) = 0$ for each $m \in M$.

Furthermore, we know that $J_n(M)$ is generated by the set

$$\{\Delta_n(m) : m \in M\}$$

as an R -module and α is an R -module homomorphism. Then we get

$$\alpha(J_n(M)) = 0.$$

So, $\alpha = 0$ which means that ψ is one-to-one. ■

Corollary 2.2.13 *Let $M = N = R$ in the proposition (2.2.12). Then*

$$\text{Hom}_R(J_n(R), R) \cong D^n(R)$$

is an R -module isomorphism.

Let M be an R -module. In the following theorem, we give the relation between $J_n(R) = R \otimes_k R / I^{n+1}$ and $J_n(M) = R \otimes_k M / I^{n+1}(R \otimes_k M)$.

Theorem 2.2.14 *Let M be an R -module. Assume $r, s \in R$ and $m \in M$. Then the map*

$$J_n(M) \xrightarrow{\gamma} J_n(R) \otimes_R M$$

given by

$$\gamma(r \otimes sm + I^{n+1}(R \otimes_k M)) = (r \otimes s + I^{n+1}) \otimes_R m$$

is an R -module isomorphism.

Proof. Consider the natural isomorphism

$$p : M \rightarrow R \otimes_R M.$$

This map induces the following isomorphism

$$J_n(M) \cong J_n(R \otimes_R M).$$

By the definition, we have

$$J_n(R \otimes_R M) = R \otimes_k (R \otimes_R M) / I^{n+1}(R \otimes_k (R \otimes_R M)).$$

On the other hand, we get

$$J_n(R \otimes_R M) = (R \otimes_k R) \otimes_R M / I^{n+1} \otimes_R M$$

since

$$I^{n+1}(R \otimes_k (R \otimes_R M)) = I^{n+1} \otimes_R M.$$

Then we conclude

$$J_n(M) \cong J_n(R) \otimes_R M.$$

as desired. ■

Corollary 2.2.15 *Let $\{M_i\}_{i \in I}$ and N be R -modules. Then the followings hold:*

$$(i) \ J_n\left(\bigoplus_i M_i\right) \cong \bigoplus_i J_n(M_i).$$

(ii) *Let $\{M_i\}$ be a finite family of R -modules. Then*

$$D_R^n\left(\bigoplus_i M_i, N\right) \cong \bigoplus_i D_R^n(M_i, N).$$

Proof.

(i) We have $J_n\left(\bigoplus_i M_i\right) \cong J_n(R) \otimes_R \left(\bigoplus_i M_i\right)$. Then

$$J_n(R) \otimes_R \left(\bigoplus_i M_i\right) \cong \bigoplus_i (J_n(R) \otimes_R M_i) \cong \bigoplus_i J_n(M_i).$$

(ii) Let $\{M_i\}$ be a finite family of R -modules. Then

$$D_R^n\left(\bigoplus_i M_i, N\right) \cong \text{Hom}_R\left(J_n\left(\bigoplus_i M_i\right), N\right).$$

By (i), we get

$$\begin{aligned} D_R^n\left(\bigoplus_i M_i, N\right) &\cong \text{Hom}_R\left(\bigoplus_i J_n(M_i), N\right) \\ &\cong \bigoplus_i \text{Hom}_R(J_n(M_i), N) \\ &\cong \bigoplus_i D_R^n(M_i, N) \end{aligned}$$

as required. ■

2.3 Universal Modules of High Order Derivations

Definition 2.3.1 Let M be an R -module. An n^{th} order differential operator d ,

$$d : R \longrightarrow M$$

such that $d(1) = 0$ is called a derivation of order n .

Definition 2.3.2 Let M be an R -module. Then the set

$$\{d \in D^n(R, M) : d(1) = 0\}$$

is called the module of derivations of order n and is denoted by $Der^n(R, M)$.

Lemma 2.3.3 Let $J_n(R)$ be the universal module of differential operators of order n of R . Then $R\Delta_n(1)$ is a direct summand of $J_n(R)$.

Proof. Let $1_R : R \longrightarrow R$ be the identity map and

$$\Delta_n : R \longrightarrow J_n(R)$$

be the universal differential operator of order n of R . By using the proposition (2.1.4), we say that $1_R \in D^n(R)$ and by universality, there exists an R -module homomorphism

$$\alpha : J_n(R) \longrightarrow R$$

such that $\alpha\Delta_n = 1_R$. Since α is an R -module homomorphism, we get α is surjective.

Let us define a map

$$\begin{aligned} \beta : R &\rightarrow J_n(R) \\ r &\mapsto r\Delta_n(1). \end{aligned}$$

This map is an R -module homomorphism and it satisfies $\alpha\beta = 1_R$.

Claim 1. $J_n(R) = \ker(\alpha) + R\Delta_n(1)$.

Proof of Claim 1. Let $x \in J_n(R)$. Then we can rewrite x as following:

$$x = (x - \beta\alpha(x)) + \beta\alpha(x).$$

Since $\alpha\beta = 1_R$, we have $\alpha(x - \beta\alpha(x)) = 0$ and hence, $x - \beta\alpha(x) \in \ker(\alpha)$.

On the other hand, by the definition of β we see that

$$\beta(\alpha(x)) = \alpha(x)\Delta_n(1).$$

Therefore we get, $x \in \ker(\alpha) + R\Delta_n(1)$.

Claim 2. $\ker(\alpha) \cap R\Delta_n(1) = 0$.

Proof of Claim 2. Assume $x \in \ker(\alpha) \cap R\Delta_n(1)$. Then $\alpha(x) = 0$ and x is of the form $r\Delta_n(1)$ for some $r \in R$. So, we get

$$0 = \alpha(x) = \alpha(r\Delta_n(1)) = r.$$

Thus, we have

$$J_n(R) = \ker(\alpha) \oplus R\Delta_n(1),$$

as desired. ■

Definition 2.3.4 Let R be a k -algebra. Then the factor module

$$\ker\alpha := J_n(R)/R\Delta_n(1)$$

is called the universal module of derivations of order n and is denoted by $\Omega_n(R)$. Universal derivation of order n is denoted by δ_n and is defined as the composition of the following maps:

$$\begin{aligned} \delta_n : R &\rightarrow J_n(R) \rightarrow \Omega_n(R) \\ r &\mapsto \Delta_n(r) \mapsto \Delta_n(r) + R\Delta_n(1). \end{aligned}$$

Proposition 2.3.5 Let R be a k -algebra. Then the map

$$\delta_n : R \longrightarrow \Omega_n(R)$$

is a derivation of order n .

Proof. Let p be the natural epimorphism

$$p : J_n(R) \longrightarrow \Omega_n(R).$$

We know that $p \in D_R^0(J_n(R), \Omega_n(R))$. Since $\delta_n = p\Delta_n$, by using the proposition (2.1.5), we get $\delta_n \in D^n(R, \Omega_n(R))$. Besides, $\delta_n(1) = p\Delta_n(1) = 0$. Then we conclude that $\delta_n \in \text{Der}^n(R, \Omega_n(R))$. ■

Proposition 2.3.6 *Let M be an R -module and*

$$d : R \longrightarrow M$$

be a derivation of order n . Then there exists a unique R -module homomorphism

$$\rho : \Omega_n(R) \rightarrow M$$

such that

$$\begin{array}{ccc} R & \xrightarrow{\delta_n} & \Omega_n(R) \\ d \downarrow & & \downarrow \rho \\ M & \xrightarrow{1_R} & M \end{array}$$

the diagram commutes. In other words, $\Omega_n(R)$ and δ_n are universal.

Proof. Assume $d \in \text{Der}^n(R, M)$. Then by the definition, we know that $d \in D^n(R, M)$. So, by the proposition (2.2.10), there exists a unique R -module homomorphism

$$\beta : J_n(R) \rightarrow M$$

such that $\beta\Delta_n = d$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{d} & M \\ \Delta_n \downarrow & & \parallel \\ J_n(R) & \xrightarrow{\beta} & M. \end{array}$$

Moreover, we have

$$\beta(\Delta_n(1)) = d(1) = 0.$$

Hence, we can induce a unique R -homomorphism

$$\rho : \Omega_n(R) \longrightarrow M$$

such that the diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{d} & M \\ \downarrow & & \parallel \\ J_n(R) & \xrightarrow{\beta} & M \\ \downarrow & & \parallel \\ \Omega_n(R) & \xrightarrow{\rho} & M. \end{array}$$

Then we have the desired result. ■

Proposition 2.3.7 *Let $\Omega'_n(R)$ and δ'_n be any other universal module and universal derivation of order n of R , respectively. Then there exists an R -module isomorphism,*

$$\gamma : \Omega_n(R) \longrightarrow \Omega'_n(R)$$

such that $\delta'_n = \gamma\delta_n$.

Proof. By universality of δ_n , we have the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\delta'_n} & \Omega'_n(R) \\ \delta_n \downarrow & & \parallel \\ \Omega_n(R) & \xrightarrow{\gamma} & \Omega'_n(R). \end{array}$$

Since the map

$$\delta'_n : R \rightarrow \Omega'_n(R)$$

is universal, in the same manner we get:

$$\begin{array}{ccc} R & \xrightarrow{\delta_n} & \Omega_n(R) \\ \delta'_n \downarrow & & \parallel \\ \Omega'_n(R) & \xrightarrow{\alpha} & \Omega_n(R). \end{array}$$

And by commutativity they both satisfy,

$$\gamma\delta_n = \delta'_n \quad \text{and} \quad \alpha\delta'_n = \delta_n.$$

So, we have

$$\alpha\gamma\delta_n(r) = \delta_n(r) \quad \text{and} \quad \gamma\alpha\delta'_n(r) = \delta'_n(r)$$

for all $r \in R$. On the other hand, we have

$$1_{\Omega_n(R)}\delta_n(r) = \delta_n(r) \quad \text{and} \quad 1_{\Omega'_n(R)}\delta'_n(r) = \delta'_n(r),$$

for all $r \in R$. Then by uniqueness, we obtain

$$\gamma\alpha = 1_{\Omega'_n(R)} \quad \text{and} \quad \alpha\gamma = 1_{\Omega_n(R)}.$$

Therefore, $\gamma : \Omega_n(R) \rightarrow \Omega'_n(R)$ is an isomorphism of R -modules. ■

Next, we will prove the relation between $\Omega_n(R)$ and $J_n(R)$.

Proposition 2.3.8 *$J_n(R)$ is projective if and only if $\Omega_n(R)$ is projective.*

Proof. Let $J_n(R)$ be a projective R -module. Then there exists a free R -module F and a projective R -module P such that

$$F = P \oplus J_n(R).$$

By the lemma (2.3.3), we have $\Omega_n(R)$ is a direct summand of the free module F . Hence, it is a projective R -module. Conversely, assume that $\Omega_n(R)$ is projective. Then there exists a free R -module Q and projective R -module K such that

$$Q = \Omega_n(R) \oplus K.$$

Then, we get

$$Q \oplus R = \Omega_n(R) \oplus R \oplus K.$$

By using the lemma (2.3.3), we obtain $J_n(R)$ is projective. ■

Proposition 2.3.9 *Let M be an R -module. Then*

$$D^n(R, M) \cong \text{Der}^n(R, M) \oplus M.$$

Proof. By the proposition (2.2.12), we have

$$D^n(R, M) \cong \text{Hom}_R(J_n(R), M).$$

And, by the lemma (2.3.3), we get

$$D^n(R, M) \cong \text{Hom}_R(\Omega_n(R), M) \oplus \text{Hom}_R(R, M).$$

By considering the isomorphism

$$\text{Hom}_R(R, M) \cong M,$$

we obtain

$$D^n(R, M) \cong \text{Der}^n(R, M) \oplus M.$$

as desired. ■

2.4 Universal Modules of Local Rings

Lemma 2.4.1 (*Uniqueness Lemma*) *Let R and S be k -algebras and let M be an R -module. Let $f : S \rightarrow R$ be an algebra homomorphism. M is considered as an S -module by means of f . Suppose that:*

if $d : R \rightarrow M$ is a derivation with $df = 0$, then $d = 0$.

Then if $d_1, d_2 : R \rightarrow M$ are differential operators of any order with $d_1f = d_2f$, then $d_1 = d_2$.

Proof. ([17], Lemma 13.1). ■

Theorem 2.4.2 (*Local Extension Lemma*) *Let R be a k -algebra and let S be a multiplicatively closed subset of R . Let $\psi : R \rightarrow R_S$ be the natural map. If M is an R_S -module and d is a differential operator from R into the R_S -module M , then there is a unique differential operator d_S from R_S into M such that $d_S\psi = d$.*

Proof. ([17], Lemma 13.2). ■

Lemma 2.4.3 *Let d be a differential operator of order n on R_S into an R_S -module M and assume that $d(r/1) = 0$ for all $r \in R$. Then $d = 0$.*

Proof. We prove it by induction on n . Let $n = 0$. Then d is an R_S -module homomorphism. So,

$$d(r/s) = r/sd(1) = 0$$

for all $r \in R$ and $s \in S$ which means that $d = 0$. Assume that the lemma is true for differential operators of order less than n . Now, we prove it for n . Let d be a differential operator of order n and let $r \in R$, $s \in S$. Then

$$([d, r/1]1/s - r/s[d, s/1]1/s)(1) = d(r/s) - r/sd(1) = [d, r/s](1). \quad (2)$$

On the other hand, we have

$$[d, r/1](s/1) = d(rs/1) - r/1d(s/1) = 0.$$

As $[d, r/1]$ is a differential operator of order $n - 1$, by the induction hypothesis, we get $[d, r/1] = 0$ for all $r \in R$.

By using the equation given in (2), we obtain

$$[d, r/s](1) = 0.$$

Hence, $d = 0$. ■

Theorem 2.4.4 *Let R be a k -algebra and let S be a multiplicatively closed subset of R . Let $J_n(R_S)$ be the universal module of order n of R_S . Then*

$$J_n(R_S) \cong R_S \otimes_R J_n(R).$$

Proof. Let $\Delta_n : R \rightarrow J_n(R)$ be the universal differential operator of order n of R and let

$$\begin{aligned} \psi : J_n(R) &\rightarrow J_n(R)_S \\ m &\mapsto m/1 \end{aligned}$$

be the canonical map where $m \in J_n(R)$.

Claim. $J_n(R)_S$ is the universal module of differential operators of order n of R_S .

Proof of Claim. We have the following maps:

$$R \xrightarrow{\Delta_n} J_n(R) \xrightarrow{\psi} J_n(R)_S.$$

By the proposition (2.1.5), $\psi\Delta_n$ is a differential operator of order n of R . Since $J_n(R)_S$ is an R_S -module, by the local extension lemma given in (2.4.2), there exists a differential operator δ of order n

$$\delta : R_S \rightarrow J_n(R)_S$$

such that $\delta(r/1) = \psi\Delta_n(r)$ for all $r \in R$. Our aim is to show that δ satisfies the universal property. Let N be an R_S -module and let D be a differential operator of order n of R_S into N . Let us define a map

$$d : R \rightarrow N$$

such that $d(r) = D(r/1)$ for all $r \in R$. By the definition of the map d , we can see that $d \in D^n(R, N)$.

By the universality of $J_n(R)$, there exists a unique R -module homomorphism

$$\alpha : J_n(R) \rightarrow N$$

such that $\alpha\Delta_n = d$. Since N is an R_S -module, α induces a unique R_S -module homomorphism

$$\alpha_S : J_n(R)_S \rightarrow N$$

such that $\alpha_S(m/1) = \alpha(m)$ for all $m \in J_n(R)$. Then for any $r \in R$, we have

$$D(r/1) - \alpha_S\delta(r/1) = D(r/1) - \alpha_S\psi\Delta_n(r) = d(r) - \alpha\Delta_n(r) = 0.$$

That is, $(D - \alpha_S\delta)(r/1) = 0$. By using the lemma (2.4.3), we obtain $D = \alpha_S\delta$.

Therefore, the following diagram commutes:

$$\begin{array}{ccc} R_S & \xrightarrow{D} & N \\ \delta \downarrow & & \parallel \\ J_n(R)_S & \xrightarrow{\alpha_S} & N. \end{array}$$

Thus, δ is the universal differential operator of order n of R_S . By the uniqueness of the universal module, we obtain

$$J_n(R)_S \cong J_n(R_S).$$

On the other hand, by considering the following isomorphism

$$J_n(R)_S \cong R_S \otimes_R J_n(R)$$

we obtain the desired result $J_n(R_S) \cong R_S \otimes_R J_n(R)$. ■

Corollary 2.4.5 *Let R be a k -algebra and let S be a multiplicatively closed subset of R . Let M be an R -module. Then*

$$J_n(M_S) \cong J_n(M)_S.$$

Proof. By the theorems (2.2.14) and (2.4.4), we have

$$\begin{aligned} J_n(M_S) &\cong M_S \otimes_{R_S} J_n(R_S) \cong M_S \otimes_{R_S} (R_S \otimes_R J_n(R)) \\ &\cong M_S \otimes_R J_n(R) \cong R_S \otimes_R (M \otimes_R J_n(R)) \\ &\cong R_S \otimes_R J_n(M) \cong J_n(M)_S \end{aligned}$$

as required. ■

Corollary 2.4.6 *Let R be a k - algebra and S be a multiplicatively closed subset of R . Let $\Omega_n(R_S)$ be the universal module of derivations of order n of R_S . Then*

$$\Omega_n(R_S) \cong R_S \otimes_R \Omega_n(R).$$

Corollary 2.4.7 *Let R be a k -algebra and S be a multiplicatively closed subset of R . Let M be an R -module. Then*

$$\Omega_n(M_S) \cong \Omega_n(M)_S.$$

2.5 Examples of Differential Operators and Their Universal Modules

Let \mathbb{N} be the set of natural numbers and let s be a fixed natural number. Suppose that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_s) \in \mathbb{N}^s$. We shall set the followings:

$$|\alpha| = \alpha_1 + \dots + \alpha_s \text{ and } \alpha! = \alpha_1! \dots \alpha_s!.$$

We say $\alpha \leq \beta$, if $\alpha_i \leq \beta_i$ for all $i = 1, \dots, s$. Let x_1, x_2, \dots, x_s be elements in R where R is a k -algebra, then we write

$$x^\alpha := x_1^{\alpha_1} \dots x_s^{\alpha_s}.$$

Example 2.5.1 Let $R = k[x_1, \dots, x_s]$ be a polynomial algebra with s variables over k . Consider the map

$$\partial_i := \frac{\partial}{\partial x_i} : R \rightarrow R \text{ with } \partial_i(x_j) = \delta_{i,j}$$

for $i, j = 1, \dots, s$ where $\delta_{i,j}$ denotes Kronecker delta function. For any monomial $x^\beta := x_1^{\beta_1} \dots x_s^{\beta_s} \in R$, the partial derivation of order $|\alpha|$ is given by the formula,

$$\partial^\alpha(x^\beta) = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} x^{\beta-\alpha} & \text{if } \beta \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

So, we can conclude that ∂^α is a differential operator of order $|\alpha|$ of R .

The next example shows the relationship between $Der^n(R, A)$ and $D^n(R, A)$.

Example 2.5.2 Let R be a k -algebra and let A be an R -module. Assume that $D \in Hom_k(R, A)$. Then we have

$$D \in D^1(R, A) \text{ if and only if } D - D(1)_R \in Der^1(R, A)$$

where $D(1)_R$ denotes the multiplication map from R into A and is defined by

$$D(1)_R(x) := xD(1).$$

Assume that $D \in D^1(R, A)$. By the definition (2.1.1), we have

$$0 = [D, a_0, a_1] = Da_0a_1 - a_0Da_1 - a_1Da_0 + a_0a_1D.$$

Hence, we see that

$$\begin{aligned} D(a_0a_1) &= a_0D(a_1) + a_1D(a_0) - a_0a_1D(1) \\ &= \sum_{\substack{T \neq \emptyset \\ T \subseteq \{0,1\}}} (-1)^{|T|+1} a_T D(a_{T'}) \end{aligned} \quad (3)$$

where $a_T = \prod_{k \in T} a_k$; T' is the complement of T in $\{0, 1\}$ and $a_0, a_1 \in R$. On the other hand, by using the equality in (3), we see that

$$(D - D(1)_R)(a_0a_1) = a_0(D - D(1)_R)(a_1) + a_1(D - D(1)_R)(a_0)$$

for any $a_0, a_1 \in R$. Hence, $D - D(1)_R$ is a derivation of R into A . Conversely, assume that $D - D(1)_R$ is a derivation. Then by using the equality,

$$(D - D(1)_R)(a_0a_1) = a_0(D - D(1)_R)(a_1) + a_1(D - D(1)_R)(a_0)$$

we obtain that

$$D(a_0a_1) = a_0D(a_1) + a_1D(a_0) - a_0a_1D(1)$$

which is the desired result. Moreover, this result can be generalized as:

$$D \in D^n(R, A) \text{ if and only if } D - D(1)_R \in \text{Der}^n(R, A)$$

(see [18, Lemma 1.2.1]).

Example 2.5.3 Let $R = k[x_1, x_2, \dots, x_s]$ be a polynomial algebra over k with s variables and let D be a differential operator of order n of R . Assume that I is an ideal of R such that $D(I) \subseteq I$. Then D induces a differential operator

$$\bar{D} : R/I \rightarrow R/I$$

of order n . Notice that \bar{D} is defined as $\bar{D}(r + I) = D(r) + I$ and since $D(I) \subseteq I$, it can be easily seen that \bar{D} is well-defined. Furthermore, we have

$$[\bar{D}, \bar{r}_0, \dots, \bar{r}_n] = [D, r_0, \dots, r_n] + I.$$

So, \bar{D} is a differential operator of order n .

Example 2.5.4 [18, Remark 1.1] Let $\{a_i\}_{i \in I}$ be a set of k -algebra generators of R and assume that $D \in \text{Hom}_k(R, A)$ and $[D, a_i] \in D^{n-1}(R, A)$ for every $i \in I$. Then by using the following equality

$$[D, a_1^{\alpha_1} \dots a_n^{\alpha_n}] = \sum_{i=1}^n \frac{\partial(a_1^{\alpha_1} \dots a_n^{\alpha_n})}{\partial a_i} [D, a_i]$$

we conclude that $D \in D^n(R, A)$.

Example 2.5.5 [19, Lemma 1] Let $R = k[x_1, \dots, x_s]/P^{n+1}$ where P is a prime ideal of the polynomial algebra $k[x_1, \dots, x_s]$. Let σ be an automorphism of R such that σ induces the identity on $R = k[x_1, \dots, x_s]/P$.

Claim. σ is a differential operator of order n of R .

Since σ induces the identity on $R = k[x_1, \dots, x_s]/P$, we obtain

$$\sigma(r) - r \in P$$

for any $r \in R$. Hence, we have the following:

$$[\sigma, r_0, \dots, r_n](1) = (\sigma(r_0) - r_0) \dots (\sigma(r_n) - r_n) = 0$$

where $r_0, r_1, \dots, r_n \in R$. So, σ is a differential operator of order n , as required.

Next, we give some examples about universal modules of differential operators of order n .

Example 2.5.6 [1, Prop. 2] Let k be a commutative ring with identity and $A = k[x_\lambda : \lambda \in \Lambda]$ be a polynomial algebra over k with indeterminates $\{x_\lambda : \lambda \in \Lambda\}$. In this case, $A \otimes_k A$ is again a polynomial ring with indeterminates $1 \otimes x_\lambda$ and $x_\lambda \otimes 1$ on the same index set Λ . If we set

$$y_\lambda := 1 \otimes x_\lambda - x_\lambda \otimes 1$$

and identify $x_\lambda \otimes 1$ with x_λ , then $A \otimes_k A$ is a polynomial ring $k[x_\lambda, y_\lambda : \lambda \in \Lambda]$.

So, the kernel of the homomorphism

$$\varphi : A \otimes A \rightarrow A$$

is generated by $\{y_\lambda\}$.

Hence, $\Omega_n(A) = I/I^{n+1}$ is a free module over A with basis

$$\delta_n x_\lambda, \delta_n x_\lambda \delta_n x_\mu, \dots, \delta_n x_{\lambda_1} \dots \delta_n x_{\lambda_n}$$

where δ_n denotes the universal differential operator of order n of A . For any polynomial $f \in A$, we obtain

$$\begin{aligned} \delta_n(f) &= \sum_{\lambda} (\Delta_{\lambda} f) \delta_n x_{\lambda} + \sum_{\lambda, \mu} (\Delta_{\lambda \mu} f) \delta_n x_{\lambda} \delta_n x_{\mu} \\ &\quad \dots + \sum_{\lambda_1 \dots \lambda_n} (\Delta_{\lambda_1 \dots \lambda_n} f) \delta_n x_{\lambda_1} \dots \delta_n x_{\lambda_n}. \end{aligned}$$

On the other hand, by using the given equality

$$\begin{aligned} \delta_n(x_{\lambda_1} \dots x_{\lambda_n}) &= \sum_i x_{\lambda_1} \dots \hat{x}_{\lambda_i} \dots x_{\lambda_n} \delta_n(x_{\lambda_i}) + \\ &\quad \dots + \sum_{i < j} x_{\lambda_1} \dots \hat{x}_{\lambda_i} \dots \hat{x}_{\lambda_j} \dots x_{\lambda_n} \delta_n(x_{\lambda_i}) \delta_n(x_{\lambda_j}) + \dots \end{aligned}$$

we can solve $\delta_n x_\lambda \delta_n x_\mu, \dots, \delta_n x_{\lambda_1} \dots \delta_n x_{\lambda_n}$ in terms of

$$\delta_n(x_\lambda), \delta_n(x_\lambda x_\mu), \dots, \delta_n(x_{\lambda_1} \dots x_{\lambda_n}).$$

Hence,

$$\{\delta_n(x_\lambda), \delta_n(x_\lambda x_\mu), \dots, \delta_n(x_{\lambda_1} \dots x_{\lambda_n})\}$$

forms a basis for $\Omega_n(k[x_\lambda, \lambda \in \Lambda])$. Note that this result is also true for $J_n(A)$, in other words, if A is given as above, then $J_n(A)$ is a free A -module with basis

$$\{\Delta_n(x^\alpha) : |\alpha| \leq n\}$$

where $\Delta_n : A \rightarrow J_n(A)$ be the universal differential operator of order n of A .

Example 2.5.7 Let $K = k(x_1, \dots, x_s)$ be the field of fractions of $k[x_1, \dots, x_s]$. Then by the following isomorphism given in (2.4.4)

$$J_n(K) \cong K \otimes_R J_n(R),$$

we obtain that $J_n(K)$ is a K -vector space with basis

$$\{\Delta_n(x^\alpha) : |\alpha| \leq n\}$$

where $\Delta_n : K \rightarrow J_n(K)$ is the universal differential operator of order n of K .

Example 2.5.8 Let F be a free module of finite rank over a polynomial algebra $R = k[x_1, \dots, x_s]$ with basis e_1, \dots, e_t . Let M be a free R -module with basis

$$\{m_{\alpha,i} : i = 1, \dots, t \text{ and } |\alpha| \leq n\}.$$

Let $\Delta_n : F \rightarrow M$ be a k -linear transformation defined by $\Delta_n(x^\alpha e_i) = m_{\alpha,i}$. Suppose that N is the submodule of M generated by all the relations

$$\{[\Delta_n, r_0, r_1, \dots, r_n](e_i) : r_j \in R, i = 1, \dots, t\}$$

and we have the natural map

$$\pi : M \rightarrow M/N.$$

Claim. The composition map

$$\pi\Delta_n : F \rightarrow M/N$$

is a differential operator of order n .

Proof of Claim. We need to show that

$$[\pi\Delta_n, r_0, \dots, r_n] = 0$$

for any $r_0, \dots, r_n \in R$. Notice that we have the following equality

$$[\pi\Delta_n, r_0, \dots, r_n](e_i) = \pi[\Delta_n, r_0, \dots, r_n](e_i),$$

and by considering the definition of N , we get

$$[\pi\Delta_n, r_0, \dots, r_n](e_i) = 0$$

for each $i = 1, \dots, t$. Moreover, M/N is the universal module of differential operators of order n of F and the composite map $\pi\Delta_n$ is the universal differential operator of F . So, $J_n(F) = M/N$. On the other hand, we have the following isomorphism

$$J_n(F) \cong J_n(R) \otimes_R F$$

given by $\pi\Delta_n(x^\alpha e_i) = \delta_n(x^\alpha) \otimes e_i$ where δ_n is the universal differential operator

$$\delta_n : R \rightarrow J_n(R)$$

and we know by the example (2.5.6) that $J_n(R)$ is a free R -module. So, $J_n(R) \otimes F$ is a free F -module with basis

$$\{\delta_n(x^\alpha) \otimes e_i : i = 1, \dots, n \text{ and } |\alpha| \leq n\}.$$

Hence, $J_n(F)$ is a free F -module with basis

$$\{\pi\Delta_n(x^\alpha e_i) : i = 1, \dots, t \text{ and } |\alpha| \leq n\}.$$

Now, we give some examples on the module of differential operators of order n .

Example 2.5.9 Let $R = k[x_1, \dots, x_s]$ be a polynomial algebra with s variables over k . Then

$$D^n(R) = \bigoplus_{|\alpha| \leq n} R \partial^\alpha$$

where ∂^α is defined as in the example (2.5.1).

Example 2.5.10 Let $K = k(x_1, \dots, x_s)$ be the field of fractions of $k[x_1, \dots, x_s]$. Then $D^n(K)$ is a K -vector space with basis

$$\{\partial^\alpha : |\alpha| \leq n\}.$$

Example 2.5.11 Let $R = k[x_1, \dots, x_s]$ be a polynomial algebra and let $S = R/I$. Then there is a well-defined map

$$\phi : \{D \in D^n(R) : D(I) \subseteq I\} \rightarrow D^n(S)$$

where $\phi(D)(\bar{r}) = \overline{D(r)}$. If $\phi(D) = 0$, then $\overline{D(r)} = 0$ and this means that $D(r) \in I$ for all $r \in R$. Hence, $D(R) \subseteq I$. Now, assume that $f \in D^n(S)$ and consider the natural map $\pi : R \rightarrow S$. Then by the proposition (2.1.5), we have

$$f\pi : R \rightarrow S$$

is a differential operator of order n of R . By the universality of $J_n(R)$, there exists a unique R -module homomorphism

$$\alpha : J_n(R) \rightarrow S$$

such that $\alpha \Delta_n = f\pi$. Notice that we have the following diagram

$$\begin{array}{ccc} & J_n(R) & \\ & \downarrow \alpha & \\ R & \xrightarrow{\pi} & S \end{array}$$

and by the example (2.5.6), we know that $J_n(R)$ is a free R -module.

So, there exists an R -module homomorphism

$$\beta : J_n(R) \rightarrow R$$

such that $\pi\beta = \alpha$. Moreover, we have

$$\pi\beta\Delta_n(I) = \alpha\Delta_n(I) = f\pi(I) = 0$$

which illustrates that $\beta\Delta_n(I) \subseteq I$. On the other hand, we obtain

$$\phi(\beta\Delta_n)(\bar{r}) = \overline{\beta\Delta_n(r)} = \alpha\Delta_n(r) = f\pi(r) = f(\bar{r}).$$

Hence, $\phi(\beta\Delta_n) = f$ and this ensures that ϕ is surjective. Therefore, we obtain the following important isomorphism:

$$\{D \in D^n(R) : D(I) \subseteq I\} / \{D \in D^n(R) : D(R) \subseteq I\} \cong D^n(S).$$

2.6 Universal Modules of Factor Rings

Let R and S be k -algebras and let

$$h : R \rightarrow S$$

be a k -algebra homomorphism. Assume that $J_n(R)$ and $J_n(S)$ are universal modules of order n of R and S , respectively and let

$$\Delta_n : R \rightarrow J_n(R)$$

and

$$\delta_n : S \rightarrow J_n(S)$$

be the universal differential operators of order n of R and S . By the k -algebra homomorphism h , we can regard $J_n(S)$ as an R -module. By the proposition (2.1.5), we know

$$\delta_n h \in D^n(R, J_n(S)).$$

By the universal property of $J_n(R)$, there exists a unique R -module homomorphism

$$h^* : J_n(R) \rightarrow J_n(S)$$

such that $h^* \Delta_n = \delta_n h$, that is, the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{h} & S \\ \Delta_n \downarrow & & \delta_n \downarrow \\ J_n(R) & \xrightarrow{h^*} & J_n(S). \end{array}$$

Since $J_n(S)$ is an S -module, we can define the following S -module homomorphism:

$$\theta : S \otimes_R J_n(R) \rightarrow J_n(S)$$

such that

$$\theta(\sum_i s_i \otimes \Delta_n(r_i)) = \sum_i s_i \delta_n(h(r_i))$$

where $r_i \in R$ and $s_i \in S$.

Let $h : R \rightarrow R/I$ where I is an ideal of R . If we consider the isomorphism

$$R/I \otimes_R J_n(R) \cong J_n(R)/IJ_n(R),$$

then θ can be defined as following:

$$\theta(\overline{\sum_i r_i \Delta_n(x_i)}) = \sum_i \bar{r}_i \delta_n(\bar{x}_i)$$

where $\bar{r}_i \in R/I$ and $x_i \in R$.

Proposition 2.6.1 *Let R be a k -algebra and let I be an ideal of R . Suppose that N is a submodule of $J_n(R)$ generated by the elements of the form*

$$\{\Delta_n(x) : x \in I\}.$$

Then we have the following short exact sequence of R/I -modules:

$$0 \rightarrow \frac{N + IJ_n(R)}{IJ_n(R)} \rightarrow \frac{J_n(R)}{IJ_n(R)} \xrightarrow{\theta} J_n(R/I) \rightarrow 0. \quad (4)$$

Proof. By the definition of the map θ , it is easy to see that it is surjective. To prove the exactness of the sequence in (4), we need to show that $\ker\theta = \frac{N + IJ_n(R)}{IJ_n(R)}$. For any $x \in I$, we have $\theta(\overline{\Delta_n(x)}) = \delta_n(\bar{x}) = 0$ which shows that

$$\frac{N + IJ_n(R)}{IJ_n(R)} \subseteq \ker\theta.$$

Then θ induces a unique R/I -module homomorphism

$$\frac{J_n(R)/IJ_n(R)}{N + IJ_n(R)/IJ_n(R)} \xrightarrow{\bar{\theta}} J_n(R/I)$$

and $\ker\bar{\theta} = \frac{\ker\theta}{N + IJ_n(R)/IJ_n(R)}$.

Claim. $\bar{\theta}$ is one-to-one.

Proof of Claim. Let us consider the following maps:

$$R \xrightarrow{\Delta_n} J_n(R) \xrightarrow{\pi_1} \frac{J_n(R)}{IJ_n(R)} \xrightarrow{\pi_2} \frac{J_n(R)/IJ_n(R)}{N + IJ_n(R)/IJ_n(R)}$$

where π_1 and π_2 are natural maps.

By the proposition (2.1.5), we have

$$\pi_2\pi_1\Delta_n \in D^n\left(R, \frac{J_n(R)/IJ_n(R)}{N+IJ_n(R)/IJ_n(R)}\right).$$

Since $\pi_2\pi_1\Delta_n(I) = 0$, it reduces a unique map $\overline{\Delta_n}$,

$$R/I \xrightarrow{\overline{\Delta_n}} \frac{J_n(R)/IJ_n(R)}{N+IJ_n(R)/IJ_n(R)}.$$

Besides $\overline{\Delta_n}$ is a differential operator of order n . By the universal property of $J_n(R/I)$, there exists a unique R/I -module homomorphism

$$\Psi : J_n(R/I) \rightarrow \frac{J_n(R)/IJ_n(R)}{N+IJ_n(R)/IJ_n(R)}$$

such that $\Psi\bar{\theta} = 1$. So, $\bar{\theta}$ is one-to-one. Then we obtain

$$\ker\theta \subseteq N + IJ_n(R)/IJ_n(R)$$

and it is the desired result. ■

Corollary 2.6.2 *Let R be a k -algebra and let I be an ideal of R . Suppose that N is the submodule of $\Omega_n(R)$ generated by the elements of the form*

$$\{\delta_n(x) : x \in I\}.$$

Then we have the following short exact sequence of R/I -modules:

$$0 \rightarrow \frac{N + I\Omega_n(R)}{I\Omega_n(R)} \rightarrow \frac{\Omega_n(R)}{I\Omega_n(R)} \xrightarrow{\theta} \Omega_n(R/I) \rightarrow 0. \quad (5)$$

Proposition 2.6.3 *Let $R = k[x_1, x_2, \dots, x_s]$ be a polynomial k -algebra with s variables and I be an ideal of R generated by the set $\{f_1, \dots, f_t\}$ and let*

$$\Delta_n : R \rightarrow J_n(R)$$

be the universal differential operator of order n of R . Assume that L is the submodule of $J_n(R)$ generated by the set

$$\{\Delta_n(x^\alpha f_i) : 0 \leq |\alpha| < n, i = 1, \dots, t\}$$

where $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s}$ and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_s$. Then

$$R\Delta_n(I) \subseteq L + IJ_n(R).$$

Proof. It is known that Δ_n is k -linear. We only need to show

$$\Delta_n(f_i g) \in L + IJ_n(R)$$

where $g \in R$. We can write g as following:

$$g = \sum_{\alpha} a_{\alpha} x^{\alpha} + \sum_{\beta} b_{\beta} x^{\beta}, \quad |\alpha| \geq n, \quad |\beta| < n$$

where $a_{\alpha}, b_{\beta} \in k$. Then

$$\Delta_n(g f_i) = \sum_{\alpha} a_{\alpha} \Delta_n(f_i x^{\alpha}) + \sum_{\beta} b_{\beta} \Delta_n(f_i x^{\beta}).$$

Since $|\alpha| \geq n$, we can write $\Delta_n(f_i x^{\alpha})$ as

$$\Delta_n(f_i x^{\alpha}) = \sum_{\mu} c_{\mu} \Delta_n(f_i x^{\mu}) + f_i \left(\sum_{\gamma} d_{\gamma} \Delta_n(x^{\gamma}) \right), \quad |\mu| < n, \quad |\gamma| \leq n$$

where $c_{\mu}, d_{\gamma} \in R$. By considering the above equations, we get

$$\Delta_n(g f_i) = \sum_{\alpha} \sum_{\mu} a_{\alpha} c_{\mu} \Delta_n(f_i x^{\mu}) + \sum_{\beta} b_{\beta} \Delta_n(f_i x^{\beta}) + f_i \sum_{\alpha} \sum_{\gamma} a_{\alpha} d_{\gamma} \Delta_n(x^{\gamma}) \in L + IJ_n(R).$$

This ensures that $R\Delta_n(I) \subseteq L + IJ_n(R)$. ■

Proposition 2.6.4 $\frac{R\Delta_n(I) + IJ_n(R)}{IJ_n(R)}$ is generated by the set

$$\{\Delta_n(f_i x^{\alpha}) + IJ_n(R) : |\alpha| < n, \quad i = 1, \dots, t\}$$

as an R/I -module.

Proof. Let L be as above. Then $\frac{L + IJ_n(R)}{IJ_n(R)}$ is generated by

$$\{\overline{\Delta_n(f_i x^{\alpha})} : |\alpha| < n, \quad i = 1, \dots, t\}.$$

On the other hand, by proposition (2.6.3), we know $R\Delta_n(I) \subseteq L + IJ_n(R)$. Hence, we see

$$\frac{R\Delta_n(I) + IJ_n(R)}{IJ_n(R)} = \frac{L + IJ_n(R)}{IJ_n(R)}$$

as stated. ■

Corollary 2.6.5 $J_n(R/I)$ is generated by the set

$$\{\delta_n(x^\alpha + I) : |\alpha| \leq n\}$$

with relations

$$\theta(\Delta_n(f_i x^\alpha) + IJ_n(R))$$

where $\delta_n : R/I \rightarrow J_n(R/I)$ is the universal differential operator of order n of R/I .

Proof. By the example (2.5.6), $J_n(R)$ is a free R -module with basis

$$\{\Delta_n(x^\alpha) : |\alpha| \leq n\}.$$

By considering the isomorphism

$$\frac{J_n(R)}{IJ_n(R)} \cong \frac{R}{I} \otimes_R J_n(R),$$

we obtain that $\frac{J_n(R)}{IJ_n(R)}$ is a free R/I -module with basis

$$\{\overline{\Delta_n(x^\alpha)} : |\alpha| \leq n\}.$$

Moreover, we have the following exact sequence

$$\frac{J_n(R)}{IJ_n(R)} \xrightarrow{\theta} J_n(R/I) \rightarrow 0.$$

So, $J_n(R/I)$ is generated by the set

$$\{\theta(\overline{\Delta_n(x^\alpha)}) : |\alpha| \leq n\}$$

and this set equals to $\{\delta_n(x^\alpha + I) : |\alpha| \leq n\}$. The relations are determined by the generators of $\ker\theta$. Hence, by the proposition (2.6.4), we get the result. ■

2.7 Relation between Universal Modules and Vector Spaces

In this subsection, we give some relations between universal modules and vector spaces. Our aim is to prove that $J_n(R) \otimes_R k \cong R/m^{n+1}$ where R is a k -algebra, \mathfrak{m} is a maximal ideal of R and $R/\mathfrak{m} \cong k$.

Lemma 2.7.1 *Let R be a k -algebra and \mathfrak{m} be an ideal of R . Let M and N be R -modules. Then*

$$D_R^n(M, N)(\mathfrak{m}^{n+i}M) \subseteq \mathfrak{m}^i N.$$

Proof. We prove it by induction on n . Firstly, assume that $n = 0$. Then we have

$$D_R^0(M, N)(\mathfrak{m}^i M) = \text{Hom}_R(M, N)(\mathfrak{m}^i M) \subseteq \mathfrak{m}^i N.$$

Suppose that the result is true for all values less than n . We need to prove it for n , that is, we need to show the following:

$$D_R^n(M, N)(\mathfrak{m}^{n+i}M) \subseteq \mathfrak{m}^i N.$$

To show it we use induction on i . Let $i = 0$. Then the result is obvious. Assume that the result is true for i . Let $\theta \in D_R^n(M, N)$ and by the definition of differential operator, we know that

$$[\theta, r] \in D_R^{n-1}(M, N)$$

for any $r \in R$. So, we have

$$\begin{aligned} \theta(\mathfrak{m}^{n+i+1}M) &\subseteq \mathfrak{m}^{i+1}N + \mathfrak{m}\theta(\mathfrak{m}^{n+i}M) \text{ (by induction hypothesis on } n) \\ &\subseteq \mathfrak{m}^{i+1}N + \mathfrak{m}(\mathfrak{m}^i N) \text{ (by induction hypothesis on } i) \\ &\subseteq \mathfrak{m}^{i+1}N. \end{aligned}$$

Then

$$D_R^n(M, N)(\mathfrak{m}^{n+i+1}M) \subseteq \mathfrak{m}^i N$$

as stated. ■

Lemma 2.7.2 *Let R be a k -algebra and let m be a maximal ideal of R such that $R/m \cong k$. Then we have the following isomorphism of k -vector spaces:*

$$\begin{aligned} \phi : D^n(R, k) &\rightarrow \text{Hom}_k(R/m^{n+1}, k) \\ D &\mapsto \tilde{D} \end{aligned}$$

where $\tilde{D}(r + m^{n+1}) = D(r)$ for any $r \in R$.

Proof. Let $D \in D^n(R, k)$. Then $D(m^{n+1}) = 0$. So, D induces a k -linear map

$$\tilde{D} : R/m^{n+1} \rightarrow k$$

such that $\tilde{D}\pi = D$ where π is the natural surjection

$$\pi : R \rightarrow R/m^{n+1}.$$

Hence, we have a well-defined map

$$\begin{aligned} \phi : D^n(R, k) &\rightarrow \text{Hom}_k(R/m^{n+1}, k) \\ D &\mapsto \tilde{D}. \end{aligned}$$

Our aim is to show that ϕ is a k -vector space isomorphism. Assume $\phi(D) = 0$. Then

$$0 = \tilde{D}(\bar{r}) = \tilde{D}\pi(r) = D(r)$$

for any $r \in R$. Hence, $D = 0$ which means that ϕ is injective. Let $\alpha \in \text{Hom}_k(R/m^{n+1}, k)$.

Then the composite of the following maps

$$R \xrightarrow{\pi} R/m^{n+1} \xrightarrow{\alpha} k$$

is an element of $\text{Hom}_k(R, k)$. Let $r_0, r_1, \dots, r_n \in R$. Then $r_i = x_i + l_i$ where $x_i \in m$ and $l_i \in k$.

$$\begin{aligned} [\alpha\pi, r_0, \dots, r_n](R) &= [\alpha\pi, x_0, \dots, x_n](R) \\ &\subseteq (\alpha\pi x_0 \dots x_n)(R) + m\alpha\pi(R) \\ &\subseteq \alpha\pi(m^{n+1}R) + m\alpha\pi(R) \\ &= 0. \end{aligned}$$

So, $\alpha\pi \in D^n(R, k)$ and $\phi(\alpha\pi) = \alpha$. And, this ensures that ϕ is surjective. ■

Corollary 2.7.3 *Let R be a Noetherian k -algebra and let m be a maximal ideal of R such that $R/m \cong k$. Then*

$$J_n(R) \otimes_R k \cong R/m^{n+1}$$

as k -vector spaces.

Proof. By the proposition (2.2.12), we have the following isomorphism

$$D^n(R, k) \cong \text{Hom}_R(J_n(R), k).$$

Since \otimes and Hom functors are adjoint operators, we get

$$\text{Hom}_R(J_n(R), k) \cong \text{Hom}_k(J_n(R) \otimes_R k, k).$$

So, by using the lemma (2.7.2),

$$\text{Hom}_k(R/m^{n+1}, k) \cong \text{Hom}_k(J_n(R) \otimes_R k, k).$$

Since R is Noetherian, R/m^{n+1} is a finite dimensional k -vector space. Therefore,

$$J_n(R) \otimes_R k \cong R/m^{n+1}$$

as required. ■

2.8 Universal Modules of Field Extensions

Let L and K be field extensions of k such that $K \subseteq L$. In this subsection, we give the relations between $J_n(K)$ and $J_n(L)$.

Theorem 2.8.1 *Let L and K be field extensions of k such that $K \subseteq L$. Let $\delta_n : K \rightarrow J_n(K)$ and $\Delta_n : L \rightarrow J_n(L)$ be the universal differential operators of order n of K and L , respectively. If L is a finite dimensional extension of K , that is, $\dim_K L$ is finite, then*

$$\begin{aligned} \theta : L \otimes_K J_n(K) &\rightarrow J_n(L) \\ \sum_i l_i \otimes_K \delta_n(x_i) &\mapsto \sum_i l_i \Delta_n(x_i) \end{aligned}$$

is an isomorphism of L -spaces.

Proof. ([17], Theorem (13.12)). ■

Proposition 2.8.2 *Let L be the field of fractions of an affine domain over a field k with transcendence basis $\{x_1, \dots, x_s\}$. Then $J_n(L)$ is an L -vector space with basis*

$$\{\Delta'_n(x^\alpha) : |\alpha| \leq n, x^\alpha = x_1^{\alpha_1} \dots x_s^{\alpha_s}\}$$

where $\Delta'_n : L \rightarrow J_n(L)$ is the universal operator of order n of L .

Proof. Let $K = k(x_1 \dots x_s)$. Then L is a finite dimensional extension of K . By the example (2.5.7), $J_n(K)$ is a K -vector space with basis

$$\{\Delta_n(x^\alpha) : |\alpha| \leq n, x^\alpha = x_1^{\alpha_1} \dots x_s^{\alpha_s}\}$$

where

$$\Delta_n : K \rightarrow J_n(K)$$

is the universal differential operator of order n of K . Therefore, $L \otimes_K J_n(K)$ is an L -vector space with basis

$$\{1 \otimes \Delta_n(x^\alpha) : |\alpha| \leq n, x^\alpha = x_1^{\alpha_1} \dots x_s^{\alpha_s}\}.$$

By the theorem (2.8.1), we know the following isomorphism:

$$L \otimes_K J_n(K) \cong J_n(L).$$

So, $J_n(L)$ is an L -vector space with basis

$$\{\Delta'_n(x^\alpha) : |\alpha| \leq n, x^\alpha = x_1^{\alpha_1} \dots x_s^{\alpha_s}\}$$

where Δ'_n is the universal differential operator of order n of L . ■

Corollary 2.8.3 *Suppose L is the field of fractions of an affine domain such that $\{x_1, \dots, x_s\}$ is a transcendence basis of L over k . Then*

$$L \otimes_R D^n(K) \cong D^n(L)$$

where $K = k(x_1 \dots x_s)$.

Proof. By the proposition (2.2.12), we have the following isomorphisms

$$D^n(K) \cong \text{Hom}_K(J_n(K), K) \text{ and } D^n(L) \cong \text{Hom}_L(J_n(L), L).$$

Since $J_n(K)$ is a finite dimensional vector space over K , we have

$$L \otimes_K \text{Hom}_K(J_n(K), K) \cong \text{Hom}_L(J_n(L), L).$$

Therefore, we get

$$L \otimes_R D^n(K) \cong D^n(L)$$

as required. ■

Lemma 2.8.4 *Let K and L be field extensions of k such that $K \subseteq L$ and L is algebraic over K . Let M be an L -module and let*

$$\delta : L \rightarrow M$$

be a differential operator on L . If δ is K -linear, then δ is L -linear.

Proof. Assume that $\delta \in D^n(L, M)$ is of the smallest degree which is K -linear but not L -linear. Since δ is K -linear, $[\delta, x]$ is K -linear for any $x \in L$. By the assumption, we know that L is algebraic over K . So, there exists a minimal polynomial

$$p(t) = \sum_n a_n t^n$$

with $a_n \in K$ such that $p(x) = 0$ for all $x \in L$.

Since $[\delta, x]$ is L -linear, we have the following equalities:

$$\begin{aligned}
[\delta, x^r] &= \delta x^r - x^r \delta \\
&= \delta x^r - x \delta x^{r-1} + x \delta x^{r-1} - x^r \delta \\
&= [\delta, x] x^{r-1} + x [\delta, x^{r-1}] \\
&= x^{r-1} [\delta, x] + x [\delta, x^{r-1}].
\end{aligned}$$

Hence, by induction we have

$$[\delta, x^n] = n x^{n-1} [\delta, x].$$

Then

$$\begin{aligned}
0 &= [\delta, p(x)] = [\delta, \sum_n a_n x^n] \\
&= \sum_n a_n [\delta, x^n] \\
&= \sum_n n a_n x^{n-1} [\delta, x] \\
&= p'(x) [\delta, x].
\end{aligned}$$

By the minimality of $p(x)$, $p'(x) \neq 0$. Therefore, $[\delta, x] = 0$ which is a contradiction as we assume that δ is not L -linear. Hence, we get δ is L -linear as claimed. ■

Proposition 2.8.5 *Let K and L be field extensions of k such that $K \subseteq L$ and L is algebraic over K . Let M be an L -module and let δ is a differential operator of L into M . If $\delta(K) = 0$, then $\delta(L) = 0$.*

Proof. Let $\delta \in D^n(L, M)$ is of the smallest degree such that it is non-zero, but its restriction to K is zero. So, we have $[\delta, x](K) = 0$ for each $x \in K$. By the minimality, $[\delta, x] = 0$ which means that δ is K -linear. By the lemma (2.8.4), δ is L -linear. Since $\delta(1) = 0$, we get $\delta(L) = 0$. ■

2.9 Universal Modules of Regular Algebras

Firstly, we develop basic tools of regular algebras. The following definitions and results can be found in [20, Chapter 15].

Definition 2.9.1 *Let R be a non-trivial commutative ring. An expression*

$$P_0 \subset P_1 \subset \dots \subset P_n$$

in which P_0, \dots, P_n are prime ideals of R , is called a chain of prime ideals of R ; the length of such a chain is the number of 'links'.

Definition 2.9.2 *The dimension of R , denoted by $\dim R$, is defined to be*

$$\sup\{n \in \mathbb{N} : \text{there exists a chain of prime ideals of } R \text{ of length } n\}$$

if this supremum exists, and ∞ otherwise.

Definition 2.9.3 *Let $P \in \text{Spec}(R)$. Then the height of P , denoted by htP , is defined to be the supremum of lengths of chains*

$$P_0 \subset P_1 \subset \dots \subset P_n$$

of prime ideals of R for which $P_n = P$ if this supremum exists, and ∞ otherwise.

Definition 2.9.4 *Let R be a Noetherian local ring with maximal ideal m . Then R is said to be regular if*

$$\dim R = v\dim_{R/m} m/m^2$$

where $v\dim$ denotes the vector space dimension.

Remark 2.9.5 *Let R be a Noetherian local ring with maximal ideal m . Then R is regular precisely when m can be generated by $\dim R$ elements.*

Example 2.9.6 *Let R be a commutative Noetherian ring, and suppose that there exists a prime ideal P which has height n and can be generated by $htP = n$ elements $\{a_1, \dots, a_n\}$. Then the localization of R at P , R_P , is a regular local ring of dimension n , because by [20, Remarks 14.18 (iv) and (v)] we have*

$$\dim R_P = ht_{R_P} PR_P = htP = n,$$

and its maximal ideal

$$PR_P = \left(\sum_{i=1}^n Ra_i\right)R_P = \sum_{i=1}^n R_P \frac{a_i}{1}$$

can be generated by n elements.

By this example, we get a way to construct substantial supply of examples of regular local rings.

Example 2.9.7 Let p be a prime number. Then, $p\mathbb{Z}$ is a prime ideal in the ring \mathbb{Z} and we have $ht(p\mathbb{Z}) = 1$. Besides, it is generated by 1 element, it follows from the example (2.9.6) that $\mathbb{Z}_{p\mathbb{Z}}$ is a regular local ring of dimension 1. Hence, \mathbb{Z} is a regular ring of dimension 1.

Definition 2.9.8 Let R be a Noetherian regular local ring with maximal ideal m . A regular system of parameters for R is a set of $\dim R$ elements which generate m .

Note that, from now on, we'll consider R to be a Noetherian local k -algebra with maximal ideal m such that $R/m \cong k$ under the natural map. Since R is Noetherian, then for each $i \geq 0$, m^i/m^{i+1} is a finite dimensional k -vector space. Let us denote $m^0 = R$. It is clear that, for each $i \geq 0$, we have the following short exact sequence of R -modules:

$$0 \rightarrow m^i/m^{i+1} \rightarrow R/m^{i+1} \rightarrow R/m^i \rightarrow 0.$$

Inductively, we obtain that

$$\dim_k R/m^{n+1} = \sum_{i=0}^n \dim_k m^i/m^{i+1}.$$

Theorem 2.9.9 Let R be a regular local k -algebra such that $R/m \cong k$ under the natural map and let $\{x_1, \dots, x_s\}$ be a regular system of parameters for R . Then

$$\dim_k R/m^{n+1} = \binom{n+s}{s}.$$

Proof. [21, Theo. 2.9, p.119] ■

Lemma 2.9.10 Let R be a domain and let L be the field of fractions of R . If M is a finitely generated R -module, then M is free if and only if

$$\dim_L L \otimes_R M = \mu(M)$$

where $\mu(M)$ denotes the number of elements in the minimal generating set of M .

Proof. Firstly, note that the dimension of $L \otimes_R M$ is called the *rank* of M . Assume that M is a free R -module. Then for some n ,

$$M \cong \bigoplus_n R.$$

Hence, we get

$$\dim_L L \otimes_R M \cong \dim_L \bigoplus_n L = n.$$

Therefore, we obtain $\dim_L L \otimes_R M = \mu(M) = n$. Conversely, let $\mu(M) = t$ and M is generated by the elements $\{m_1, m_2, \dots, m_t\}$. Then we have

$$0 \rightarrow \ker\theta \rightarrow R^t \xrightarrow{\theta} M \rightarrow 0 \quad (6)$$

short exact sequence of R -modules where the map θ is defined as following:

$$\begin{aligned} \theta : R^t &\rightarrow M \\ e_i &\mapsto m_i \end{aligned}$$

and $\{e_1, \dots, e_t\}$ forms a free basis for R^t . If we tensor the exact sequence given in (6) by L and consider the fact that L is a flat R -module, then we obtain the following short exact sequence of vector spaces:

$$0 \rightarrow L \otimes_R \ker\theta \rightarrow L \otimes_R R^t \rightarrow L \otimes_R M \rightarrow 0. \quad (7)$$

By the assumption,

$$\dim_L L \otimes_R M = \mu(M) = t.$$

Then we get $L \otimes_R \ker\theta = 0$ and so, $\ker\theta$ is a torsion submodule of R^t . Since R^t is a free R -module, we have $\ker\theta = 0$ and this ensures that M is a free R -module. ■

Lemma 2.9.11 *Let R be a commutative Noetherian ring and let M be a finitely generated R -module. Then M is projective if and only if M is locally projective.*

Proof. We denote the projective dimension of M by $pd(M)$. If M is projective, then $pd(M) = 0$. By using the following fact:

$$pd(M) = \sup_m \{pd(M_m) : m \text{ maximal ideal of } R\}$$

we get $pd(M_m) = 0$ for every maximal ideal m , and this means that M is locally projective. The other side can be proved similarly, by using the above fact. ■

Proposition 2.9.12 *Let R be an s -dimensional regular local k -algebra with maximal ideal m such that $R/m \cong k$. Let $\{x_1, \dots, x_s\}$ be a regular system of parameters for R . Then*

$$J_n(R) = \bigoplus_{|\alpha| \leq n} R\Delta_n(x^\alpha)$$

where $x^\alpha = x_1^{\alpha_1} \dots x_s^{\alpha_s}$ and $|\alpha| = \alpha_1 + \dots + \alpha_s$ and Δ_n is the universal differential operator of order n of R . Therefore, $J_n(R)$ is a free R -module.

Proof. Let $\{x_1, \dots, x_s\}$ be a regular system of parameters for R . Then we can conclude, as a result of the theorem (2.9.9), that the set

$$\{\vartheta + m^{n+1} : \vartheta \text{ is a monomial in } x_1, \dots, x_s; 0 \leq \deg \vartheta \leq n\}$$

forms a k -basis of R/m^{n+1} . By the corollary (2.7.3), we have the following k -vector space isomorphism:

$$J_n(R) \otimes_R k \cong R/m^{n+1}.$$

So, $J_n(R) \otimes_R k$ is a k -vector space with basis

$$\{\Delta_n(\vartheta) \otimes 1 : \vartheta \text{ is a monomial in } x_1, \dots, x_s; 0 \leq \deg \vartheta \leq n\}$$

where $\Delta_n : R \rightarrow J_n(R)$ is the universal differential operator of order n of R . By using the isomorphism

$$J_n(R) \otimes_R R/m \cong J_n(R)/mJ_n(R),$$

we have

$$\{\Delta_n(\vartheta) + mJ_n(R) : \vartheta \text{ is a monomial in } x_1, \dots, x_s; 0 \leq \deg \vartheta \leq n\}$$

is a k -basis for $J_n(R)/mJ_n(R)$. Therefore, by Nakayama's lemma we obtain

$$\{\Delta_n(\vartheta) : \vartheta \text{ is a monomial in } x_1, \dots, x_s; 0 \leq \deg \vartheta \leq n\}$$

is a minimal set of generators of $J_n(R)$. Our aim is to show that

$$\mu(J_n(R)) = \text{rank} J_n(R).$$

Then by the lemma (2.9.10), we can conclude that $J_n(R)$ is a free R -module. Since R is a regular local k -algebra, by [20, Theo. 15.34], R is an integral domain.

Let L be the field of fractions of R , then we have the following isomorphism:

$$\theta : L \otimes_R J_n(R) \rightarrow J_n(L).$$

Hence,

$$\text{rank} J_n(R) = \dim_L L \otimes_R J_n(R) = \dim_L L.$$

Since L is an algebraic extension of $k(x_1, \dots, x_s)$, by the proposition (2.8.2), we get

$$\dim_L L = \binom{n+s}{s}.$$

Therefore, $\mu(J_n(R)) = \text{rank} J_n(R)$ as required. ■

Corollary 2.9.13 *Let R be an s -dimensional regular local k -algebra with maximal ideal m such that $R/m \cong k$ and let F be a free R -module. Then $J_n(F)$ is free.*

Proof. Let $\{x_1, \dots, x_s\}$ be a regular system of parameters for R and let $\{e_i\}$ be a basis for F . As a consequence of the proposition (2.9.12), $J_n(R) \otimes_R F$ is a free module with basis

$$\{\Delta_n(x^\alpha) \otimes e_i : |\alpha| \leq n, i = 1, \dots, t\}$$

where Δ_n is the universal operator of order n of R . Notice that by the theorem (2.2.14), we have the following isomorphism:

$$J_n(R) \otimes_R F \cong J_n(F).$$

So, we obtain $J_n(F)$ is free. ■

Corollary 2.9.14 *Let R be a regular affine k -algebra such that for each maximal ideal m of R , $R_m/mR_m \cong k$. Then $J_n(R)$ is a projective R -module.*

Proof. For each maximal ideal m of R , R_m is a regular local ring. Then by the theorem (2.9.12), $J_n(R_m)$ is a free R_m -module. By the theorem (2.2.14), we have the following isomorphism

$$J_n(R_m) \cong R_m \otimes_R J_n(R).$$

So, we get $J_n(R)$ is locally projective. Hence, by using (2.9.11), we get $J_n(R)$ is projective. ■

Corollary 2.9.15 *Let R be a regular affine k -algebra such that for each maximal ideal m of R , $R_m/mR_m \cong k$ and let F be a finitely generated projective R -module. Then so is $J_n(F)$.*

Proof. Since F is finitely generated projective over R , then by [29, Corol. 3.5], F_m is finitely generated free over R_m . So, by the corollary (2.9.13), $J_n(F_m)$ is a free R_m -module for each maximal ideal m of R . Hence, by the lemma (2.9.11) we get the result.

■

3 PROJECTIVE DIMENSION OF THE UNIVERSAL MODULE OF DIFFERENTIAL OPERATORS

In this section, we give some important and well-known theorems to estimate the projective dimension of the universal module of differential operators of order n . Actually, we see that there exists an upper bound for the projective dimension, if R is of the form $k[x_1, \dots, x_s]/(f)$. And next, we provide some examples to illustrate these results. Moreover, in contrast to the given case, we see in the example (3.1.5) that it is difficult to find an upper bound for the universal module of differential operators if R is not of the form $k[x_1, \dots, x_s]/(f)$.

3.1 Characterizing the Projective Dimension of the Universal Module of Differential Operators

If R is a regular affine algebra, then by using the result given in (2.9.14) we can conclude that $J_n(R)$ is a projective R -module. Hence, $pd(J_n(R)) = 0$.

Theorem 3.1.1 [9] *Let S be an affine domain represented by*

$$S = k[x_1, \dots, x_s]/(f).$$

Then

$$pd(J_n(S)) \leq 1.$$

Proof. Let $R = k[x_1, \dots, x_s]$ and let $\Delta_n : R \rightarrow J_n(R)$ be the universal differential operator of order n of R . Then we have

$$0 \rightarrow \frac{N + IJ_n(R)}{IJ_n(R)} \rightarrow \frac{J_n(R)}{IJ_n(R)} \xrightarrow{\theta} J_n(S) \rightarrow 0 \quad (8)$$

short exact sequence of S -modules where $I = (f)$. We want to show that the exact sequence given in (8) is also a projective resolution for $J_n(S)$. Since R is a polynomial algebra, by the example (2.5.6), we see that $J_n(R)$ is a free R -module of rank $\binom{n+s}{s}$.

By using the isomorphism

$$R/I \otimes_R J_n(R) \cong \frac{J_n(R)}{IJ_n(R)}$$

we get that $\frac{J_n(R)}{IJ_n(R)}$ is a free S -module of rank $\binom{n+s}{s}$. Let \bar{m} be any maximal ideal of S . Then $\left(\frac{J_n(R)}{IJ_n(R)}\right)_{\bar{m}}$ is a free $S_{\bar{m}}$ -module of the same rank $\binom{n+s}{s}$.

We need to show that $(\ker \theta)_{\bar{m}}$ is a free $S_{\bar{m}}$ -module for any maximal ideal \bar{m} of S . If we tensor the exact sequence given in (8) by $S_{\bar{m}}$ and if we consider the following isomorphism

$$J_n(S_{\bar{m}}) \cong S_{\bar{m}} \otimes_S J_n(S) \cong J_n(S)_{\bar{m}}$$

we get the following short exact sequence of $S_{\bar{m}}$ -modules:

$$0 \rightarrow \left(\frac{N + IJ_n(R)}{IJ_n(R)}\right)_{\bar{m}} \rightarrow \left(\frac{J_n(R)}{IJ_n(R)}\right)_{\bar{m}} \xrightarrow{\theta_{\bar{m}}} J_n(S_{\bar{m}}) \rightarrow 0. \quad (9)$$

On the other hand, since S is a domain of dimension $s - 1$, we obtain $S_{\bar{m}}$ is a domain of dimension $s - 1$. Let L be the field of fractions of $S_{\bar{m}}$. Then $\text{Tr deg } L = s - 1$. By tensoring the exact sequence given in (9) by L , we have

$$0 \rightarrow L \otimes_{S_{\bar{m}}} \left(\frac{N + IJ_n(R)}{IJ_n(R)}\right)_{\bar{m}} \rightarrow L \otimes_{S_{\bar{m}}} \left(\frac{J_n(R)}{IJ_n(R)}\right)_{\bar{m}} \xrightarrow{\theta_{\bar{m}}} L \otimes_{S_{\bar{m}}} J_n(S_{\bar{m}}) \rightarrow 0$$

exact sequence of L -vector spaces. By using the equalities

$$\text{rank } (J_n(S_{\bar{m}})) = \dim L \otimes_{S_{\bar{m}}} J_n(S_{\bar{m}}) = \dim J_n(L) = \binom{n+s-1}{s-1}$$

and

$$\dim L \otimes_{S_{\bar{m}}} \left(\frac{J_n(R)}{IJ_n(R)}\right)_{\bar{m}} = \binom{n+s}{s},$$

we obtain

$$\text{rank}(\ker \theta)_{\bar{m}} = \dim L \otimes_{S_{\bar{m}}} \left(\frac{N + IJ_n(R)}{IJ_n(R)}\right)_{\bar{m}} = \binom{n+s}{s} - \binom{n+s-1}{s-1} = \binom{n+s-1}{s}.$$

Moreover, by using the proposition given in (2.6.4), $\ker \theta$ is generated by the set

$$\{\Delta_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) : 0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_s \leq n - 1\}$$

and this set contains $\binom{n+s-1}{s}$ elements. Hence, $(\ker \theta)_{\bar{m}}$ is generated by the images of these elements. Since the number of elements in the minimal generating set is equal to its rank, by using the lemma (2.9.10), we can conclude that $(\ker \theta)_{\bar{m}}$ is a free $S_{\bar{m}}$ -module. So, $\ker \theta$ is a projective S -module. ■

Definition 3.1.2 Let A_k^t be an affine t -space and let $k[U]$ denote the coordinate ring corresponding to an algebraic set U in A_k^t . We say that an algebraic set U is a reduced hypersurface if the coordinate ring $T = k[U]$ is a reduced k -algebra and T is presented by $R/(f)$ where R is a polynomial ring.

Theorem 3.1.3 [10, Theorem 3] Let U be a reduced hypersurface and A_k^t be an affine t -space. Suppose that $k[U \times A_k^t]$ is the coordinate ring of the product of U and A_k^t . Then the projective dimension of

$$J_n(k[U \times A_k^t])$$

is at most one.

Example 3.1.4 Let $R = k[x, y, z]$ and let I be an ideal generated by the polynomial $x^3 - yz$. Assume that $S = R/I$. Our aim is to find $J_1(S)$, $J_2(S)$ and $J_3(S)$.

(i) By using the corollary (2.6.5), we see that $J_1(S) \cong F/N$ where F is a free S -module with basis

$$\{\Delta_1(x), \Delta_1(y), \Delta_1(z), \Delta_1(1)\}$$

and let N be the submodule of F generated by the element

$$\Delta_1(f) = 3x^2\Delta_1(x) - z\Delta_1(y) - y\Delta_1(z) - x^3\Delta_1(1).$$

So, we have the short exact sequence of S -modules:

$$0 \rightarrow N \xrightarrow{\phi} F \rightarrow J_1(S) \rightarrow 0$$

where ϕ is given by the matrix

$$\begin{bmatrix} 3x^2 \\ -z \\ -y \\ -x^3 \end{bmatrix}.$$

Moreover, we know that $\text{rank} J_1(S) = \binom{1+2}{2} = 3$ and hence,

$$\text{rank} N = \text{rank} F - \text{rank} J_1(S) = 4 - 3 = 1.$$

As we obtain $\mu(N) = \text{rank}N$, by using the lemma (2.9.10), we conclude that N is a free S -module and the exact sequence given above is actually a free resolution of $J_1(S)$. Then $\text{pd}(J_1(S)) \leq 1$.

(ii) By using the corollary (2.6.5), we say that $J_2(S) \cong F'/N'$ where F' is a free S -module with basis

$$\{\Delta_2(x^2), \Delta_2(y^2), \Delta_2(z^2), \Delta_2(xy), \Delta_2(xz), \Delta_2(yz), \Delta_2(x), \Delta_2(y), \Delta_2(z), \Delta_2(1)\}$$

and N' is a submodule of F' generated by the elements

$$\{\Delta_2(f), \Delta_2(xf), \Delta_2(yf), \Delta_2(zf)\}.$$

If we compute these expressions, we obtain

$$\begin{aligned} \Delta_2(f) &= 3x\Delta_2(x^2) - 3x^2\Delta_2(x) + x^3\Delta_2(1) - \Delta_2(yz), \\ \Delta_2(xf) &= 6x^2\Delta_2(x^2) - 7yz\Delta_2(x) - x\Delta_2(yz) - y\Delta_2(xz) \\ &\quad - z\Delta_2(xy) + xy\Delta_2(z) + xz\Delta_2(y) + 2x^4\Delta_2(1), \\ \Delta_2(yf) &= 3xy\Delta_2(x^2) + 3x^2\Delta_2(xy) - 2y\Delta_2(yz) - z\Delta_2(y^2) \\ &\quad + y^2\Delta_2(z) - 6x^2y\Delta_2(x) + 2x^3y\Delta_2(1), \\ \Delta_2(zf) &= 3x^2\Delta_2(xz) + 3xz\Delta_2(x^2) - 6x^2z\Delta_2(x) \\ &\quad - 2z\Delta_2(yz) - y\Delta_2(z^2) + z^2\Delta_2(y) + 2x^3z\Delta_2(1). \end{aligned}$$

So, we have the exact sequence of S -modules

$$0 \rightarrow N' \xrightarrow{\phi'} F' \rightarrow J_2(S) \rightarrow 0$$

where ϕ' is given by the matrix

$$\begin{bmatrix} 3x & 6x^2 & 3xy & 3xz \\ 0 & 0 & -z & 0 \\ 0 & 0 & 0 & -y \\ 0 & -z & 3x^2 & 0 \\ 0 & -y & 0 & 3x^2 \\ -1 & -x & -2y & -2z \\ -3x^2 & -7yz & -6x^2y & -6x^2z \\ 0 & xz & 0 & z^2 \\ 0 & xy & y^2 & 0 \\ x^3 & 2x^4 & 2x^3y & 2x^3z \end{bmatrix}.$$

Furthermore, we have $\text{rank}J_2(S) = \binom{2+2}{2} = 6$ and so,

$$\text{rank}N' = \text{rank}F' - \text{rank}J_2(S) = 4.$$

Thus, we conclude that N' is a free S -module, as $\text{rank}N' = \mu(N')$. And, this ensures that the exact sequence given above is a free resolution of $J_2(S)$. Then $\text{pd}(J_2(S)) \leq 1$.

(iii) We know that $J_3(S) \cong F''/N''$ where F'' is a free S -module with basis

$$\begin{aligned} &\{\Delta_3(x^3), \Delta_3(y^3), \Delta_3(z^3), \Delta_3(x^2y), \Delta_3(xy^2), \Delta_3(xz^2), \Delta_3(yz^2), \Delta_3(x^2z), \\ &\Delta_3(y^2z), \Delta_3(xyz), \Delta_3(x^2), \Delta_3(y^2), \Delta_3(z^2), \Delta_3(xy), \Delta_3(xz), \Delta_3(yz), \\ &\Delta_3(x), \Delta_3(y), \Delta_3(z), \Delta_3(1)\} \end{aligned}$$

and N'' is a submodule of F'' generated by the elements

$$\begin{aligned} &\{\Delta_3(x^2f), \Delta_3(y^2f), \Delta_3(z^2f), \Delta_3(xyf), \Delta_3(xzf), \\ &\Delta_3(yzf), \Delta_3(xf), \Delta_3(yf), \Delta_3(zf), \Delta_3(f)\}. \end{aligned}$$

If we compute these expressions, we obtain

$$\begin{aligned} \Delta_3(f) &= \Delta_3(x^3) - \Delta_3(yz), \\ \Delta_3(xf) &= 4x\Delta_3(x^3) - 6x^2\Delta_3(x^2) + 4x^3\Delta_3(x) - \Delta_3(xyz) - x^4\Delta_3(1), \\ \Delta_3(yf) &= 3x\Delta_3(x^2y) + y\Delta_3(x^3) - 3x^2\Delta_3(xy) - 3xy\Delta_3(x^2) \\ &\quad + x^3\Delta_3(y) + 3x^2y\Delta_3(x) - \Delta_3(y^2z) - x^3y\Delta_3(1), \\ \Delta_3(zf) &= 3x\Delta_3(x^2z) + z\Delta_3(x^3) - 3x^2\Delta_3(xz) - 3xz\Delta_3(x^2) \\ &\quad + x^3\Delta_3(z) + 3x^2z\Delta_3(x) - \Delta_3(yz^2) - x^3z\Delta_3(1), \\ \Delta_3(x^2f) &= 10x^2\Delta_3(x^3) - 20x^3\Delta_3(x^2) + 15x^4\Delta_3(x) - 2x\Delta_3(xyz) \\ &\quad - y\Delta_3(x^2z) - z\Delta_3(x^2y) + x^2\Delta_3(yz) + 2xy\Delta_3(xz) \\ &\quad + 2xz\Delta_3(xy) + yz\Delta_3(x^2) - x^2y\Delta_3(z) - x^2z\Delta_3(y) \\ &\quad - 2xyz\Delta_3(x) - 3x^5\Delta_3(1), \\ \Delta_3(y^2f) &= 3x^2\Delta_3(xy^2) + x^3\Delta_3(y^2) + 6xy\Delta_3(x^2y) - 12x^2y\Delta_3(xy) \\ &\quad + 3x^3y\Delta_3(y) - 6xy^2\Delta_3(x^2) + 9x^2y^2\Delta_3(x) - y^3\Delta_3(z) \\ &\quad + y^2\Delta_3(x^3) - 3y\Delta_3(y^2z) - z\Delta_3(y^3) + 3y^2\Delta_3(yz) - 3x^3y^2\Delta_3(1), \end{aligned}$$

$$\begin{aligned}
\Delta_3(z^2f) &= 3x^2\Delta_3(xz^2) + x^3\Delta_3(z^2) + 6xz\Delta_3(x^2z) - 12x^2z\Delta_3(xz) \\
&\quad - 6xz^2\Delta_3(x^2) + 9x^2z^2\Delta_3(x) + z^2\Delta_3(x^3) - 3z\Delta_3(z^2y) \\
&\quad - y\Delta_3(z^3) + 3z^2\Delta_3(yz) - z^3\Delta_3(y) + 3x^3z\Delta_3(z) - 3x^3z^2\Delta_3(1),
\end{aligned}$$

$$\begin{aligned}
\Delta_3(xyf) &= 6x^2\Delta_3(x^2y) + 4xy\Delta_3(x^3) - 6x^3\Delta_3(xy) - 12x^2y\Delta_3(x^2) \\
&\quad + x^4\Delta_3(y) + 11x^3y\Delta_3(x) - x\Delta_3(y^2z) - 2y\Delta_3(xyz) \\
&\quad - z\Delta_3(xy^2) + y^2\Delta_3(xz) + xz\Delta_3(y^2) + 2xy\Delta_3(yz) \\
&\quad - xy^2\Delta_3(z) - 3x^4y\Delta_3(1),
\end{aligned}$$

$$\begin{aligned}
\Delta_3(xzf) &= 6x^2\Delta_3(x^2z) + 4xz\Delta_3(x^3) - 6x^3\Delta_3(xz) - 12x^2z\Delta_3(x^2) \\
&\quad + x^4\Delta_3(z) + 11x^3z\Delta_3(x) - x\Delta_3(yz^2) - y\Delta_3(xz^2) - 2z\Delta_3(xyz) \\
&\quad + xy\Delta_3(z^2) + 2xz\Delta_3(yz) + z^2\Delta_3(xy) - xz^2\Delta_3(y) - 3x^4z\Delta_3(1),
\end{aligned}$$

$$\begin{aligned}
\Delta_3(yzf) &= 3x^2\Delta_3(xyz) + 3xy\Delta_3(x^2z) + 3xz\Delta_3(x^2y) + 2x^3\Delta_3(yz) \\
&\quad - 6x^2y\Delta_3(xz) - 6x^2z\Delta_3(xy) - 6x^4\Delta_3(x^2) + x^3y\Delta_3(z) + x^3z\Delta_3(y) \\
&\quad + 9x^2yz\Delta_3(x) + yz\Delta_3(x^3) - 2z\Delta_3(y^2z) + z^2\Delta_3(y^2) \\
&\quad - 2y\Delta_3(yz^2) + y^2\Delta_3(z^2) - 3x^6\Delta_3(1).
\end{aligned}$$

Moreover, we know that $\text{rank}J_3(S) = \binom{3+2}{2} = 10$ and

$$\text{rank}N'' = 20 - 10 = 10.$$

Since $\text{rank}N'' = \mu(N'')$, we obtain N'' is a free S -module and the short exact sequence

$$0 \rightarrow N'' \xrightarrow{\phi''} F'' \rightarrow J_3(S) \rightarrow 0$$

is a free resolution of $J_3(S)$. Thus, $\text{pd}J_3(S) \leq 1$.

Example 3.1.5 [30] Let $R = k[x, y, z]$ be a polynomial algebra and let I be an ideal of R generated by the polynomials

$$f = y^2 - xz, \quad g = yz - x^3 \quad \text{and} \quad h = z^2 - x^2y.$$

Let $S = R/I$. We know that $\Omega_1(S) \cong F/N$ where F is a free S -module with basis

$$\{d_1(x), d_1(y), d_1(z)\}$$

and N is a submodule of F generated by the elements

$$\{d_1(f), d_1(g), d_1(h)\}.$$

We have the followings:

$$\begin{aligned} d_1(f) &= d_1(y^2 - xz) = 2yd_1(y) - xd_1(z) - zd_1(x) \\ d_1(g) &= d_1(yz - x^3) = yd_1(z) + zd_1(y) - 3x^2d_1(x) \\ d_1(h) &= d_1(z^2 - x^2y) = 2zd_1(z) - x^2d_1(y) - 2xyd_1(x). \end{aligned}$$

Then

$$0 \rightarrow N \xrightarrow{\phi} F \rightarrow \Omega_1(S) \rightarrow 0$$

is an exact sequence of S -modules and ϕ is given by the matrix

$$\begin{bmatrix} -z & -3x^2 & -2xy \\ 2y & z & -x^2 \\ -x & y & 2z \end{bmatrix}.$$

If we apply elementary row operations to this matrix, then we obtain

$$\begin{bmatrix} 0 & x & y \\ y & 0 & -x^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

And, we get the equations

$$\begin{aligned} xr_2 + yr_3 &= 0 \\ yr_1 - x^2r_3 &= 0. \end{aligned}$$

The solution set of these equations is

$$\{m_1 = (-x^2, z, -y), m_2 = (-xy, x^2, -z), m_3 = (-z, y, -x)\}.$$

So, $N = \langle m_1, m_2, m_3 \rangle$. Let N' be the kernel of the map $S^3 \rightarrow N$, then we see that $N' = \langle x, y, z \rangle$ which is a maximal ideal of S . Hence, we have

$$0 \rightarrow m \rightarrow S^3 \rightarrow F \rightarrow \Omega_1(S) \rightarrow 0$$

the exact sequence of S -modules and since $\text{pdm} = \infty$, we conclude $\text{pd}(\Omega_1(S)) = \infty$.

3.2 Some Results On Universal Modules of Differential Operators

In [11, Theorem 1.1], the following problem is examined:

Let R be any finitely generated k -algebra where k is a field of characteristic zero and let n be a positive integer. Let us consider the map

$$\Omega_n(R) \rightarrow \Omega_1(R).$$

Then what are the generators of the kernel of this map?

The purpose of this section is to answer the following question which generalizes the above result:

Let R be any k -algebra where k is a field of characteristic zero and let m and n be positive integers such that $m < n$. Then how can we characterize the generators of the kernel of the map

$$\Omega_n(R) \xrightarrow{\theta} \Omega_m(R)?$$

Then we give some examples which illustrate our result. By universality, we know the existence of the map

$$J_n(R) \xrightarrow{\alpha} J_{n-1}(\Omega_1(R)).$$

Moreover, we prove some results on *kernel* and *cokernel* of this map.

Theorem 3.2.1 *Let R be a k -algebra and m, n be positive integers such that $m < n$. Assume δ_n and δ_m denote the universal differential operators of order n and m , respectively. Then we have the following short exact sequence of R -modules:*

$$0 \rightarrow \ker\theta \rightarrow \Omega_n(R) \xrightarrow{\theta} \Omega_m(R) \rightarrow 0.$$

Moreover, $\ker\theta$ is generated by the set

$$\{\delta_n(r_0 \dots r_m) + \sum_{\substack{T \neq \emptyset \\ T \subseteq \{0, \dots, m\}}} (-1)^{|T|} r_T \delta_n(r_{T'})\}$$

where $r_i \in R$ for $i = 0, \dots, m$; T' is the complement of T in the set $\{0, \dots, m\}$ and

$$r_T = \prod_{\substack{k \in T \\ T \subseteq \{0, \dots, m\}}} r_k.$$

Proof. By using the universal property of $\Omega_n(R)$ and by using the proposition (2.1.4), there exists a map

$$\theta : \Omega_n(R) \rightarrow \Omega_m(R).$$

Moreover, θ is surjective since $m < n$. Then we obtain

$$0 \rightarrow \ker\theta \rightarrow \Omega_n(R) \xrightarrow{\theta} \Omega_m(R) \rightarrow 0$$

short exact sequence of R -modules. Let N be the submodule of $\Omega_n(R)$ generated by the set

$$\{\delta_n(r_0 \dots r_m) + \sum_{\substack{T \neq \emptyset \\ T \subseteq \{0, \dots, m\}}} (-1)^{|T|} r_T \delta_n(r_{T'})\}.$$

We consider the composite of the following maps

$$R \xrightarrow{\delta_n} \Omega_n(R) \xrightarrow{\pi} \Omega_n(R)/N.$$

By the proposition (2.1.5), we know that $\pi\delta_n$ is a differential operator of order n .

Claim 1. $\pi\delta_n$ is also a differential operator of order m .

Proof of Claim 1. Let $r_0, r_1, \dots, r_m \in R$ and by the definition of N , we obtain

$$[\pi\delta_n, r_0, r_1, \dots, r_m](1) = \pi(\delta_n(r_0 \dots r_m) + \sum_{\substack{T \neq \emptyset \\ T \subseteq \{0, \dots, m\}}} (-1)^{|T|} r_T \delta_n(r_{T'})) = 0.$$

Hence, $\pi\delta_n \in D^m(R, \Omega_n(R)/N)$. So, by universality there exists a unique R -module homomorphism

$$\Omega_m(R) \xrightarrow{\beta} \Omega_n(R)/N$$

such that $\beta\delta_m = \pi\delta_n$.

Claim 2. $N = \ker\theta$.

Proof of Claim 2. By using the definition of θ and by using the properties of m th order differential operators we get:

$$\theta(\delta_n(r_0 \dots r_m) + \sum_{\substack{T \neq \emptyset \\ T \subseteq \{0, \dots, m\}}} (-1)^{|T|} r_T \delta_n(r_{T'})) = \delta_m(r_0 \dots r_m) + \sum_{\substack{T \neq \emptyset \\ T \subseteq \{0, \dots, m\}}} (-1)^{|T|} r_T \delta_m(r_{T'}) = 0.$$

So, this illustrates that $N \subseteq \ker\theta$. Conversely, let $x \in \ker\theta$. Then we have

$$\beta\theta(x) = \pi(x) = 0.$$

And, this ensures that $x \in N$. ■

Example 3.2.2 Let $R = k[x, y]$ be a polynomial algebra over k . Then we have

$$0 \rightarrow \ker\theta \rightarrow \Omega_3(R) \xrightarrow{\theta} \Omega_2(R) \rightarrow 0$$

short exact sequence of R -modules. Here, $\Omega_3(R)$ is generated by the set

$$\{\delta_3(x^3), \delta_3(y^3), \delta_3(x^2y), \delta_3(xy^2), \delta_3(x^2), \delta_3(y^2), \delta_3(xy), \delta_3(x), \delta_3(y)\}$$

and $\Omega_2(R)$ is generated by the set

$$\{\delta_2(x^2), \delta_2(y^2), \delta_2(xy), \delta_2(x), \delta_2(y)\}.$$

Let us set the followings:

$$\begin{aligned} \epsilon_1 &= \delta_3(x^3) - 3x\delta_3(x^2) + 3x^2\delta_3(x), \\ \epsilon_2 &= \delta_3(y^3) - 3y\delta_3(y^2) + 3y^2\delta_3(y), \\ \epsilon_3 &= \delta_3(x^2y) - y\delta_3(x^2) - 2x\delta_3(xy) + 2xy\delta_3(x) + x^2\delta_3(y), \\ \epsilon_4 &= \delta_3(xy^2) - x\delta_3(y^2) - 2y\delta_3(xy) + y^2\delta_3(x) + 2xy\delta_3(y). \end{aligned}$$

Our aim is to show that $\ker\theta$ is generated by the set

$$\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}.$$

It is easy to see that $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\} \subseteq \ker\theta$. Conversely, let x be any element of $\ker\theta$.

So, x is of the form

$$\begin{aligned} x &= \alpha_{10}\delta_3(x) + \alpha_{20}\delta_3(x^2) + \alpha_{30}\delta_3(x^3) + \alpha_{11}\delta_3(xy) + \alpha_{12}\delta_3(xy^2) \\ &\quad + \alpha_{21}\delta_3(x^2y) + \alpha_{01}\delta_3(y) + \alpha_{02}\delta_3(y^2) + \alpha_{03}\delta_3(y^3). \end{aligned}$$

And, we get

$$\begin{aligned} 0 = \theta(x) &= \alpha_{10}\delta_2(x) + \alpha_{20}\delta_2(x^2) + \alpha_{30}\delta_2(x^3) + \alpha_{11}\delta_2(xy) + \alpha_{12}\delta_2(xy^2) \\ &\quad + \alpha_{21}\delta_2(x^2y) + \alpha_{01}\delta_2(y) + \alpha_{02}\delta_2(y^2) + \alpha_{03}\delta_2(y^3). \end{aligned}$$

By the properties of second order derivations, we obtain the following equalities:

$$\begin{aligned} \delta_2(x^3) &= 3x\delta_2(x^2) - 3x^2\delta_2(x), \\ \delta_2(y^3) &= 3y\delta_2(y^2) - 3y^2\delta_2(y), \\ \delta_2(xy^2) &= 2y\delta_2(xy) + x\delta_2(y^2) - y^2\delta_2(x) - 2xy\delta_2(y), \\ \delta_2(x^2y) &= 2x\delta_2(xy) + y\delta_2(x^2) - x^2\delta_2(y) - 2xy\delta_2(x). \end{aligned}$$

Hence, we see

$$\begin{aligned}
0 = \theta(x) &= \alpha_{10}\delta_2(x) + \alpha_{20}\delta_2(x^2) + \alpha_{30}(3x\delta_2(x^2) - 3x^2\delta_2(x)) \\
&+ \alpha_{11}\delta_2(xy) + \alpha_{12}(2y\delta_2(xy) + x\delta_2(y^2) - y^2\delta_2(x) - 2xy\delta_2(y)) \\
&+ \alpha_{21}(2x\delta_2(xy) + y\delta_2(x^2) - x^2\delta_2(y) - 2xy\delta_2(x)) + \alpha_{01}\delta_2(y) \\
&+ \alpha_{02}\delta_2(y^2) + \alpha_{03}(3y\delta_2(y^2) - 3y^2\delta_2(y)).
\end{aligned}$$

If we rewrite the above expression, we have

$$\begin{aligned}
0 &= (\alpha_{10} - 3\alpha_{30}x^2 - \alpha_{12}y^2 - 2\alpha_{21}xy)\delta_2(x) \\
&+ (\alpha_{20} + 3\alpha_{30}x + \alpha_{21}y)\delta_2(x^2) + (\alpha_{02} + 3\alpha_{03}y + \alpha_{12}x)\delta_2(y^2) \\
&+ (\alpha_{01} - 3\alpha_{03}y^2 - \alpha_{21}x^2 - 2\alpha_{12}xy)\delta_2(y) \\
&+ (\alpha_{11} + 2\alpha_{12}y + 2\alpha_{21}x)\delta_2(xy).
\end{aligned}$$

On the other hand, since $\Omega_2(R)$ is a free R -module with basis

$$\{\delta_2(x^2), \delta_2(y^2), \delta_2(xy), \delta_2(x), \delta_2(y)\},$$

we get

$$\begin{aligned}
\alpha_{10} &= 3\alpha_{30}x^2 + \alpha_{12}y^2 + 2\alpha_{21}xy, \\
\alpha_{20} &= -3\alpha_{30}x - \alpha_{21}y, \\
\alpha_{02} &= -3\alpha_{03}y - \alpha_{12}x, \\
\alpha_{01} &= 3\alpha_{03}y^2 + \alpha_{21}x^2 + 2\alpha_{12}xy, \\
\alpha_{11} &= -2\alpha_{12}y - 2\alpha_{21}x.
\end{aligned}$$

And, these results enable us

$$\begin{aligned}
x &= (3\alpha_{30}x^2 + \alpha_{12}y^2 + 2\alpha_{21}xy)\delta_3(x) + (-3\alpha_{30}x - \alpha_{21}y)\delta_3(x^2) \\
&+ \alpha_{30}\delta_3(x^3) + (-2\alpha_{12}y - 2\alpha_{21}x)\delta_3(xy) + \alpha_{12}\delta_3(xy^2) \\
&+ \alpha_{21}\delta_3(x^2y) + (3\alpha_{03}y^2 + \alpha_{21}x^2 + 2\alpha_{12}xy)\delta_3(y) \\
&+ (-3\alpha_{03}y - \alpha_{12}x)\delta_3(y^2) + \alpha_{03}\delta_3(y^3).
\end{aligned}$$

Thus,

$$\begin{aligned}
x &= \alpha_{30}(3x^2\delta_3(x) - 3x\delta_3(x^2) + \delta_3(x^3)) \\
&+ \alpha_{12}(y^2\delta_3(x) - 2y\delta_3(xy) - x\delta_3(y^2) + 2xy\delta_3(y) + \delta_3(xy^2)) \\
&+ \alpha_{21}(2xy\delta_3(x) - y\delta_3(x^2) - 2x\delta_3(xy) + x^2\delta_3(y) + \delta_3(x^2y)) \\
&+ \alpha_{03}(3y^2\delta_3(y) - 3y\delta_3(y^2) + \delta_3(y^3)).
\end{aligned}$$

Hence, $x \in \langle \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \rangle$ as claimed.

Example 3.2.3 Let $R = k[x]$ be a polynomial algebra over k with one variable. Then we have

$$0 \rightarrow \ker\theta \rightarrow \Omega_5(R) \xrightarrow{\theta} \Omega_3(R) \rightarrow 0$$

short exact sequence of R -modules. $\Omega_5(R)$ is generated by the set

$$\{\delta_5(x^5), \delta_5(x^4), \delta_5(x^3), \delta_5(x^2), \delta_5(x)\}$$

and $\Omega_3(R)$ is generated by the set

$$\{\delta_3(x^3), \delta_3(x^2), \delta_3(x)\}.$$

We set the followings:

$$\begin{aligned} \epsilon_1 &= \delta_5(x^5) - 5x\delta_5(x^4) + 10x^2\delta_5(x^3) - 10x^3\delta_5(x^2) + 5x^4\delta_5(x), \\ \epsilon_2 &= \delta_5(x^4) - 4x\delta_5(x^3) + 6x^2\delta_5(x^2) - 4x^3\delta_5(x). \end{aligned}$$

Let x be an arbitrary element in $\ker\theta$. Then, we can write x as

$$x = \alpha_1\delta_5(x) + \alpha_2\delta_5(x^2) + \alpha_3\delta_5(x^3) + \alpha_4\delta_5(x^4) + \alpha_5\delta_5(x^5).$$

And, we have

$$0 = \theta(x) = \alpha_1\delta_3(x) + \alpha_2\delta_3(x^2) + \alpha_3\delta_3(x^3) + \alpha_4\delta_3(x^4) + \alpha_5\delta_3(x^5).$$

By considering the properties of differential operators of order 3, we obtain the following results:

$$\begin{aligned} \delta_3(x^4) &= 4x\delta_3(x^3) - 6x^2\delta_3(x^2) + 4x^3\delta_3(x), \\ \delta_3(x^5) &= 10x^2\delta_3(x^3) - 20x^3\delta_3(x^2) + 15x^4\delta_3(x). \end{aligned}$$

Hence,

$$\begin{aligned} 0 &= \alpha_1\delta_3(x) + \alpha_2\delta_3(x^2) + \alpha_3\delta_3(x^3) + \alpha_4(4x\delta_3(x^3) - 6x^2\delta_3(x^2) + 4x^3\delta_3(x)) \\ &\quad + \alpha_5(10x^2\delta_3(x^3) - 20x^3\delta_3(x^2) + 15x^4\delta_3(x)). \end{aligned}$$

If we rewrite the expression above, we get

$$\begin{aligned} 0 &= (\alpha_1 + 4\alpha_4x^3 + 15\alpha_5x^4)\delta_3(x) + (\alpha_2 - 6\alpha_4x^2 - 20\alpha_5x^3)\delta_3(x^2) \\ &\quad + (\alpha_3 + 4\alpha_4x + 10\alpha_5x^2)\delta_3(x^3). \end{aligned}$$

Then

$$\begin{aligned}\alpha_1 &= -4\alpha_4x^3 - 15\alpha_5x^4, \\ \alpha_2 &= 6\alpha_4x^2 + 20\alpha_5x^3, \\ \alpha_3 &= -4\alpha_4x - 10\alpha_5x^2.\end{aligned}$$

So, this ensures that

$$\begin{aligned}x &= \alpha_4(-4x^3\delta_5(x) + 6x^2\delta_5(x^2) - 4x\delta_5(x^3) + \delta_5(x^4)) \\ &\quad + \alpha_5(-15x^4\delta_5(x) + 20x^3\delta_5(x^2) - 10x^2\delta_5(x^3) + \delta_5(x^5)).\end{aligned}$$

On the other hand, observe that

$$\epsilon_1 + 5x\epsilon_2 = -15x^4\delta_5(x) + 20x^3\delta_5(x^2) - 10x^2\delta_5(x^3) + \delta_5(x^5).$$

Therefore, $x \in \langle \epsilon_1, \epsilon_2 \rangle$.

3.3 Homomorphisms between Universal Modules

Let R be a k -algebra. Then consider

$$0 \rightarrow \ker\alpha \rightarrow J_n(R) \xrightarrow{\alpha} J_{n-1}(\Omega_1(R)) \rightarrow \operatorname{coker}\alpha \rightarrow 0 \quad (10)$$

exact sequence of R -modules. Firstly, we show the existence of α .

Let $d_1 : R \xrightarrow{d_1} \Omega_1(R)$ be the universal derivation and let Δ_{n-1} be the universal differential operator of order $n - 1$ of $\Omega_1(R)$. Consider the composite of the following maps:

$$R \xrightarrow{d_1} \Omega_1(R) \xrightarrow{\Delta_{n-1}} J_{n-1}(\Omega_1(R)).$$

Then by using the proposition (2.1.5), $\Delta_{n-1}d_1 \in D^n(R, J_{n-1}(\Omega_1(R)))$. By the universality of $J_n(R)$, there exists a unique R -module homomorphism

$$J_n(R) \xrightarrow{\alpha} J_{n-1}(\Omega_1(R)).$$

Hence, we obtain the exact sequence of R -modules given in (10).

Theorem 3.3.1 *Let R be a domain of dimension 1. Consider the following exact sequence of R -modules:*

$$0 \rightarrow \ker\alpha \rightarrow J_n(R) \xrightarrow{\alpha} J_{n-1}(\Omega_1(R)) \rightarrow \operatorname{coker}\alpha \rightarrow 0. \quad (11)$$

Then $\operatorname{coker}\alpha$ is a torsion R -module.

Proof. Let L be the field of fractions of R . By tensoring the exact sequence given in (11) by L , we get

$$0 \rightarrow L \otimes_R \ker\alpha \rightarrow L \otimes_R J_n(R) \xrightarrow{1 \otimes \alpha} L \otimes_R J_{n-1}(\Omega_1(R)) \rightarrow L \otimes_R \operatorname{coker}\alpha \rightarrow 0 \quad (12)$$

exact sequence of L -vector spaces. And notice that we have the following isomorphisms of L -modules:

$$L \otimes_R J_n(R) \cong J_n(L) \quad \text{and} \quad L \otimes_R J_{n-1}(\Omega_1(R)) \cong J_{n-1}(\Omega_1(L)).$$

Since $\dim R = 1$, we obtain

$$\dim J_n(L) = \binom{n+1}{1} = n + 1 \text{ and } \dim J_{n-1}(\Omega_1(L)) = \binom{n}{1} = n.$$

So, we get

$$L \otimes_R \operatorname{coker} \alpha = 0$$

and this means that $\operatorname{coker} \alpha$ is a torsion R -module. ■

Theorem 3.3.2 *Let R be an affine domain of dimension 1. Then for the following exact sequence*

$$0 \rightarrow \ker \alpha \rightarrow J_n(R) \xrightarrow{\alpha} J_{n-1}(\Omega_1(R)) \rightarrow \operatorname{coker} \alpha \rightarrow 0 \quad (13)$$

$\operatorname{coker} \alpha$ is of finite length.

Proof. By the theorem (3.3.1), we know that $\operatorname{coker} \alpha$ is a torsion R -module. Then the set

$$S := \{\operatorname{ann}(x) : 0 \neq x \in \operatorname{coker} \alpha\}$$

is non-empty. It is known that the maximal element of this set is a prime ideal. Let us denote this prime ideal by P_1 . We consider the following map:

$$R \rightarrow \operatorname{coker} \alpha, r \mapsto rx.$$

Then we have

$$R/P_1 \cong N_1$$

where $Rx = N_1$. If $\operatorname{coker} \alpha = N_1$, then we get the result. Let $N_1 \neq \operatorname{coker} \alpha$. Now, consider the set

$$S' := \{\operatorname{ann}(\bar{x}) : 0 \neq \bar{x} \in \operatorname{coker} \alpha/N_1\}$$

and denote its maximal element by P_2 . Let us define the map

$$R \rightarrow \operatorname{coker} \alpha/N_1, r \mapsto r\bar{x}.$$

Then, we get $R/P_2 \cong R\bar{x}$. Since $R\bar{x}$ is a submodule of $\operatorname{coker} \alpha/N_1$, it is of the form N_2/N_1 where N_2 is a submodule of $\operatorname{coker} \alpha$ containing N_1 . Hence, $R/P_2 \cong N_2/N_1$.

Since $\text{coker}\alpha$ is finitely generated, there exists $i_0 \geq 0$ such that $N_{i_0} = \text{coker}\alpha$. By continuing on this way, we have the following chain

$$0 \subset N_1 \subset N_2 \subset \dots \subset N_{i_0} \quad (14)$$

of submodules of $\text{coker}\alpha$. Moreover, we know that $N_{i+1}/N_i \cong R/P_{i+1}$ and $\dim R = 1$. This ensures that N_{i+1}/N_i is simple. So, the chain given in (14) is a composition series for $\text{coker}\alpha$. ■

Theorem 3.3.3 *Let R be a domain of dimension 2. Then for the following exact sequence of R -modules*

$$0 \rightarrow \ker\alpha \rightarrow J_2(R) \xrightarrow{\alpha} J_1(\Omega_1(R)) \rightarrow \text{coker}\alpha \rightarrow 0 \quad (15)$$

$\ker\alpha$ and $\text{coker}\alpha$ are torsion R -modules.

Proof. The exact sequence given in (15) is just a particular case of the sequence in (10), namely for $n = 2$. If we tensor this exact sequence by L , then we get

$$0 \rightarrow L \otimes_R \ker\alpha \rightarrow L \otimes_R J_2(R) \xrightarrow{\alpha} L \otimes_R J_1(\Omega_1(R)) \rightarrow L \otimes_R \text{coker}\alpha \rightarrow 0$$

the exact sequence of L -vector spaces. On the other hand, we know that

$$\dim J_2(L) = \dim J_{n-1}(\Omega_1(L)).$$

So, we conclude $\ker\alpha$ and $\text{coker}\alpha$ are torsion R -modules. ■

4 BETTI SERIES OF THE UNIVERSAL MODULE OF DERIVATIONS

In [27, Erdoğan], it is proved that under some conditions the Betti series of the universal module of second order derivations, $B(\Omega_2(R_m), t)$, is rational where R is the coordinate ring of an affine irreducible curve represented by $\frac{k[x,y]}{(f)}$ and m is a maximal ideal of R . It is proved in [13, Prop. 3.4.2] that if R is a regular ring of dimension one, then

$$\Omega_{n+1}(R) \cong J_n(\Omega_1(R))$$

but it is not true in the general case. Further, notice that while trying to generalize the dimension of R in the theorem (3.3.1), we obtain in (3.3.3) that the dimension of R must be two and n must be two in the sequence (11). So, there is two natural questions arise from these results.

Is the Betti series of $\Omega_2(R_m)$ rational where R is the coordinate ring of an affine irreducible curve represented by

$$\frac{k[x_1, x_2, \dots, x_s]}{(f)}?$$

In other words, can we generalize the dimension of R ? More generally, can we generalize this result for $\Omega_n(R_m)$ where R and m are defined as above?

4.1 Some Homological Background

The aim of the present subsection is to construct a framework for further investigation. Thus, we recall some concepts of homology and derived functors, such as *Ext* functor, which will play a role in examining the rationality of the Betti series of the universal module of derivations of order n . The following definitions, examples and results can be found in [22], [23], [24], [25] and [26].

Definition 4.1.1 *Let A be an R -module. An exact sequence*

$$\mathcal{P} : \dots \rightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \dots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \rightarrow 0$$

in which every P_n is projective is called a projective resolution of A .

Remark 4.1.2 *It is a well-known fact that every R -module A has a projective resolution.*

Example 4.1.3 Let G be a finite cyclic group of order n . Then G is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ where \mathbb{Z} is the additive group of integers. Then we have the following \mathbb{Z} -projective resolution

$$\dots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \dots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} G \rightarrow 0$$

of G where $P_1 = P_0 = \mathbb{Z}$, $P_n = 0$ for $n \geq 2$, ε is the natural projection and d_1 is the multiplication map by n .

Let A and D be R -modules. For any projective resolution of A

$$\dots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \rightarrow \dots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \rightarrow 0$$

let us consider the following sequence:

$$0 \rightarrow \text{Hom}_R(A, D) \xrightarrow{\varepsilon} \text{Hom}_R(P_0, D) \xrightarrow{d_1} \text{Hom}_R(P_1, D) \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} \text{Hom}_R(P_{n-1}, D) \xrightarrow{d_n} \text{Hom}_R(P_n, D) \xrightarrow{d_{n+1}} \dots$$

where to simplify the notation, we denoted the induced maps in the same way.

Definition 4.1.4 Let A and D be R -modules. For any projective resolution of A let

$$d_n : \text{Hom}_R(P_{n-1}, D) \rightarrow \text{Hom}_R(P_n, D).$$

Define

$$\text{Ext}_R^n(A, D) = \ker d_{n+1} / \text{im} d_n$$

where $\text{Ext}_R^0(A, D) = \ker d_1$. The group $\text{Ext}_R^n(A, D)$ is called the n th cohomology group derived from the functor $\text{Hom}_R(-, D)$.

Note that these cohomology groups depend only on A and D , that is, they are independent on the choice of projective resolution of A . And, in the following remark we see that we can identify the 0^{th} cohomology group.

Remark 4.1.5 For any R -module A we have $\text{Ext}_R^0(A, D) \cong \text{Hom}_R(A, D)$.

Example 4.1.6 Let the abelian group $A = \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 2$. By the remark given above, $\text{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/m\mathbb{Z}, D) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D)$.

Consider the projective resolution

$$0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 0$$

where m denotes the multiplication by m on \mathbb{Z} .

Then we have

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD$$

and

$$\text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/m\mathbb{Z}, D) = 0 \text{ for all } n \geq 2.$$

Definition 4.1.7 A free resolution of $\Omega_n(R)$ where R is a local k -algebra with maximal ideal m is called a minimal resolution if the followings are satisfied:

$$\dots \rightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \Omega_n(R) \rightarrow 0$$

F_i 's are free R -modules of finite rank for all i and $\partial_n(F_n) \subseteq mF_{n-1}$ for all $n \geq 1$ (see [29] for definition).

Remark 4.1.8 Let (R, m) be a local ring. Every finitely generated R -module has a minimal resolution. ([24, Prop. 11.184])

Definition 4.1.9 Let (R, m) be a local ring. The Betti series of $\Omega_n(R)$ is defined to be the series

$$B(\Omega_n(R), t) = \sum_{i \geq 0} \dim_{R/m} \text{Ext}_R^i(\Omega_n(R), \frac{R}{m}) t^i \text{ for all } n \geq 1.$$

Example 4.1.10 Let $R = k[x_1, \dots, x_s]$ be a polynomial algebra over k with s variables and let m be any maximal ideal of R . By the example (2.5.6), we know that $\Omega_n(R)$ is a free R -module. Then $\Omega_n(R_m)$ is a free R_m -module with basis

$$\{\delta_n(x^\alpha) : 0 < |\alpha| \leq n\}$$

where $\delta_n : R_m \rightarrow \Omega_n(R_m)$ is the universal derivation of order n of R_m . Since $\Omega_n(R_m)$ is a free R_m -module, we have $\text{Ext}_{R_m}^n(\Omega_n(R_m), R_m/mR_m) = 0$ for all $n \geq 1$.

On the other hand,

$$\begin{aligned} \text{Ext}_{R_m}^0(\Omega_n(R_m), R_m/mR_m) &\cong \text{Hom}_{R_m}(\Omega_n(R_m), R_m/mR_m) \\ &\cong \bigoplus_1^d R_m/mR_m \end{aligned}$$

where $d = \binom{n+s}{s} - 1$. Hence, $B(\Omega_n(R_m), t) = d$.

Next, we will give a well-known fact.

Lemma 4.1.11 *Let R be a local ring with maximal ideal m and M be a finitely generated R -module. Suppose that*

$$0 \rightarrow F_1 \xrightarrow{\partial} F_0 \rightarrow M \rightarrow 0$$

is a minimal resolution of M . Then $\text{Ext}_R^1(M, R/m)$ is not zero.

Proof. Assume that the following exact sequence is a minimal resolution of M

$$0 \rightarrow F_1 \xrightarrow{\partial} F_0 \rightarrow M \rightarrow 0$$

in other words, F_i is of finite rank for $i = 0, 1$ and $\partial(F_1) \subseteq mF_0$. Then we have the complex

$$0 \rightarrow \text{Hom}_R(M, R/m) \rightarrow \text{Hom}_R(F_0, R/m) \xrightarrow{\partial^*} \text{Hom}_R(F_1, R/m) \rightarrow 0$$

of R/m -vector spaces. Therefore, ∂^* has a matrix representation.

Claim 1. All the entries of this matrix belong to m , that is, $\partial^* = 0$.

Proof of Claim 1. Assume $F \in \text{Im} \partial^*$. Then there exists $f \in \text{Hom}_R(F_0, R/m)$ such that $\partial^*(f) = F$, that is, $f\partial = F$. Hence, we obtain

$$F(F_1) = f\partial(F_1) \subseteq f(mF_0) = mf(F_0) = 0.$$

This ensures that $\text{Im} \partial^* = 0$. So, $\partial^* = 0$.

Claim 2. We have $\text{Hom}_R(F_1, R/m) \neq m\text{Hom}_R(F_1, R/m)$.

Proof of Claim 2. Conversely, assume that $\text{Hom}_R(F_1, R/m) = m\text{Hom}_R(F_1, R/m)$. Then by Nakayama's Lemma, we get $\text{Hom}_R(F_1, R/m) = 0$. So, we get $M \cong F_0$ and this contradicts the minimality of the sequence. Hence, we conclude

$$\text{Ext}^1(M, R/m) = \frac{\text{Hom}_R(F_1, R/m)}{m\text{Hom}_R(F_1, R/m)} \neq 0$$

as desired. ■

4.2 Some Results on Rationality of Betti Series

In this subsection, we prove some results on rationality of Betti series of $\Omega_2(R_m)$ where R is a coordinate ring of an affine irreducible curve represented by $\frac{k[x_1, x_2, \dots, x_s]}{(f)}$ and m is a maximal ideal of R containing f . Then we generalize these results for $\Omega_n(R_m)$. Next, we provide some examples which illustrate our results.

Lemma 4.2.1 *Let $k[x_1, x_2, \dots, x_s]$ be a polynomial algebra over k with s variables and let m be a maximal ideal of $k[x_1, x_2, \dots, x_s]$ containing f . Let*

$$d_2 : k[x_1, x_2, \dots, x_s] \rightarrow \Omega_2(k[x_1, x_2, \dots, x_s])$$

be the universal derivation of second order. Suppose that $d_2(f)$ and $d_2(x_i f)$ belong to $m\Omega_2(k[x_1, x_2, \dots, x_s])$ for all $i = 1, \dots, s$. Then a module generated by

$$\{d_2(g) : g \in fk[x_1, x_2, \dots, x_s]\}$$

is a submodule of $m\Omega_2(k[x_1, x_2, \dots, x_s])$.

Proof. It suffices to show that $d_2(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) \in m\Omega_2(k[x_1, x_2, \dots, x_s])$.

By the properties of d_2 , we have

$$\begin{aligned} d_2(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) &= a_1(x_1, \dots, x_s) d_2(x_1 f) + \dots + a_s(x_1, \dots, x_s) d_2(x_s f) \\ &\quad + a_{s+1}(x_1, \dots, x_s) d_2(f) + f \left(\sum_{\gamma, \beta} \gamma(x_1, \dots, x_s) d_2(x_1^{\beta_1} x_2^{\beta_2} \dots x_s^{\beta_s}) \right) \end{aligned}$$

where $0 < \beta = \beta_1 + \beta_2 + \dots + \beta_s \leq 2$ and $\gamma, a_i \in k[x_1, x_2, \dots, x_s]$ for all $i = 1, \dots, s + 1$.

On the other hand, we have

$$d_2(x_i f), d_2(f) \in m\Omega_2(k[x_1, x_2, \dots, x_s])$$

for all $i = 1, \dots, s$ and $f \in m$ and this ensures that

$$d_2(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) \in m\Omega_2(k[x_1, x_2, \dots, x_s]).$$

Hence, the result follows. ■

Proposition 4.2.2 *Let $k[x_1, x_2, \dots, x_s]$ be a polynomial algebra over k with s variables and let m be a maximal ideal of $k[x_1, x_2, \dots, x_s]$ containing an irreducible element f . If $d_2(f)$ and $d_2(x_i f)$ are elements of $m\Omega_2(k[x_1, x_2, \dots, x_s])$ for all $i = 1, \dots, s$ then*

$$\Omega_2\left(\left(\frac{k[x_1, x_2, \dots, x_s]}{(f)}\right)_{\bar{m}}\right)$$

admits a minimal resolution of $\left(\frac{k[x_1, x_2, \dots, x_s]}{(f)}\right)_{\bar{m}}$ -modules where $\bar{m} = m/(f)$ is a maximal ideal of $\frac{k[x_1, x_2, \dots, x_s]}{(f)}$.

Proof. Let $R = \frac{k[x_1, x_2, \dots, x_s]}{(f)}$ and \bar{m} be a maximal ideal of R . Then we have the following exact sequence of $R_{\bar{m}}$ -modules:

$$0 \rightarrow \ker \alpha_{\bar{m}} \rightarrow \left(\frac{\Omega_2(k[x_1, x_2, \dots, x_s])}{f\Omega_2(k[x_1, x_2, \dots, x_s])}\right)_{\bar{m}} \xrightarrow{\alpha_{\bar{m}}} \Omega_2(R_{\bar{m}}) \rightarrow 0. \quad (16)$$

We claim that this exact sequence is a minimal resolution of $\Omega_2(R_{\bar{m}})$.

We know that $\ker \alpha$ is of the form

$$\frac{N + f\Omega_2(k[x_1, x_2, \dots, x_s])}{f\Omega_2(k[x_1, x_2, \dots, x_s])}$$

where N is a submodule of $\Omega_2(k[x_1, x_2, \dots, x_s])$ generated by the elements

$$\{d_2(g) : g \in fk[x_1, x_2, \dots, x_s]\}.$$

Then it is easy to see that

$$\ker \alpha_{\bar{m}} \subseteq \bar{m} \left(\frac{\Omega_2(k[x_1, x_2, \dots, x_n])}{f\Omega_2(k[x_1, x_2, \dots, x_n])}\right)_{\bar{m}}.$$

Now, we need to show that $\ker \alpha_{\bar{m}}$ is a free $R_{\bar{m}}$ module. We know that

$$\left(\frac{\Omega_2(k[x_1, x_2, \dots, x_s])}{f\Omega_2(k[x_1, x_2, \dots, x_s])}\right)_{\bar{m}}$$

is a free module of rank $\binom{s+2}{s} - 1$. On the other hand, the Krull dimension of $R_{\bar{m}}$ is $s - 1$ and let K be the field of fractions of $R_{\bar{m}}$. Then $\text{Tr deg } K = s - 1$. Note that

$$\dim_K \Omega_2(R_{\bar{m}}) \otimes_{R_{\bar{m}}} K = \dim_K \Omega_2(K) = \binom{s+1}{s-1} - 1.$$

By tensoring the exact sequence given in (16) with K , we obtain an exact sequence of K -vector spaces.

Therefore, we get

$$\begin{aligned} \dim_K \ker \alpha_{\bar{m}} \otimes_{R_{\bar{m}}} K &= \dim_K \left(\frac{\Omega_2(k[x_1, x_2, \dots, x_s])}{f\Omega_2(k[x_1, x_2, \dots, x_s])} \right)_{\bar{m}} \otimes_{R_{\bar{m}}} K - \dim_K \Omega_2(K) \\ &= \binom{s+2}{s} - \binom{s+1}{s-1} = s+1. \end{aligned}$$

Since $\ker \alpha$ is generated by the elements $d_2(f), d_2(x_1f), \dots, d_2(x_sf)$ as an R -module, $\ker \alpha_{\bar{m}}$ is generated by the images of these elements in $R_{\bar{m}}$. Therefore, by using the lemma (2.9.10) we get $\ker \alpha_{\bar{m}}$ is a free $R_{\bar{m}}$ module. ■

Let R be a finitely generated regular algebra and m be a maximal ideal of R . Then we know that $\Omega_2(R_m)$ is a free R_m -module and so, $\text{Ext}_{R_m}^n(\Omega_2(R_m), R_m/mR_m) = 0$ for $n \geq 1$. Hence, we can conclude that $B(\Omega_2(R_m), t)$ is rational.

Theorem 4.2.3 *Let $k[x_1, x_2, \dots, x_s]$ be a polynomial algebra over k with s variables and let m be a maximal ideal of $k[x_1, x_2, \dots, x_s]$ containing an irreducible polynomial f . Suppose that $R = \frac{k[x_1, x_2, \dots, x_s]}{(f)}$ is not a regular ring at $\bar{m} = \frac{m}{(f)}$. Let $d_2(f)$ and $d_2(x_i f)$ be the elements of $m\Omega_2(k[x_1, x_2, \dots, x_s])$ for all $i = 1, \dots, s$. Then $B(\Omega_2(R_{\bar{m}}), t)$ is a rational function.*

Proof. By the proposition (4.2.2), we have that

$$0 \rightarrow \ker \alpha_{\bar{m}} \rightarrow \left(\frac{\Omega_2(k[x_1, x_2, \dots, x_s])}{f\Omega_2(k[x_1, x_2, \dots, x_s])} \right)_{\bar{m}} \xrightarrow{\alpha_{\bar{m}}} \Omega_2(R_{\bar{m}}) \rightarrow 0$$

is a minimal resolution of $\Omega_2(R_{\bar{m}})$ and we know $\text{Ext}^1(\Omega_2(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) \neq 0$. Therefore, we get the result. ■

Next, we will generalize these results for the universal module of derivations of order n . Before proving them, it is worth to point out the difficulties encountered in proving the results for the n th order case. Let us give some examples:

Example 4.2.4 [13, Example 3.1.6 and example 3.4.7] *Let $R = k[x, y, z]$ be the polynomial algebra over k and let I be an ideal of R generated by $f = z^2 - x^3$ and $g = y^2 - xz$. Suppose $S = R/I$. Then $\text{pd}(\Omega_1(S)) \leq 1$ but $\text{pd}(\Omega_2(S))$ is not finite.*

Example 4.2.5 [30, Proposition 4.2.1] Let $R = k[x_1, \dots, x_s]$ and $S = k[y_1, \dots, y_t]$ be polynomial algebras and let I be an ideal of R generated by the elements $\{f_1, \dots, f_m\}$. Assume that R/I is an affine k -algebra with dimension $s - m$ and $\text{pd}(J_2(R/I)) \leq 1$. Then

$$\text{pd}(J_2(R/I \otimes_k S)) \leq 1.$$

But note that this result fails even for the case $n = 3$.

Proposition 4.2.6 Let $k[x_1, x_2, \dots, x_s]$ be a polynomial algebra and m be a maximal ideal of $k[x_1, x_2, \dots, x_s]$ containing an irreducible element f . If the elements

$$d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f)$$

belong to $m\Omega_n(k[x_1, x_2, \dots, x_s])$ whenever $0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_s \leq n - 1$, then $\Omega_n\left(\frac{k[x_1, x_2, \dots, x_s]}{(f)}\right)_{\bar{m}}$ admits a minimal resolution of $\left(\frac{k[x_1, x_2, \dots, x_s]}{(f)}\right)_{\bar{m}}$ -modules where $\bar{m} = m/(f)$ is a maximal ideal of $\frac{k[x_1, x_2, \dots, x_s]}{(f)}$.

Proof.

Let $R = S/I = \frac{k[x_1, x_2, \dots, x_s]}{(f)}$ and \bar{m} be a maximal ideal of R . Then we have the following short exact sequence of R -modules:

$$0 \longrightarrow \frac{N + I\Omega_n(S)}{I\Omega_n(S)} \longrightarrow \frac{\Omega_n(S)}{I\Omega_n(S)} \xrightarrow{\alpha} \Omega_n(R) \longrightarrow 0 \quad (17)$$

where N is a submodule of $\Omega_n(S)$ generated by the elements of the form

$$\{d_n(g) : g \in fk[x_1, x_2, \dots, x_s]\}.$$

By localizing (17) at \bar{m} , we get the following exact sequence of $R_{\bar{m}}$ -modules:

$$0 \longrightarrow \left(\frac{N + I\Omega_n(S)}{I\Omega_n(S)}\right)_{\bar{m}} \longrightarrow \left(\frac{\Omega_n(S)}{I\Omega_n(S)}\right)_{\bar{m}} \xrightarrow{\alpha_{\bar{m}}} \Omega_n(R)_{\bar{m}} \longrightarrow 0. \quad (18)$$

Step 1. A module generated by the set

$$\{d_n(g) : g \in fk[x_1, x_2, \dots, x_s]\}$$

is a submodule of $m\Omega_n(k[x_1, x_2, \dots, x_s])$.

Proof of Step 1. Since d_n is k -linear, it suffices to show

$$d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) \in m\Omega_n(k[x_1, x_2, \dots, x_s]).$$

By using the properties of d_n , we get

$$\begin{aligned} d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) &= \sum_{\gamma} a_{\gamma}(x_1, x_2, \dots, x_s) d_n(x_1^{\gamma_1} x_2^{\gamma_2} \dots x_s^{\gamma_s} f) \\ &\quad + f \sum_{\beta} a'_{\beta}(x_1, x_2, \dots, x_s) d_n(x_1^{\beta_1} x_2^{\beta_2} \dots x_s^{\beta_s}) \end{aligned}$$

where $a_{\gamma}(x_1, x_2, \dots, x_s), a'_{\beta}(x_1, x_2, \dots, x_s) \in k[x_1, x_2, \dots, x_s]$, $0 \leq \gamma_1 + \gamma_2 + \dots + \gamma_s \leq n - 1$, $0 < \beta_1 + \beta_2 + \dots + \beta_s \leq n$. By the assumption, we know

$$d_n(x_1^{\gamma_1} x_2^{\gamma_2} \dots x_s^{\gamma_s} f) \in m\Omega_n(k[x_1, x_2, \dots, x_s])$$

whenever $0 \leq \gamma_1 + \gamma_2 + \dots + \gamma_s \leq n - 1$ and $f \in m$, then the result follows.

Step 2. $(\frac{N+I\Omega_n(S)}{I\Omega_n(S)})_{\bar{m}} \subseteq \bar{m}(\frac{\Omega_n(S)}{I\Omega_n(S)})_{\bar{m}}$.

Proof of Step 2. By step 1, we know $N \subseteq m\Omega_n(S)$ and the rest is clear.

Step 3. $(\frac{N+I\Omega_n(S)}{I\Omega_n(S)})_{\bar{m}}$ is generated by $\binom{n+s-1}{s}$ elements.

Proof of Step 3. It is known that $\frac{N+I\Omega_n(S)}{I\Omega_n(S)}$ is generated by the set

$$\{d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) + I\Omega_n(S) : 0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_s \leq n - 1\}.$$

And, it has $\binom{n+s-1}{s}$ elements.

Step 4. $(\frac{N+I\Omega_n(S)}{I\Omega_n(S)})_{\bar{m}}$ is a free $R_{\bar{m}}$ -module.

Proof of Step 4. The Krull dimension of $R_{\bar{m}}$ is $s - 1$ and let K be the field of fractions of $R_{\bar{m}}$. Then by tensoring the exact sequence in (18) by K , we get

$$0 \longrightarrow K \otimes_{R_{\bar{m}}} (\frac{N+I\Omega_n(S)}{I\Omega_n(S)})_{\bar{m}} \longrightarrow K \otimes_{R_{\bar{m}}} (\frac{\Omega_n(S)}{I\Omega_n(S)})_{\bar{m}} \xrightarrow{\alpha_{\bar{m}}} K \otimes_{R_{\bar{m}}} \Omega_n(R)_{\bar{m}} \longrightarrow 0. \quad (19)$$

We know that $(\frac{\Omega_n(S)}{I\Omega_n(S)})_{\bar{m}}$ is a free $R_{\bar{m}}$ -module of rank $\binom{n+s}{s} - 1$.

By using the isomorphism

$$K \otimes_{R_m} \Omega_n(R_m) \cong \Omega_n(K),$$

we have

$$\begin{aligned} \dim K \otimes_{R_m} \left(\frac{N+I\Omega_n(S)}{I\Omega_n(S)} \right)_{\bar{m}} &= \dim K \otimes_{R_m} \left(\frac{\Omega_n(S)}{I\Omega_n(S)} \right)_{\bar{m}} - \dim \Omega_n(K) \\ &= \binom{n+s}{s} - \binom{n+s-1}{s-1} = \binom{n+s-1}{s}. \end{aligned}$$

Hence, $\left(\frac{N+I\Omega_n(S)}{I\Omega_n(S)} \right)_{\bar{m}}$ is a free R_m -module. Therefore, the short exact sequence given in (18) is a minimal resolution for $\Omega_n(R_m)$. ■

Let R be a finitely generated regular k -algebra and m be a maximal ideal of R . Then $\Omega_n(R_m)$ is a free R_m -module. Hence, by a similar argument for the second order case we can conclude that $B(\Omega_n(R_m), t)$ is rational.

Theorem 4.2.7 *Let $k[x_1, x_2, \dots, x_s]$ be a polynomial algebra and m be a maximal ideal of $k[x_1, x_2, \dots, x_s]$ containing an irreducible element f . Let*

$$d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} f) \in m\Omega_n(k[x_1, x_2, \dots, x_s])$$

for $0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_s \leq n-1$. Assume that $R = \frac{k[x_1, x_2, \dots, x_s]}{(f)}$ is not a regular ring at $\bar{m} = m/(f)$. Then $B(\Omega_n(R_m), t)$ is a rational function.

Proof. By the previous proposition, the exact sequence of R_m -modules in (18) is a minimal resolution of $\Omega_n(R_m)$. And we get the result. ■

Example 4.2.8 *Let R be a k -algebra represented by $k[x, y, z]/(f)$ where $f = y^4 - x^4z$. Then it is known that R is not regular at the origin. Let us compute the Betti series of $\Omega_3(R_{\bar{m}})$ where $\bar{m} = m/(f)$ is the maximal ideal of R with $m = (x, y, z)$. Since*

$$d_3(x^\alpha y^\beta z^\gamma f) \in m\Omega_3(k[x, y, z])$$

where $0 \leq \alpha + \beta + \gamma \leq 2$, we get that $\text{pd } \Omega_3\left(\frac{k[x, y, z]}{(y^4 - x^4z)}\right) = 1$ and let

$$0 \rightarrow F_1 \xrightarrow{\partial} F_0 \rightarrow \Omega_3(R) \rightarrow 0 \quad (20)$$

be the projective resolution (also free resolution) for $\Omega_3(R)$.

Then

$$0 \rightarrow (F_1)_{\bar{m}} \xrightarrow{\partial} (F_0)_{\bar{m}} \rightarrow \Omega_3(R_{\bar{m}}) \rightarrow 0$$

is a free resolution of $R_{\bar{m}}$ -modules for $\Omega_3(R_{\bar{m}})$. If we apply the contravariant functor $\text{Hom}_{R_{\bar{m}}}(-, R_{\bar{m}}/\bar{m}R_{\bar{m}})$, then we get the following complex

$$\begin{aligned} 0 \rightarrow \text{Hom}_{R_{\bar{m}}}(\Omega_3(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) &\rightarrow \text{Hom}_{R_{\bar{m}}}((F_0)_{\bar{m}}, R_{\bar{m}}/\bar{m}R_{\bar{m}}) \xrightarrow{\partial^*} \\ &\text{Hom}_{R_{\bar{m}}}((F_1)_{\bar{m}}, R_{\bar{m}}/\bar{m}R_{\bar{m}}) \rightarrow 0. \end{aligned}$$

So, we obtain

$$\text{Ext}_{R_{\bar{m}}}^1(\Omega_3(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) = \bigoplus_1^{10}(R_{\bar{m}}/\bar{m}R_{\bar{m}})$$

and this ensures that $\dim_{R_{\bar{m}}/\bar{m}R_{\bar{m}}}\text{Ext}_{R_{\bar{m}}}^1(\Omega_3(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) = 10$.

On the other hand,

$$\text{Ext}_{R_{\bar{m}}}^0(\Omega_3(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) \cong \text{Hom}_{R_{\bar{m}}}(\Omega_3(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}).$$

Observe that

$$\dim \text{Hom}_{R_{\bar{m}}}((F_0)_{\bar{m}}, R_{\bar{m}}/\bar{m}R_{\bar{m}}) = 19.$$

And, by considering the facts that the sequence given in (20) is a minimal resolution and $\text{Hom}_{R_{\bar{m}}}(\Omega_3(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}})$ is a subspace of $\text{Hom}_{R_{\bar{m}}}((F_0)_{\bar{m}}, R_{\bar{m}}/\bar{m}R_{\bar{m}})$ we obtain that

$$\dim \text{Hom}_{R_{\bar{m}}}(\Omega_3(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) = d$$

where $1 \leq d < 19$. So, we have $B(\Omega_3(R_{\bar{m}}), t) = d + 10t$ which is a rational function.

Example 4.2.9 Let R be a k -algebra represented by $k[x, y, z]/(f)$ where $f = x^3 - y^2z$. We know that R is not regular at $\bar{m} = m/(f)$ where $m = (x, y, z)$ is the maximal ideal of $k[x, y, z]$. Now, we compute the Betti series for $\Omega_2(R_{\bar{m}})$. By a similar argument as above,

$$0 \rightarrow (F_1)_{\bar{m}} \xrightarrow{\partial} (F_0)_{\bar{m}} \rightarrow \Omega_2(R_{\bar{m}}) \rightarrow 0$$

is a free resolution of $R_{\bar{m}}$ -modules of $\Omega_2(R_{\bar{m}})$ with

$$\text{rank}(F_0)_{\bar{m}} = 9 \text{ and } \text{rank}(F_1)_{\bar{m}} = 4.$$

If we apply the contravariant functor $\text{Hom}_{R_{\bar{m}}}(-, R_{\bar{m}}/\bar{m}R_{\bar{m}})$, then we get the following complex

$$0 \rightarrow \text{Hom}_{R_{\bar{m}}}(\Omega_2(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) \rightarrow \text{Hom}_{R_{\bar{m}}}((F_0)_{\bar{m}}, R_{\bar{m}}/\bar{m}R_{\bar{m}}) \xrightarrow{\partial^*} \\ \text{Hom}_{R_{\bar{m}}}((F_1)_{\bar{m}}, R_{\bar{m}}/\bar{m}R_{\bar{m}}) \rightarrow 0.$$

Hence, we obtain

$$\text{Ext}_{R_{\bar{m}}}^1(\Omega_2(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) = \bigoplus_1^4 R_{\bar{m}}/\bar{m}R_{\bar{m}}$$

and this ensures that $\dim_{R_{\bar{m}}/\bar{m}R_{\bar{m}}} \text{Ext}_{R_{\bar{m}}}^1(\Omega_2(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) = 4$. We know that

$$\text{Ext}_{R_{\bar{m}}}^0(\Omega_2(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) \cong \text{Hom}_{R_{\bar{m}}}(\Omega_2(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}).$$

By considering the fact that $\text{Hom}_{R_{\bar{m}}}(\Omega_2(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}})$ is a subspace of $\text{Hom}_{R_{\bar{m}}}((F_0)_{\bar{m}}, R_{\bar{m}}/\bar{m}R_{\bar{m}})$, we get

$$\dim \text{Hom}_{R_{\bar{m}}}(\Omega_2(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) = d$$

where $1 \leq d < 9$. So, we have $B(\Omega_2(R_{\bar{m}}), t) = d + 4t$ which is a rational function.

For the affine t -space A_k^t , we know that the coordinate ring of A_k^t is denoted by $k[A_k^t]$ and is of the form $k[y_1, \dots, y_t]$ and if U is a reduced hypersurface, then the coordinate ring of U is of the form $k[U] = k[x_1, \dots, x_s]/(f)$.

Theorem 4.2.10 *Let U be a reduced hypersurface and A_k^t be an affine t -space. Suppose that $k[U \times A_k^t]$ is the coordinate ring of the product of U and A_k^t . Let m be a maximal ideal of $k[x_1, \dots, x_s, y_1, \dots, y_t]$ containing the irreducible element f . Let*

$$d_n(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s} y_1^{\beta_1} y_2^{\beta_2} \dots y_t^{\beta_t} f) \in m \Omega_n(k[x_1, x_2, \dots, x_s, y_1, \dots, y_t])$$

for $0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_s + \beta_1 + \dots + \beta_t \leq n - 1$. And assume that $k[U \times A_k^t]$ is not a regular ring at $\bar{m} = m/(f)$. Then the Betti Series of

$$\Omega_n((k[U \times A_k^t])_{\bar{m}})$$

is a rational function.

Proof. Notice that, we have the following isomorphism

$$k[U \times A_k^t] \cong k[A_k^t] \otimes k[U] \cong k[x_1, \dots, x_s, y_1, \dots, y_t]/(f).$$

And, by using the theorems (3.1.3) and (4.2.7) we get the desired result. ■

Example 4.2.11 Let R be a k -algebra represented by $k[x, y, z]/(f)$ where $f = y^7 - x^6$. Then R is not regular at the origin. We will compute the Betti series of $\Omega_5(R_{\bar{m}})$ where $\bar{m} = m/(f)$ is the maximal ideal of R with $m = (x, y, z)$. We know that $\text{pd } \Omega_5\left(\frac{k[x, y, z]}{(y^7 - x^6)}\right) = 1$ and so,

$$0 \rightarrow (F_1)_{\bar{m}} \rightarrow (F_0)_{\bar{m}} \rightarrow \Omega_5(R_{\bar{m}}) \rightarrow 0$$

is a free resolution of $\Omega_5(R_{\bar{m}})$. If we apply the contravariant functor $\text{Hom}_{R_{\bar{m}}}(-, R_{\bar{m}}/\bar{m}R_{\bar{m}})$, then we obtain

$$\text{Ext}_{R_{\bar{m}}}^1(\Omega_5(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) = \bigoplus_1^{35}(R_{\bar{m}}/\bar{m}R_{\bar{m}})$$

and this ensures that $\dim_{R_{\bar{m}}/\bar{m}R_{\bar{m}}} \text{Ext}_{R_{\bar{m}}}^1(\Omega_5(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) = 35$. And, we know that

$$\text{Ext}_{R_{\bar{m}}}^0(\Omega_5(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) \cong \text{Hom}_{R_{\bar{m}}}(\Omega_5(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}).$$

Since $\text{Hom}_{R_{\bar{m}}}(\Omega_5(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}})$ is a subspace of $\text{Hom}_{R_{\bar{m}}}((F_0)_{\bar{m}}, R_{\bar{m}}/\bar{m}R_{\bar{m}})$, we obtain that

$$\dim \text{Hom}_{R_{\bar{m}}}(\Omega_5(R_{\bar{m}}), R_{\bar{m}}/\bar{m}R_{\bar{m}}) = d$$

where $1 \leq d < 55$. So, we have $B(\Omega_5(R_{\bar{m}}), t) = d + 35t$ which is a rational function.

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1. Erdoğan, A. and Tekin Akçin, H.M., On Betti Series of the Universal Modules of Second Order Derivations of $k[x_1, \dots, x_s]/(f)$, Turkish Journal of Mathematics, 38, 25-28, 2014.

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