SAKARYA UNIVERSITY
INSTITUTE OF SCIENCE AND TECHNOLOGY

FIXED POINT THEOREMS WITH DIFFERENT TYPES OF CONTRACTIONS IN METRIC SPACES INVOLVING A GRAPH

M.Sc. THESIS

Ekber GİRGİN

Department	:	MATHEMATICS
Field of Science	:	TOPOLOGY
Supervisor	:	Assist. Prof. Dr. Mahpeyker ÖZTÜRK

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This thesis has been accepted unanimously by the examination committee on 02.01.2014

Prof. Dr. Metin Başarır Head of Jury	Assist. Prof. Dr. Mahpeyker Öztürk Jury Member	Assist. Prof. Dr. Betül Usta Jury Member

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LIST OF SYMBOLS AND ABBREVIATIONS

B	: Real Banach Space
$E(G)$	: The set of all edges of G
$F(T)$	: The set of all fixed points of T
G	: Graph
G_x	: Component of G containing x
$\operatorname{int} K$	: Interior of K
K	: Cone
$\mathbb{N}$	: The set of natural numbers
$\mathbb{N}^*$	: $\mathbb{N}-\{0\}$
PO	: Picard operator
$\mathbb{R}$	: The set of real numbers
$\mathbb{R}^+$	: The set of positive real numbers
$T^n x$	: n^{th} iterate of x under T
$V(G)$	: Set of all vertices of G
WPO	: Weakly picard operator
$[x]_G$	: Equivalent class relation which consist of vertices of G_x
X_T	: $X_T=\{x\in X : (x,Tx)\in E(G)\}$
X^T	: $X^T=\{x\in X : (x,Tx)\in E(G) \text{ or } (Tx,x)\in E(G)\}$
$x \ll y$	: $y-x\in \operatorname{int} K$
$\leq$	: partially ordered relation
Δ	: Set of all loops.

LIST OF FIGURES

SUMMARY

Key Words: Graph Theory, Fixed Points, Contraction Mappings, Metric Space.

This thesis consists of eight chapters. In the first chapter, literature notices, some fundamental definitions and theorems which will be used in the later chapters were given.

In the second chapter, some properties were examined by using the structure of a graph with different contractions.

In the third chapter, $(G,\varphi)-$graphic contractions were defined by using a comparison function and studied the existence of fixed points. Also, the Hardy-Rogers $G-$graphic contractions were introduced and some fixed point theorems were proved.

In the fourth chapter, $(G,\psi)-$contraction and $(G,\psi)-$graphic contraction were introduced in a metric space by using a graph. Furthermore, existence and uniqueness of fixed point was examined by applying the connectivity of the graph in both cases.

In the fifth chapter, $\psi-$type contractions were defined on complete metric space involving with a graph. Also, fixed point results were given for such contractions.

In the sixth chapter, $(G,\psi,\varphi)-$contractions were defined and some fixed point theorems were obtained in metric space with a graph. Also, some results were obtained which were extensions of some recent results.

In the seventh chapter, $(G_c,\varphi)-$contractions were defined on cone metric space endowed with a graph without assuming the normality condition of cone and fixed point results were investigated.

In the last chapter, the main results which were obtained summarised.

GRAF İÇEREN METRİK UZAYLARDA FARKLI TİPLERDE DARALMA DÖNÜŞÜMLERİ İLE SABİT NOKTA TEOREMLERİ

ÖZET

Anahtar kelimeler: Graf Teori, Sabit Nokta, Daralma Dönüşümü, Metrik Uzay.

Sekiz bölüm olarak hazırlanan bu çalışmanın birinci bölümünde daha sonraki bölümlerde kullanılacak olan bazı temel tanım ve teoremler verildi.

İkinci bölümde, graf yapısı kullanılarak daha önceden yapılan bazı çalışmalar incelendi.

Üçüncü bölümde, karşılaştırmalı fonksiyon kullanılarak $(G,\varphi)-$grafik daralma dönüşümü tanımlandı ve sabit noktanın varlığı çalışıldı. Ayrıca, Hardy Rogers $G-$grafik daralma dönüşümü tanımlanarak sabit nokta teoremleri ispatlandı.

Dördüncü bölümde, metrik uzayda graf yapısı kullanılarak $(G,\psi)-$daralma ve $(G,\psi)-$grafik daralma dönüşümlerini tanımlandı. Ayriyetten, grafın bağlantılılığı kullanılarak sabit noktanın varlığı ve tekliği incelendi.

Beşinci bölümde, grafla donatılmış tam metrik uzayda $\psi-$daralma dönüşümleri tanımlandı. Aynı zamanda, bu dönüşümler için sabit nokta sonuçları verildi.

Altıncı bölümde, $(G,\varphi,\psi)-$daralma dönüşümü tanımlanarak grafla donatılmış metrik uzayda bazı sabit nokta teoremleri ispalandı ve bazı sonuçların genelleştirilmesi olduğu elde edildi.

Yedinci bölümde, koninin normallik şartı kaldırılarak grafla donatılmış konik metrik metrik uzayda $(G_c,\varphi)-$daralma dönüşümü tanımlanarak sabit noktanın varlığı ve tekliği incelendi.

Son bölümde ise bazı genel sonuçlar ve öneriler verildi.

CHAPTER 1. INTRODUCTION

1.1. Basic Facts and Definitions

Definition 1.1.1. [1] Let X be a non-empty set. A function

$$\begin{aligned} d: X \times X &\to \mathbb{R}^+ \\ (x, y) &\to d(x, y) \end{aligned}$$

is said to be a metric on X if it satisfies the following conditions:

d1. $d(x, y) \geq 0$, $\forall x, y \in X$

d2. $d(x, y) = 0 \Leftrightarrow x = y$, $\forall x, y \in X$

d3. $d(x, y) = d(y, x)$, $\forall x, y \in X$ (symmetry)

d4. $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y \in X$ (triangle inequality).

The ordered pair (X, d) is called a metric space. If there is no confusion likely to occur we, sometimes, denote the metric space (X, d) by X.

Example 1.1.2. [1] Let $X = \mathbb{R}$, the set of all real numbers. For $x, y \in X$, define $d(x, y) = |x - y|$. Then (X, d) is a metric space. This is called the metric space $\mathbb{R}$ with the usual metric.

Example 1.1.3. [2] Let X be an arbitrary non-empty set. For $x, y \in X$, define d by

$$d(x,y)=\begin{cases}0, & x=y\\ 1, & x\neq y\end{cases}$$

Then (X,d) is a metric space. The metric d is called the discrete metric and the space (X,d) is called discrete metric space.

Example 1.1.4. [3] The metric space $\mathbb{R}^2$, called the Euclidean plane, is obtained if we take the set of ordered pairs of real numbers, written $x=(x_1,x_2)$, $y=(y_1,y_2)$, and the Euclidean metric defined by $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$.

Example 1.1.5. [3] As a set X we take the set of all real-valued functions $x,y...$ which are functions of an independent real variable t and are defined and continuous on a given closed interval $J=[a,b]$. Choosing the metric defined by $d(x,y)=\max\limits_{t\in J}|x(t)-y(t)|$, we obtain a metric space which is denoted by $C[a,b]$. This is a function space because every point of $C[a,b]$ is a function.

Definition 1.1.6. [2] Let (X,d) be a metric space. $x=(x_n)$ is called convergent (with limit x_0) if and only if, for every $\varepsilon>0$ there exists $N=N(\delta,\varepsilon)$ such that $d(x_n,x_0)<\varepsilon,$ for all $n\geq N$. We write $x_n\to x_0$ $(n\to\infty)$, or $\lim x_n=x_0$, and denote the set of all convergent sequences by c .

Definition 1.1.7. [2] Let (X,d) be a metric space. $x=(x_n)$ is called a Cauchy sequence if and only if $d(x_n,x_m)\to 0,$ $(n,m\to\infty)$, i.e. for all $\varepsilon>0$, there exists $N=N(\varepsilon)$ such that $d(x_n,x_m)<\varepsilon$, for all $n,m>N$.

A convergent sequence has a unique limit. Every convergent sequence is also a Cauchy sequence, but not conversely, in general. If a Cauchy sequence has a convergent subsequence then the whole sequence is convergent.

Definition 1.1.8. [2] A metric space (X,d) is called complete if and only if every Cauchy sequence converges (to a point of X). Explicitly, we require that if

$$d(x_n,x_m)\to 0,\ \text{as}\ n,m\to\infty,$$

then there exists $x\in X$ such that $d(x_n,x)\to 0,$ as $n\to\infty$.

Example 1.1.9. [2] The real numbers $\mathbb{R}$ with the usual metric form a complete metric space.

Definition 1.1.10. [27] Let (X,d) and (Y,ρ) be metric spaces. Then $T:X\to Y$ is called continuous function on X if and only if for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,x_0)>0$ such that $d(x,x_0)<\delta$ implies $\rho\big(T(x),T(x_0)\big)<\varepsilon$, where $x,x_0\in X$.

Definition 1.1.11. [4] Let T be a mapping from a metric space (X,d) into another metric space (Y,ρ). Then T is said to be uniformly continuous on X if for given $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that $\rho\big(T(x),T(x_0)\big)<\varepsilon$ whenever $d(x,y)<\delta$ for all $x,y\in X$.

Definition 1.1.12. [5] A mapping $T:X\to X$ is called orbitally continuous if for all $x,y\in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers,

$$T^{k_n}x\to y\ \text{implies}\ T\left(T^{k_n}x\right)\to Ty\ \text{as}\ n\to\infty.$$

Definition 1.1.13. [5] Let (X,d) be a metric space. We say that sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, elements of X, are Cauchy equivalent if each of them is a Cauchy sequences and $d(x_n,y_n)\to 0$.

1.2. The Banach Contraction Principle and Some Basic Notations of Fixed Point Theory

The Fixed Point Theory is one of the most powerful and productive tools from the nonlinear analysis and it can be considered the kernel of nonlinear analysis. The best known result from the Fixed Point Theory is Banach's Contraction Principle (1922), which can be considered the beginning of this theory. In a metric space setting it can be briefly stated as follows:

Definition 1.2.1. [6] Let X be a nonempty set and $T : X \to X$ a selfmap. We say that $x \in X$ is a fixed point of T if $T(x) = x$ and denoted by $F(T)$ or $Fix(T)$ the set of all fixed points of T.

Example 1.2.2. [6]

i. If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $F(T) = \{-2\}$;
ii. If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then $F(T) = \{0, 2\}$;
iii. If $X = \mathbb{R}$ and $T(x) = x + 2$, then $F(T) = \theta$;
iv. If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$.

Let X be any nonempty set and $T : X \to X$ be a selfmap. For any given $x \in X$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we call $T^n(x)$ the n^{th} iterate of x under T. In order to simplify the notions we will often use Tx instead of $T(x)$.

The mapping $T^n\ (n \geq 1)$ is called the n^{th} iterate of T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by $x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \ldots$ is called the sequence of successive approximations with the initial value x_0. It is also known as the Picard iteration starting at x_0.

For a given selfmap the following properties obviously hold:

i. $F(T) \subset F(T^n)$, for each $n \in \mathbb{N}^*$;

ii. $F(T^n) = \{x\}$, for some $n \in \mathbb{N}^*$ $\Rightarrow$ $F(T) = \{x\}$;

The inverse of (ii) is not true, in general, as shown by the next example.

Example 1.2.3. [6] Let $T:\{1,2,3\} \to \{1,2,3\}$, $T(1)=3$, $T(2)=2$ and $T(3)=1$. Then $F(T^2)=\{1,2,3\}$ but $F(T)=\{2\}$.

Definition 1.2.4. [7] Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be Lipschitzian if there exists a constant $k \geq 0$ such that for all $x, y \in X$

$$d(Tx,Ty) \leq kd(x,y).$$

The smallest number k is called the Lipschitz constant of T.

Definition 1.2.5. [7] A Lipschitzian mapping $T: X \to X$ with Lipschitz constant $k < 1$ is said to be a contraction mapping.

Definition 1.2.6. [7] A Lipschitzian mapping $T: X \to X$ with Lipschitz constant $k = 1$ is said to be a nonexpansive mapping.

Definition 1.2.7. [7] Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be contractive mapping if

$$d(Tx,Ty) < d(x,y), \quad \text{for all } x, y \in X.$$

Remark 1.2.8. [3] T contraction $\Rightarrow$ T contractive $\Rightarrow$ T nonexpansive $\Rightarrow$ T Lipschitzian.

Remark 1.2.9. [6] If T is a Lipschitzian mapping, then T is a uniformly continuous.

Theorem 1.2.10. [7] **(Banach's Contraction Mapping Principle**) Let (X,d) be a complete metric space and let $T:X \to X$ be a contraction. Then T has a unique fixed point x_0 in X. Moreover, for each $x \in X$,

$$\lim_{n\to\infty}\left(T^n x\right) = x_0$$

and in fact for each $x \in X$,

$$d\left(T^n x, x_0\right) \le \frac{k^n}{1-k} d\left(x, Tx\right), \quad n = 1, 2, \dots .$$

Example 1.2.11 [1] Take $X = \left(0, \frac{1}{2}\right]$ equipped with the usual metric. This is clearly an incomplete metric space. Note that the mapping $T:X \to X$ given by $Tx = x^2$ is a contraction but T has no fixed point.

Example 1.2.12. [1] Consider the complete metric space $X = [0,\infty)$ with the usual metric and $T:X \to X$ given by $Tx = \dfrac{1}{1+x^2}$. Then;

i. The mapping T satisfies $d\left(Tx,Ty\right) < d\left(x,y\right)$ and hence T is a contractive mapping, while T is a not a contraction.
ii. T has no fixed point.

Let define the class $\Phi = \left\{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+\right\}$ as follows.

Definition 1.2.12. [6] A function $\varphi \in \Phi$ is said to be a comparison function if following conditions hold;

i. φ is monotone increasing, i.e., $t_1 \le t_2$ implies $\varphi(t_1) \le \varphi(t_2)$;

ii. $\left(\varphi^n(t)\right)_{n\in\mathbb{N}}$ converges to 0 for all $t>0$;

Definition 1.2.13. [6] A function $\varphi\in\Phi$ is said to be a (c)-comparison function if following conditions hold;

i. φ is monotone increasing, i.e., $t_1\leq t_2$ implies $\varphi(t_1)\leq\varphi(t_2)$;

ii. $\sum_{n=0}^{\infty}\varphi^n(t)$ converges for all $t>0$;

Remark 1.2.14. [6] Any (c)-comparison function is a comparison function.

Definition 1.2.14. [8] Let $\varphi\in\Phi$ be a function.

i. φ is monotone increasing, i.e., $t_1\leq t_2$ implies $\varphi(t_1)\leq\varphi(t_2)$;

ii. $\left(\varphi^n(t)\right)_{n\in\mathbb{N}}$ converges to 0 for all $t>0$;

iii. $\sum_{n=0}^{\infty}\varphi^n(t)$ converges for all $t>0$;

If the conditions (i-iii) hold then φ is called a strong comparison function.

Remark 1.2.15. [8] Any strong comparison function is a comparison function.

Remark 1.2.16. [8] If $\varphi\in\Phi$ is a comparison function, then $\varphi(t)\leq t$, for all $t>0$, $\varphi(0)=0$ and φ is right continuous at 0.

Example 1.2.17. [6] $\varphi\in\Phi$, $\varphi(t)=\dfrac{t}{1+t}$ is a comparison function but it is not a (c)-comparison function.

Definition 1.2.18. [6] Let (X,d) be a metric space. The mapping $T:X\to X$ is said to be a $\varphi-$contraction if there exists a comparison function $\varphi\in\Phi$ such that

$$d(Tx,Ty)\le\varphi\big(d(x,y)\big) \quad \text{for all } x,y\in X .$$

Remark 1.2.19. [8] Let define the class $\Psi=\left\{\psi:\mathbb{R}^+\to\mathbb{R}^+ \middle| \ \psi \text{ is nondecreasing}\right\}$ which the following conditions hold;

ψ_1. $\psi(\omega)=0$ iff $\omega=0$;

ψ_2. for every $(\omega_n)\in\mathbb{R}^+$, $\psi(\omega_n)\to 0$ iff $\omega_n\to 0$;

ψ_3. for every $\omega_1,\omega_2\in\mathbb{R}^+$, $\psi(\omega_1+\omega_2)\le\psi(\omega_1)+\psi(\omega_2)$.

Definition 1.2.20. [9] Let (X,d) be a metric space. The mapping $T:X\to X$ is said to be a Kannan operator if there exists $\alpha\in\left[0,\frac{1}{2}\right)$ such that:

$$d(Tx,Ty)\le\alpha\left[d(x,Tx)+d(y,Ty)\right],$$

for all $x,y\in X$.

Definition 1.2.21. [10] Let (X,d) be a metric space. The operator $T:X\to X$ is said to be a Ciric-Reich-Rus operator if there exists nonnegative number α,β,γ with $\alpha+\beta+\gamma<1$ such that ;

$$d(Tx,Ty)\le\alpha d(x,y)+\beta d(x,Tx)+\gamma d(y,Ty),$$

for all $x,y\in X$.

Definition 1.2.22. [11] Let (X,d) be a metric space. The operator $T: X \to X$ is called Hardy-Rogers contraction if there exist nonnegative numbers $\alpha, \beta, \gamma, \delta, \eta$ with $\alpha+\beta+\gamma+\delta+\eta<1$, such that

$$d(Tx,Ty) \le \alpha d(x,Tx)+\beta d(y,Ty)+\gamma d(x,Ty)+\delta d(y,Tx)+\eta d(x,y),$$

for all $x, y \in X$.

Definition 1.2.23. [8] Let (X,d) be a metric space. The mapping $T: X \to X$ is a graphic contraction if there exists $\alpha \in [0,1)$ such that:

$d\left(Tx,T^2x\right) \le \alpha d\left(x,Tx\right)$ for all $x \in X$.

Definition 1.2.24. [8] Let T be a selfmap of a metric space (X,d). We say that T is a Picard operator (abbr., PO) if T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$ for all $x^* \in X$ and T is a weakly Picard operator (abbr., WPO) if the sequence $\left(T^n x\right)_{n\in\mathbb{N}}$ converges, for all $x \in X$ and the limit (which depends on x) is a fixed point of T.

1.3. Graph Theory

Although the first paper in graph theory goes back to 1736 (Example 1.3.9.) and several important results in graph theory were obtained in the nineteenth century, it is only since the 1920s that there has been a sustained, widespread, intense interest in graph theory. Indeed, the first text on graph theory ([König]) appeared in 1936. Undoubtedly, one of the reasons for recent interest in graph theory is its applicability in many diverse fields, including computer science, chemistry, operations research, electrical engineering, linguistics and economics.

We begin with some basic graph terminology and examples. Then we discuss some important concepts in graph theory, including connectivity.

Definition 1.3.1. [12] A graph (or undirected graph) G consist of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an unordered pair of vertices. If there is a unique edge e associated with the vertices v and w, we write $e = (v, w)$ or $e = (w, v)$. In this context, (v, w) denotes an edge between v and w in an undirected graph and not an ordered pair.

A directed graph (or digraph) G consist of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an ordered pair of vertices. If there is a unique edge e associated with the ordered pair (v, w) of vertices, we write $e = (v, w)$, which denotes an edge from v to w.

An edge e in a graph (undirected or directed) that is associated with the pair of vertices v and w is said to be incident on v and w, and v and w are said to be incident on e and to be a adjacent vertices

If G is a graph (undirected or directed) with vertices V and edges E, we write $G = (V, E)$. Unless specified otherwise, the sets E and V are assumed to be finite and V is assumed to be nonempty.

Example 1.3.2. [12] A directed graph is shown in Figure 1.3.1. The directed edges are indicated by arrows. Edge e_1 is associated with the ordered pair (v_2, v_1) of vertices, and edge e_7 is associated with the ordered pair (v_6, v_6) of vertices. Edge e_1 is denoted (v_2, v_1), and edge e_7 is denoted (v_6, v_6).

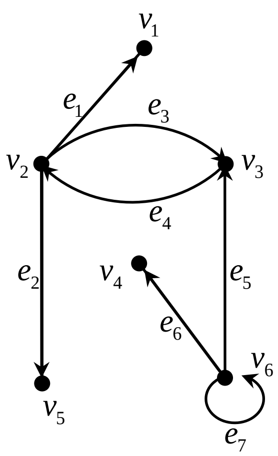

Figure 1.3.1. A directed graph.

Definition 1.3.1. allows distinct edges to be associated with the same pair of vertices. For example, in Figure 1.3.2, edges e_1 and e_2 are both associated with the vertex pair $\{v_1, v_2\}$. Such edges are called parallel edges. An edge incident on a single vertex is called a loop. For example, in Figure 1.3.2, edge $e_3 = (v_2, v_2)$ is a loop. A vertex, such as vertex v_4 in Figure 1.3.1, that is not incident on any edge is called an isolated vertex. A graph with neither loops nor paralel edges is called a simple graph.

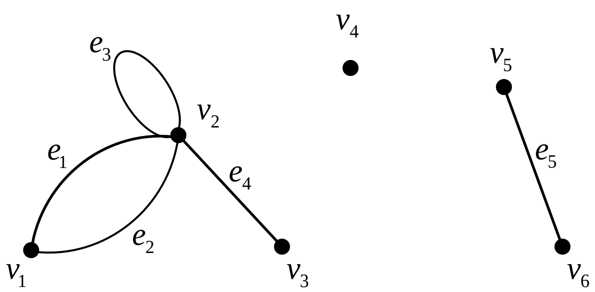

Figure 1.3.2. A graph with parallel edges and loops.

If we think of the vertices in a graph as cities and the edges as roads, a path corresponds to a trip beginning at some city, passing through several cities, and terminating at some city. We begin by giving a formal definition of path.

Definition 1.3.3. [12] Let v_0 and v_n be vertices in a graph. A path from v_0 to v_n of length n is an alternating sequence of $n+1$ vertices and n edges beginning with vertex v_0 and ending with vertex v_n, $(v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n)$, in which edge e_i is incident on vertices v_{i-1} and v_i for $i = 1, \ldots, n$.

Example 1.3.4. [12] In the graph of Figure 1.3.3, $(1, e_1, 2, e_2, 3, e_3, 4, e_4, 2)$ is a path of length 4 from vertex 1 to vertex 2.

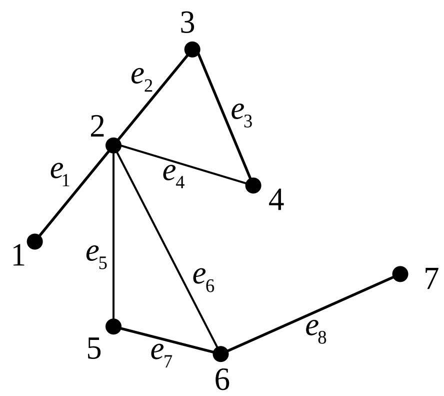

Figure 1.3.3. A connected graph

Definition 1.3.5. [12] A graph G is connected if given any vertices v and w in G, there is path from v to w.

Example 1.3.6. [12] The graph G of Figure 1.3.3 is connected since, given any vertices v and w in G, there is a path from v to w.

Example 1.3.7. [12] The graph G of Figure 1.3.4 is not connected since, for example, there is no path from vertex v_2 to vertex v_5.

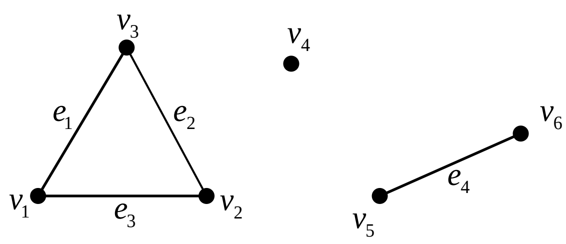

Figure 1.3.4. A graph that is not connected.

Definition 1.3.8. [12] Let v and w be vertices in a graph G. A simple path from v to w is a path from v to w with no repeated vertices. A cycle (or circuit) is a path of nonzero length from v to v with no repeated edges. A simple cycle is a cycle from v to v in which, except for the begining and ending vertices that are both equal to v, there are no repeated vertices.

Example 1.3.9. [12] **(Königsberg Bridge Problem)** The first paper in graph theory was Leonhard Euler's in 1736. The paper presented a general theory that included a solution to what is now called the Königsberg bridge problem.

Two islands lying in the Pregel River in Königsberg (now Kaliningrad in Russia) were connected to each other and the river banks by bridges, as shown in Figure 1.3.5. The problem is to start at any location A, B, C or D; walk over each bridge exactly once; then return to the starting location.

The bridge configuration can be modelled as a graph, as shown in Figure 1.3.6. The vertices represent the locations and the edges represent the bridges. The Königsberg bridge problem is now reduced of finding a cycle in the graph of Figure 1.3.6 that includes all of the edges and all of the vertices. In honor of Euler, a cycle in a grap G that includes all of the edges and all of the vertices of G is called an Euler cycle.

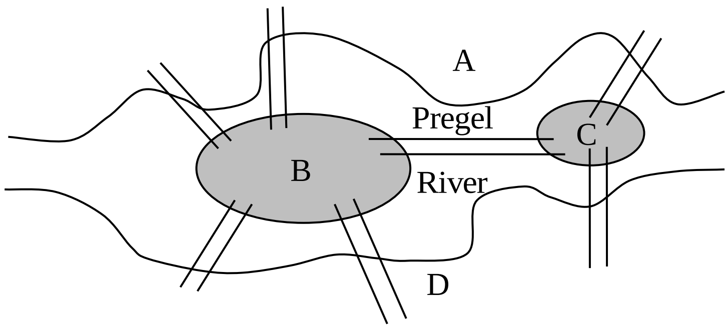

Figure 1.3.5.

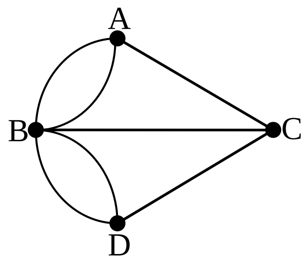

Figure 1.3.6.

Throughout this thesis we suppose following notations:

Let (X, d) be a metric space and Δ denote the diagonal of the Cartesian product XxX. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. Assume that G has no parallel edges, so one can identify G with the pair $(V(G), E(G))$.

The conversion of a graph G is denoted by G^{-1} and which is a graph obtained from G by reversing the direction of edges. Hence

$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Also, $V\left(G^{-1}\right) = V\left(G\right)$.

By $\tilde{G}$, we denote the undirected graph obtained from G by omitting the direction of edges. Indeed; it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention, we have

$$E\left(\tilde{G}\right) = E\left(G\right) \cup E\left(G^{-1}\right).$$

For any $x, y \in V'$, $(x, y) \in E'$ such that $V' \subseteq V(G)$, $E' \subseteq E(G)$, then (V', E') is called a subgraph of G.

If x and y are vertices in a graph G, then a path from x to y of length $N (N \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x,\ x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, ..., N$.

A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G) = [x]_G$ where $[x]_G$ denotes the equivalence class of relation $\Re$ defined on $V(G)$ by the rule: $y \Re z$ if there is a path in G from y to z. Clearly, G_x is connected.

Definition 1.3.10. [13] Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a mapping. We say that the graph G is $T-$connected if for all vertices x, y of G with $(x, y) \notin E(G)$, there exists a path in G, $(x_i)_{i=0}^{N}$ from x to y such that $x_0 = x$, $x_N = y$ and $(x_i, Tx_i) \in E(G)$ for all $i = 1, 2, ..., N-1$. A graph G is weakly $T-$connected if $\tilde{G}$ is $T-$connected.

Now, we give some definition related to types of continuity of mappings.

Definition 1.3.11. [5] A mapping $T : X \to X$ is called $G-$continuous if given $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ imply $Tx_n \to Tx$.

Definition 1.3.12. [5] A mapping $T : X \to X$ is called orbitally $G-$continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers,

$$T^{k_n} x \to y,\ \left(T^{k_n} x, T^{k_{n+1}} x\right) \in E(G) \text{ imply } T\left(T^{k_n} x\right) \to Ty \text{ as } n \to \infty.$$

Remark 1.3.13. [5] Clearly, we have the following relations:

continuity $\Rightarrow$ $G-$continuity $\Rightarrow$ orbital $G-$continuity;
continuity $\Rightarrow$ orbital continuity $\Rightarrow$ orbital $G-$continuity.

1.4. Cone Metric Space

Definition 1.4.1. [14] Let B be a real Banach space and K be a subset of B. K is called a cone if and only if:

i. K is closed, nonempty and $K \neq \{0\}$,
ii. $a,b \in R$; $a,b \geq 0$; $x,y \in K \Rightarrow ax+by \in K$,
iii. $x \in K$ and $-x \in K \Rightarrow x = 0$.

Given a cone $K \subset B$, we define a partial ordering $\leq$ with respect to K by $x \leq y$ if and only if $y-x \in K$. We write $x < y$ if $x \leq y$ but $x \neq y$; $x \ll y$ if $y-x \in \operatorname{int} K$, where $\operatorname{int} K$ is the interior of K. The cone K is a normal cone if

$$\inf\{\|x+y\| : x,y \in K \text{ and } \|x\| = \|y\| = 1\} > 0 \qquad \textbf{(1.1)}$$

or equivalently, if there is a number $M > 0$ such that for all $x,y \in B$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq M\|y\|. \qquad \textbf{(1.2)}$$

The least positive number satisfying (1.2) is called normal constant of K. From (1.1) one can conclude that K is a non normal if and only if there exist sequences $x_n, y_n \in K$ such that

$$0 \leq x_n \leq x_n + y_n, \lim_{n\to\infty}(x_n + y_n) = 0, \text{ but } \lim_{n\to\infty} x_n \neq 0$$

Rezapour and Hamlbarini [15] proved that there are no normal cones with constants $M<1$ and for each $k>1$ there are cones with normal constants $M>k$.
Huang and Zhang [14] redefined cone metric spaces as follows:

Definition 1.4.2. Let X be nonempty set, B be a real Banach space and $K\subset B$ be a cone. Suppose the mapping $d:X\times X\to B$ satisfies:

i. $0<d(x,y)$ for all $x,y\in X$ and $d(x,y)=0$ if and only if $x=y$;
ii. $d(x,y)=d(y,x)$ for all $x,y\in X$;
iii. $d(x,y)\leq d(x,z)+d(z,y)$ for all $x,y,z\in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space. It is obvious that the concept of a cone metric space is more general than a metric space.

Example 1.4.3. [14] Let $B=\mathbb{R}^2$, $K=\{(x,y)\in B:\ x,y\geq 0\}\subset\mathbb{R}^2$, and $d:XxX\to B$ such that $d(x,y)=(|x-y|,\alpha|x-y|)$, where $\alpha\geq 0$ is a constant. Then (X,d) is a cone metric space.

Let $\{x_n\}$ be a sequence in a cone metric space X and $x\in X$. If for every $c\in B$ with $\theta\ll c$ there is $n_0\in N$ such that for all $n>n_0$, $d(x_n,x)\ll c$ then x_n is called convergent sequence. If for every $c\in B$ with $\theta\ll c$ there is $n_0\in N$ such that for all $n,m>n_0$, $d(x_n,x_m)\ll c$ then x_n is called a Cauchy sequence in X . A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.
It is known that $\{x_n\}$ converges to $x\in X$ if and only if $d(x_n,x)\to 0$ as $n\to\infty$.

The following lemma has been given in [16] that we utilize them to prove our theorems.

Lemma 1.4.4. Let (X,d) be a cone metric space, $u,v,w\in X$. Then

1. If $u\ll v$ and $v\ll w$, then $u\ll w$.

2. If $u \leq v$ and $v \ll w$, then $u \ll w$.
3. If $\theta \leq u \ll c$ for each $c \in \text{int}K$, then $u = \theta$.
4. If $c \in \text{int}K$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$, it follows that $a_n \ll c$.

Definition 1.4.5. [15] Let K be a cone defined as above. A nondecreasing function $\varphi : \text{int}K \to \text{int}K$, which satisfies the following conditions;

φ_1. $\varphi(\theta) = \theta$ and $\theta < \varphi(z) < z$ for $z \in K - \{\theta\}$;

φ_2. $z \in \text{int}\, K$ implies $z - \varphi(z) \in \text{int}\, K$;

φ_3. $\lim_{n\to\infty} \varphi^n(z) = \theta$ for every $z \in K - \{\theta\}$;

φ_4. $\sum_{n=0}^{\infty} \varphi^n(z)$ converges for all $z \in K - \{\theta\}$.

CHAPTER 2. SOME FIXED POINT THEOREMS ON METRIC SPACE ENDOWED WITH A GRAPH

Metric fixed point theory has been researched extensively in the past two decades. Particularly, works have been proved in a metric space endowed with a partial ordering and many results have appeared, giving sufficient conditions of a mapping to be a Picard operator came into prominence. The Banach Contraction Principle and the Knaster-Tarski Theorem [5] are celebrated theorems for these concepts. Jachymski [5] used the platform of graph theory instead of partially ordering in metric space. Also, a mapping on a complete metric space still has a fixed point as long as the mapping satisfies the contraction condition for pairs of points which from edges in the graph. Subsequently Beg [17] established set valued mappings version of the main results of Jachymski [5]. Later, Bojor [13, 18, 19] obtained some results in such settings by weakening the condition of Banach $G-$contraction and introducing some new type of connectivity of a graph, and also Petruşel and Chifu [20] found generalized contractions of Banach $G-$contraction defining some new contractions in metric space endowed with a graph.

2.1. The Contraction Principle for Mappings on a Metric Space Endowed with a Graph

Definition 2.1.1. [5] We say that a mapping $T: X \to X$ is a Banach $G-$contraction or simply $G-$contraction if

i. T preserves edges of G, i.e.,

$$\forall_{x,y\in X}\ \big((x,y)\in E(G)\ \Rightarrow\ (Tx,Ty)\in E(G)\big),$$

ii. T decreases weights of edges of G in the following way:

$$\exists_{\alpha\in(0,1)}\ \forall_{x,y\in X}\quad \left((x,y)\in E(G)\ \Rightarrow\ d(Tx,Ty)\le \alpha d(x,y)\right).$$

Example 2.1.2. [5] Any constant function $T:X\to X$ is a Banach $G-$contraction since $E(G)$ contains all loops.

Example 2.1.3. [5] Any Banach contraction is a G_0-contraction, where G_0 is defined by $E(G_0):=XxX$.

Proposition 2.1.4. [5] If a mapping $T:X\to X$ is a $G-$contraction, then T is both a $G^{-1}-$contraction and a $\tilde{G}-$contraction.

Lemma 2.1.5. [5] Let $T:X\to X$ be a $G-$contraction with a constant α . Then, given $x\in X$ and $y\in[x]_{\tilde{G}}$, there is $r(x,y)\ge 0$ such that

$$d\left(T^n x,T^n y\right)\le \alpha^n\ r(x,y),\quad \text{for all } n\in\mathbb{N}.$$

2.2. Fixed Point of $\varphi-$Contraction in Metric Spaces Endowed with a Graph

Definition 2.2.1. [19] Let (X,d) be a metric space and G be a graph. The mapping $T:X\to X$ is said to be a $(G,\varphi)-$contraction if:

i. $\forall_{x,y\in X}\ \left((x,y)\in E(G)\ \Rightarrow\ (Tx,Ty)\in E(G)\right),$

ii. there exists a comparison function $\varphi\in\Phi$ such that

$$d(Tx,Ty)\le\varphi\left(d(x,y)\right)\quad \text{for all } (x,y)\in E(G).$$

Remark 2.2.2. [19] If a mapping $T: X \to X$ is a $(G,\varphi)-$contraction, then T is both a $(G^{-1},\varphi)-$contraction and a $(\tilde{G},\varphi)-$contraction.

Example 2.2.3. [19] Any $\varphi-$contraction is a $(G_0,\varphi)-$contraction, where the graph G_0 is defined by $E(G_0) := XxX$.

Example 2.2.4. [19] Any $G-$contraction is a $(G,\varphi)-$contraction, where the comparison function is $\varphi \in \Phi$, $\varphi(t) = \alpha t$.

2.3. Fixed Points of Kannan Mappings in Metric Spaces Endowed with a Graph

Definition 2.3.1. [13] Let (X,d) be a metric space. The mapping $T: X \to X$ is said to be a $G-$Kannan mapping if:

i. T preserves edges of G, i.e., $(x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G)$,

ii. There exists $\alpha \in \left[0,\frac{1}{2}\right)$ such that:

$$d(Tx,Ty) \leq \alpha\left[d(x,Tx)+d(y,Ty)\right], \quad \text{for all } (x,y) \in E(G).$$

Remark 2.3.2. [13] If a mapping $T: X \to X$ is a $G-$Kannan mapping, then T is both a $G^{-1}-$Kannan mapping and a $\tilde{G}-$Kannan mapping.

Example 2.3.3. [13] Any Kannan mapping is a G_0-Kannan contraction, where the graph G_0 is defined by $E(G_0) := XxX$.

Example 2.3.4. [13] Let $X = \{0,1,3\}$ and the Euclidean metric

$d(x,y)=|x-y|, \forall x,y\in X$.

The mapping $T:X\to X$,

$$Tx=\begin{cases} Tx=0, & \text{if } x\in\{0,1\} \\ Tx=1, & \text{if } x=3. \end{cases}$$

is a $G-$Kannan mapping with constant

$\alpha=\frac{1}{3}$, where $\mathrm{E}(G)=\{\ (0,1);(1,3);(0,0);(1,1);(3,3)\ \}$,

but is not a Kannan mapping because $d(T0,T3)=1$ and $d(0,T0)+d(3,T3)=2$.

Lemma 2.3.5. [13] Let (X,d) be a metric space endowed with a graph G and $T:X\to X$ be a $G-$Kannan mapping with constant α. If the graph G is weakly $T-$connected, then given $x,y\in X$, there is $r(x,y)\geq 0$ such that

$$d(T^n x,T^n y)\leq \alpha d(T^{n-1}x,T^n x)+\left(\frac{\alpha}{1-\alpha}\right)^n r(x,y)+\alpha d(T^{n-1}y,T^n y)$$

for all $n\in\mathbb{N}^*$.

2.4. Fixed Point Theorems for Reich Type Contractions on Metric Spaces with a Graph

Definition 2.4.1. [18] Let (X,d) be a metric space. The operator $T:X\to X$ is said to be a $G-$ Ciric-Reich-Rus operator if:

i. T preserves edges of G, i.e., $(x,y)\in E(G)\Rightarrow(Tx,Ty)\in E(G)$,

ii. There exists nonnegative number α,β,γ with $\alpha+\beta+\gamma<1$ such that for each $(x,y)\in E(G)$, we have;

$$d(Tx,Ty)\le \alpha d(x,y)+\beta d(x,Tx)+\gamma d(y,Ty).$$

Example 2.4.2. [18] Any Ciric-Reich-Rus operator is a G_0- Ciric-Reich-Rus operator, where the graph G_0 is defined by $E(G_0):=XxX$.

Example 2.4.3. [18] Let $X=\{0,1,2,3\}$ and the Euclidean metric

$$d(x,y)=|x-y|, \forall x,y\in X .$$

The mapping $T:X\to X$,

$$Tx=\begin{cases} Tx=0, & \text{if } x\in\{0,1\} \\ Tx=1, & \text{if } x{=}3. \end{cases}$$

is a $G-$Ciric-Reich-Rus operator with constants $\alpha=\frac{1}{3},\ \beta=0,\ \gamma=\frac{1}{3}$, where the edges of G defined by $\mathrm{E}(G)=\{(0,1);(0,2);(2,3);(0,0);(1,1);(2,2);(3,3)\}$, but is not a Ciric-Reich-Rus operator because

$$d(T1,T2)=1, d(1,2)=1, d(1,T1)=1 \text{ and } d(2,T2)=1.$$

Lemma 2.4.4. [7] Let (X,d) be a metric space endowed with a graph G and $T:X\to X$ be a $G-$Ciric-Reich-Rus operator. If $x\in X$ satisfies the property $(x,Tx)\in E(G)$, then we have

$$d(T^n x, T^{n+1}x)\le c^n d(x,Tx),$$

for all $n \in \mathbb{N}$, where $c = \dfrac{\alpha + \beta}{1 - \gamma}$.

Lemma 2.4.5. [18] Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a $G-$Ciric-Reich-Rus operator such that the graph G is $T-$ connected. For all $x \in X$ the subsequence $(T^n x)_{n \in \mathbb{N}}$ is a Cauchy sequence.

2.5. Generalized Contractions in Metric Spaces Endowed with a Graph

Definition 2.5.1. [20] We say that a mapping $T : X \to X$ is a $G-$graphic contraction if

i. T preserves edges of G, i.e., $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$,

ii. there exists $\alpha \in [0,1)$ such that

$$d(Tx, T^2 x) \le \alpha d(x, Tx)$$

for all $x \in X^T$, where $X^T = \{x \in X : (x, Tx) \in E(G) \text{ or } (Tx, x) \in E(G)\}$.

Lemma 2.5.2. [20] Let (X, d) be a metric space endowed with a graph G. If a mapping $T : X \to X$ is a $G-$graphic contraction, then T is both a $G^{-1}-$graphic contraction and a $\tilde{G}-$graphic contraction.

Lemma 2.5.3. [20] Let $T : X \to X$ be a $G-$graphic contraction with a constant α. Then, given $x \in X^T$, there is $r(x) \ge 0$ such that

$$d(T^n x, T^{n+1} x) \le \alpha^n r(x,), \quad \text{for all } n \in \mathbb{N}.$$

Lemma 2.5.4. [20] Let (X,d) be a complete metric space endowed with a graph G. Suppose that: $T:X\to X$ is a $G-$graphic contraction. Then for each $x\in X^T$, there exists $x^*\in X$ such that the sequence $(T^n x)_{n\in\mathbb{N}}$ converges to x^* as $n\to\infty$.

Lemma 2.5.5. [20] Let (X,d) be metric space endowed with a graph G. Assume that $T:X\to X$ is a $G-$graphic contraction such that for some $x_0\in X$, $Tx_0\in[x_0]_{\tilde{G}}$. Let $\tilde{G}_{x_0}$ be a component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is $T-$invariant and $T\big|_{[x_0]_{\tilde{G}}}$ is a $\tilde{G}_{x_0}-$graphic contraction.

Example 2.5.6. [20] Let $X=[0,1]$ be endowed with the usual metric. Consider

$$E(G)=\{\,(0,0)\,\}\cup\{\,(0,x):x\geq 1/2\,\}\cup\{\,(x,y):x,y\in(0,1]\,\},$$

and $T:X\to X$

$$Tx=\begin{cases}\dfrac{x}{2}, & \text{if } x\in(0,1];\\ \dfrac{3}{4}, & \text{if } x=0;\\ 1, & \text{if } x=1.\end{cases}$$

Then G is weakly connected, X^T is nonempty and T is a $G-$graphic contraction but is not $G-$contraction. Moreover, $F(T)=\{1\}$.

Definition 2.5.7. [20] Let (X,d) be a metric space. The operator $T:X\to X$ is said to be a Ciric-Reich-Rus $G-$contraction if:

i. T preserves edges of G, i.e., $(x,y)\in E(G)\Rightarrow(Tx,Ty)\in E(G)$,

ii. There exists nonnegative number α, β, γ with $\alpha+\beta+\gamma<1$ such that for each $x, y \in X$, we have

$$\left((x, y) \in E(G) \text{ implies } d(Tx, Ty) \leq \alpha d(x, y)+\beta d(x, Tx)+\gamma d(y, Ty)\right).$$

Lemma 2.5.8. [20] Let (X, d) be a metric space endowed with a graph G. If a mapping $T: X \rightarrow X$ is a Ciric-Reich-Rus $G-$contraction, then T is both a Ciric-Reich-Rus $G^{-1}-$contraction and a Ciric-Reich-Rus $\tilde{G}-$contraction.

Lemma 2.5.9. [20] Let (X, d) be a metric space endowed with a graph G and $T: X \rightarrow X$ be a Ciric-Reich-Rus $G-$contraction with constants α, β, γ. Then, given $x \in X^{T}$, there exists a $r(x) \geq 0$ such that, then we have

$$d\left(T^{n} x, T^{n+1} x\right) \leq c^{n} r(x),$$

for all $n \in \mathbb{N}$, where $c=\dfrac{\alpha+\beta}{1-\gamma}$.

Lemma 2.5.10. [20] Let (X, d) be a complete metric space endowed with a graph G. Suppose that $T: X \rightarrow X$ is a Ciric-Reich-Rus $G-$contraction. Then for each $x \in X^{T}$, there exists $x^{*} \in X$ such that the sequence $\left(T^{n} x\right)_{n \in \mathbb{N}}$ converges to x^{*} as $n \rightarrow \infty$.

CHAPTER 3. SOME FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS IN METRIC SPACE WITH A GRAPH

In the present section $(G,\varphi)-$graphic contractions have been defined by using a comparison function and studied the existence of fixed points. Also, the Hardy-Rogers $G-$graphic contractions have been introduced and some fixed point theorems have been proved. Some results in the literature are also generalized and extended. Moreover, we give some examples to support the usability of our results.

3.1. $(G,\varphi)-$ Graphic Contraction and Fixed Point Theorems

We study the existence of fixed points in metric spaces with a graph by defining $(G,\varphi)-$ graphic contraction. Also, we will consider that the function φ is a strong comparison function.

Definition 3.1.1. Let (X,d) be a metric space and G a graph. The mapping $T: X \to X$ is called a $(G,\varphi)-$graphic contraction if the following conditions hold;

i. T preserves edges of G; $(x,y)\in E(G) \Rightarrow (Tx,Ty)\in E(G)$ for all $x,y\in X$,

ii. there exist a comparison function $\varphi \in \Phi$ such that

$$d\left(Tx,T^2x\right)\le \varphi\left(d\left(x,Tx\right)\right) \text{ for all } x\in X^T .$$

Remark 3.1.2. If T is a $(G,\varphi)-$graphic contraction, then T is both a $(G^{-1},\varphi)-$ graphic contraction and a $(\widetilde{G},\varphi)-$ graphic contraction.

Example 3.1.3. Any $G-$graphic contraction is a $(G,\varphi)-$graphic contraction, if the comparison function is given as $\varphi \in \Phi$.

The following example shows that $(G,\varphi)-$graphic contraction is an extension of $(G,\varphi)-$contraction given in [19].

Example 3.1.4. Let $X=[0,1]$ be endowed with the usual metric. Take

$$E(G)=\{\,(0,0)\,\}\cup\{\,(0,1)\,\}\cup\{\,(x,y)\in(0,1]\times(0,1]:x\geq y\,\},$$

and $T:X\to X$ as follows:

$$Tx=\begin{cases}\dfrac{x}{4}, & \text{if } x\in(0,1);\\ \dfrac{1}{4}, & \text{if } x=0;\\ 1, & \text{if } x=1.\end{cases}$$

Then G is weakly connected and X^T is nonempty and T is a $(G,\varphi)-$graphic contraction with $\varphi(t)=\dfrac{3t}{4}$ which is not a $(G,\varphi)-$contraction. Moreover; $F(T)=\{1\}$.

Proof. It is obvious that G is weakly connected and $X^T\neq\varnothing$. It can be easily seen that T is a $(G,\varphi)-$graphic contraction. Take $d\left(T1,T\dfrac{1}{2}\right)\leq\varphi\left(d\left(1,\dfrac{1}{2}\right)\right)\Rightarrow\dfrac{7}{8}\leq\dfrac{3}{8}$, which is a contradiction. Thereby, T is not $(G,\varphi)-$contraction.

Lemma 3.1.5. Let (X,d) be a metric space endowed with a graph G. Let $T:X\to X$ be a $(G,\varphi)-$graphic contraction. If $x\in X^T$ then, there exists $r(x)\geq 0$ such that

$$d\left(T^n x, T^{n+1}x\right)\leq \varphi^n\left(r(x)\right)$$

for all $n\in\mathbb{N}$, where $r(x)=d(x,Tx)$.

Proof. Take $x\in X^T$, that is, $(x,Tx)\in E(G)$ or $(Tx,x)\in E(G)$. If $(x,Tx)\in E(G)$, then by induction we have $\left(T^n x, T^{n+1}x\right)\in E(G)$ for each $n\in\mathbb{N}$. Thus

$$\begin{aligned} d\left(T^n x, T^{n+1}x\right) &\leq \varphi\left(d\left(T^{n-1}x, T^n x\right)\right)\leq \varphi^2\left(d\left(T^{n-2}x, T^{n-1}x\right)\right)\\ &\;\;\vdots\\ &\leq \varphi^n\left(d\left(x,Tx\right)\right)=\varphi^n\left(r(x)\right). \end{aligned}$$

If $(Tx,x)\in E(G)$, again by induction, we have that $\left(T^{n+1}x, T^n x\right)\in E(G)$ for each $n\in\mathbb{N}$. Hence

$$\begin{aligned} d\left(T^n x, T^{n+1}x\right) &\leq \varphi\left(d\left(T^{n-1}x, T^n x\right)\right)\leq \varphi^2\left(d\left(T^{n-2}x, T^{n-1}x\right)\right)\\ &\;\;\vdots\\ &\leq \varphi^n\left(d\left(x,Tx\right)\right)=\varphi^n\left(r(x)\right). \end{aligned}$$

Lemma 3.1.6. Let (X,d) be a complete metric space endowed with a graph G. Assume that $T:X\to X$ is a $(G,\varphi)-$graphic contraction. Then, for each $x\in X^T$, there exists $x^*\in X$ such that the sequence $\left(T^n x\right)_{n\in\mathbb{N}}$ converges x^* as $n\to\infty$.

Proof. Let $x\in X^T$. By Lemma 1.3.5., we obtain

$$d\left(T^n x, T^{n+1}x\right)\leq \varphi^n\left(r(x)\right),$$

for all $n \in \mathbb{N}$. Hence $\sum_{n=0}^{\infty} d\left(T^n x, T^{n+1} x\right) < \infty$ and the sequence $\left(T^n x\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. Because (X,d) is a complete metric space, $\left(T^n x\right)_{n\in\mathbb{N}}$ is convergence sequence and say limit is $x^* \in X$.

In the following example shows that above the lemma does not satisfy unless the function φ is not strong comparison.

Example 3.1.7. Recall that $\varphi(t) = \dfrac{t}{t+1}$, $t \geq 0$ is a comparison function but not a strong comparison function. If we use $\varphi(t) = \dfrac{t}{t+1}$, $t \geq 0$ in the previous lemma, we have

$$\sum_{n=1}^{\infty} \varphi^n\left(d\left(x,Tx\right)\right) = \sum_{n=1}^{\infty} \frac{d\left(x,Tx\right)}{n\, d\left(x,Tx\right)+1}$$

diverges if $d(x,Tx) > 0$. Thus, this shows that it is necessary to use a strong comparison function.

Lemma 3.1.8. Let (X,d) be a complete metric space endowed with a graph G, $T: X \to X$ is a $(G,\varphi)-$graphic contraction for which there exists $x_0 \in X$ such that $Tx_0 \in [x_0]_{\tilde{G}}$. Let $\tilde{G}_{x_0}$ be the component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is $T-$invariant and $T\big|_{[x]_{\tilde{G}}}$ is a $\left(\tilde{G}_{x_0}, \varphi\right)-$graphic contraction.

Proof. Choose $x \in [x_0]_{\tilde{G}}$. Then there exists a path $(x_i)_{i=0}^{N}$ in $\tilde{G}$ from x_0 to x, i.e., $x_N = x$ and $\left(x_{i-1}, x_i\right) \in E\left(\tilde{G}\right)$ for $i = 1,2,...,N$. But T is a $(G,\varphi)-$graphic contraction which yields $\left(Tx_{i-1}, Tx_i\right) \in E\left(\tilde{G}\right)$ for $i = 1,2,...,N$, this means that $(Tx)_{i=0}^{N}$ is a path in $\tilde{G}$ from Tx_0 to Tx. Hence $Tx \in [Tx_0]_{\tilde{G}}$. Since, by the hypothesis,

$Tx_0 \in [x_0]_{\tilde{G}}$, that is, $[Tx_0]_{\tilde{G}} \in [x_0]_{\tilde{G}}$, we conclude $Tx \in [x_0]_{\tilde{G}}$ and consequently $[x_0]_{\tilde{G}}$ is $T-$invariant. Let $(x,y) \in E(\tilde{G}_{x_0})$, then there exists a path $(x_i)_{i=0}^{N}$ in $\tilde{G}$ from x_0 to y such that $x_{N-1} = x$ and $(y_i)_{i=0}^{M}$ be a path in $\tilde{G}$ from x_0 to Tx_0. With the same argument as the first part of the proof, we deduce that $(y_0, y_1, ..., y_M, Tx_1, Tx_2, ..., Tx_N)$ is a path in $\tilde{G}$ from x_0 to Ty; especially $(Tx_{N-1}, Tx_N) \in E(\tilde{G}_{x_0})$, i.e., $(Tx,Ty) \in E(\tilde{G}_{x_0})$. Also, T is a $(\tilde{G}_{x_0}, \varphi)-$graphic contraction. Since $E(\tilde{G}_{x_0}) \in E(\tilde{G})$, and T is a $(\tilde{G}, \varphi)-$graphic contraction.

Theorem 3.1.9. Let (X,d) be a complete metric space and G be a directed graph. Let the triple (X,d,G) has the following condition;

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$

$\big($or respectively $(x_{n+1}, x_n) \in E(G)\big)$ for all $n \in \mathbb{N}$, then there is a subsequence

$$\left(x_{k_n}\right)_{n\in\mathbb{N}} \text{ with } \left(x_{k_n}, x\right) \in E(G) \ \left(\text{or respectively } \left(x, x_{k_n}\right) \in E(G)\right) \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Let $T : X \to X$ be a $(G, \varphi)-$graphic contraction which is orbitally $G-$continuous. Then the following statements hold:

i. $F(T) \neq \emptyset$ iff $X^T \neq \emptyset$.

ii. If $X^T \neq \emptyset$ and G is weakly connected, then T is a weakly Picard operator.

iii. For any $x \in X^T$, we have that $T\big|_{[x]_{\tilde{G}}}$ is a weakly Picard operator.

Proof. We begin with the statement (iii). Let $x \in X^T$. Hence, there exists $r(x) \geq 0$ such that

$$d\left(T^n x, T^{n+1} x\right) \leq \alpha^n r(x),$$

for all $n \in \mathbb{N}$. So, there exists $x^* \in X$ such that $\lim_{n \to \infty} \left(T^n x\right) = x^*$. Since $x \in X^T$ in Definition 3.1.1. implies that $T^n x \in X^T$ for every $n \in \mathbb{N}$. Now assume that $(x, Tx) \in E(G)$. (This can be done if $(Tx, x) \in E(G)$). By using (3.1), a subsequence $\left(T^{k_n} x\right)_{n \in \mathbb{N}}$ of $\left(T^n x\right)_{n \in \mathbb{N}}$ such that $\left(T^{k_n} x, x^*\right) \in E(G)$ for each $n \in \mathbb{N}$. A path in G can be formed by using the points $x, Tx, ..., T^{k_1} x, x^*$ and hence $x^* \in [x]_{\tilde{G}}$. Since T is orbitally $G-$continuous, we obtain that x^* is a fixed point for $T\big|_{[x]_{\tilde{G}}}$.

To prove (i), using (iii) we have $F(T) \neq \varnothing$ if $X^T \neq \varnothing$. Suppose that $F(T) \neq \varnothing$. By using the assumption that $\Delta \subseteq E(G)$, we immediately obtain that $X^T \neq \varnothing$. Hence (i) holds.

For proving (ii), let $x \in X^T$. If we use weak connectivity of G, we have that $X = [x]_{\tilde{G}}$ and by applying (iii), we obtain the desired result.

The next example shows that for any $(G, \varphi)-$graphic contraction $T : X \to X$, being orbitally $G-$continuous, is a necessary condition to be a weakly Picard operator.

Example 3.1.10. Let $X = [0,1]$ be endowed with the usual metric. Consider

$$E(G) = \{ (0,0) \} \cup \{ (0,x) : x \geq 1/2 \} \cup \{ (x,y) : x, y \in (0,1] \},$$

and $T : X \to X$

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in (0,1]; \\ \dfrac{1}{2}, & \text{if } x = 0. \end{cases}$$

Then G is weakly connected, X^T is nonempty and T is a $(G,\varphi)-$graphic contraction with $\varphi(t)=\frac{t}{2}$, but is not orbitally $G-$continuous. Thus T has not a fixed point.

The example which is given below satisfies all conditions and statements (i-iii) of Theorem 3.1.9.

Example 3.1.11. Let $X=[0,1]$ be endowed with the usual metric. Consider

$$E(G)=\{\,(0,0)\,\}\cup\{\,(0,x):x\geq 1/2\,\}\cup\{\,(x,y):x,y\in(0,1]\,\},$$

and $T:X\to X$,

$$Tx=\begin{cases}\frac{x}{2}, & \text{if } x\in(0,1);\\ 0, & \text{if } x=0;\\ 1, & \text{if } x=1.\end{cases}$$

Then G is weakly connected, X^T is nonempty and T is a $(G,\varphi)-$graphic contraction with $\varphi(t)=\frac{t}{2}$ and also, T is orbitally $G-$continuous. Moreover; $F(T)=\{0,1\}$.

3.2. Hardy-Rogers $G-$Graphic Contraction and Fixed Point Theorems

Definition 3.2.1. The mapping $T:X\to X$ is a Hardy-Rogers $G-$graphic contraction if the following conditions hold:

i. T preserves edges of G; $(x,y)\in E(G)\Rightarrow(Tx,Ty)\in E(G)$,

ii. there exist $\alpha,\beta,\gamma,\delta,\eta$ nonnegative real numbers and $\alpha+\beta+\gamma+\delta+\eta<1$ such that

$$d\left(Tx,T^2x\right)\le \alpha d\left(x,Tx\right)+\beta d\left(Tx,T^2x\right)+\gamma d\left(x,T^2x\right)+\delta d\left(Tx,Tx\right)+\eta d\left(x,Tx\right)$$

for all $x\in X^T$.

Remark 3.2.2. If T is a Hardy-Rogers $G-$graphic contraction, then T is both a Hardy-Rogers $G^{-1}-$graphic contraction and a Hardy-Rogers $\tilde{G}-$graphic contraction.

Remark 3.2.3. Any $G-$graphic contraction is a Hardy-Rogers $G-$graphic contraction where $\alpha=\beta=\gamma=\delta=0$.

Example 3.2.4. Let $X=\{0,1,2,3\}$ and $d(x,y)=|x-y|$ for all $x,y\in X$. Define the operator $T:X\to X$ as;

$$Tx=\begin{cases}0, \text{ if } x\in\{0,1\};\\ 1, \text{ if } x\in\{2,3\}.\end{cases}$$

T is a Hardy-Rogers $G-$graphic contraction with constants $\alpha=\beta=\gamma=\eta=\frac{1}{5}$ and $\delta=0$, where $E(G)=\{\,(0,1);(0,2);(2,3);(0,0);(1,1);(2,2);(3,3)\,\}$, but it is not a Hardy-Rogers contraction

$$d\left(T1,T2\right)\le \alpha d\left(1,T1\right)+\beta d\left(2,T2\right)+\gamma d\left(1,T2\right)+\delta d\left(2,T1\right)+\eta d\left(1,2\right),$$

it is a contradiction since $1\le\frac{3}{5}$.

Lemma 3.2.5. Let (X,d) be a metric space endowed with a graph G. Let $T:X\to X$ be a Hardy-Rogers $G-$graphic contraction with $\alpha,\beta,\gamma,\delta,\eta$ nonnegative real numbers and $\alpha+\beta+\gamma+\delta+\eta<1$. If $x\in X^T$ then there exists $r(x)\geq 0$ such that

$$d\left(T^n x,T^{n+1}x\right)\leq \lambda^n r\left(x\right)$$

for all $n\in\mathbb{N}$, where $\lambda=\dfrac{\alpha+\gamma+\eta}{1-\beta-\gamma}<1$.

Proof. Take $x\in X^T$, then $\left(x,Tx\right)\in E\left(G\right)$ or $\left(Tx,x\right)\in E\left(G\right)$. If $\left(x,Tx\right)\in E\left(G\right)$ then by induction we get $\left(T^n x,T^{n+1}x\right)\in E\left(G\right)$ for each $n\in\mathbb{N}$. Therefore,

$$\begin{aligned} d\left(T^n x,T^{n+1}x\right) &\leq \alpha d\left(T^{n-1}x,T^n x\right)+\beta d\left(T^n x,T^{n+1}x\right)+\gamma d\left(T^{n-1}x,T^{n+1}x\right) \\ &\quad +\delta d\left(T^n x,T^n x\right)+\eta d\left(T^{n-1}x,T^n x\right) \\ &\leq \alpha d\left(T^{n-1}x,T^n x\right)+\beta d\left(T^n x,T^{n+1}x\right)+\gamma d\left(T^{n-1}x,T^n x\right) \\ &\quad +\gamma d\left(T^n x,T^{n+1}x\right)+\eta d\left(T^{n-1}x,T^n x\right). \end{aligned}$$

$$d\left(T^n x,T^{n+1}x\right)\leq \lambda d\left(T^{n-1}x,T^n x\right)$$

where $\lambda=\dfrac{\alpha+\gamma+\eta}{1-\beta-\gamma}<1$. Hence, we obtain that

$$d\left(T^n x,T^{n+1}x\right)\leq \lambda d\left(T^{n-1}x,T^n x\right)\leq \ldots \leq \alpha^n d\left(x,Tx\right)=\lambda^n r\left(x\right).$$

If $(Tx,x)\in E(G)$ then we can also prove $(T^{n+1}x,T^{n}x)\in E(G)$ for each $n\in\mathbb{N}$ by induction. Consequently, we obtain the proof.

Lemma 3.2.6. Let (X,d) be a complete metric space endowed with a graph G. Suppose that $T:X\to X$ is a Hardy-Rogers $G-$graphic contraction with constant $\alpha,\beta,\gamma,\delta,\eta$ with $\alpha+\beta+\gamma+\delta+\eta<1$. Then for each $x\in X^{T}$, there exists $x^{*}\in X$ such that the sequence $(T^{n}x)_{n\in\mathbb{N}}$ converges to x^{*} as $n\to\infty$.

Proof. Choose an element x in X^{T} then by Lemma 3.2.5. we have

$$d(T^{n}x,T^{n+1}x)\le\lambda^{n}r(x)$$

for all $n\in\mathbb{N}$, where $r(x)=d(x,Tx)$.

Hence $\sum_{n=0}^{\infty}d(T^{n}x,T^{n+1}x)<\infty$ and by using same arguments we obtain that $(T^{n}x)_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of X, there exists $x^{*}\in X$ such that $(T^{n}x)_{n\in\mathbb{N}}$ converges to x^{*} as $n\to\infty$.

Lemma 3.2.7. Let (X,d) be a complete metric space endowed with a graph G. The self mapping T is a Hardy-Rogers $G-$graphic contraction for which there exists $x_{0}\in X$ such that $Tx_{0}\in[x_{0}]_{\tilde{G}}$. Then the set $[x_{0}]_{\tilde{G}}$ invariant with respect to T and $T\big|_{[x_{0}]_{\tilde{G}}}$ is a Hardy-Rogers $\tilde{G}_{x_{0}}-$graphic contraction, where $\tilde{G}_{x_{0}}$ is the component of $\tilde{G}$ containing x_{0}.

Proof. Let x be an element in $x\in[x_{0}]_{\tilde{G}}$. Then there exist $(x_{i})_{i=0}^{N}$ in $\tilde{G}$ from x_{0} to x, i.e., $x_{N}=x$ and $(x_{i-1},x_{i})\in E(\tilde{G})$ for $i=1,2,...,N$. Since T is a Hardy-Rogers $G-$ graphic contraction we get that $(Tx_{i-1},Tx_{i})\in E(\tilde{G})$ for $i=1,2,...,N$. So we have a path from Tx_0 to Tx. Therefore $Tx \in [Tx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$ since $Tx_0 \in [x_0]_{\tilde{G}}$. Consequently $[x_0]_{\tilde{G}}$ is invariant with respect to T. Take $(x, y) \in E(\tilde{G}_{x_0})$, then there is a path $(x_i)_{i=0}^{N}$ in $\tilde{G}$ from x_0 to y such that $x_{N-1} = x$. Also, let $(y_i)_{i=0}^{M}$ be a path in $\tilde{G}$ from x_0 to Tx_0. By using the argument from the first part of the proof, we realize

$$(y_0, y_1, ..., y_M, Tx_1, Tx_2, ..., Tx_{N-1} = Tx, Tx_N = Ty)$$

is a path in $\tilde{G}$ from x_0 to Ty such that $(Tx, Ty) \in E(\tilde{G}_{x_0})$. Furthermore, T is a Hardy-Rogers $\tilde{G}_{x_0}$ – graphic contraction because $E(\tilde{G}_{x_0}) \subseteq E(\tilde{G})$ and T is a Hardy-Rogers $\tilde{G}$ – graphic contraction.

Theorem 3.2.8. Let (X, d) be a complete metric space and G be a directed graph such that the triple (X, d, G) has the following condition;

$$\begin{gathered}\text{for any } (x_n)_{n\in\mathbb{N}} \text{ in } X, \text{ if } x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G) \\ \big(\text{or respectively } (x_{n+1}, x_n) \in E(G)\big) \text{ for all } n \in \mathbb{N}, \text{ then there is a subsequence} \\ \left(x_{k_n}\right)_{n\in\mathbb{N}} \text{ with } \left(x_{k_n}, x\right) \in E(G) \ \left(\text{or respectively } \left(x, x_{k_n}\right) \in E(G)\right) \text{ for all } n \in \mathbb{N}.\end{gathered} \tag{3.2}$$

Let $T : X \to X$ be Hardy-Rogers G – graphic contraction with nonnegative constants $\alpha, \beta, \gamma, \delta, \eta$ with $\alpha + \beta + \gamma + \delta + \eta < 1$ such that T is orbitally G – continuous. Then we have the following statements:

i. $F(T) \neq \varnothing$ iff $X^T \neq \varnothing$,

ii. if $X^T \neq \varnothing$ and G is weakly connected, then T is weakly Picard operator.

iii. for any $x \in X^T$, we have that $T\big|_{[x]_{\tilde{G}}}$ is weakly Picard operator.

Proof. We first prove the statement (iii). Let x be an arbitrary element in X^T, then there exists $r(x) \geq 0$ such that

$$d\left(T^n x, T^{n+1} x\right) \leq \lambda^n r\left(x\right), \text{ for all } n \in \mathbb{N}.$$

This gives that there exists $x^* \in X$ such that $\lim_{n\to\infty}\left(T^n x\right) = x^*$ Since $x \in X^T$, in Definition 3.2.1. implies that $T^n x \in X^T$ for every $n \in \mathbb{N}$. Now let us suppose that $\left(x, Tx\right) \in E\left(G\right)$. (If we use $\left(Tx, x\right) \in E\left(G\right)$, a similar deduction can be done.) If we use (3.2), then there exists a subsequence $\left(T^{k_n} x\right)_{n\in\mathbb{N}}$ of $\left(T^n x\right)_{n\in\mathbb{N}}$ such that $\left(T^{k_n} x, x^*\right) \in E\left(G\right)$ for each $n \in \mathbb{N}$. Then there is a path in G formed by the points $x, Tx, \ldots, T^{k_1} x, x^*$, and hence $x^* \in \left[x\right]_{\tilde{G}}$. Since T is orbitally $G-$continuous, we obtain that x^* is a fixed point for $T\big|_{[x]_{\tilde{G}}}$.

To prove (i) and (ii) we can use the similar method which we use in last part of Theorem 3.1.9. By this way we complete the proof.

Remark 3.2.9. In Definition 3.2.1., if we take $\gamma = \delta = 0$, we get that T is Ciric-Reich-Rus $G-$contraction and our results are extensions of results which given in [20].

CHAPTER 4. FIXED POINT THEOREMS FOR $\psi-$ CONTRACTIONS IN METRIC SPACE INVOLVING A GRAPH

We introduce $(G,\psi)-$contraction and $(G,\psi)-$graphic contraction in a metric space by using a graph. We explain some conditions for a mapping which is a $(G,\psi)-$ contraction to have a unique fixed point and also we give conditions about the existence of a fixed point for $(G,\psi)-$graphic contraction by applying the connectivity of the graph in both cases.

4.1. $(G,\psi)-$Contraction and Fixed Point Theorems

Definition 4.1.1. We say that a mapping $T:X\to X$ is a $(G,\psi)-$ contraction if the followings hold;

i. T preserves edges of G, i.e.

$$\left((x,y)\in E(G)\Rightarrow(Tx,Ty)\in E(G)\right),\ \forall x,y\in X,$$

ii. T decreases weight of edges of G that is there exists $c\in(0,1)$ such that

$$(x,y)\in E(G)\Rightarrow\psi\left(d\left(Tx,Ty\right)\right)\le c\psi\left(d\left(x,y\right)\right),\text{ for all } x,y\in X.$$

Lemma 4.1.2. If $T:X\to X$ is a $(G,\psi)-$contraction, then T is both $(G^{-1},\psi)-$ contraction and $(\tilde{G},\psi)-$contraction.

Proof. The proof can be obtained by the symmetry of d and the definition of $(\tilde{G},\psi)-$contraction.

Lemma 4.1.3. Let $T:X\to X$ be a $(G,\psi)-$contraction with constant $c\in(0,1)$, for a given $x\in X$ and $y\in[x]_{\tilde{G}}$, there exists $r(x,y)\geq 0$ such that

$$\psi\left(d\left(T^n x,T^n y\right)\right)\leq c^n\, r\left(x,y\right).$$

Proof. Let $x\in X$ and $y\in[x]_{\tilde{G}}$. Then there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x to y, this means; $x_0=x$, $x_N=y$ and $(x_{i-1},x_i)\in E(\tilde{G})$ for $i=1,2,...,N$. By Lemma 4.1.2., T is a $(\tilde{G},\psi)-$contraction. With an easy induction we have, $\left(T^n x_{i-1},T^n x_i\right)\in E\left(\tilde{G}\right)$ and

$$\psi\left(d\left(T^n x_{i-1},T^n x_i\right)\right)\leq c\psi\left(d\left(T^{n-1}x_{i-1},T^{n-1}x_i\right)\right)$$

$$\leq c\left(c\psi\left(d\left(T^{n-2}x_{i-1},T^{n-2}x_i\right)\right)\right)\leq ...\leq c^n\psi\left(d\left(x_{i-1},x_i\right)\right)$$

for all $n\in\mathbb{N}$ and $i=1,2,...,N$. Hence using triangle inequality, we get

$$\psi\left(d\left(T^n x,T^n y\right)\right)\leq\sum_{i=1}^{N}\psi\left(d\left(T^n x_{i-1},T^n x_i\right)\right)\leq c^n\sum_{i=1}^{N}\psi\left(d\left(x_{i-1},x_i\right)\right).$$

So it qualifies to set $r\left(x,y\right):=\sum_{i=1}^{N}\psi\left(d\left(x_{i-1},x_i\right)\right)$.

Lemma 4.1.4. Let (X,d) be a complete metric space endowed with a graph G, $T:X\to X$ is a $(G,\psi)-$contraction for which there exists $x_0\in X$ such that $Tx_0\in[x_0]_{\tilde{G}}$. Let $\tilde{G}_{x_0}$ be the component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is

$T-$invariant and $T|_{[x]_{\tilde{G}}}$ is a $(\tilde{G}_{x_0},\psi)-$contraction. Furthermore, if $x,y\in[x_0]_{\tilde{G}}$, and the sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent.

Proof. The proof of this lemma can obtained by using similar arguments given in [19]. So we omitted the proof.

The following result shows us that there is a close relation between convergence of iteration sequence which obtained by using a $(G,\psi)-$contraction mapping and connectivity of the graph.

Theorem 4.1.5. Let (X,d) be a metric space endowed with a graph G and $T:X\to X$ be a $(G,\psi)-$contraction, then the following statements are equivalent:

i. G is weakly connected;
ii. for given $x,y\in X$, the sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent;
iii. $\operatorname{card}(F(T))\le 1$.

Proof. (i)$\Rightarrow$(ii) Let T be a $(G,\psi)-$contraction and $x,y\in X$. By hypothesis, $[x]_{\tilde{G}}=X$, so $Tx\in[x]_{\tilde{G}}$. By Lemma 4.1.3., we get

$$\psi\left(d\left(T^n x,T^{n+1}x\right)\right)\le c^n\ r\left(x,Tx\right),$$

for all $n\in\mathbb{N}$. Hence

$$\sum_{n=0}^{\infty}\psi\left(d\left(T^n x,T^{n+1}x\right)\right)<\infty$$

and if a standard argument is used then $\left(T^n x\right)_{n\in\mathbb{N}}$ is obtained as a Cauchy sequence. Since also, $y \in [x]_{\tilde{G}}$, Lemma 4.1.3. provides $\psi\left(d\left(T^n x, T^n y\right)\right) \leq c^n r(x, y)$. Therefore, $\left(T^n x\right)_{n\in\mathbb{N}}$ and $\left(T^n y\right)_{n\in\mathbb{N}}$ are equivalent. Clearly, because $\left(T^n x\right)_{n\in\mathbb{N}}$ is a Cauchy sequence, so $\left(T^n y\right)_{n\in\mathbb{N}}$.

(ii) $\Rightarrow$ (iii) Let T be a $(G,\psi)-$contraction and $x, y \in F(T)$. By (ii), $\left(T^n x\right)_{n\in\mathbb{N}}$ and $\left(T^n y\right)_{n\in\mathbb{N}}$ are equivalent which yields that $x = y$.

(iii) $\Rightarrow$ (ii) Suppose, on the contrary, G is not weakly connected, that is, $\tilde{G}$ is disconnected. Let $x_0 \in X$. Then both the sets $[x_0]_{\tilde{G}}$ and $X - [x_0]_{\tilde{G}}$ are nonempty. Let $y_0 \in X - [x_0]_{\tilde{G}}$ and define

$$Tx = \begin{cases} x_0, & \text{if } x \in [x_0]_{\tilde{G}}, \\ y_0, & \text{if } x \in X - [x_0]_{\tilde{G}}. \end{cases}$$

Obviously, $F(T) = \{x_0, y_0\}$. We show T is $(G,\psi)-$contraction. Let $(x, y) \in E(G)$. Then $[x]_{\tilde{G}} = [y]_{\tilde{G}}$, so either $x, y \in [x_0]_{\tilde{G}}$ or $x, y \in X - [x_0]_{\tilde{G}}$. Hence in both cases $Tx = Ty$, so $(Tx, Ty) \in E(G)$ as $E(G) \supseteq \Delta$, and $\psi\left(d\left(Tx, Ty\right)\right) = 0$. Thereby, T is $(G,\psi)-$contraction having two fixed points which violates.

The result which given in the following is an easy consequence of Theorem 4.1.5.

Corollary 4.1.6. Let (X, d) be a complete metric space endowed with a graph G and $T : X \to X$ be a $(G,\psi)-$contraction, then the following statements are equivalent:

i. G is weakly connected;
ii. there is $x^* \in X$ such that $\lim_{n\to\infty}\left(T^n x\right) = x^*$ for all $x \in X$.

Now, we give an example which T is $(G,\psi)-$contraction and this example shows that we could not add that x^* is a fixed point of T in Corollary 4.1.6.

Example 4.1.7. Let $X=[0,1]$ be endowed with the usual metric. Take

$$E(G)=\{\,(0,0)\,\}\cup\{\,(0,1)\,\}\cup\{\,(x,y)\in(0,1]\times(0,1]:x\geq y\,\},$$

and $T:X\to X$ as follows:

$$Tx=\begin{cases}\dfrac{x}{3}, & \text{if } \ x\in(0,1],\\[2mm] \dfrac{1}{2}, & \text{if } \ x=0.\end{cases}$$

Then T is $(G,\psi)-$contraction where $\psi(\omega)=\dfrac{\omega}{\omega+1}$.

Proof. It can be easily seen that G is a weakly connected graph and T is a $(G,\psi)-$contraction where $\psi(\omega)=\dfrac{\omega}{\omega+1}$. It is the fact that $T^n x\to 0$, for all $x\in X$ but T has no fixed point.

For any mapping which satisfy the condition of Corollary 4.1.6. to have a fixed point we need to add condition (4.1), given in the following Theorem.

Theorem 4.1.8. Let (X,d) be a complete metric space and the triple (X,d,G) have the following condition:

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n\to x$ and $(x_n,x_{n+1})\in E(G)$ for $n\in\mathbb{N}$, then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n},x)\in E(G)$ for $n\in\mathbb{N}$. **(4.1)**

Let $T:X\to X$ be a $(G,\psi)-$contraction, then the following statements hold.

i. $\operatorname{card}\left(F\left(T\right)\right) = \operatorname{card}\left\{ \left[x\right]_{\tilde{G}} : x \in X_T \right\}$.

ii. $F\left(T\right) \neq \varnothing$ iff $X_T \neq \varnothing$.

iii. T has a unique fixed point iff there exists $x_0 \in X_T$ such that $X_T \subseteq \left[x_0\right]_{\tilde{G}}$.

iv. For any $x \in X_T$, $T\big|_{[x]_{\tilde{G}}}$ is a Picard operator.

v. If $X_T \neq \varnothing$ and G is weakly connected, then T is a Picard operator.

vi. If $X' := \bigcup\left\{\left[x\right]_{\tilde{G}} : x \in X_T\right\}$, then $T\big|_{X'}$ is a weakly Picard operator.

vii. If $T \subseteq E(G)$, then T is a weakly Picard operator.

Proof. Initially, we prove the items (iv) and (v). Take $x \in X_T$ and then $Tx \in \left[x\right]_{\tilde{G}}$, so by Lemma 4.1.4., if $y \in \left[x\right]_{\tilde{G}}$, then $\left(T^n x\right)_{n\in\mathbb{N}}$ and $\left(T^n y\right)_{n\in\mathbb{N}}$ are Cauchy equivalent. Since X is complete, $\left(T^n x\right)_{n\in\mathbb{N}}$ converges to some $x^* \in X$. It is obvious that $\lim_{n\to\infty}\left(T^n y\right) = x^*$. Then by using induction we get

$$\left(T^n x, T^{n+1} x\right) \in E\left(G\right) \tag{4.2}$$

for all $n \in \mathbb{N}$, since $(x, Tx) \in E(G)$. By (4.1), there is a subsequence $\left(T^{k_n} x\right)_{n\in\mathbb{N}}$ such that $\left(T^{k_n} x, x^*\right) \in E\left(G\right)$ for all $n \in \mathbb{N}$. If we use (4.2), we conclude that $\left(x, Tx, T^2 x, \ldots, T^{k_1}, x^*\right)$ is a path in G and also in $\tilde{G}$ from x to x^*, this means that $x^* \in \left[x\right]_{\tilde{G}}$. Since T is $\left(G,\psi\right)-$contraction we have,

$$\psi\left(d\left(T^{k_{n+1}} x, Tx^*\right)\right) \leq c\,\psi\left(d\left(T^{k_n} x, x^*\right)\right),$$

for all $n \in \mathbb{N}$. By taking limit as $n \to \infty$, we deduce $Tx^* = x^*$. Thereby, $T\big|_{[x]_{\tilde{G}}}$ is a Picard operator. Also, we conclude that T is a Picard operator, when $\left[x\right]_{\tilde{G}} = X$, since weakly connectedness of G.

(vi) is obvious from (iv). For proof of (vii), if $T \subseteq E(G)$ then $X_T = X$ and so $X' = X$ holds. Thus T is a weakly Picard operator because of (vi).

Let define a mapping to prove (i) as; $\rho(x) = [x]_{\tilde{G}}$ for all $x \in F(T)$. It is sufficient to show that $\rho : F(T) \to C = \{ [x]_{\tilde{G}} : x \in X_T \}$, is a bijection. Because $E(G) \supseteq \Delta$, we deduce $F(T) \subseteq X_T$ and then $\rho(F(T)) \subseteq C$. Beside, if $x \in X_T$, then by (iv), $\lim_{n\to\infty}(T^n x) \in [x]_{\tilde{G}} \cap F(T)$ which implies $\rho\left(\lim_{n\to\infty}(T^n x)\right) = [x]_{\tilde{G}}$ and so ρ is a surjective mapping. We show that T is injective. Take $x_1, x_2 \in F(T)$ which are such that $\rho(x_1) = \rho(x_2) \Rightarrow [x_1]_{\tilde{G}} = [x_2]_{\tilde{G}}$, then $x_2 \in [x_1]_{\tilde{G}}$ and so by (i),

$$\lim_{n\to\infty}(T^n x_2) \in [x_1]_{\tilde{G}} \cap F(T) = \{x_1\},$$

which gives $x_1 = x_2$. Thus, T is injective and this is the desired result.
Finally, one can see that (ii) and (iii) are easy consequences of (i).

Corollary 4.1.9. Let (X, d) be complete metric space and (X, d, G) have property (4.1). The followings are equivalent:

i. G is weakly connected;
ii. for every $(G,\psi)-$contraction $T : X \to X$ such that $(x_0, Tx_0) \in E(G)$ for some $x_0 \in X$, is a Picard operator;
iii. for any $(G,\psi)-$contraction, $\text{card}(F(T)) \le 1$.

Proof. $(\text{i}) \Rightarrow (\text{ii})$: This can be obtained directly from Theorem 4.1.8., (v).

$(\text{ii}) \Rightarrow (\text{iii})$: Let $T : X \to X$ be a $(G,\psi)-$contraction. If X_T is empty, so is $F(T)$ because $F(T)$ is subset of X_T. If X_T is nonempty, then by (ii), $F(T)$ is singleton. In these two cases, $\text{card}(F(T)) \le 1$.

(iii) $\Rightarrow$ (i): This implication follows from Theorem 4.1.5.

Remark 4.1.10. In the above results by taking $\psi(\omega)=\omega$, one can we obtain the results, given in [5].

4.2. $(G,\psi)-$Graphic Contraction and Fixed Point Theorems

In this section, we define $(G,\psi)-$graphic contraction and give some results and examples.

Definition 4.2.1. Let (X,d) be a metric space and G be a graph. The mapping $T:X\to X$ is called a $(G,\psi)-$graphic contraction if the following conditions hold:

i. $(x,y)\in E(G)$ implies $(Tx,Ty)\in E(G)$, (T is edge preserving);

ii. there exists a $\psi:\mathbb{R}^+\to\mathbb{R}^+$ function with constants $c\in[0,1)$ such that

$$\psi\left(d\left(Tx,T^2x\right)\right)\le c\,\psi\left(d\left(x,Tx\right)\right), \text{ for all } x\in X^T.$$

Lemma 4.2.2. If $T:X\to X$ is a $(G,\psi)-$graphic contraction, then T is both $(G^{-1},\psi)-$graphic contraction and $(\widetilde{G},\psi)-$graphic contraction.

Lemma 4.2.3. Let $T:X\to X$ be a $(G,\psi)-$graphic contraction with constant $c\in[0,1)$. Then, given $x\in X^T$, there exists $r(x)\ge 0$ such that

$$\psi\left(d\left(T^nx,T^{n+1}x\right)\right)\le c^n\,r\left(x\right),$$

for all $n\in\mathbb{N}$, where $r\left(x\right):=\psi\left(d\left(x,Tx\right)\right)$.

Lemma 4.2.4. Suppose that $T: X \to X$ is a $(G,\psi)-$graphic contraction. Then for each $x \in X^T$, there exists $x^* \in X$ such that the sequence $\left(T^n x\right)_{n\in\mathbb{N}}$ converges to x^* as $n \to \infty$.

Proof. Take an arbitrary element x in X^T. By Lemma 4.2.3., we obtain that

$$\psi\left(d\left(T^n x, T^{n+1}x\right)\right) \le c^n r(x),$$

for all $n \in \mathbb{N}$. Therefore,

$$\sum_{n=0}^{\infty} \psi\left(d\left(T^n x, T^{n+1}x\right)\right) < \infty,$$

and so $\psi\left(d\left(T^n x, T^{n+1}x\right)\right) \to 0$, consequently using property of ψ we have $d\left(T^n x, T^{n+1}x\right) \to 0$. Then we say that $\left(T^n x\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $\left(T^n x\right)_{n\in\mathbb{N}}$ converges as $n \to \infty$.

Lemma 4.2.5. The self mapping T is a $(G,\psi)-$graphic contraction for which there exists $x_0 \in X$ such that $Tx_0 \in [x_0]_{\tilde{G}}$. Then the set $[x_0]_{\tilde{G}}$ invariant with respect to T and $T\big|_{[x_0]_{\tilde{G}}}$ is a $\left(\tilde{G}_{x_0}, \psi\right)-$graphic contraction, where $\tilde{G}_{x_0}$ is the component of $\tilde{G}$ containing x_0.

Proof. Let x be an element in $[x_0]_{\tilde{G}}$. Then there exist $(x_i)_{i=0}^N$ in $\tilde{G}$ from x_0 to x, i.e., $x_N = x$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1, 2, ..., N$. Since T is a $(G,\psi)-$graphic contraction we get that $(Tx_{i-1}, Tx_i) \in E(\tilde{G})$ for $i = 1, 2, ..., N$. So we have a path from Tx_0 to Tx. Therefore, $Tx \in [Tx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$ since $Tx_0 \in [x_0]_{\tilde{G}}$. Consequently $[x_0]_{\tilde{G}}$ is invariant with respect to T. Take $(x, y) \in E(\tilde{G}_{x_0})$, then there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$

from x_0 to y such that $x_{N-1}=x$. Also let $(y_i)_{i=0}^{M}$ be a path in $\tilde{G}$ from x_0 to Tx_0. Then, we realize

$$\left(y_0, y_1, ..., y_M, Tx_1, Tx_2, ..., Tx_{N-1}=Tx, Tx_N=Ty\right)$$

is a path in $\tilde{G}$ from x_0 to Ty such that $(Tx,Ty)\in E\left(\tilde{G}_{x_0}\right)$. Furthermore, T is a $\left(\tilde{G}_{x_0},\psi\right)-$graphic contraction because $E\left(\tilde{G}_{x_0}\right)\subseteq E\left(\tilde{G}\right)$ and T is a $\left(\tilde{G},\psi\right)-$graphic contraction.

Theorem 4.2.6. Let (X,d) be a complete metric space and let the triple (X,d,G) have the following condition:

$$\text{for any } (x_n)_{n\in\mathbb{N}} \text{ in } X, \text{ if } x_n\to x \text{ and } (x_n,x_{n+1})\in E(G)$$
$$\left(\text{or respectively } (x_{n+1},x_n)\in E(G)\right) \text{ for all } n\in\mathbb{N}, \text{then there is a subsequence}$$
$$\left(x_{k_n}\right)_{n\in\mathbb{N}} \text{ with } \left(x_{k_n},x\right)\in E(G) \left(\text{or respectively } \left(x,x_{k_n}\right)\in E(G)\right) \text{ for all } n\in\mathbb{N}. \quad (4.3)$$

Let $T:X\to X$ be a $(G,\psi)-$graphic contraction and T is orbitally $G-$continuous. Then the following statements hold:

i. $F(T)\neq\varnothing$ if and only if $X^T\neq\varnothing$;

ii. if $X^T\neq\varnothing$ and G is weakly connected, then T is a weakly Picard operator;

iii. for any $x\in X^T$, we have that $T\big|_{[x]_{\tilde{G}}}$ is a weakly Picard operator.

Proof. We begin with the statement (iii) Let $x\in X^T$. By Lemma 4.2.3., there exists $r(x)\geq 0$ such that

$$\psi\left(d\left(T^n x, T^{n+1} x\right)\right) \leq c^n\, r(x)$$

for all $n \in \mathbb{N}$. This gives, as in the proof of Lemma 4.2.4., there exists $x^* \in X$ such that $\lim_{n\to\infty}\left(T^n x\right) = x^*$. Since $x \in X^T$ in Definition 4.2.1. implies that $T^n x \in X^T$ for every $n \in \mathbb{N}$. Now assume that $(x, Tx) \in E(G)$. A similar deduction can be made if $(Tx, x) \in E(G)$. By condition (4.3), a subsequence $\left(T^{k_n} x\right)_{n\in\mathbb{N}}$ of $\left(T^n x\right)_{n\in\mathbb{N}}$ such that $\left(T^{k_n} x, x^*\right) \in E(G)$ for each $n \in \mathbb{N}$. A path in G can be formed by using the points $x, Tx, \ldots, T^{k_1} x, x^*$ and hence $x^* \in [x]_{\tilde{G}}$. Since T is orbitally $G-$ continuous, we obtain that x^* is a fixed point for $T\big|_{[x]_{\tilde{G}}}$.

To prove (i), using (iii) we have $F(T) \neq \varnothing$ if $X^T \neq \varnothing$. Suppose that $F(T) \neq \varnothing$. By using the assumption that $\Delta \subseteq E(G)$, we immediately obtain that $X^T \neq \varnothing$. Hence (i) holds.

For proving (ii) let $x \in X^T$. If we use weak connectivity of G, we have that $X = [x]_{\tilde{G}}$ and by applying (iii) we obtain the desired result.

The next example illustrates that T must be orbitally $G-$continuous in order to obtain statements which are given above theorem.

Example 4.2.7. Let $X = [0,1]$ be endowed with the usual metric. Consider

$$E(G) = \{\,(0,0)\,\} \cup \{\,(0,x) : x \geq 1/2\,\} \cup \{\,(x,y) : x, y \in (0,1]\,\},$$

and $T : X \to X$,

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in (0,1]; \\ \dfrac{1}{2}, & \text{if } x = 0. \end{cases}$$

Then G is weakly connected, X^T is nonempty and T is a $(G,\psi)-$graphic contraction where $\psi(\omega) = \dfrac{\omega}{3}$ but is not orbitally $G-$continuous. Thus, T does not have a fixed point.

Remark 4.2.8. In the Theorem 4.2.6., by replacing the condition that the triple (X,d,G) satisfies (4.3) and T is orbitally $G-$continuous with the mapping T is orbitally continuous, we have the above result, too.

Example 4.2.9. Let $X = [0,1]$ be endowed with the usual metric. Take

$$E(G) = \{ (0,0) \} \cup \{ (0,1) \} \cup \{ (x,y) \in (0,1] \times (0,1] : x \geq y \},$$

and $T : X \to X$ as follows:

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in (0,1], \\ \dfrac{3}{4}, & \text{if } x = 0. \end{cases}$$

Then G is weakly connected and X^T is nonempty and T is a $(G,\psi)-$graphic contraction with $\psi(\omega) = \dfrac{\omega}{2}$ which is not a $(G,\psi)-$contraction.

Proof. It is clear that G is weakly connected, $X^T \neq \varnothing$ and with simple calculations it can be easily seen that T is a $(G,\psi)-$graphic contraction. Take

$$\psi\left(d\left(T0,T\frac{1}{2}\right)\right)\leq c\psi\left(d\left(0,\frac{1}{2}\right)\right)\Rightarrow\frac{1}{4}\leq c\frac{1}{4},$$

which is a contradiction since $c\in[0,1)$. Thus, T is not $(G,\psi)-$contraction.

Remark 4.2.10. In Theorem 4.2.6., if we take $\psi(\omega)=\omega$ then we get the Theorem 2.1 which given in [20].

CHAPTER 5. FIXED POINT RESULTS FOR ψ −TYPE CONTRACTIONS IN METRIC SPACE INVOLVING A GRAPH

We consider $\psi-$type contractions defined on a complete metric space endowed with a graph. We establish fixed point results for such contractions. Also, our results improve and extend several known results in the existing literature. Furthermore, we give some examples to support our results.

5.1. $(G,\psi)-$ Ciric-Reich-Rus Contraction and Fixed Point Theorems

Definition 5.1.1. Let (X,d) be a metric space and G a graph. The mapping $T: X \to X$ is called $(G,\psi)-$Ciric-Reich-Rus contraction if the following conditions hold;

i. T preserves the edges of G, $(x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G)$, for all $x, y \in X$,

ii. there exists nonnegative numbers α, β, γ with $\alpha + \beta + \gamma < 1$, such that

$$\psi\left(d\left(Tx,Ty\right)\right) \le \alpha\, \psi\left(d\left(x,y\right)\right) + \beta\, \psi\left(d\left(x,Tx\right)\right) + \gamma\, \psi\left(d\left(y,Ty\right)\right),$$

for all $(x,y) \in E(G)$.

Remark 5.1.2. Let (X,d) be a metric space endowed with a graph G. If $T: X \to X$ is a $(G,\psi)-$Ciric-Reich-Rus contraction, then T is both a $(G^{-1},\psi)-$ Ciric-Reich-Rus contraction and a $(\widetilde{G},\psi)-$ Ciric-Reich-Rus contraction.

Remark 5.1.3. Any $(G,\psi)-$Ciric-Reich-Rus contraction is G-Ciric-Reich-Rus operator with $\psi(\omega)=\omega$.

The following lemma is useful tool to obtain our results.

Lemma 5.1.4. Let (X,d) be a metric space with a graph G and $T:X\to X$ be a $(G,\psi)-$Ciric-Reich-Rus contraction. If $x\in X$ satisfies the condition $(x,Tx)\in E(G)$, then we have

$$\psi\left(d\left(T^n x,T^{n+1}x\right)\right)\le c^n\,\psi\left(d\left(x,Tx\right)\right),$$

for all $n\in\mathbb{N}$, where $c=\dfrac{\alpha+\beta}{1-\gamma}$.

Proof. Let $x\in X$ with $(x,Tx)\in E(G)$. So, $(T^n x,T^{n+1}x)\in E(G)$, for all $n\in\mathbb{N}$.
Then for $n\in\mathbb{N}^*$,

$$\psi\left(d\left(T^n x,T^{n+1}x\right)\right)\le\alpha\,\psi\left(d\left(T^{n-1}x,T^n x\right)\right)+\beta\,\psi\left(d\left(T^{n-1}x,T^n x\right)\right)+\gamma\,\psi\left(d\left(T^n x,T^{n+1}x\right)\right),$$

which implies

$$\psi\left(d\left(T^n x,T^{n+1}x\right)\right)\le c\,\psi\left(d\left(T^{n-1}x,T^n x\right)\right),$$

where $c=\dfrac{\alpha+\beta}{1-\gamma}$, so we get

$$\psi\left(d\left(T^n x,T^{n+1}x\right)\right)\le c^n\,\psi\left(d\left(x,Tx\right)\right),$$

for all $n\in\mathbb{N}$.

Lemma 5.1.5. Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be a $(G,\psi)-$Ciric-Reich-Rus contraction such that the graph G is $T-$ connected. For all $x \in X$ the subsequence $\left(T^n x\right)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof. Let $x \in X$ fixed. Then:

1. If $(x,Tx) \in E(G)$, then we have

$$\psi\left(d\left(T^n x, T^{n+1}x\right)\right) \le c^n\, \psi\left(d\left(x,Tx\right)\right),$$

for all $n \in \mathbb{N}^*$ where $c = \dfrac{\alpha+\beta}{1-\gamma}$. Because $c<1$, we get

$$\sum_{n=0}^{\infty} \psi\left(d\left(T^n x, T^{n+1}x\right)\right) \le \frac{1}{1-c}\, \psi\left(d\left(x,Tx\right)\right) < \infty,$$

and a standard argument shows $\left(T^n x\right)_{n\in\mathbb{N}}$ is a Cauchy sequence.

2. If $(x,Tx) \notin E(G)$. Then there is a path in G, $(x_i)_{i=0}^{N}$ from x to Tx such that $x_0 = x$, $x_N = Tx$ with $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, 2, ..., N$ and $(x_i, Tx_i) \in E(G)$ for all $i = 1, 2, ..., N-1$. Then by the triangle inequality and Lemma 5.1.5., we obtain that

$$\psi\left(d\left(T^n x, T^{n+1}x\right)\right) \le \sum_{i=1}^{N} \psi\left(d\left(T^n x_{i-1}, T^n x_i\right)\right)$$

$$\le \alpha \sum_{i=1}^{N} \psi\left(d\left(T^{n-1}x_{i-1}, T^n x_i\right)\right) + \beta \sum_{i=1}^{N} \psi\left(d\left(T^{n-1}x_{i-1}, T^n x_{i-1}\right)\right) + \gamma \sum_{i=1}^{N} \psi\left(d\left(T^{n-1}x_i, T^n x_i\right)\right)$$

$$\le \alpha \sum_{i=1}^{N} \psi\left(d\left(T^{n-1}x_{i-1}, T^n x_i\right)\right) + \beta\, \psi\left(d\left(T^{n-1}x, T^n x\right)\right) + \beta\, c^{n-1} \sum_{i=1}^{N} \psi\left(d\left(x_{i-1}, Tx_{i-1}\right)\right)$$

$$+\gamma\, \psi\left(d\left(T^n x, T^{n+1}x\right)\right) + \gamma\, c^{n-1} \sum_{i=1}^{N} \psi\left(d\left(x_i, Tx_i\right)\right),$$

let us denote

$$x_n = \sum_{i=1}^{N} \psi\left(d\left(T^n x_{i-1}, T^n x_i\right)\right),$$

$n \in \mathbb{N}$ and set

$$r(x) = (\beta+\gamma) \sum_{i=2}^{N} \psi\left(d\left(x_{i-1}, Tx_{i-1}\right)\right);$$

Then we get

$$x_n \le (\alpha+\beta)\, x_{n-1} + (\beta+\gamma)\, c^{n-1} r(x) + \gamma\, x_n,$$

hence

$$x_n \le c\, x_{n-1} + \frac{\beta+\gamma}{1-\gamma}\, c^{n-1}\, r(x), \tag{5.1}$$

where, $c = \dfrac{\beta+\gamma}{1-\gamma}$. Using relation (5.1) and elementary computations, we have

$$x_n \le n\,\frac{\beta+\gamma}{1-\gamma}\,c^{n-1}\,r(x), \tag{5.2}$$

for all $n \in \mathbb{N}$. Because $c \in [0,1)$ and using (5.2), we obtain

$$\sum_{n=0}^{\infty} \psi\left(d\left(T^n x, T^{n+1} x\right)\right) \le \sum_{n=0}^{\infty} x_n \le x_n \le \frac{\beta+\gamma}{1-\gamma}\,r(x)\sum_{n=0}^{\infty} n\,c^{n-1} = \frac{\beta+\gamma}{(1-\gamma)(1-c)^2}\,r(x) < \infty,$$

and a standart argument shows that $\left(T^n x\right)_{n\ge 0}$ is a Cauchy sequence.

The main result of this section is given by the following theorem.

Theorem 5.1.6. Let (X,d) be a complete metric space endowed with a graph G and $T: X \to X$ be a $(G,\psi)-$Ciric-Reich-Rus contraction. We suppose that G is weakly $T-$connected and the triple (X,d,G) satisfies the condition:

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $\left(x_{k_n}\right)_{n\in\mathbb{N}}$ with $\left(x_{k_n}, x\right) \in E(G)$ for $n \in \mathbb{N}$. **(5.3)**

Then T is a PO.

Proof. From Lemma 5.1.5., $\left(T^n x\right)_{n\ge 0}$ is a Cauchy sequence for all $x \in X$, and by hypothesis, we obtain that $\left(T^n x\right)_{n\ge 0}$ is convergent. Let $x, y \in X$ then $\left(T^n x\right)_{n\ge 0} \to x^*$ and $\left(T^n y\right)_{n\ge 0} \to y^*$, as $n \to \infty$.

1. If $(x,y) \in E(G)$, we get $\left(T^n x, T^n y\right) \in E(G)$, for all $n \in \mathbb{N}$, then

$$\psi\left(d\left(T^n x, T^n y\right)\right) \le \alpha\psi\left(d\left(T^{n-1}x, T^{n-1}y\right)\right) + \beta\psi\left(d\left(T^{n-1}x, T^n x\right)\right) + \gamma\,\psi\left(d\left(T^{n-1}y, T^n y\right)\right),$$

for all $n \in \mathbb{N}^*$. Letting $n \to \infty$ we obtain that $\psi(d(x^*, y^*)) \le \alpha\psi(d(x^*, y^*))$ and because $\alpha \in [0,1)$ we obtain $\psi(d(x^*, y^*)) = 0 \Leftrightarrow d(x^*, y^*) = 0 \Leftrightarrow x^* = y^*$.

2. If $(x, y) \notin E(G)$, then there is a path in G, $(x_i)_{i=0}^N$ from x to y such that $x_0 = x$, $x_N = y$ with $(x_{i-1}, x_i) \in E(G)$ for all $i = 1, 2, ..., N$ and $(x_i, Tx_i) \in E(G)$ for all $i = 1, 2, ..., N-1$. Then $(T^n x_{i-1}, T^n x_i) \in E(G)$ for all $n \in \mathbb{N}$ and $i = 1, 2, ..., N$ and by the triangle inequality, we have

$$\psi\left(d\left(T^n x, T^n y\right)\right) \le \sum_{i=1}^{N} \psi\left(d\left(T^n x_{i-1}, T^n x_i\right)\right)$$

$$\le \alpha \sum_{i=1}^{N} \psi\left(d\left(T^{n-1} x_{i-1}, T^n x_i\right)\right) + \beta \sum_{i=1}^{N} \psi\left(d\left(T^{n-1} x_{i-1}, T^n x_{i-1}\right)\right) + \gamma \sum_{i=1}^{N} \psi\left(d\left(T^{n-1} x_i, T^n x_i\right)\right)$$

From previous lemma and hypothesis, we get that the sequence $(T^n x)_{n\ge 0}$ is convergent and using the continuity of distance we obtain, the sequence $\left(\psi\left(d\left(T^n x_{i-1}, T^n x_i\right)\right)\right)_{n\in\mathbb{N}}$ is convergent and $\lim\limits_{n\to\infty}\left(\psi\left(d\left(T^n x_{i-1}, T^n x_i\right)\right)\right) = l_i$ for all $i = 1, 2, ..., N$. Letting $n \to \infty$ we obtain $l_i = 0$ for all $i = 1, 2, ..., N$ that is $\psi(d(x^*, y^*)) \le 0$ and so $x^* = y^*$. Therefore, for all $x \in X$ there exists a unique $x^* \in X$ such that $\lim\limits_{n\to\infty} T^n x = x^*$.

Now we will prove that $x^* \in F(T)$. Since the graph G is weakly $T-$connected, there exists at least $x_0 \in X$ such that $(x_0, Tx_0) \in E(\tilde{G})$ so $(T^n x_0, T^{n+1} x_0) \in E(\tilde{G})$ for all $n \in \mathbb{N}$. But $\lim\limits_{n\to\infty} T^n x_0 = x^*$, then by condition (5.3), there is a subsequence $\left(T^{k_n} x_0\right)_{n\in\mathbb{N}}$ with $\left(T^{k_n} x_0, x^*\right) \in E(\tilde{G})$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, we have

$$\psi\left(d\left(x^*, Tx^*\right)\right) \le \psi\left(d\left(x^*, T^{k_{n+1}} x_0\right)\right) + \psi\left(d\left(T^{k_{n+1}} x_0, Tx^*\right)\right)$$

$$\leq \psi\left(d\left(x^*, T^{k_{n+1}}x_0\right)\right) + \alpha\,\psi\left(d\left(T^{k_n}x_0, x^*\right)\right) + \beta\,\psi\left(d\left(T^{k_n}x_0, T^{k_{n+1}}x_0\right)\right) + \gamma\,\psi\left(d\left(x^*, Tx^*\right)\right),$$

Now letting $n \to \infty$, we obtain $\psi\left(d\left(x^*, Tx^*\right)\right) \leq \gamma\,\psi\left(d\left(x^*, Tx^*\right)\right) \Rightarrow x^* = Tx^*$, that is $x^* \in F(T)$. If we have $Ty = y$ for some $y \in X$, then from above, we must have $T^n y \to x^*$, so $y = x^*$. Thus T is a PO.

The next example shows that the graph G must be weakly $T-$connected in order that the $(G,\psi)-$Ciric-Reich-Rus contraction T is a PO.

Example 5.1.7. Let $X = [0,\infty)$ be endowed with the Euclidean metric $d(x,y) = |x-y|$, and $T: X \to X$, $Tx = x + a,\ a \in \left[\frac{5}{4}, \infty\right)$. Define the graph G by $V(G) = X$, $E(G) = \{\,(x, x+b),\ (x+b, x):\ x \in X,\ b \in [0,1]\,\}$. Then (X,d) is a complete metric space but not weakly $T-$connected since $(x, Tx) \notin E(G)$ for all $x \in X$. The self mapping T is a $(G,\psi)-$Ciric-Reich-Rus contraction with $\psi(\omega) = \frac{\omega}{2}$ and $\alpha = \frac{1}{6}$, $\beta = \gamma = \frac{1}{3}$. Straightforwardly, $(T^n x)$ does not converge for all $x \in X$ and T has no fixed point.

The next example illustrates that condition (5.3) is a necessary condition inasmuch as the $(G,\psi)-$Ciric-Reich-Rus contraction T is a PO.

Example 5.1.7. Let $X = [0,1]$ be endowed with the usual metric. Take

$$E(G) = \{\,(0,0)\,\} \cup \{\,(0,1)\,\} \cup \{\,(x,y) \in (0,1] \times (0,1] : x \geq y\,\},$$

and $T: X \to X$ as follows:

$$Tx = \begin{cases} \frac{x}{3}, & \text{if} \quad x \in (0,1], \\ 1, & \text{if} \quad x = 0. \end{cases}$$

Then (X,d) is a complete metric space, G is weakly $T-$connected and T is $(G,\psi)-$Ciric-Reich-Rus contraction where $\psi(\omega) = \frac{\omega}{3}$ and $\alpha = \beta = \frac{1}{3}$, $\gamma = \frac{1}{4}$. Obviously $T^n x \to 0$, for all $x \in X$ but T has no fixed point.

Definition 5.1.8. Let (X,d) be a metric space and G be a graph. The self mapping T is called $(G,\psi)-$Kannan contraction if the following conditions hold;

i. T preserves the edges of G, $(x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G)$, for all $x, y \in X$,

ii. there exists $\delta \in \left[0, \frac{1}{2}\right)$ such that

$$\psi(d(Tx,Ty)) \leq \delta\left[\psi(d(x,Tx)) + \psi(d(y,Ty))\right], \text{ for all } (x,y) \in E(G).$$

Remark 5.1.9. If T is a $(G,\psi)-$Kannan contraction, then T is both a $(G^{-1},\psi)-$Kannan contraction and a $(\tilde{G},\psi)-$Kannan contraction.

Remark 5.1.10. Any $(G,\psi)-$Kannan contraction is $G-$Kannan mapping with $\psi(\omega) = \omega$.

Remark 5.1.11. If T is a $(G,\psi)-$Kannan contraction with constant δ, then T is a $(G,\psi)-$Ciric-Reich-Rus contraction with $\alpha = 0, \beta = \gamma = \delta$.

Lemma 5.1.12. Let (X,d) be a metric space endowed with a graph G and $T:X\to X$ be a $(G,\psi)-$Kannan contraction. If the graph G is weakly $T-$connected, then given $x,y\in X$, there is $r(x,y)\geq 0$ such that

$$\psi\left(d\left(T^n x,T^n y\right)\right)\leq \delta\ \psi\left(d\left(T^{n-1}x,T^n x\right)\right)+\left(\frac{\delta}{1-\delta}\right)^n r\left(x,y\right)+\delta\,\psi\left(d\left(T^{n-1}y,T^n y\right)\right)$$

for all $n\in\mathbb{N}^*$.

Proof. The proof can be obtained by using a method similar to that used in [13].

Theorem 5.1.13. Let (X,d) be a complete metric space endowed with a graph G and $T:X\to X$ be a $(G,\psi)-$Kannan contraction. We suppose that G is weakly $T-$connected and the triple (X,d,G) satisfies the condition (5.3), then T is a PO.

Proof. The proof is acquired by using analogue method which is given in Theorem 5.1.6.

Corollary 5.1.14. Let (X,d) be a complete metric space endowed with a graph G. T is a self mapping and the triple (X,d,G) satisfies the condition (5.3). We suppose that:

i. G is weakly T connected,
ii. there is nonnegative numbers α and β satisfying $\alpha+2\beta<1$ such that

$$\psi\left(d\left(Tx,Ty\right)\right)\leq\alpha\,\psi\left(d\left(x,y\right)\right)+\beta\left[\psi\left(d\left(x,Tx\right)\right)+\psi\left(d\left(y,Ty\right)\right)\right],$$

for all $(x,y)\in E(G)$.

Then T is a PO.

Proof. It is obvious that the self mapping T is a $(G,\psi)-$Ciric-Reich-Rus contraction with $\beta=\gamma$, so the conclusion arrives Theorem 5.1.6.

Remark 5.1.15. In corollary 5.1.14., if we take $\psi(\omega)=\omega$, then we get Corollary 1 in [18].

Corollary 5.1.16. Let (X,d) be a complete metric space endowed with a graph G and $T:X\to X$ be a $(G,\psi)-$contraction. We suppose that G is weakly $T-$connected and the triple (X,d,G) satisfies the condition (5.3), then T is a PO.

Proof. If T is a $(G,\psi)-$contraction with constant $c\in(0,1)$, then T is a $(G,\psi)-$Ciric-Reich-Rus contraction with $\alpha=c$ and $\beta=\gamma$ from Theorem 5.1.6., T is a PO.

Remark 5.1.17. In corollary 5.1.16., if we take $\psi(\omega)=\omega$, then we get Corollary 2 in [18].

5.2. $(G,\psi)-$Ciric-Reich-Rus Graphic Contraction and Fixed Point Theorems

Definition 5.2.1. Let (X,d) be a metric space and G a graph. The mapping $T:X\to X$ is called $(G,\psi,)-$Ciric-Reich-Rus graphic contraction if the following conditions hold;

i. T preserves the edges of G, $(x,y)\in E(G)\Rightarrow(Tx,Ty)\in E(G)$, for all $x,y\in X$,

ii. there exists nonnegative numbers α,β,γ with $\alpha+\beta+\gamma<1$, such that

$$\psi\left(d\left(Tx,T^2x\right)\right)\leq\alpha\,\psi\left(d\left(x,Tx\right)\right)+\beta\,\psi\left(d\left(x,Tx\right)\right)+\gamma\,\psi\left(d\left(Tx,T^2x\right)\right)$$

for all $x\in X^T$.

Remark 5.2.2. Let (X,d) be a metric space endowed with a graph G. If $T:X\to X$ is a $(G,\psi)-$Ciric-Reich-Rus graphic contraction, then T is both a $(G^{-1},\psi)-$Ciric-Reich-Rus graphic contraction and a $(\tilde{G},\psi)-$Ciric-Reich-Rus graphic contraction.

Remark 5.2.3. Any $(G,\psi)-$Ciric-Reich-Rus graphic contraction is a $(G,\psi)-$graphic contraction with $\beta=\gamma=0$.

Remark 5.2.4. If T is a $(G,\psi)-$Ciric-Reich-Rus graphic contraction, then T is a $G-$graphic contraction with $\psi(\omega)=\omega$ and $\beta=\gamma=0$.

Example 5.2.5. Let $X=\{0,1,2,3\}$ and $d(x,y)=|x-y|$ for all $x,y\in X$. Define the mapping $T:X\to X$ as;

$$Tx=\begin{cases}0, \text{ if } x\in\{0,1\},\\ 1, \text{ if } x\in\{2,3\}.\end{cases}$$

T is a $(G,\psi)-$Ciric-Reich-Rus graphic contraction with $\psi(\omega)=\dfrac{\omega}{3}$ and $\alpha=\beta=\gamma=\dfrac{1}{4}$, where

$$E(G)=\{\,(0,1);(0,2);(2,3);(0,0);(1,1);(2,2);(3,3);(1,2)\,\}.$$

But it is not a $(G,\psi)-$Ciric-Reich-Rus contraction

$$\psi(d(T1,T2))\le\alpha\,\psi(d(1,2))+\beta\,\psi(d(1,T1))+\gamma\,\psi(d(2,T2)),$$

it is a contradiction since $\dfrac{1}{3}\le\dfrac{1}{4}$ and what is worse $F(T)=\{1\}$.

Lemma 5.2.6. Let (X,d) be a metric space endowed with a graph G and $T:X\to X$ be a $(G,\psi)-$Ciric-Reich-Rus graphic contraction. Then, given $x\in X^T$, there exists $r(x)\geq 0$ such that

$$\psi\left(d\left(T^n x,T^{n+1}x\right)\right)\leq c^n\, r\left(x\right)$$

for all $n\in\mathbb{N}$, where $c=\dfrac{\alpha+\beta}{1-\gamma}$, $r\left(x\right)=\psi\left(d\left(x,Tx\right)\right)$

Proof. Let $x\in X^T$ i.e., $(x,Tx)\in E(G)$ or $(Tx,x)\in E(G)$. If $(x,Tx)\in E(G)$, then by easy induction, we have that $\left(T^n x,T^{n+1}x\right)\in E(G)$ for each $n\in\mathbb{N}$. Therefore,

$$\psi\left(d\left(T^n x,T^{n+1}x\right)\right)\leq\frac{\alpha+\beta}{1-\gamma}\,\psi\left(d\left(T^{n-1}x,T^n x\right)\right)$$

$$\leq c\,\psi\left(d\left(T^{n-1}x,T^n x\right)\right)\leq\ldots\leq c^n\,\psi\left(d\left(x,Tx\right)\right)=c^n\,r\left(x\right).$$

If $(Tx,x)\in E(G)$, again by induction, we obtain that $\left(T^{n+1}x,T^n x\right)\in E(G)$. So, we get the same relation as before.

Lemma 5.2.7. Let (X,d) be a complete metric space endowed with a graph G suppose that $T:X\to X$ be a $(G,\psi,)-$Ciric-Reich-Rus graphic contraction. Then, for all $x\in X^T$, there exists $x^*\in X$ such that the sequence $\left(T^n x\right)_{n\in\mathbb{N}}$ converges to x^* as $n\to\infty$.

Proof. Let $x\in X^T$. From lemma 5.2.6., we get that

$$\psi\left(d\left(T^n x,T^{n+1}x\right)\right)\leq c^n\, r\left(x\right),$$

for all $n \in \mathbb{N}$, where $r(x) = \psi(d(x,Tx))$ and $c = \dfrac{\alpha+\beta}{1-\gamma}$. Thus,

$$\sum_{n=0}^{\infty} \psi\left(d\left(T^n x, T^{n+1}x\right)\right) < \infty,$$

that is, $d\left(T^n x, T^{n+1}x\right) \to 0$. So, the sequence $\left(T^n x\right)_{n\in\mathbb{N}}$ is a Cauchy Sequence. Due to completeness of X, we obtain that there exists $x^* \in X$ such that the sequence $\left(T^n x\right)_{n\in\mathbb{N}}$ converges to x^* as $n \to \infty$.

Theorem 5.2.8. Let (X,d) be a complete metric space and the triple (X,d,G) satisfies the condition:

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$

$\big($or respectively $(x_{n+1}, x_n) \in E(G)\big)$ for all $n \in \mathbb{N}$, then there is a subsequence

$\left(x_{k_n}\right)_{n\in\mathbb{N}}$ with $\left(x_{k_n}, x\right) \in E(G)$ $\left(\text{or respectively } \left(x, x_{k_n}\right) \in E(G)\right)$ for all $n \in \mathbb{N}$. $\quad (5.4)$

Let $T : X \to X$ be a $(G,\psi)-$Ciric-Reich-Rus graphic contraction and be orbitally $G-$continuous. Then the following statements hold:

i. $F(T) \neq \emptyset$ iff $X^T \neq \emptyset$;

ii. if $X^T \neq \emptyset$ and G is weakly connected, then T is a weakly Picard operator;

iii. for any $x \in X_T$, we have that $T\big|_{[x]_{\tilde{G}}}$ is a weakly Picard operator.

Proof. We start by proving statement (iii). Let $x \in X^T$. From lemma 5.1.12., we obtain that there exists $r(x) \geq 0$ such that

$\psi\left(d\left(T^n x, T^{n+1} x\right)\right) \leq c^n\, r\left(x\right),$

for all $n \in \mathbb{N}$, and also by lemma 5.2.6., there exists $x^* \in X$ such that $\lim_{n\to\infty}\left(T^n x\right) = x^*$. Owing to $x \in X^T$, $T^n x \in X^T$ for all $n \in \mathbb{N}$. Now assume that $(x, Tx) \in E(G)$. (A similar deduction can be done if $(Tx, x) \in E(G)$). From condition (5.4), there exists a subsequence $\left(T^{k_n} x\right)_{n\in\mathbb{N}}$ of $\left(T^n x\right)_{n\in\mathbb{N}}$ such that $\left(T^{k_n} x, x^*\right) \in E\left(G\right)$ for all $n \in \mathbb{N}$. Then the points $x, Tx, T^2 x, ..., T^{k_1} x, x^*$ from a path in G, and so $x^* \in [x]_{\tilde{G}}$. Because T is orbitally $G-$continuous, we get that x^* is a fixed point of $T\big|_{[x]_{\tilde{G}}}$.

To prove (i), notice that from (iii), it follows that $F\left(T\right) \neq \varnothing$ if $X^T \neq \varnothing$. Suppose that $F\left(T\right) \neq \varnothing$ because of $E\left(G\right) \supseteq \Delta$, directly obtain that $X^T \neq \varnothing$. Thus, also (i) holds.

To prove (ii), let $x \in X^T$. Since G is weakly connected, we obtain that $X = [x]_{\tilde{G}}$, and just need to apply (iii).

Remark 5.2.9. In Theorem 5.2.8.,

- **i.** if we take $\psi(\omega) = \omega$, then we obtain Theorem 2.2 in [20].
- **ii.** if we take $\beta = \gamma = 0$, then we get Theorem 3 in [21].
- **iii.** if we take $\psi(\omega) = \omega$ and $\beta = \gamma = 0$, then we have Theorem 2.1 in [20].

CHAPTER 6. FIXED POINT THEOREMS FOR GENERALİZED φ −CONTRACTIONS IN METRIC SPACE WİTH A GRAPH

$(G,\psi,\varphi)-$contractions have been defined and some fixed point theorems have been obtained in a metric space with a graph. Also some results have been given which are extensions of some recent results. Moreover, we give some examples to support our results.

6.1. Fixed Point Theorems for $(G,\psi,\varphi)-$ Contraction

Definition 6.1.1. Let (X,d) be a metric space and G a graph. The mapping $T: X \to X$ is called $(G,\psi,\varphi)-$contraction if the following conditions hold;

i. T preserves the edges of G, $(x,y)\in E(G) \Rightarrow (Tx,Ty)\in E(G)$, for all $x,y\in X$,

ii. there exists a $\varphi\in\Phi$ and $\psi\in\Psi$ such that

$$\psi\left(d\left(Tx,Ty\right)\right)\leq\varphi\left(\psi\left(d\left(x,y\right)\right)\right),\ \text{for all } \left(x,y\right)\in E\left(G\right).$$

Lemma 6.1.2. If $T: X \to X$ is a $(G,\psi,\varphi)-$contraction, then T is both a $(G^{-1},\psi,\varphi)-$contraction and a $(\tilde{G},\psi,\varphi)-$ contraction.

Theorem 6.1.3. Let (X,d) be a complete metric space and G be weakly connected. T is a self mapping on X. We suppose that:

i. for any sequence $(x_n)_{n\in\mathbb{N}} \in X$ with $d(x_n, x_{n+1}) \to 0$ there exists $k, n_0 \in \mathbb{N}$ such that $(x_{k_n}, x_{k_m}) \in E(G)$ for $m,n \in \mathbb{N}$, $m,n \geq 0$;

ii$_a$. T is orbitally continuous

or

ii$_b$. T is orbitally $G-$continuous and there exists a subsequence $(T^{k_n}x_0)_{k\in\mathbb{N}}$ of $(T^n x_0)_{n\in\mathbb{N}}$ such that $(T^{k_n}x_0, x^*)_{k\in\mathbb{N}} \in E(G)$ for each $k \in \mathbb{N}$;

iii. There exist a $\varphi \in \Phi$ and a $\psi \in \Psi$ such that T is a $(G,\psi,\varphi)-$contraction.

Then T is a PO.

Proof. Take an arbitrary element $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. By using the definition of a $(G,\psi,\varphi)-$contraction and a standard induction argument we get $(T^n x_0, T^{n+1}x_0) \in E(G)$ and for all $n \in \mathbb{N}$,

$$\psi\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) \leq \varphi^n\left(\psi\left(d\left(x_0, Tx_0\right)\right)\right).$$

Then $\lim\limits_{n\to\infty}\psi\left(d\left(T^n x_0, T^{n+1}x_0\right)\right) = 0$ and, using property of ψ_2 we have

$$\lim_{n\to\infty} d\left(T^n x_0, T^{n+1}x_0\right) = 0.$$

Hence from (i), there exist $k, n_0 \in \mathbb{N}$ such that $\left(T^{k_n}x_0, T^{k_m}x_0\right) \in E(G)$ for all $m,n \in \mathbb{N}$; $m,n \geq n_0$. Since $d\left(T^{k_n}x_0, T^{k_{(n+1)}}x_0\right) \to 0$, for $\varepsilon > 0$, there exists $N \in \mathbb{N}$, $N \geq n_0$ such that

$$\psi\left(d\left(T^{k_n}x_0, T^{k_{(n+1)}}x_0\right)\right) < \varepsilon - \varphi(\varepsilon), \qquad \textbf{(6.1)}$$

for each $n \geq N$. As $(T^{k_n}x_0, T^{k_{(n+1)}}x_0) \in E(G)$ and by using the triangle inequality, property ψ_3 and (6.1), we get that, for any $n \geq N$

$$\psi\left(d\left(T^{k_n}x_0,T^{k_{(n+2)}}x_0\right)\right)\le\psi\left(d\left(T^{k_n}x_0,T^{k_{(n+1)}}x_0\right)\right)+\psi\left(d\left(T^{k_{(n+1)}}x_0,T^{k_{(n+2)}}x_0\right)\right)$$

$$<\varepsilon-\varphi(\varepsilon)+\varphi\left(\psi\left(d\left(T^{k_{(n+1)}}x_0,T^{k_{(n+2)}}x_0\right)\right)\right),$$

and, since φ is monotone increasing,

$$\psi\left(d\left(T^{k_n}x_0,T^{k_{(n+2)}}x_0\right)\right)<\varepsilon. \qquad \textbf{(6.2)}$$

By (6.2), we have $\left(T^{k_n}x_0,T^{k_{(n+2)}}x_0\right)\in E(G)$ and, for any $n\ge N$,

$$\psi\left(d\left(T^{k_n}x_0,T^{k_{(n+3)}}x_0\right)\right)\le\psi\left(d\left(T^{k_n}x_0,T^{k_{(n+1)}}x_0\right)\right)+\psi\left(d\left(T^{k_{(n+1)}}x_0,T^{k_{(n+3)}}x_0\right)\right)$$

$$<\varepsilon-\varphi(\varepsilon)+\varphi\left(\psi\left(d\left(T^{k_n}x_0,T^{k_{(n+2)}}x_0\right)\right)\right)<\varepsilon.$$

Using an easy induction, we obtain for any $m\in\mathbb{N}$ and $n\ge N$,

$$\psi\left(d\left(T^{k_n}x_0,T^{k_{(n+m)}}x_0\right)\right)<\varepsilon.$$

Then property ψ_2 yields that $d\left(T^{k_n}x_0,T^{k_{(n+m)}}x_0\right)<\varepsilon$. Hence, $\left(T^{k_n}x_0\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. By the completeness of X, there exists a $x^*\in X$ such that $T^{k_n}x_0\to x^*$ as $n\to\infty$. Because $\psi\left(d\left(T^{k_n}x_0,T^{k(n+1)}x_0\right)\right)\to 0$, so also $d\left(T^{k_n}x_0,T^{k_{(n+1)}}x_0\right)\to 0$, and we get $T^n x_0\to x^*$ as $n\to\infty$. Take an arbitrary element $x\in X$. Then:

1. If $(x,x_0)\in E(G)$, then $\left(T^n x,T^n x_0\right)\in E(G)$, for all $n\in\mathbb{N}$. Hence

$$\psi\left(d\left(T^n x,T^n x_0\right)\right)\le\varphi^n\left(\psi\left(d\left(x,x_0\right)\right)\right),\ \ \forall n\in\mathbb{N}.$$

Letting $n \to \infty$, we obtain $\psi\left(d\left(T^n x, T^n x_0\right)\right) \to 0$ So, by ψ_2 we get $d\left(T^n x, x^*\right) \to 0$.

2. If $(x, x_0) \notin E(G)$, then, since G is weakly connected, we have a path $(x_i)_{i=0}^M$ in $\widetilde{G}$ from x_0 to x; that is, $x_M = x$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$ for $i = 1,2,...,M$. With an easy induction we obtain $\left(T^n x_{i-1}, T^n x_i\right) \in E(\widetilde{G})$ for $i = 1,2,...,M$ and

$$\psi\left(d\left(T^n x_0, T^n x\right)\right) \le \sum_{i=1}^{M} \varphi^n\left(\psi\left(d\left(x_{i-1}, x_i\right)\right)\right).$$

Letting $n \to \infty$, we conclude that $\psi\left(d\left(T^n x_0, T^n x\right)\right) \to 0$ and, from ψ_2, $d\left(T^n x_0, T^n x\right) \to 0$, which yields $T^n x \to x^*$.

Now we are in the position to prove that $x^* \in F(T)$. It is obvious that $x^* \in F(T)$, if $(\text{ii})_a$ holds. If $(\text{ii})_b$ occurs, then, since $\left(T^{k_n} x_0\right)_{k \in \mathbb{N}} \to x^*$ and $\left(T^{k_n} x_0, x^*\right) \in E(G)$ for all $k \in \mathbb{N}$, we attain, using the orbitally $G-$continuity of T, that $T^{k_{(n+1)}} x_0 \to Tx^*$ as $k \to \infty$. Thus $x^* = Tx^*$. Let $Ty = y$, for some $y \in X$, then we have $T^n y \to x^*$. But it must be the case that $y = x^*$.

Remark 6.1.4. In Theorem 6.1.3., if we take $\psi(\omega) = \omega$, we get Theorem 2.2 in [19].

The next example shows that T must be either orbitally continuous or orbitally $G-$continuous to be a PO.

Example 6.1.5. Let $X = [0,1]$ be endowed with the usual metric. Consider

$$E(G) = \{\, (0,0) \,\} \cup \{\, (0,x) : x \ge 1/2 \,\} \cup \{\, (x,y) : x, y \in (0,1] \,\},$$

and $T : X \to X$,

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in (0,1]; \\ \dfrac{1}{2}, & \text{if } x = 0. \end{cases}$$

Then G is weakly connected, X_T is nonempty and T is a $(G,\psi,\varphi)-$contraction where $\varphi(t)=\dfrac{t}{2}$, $\psi(\omega)=\dfrac{\omega}{3}$ but is neither orbitally continuous nor orbitally $G-$continuous. Thus T does not have a fixed point.

The next example shows that, in Theorem 6.1.3., all conditions are necessary for the mapping T to be a PO.

Example 6.1.6. Let $X=[0,1]$ be endowed with the usual metric. Consider

$$E(G)=\{\,(0,0)\,\}\cup\{\,(0,x):x\geq 1/2\,\}\cup\{\,(x,y):x,y\in(0,1]\,\},$$

and $T:X\to X$,

$$Tx=\frac{x}{2}, \quad \text{if } x\in[0,1].$$

Then G is weakly connected, X_T is nonempty and T is a $(G,\psi,\varphi)-$contraction where $\varphi(t)=\dfrac{t}{2}$, $\psi(\omega)=\dfrac{\omega}{3}$. Also T is both orbitally continuous and orbitally $G-$continuous. Thus the conditions of theorem 6.1.3. holds; i.e., T is a PO.

There is a close relation between the convergence of iteration sequences, obtained by using the $(G,\psi,\varphi)-$contraction and the connectivity of graph G.

Theorem 6.1.7. Let (X,d) be a metric space endowed with a graph G and $T:X\to X$ be a $(G,\psi,\varphi)-$contraction, then the following statements are equivalent:

i. G is weakly connected;

ii. for given $x, y \in X$, the sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent;

iii. $\mathrm{card}(F(T)) \le 1$.

Proof. (i)$\Rightarrow$(ii) Let T be a $(G,\psi,\varphi)-$contraction and $x, y \in X$. By hypothesis, $[x]_{\tilde{G}} = X$, so $y \in [x]_{\tilde{G}}$. Then there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x to y, which means, $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1,2,...,N$. If we apply an easy induction, we have $(T^n x_{i-1}, T^n x_i) \in E(\tilde{G})$ for $i = 1,2,...,N$ and

$$\psi\left(d\left(T^n x, T^n y\right)\right) \le \sum_{i=1}^{N} \varphi^n\left(\psi\left(d\left(x_{i-1}, x_i\right)\right)\right),$$

so, as $n \to \infty$, we have $\psi(d(T^n x, T^n y)) \to 0$ and hence, from property ψ_2,

$$d(T^n x, T^n y) \to 0.$$

Likewise, there is a path $(w_i)_{i=0}^M$ in $\tilde{G}$ from x to Tx; that is, $w_0 = x$, $w_N = Tx$ and $(w_{i-1}, w_i) \in E(\tilde{G})$ for $i = 1,2,...,M$. Then by ψ_3, the triangle inequality and the definition of $(G,\psi,\varphi)-$contraction, we have

$$\psi\left(d\left(T^n x, T^{n+1} x\right)\right) \le \sum_{i=1}^{M} \varphi^n\left(\psi\left(d\left(w_{i-1}, w_i\right)\right)\right).$$

Hence

$$\sum_{n=0}^{\infty} \psi\left(d\left(T^n x, T^{n+1} x\right)\right) \le \sum_{i=1}^{M} \sum_{n=0}^{\infty} \varphi^n\left(\psi\left(d\left(w_{i-1}, w_i\right)\right)\right) < \infty,$$

and this implies that $\left(T^n x\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. So, $\left(T^n y\right)_{n\in\mathbb{N}}$ is a Cauchy sequence.

(ii) $\Rightarrow$ (iii) Let T be a $(G,\psi,\varphi)-$contraction and $x, y \in F(T)$. By (ii), $\left(T^n x\right)_{n\in\mathbb{N}}$ and $\left(T^n y\right)_{n\in\mathbb{N}}$ are Cauchy equivalent, from which one concludes that $x = y$.

(iii) $\Rightarrow$ (i) Conversely, let G is not weakly connected; that is, $\tilde{G}$ is disconnected. Let $x_0 \in X$. Then both the sets $[x_0]_{\tilde{G}}$ and $X-[x_0]_{\tilde{G}}$ are nonempty. Let $y_0 \in X-[x_0]_{\tilde{G}}$ and define

$$Tx = \begin{cases} x_0, & \text{if } x \in [x_0]_{\tilde{G}}, \\ y_0, & \text{if } x \in X-[x_0]_{\tilde{G}}. \end{cases}$$

Obviously, $F(T) = \{x_0, y_0\}$. We prove that T is a $(G,\psi,\varphi)-$contraction. Let $(x,y)\in E(G)$. Then $[x]_{\tilde{G}} = [y]_{\tilde{G}}$, so either $x, y \in [x_0]_{\tilde{G}}$ or $x, y \in X-[x_0]_{\tilde{G}}$. Hence in both cases $Tx = Ty$, so $(Tx,Ty)\in E(G)$, because $E(G)\supseteq\Delta$, and $d(Tx,Ty)=0$. Then, from ψ_1 we get

$$\psi\left(d\left(Tx,Ty\right)\right) = 0 \le \varphi\left(\psi\left(d\left(x,y\right)\right)\right).$$

Therefore, T is a $(G,\psi,\varphi)-$contraction having two fixed points, which conflicts with (iii).

The following result can be obtained from Theorem 6.1.7., directly.

Corollary 6.1.8. Let (X,d) be a complete metric space and G is a weakly connected graph. If $T: X \to X$ is a $(G,\psi,\varphi)-$contraction, then there is $x^* \in X$ such that $\lim_{n\to\infty}\left(T^n x\right) = x^*$ for all $x \in X$.

Proposition 6.1.9. Let us suppose that $T: X \to X$ is a $(G,\psi,\varphi)-$contraction with $\varphi \in \Phi$ and $\psi \in \Psi$, for which there exists an $x_0 \in X$ such that $Tx_0 \in [x_0]_{\tilde{G}}$. Let $\tilde{G}_{x_0}$ be the component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is $T-$invariant and $T|_{[x]_{\tilde{G}}}$ is a $(\tilde{G}_{x_0},\psi,\varphi)-$ contraction. Moreover, if $x, y \in [x_0]_{\tilde{G}}$, then $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent.

Proof. The proof can be obtained by using a method similar to that used in [5].

In Theorem 6.1.3. the second statement follows from the first one because $\tilde{G}_{x_0}$ is connected.

Theorem 6.1.10. Let (X,d) be a complete metric space and the triple (X,d,G) have the following condition:

for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$. **(6.3)**

Let $T: X \to X$ be a $(G,\psi,\varphi)-$contraction. Then the following statements hold:

i. $\operatorname{card}(F(T)) = \operatorname{card}\{ [x]_{\tilde{G}} : x \in X_T \}$.

ii. $F(T) \neq \varnothing$ iff $X_T \neq \varnothing$.

iii. T has a unique fixed point iff there exists $x_0 \in X_T$ such that $X_T \subseteq [x_0]_{\tilde{G}}$.

iv. For any $x \in X_T$, $T|_{[x]_{\tilde{G}}}$ is a PO.

v. If $X_T \neq \varnothing$ and G is weakly connected, then T is a PO.

vi. If $X' := \bigcup\{ [x]_{\tilde{G}} : x \in X_T \}$, then $T|_{X'}$ is a WPO.

vii. If $T \subseteq E(G)$, then T is a WPO.

Proof. Initially we prove the items (iv) and (v). Take $x \in X_T$. Then $Tx \in [x]_{\tilde{G}}$, so by Proposition 6.1.9., if $y \in [x]_{\tilde{G}}$, then $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent. Since X is complete, $(T^n x)_{n\in\mathbb{N}}$ converges to some $x^* \in X$. It is obvious that $\lim_{n\to\infty}(T^n y) = x^*$. Using induction we get

$$(T^n x, T^{n+1}x) \in E(G) \tag{6.4}$$

for all $n \in \mathbb{N}$, since $(x, Tx) \in E(G)$. By (6.3), there is a subsequence $(T^{k_n} x)_{n\in\mathbb{N}}$ such that $(T^{k_n} x, x^*) \in E(G)$ for all $n \in \mathbb{N}$. If we use (6.4), we conclude that $(x, Tx, T^2 x, ..., T^{k_1}, x^*)$ is a path in G and also in $\tilde{G}$ from x to x^*. This means that $x^* \in [x]_{\tilde{G}}$. Since T is a $(G, \psi, \varphi)-$contraction we have,

$$\psi\left(d\left(T^{k_{n+1}}x, Tx^*\right)\right) \leq \varphi\left(\psi\left(d\left(T^{k_n}x, x^*\right)\right)\right)$$

for all $n \in \mathbb{N}$. If n tends to ∞ we deduce that $Tx^* = x^*$. Thereby, $T|_{[x]_{\tilde{G}}}$ is a PO. Also, we conclude that T is a PO when $[x]_{\tilde{G}} = X$, since G is weakly connected.

(vi) is obvious from (iv). For the proof of (vii), if $T \subseteq E(G)$ then $X_T = X$ and so $X' = X$ holds. Thus T is a WPO because of (vi).

To prove (i) define the mapping; $\rho(x) = [x]_{\tilde{G}}$ for all $x \in F(T)$. It is sufficient to show that $\rho : F(T) \to C = \{ [x]_{\tilde{G}} : x \in X_T \}$, is a bijection. Because $E(G) \supseteq \Delta$, we deduce that $F(T) \subseteq X_T$, and then $\rho(F(T)) \subseteq C$. Beside if $x \in X_T$, then by (iv), $\lim_{n\to\infty}(T^n x) \in [x]_{\tilde{G}} \cap F(T)$, which implies that $\rho\left(\lim_{n\to\infty}(T^n x)\right) \in [x]_{\tilde{G}}$, and so ρ is a surjective mapping. We now show that T is injective. Take $x_1, x_2 \in F(T)$ such that

$\rho(x_1)=\rho(x_2) \;\Rightarrow\; [x_1]_{\tilde{G}}=[x_2]_{\tilde{G}}$.

Then $x_2 \in [x_1]_{\tilde{G}}$ and so, by (i), $\lim_{n\to\infty}(T^n x_2)\in[x_1]_{\tilde{G}}\cap F(T)=\{x_1\}$, which gives $x_1 = x_2$. Thus T is injective, and this is the desired result.

Finally, one can see that (ii) and (iii) are easy consequences of (i).

Corollary 6.1.11. Let (X,d) be a complete metric space and that the triple (X,d,G) satisfies condition (6.3). Then the following statements are equivalent:

i. G is weakly connected.
ii. Every $(G,\psi,\varphi)-$contraction $T:X\to X$ is such that $(x_0,Tx_0)\in E(G)$ for some $x_0\in X$, is a PO.
iii. For any $(G,\psi,\varphi)-$contraction $T:X\to X$, with $\mathrm{card}(F(T))\le 1$.

Proof. (i)$\Rightarrow$(ii) This implication follows directly from Theorem 6.1.10 (v).

(ii)$\Rightarrow$(iii) Let $T:X\to X$ be a $(G,\psi,\varphi)-$contraction. If X_T is empty, so, $F(T)$ since $F(T)\subseteq X_T$. If $X_T\neq\varnothing$, then, by (ii), $F(T)$ is a singleton. In both cases $\mathrm{card}(F(T))\le 1$.

(iii)$\Rightarrow(i)$ This can be obtained from Theorem 6.1.3.

Corollary 6.1.12. Let (X,d) be a complete metric space and the triple (X,d,G) satisfies condition (6.3). Then the following statements are equivalent:

i. G is weakly connected.
ii. Every $(G,\varphi)-$contraction $T:X\to X$ is such that $(x_0,Tx_0)\in E(G)$ for some $x_0\in X$, is a PO.

iii. For any $(G,\varphi)-$contraction $T:X\to X$, with $\mathrm{card}(F(T))\leq 1$.

Proof. If we take $\psi(\omega)=\omega$ in Corollary 6.1.11., we obtain Theorem 2.2 in [19].

Corollary 6.1.13. Let (X,d) be a complete metric space and the triple (X,d,G) satisfies condition (6.3). Then the following statements are equivalent:

i. G is weakly connected.
ii. Every $(G,\psi)-$contraction $T:X\to X$ is such that $(x_0,Tx_0)\in E(G)$ for some $x_0\in X$, is a PO.
iii. For any $(G,\psi)-$contraction $T:X\to X$, with $\mathrm{card}(F(T))\leq 1$.

Proof. If we regard $\varphi(t)=ct$ for $c\in(0,1)$ in Corollary 6.1.11., then we obtain Corollary 2 in [21].

Corollary 6.1.14. Let (X,d) be a complete metric space and the triple (X,d,G) satisfies condition (6.3). Then the following statements are equivalent:

i. G is weakly connected.
ii. Every Banach G-contraction $T:X\to X$ is such that $(x_0,Tx_0)\in E(G)$ for some $x_0\in X$, is a PO.
iii. For any Banach G-contraction $T:X\to X$, with $\mathrm{card}(F(T))\leq 1$.

Proof. If $\psi(\omega)=\omega$, $\varphi(t)=ct$ for $c\in(0,1)$ is applied to Corollary 6.1.11., then Theorem 3.1 in [5] is obtained.

Example 6.1.15. Let $X=[0,1]$ be endowed with the usual metric. Take

$$E(G)=\{\ (0,0)\ \}\cup\{\ (0,1)\ \}\cup\{\ (x,y)\in(0,1]\times(0,1]:x\geq y\ \},$$

and $T: X \to X$ as follows:

$$Tx = \begin{cases} \dfrac{x}{3}, & \text{if } \ x \in (0,1], \\ \dfrac{2}{3}, & \text{if } \ x = 0. \end{cases}$$

Then T is $(G,\psi,\varphi)-$contraction where $\psi(\omega) = \dfrac{\omega}{2}$ and $\varphi(t) = \dfrac{t}{1-t}$ where $t \in [0,1)$. But it is not a $(G,\psi)-$contraction.

Proof. It is easy to confirm that G is weakly connected, X_T is nonempty, and that T is a $(G,\psi,\varphi)-$contraction. The following easy calculations show that T is not a $(G,\psi)-$contraction realize the following easy calculations;

$$\psi\left(d\left(T0, T\frac{1}{2}\right)\right) \le c\,\psi\left(d\left(0,\frac{1}{2}\right)\right) \Rightarrow \frac{1}{4} \le c\frac{1}{4}$$

which implies $c \ge 1$, a contradiction of the definition of a $(G,\psi)-$contraction.

Example 6.1.16. Take X and $E(G)$ as above and define $T: X \to X$ as follows:

$$Tx = \begin{cases} \dfrac{x}{3}, & \text{if } \ x \in (0,1], \\ \dfrac{1}{2}, & \text{if } \ x = 0. \end{cases}$$

Then T is a $(G,\psi,\varphi)-$contraction, where $\psi(\omega) = \dfrac{\omega^2}{2}$ and $\varphi(t) = \dfrac{t}{4}$, but it is not $(G,\varphi)-$contraction.

Proof. To see that T is not a $(G,\varphi)-$contraction with $\varphi(t)=\frac{t}{4}$, notice that

$$d\left(T\frac{3}{4},T\frac{1}{2}\right)\le\varphi\left(d\left(\frac{3}{4},\frac{1}{2}\right)\right)\Rightarrow\ \frac{1}{12}\le\frac{1}{16}.$$

Also, $T^n x\to 0$ for all $x\in X$, but T has no fixed point.

CHAPTER 7. ON SOME FIXED POINT THEOREMS WITH $\varphi-$CONTRACTIONS IN CONE METRIC SPACE INVOLVINGA GRAPH

We introduce $\varphi-$ contractions defined on a cone metric space endowed with a graph without assuming the normality condition of cone. We establish fixed point results for such contractions which are extension of several known results. Also, an example have been given which satisfies our main result.

Throughout the section, we assume that X is a nonempty set, G is a directed graph and B is a real Banach space and K is a cone in B with $\operatorname{int} K \neq \emptyset$. By this way, we uniquely determine the limit of a sequence.

7.1. Fixed Point Theorems for $(G_c, \varphi)-$Contraction

Definition 7.1.1. Let (X, d) be a cone metric space and G be a graph. The mapping $T : X \to X$ is called as $(G_c, \varphi)-$contraction if the following conditions hold;

i. T preserves the edges of G; $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$ for all $x, y \in X$,

ii. there exists a function $\varphi : K \to K$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)), \text{ for all } (x, y) \in E(G).$$

Remark 7.1.2. Let (X,d) be a cone metric space endowed with a graph G. If $T:X\to X$ is a $(G_c,\varphi)-$contraction, then T is both a $(G_c^{-1},\varphi)-$contraction and a $(\widetilde{G}_c,\varphi)-$contraction.

Theorem 7.1.3. Let (X,d) be a complete cone metric space and G be weakly connected. T is a self mapping on X. We suppose that:

i. For any sequence $(x_n)_{n\in\mathbb{N}}\in X$ with $d(x_n,x_{n+1})\ll c$ for every $\theta\ll c$ there exist $k,n_0\in\mathbb{N}$ such that $(x_{k_n},x_{k_m})\in E(G)$ for all $m,n\in\mathbb{N}$ and $m,n\geq 0$;

ii$_a$ T is orbitally continuous

or

ii$_b$ T is orbitally $G-$continuous and there exists a subsequence $\left(T^{k_n}x_0\right)_{k\in\mathbb{N}}$ of $\left(T^n x_0\right)_{n\in\mathbb{N}}$ such that $\left(T^{k_n}x_0,x^*\right)_{k\in\mathbb{N}}\in E(G)$ for each $k\in\mathbb{N}$

iii. T is a $(G_c,\varphi)-$contraction.

Then T is a PO.

Proof. Take an arbitrary element $x_0\in X$ such that $(x_0,Tx_0)\in E(G)$. By using the definition of a $(G_c,\varphi)-$contraction and an easy induction we get $\left(T^n x_0,T^{n+1}x_0\right)\in E(G)$ and

$$d\left(T^n x_0,T^{n+1}x_0\right)\leq\varphi^n\left(d\left(x_0,Tx_0\right)\right)$$

for all $n\in\mathbb{N}$. Given $\theta\ll c$ and we choose a positive real number δ such that $c-\varphi(c)+N(\theta+\delta)\subset\operatorname{int}K$ where $N(\theta+\delta)=\{\, y\in B:\|y\|<\delta \,\}$. Also choose a natural number N such that

$$\varphi^m\left(d\left(x_0,Tx_0\right)\right)\ll c-\varphi(c)$$

for all $m \geq N$. Consequently, since $\left(T^m x_0, T^{m+1} x_0\right) \in E(G)$ for all $m \in \mathbb{N}$ then we have

$$d\left(T^m x_0, T^{m+1} x_0\right) \ll c - \varphi(c)$$

for all $m \geq N$. Fix $m \geq N$ and we prove

$$d\left(T^m x_0, T^{n+1} x_0\right) \ll c \qquad \textbf{(7.1)}$$

and from (i) there exists $m, n+1 \in \mathbb{N}$ such that $\left(T^m x_0, T^{n+1} x_0\right) \in E(G)$ for all $n \geq m$. If we take $n = m$, then (7.1) holds. Now, we assume that (7.1) holds for some $n \geq m$. Since $\left(T^m x_0, T^{m+1} x_0\right) \in E(G)$ for any $m \geq N$, we have that

$$\begin{aligned} d\left(T^m x_0, T^{n+2} x_0\right) &\leq d\left(T^m x_0, T^{m+1} x_0\right) + d\left(T^{m+1} x_0, T^{n+2} x_0\right) \\ &\leq d\left(T^m x_0, T^{m+1} x_0\right) + \varphi\left(d\left(T^m x_0, T^{n+1} x_0\right)\right) \\ &\leq c - \varphi(c) + \varphi(c) = c. \end{aligned}$$

Therefore, (7.1) holds when $m = n+1$. By induction, we deduce (7.1) holds for all $m, n \geq N$ Thus, $\left(T^m x_0\right)_{m \in \mathbb{N}}$ is a Cauchy sequence in X and by the completeness of X, there exists a $x^* \in X$ such that $T^m x_0 \to x^* \in X$ as $n \to \infty$. Since $d\left(T^n x_0, T^{n+1} x_0\right) \ll c$, we get $T^n x_0 \to x^*$ as $n \to \infty$. Take an arbitrary element $x \in X$. Then, if $(x, x_0) \in E(G)$, then $\left(T^n x, T^n x_0\right) \in E(G)$, for all $n \in \mathbb{N}$. Hence, from φ_3

$$\begin{aligned} d\left(T^n x, x^*\right) &\leq d\left(T^n x, T^n x_0\right) + d\left(T^n x_0, x^*\right) \\ &\leq \varphi^n\left(d\left(x, x_0\right)\right) + d\left(T^n x_0, x^*\right) \ll c. \end{aligned}$$

Letting $n \to \infty$, we obtain that $d\left(T^n x, x^*\right) \ll c$. That is $T^n x \to x^*$. If $(x, x_0) \notin E(G)$, then, since G is weakly connected, we have a path $(x_i)_{i=0}^{M}$ in $\tilde{G}$ from x_0 to x; that is, $x_M = x$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1,2,...,M$. With an easy induction we obtain $(T^n x_{i-1}, T^n x_i) \in E(\tilde{G})$ for $i = 1,...,M$ and

$$d\left(T^n x_0, T^n x\right) \leq \sum_{i=1}^{M} \varphi^n\left(d\left(x_{i-1}, x_i\right)\right).$$

So, letting $n \to \infty$ from φ_3 we conclude that $d\left(T^n x, x^*\right) \ll c$. That is $\left(T^n x\right) \to x^*$.

Now we are in the position to proved that $x^* \in F(T)$. It is obvious that $x^* \in F(T)$, if $(\text{ii})_a$ holds. If $(\text{ii})_b$ occurs since $d\left(T^{k_n} x_0, x^*\right) \ll c$, that is, $T^{k_n} x_0 \to x^*$ and $\left(T^{k_n} x_0, x^*\right) \in E(G)$ for all $n \in \mathbb{N}$, we attain, using the orbitally $G-$continuity of T, that $d\left(T^{k_{n+1}} x_0, Tx^*\right) \ll c$ for all $n \in \mathbb{N}$.That is, $T^{k_{n+1}} x_0 \to Tx^*$ Thus $x^* = Tx^*$. Let $Ty^* = y^*$, for some $y^* \in X$, then we have $T^n y \to x^*$. But it must be the case that $y^* = x^*$.

The next example illustrates that, in Theorem 7.1.3., all conditions are necessary for the mapping T to be a PO.

Example 7.1.4. Let $X = [0,1]$, $K = \{x \in B : x \geq 0\}$ and $B = \mathbb{R}^2$ with the metric

$$\begin{aligned} d &: XxX \to E \\ &(x, y) \to d(x, y) = \left(|x-y|, \alpha|x-y| \right), \quad \alpha \geq 0. \end{aligned}$$

Consider

$$E(G) = \{(0,0)\} \cup \{(0,x) : x \geq 1/2\} \cup \{ (x,y) : x, y \in (0,1] \},$$

and $T : X \to X$,

$$Tx = \frac{x}{2}, \text{ if } x \in [0,1].$$

Then G is weakly connected, X_T is nonempty and T is a $(G_c, \varphi)-$contraction where $\varphi(t) = \frac{t}{2}$. Also, T is both orbitally continuous and orbitally $G-$continuous. Thus the conditions of Theorem 7.1.7. holds; that is, T is a PO.

There is a close relation between the convergence of iteration sequences, obtained by using the $(G_c, \varphi)-$contraction and the connectivity of graph G.

Theorem 7.1.5. Let (X, d) be a cone metric space endowed with a graph G and $T : X \to X$ be a $(G_c, \varphi)-$contraction, then the following statements are equivalent:

i. G is weakly connected;

ii. for given $x, y \in X$, the sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent;

iii. $\text{card}(F(T)) \leq 1$.

Proof. (i) $\Rightarrow$ (ii) Let T be a $(G_c, \varphi)-$contraction and $x, y \in X$. By hypothesis, $[x]_{\tilde{G}} = X$, so $y \in [x]_{\tilde{G}}$. Then there is a path $(x_i)_{i=0}^{N}$ in $\tilde{G}$ from x to y, which means, $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1, \ldots, N$. If we apply an easy induction, we have $(T^n x_{i-1}, T^n x_i) \in E(\tilde{G})$ for $i = 1, \ldots, N$ and

$$d(T^n x, T^n y) \leq \sum_{i=1}^{N} \varphi^n (d(x_{i-1}, x_i)),$$

as $n \to \infty$, from property φ_3, we obtain $d(T^n x, T^n y) \to 0$. Likewise, there is a path $(w_i)_{i=0}^{M}$ in $\tilde{G}$ from x to Tx; that is, $w_0 = x$, $w_M = Tx$ and $(w_{i-1}, w_i) \in E(\tilde{G})$ for $i = 1,...,M$. Then by φ_4, the triangle inequality and the definition of $(G_c, \varphi)-$contraction, we have

$$d\left(T^n x, T^{n+1} x\right) \leq \sum_{i=1}^{M} \varphi^n \left(d\left(w_{i-1}, w_i\right)\right).$$

Hence,

$$\sum_{n=0}^{\infty} d\left(T^n x, T^{n+1} x\right) \leq \sum_{i=1}^{M} \sum_{n=0}^{\infty} \varphi^n \left(d\left(w_{i-1}, w_i\right)\right) < \infty$$

and this implies that $\left(T^n x\right)_{n \in \mathbb{N}}$ is a Cauchy sequence. So, $\left(T^n y\right)_{n \in \mathbb{N}}$ is a Cauchy sequence.

(ii) $\Rightarrow$ (iii) Let T be a $(G_c, \varphi)-$contraction and $x, y \in F(T)$. By (ii), $\left(T^n x\right)_{n \in \mathbb{N}}$ and $\left(T^n y\right)_{n \in \mathbb{N}}$ are Cauchy equivalent, from which one concludes that $x = y$.

(iii) $\Rightarrow$ (i) Conversely, let G is not weakly connected; that is, $\tilde{G}$ is disconnected. Let $x_0 \in X$. Then both the sets $[x_0]_{\tilde{G}}$ and $X - [x_0]_{\tilde{G}}$ are nonempty. Let $y_0 \in X - [x_0]_{\tilde{G}}$ and define

$$Tx = \begin{cases} x_0, \text{ if } x \in [x_0]_{\tilde{G}}, \\ \\ y_0, \text{ if } x \in X - [x_0]_{\tilde{G}}. \end{cases}$$

Obviously, $F(T) = \{x_0, y_0\}$. We prove that T is a $(G_c, \varphi)-$contraction. Let $(x, y) \in E(G)$. Then $[x]_{\tilde{G}} = [y]_{\tilde{G}}$, so either $x, y \in [x_0]_{\tilde{G}}$ or $x, y \in X - [x_0]_{\tilde{G}}$. Hence in both cases $Tx = Ty$, so $(Tx, Ty) \in E(G)$, because $E(G) \supseteq \Delta$, and $d(Tx, Ty) = \theta$. Then, we get

$$d(Tx, Ty) = 0 \leq \varphi(d(x, y)).$$

Therefore, T is a $(G_c, \varphi)-$contraction having two fixed points, which conflicts with (iii).

The following result can be obtained from Theorem 7.1.5., directly.

Corollary 7.1.6. Let (X, d) be a complete cone metric space and G be a weakly connected graph. If $T : X \to X$ is a $(G_c, \varphi)-$contraction, then there is $x^* \in X$ such that $\lim_{n \to \infty}(T^n x) = x^*$ for all $x \in X$.

CHAPTER 8. RESULTS AND SUGGESTIONS

In this section we summarize some results which are obtained in previous sections.

In the chapter 3., $(G,\varphi)-$graphic contractions have been defined by using a comparison function and studied the existence of fixed points. Also, the Hardy-Rogers $G-$graphic contractions have been introduced and some fixed point theorems have been proved. Some results in the literature are also generalized and extended.

In the chapter 4., we define $(G,\psi)-$contractions and $(G,\psi)-$graphic contractions which are extensions of some contractions given in the literature. Also we prove some fixed point theorems in a metric space by using connectivity of graph.

In the chapter 5., we motivated by the work of Jachymski [5], Bojor [18] and Petruşel [20], we introduced new contractions the mappings on complete metric space and obtained some fixed point theorems. Our results generalize and unify some results which is given [5, 18, 20, 21].

In the chapter 6., $(G,\psi,\varphi)-$contractions have been defined and some fixed point theorems have been obtained in a metric space with a graph. Also some results have been given which are extensions of some recent results which is given [19] and [21]. Moreover, we give some examples to support our reults.

In the chapter 7., we introduce $\varphi-$contraction defined on a cone metric space endowed with a graph without assuming the normality condition of cone. We establish fixed point results for such contractions which are extension of several known results. Also, an example have been given which satisfies our main result.

In addition these studies, it can be extended by using some various contractive conditions in different types of metric spaces endowed with a graph.

REFERENCES

[1] JAIN, P.K., Metric Spaces, Narosa Publishing House, New Delhi, 2009.

[2] MADDOX, I.J., Elements of Functıonal Analysis, Cambridge University press, 1970.

[3] KREYZIG, E., Introductory Functional Analysis with Applications, Jhon Wiley and Sons, New York, 1978.

[4] AGARWAL, R.P., SAHU, D.R., O'REGAN, D., Fixed Point Theory for Lipschitzian-type Mappings with Applications, Springer, 2000.

[5] JACHYMSKI, J., The contraction principle for mappings on a metric space endowed with a graph, Proc. Amer. Math. Soc., 136, 1359-1373, 2008.

[6] BERINDE, V., Iterative Approximation of Fixed Points, Springer, 2007.

[7] KIRK, W.A., SIMS, B., Handbook of Metric Fixed Point Theory, Kluwer Acad. Publ., Dordrecht, 2001.

[8] RUS, I.A., PETRUSEL, A., PETRUSEL, G., Fixed point theory, Cluj Univ. Press, 2008.

[9] KANNAN, R., Some reults on fixed points, Bull. Calcutta Math. Soc. 60, 71-76, 1968.

[10] CIRIC, L.B., A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45, 267-273, 1974.

[11] HARDY, G.E., ROGERS, T.D., A generalization of a fixed point theorem of Reich, Canad. Math. Bull., 16, 201-206, 1973.

[12] JOHNSONBAUGH, R., Discrete Mathematics,Prentice-Hall, Inc., New Jersey, 1997.

[13] BOJOR, F., Fixed points of Kannan mappings in metric spaces endowed with a graph, An. S t. Univ. Ovidius Constanta, 20(1), 31-40, 2012.

[14] HUANG, L.-G., ZHANG, X., Cone Metric Spaces and Fixed Point Theorems of Contractive mappings, J.Math. Anal. Appl., 332, 1468-1476, 2007.

[15] REZAPOUR, SH., HAMLBARANI, R., Some Notes on Paper Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings, J. Math. Anal. Appl., 345, 719-724, 2008

[16] JUNGCK, G., RADENOVIC, S., RADOJEVIC, S., RAKOCEVIC, V., Common fixed point theorems for weakly compatible pairs on cone metric spaces, Fixed Point Theory Appl., 13 pages, Article ID 643840, 2009.

[17] BEG, I., BUTT, R.A., RADOJEVIC, S., The contraction principle for set valued mappings on a metric space with a graph, Comput. Math. Appl., 60, 1214-1219, 2010.

[18] BOJOR, F., Fixed point theorems for Reich type contractions on metric spaces with a graph, Nonlinear Anal., 75, 3895-3901, 2012.

[19] BOJOR, F., Fixed point of $\varphi -$contraction in metric spaces endowed with a graph, Annals of the Uni. Craiova, Math. Comput. Sci. Series, 37(4), 85-92, 2010.

[20] PETRUSEL, G.R., CHIFU, C.I., Generalized contractions in metric spaces endowed with a graph, Fixed Point Theory and Appl., (161), 1-9, 2012.

[21] ÖZTÜRK, M., GİRGİN, E., Fixed Point Theorems for $\psi-$Contractions in Metric Space Involving a Graph, Under Press.

[22] ÖZTÜRK, M., GİRGİN, E., On Some Fixed Point Theorems with $\varphi-$Contraction in Cone Metric Spaces Involving a Graph, Int. Journal of Math. Analysis, 7, 63, 3109-3117, 2013.

[23] DUE, W.S., Some results and generalizations in metric fixed point theory, Nonlinear Analysis, Theory-Methods ξ Applications, 73, 5, 1439-1446, 2010.

[24] ENJOUJI, Y., NAKANISHI, M., SUZUKI, T., A generalizaton of Kannans fixed point theorem Fixed Point Theory and Applications, Article Number 192872, DOI:10.1155/2009/192872, 2010.

[25] GNANA, B.T., LAKSHMIKANTHAM, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65, 1379-1393. MR2245511 (2007c:47067), 2006.

[26] GRANAS, A., DUGUNDJI, J., Fixed Point Theory, Springer Verlag, New York, 2003.

[27] GWOZDZ-LUKAWSKA, G., JACHYMSKI, J., IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem, J. Math. Anal. Appl., 356, 453-463, 2009.

[28] HARJANI, J., SADARANGANI, K., Fixed point theorems for weakly contractive mappings in partially ordered sets. Nonlinear Anal. 71.3403-3410, 2009.

[29] JACHYMSKI, J., Order-theoretic aspects of metric fixed point theory , Handbook of Metric Fixed Point Theory (eds., Kirk B.A. and Sims B.), Kluwer Acad. Publ., Dordrecht, MR1904289(2003f:54094), 2001.

[30] BERINDE, V., Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum 9, 43-53, 2004.

[31] KANNAN, R., Some reults on fixed points-II. Amer. Math. Monthy. 76, 405-408, 1969.

[32] NIETO, J.J., RODRIGUEZ-LOPEZ, R., Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math. Sinica, English Ser., 23, 2205-2212, 2007.

[33] NIETO, J.J., RODRIGUEZ-LOPEZ, R., Contractive mapping theorems im partially ordered sets and applications to ordinary differential equatios, Order, 22, 223-239, 2007.

[34] NIETO, J.J., POUSO, R.L., RODRIGUEZ-LOPEZ, R., Fixed point theorems in ordered abstract spaces, Proc. Amer. Math. Soc., 135, 2505-2517, 2007.

[35] O’REGAN, D., RUS, I.A:, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. 341, 1241-1252, 2008.

[36] PETRIC, M.A., Some remarks concerning Ciric-Reich-Rus operators, Creat. Math Inform. 18(2), 188-193, 2009.

[37] PETRIC, M.A., ZLATANOV, B.G., Fixed Point Theorems of Kannan type for Cyclical Conractive Conditions, Proceedings of the Anniversary International Conference REMIA, Plovdiv, Bulgaria 187-194, 2010.

[38] PETRUSEL, A., RUS, I.A., Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc. 134, 411-418, 2006.

[39] RAN, A.C.M., REURINGS, M.C.B., A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132, 1435-1443, 2004.

[40] REICH, S., Fixed points of contractive functions, Boll. Unione Mat.Ital. 5, 26-42, 1972.

[41] RHOADES, B.E., A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226, 257-290, 1977.

[42] RUS, I.A., Generalized Contractions and Applications, Cluj Univ. Press, 2001.

[43] RUS, I.A., Picard operations and applications. Sci. Math. Jon. 58, 191-219, 2003.

[44] BERINDE, V., Some remarks on a fixed point theorem for Ciric –type almost contractions Carpath J. Math. 25(2),157-162, 2009.

[45] UME, J.S., Existence theorems for generalized distance on complete metric spaces Fixed Point Theory and Applications, Article Number 397150, DOI: 10.1155/2010/397150, 2010.

BIOGRAPHY

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