NUMERICAL AND ALGEBRAIC TREATMENT OF DYNAMICAL SYSTEM EQUATIONS AND SOLVABLE POTENTIALS

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ABSTRACT

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In this thesis, the solution of the dynamic system equations of physics have been studied by using algebraic and numerical methods. In recent times Lie algebraic techniques have been used to construct complex quasi exactly solvable potentials with real spectrum. In our study, we find the solution of the several different Schrödinger equations using the method of Lie algebra and show how the use of Lie algebra helps in simplifying the eigenvalue problem. In the present study, we begin with a specific differential realization of the $SU(1,1) \approx SO(2,1)$ algebra which can be used to derive the second order differential equation. Then apply variable and similarity transformations to the group generators in order to recover the Schrödinger equations for various potentials. We also demonstrated that non-Hermition PT symmetric Hamiltonians have real eigenvalues. In addition, the two well-known non-linear equations, the Heat and Lorenz equations are solved by numerical Runge Kutta 4 method and via Mathematica in physics. The effects of parameters and the initial conditions are examined. Finally, we review the topological analysis of data generated by a dynamical system operating a chaotic regime.

Key Words: Lie Algebra, Realization of SO(2,1), PT Symmetric Hamiltonians and Dynamical System.

ÖZET

ÇÖZÜLEBİLEN POTANSİYELLER VE DİNAMİK DENKLEM SİSTEMLERİNİN SAYISAL VE CEBİRSEL ÇÖZÜMLERİ

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Bu tezde fizikteki dinamik denklem sistemleri cebirsel ve sayısal metodlar kullanılarak çözüldü. Son zamanlarda, Lie cebirsel tekniği kullanılarak kompleks ve tam çözüme haiz olmayan potensiyellerin gerçek sepekturumlara sahip olduğu gösterildi. Bu çalışmada bazı Schrödinger denklemlerinin çözümünü Lie cebiri ile bulunmaya çalışıldı ve Lie cebirinin yardımı ile özdeğer problemlerinin nasıl basite indirgendiği gösterildi. Bu çalışmada ilk olarak ikinci dereceden diferansiyel denklemleri çözmek için SO(2,1) cebirinin diferansiyel realizasyonu yapıldı. Schrödinger denkleminin farklı potensiyelleri için değişken ve benzerlik dönüşümleri uygulanarak jeneratörler yeniden elde edildi. Kompleks PT simetrik Hamilton tipi denklemlerin gerçek özdeğere sahip olduğu gösterildi. Fizikte iki ünlü denklem olan ısı ve Lorenz denklemleri sayısal Runge Kutta 4 metodu ve Mathematica poroğramı kullanılarak çözüldü. Son olarak kaotik bölgelerde çalışan dinamik sistemler ile elde edilen verilerin topologic olarak analizi yapıldı.

Anahtar Kelimeler: Lie Cebiri, SO(2,1) Realizasyonu, PT Simetrik Hamiltonian ve Dinamik Sistemler

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LIST OF SYMBOLS

Symbols

$J_{\pm,0}$	: Ladder operators
J^2	: Casimir operator
$\sigma_i(x)$	: Arbitrary functions
$\eta_i(x)$	: Arbitrary functions
α_i	: Arbitrary constants
γ_i	: Arbitrary constants
$q_{\pm,0}$	: Arbitrary constants
β_i	: Arbitrary constants
x_i	: Eigenvectors of an operator H
$\varepsilon_i^{\pm,0}$	: Eigenvalues of an operator H
S	: Transformation operator
H_m	: Hermite polynomial
δ_{nm}	: Kronicer delta
$L_m(x)$	: Laguerre polynomial
σ	: Positive numbers
ε	: Positive numbers
A	: Size of the population
σ	: Lorenz parameters
β	: Lorenz parameters
r	: Lorenz parameters

CHAPTER 1

INTRODUCTION

Elementary mechanics, both classical and quantum, has become growth industry in the last decade. A century ago Mathematicians discovered that some apparently simple physical systems can have very complicated motions. The physicists have finally realized that most dynamical systems don't follow simple, regular, and predictable patterns, but run along a seemingly random, yet well defined trajectory. Physicists and Mathematicians have put a lot of effort into solving equations of nonlinear dynamical systems. Most nonlinear problems in dynamics remained unsolved because there existed no general mathematical method, and each physical system has different kind of chaotic behaviour [1,2]. Because of the absence of a universal method, each problem seemed to be a law onto itself.

Many different approaches have been tried to solve nonlinear dynamical problems. Importance of two of them have been arising with the very rapidly improving computer technology [3]. One of them is group theoretical methods and the other is numerical analysis methods.

Over the years, a variety of the group theoretic methods for the study of differential equations have been devised [4,5,6]. One of the most important problems of the theory of nonlinear dynamical systems is how to study the system on its invariant manifold. In the group theoretical approach, invariant manifolds of a physical system are easily determined. The ready availability of computers has led to many interesting numerical results, with as many intuitive in preparations, all in need of further sorting to find relevant ideas.

This thesis covers two main topics. One of them the solution of the non-linear differential equations which have solved Lorenz equation and heat equation by using the method of numerical analysis and by developing an analytical method respectively. Another subjects is to obtain a realization of the generators of the noncompact Lie Algebra $SU(1,1)\approx SO(2,1)$ in terms of a single variable, and then show that many of the second order differential equations that can be expressed in terms of these generators.

1.1. Outline of the Study

The arrangement of this thesis is as follows. In chapter 2 several realizations of SO(2,1) algebra are constructed to obtain eigenvalues and eigenvectors of the Schrödinger equation. Some well known realizations are given and it is found that most of them are reproduced by using our realization. At the end of the chapter 2 a general method which is developed to diagonalize linear combinations of algebraic operators and to obtain eigenvalues and eigenvectors is given.

In chapter 3, exactly solvable quantum mechanical potentials have been investigated and some homogeneous second order differential equations have been solved by using operators of SO(2,1) algebra. The realization introduced in chapter 2 have been applied to solve the differential equations which have different potentials.

In chapter 4, non-hermitian Hamiltonians with complex potentials have been analysed. A method is devoted to comlexify the potentials and the required conditions are investigated in obtaining real eigenvalues. The Hamiltonian is expressed in terms of operators of SO(2,1) algebra.

Chapter 5 is devoted to the study of non-linear equations of classical mechanics. The details of Lorenz model have been studied.

Finally the concluding our remarks and a computer program developed to construct realization of a Lie algebra are given in Chapter 6 and Appendix section respectively.

1.2. Algebraic Structure of Second Order Differential Equations

In general, it is difficult to find the exact eigenfunctions and eigenvalues of an operator , except the case of exactly solvable potentials. However, the problem can be considerably simplified using some algebraic methods. In our study, we deal with the solutions of various Schrödinger equations employing the methods of Lie algebra and show how the use of Lie Algebra leads to the simplification of the eigenvalue problem.

Lie algebra's is one of the basic notions of mathematics. Being non-associative algebra's, they are connected with many branches of mathematics. The beautiful classical theory of Lie algebra's and Lie groups was developed by the middle of this century it is connected with the names S. Lie, W. Killing, H. Cartan, H.Weyl [7]. In the last several years the relationship between mathematics and fundamental physics has reached the most significant stage of which developments in one science yield important result for the other. Lie algebra's (especially the infinite dimension ones) play a crucial role in this process, for example in string theory and conformable field theory [8,9].

Lie algebra is an incredibly exciting and interesting place. There are large numbers of important unsolved questions, there is a rich theory in place, and there are hards of applications. Recently, Lie algebra has spawned numerous variants, including Barcher's algebra's, colour algebra's and the enormously popular if somewhat misnamed, quantum groups [10].

Also symmetry plays an important role in all of modern physics. The most successful applications of symmetry to physics have employed Lie Groups and Lie Algebra's [11,12]. Lie Algebra's and Lie Groups utilise the concept of a dynamical symmetry (the Hamiltonian of the system can be expressed in terms of polynomials in the Casimir invariant of a subgroup chain of some highest symmetry). In the algebraic structure theory, there is a rich set of symmetry limits in which analytical solutions to complex many body problems may be obtained.

In recent times Lie algebraic techniques have been used to construct complex quasi exactly potentials with real spectrum [13,14]. The problem of interest can usually be approached in this manner when the Hamiltonian can be expressed in terms of generators of algebra. As a consequence, the solution of Schrödinger equations then becomes an algebraic problem, which can be solved using the tools of Lie algebra. At this point note that, the choice of differential realization of generators becomes important, because Hamiltonian will take different forms in different realization of the generators. On the other word the solution of Schrödinger equation is greatly simplified if appropriate realization is constructed to express the Hamiltonian as a linear function of generators. It may then possible to prove employ purely algebraic methods to construct a similarity transformation S, which makes SHS^{-1} diagonal. These transformation yield a wave function of the form, $\psi = S^{-1}x^{m}$. This algebraic approach involves a number of steps:

1) first, choose an appropriate Lie Algebra,
2) next, determine a realization of the generator of the chosen algebra in order to
3) express the given system Hamiltonian H as a simple function of the generator
4) finally, construct the similarity transformation S to diagonalize H.

We have applied this procedure, and found the solution of the several different eigenvalue problem. Our realizations can be adopted for special functions. For example; Morse potential, Kepler problem, Laguerre's differential equations etc.

Also we have analysed some non-Hermition PT invariant Hamiltonians which have real spectrum. Adopting a most general differential realizations of the SO(2,1) algebra, we have demonstrate how new complex potentials can be generated, which are not necessarily PT symmetric but possess common real eigenvalues [15,16].

1.3. Non-linear Dynamical Systems

Nowadays it is well known that the Lorenz model is a paradigm one for low dimensional chaos in dynamical systems are widely investigated in connection with modelling purposes in meteorology, hydrodynamics, laser physics, superconductivity, electronics, oil industry etc. From the mathematical point of view, the Lorenz model is a system of non-linear equations. Needless to say that in general it is virtually impossible to find a closed analytical solution to the most of the nonlinear equations. In this thesis we apply different methods to solve the Lorenz equation.

In the second part of thesis, we will deal with two famous non-linear equations of physics and topological analysis of chaotic dynamical systems. These equations are heat diffusion and Lorenz equations. The equations have been solved by the help of Mathematica.

Non-linear dynamics and its subdiscipline "chaos theory" have swept over the landscape of science, mathematics, and engineering in the past two decades. Non-linear dynamics can appear at several points in the undergraduate or graduate curriculum. It can be included as a special topic in a standard course in classical mechanics or differential equations. Courses focusing on non-linear dynamics at the junior, senior, or beginning graduate level are also becoming common [17].

There are two important themes in non-linear dynamics [18]. One emphasises the temporal behaviour of systems, most with no significant spatial variation. The other emphasises spatial structures and the formation of spatial patterns. Most beginning courses in non-linear dynamics focus on the temporal behaviour of "low-dimensional" systems, that is, those governed by a few degrees of freedom, for which there now exists a fairly complete body of theoretical results and experimental techniques. Methods of non-linear time-series analysis, in which a sequence of values of a single dynamical variable is used to determine the qualitative and quantitative measures of the temporal dynamical behaviour of the entire system, have dominated both the theoretical and experimental methodologies and are already finding their way into many practical applications.

The extension of these notions to higher-dimensional systems, to problems of "spatio-temporal" chaos, and to the question of turbulence is still a subject of active debate and inquiry. So too is the question of the correspondence between chaos in classical and quantum-mechanical systems [19]. Both issues, though, have motivated some exciting new experimental and theoretical work.

The crucial theoretical construct in non-linear dynamics is phase space (also called state space). Each of the (independent) dynamical variables is used as a coordinate to construct the state space for the system. For a deterministic system, the future behaviour is determined by the current state of the system, represented as a point in state space. As a system evolves in time, its state space representation maps out a trajectory in that space. Sets of trajectories form a phase portrait. These phase portraits often have interesting geometric properties. For example, chaotic systems can have phase portraits with a fractal geometric structure. The important and distinctive features of non-linear behaviour are: Symmetry-breaking, either temporal or spatial. The temporal response of a system need not be the same as that of the "driving force." Even autonomous systems (those with no explicit time-dependent forcing) can spontaneously develop complex temporal behaviour. The most dramatic of these broken symmetries is chaotic behaviour, which is bounded, completely a periodic behaviour. For non-linear systems with significant spatial variation, the spatial patterns may be independent of the boundary conditions.

Dramatic changes in behaviour, called bifurcation's, which occur over extremely small parameter ranges. Sensitive dependence on initial conditions (the so-called Butterfly Effect): Small changes in initial conditions may lead to qualitatively and quantitatively different long-term behaviour. Such sensitivity leads to the loss of long-term predictability even if the system is completely deterministic [20]. Universal scaling laws for the transitions between chaotic and regular behaviour.

Much of the analysis of the temporal behaviour of non-linear systems is carried out using time-series data from a single dynamical variable. The series may be generated by using a "stroboscopic" technique: every time the state space trajectory intersects a plane in the state space, forming a so-called Poincaré section, a data point is recorded. From such a series, the topology and dynamics of the entire attractor can be reconstructed. Topological methods have recently been developed for the analysis of dissipative dynamical systems that operate in the chaotic regime. They were originally developed for three-dimensional dissipative dynamical systems, but they are applicable to all low dimensional dynamical system. Topological methods supplement methods previously developed the determine the values of metric and dynamical invariant. However, topological methods possess three additional features; they describe how to model the dynamics; they allow validation of the models so developed; and the topological invariant are robust under changes in control parameter values. The topological analysis procedure depends on identifying the stretching and squeezing mechanisms that act to create a strange attractor and organise all the unstable periodic orbits in this attractor in a unique way. The stretching and squeezing mechanisms are represented by a caricature, a branched manifold, which is also called a template or knot holder.

CHAPTER 2

REALIZATION OF SO(2,1) LIE ALGEBRA AND A METHOD TO SOLVE EIGENVALUE PROBLEM

Lie groups and their associated algebreas are extensively used in the solution of physical problems. In this thesis, realization of $SO(2,1) \approx SU(1,1)$ have been used to obtain eigenvalues and eigenvectors of many quantum mechanical problems. Main purpose of this chapter is to construct a useful realization for obtaining eigenvectors and eigenvalues.

2.1. Some Properties of the Lie Algebra

The SO(2,1) Lie algebra are described by the commutation relation

$$[J_0, J_+] = J_+, \quad [J_0, J_-] = -J_-, \quad [J_+, J_-] = -g(J_0) \tag{2.1}$$

where $J_\pm = J_1 \pm iJ_2$ are the well known Ladder operators. $g(J_0)$ is an arbitrary functions of operators J_0. The special choice of $g(J_0) = -2J_0$ correspond to SO(2,1) algebra. The J_1, J_2 and J_0 are cartesian generators and the commutation relations are

$$[J_1, J_2] = \frac{i}{2} g(J_0), \quad [J_2, J_0] = iJ_1, \quad [J_0, J_1] = iJ_2 \tag{2.2}$$

Note that for the special case of SO(2,1), the choice of $g(J_0) = -2J_0$, the Casimir operator corresponding to the above generators is

$$J^2 = J_0^2 \mp J_0 - J_{\pm}J_{\mp} \tag{2.3}$$

In the next section the differential realization of SO(2,1) are constructed. We show that our approach reproduces most of the previously known realization in the literature[10,11,24].

2.2. Construction of Differential Realization of J_0, J_+ and J_- (One dimensional space)

The theory of classical mechanics consists of mathematical equations of motion, rules whereby the symbols occurring in the equations can be connected with measurable physical quantities, and techniques for solving equations. It is well known that most of the physical system can be modulated in the form of second order differential equation. Here we will drive a differential realization of SO(2,1) for solving second order differential equations. We shall start with the time independent Schrödinger equation. It should be possible to write down the equation of motion in the general case in terms of the operator.

$$H = \frac{p^2}{2m} + V(\vec{r}) \tag{2.4}$$

so that the time independent equation becomes

$$H\psi = E\psi \quad \text{or} \quad -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r})\psi = E\psi \tag{2.5}$$

We shall refer to H as the Hamiltonian operator, or the Hamiltonian for short. A natural question to ask is whether one can construct differential realizations with second order derivatives. This is, in fact, possible by starting more general form of generators.

$$
\begin{aligned}
J_+ &= \alpha_+ \frac{d^2}{dx^2} + \sigma_+(x)\frac{d}{dx} + \eta_+(x) \\
J_- &= \alpha_- \frac{d^2}{dx^2} + \sigma_-(x)\frac{d}{dx} + \eta_-(x) \\
J_0 &= \alpha_0 \frac{d^2}{dx^2} + \sigma_0(x)\frac{d}{dx} + \eta_0(x)
\end{aligned}
\qquad (2.6)
$$

It is not easy to compute the arbitrary functions $\sigma_i(x)$ and $\eta_i(x)$ which satisfy the commutation relations in equation (2.1). A realization may be obtained for the SU(1,1) algebra in terms of a single variable x, by considering the existence of Casimir invariance and commutation relation given in equation (2.1). Under the given conditions and using the Mathematica program, one possible realization of the SU(1,1) algebra is derived and is found that;

$$
\begin{aligned}
J_0 &= \frac{1}{2}(x + 2\gamma_1)\frac{d}{dx} + \eta_0(x) \\
J_+ &= \frac{1}{4\gamma_0}(x + 2\gamma_1)^2 \\
J_- &= \gamma_0 \frac{d^2}{dx^2} + \frac{\gamma_0}{x + 2\gamma_1}(-1 + 4\eta_0(x))\frac{d}{dx} + \\
&\frac{1}{(x + 2\gamma_1)^2}\left[\gamma_2 + 4\gamma_0(-1 + \eta_0(x))\eta_0(x) + 2\gamma_0(x + 2\gamma_1)\eta_0'(x)\right] \\
J^2 &= -\frac{\gamma_2}{4\gamma_0}
\end{aligned}
\qquad (2.7)
$$

where γ_i are arbitrary constant and $\eta_i(x)$ is an arbitrary functions. Using in equation (2.7) we have found the solutions of the various Schrödinger equations. We begin with a general linear functions of generators.

$$
\begin{aligned}
&H = q_0 J_0 + q_+ J_+ + q_- J_- \\
&H = q_- \gamma_0 \frac{d^2}{dx^2} + \left[\frac{q_0 (x + 2\gamma_1)}{2} + \frac{q_- \gamma_0}{x + 2\gamma_1} (-1 + 4\eta_0(x)) \right] \frac{d}{dx} + \\
&\frac{q_-}{(x + 2\gamma_1)^2} [\gamma_2 + 4\gamma_0 (-1 + \eta_0(x)) \eta_0(x) + 2\gamma_0 (x + 2\gamma_1) \eta_0'(x)] - \\
&\frac{q_+}{4\gamma_0} (x + 2\gamma_1)^2 + q_0 \eta_0(x)
\end{aligned}
\tag{2.8}
$$

where q_i are arbitrary constants. An interesting way of extending the range of potentials given in H is maintained if one applies a variable or similarity transformations of generators $J_{\pm}, J_0$ and the basis $|jm\rangle$. In particular, a variable transformation y=y(x) changes the function $\eta(x)$ and the basis state in the following way

$$
\begin{aligned}
&\frac{d}{dx} \to \frac{dy}{dx} \frac{d}{dz} \\
&\frac{d^2}{dx^2} \to \left(\frac{dy}{dx} \right)^2 \frac{d^2}{dz^2} + \frac{d^2 y}{dx^2} \frac{d}{dz} \\
&\psi_{jm}(x) \to \psi_{jm}(x(y)) \\
&\eta(x) \to \eta(x(y)) \\
&\eta'(x) \to \frac{dy}{dx} \eta'(x)
\end{aligned}
\tag{2.9}
$$

We mention that the mathematical construction presented in this section can be used to solve the wide-range differential equations. In the next chapter we will try to classify the solvable potentials by using the realization given in equation (1.7). Before we give a method to obtain eigenvalues and eigenvectors (or to solve the equation),let us obtain another realization which is useful to express the Morse class potentials. Using the same analogy given before we obtain

$$
\begin{aligned}
&J_0 = \beta_1 e^{x\beta_2} \frac{d^2}{dx^2} - \frac{1}{4\beta_1} e^{-x\beta_2} \\
&J_+ = \frac{\beta_1}{\beta_0} e^{x\beta_8} \frac{d^2}{dx^2} - \frac{1}{\beta_0} \frac{d}{dx} + \frac{1}{2\beta_0} + \frac{1}{4\beta_0 \beta_1} e^{-x\beta_2} \\
&J_- = \beta_1 \beta_0 e^{x\beta_2} \frac{d^2}{dx^2} + \beta_0 \frac{d}{dx} - \frac{\beta_0}{2} + \frac{\beta_0}{4\beta_1} e^{-x\beta_2}
\end{aligned}
\tag{2.10}
$$

where β_i are the constants. It is obvious that this form of equation is not useful. If we change the basis $\psi \rightarrow e^{-\alpha x}\psi$, the new form of realization is obtained.

$$
\begin{aligned}
J_0 &= \beta_1\frac{d^2}{dx^2} - 2\beta_1\frac{d}{dx} - \frac{1}{4\beta_1}e^{-2x\beta_2} + \beta_1 \\
J_+ &= \frac{\beta_1}{\beta_0}\frac{d^2}{dx^2} - \frac{e^{-x\beta_2}+2\beta_1}{\beta_0}\frac{d}{dx} + \frac{3}{2\beta_0}e^{-x\beta_2} + \frac{1}{4\beta_1\beta_0}e^{-2x\beta_2} + \frac{\beta_1}{\beta_0} \\
J_- &= \beta_1\beta_0\frac{d^2}{dx^2} + \beta_0(e^{-x\beta_2} - 2\beta_1)\frac{d}{dx} + \frac{3\beta_0 e^{-x\beta_2}}{2} + \frac{\beta_0}{4\beta_1}e^{-2x\beta_2} + \beta_1\beta_0 \\
J^2 &= \frac{1}{2}
\end{aligned}
\qquad (2.11)
$$

In this case the Hamiltonian can be written as

$$
\begin{aligned}
H = &\left(q_0\beta_1 + q_+\frac{\beta_1}{\beta_0} + q_-\beta_1\beta_0\right)\frac{d^2}{dx^2} + \\
&\left(-2q_0\beta_1 - q_+\frac{e^{-x\beta_2}+2\beta_1}{\beta_0} + q_-\beta_0(e^{-x\beta_2} - 2\beta_1)\right)\frac{d}{dx} + q_0\beta_1 + \\
&q_+\left(\frac{3e^{-x\beta_2}}{2\beta_0} + \frac{e^{-2x\beta_2}}{4\beta_1\beta_0} + \frac{\beta_1}{\beta_0}\right) + q_-\left(\frac{3\beta_0 e^{-x\beta_2}}{2} + \frac{\beta_0 e^{-2x\beta_2}}{4\beta_1} + \beta_1\beta_0\right)
\end{aligned}
\qquad (2.12)
$$

2.3. Other One Dimensional Realizations

There are various differential realization of SO(2,1) algebra which can be found in literature. Most of them are reproduced from each others. In here some well known realizations are given.

Filho and Vaidya [5] have discussed physical applications based on the following representation of SO(2,1);

$$
J_- = 2\frac{d^2}{dy^2} - \frac{2\alpha}{y^2}, \quad J_+ = \frac{y^2}{8}, \quad J_0 = \frac{y}{2}\frac{d}{dy} + \frac{1}{4} \qquad (2.13)
$$

where α is an arbitrary constant. This realization can be reproduced from the realization given in equation (2.7), by choosing $\gamma_1 = 0, \gamma_0 = 2$, $\eta_0(x) = \frac{1}{4}$ and $\gamma_2 = 2\alpha + 3$.

Another famous differential realization of the SO(2,1) algebra was given by Barut and Bornzin [5]. Their expressions for the generators are

$$J_+ = iy^n, \quad J_- = i\left(\frac{y^{2-n}}{n^2}\frac{d^2}{dy^2} + 2\frac{y^{1-n}}{n^2}\frac{d}{dy} - \frac{\xi}{y^n}\right), \quad J_0 = \frac{y}{n}\frac{d}{dy} + \frac{n+1}{2n} \tag{2.14}$$

This representation is also derived from the representation given in equation (2.7). In these case the variable x transformed to $y^{n/2}$. Using the relation in equation (2.9) we obtain

$$\begin{aligned} &x \to y^{n/2} \\ &\frac{d}{dx} \to \frac{2}{n} y^{\frac{2-n}{2}} \frac{d}{dy} \\ &\frac{d^2}{dx^2} \to \frac{4}{n^2} y^{2-n} \frac{d^2}{dy^2} + \frac{4-2n}{n^2} y^{1-n} \frac{d}{dy} \end{aligned} \tag{2.15}$$

Substitution into equation (2.7) and choosing $\gamma_1 = 0$ gives us

$$\begin{aligned} &J_+ = \frac{y^n}{4\gamma_0} \\ &J_0 = \frac{y}{n}\frac{d}{dy} + \eta_0(x(y)) \\ &J_- = \gamma_0\left(\frac{4}{n^2} y^{2-n} \frac{d^2}{dy^2}\right) + \left(\gamma_0 \frac{4-2n}{n^2} + \frac{(-2+8\eta_0(x(y))}{n}\right) y^{1-n} \frac{d}{dy} \\ &+ \frac{\gamma_2 + 4\gamma_0(-1+4\eta_0(x(y)))\eta_0(x(y))}{y^n} \end{aligned} \tag{2.16}$$

It is obvious that $\eta_0(x(y))$ should be (n+1)/2n, $\gamma_0 = -\frac{1}{4}$ and $\gamma_2 = i\left(\xi + \frac{n^2+n}{2n}\right)$

Our magic formalism gives that most of the one-dimensional realizations. The realization given in equation (2.7) is very flexible since there is freedom in choosing $\gamma_0, \gamma_1, \gamma_2$ and $\eta_0(x)$.

The other realization was given by W. Miller [30] is useful to solve the potentials of the form

$$V(y) = \frac{\alpha}{y^2} + by^2 + c \qquad (2.17)$$

The standard form for the generators of SO(2,1) was obtained by W. Miller is given by

$$\begin{aligned}
\Gamma_0 &= \frac{d^2}{dy^2} + \frac{\alpha}{y^2} - \frac{y^2}{16} \\
\Gamma_+ &= \frac{d^2}{dy^2} + \frac{y}{2}\frac{d}{dy} + \frac{\alpha}{y^2} + \frac{y^2}{16} + \frac{1}{4} \\
\Gamma_- &= \frac{d^2}{dy^2} - \frac{y}{2}\frac{d}{dy} + \frac{\alpha}{y^2} + \frac{y^2}{16} - \frac{1}{4}
\end{aligned} \qquad (2.18)$$

Above formalism may not be expressed by using our realization in equation (2.7) but we have checked that the Hamiltonian consists of the potential given in equation (2.17) can easily be expressed by using the generators given in equation (2.7).

2.4. Two Dimensional Realizations

In this section we will introduce realizations of SO(2,1) using two coordinates. The more general form of the generators constructed by Alhassid [2], he considered two variables ϕ and x, were given by

$$\begin{aligned}
J_{\pm} &= e^{\pm i\phi}\left(\pm h(x)\frac{\partial}{\partial x} \pm g(x) + f(x)J_0 + c(x)\right) \\
J_z &= -i\frac{\partial}{\partial \phi}
\end{aligned} \qquad (2.19)$$

It is easy to show that they satisfy equation (2.1) if

$$f^2(x) - h(x)\frac{df}{dx} = 1 \tag{2.20 a}$$

and

$$h(x)\frac{dc(x)}{dx} - c(x)f(x) = 0 \tag{2.20 b}$$

hold. In terms of this realizations the Casimir operator has the form

$$\begin{aligned} J^2 = h^2(x)\frac{d^2}{dx^2} + h(x)\left(\frac{dh(x)}{dx} + 2g(x) - f(x)\right)\frac{d}{dx} - f(x)g(x) + g^2(x) - \\ h(x)\frac{dg(x)}{dx} - c^2(x) - 2c(x)f(x)J_z + (1 - f^2(x))J_z^2 \end{aligned} \tag{2.21}$$

This result can be extended to find the eigenvalues of the Hamiltonians for a wider class of potentials. In the next section we will discuss the type of potentials which can be solved by using equation (2.21).

2.5. Calculation of Eigenvalues and Eigenvectors of the Schrödinger Equations

The classical power series method of solution is often suitable for general one-dimensional problems. It is assumed that a required solution may be expressed in the form

$$\psi = \sum_{n=0}^{\infty} a_n x^n e^{-\alpha(x)} \tag{2.22}$$

Let's choose a finite set of N+1 linearly independent set of vectors x_i and apply the ladder operators J_+ and J_- with the properties

$$J_+ x_i = \varepsilon_i^+ x_{i+1}, \qquad J_- x_i = \varepsilon_i^- x_{i-1} \tag{2.23}$$

with the properties

$$\varepsilon_M^+ = \varepsilon_M^- = 0 \tag{2.24}$$

Since $J_0 = -\frac{1}{2}[J_+, J_-]$ then we have

$$J_0 x_i = -\frac{1}{2}(J_+J_- - J_-J_+) = \varepsilon_i^0 x_i \tag{2.25}$$

comparison of equations (2.23) and (2.25) gives us

$$\varepsilon_i^0 = -\frac{1}{2}(\varepsilon_{i-1}^+ \varepsilon_i^- - \varepsilon_i^+ \varepsilon_{i+1}^-) \tag{2.26}$$

where $\varepsilon_i^{\pm,0}$ are the eigenvalues of an operator H. Note that if the given set x_i are eigenvectors of an operator H, then J_0 and H commute. On the other hand if H and J_0 commute then H is a polynomial function of J_0, so that the spectrum of H can be deduced from J_0. As we see in the following section if the Hamiltonian can be expressed as a linear function, the solution greatly simplified.

2.6. A Systematic Method to Diagonalize the Hamiltonian

Consider the eigenvalue problem of a general linear function of generators

$$H = q_+J_+ + q_-J_- + q_0J_0 \tag{2.27}$$

Assuming that $\{x_i\}$ are eigenvectors of J_0. Then we can write any eigenvector of H in the form

$$x = \sum_{i=0}^{n} a_i x_i$$

Thus the eigenvalue problem (H-E)x=0 becomes

$$\sum_{i=0}^{M} [p\varepsilon_{i+1}^+ a_{i+1} + q\varepsilon_{i-1}^- a_{i-1} + (V_i - E)a_i]x_i = 0 \tag{2.28}$$

The spectrum is obtained from the roots of the polynomial equation. We will follow a different way to solve the eigenvalue problem. In general Hamiltonian is expected to involve powers of generators beyond the linear terms, so that the transformations element cannot be constructed easily. We show below that the linear function of generator $J_{\pm,0}$ can be diagonalized quite generally. Once Hamiltonian is expressed in the form of linear combinations of generators, it is easy to solve the eigenvalue problem. We begin with a general linear functions of generators given in equation (2.27). To diagonalize Hamiltonian, we introduce a similarity transformation to eliminate $J_{\pm}$ from equation (2.27).

$$\mathrm{Exp(S)HExp(S)} = H - [H,S] + \frac{1}{2!}[[H,S],S] - \frac{1}{3!}[[[H,S],S],S] + \cdots \qquad (2.29)$$

Where S is the transformation operator. From the relation equation (2.29) one can easily obtain the relations,

$$\begin{aligned}
e^{\alpha J_0} J_+ e^{-\alpha J_0} &= e^{\alpha} J_+ \\
e^{\alpha J_0} J_- e^{-\alpha J_0} &= e^{-\alpha} J_- \\
e^{\alpha J_+} J_0 e^{-\alpha J_+} &= J_0 - \alpha J_+ \\
e^{\alpha J_+} J_- e^{-\alpha J_+} &= J_- - 2\alpha J_0 + \alpha^2 J_+ \\
e^{\alpha J_-} J_0 e^{-\alpha J_-} &= J_0 + \alpha J_- \\
e^{\alpha J_-} J_+ e^{-\alpha J_-} &= J_+ + 2\alpha J_0 - \alpha^2 J_-
\end{aligned} \qquad (2.30)$$

The Hamiltonian H can be diagonalized by using the relation given in equation (2.30), such that

$$UHU^{-1} = (q_0 - 2\alpha q_-)J_0 \qquad (2.31\text{ a})$$

where $U^{-1} = e^{-\alpha J_+} e^{-\beta J_-}$

$$\alpha = \frac{q_0}{2q_-} \pm \sqrt{\frac{q_0^{\,2}}{4q_-^{\,2}} - \frac{q_+}{q_-}} \quad \text{and} \quad \beta = -\frac{q_-}{q_0 - 2\alpha q_-} \qquad (2.31\text{ b})$$

In order to use this transformation coefficient of J. should be different from zero. If the arbitrary constants q_- is equal zero then the Hamiltonian can be diagonalized by using the relation.

$$THT^{-1} = (q_0 + 2\alpha q_+)J_0 \tag{2.32 a}$$

where $T^{-1} = e^{-\alpha J_-} e^{-\beta J_+}$

$$\alpha = \frac{q_0}{2q_+} \pm \sqrt{\frac{q_0^{\ 2}}{4q_+^{\ 2}} + \frac{q_-}{q_+}} \quad \text{and} \quad \beta = \frac{q_+}{q_0 + 2\alpha q_+} \tag{2.32 b}$$

The preceding results may be readily applied to a wide range of solvable problems involving second order differential equations.

It can be shown that the spectra of H and J_0 are directly proportional, while the eigenvectors $\psi_i(x)$ of H are evidently transform of the corresponding eigenvectors x_i of J_0.

$$\psi_i = U^{-1}x_i \quad \text{or} \quad \psi_i = T^{-1}x_i \tag{2.33}$$

The results contained in equations (2.30)-(2.33) can be applied to any linear operator H constructed from generators $\{J_o, J_+, J_-\}$. In the next section the approaches mentioned above can be used to solve the Schrödinger equation and some other second order differential equations.

CHAPTER 3

INVESTIGATION OF THE POTENTIALS ASSOCIATED WITH SO(2,1)

In this section solution of several different eigenvalue problems are studied rather efficiently using the methods mentioned in previous chapter. The present chapter emphasises the role of similarity transformation. At the end of the chapter our solutions are summarised.

3.1. One Dimensional Realization of SO(2,1) and Exactly Solvable Quantum Mechanical Potentials

In general, a compact way of writing one-dimensional Schrödinger equation with $(\hbar = m = 1)$ is

$$H\psi = \left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi \tag{3.1}$$

where V(x) is the one dimensional quantum mechanical potential. In our approach first we seek to express the Hamiltonian H as a simple function of generators of Lie algebra. It may then be constructed a similarity transformation S to diagonalize H. Clearly such a program can not be completed for every given Hamiltonian, even for one dimensional problems. As we mentioned before, if Hamiltonian is expressed as a linear function of generators, then the problem can be easily solved.

Consider the linear combinations of generators $\{J_0, J_+, J_-\}$. For simplicity let's choose $\gamma_1 = 0$ then we can write

$$L = q_-\gamma_0 \frac{d^2}{dx^2} + F(x)\frac{d}{dx} + G(x) = q_-J_- + q_+J_+ + q_0J_0 \tag{3.2}$$

where $F(x) = \dfrac{q_0x^2 + 2q_-\gamma_0(1-\eta_0(x))}{2x}$ and

$$G(x) = q_+\frac{x^2}{4\gamma_0} + q_0\eta_0(x) + \frac{q_-}{x^2}(\gamma_2 + 4\gamma_0(-1+\eta_0(x))\eta_0(x) + 2\gamma_0 x\eta_0{}'(x))$$

We now proceed to express equation (3.1) in terms of generators by comparing H and L. If the equations (3.1) and (3.2) is compared, we obtain $\eta_0(x)$ by choosing F(x)=0.

Case I Harmonic Oscillator $(F(x) = 0,\ q_0 = 0)$

In this case $\eta_0(x)$ has a value 1/4. Then we can write

$$G(x) = V(x) = \frac{q_+}{4\gamma_0}x^2 + \frac{q_-(4\gamma_2 - 3\gamma_0)}{4x^2} \tag{3.3}$$

The potentials in the form of equation (3.3) can easily be solved by using the method given in Chapter 2.

As an example, consider the quantum mechanical harmonic oscillator with the potential $V = \frac{1}{2}x^2$ is given by

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 \tag{3.4}$$

For this problem appropriate values of $\gamma_2 = \frac{3}{4}\gamma_0$, $\frac{q_+}{4\gamma_0} = \frac{1}{2}$ and $q_-\gamma_0 = \frac{1}{2}$.Note that $\eta_0(x) = \frac{1}{4}$ and $\eta_0{}'(x) = 0$. Then the Hamiltonian can be expressed as

$$H = 2\gamma_0 J_+ - \frac{1}{2\gamma_0} J_- \tag{3.5}$$

where

$$J_+ = \frac{1}{4\gamma_0} x^2 \quad \text{and} \quad J_- = \gamma_0 \frac{d^2}{dx^2} \tag{3.6}$$

In order to diagonalize the H we can use equations (2.31) or (2.32).We have checked the results obtained by using equation (2.31) is simpler than the result obtained equation (2.32).

$$\alpha = \pm 2\gamma_0, \qquad \beta = \pm \frac{1}{4\gamma_0} \qquad \text{and} \qquad \psi_m(x) = e^{\frac{1}{2}x^2} e^{-\frac{1}{4}\frac{d^2}{dx^2}} x^m \tag{3.7}$$

are obtained from equation (2.31) and

$$\alpha = \pm \frac{i}{2\gamma_0} \qquad \beta = \pm i\gamma_0 \qquad \text{and} \qquad \psi_m(x) = e^{\pm\frac{i}{2}\frac{d^2}{dx^2}} e^{\pm\frac{i}{4}x^2} \tag{3.8}$$

are obtained equation (2.32). The second solution have imaginary eigenvalues. The Hamiltonian should have real eigenvalues therefore the solution of the equation (3.4) should be in the form of equation (3.7). In equation (3.7) m is an integer, and the first few values of $\psi_m(x)$ are listed in Table 3.1

Table 3.1. The wave function $\psi_m(x)$ and Hermite polynomials $H_m(x)$

m	$e^{\frac{x^2}{2}}\psi_m(x)$	$H_m(x)$
0	1	1
1	x	2x
2	$x^2 - \frac{1}{2}$	$4x^2 - 2$
3	$x^3 - \frac{3}{2}x$	$8x^3 - 12x$
4	$x^4 - 3x^2 + \frac{3}{4}$	$16x^4 - 48x^2 + 12$

It is verified that the wave function and Hermite polynomials are related

$$\psi_m(x) = e^{-\frac{x^2}{2}} e^{-\frac{1}{4}\frac{d^2}{dx^2}} x^m = e^{-\frac{x^2}{2}} \frac{H_m(x)}{2^m} \tag{3.9}$$

The solution of equation (3.8) is also solution of equation (3.4), with the complex eigenvalues. So that the solution of our equation is given in equation (3.7). The eigenvalues of the Hamiltonian is obtained by inserting the wave function $\psi_m(x)$ into the equation (3.4).

The result may be readily applied to wide range of solvable problems involving second order differential equations. In each case the relevant second order differential equations is cast into the standard form.

Case II. Three Dimensional Isotropic Harmonic Oscillator $(F(x) = 0, q_0 = 0)$

In this case the arbitrary function $\eta_0(x)$ may take some value as in the case I . The appropriate radial differential equation is

$$H=-\frac{1}{2}\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{2r^2}+\frac{1}{2}r^2 \quad (3.10)$$

choosing the $\frac{q_+}{4\gamma_0}=\frac{1}{2}$ $\frac{q_-(4\gamma_2-3\gamma_0)}{4}=\frac{\ell(\ell+1)}{2}$, the Hamiltonian can be expressed as

$$H=2\gamma_0 J_+-\frac{1}{2\gamma_0}J_- \quad (3.11)$$

where

$$J_+=\frac{1}{4\gamma_0}x^2 \quad \text{and} \quad J_-=\gamma_0\frac{d^2}{dx^2}-\gamma_0\frac{\ell(\ell+1)}{x^2} \quad (3.12)$$

one can follow the same procedure as in the previous case and by using equation (2.31), we obtain,

$$\alpha=\pm 2\gamma_0, \quad \beta=\pm\frac{1}{4\gamma_0} \quad \text{and} \quad \psi_m^\ell(x)=\left(e^{\frac{1}{2}x^2}e^{-\frac{1}{4}\frac{d^2}{dx^2}-\frac{\ell(\ell+1)}{4x^2}}\right)x^m \quad (3.13)$$

Table 3.2. The $\psi_m^\ell(x)$ for some values of ℓ and m

ℓ	m	$e^{-\frac{x^2}{2}}\psi_m^\ell(x)$
0	0	1
1	0	$1-\frac{1}{2x^2}+\frac{3}{8x^4}+\cdots$
	1	$x-\frac{1}{2x}+\frac{1}{4x^3}+\cdots$
2	0	$1-\frac{3}{2x^2}+\frac{9}{8x^4}+\cdots$
	1	$x-\frac{3}{2x^2}+\frac{3}{4x^3}+\cdots$
	2	$-\frac{1}{2}$

Few values of the wave function are given in the Table 3.2. Again the eigenvalues obtained using the wave function $\psi_m^\ell(x)$.

Case III. $F(x) = 0, \eta_0(x) = \frac{q_0}{2q_- \gamma_0} x^2 + \frac{1}{4}$

We have checked that this selection does not give a new form of potential. General form of the potential is

$$G(x) = Ax^2 + \frac{B}{x^2} + C \tag{3.14}$$

In the next section we consider some transformations which recover the trivial shape invariant potentials from the differential equations of some orthogonal polynomials.

3.2. Orthogonal Functions and SO(2,1) Algebra

The general homogeneous second order differential equations can be written in the form

$$R(x)\frac{d^2\phi}{dx^2} + p(x)\frac{d\phi}{dx} + Q(x)\phi = 0 \tag{3.15}$$

we define two solutions ϕ_m and ϕ_n to be orthogonal if

$$\langle \phi_m | \phi_n \rangle = \delta_{nm} \tag{3.16}$$

We generalize this result so that these functions are orthogonal with respect to a weighting factor w(x), if

$$\langle w\phi_m | \phi_n \rangle = \delta_{nm} \tag{3.17}$$

In here we will discuss the solution of some orthogonal functions by using the Lie algebraic method.

3.2.1. Hermite Equation

The differential equation of Hermite polynomials can be obtained from the Schrödinger equation for quantum mechanical harmonic oscillators, by substituting $\psi_m(x) = e^{-\frac{x^2}{2}}\phi(x)$. Then we obtain the following eigenvalue problem

$$h\phi = E\phi \tag{3.18}$$

where the reduced Hamiltonian

$$h = -\frac{1}{2}\frac{d^2}{dx^2} + x\frac{d}{dx} + \frac{1}{2} \tag{3.19}$$

If we compare this equation with equation (3.2), to obtain the arbitrary function $\eta_0(x)$,

$q_-\gamma_0(1-4\eta_0(x)) = x^2(2-q_0)$ or the choice $\eta_0 = \frac{1}{4}, q_0 = 2$ are suitable. The other condition G(x)=$\frac{1}{2}$. In order to supply this condition we choose $q_+ = 0, \gamma_2 = \frac{3}{4}\gamma_0$. Then the equation (3.19) can be expressed as linear function of generators

$$h = -\frac{1}{2\gamma_0}J_- + 2J_0 \tag{3.20}$$

with this selection of arbitrary constant and functions the operators are

$$J_- = \gamma_0\frac{d^2}{dx^2}, \quad J_+ = \frac{x^2}{4\gamma_0} \quad \text{and} \quad J_0 = \frac{1}{2}x\frac{d}{dx} + \frac{1}{4}$$

In order to solve equation (3.20) we can use the methods given in previous chapter. From equations (2.31) we obtain

$$\alpha = 0, \qquad \beta = \frac{1}{4\gamma_0} \quad \text{and} \quad \psi_m(x) = e^{-\frac{1}{4}\frac{d^2}{dx^2}} x^m \tag{3.21}$$

As we mentioned before (See Table (3.1)) The function $\psi_m(x)$ is usual Hermite polynomial.

3.2.2. Laguerre's Differential Equations

The differential equation of generalized Laguerre polynomials is given by

$$x\frac{d^2}{dx^2} + (1-x)\frac{d}{dx} + n = 0 \tag{3.22}$$

In order to express this equation as a linear function of generators we need to change the variable x (because in our realization the generators does not contain the term $f(x)\frac{d^2}{dx^2}$).For this purpose we can use the realization given equation (2.16). If we choose n=1 $y \rightarrow x$ then our realization takes form

$$\begin{aligned} J_+ &= \frac{y}{4\gamma_0} \\ J_0 &= y\frac{d}{dy} + \eta_0(x(y)) \\ J_- &= 4\gamma_0 y\frac{d^2}{dy^2} + (2\gamma_0 - 2 + 8\eta_0(x(y)))\frac{d}{dy} + \\ &\frac{\gamma_2 + 4\gamma_0(-1 + 4\eta_0(x(y)))\eta_0(x(y))}{y} \end{aligned} \tag{3.23}$$

Comparing the equation (3.22) and the above realization an obtain choosing appropriate values of arbitrary parameters. Then we get

$$J_+ = y$$
$$J_0 = y\frac{d}{dy} + \frac{5}{16} \tag{3.24}$$
$$J_- = y\frac{d^2}{dy^2} + \frac{d}{dy}$$

Then the Laguerre's differential equation can be expressed in terms of the generators J_0 and J_-. Before we express the Laguerre's equation let's write the equation in the form of eigenvalue equation such that

$h\psi = E\psi$;

$$h = -x\frac{d^2}{dx^2} - (1-x)\frac{d}{dx} + \frac{5}{16} \quad \text{and} \quad E = \left(n - \frac{5}{16}\right) \tag{3.25}$$

Now it is easy to write the reduced Hamiltonian in terms of generators of Lie algebra

$$h = J_0 - J_- \tag{3.26}$$

From equation (2.31) we obtain

$$\alpha = 0, -\frac{1}{4}, \quad \beta = 1,2 \quad \text{and} \quad \psi_m(x) = e^{-\alpha J_+} e^{-\beta J_-} x^m \tag{3.27}$$

choosing the first root of α we obtain an expression for the solution of $\psi_m(x)$.

$$\psi_m(x) = e^{-x\frac{d^2}{dx^2} \frac{d}{dx}} x^m \tag{3.28}$$

In table (3.3) first few values of $\psi_m(x)$ and Laguerre polynomial are given.

Table 3.3. Laguerre polynomial and $\psi_m(x)$

m	$\psi_m(x)$	$L_m(x)$
0	1	1
1	x-1	1-x
2	x^2-4x+2	1-2x+x^2/2
3	x^3-9x^2+18x-6	1-3x+3x^2/2-x^3/6

It is shown that the wave function and Laguerre's polynomials related by the equation.

$$\psi_m(x) = \mathrm{Exp}\left(-\frac{d^2}{dx^2}-\frac{d}{dx}\right)x^m = \frac{(-1)^m L_m(x)}{m!} \tag{3.30}$$

This examples emphasises the importance of the choice of realization. The realization given in equation (2.7) leads directly to a linear problem for a wide range of differential equations.

3.2.3. Hypergeometric Functions

Another, potentially more general, example of quasilinear problem is provided by Kummers equation for the confluent hypergeometric functions. This can be written as a Sturm-Lioville eigenvalue problem.

$$h\psi = E\psi \ , \quad h = x\frac{d^2}{dx^2} + (b-x)\frac{d}{dx} \tag{3.31}$$

We can adopt the realization given in equation (3.23) by choosing $\gamma_0 = \frac{1}{4}, \eta_0(x) = \frac{2b+3}{16}$. Then the generators takes form

$$\begin{aligned} J_+ &= y \\ J_0 &= y\frac{d}{dy}+\frac{2b+3}{16} \\ J_- &= y\frac{d^2}{dy^2}+b\frac{d}{dy} \end{aligned} \tag{3.32}$$

The hypergeometric function can be written as

$$h'\psi = E'\psi; \quad h' = x\frac{d^2}{dx^2}+b\frac{d}{dx}-x\frac{d}{dx}-\frac{2b+3}{16}, \quad E' = E-\frac{2b+3}{16}$$

and h' can be expressed in terms of generators such that

$$h' = J_- - J_0 \tag{3.33}$$

In usual way the eigenfunction is given by

$$\psi_m(x) = \mathrm{Exp}\left(-x\frac{d^2}{dx^2}-b\frac{d}{dx}\right)x^m \tag{3.34}$$

For integer m, $\psi_m(x)$ is a polynomial of degree m.

3.3. Solution of Kepler Problems and Morse Potential

The second order differential equation

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{\delta}{r}+\frac{\alpha}{r^2}-\beta\right)R(r)=0 \tag{3.35}$$

arises in the generalized Kepler problem for motion in three dimension. This types equations can also be solved by using the algebraic methods. It is not possible to express the equation (3.35) in terms of generators of SO(2,1) given in Chapter 2. But by changing the variable and then by inserting a useful function into the equation we can put the equation in a standard form. Equation (3.35) may be transformed into the standard form by putting

$$r = x^2 \quad \text{and} \quad R(r) = x^{-\frac{3}{2}}\psi(x) \tag{3.36}$$

to give

$$\left(\frac{d^2}{dx^2} + \frac{4u - 3/4}{x^2} - 4v^2x^2 + 4\delta\right)\psi(x) = 0 \tag{3.37}$$

The equation can be written as eigenvalue problem such that

$h\psi(x) = E\psi(x)$

where

$h = -\frac{d^2}{dx^2} - \frac{u - 3/4}{x^2} + 4v^2x^2$ and $E = 4\delta$

By choosing $F(x)=0$; $q_0 = 0$, $G(x) = \frac{q_+}{4\gamma_0}x^2 + \frac{q_-(4\gamma_2 - 3\gamma_0)}{4x^2}$, $\eta_0(x) = \frac{1}{4}$ then generators takes form

$$J_+ = \frac{x^2}{4\gamma_0} \qquad J_- = \gamma_0\frac{d^2}{dx^2} + \frac{\gamma_2 - 3/4\gamma_0}{x^2} \tag{3.38}$$

In order to obtain spectrum of operator h we can write

$$h = J_- - 16\beta^2 J_+ \tag{3.39}$$

(with $\gamma_0 = -1, \gamma_2 = -u$)

again by using equation (2.31) we obtain

$$\alpha = \pm 4v \quad \text{and} \quad \beta = \pm\frac{1}{8v} \quad \text{then} \quad \psi_m(x) = e^{-4vJ_+}e^{-\frac{1}{8v}J_-}x^m \tag{3.40}$$

In the case of Klein Gordon equation we have for a hydrogenic atom $\delta = -2Z\alpha^{12}E$, $u = Z^2\alpha^2 - \ell(\ell+1)$ and $v = (\alpha^{14}E^2 - 1)/\alpha^2$ then the wave functions takes form,

$$\psi_m(x) = \text{Exp}(-vx^2)\text{Exp}\left(-\frac{1}{8v}\left(-\frac{d^2}{dx^2} - \frac{3/4 + Z^2\alpha^{12}}{x^2}\right)\right)x^m$$
$$R(r) = r^{-3}\text{Exp}(-vr)\text{Exp}\left(-\frac{1}{8v}\left(-4r\frac{d^2}{dr^2} - 2\frac{d}{dr} - \frac{3/4 + Z^2\alpha^{12}}{r}\right)\right)r^{\frac{m}{2}} \qquad (3.41)$$

Biederhan has shown that the second order dirac equation may be written as

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\Gamma(\Gamma-1)}{r^2} - \frac{2Z\alpha^2 E}{r} + \frac{\alpha^4 E^2 - 1}{\alpha^2}\right)\phi(r) = 0 \qquad (3.42)$$

Without further calculation we obtain the wave function $\phi(r)$;

$$\phi(r) = \text{Exp}(-vr)\text{Exp}\left(-\frac{1}{8v}\left(-4r\frac{d^2}{dr^2} - 2\frac{d}{dr} - \frac{3/4 + u}{r}\right)\right)r^{\frac{m}{2}} \qquad (3.43)$$

where $v^2 = \dfrac{\alpha^4 E^2 - 1}{\alpha^2}$ and $u = \Gamma(\Gamma - 1)$

In physics the solution of Schrödinger equation with Morse potential takes an important place. The Schrödinger equation is in the form,

$$\left(\frac{d^2}{dz^2} + pe^{2\tau z} + qe^{\tau z} + k\right)R(z) = 0 \qquad (3.44)$$

The equation may be transformed into the standard form by putting

$z = \ell n x^2$ and $R(z) = x^{-\frac{1}{2}}\psi(x)$

to yield

$$\left(\frac{d^2}{dx^2} + \frac{16r + \tau^2}{4\tau^2 x^2} + \frac{4p}{\tau^2}x^2 + \frac{4q}{\tau^2}\right)\psi(x) = 0 \qquad (3.45)$$

The last equation can be solved by usual way. Here we will solve the equation by using the realization given in equations (2.10) and (2.11). Let's rewrite the Schrödinger equation;

$$h\psi = E\psi$$
$$h = -\frac{1}{2}\frac{d^2}{dr^2} + De^{-2\alpha r} - 2De^{-\alpha r} + \frac{16}{9}D \tag{3.46}$$

The appropriate values of β_i's and $q_{\pm,0}$ as follows

$$\beta_0 = -\frac{2}{3}D,\ \beta_1 = -\frac{2}{3},\ q_0 = -\frac{4}{3}D,\ q_+ = \frac{4}{9}D^2 \text{ and } q_- = 1$$

Then h takes form

$$h = -\frac{4}{3}DJ_0 + \frac{4}{9}D^2J_+ + J_- \tag{3.47}$$

Again using the identities given an equation (2.32) then we obtain

$$\alpha = \frac{3}{2D}(-1+\sqrt{5}) \quad \text{and} \quad \beta = \frac{D}{3(-2+\sqrt{5})}$$

then

$$\psi(x) = e^{-x}(\text{Exp}e^{-\alpha J_-}e^{-\beta J_+})x^m \tag{3.48}$$

where

$$J_+ = \frac{1}{D}\left[\frac{d^2}{dx^2} - \frac{(3e^{-x}-4)}{2}\frac{d}{dx} - \frac{9}{4}e^{-x} + \frac{9}{16}e^{-2x} + 1\right]$$
$$J_- = \frac{4}{9}D\left[\frac{d^2}{dx^2} - \left(\frac{3}{2}e^{-x} - 2\right)\frac{d}{dx} - \frac{9}{4}e^{-x} + \frac{9}{16}e^{-2x} + 1\right] \tag{3.49}$$

The preceding examples illustrate the importance of choice of realization of Lie algebra. A number of authors have attempted to determine the Hamiltonians of quantum-mechanical systems that can be associated with a given Lie spectrum generating algebra. There is clearly a need for much more study of spectrum generating algebras, and in particular for away to decide directly on the spectrum generating algebra appropriate to a given Hamiltonian. In this thesis we proposed to construct a realization to express the Hamiltonian (or second order differential equation) and we showed that many of the second order differential equation can be expressed by using our realization.

3.4. Search for Two-Dimensional SO(2,1) Algebraic Structure Related to Solvable Potentials

Let us begin by adopting the general realization in equation (2.19).In this section we propose a realization to solve Schrödinger equation. The most general differential realization of the SO(2,1) algebra can be adopted by the choice of appropriate values of arbitrary functions given in equation (2.19) On the other hand we can use the Casimir invariance and the combination of

$$J_+J_- = J_0^2 - J_0 - J^2 \tag{3.50}$$

In terms of the realization given in equation (2.19) we can write

$$J_+J_-\psi = \left[-\frac{d^2}{dx^2} + \left(m - m^2 - \frac{1}{2}\right)f^2(x) + c^2(x) + c'(x) + \frac{1}{2}\right]\psi \tag{3.51}$$

The last equation is obtained by using the relations;

$$\begin{aligned} &\Psi(\phi, x) = e^{im\phi}\psi(x) \\ &g(x) = \left(\frac{1}{2} - m\right) \\ &h(x) = 1 \end{aligned} \tag{3.52}$$

and the relations $f^2(x) - h(x)\frac{df(x)}{dx} = 1$ and $h(x)\frac{dc(x)}{dx} - c(x)f(x) = 0$

In the SO(2,1) case, one considers for bound states, for which

$$\begin{aligned} &J_0|\ell m\rangle = m|\ell m\rangle \\ &J^2|\ell m\rangle = \ell(\ell-1)|\ell m\rangle \end{aligned} \tag{3.53}$$

$m = \ell, \ell+1, \ell+2, \ldots$

where ℓ is positive (but not necessarily restricted to integers or half integers; as one only deals with the algebra). From equations (3.50) and (3.53) it follows that the Schrödinger equation can be written as

$$\left(-\frac{d^2}{dx^2}+V_m\right)\psi_{\ell m}(x)=(m^2-m-\ell(\ell-1))\psi_{\ell m}(x) \tag{3.54}$$

In the above equation, the one parameter potentials, denoted V_m, is represented by

$$V_m=\left[\left(m-m^2-\frac{1}{2}\right)f^2(x)+c^2(x)+c'(x)+\frac{1}{2}\right] \tag{3.55}$$

For the potentials V_m, corresponding to the energy eigenvalues

$$E_\ell^m=(m^2-m-\ell(\ell-1)) \tag{3.56}$$

Before we embark upon our detailed study of SO(2,1), let us make a few remarks on the use of equation (3.54) in the real domain. In the next chapter we will investigate the possible solvable potentials in complex domain with real eigenvalues. From the relation given in equation (2.20) we obtain the values of $f(x)$ and $c(x)$;

$$\begin{aligned} f(x)&=-\coth\left(x+\frac{1}{2}\ln k\right) \\ c(x)&=\frac{k'e^x}{-1+ke^{2x}} \end{aligned} \tag{3.57}$$

where k and k' are integrate constants. When the values of f(x) and c(x) substituted in equation (3.54) we obtain a general relation for the potential $V_m(x)$.

$$\begin{aligned} V_m(x)=&\frac{1+k^2e^{4x}-2k'e^x-2k'^2e^{2x}-2ke^{2x}(1+k'e^x)}{2(-1+ke^{2x})^2}- \\ &\frac{1}{2}\coth\left(x+\frac{\ln(k)}{2}\right)^2+\coth\left(x+\frac{\ln(k)}{2}\right)^4-\coth\left(x+\frac{\ln(k)}{2}\right)^6 \end{aligned} \tag{3.58}$$

From the these result we obtain a wide range of potentials by justifying the values of k and k'. Some of these are given in table (3.4) with their eigenvalues.

Table 3.4. The final form of potential and energy eigenvalues obtained from the two dimensional differential realization of the SO(2,1) algebra.

$V_m(x)$	E_m^ℓ
$\frac{1}{2}+\left(-\frac{1}{2}+m(m-1)\right)\coth\left(\frac{2x+Log(q)}{2}\right)^2$	$m(m-1)-\ell(\ell-1)-\frac{1}{2}$
$\frac{1}{2}+\left(-\frac{1}{2}+m(m-1)\right)\tanh(x)^2$	$m(m-1)-\ell(\ell-1)-\frac{1}{2}$
$\frac{1}{4}\left(-1+2m(m-1)(1+\cosh(2x))-2\cosh(x)\right)\operatorname{csch}(x)^2$	$\frac{1}{2}m(m-1)-\ell(\ell-1)+\frac{1}{4}$
$m(m-1)+\frac{1}{4}\operatorname{sech}(x)^2\left(3-4m(m-1)+2\sinh(x)\right)$	$-\ell(\ell-1)$
$e^{\pm x}+e^{\pm 2x}\pm m(m-1)$	$2m(m-1)-\ell(\ell-1)$ or $-\ell(\ell-1)$
$e^{\pm x}(e^{\pm x}-1)\pm m(m-1)$	$2m(m-1)-\ell(\ell-1)$ or $-\ell(\ell-1)$
$ke^{\pm x}(ke^{\pm x}-1)\pm m(m-1)$	$2m(m-1)-\ell(\ell-1)$ or $-\ell(\ell-1)$

As we mentioned before, by transforming the wave function or by changing the variable we can drive other types of potentials. In the table the potentials in the 4^{th} row is known as Scarf II type potentials or Gendenshtein potential, the potentials in the rows 5,6 and 7 are known as the Morse potential and the potential in the row 3 is the Pöschl-teller potential.

In this chapter we have made a systematic search for SO(2,1) algebraic structures related to the potentials in Schrödinger equations and several special functions of physics, by using three specific differential realization of SO(2,1) generators. In particular the generators, we constructed as a linear differential operators depending on one variable are most appropriate and easily adopted to the Hamiltonian. The differential realization which depending on two variables was inspired by Alhassid [2].

The present work can be generalised in various directions. Here we have considered the solution of a few special functions. In fact our realization can be adopted for other special functions. Therefore a systematic study of these special functions seem to be necessary.

The algebras discussed here can be embedded into some larger algebras. These are SO(2,2), SO(2,1), SO(3,1) dynamical potential algebras. Our method gives an idea to construct the realization of larger algebras. $N(3)\otimes Sp(6,R)$ or SU(3,1) which are useful to solve three dimensional Schrödinger equation.

CHAPTER 4

ANALYSIS OF SOME NON-HERMITIAN HAMILTONIANS WITH REAL EIGENVALUES

Since the paper of Bender an Boetcher [23] there has been a great deal of interest in the study of non hermitian PT-symmetric potentials with real spectrum. Several years ago, D. Bessis conjectured on the basis of numerical studies that the spectrum of the Hamiltonian $H = p^2 + x + ix^3$ is real and positive. Bender claimed that the reality of the spectrum of H is due to PT symmetry. He noted that H is invariant neither under parity P, whose effect is to make spatial reflections, $P \to -P$ and $x \to -x$, nor under time reversal T which replaces $P \to -P$, $x \to x$ and $i \to -i$. Bender showed that the Hamiltonian $p^2 + ix^3 + ix$ has PT symmetry and the spectrum is positive definite, the Hamiltonian $p^2 + ix^3 + x$ is not PT symmetric and the spectrum is complex.

We have checked that the connection between PT symmetry and positivity of a spectra. For instance, energy levels of the quantum mechanical harmonic oscillator ($H=p^2+x^2$)is 2n+1. We added ix to the H and H does not break PT symmetry and the spectrum remains positive definite: $E_n = 2n + \frac{5}{4}$. Adding -x also does not break PT symmetry, and the again the spectrum is positive definite: $E_n = 2n + \frac{3}{4}$. By contrast, adding ix-x does break symmetry, and the spectrum is now complex: $E = 2n + 1 + \frac{1}{2}i$.

There are many applications of non hermition PT invariant Hamiltonians in physics. Hamiltonians rendered non hermition by an imaginary external field have been introduced recently to study delocalization transitions in condensed matter systems such as vortex fluxtire depinnig in superconductors, or even to study population biology.

Of late, a sharp increase of interest has been noticed in searching for non hermitian Hamiltonians. Although the history of complex potentials is old, as we mentioned above, especially in relation to scattering problems. Bender a few years ago, revived interest in complex potentials by restricting a non-hermitian Hamiltonian to be PT symmetric. Subsequently the idea of PT symmetry has been pursued by several authors, who have obtained different kind potentials with real eigenvalues.

In this chapter we propose to complexify the so(2,1) algebra for the Schrödinger equation to study non-hermitian systems from a group theoretical points of view. Adopting the two dimensional differential realization of the so(2,1) algebra we demonstrate how new complex potentials can be generated and possess real eigenvalues.

4.1. Complexification of Some Potentials

We now proceed to complexify the potentials, given in table (3.1). In the table the real Scarf potential is given in the form

$$V_m^s(x) = (B^2 - A(A+1))\operatorname{sech}^2(x) + B(2A+1)\operatorname{sech}(x)\tanh(x) \qquad (4.1)$$

This potential is well known to be exactly solvable. For A>0 the associated eigenfunctions and eigenvalues are

$$\begin{aligned} &\psi_\ell(x) = N_\ell(\operatorname{sech}(x))^A \operatorname{Exp}(-b\tan^{-1}(\sinh(x))P_\ell^{(-iB-A-1/2,iB-A-1/2)}(i\sinh(x)) \\ &E_\ell = -(A-n)^2 \end{aligned} \qquad (4.2)$$

$n = 0,1,2\ldots$

where N_ℓ is a normalisation constant, and $P_\ell^{(A,B)}$ is a Jacobi polynomial. Replacing B by iB leads to the PT symmetric form of the potential in equation (4.1).

$$V_m^{cs}(x) = -(B^2 - A(A+1))\operatorname{sech}^2(x) + iB(2A+1)\operatorname{sech}(x)\tanh(x) \qquad (4.3)$$

Comparing the equation (4.3) with the potential given in table (3.4) we obtain $m = \varepsilon\left(A + \frac{1}{2}\right)$, where $\varepsilon = \pm 1$. Since by assumption m>0, we have to choose $\varepsilon = -1$. The eigenfunctions were obtained by Bi Bagchi. We just state his results, which are

$$\psi_\ell^m(x) = N_\ell^m(\operatorname{sech}(x))^{m-\frac{1}{2}} \operatorname{Exp}(-iB\arctan(\sinh(x)))P_\ell^{(B-m,-B-m)}(i\sinh(x)) \qquad (4.4)$$

The corresponding eigenvalues given that

$$E_\ell^m = \left(m - n - \frac{1}{2}\right)^2 \qquad (4.5)$$

Again from the table the Pöschl Teller potential can be written as

$$\begin{aligned} V^{CGPT}(x) &= (B^2 + A(A+1))\operatorname{cosech}^2(x - i\gamma) - \\ &B(2A+1)\operatorname{cosech}(x - i\gamma)\coth(x - i\gamma) \end{aligned} \qquad (4.6)$$

The real potential corresponding $\gamma = 0$. Using now a complex analogue of the point canonical coordinate transformation known to relate to generalized Pöschl-Teller and Pöschl-Teller II potentials, the potential can be changed into the complexified Pöschl-Teller II potential.

$$V^{CPT}(x) = \frac{(B-A)(B-A-1)}{\sinh^2(t-i\varepsilon)} - \frac{(A+B)(A+B+1)}{\cosh^2(t-i\varepsilon)} \qquad (4.6)$$

where t=x/2 and $\varepsilon = \gamma/2$. Again the spectrum of the potential has real eigenvalues.

The last type of potential, we consider the complexified Morse potential.

$$V^{CM}(x) = (B_r + iB_i)^2 e^{-2x} - (B_r + iB_i)(2A+1)e^{-x} \tag{4.7}$$

It is straightforward to show that for A and B_r positive, the potential has the same real eigenvalues as its real counterpart. (But of course different wave functions.) This provides a very simple example of non PT-symmetric complex potential with real eigenvalues.

4.2. Solution of Complex Potentials in Polynomial Form

Recently it has been found that there are large classes of non-Hermitian Hamiltonians whose spectra are real. Although they are non-Hermitian, these Hamiltonians exhibit the weaker invariance of PT symmetry. A class of these Hamiltonians,

$$H = p^2 - (ix)^N \quad N \geq 2 \tag{4.8}$$

In here we generalize the Hamiltonian to the two parameter class.

$$H = p^2 - x^4 + 2iax^3 + (a^2 - 2b)x^2 + 2i(ab - J)x \tag{4.9}$$

where a and b are real and J is a positive integer. The equation is quasi exactly solvable equation. The wave function can be written in the form

$$\psi(x) = e^{-i\frac{x^3}{3} - a\frac{x^2}{2} - ibx}\Phi(x) \tag{4.10}$$

If we apply the Hamiltonian H to $\psi(x)$ and divide of the exponential we obtain an operator h acting on the Φ; h has the form

$$h = -\frac{d^2}{dx^2} + (2ix^2 + 2ax + 2ib)\frac{d}{dx} - (2i(J-1) - b^2 - a) \tag{4.11}$$

The polynomial is solved by using series method and the results for energy eigenvalues are given in Table (4.1)

Table 4.1. Energy eigenvalues of the reduced hamiltonian h. In here $p=E-b^2-Ja$ and $K=4b+a^2$.

J	Q_J
1	$-P$
2	P^2-K
3	$P^3-4KP-16$
4	$P^4-10KP^2-96P+9K^2$
5	$P^5-20KP^3-336P^2+64K^2P+768K$

The roots of these polynomials are all real. To conclude, we have solved a complex PT- symmetric potential in the most general form by using mathematica. We have also demonstrated complexification of some potentials. We have seen that at least the quasi exactly solvable eigenvalues of the non-Hermitian PT-invariant potentials are real.

CHAPTER 5

NON-LINEAR DYNAMICAL SYSTEMS

This chapter provides an introductory guide to the solution of non-linear differential equations and analyses of the dynamical system by the topological methods. We have solved Lorenz Equation and Heat equation by using the method of numerical analysis and by developing an analytical method respectively.

5.1. Nonlinear Differential Equations

Dynamical systems are mathematical objects used to model physical phenomena whose state (or instantaneous description) changes over time. These models are used in financial and economic forecasting, environmental modelling, medical diagnosis, industrial equipment diagnosis, and a host of other applications.

For the most part, applications fall into three broad categories: predictive (also referred to as generative), in which the objective is to predict future states of the system from observations of the past and present states of the system, diagnostic, in which the objective is to infer what possible past states of the system might have led to the present state of the system (or observations leading up to the present state), and, finally, applications in which the objective is neither to predict the future nor explain the past but rather to provide a theory for the physical phenomena. These three categories correspond roughly to the need to predict, explain, and understand physical phenomena.

As an example of the later, a scientist might offer a theory for a particular chemical reaction in terms of a set of differential equations involving temperature, pressure,

and amounts of compounds. The scientist's theory might be used to predict the outcome of an experiment or explain the results of a reaction, but from the scientist's point of view the set of equations is the object of primary interest as it provides a particular sort of insight into the physical phenomena.
Not all physical phenomena can be easily predicted or diagnosed. Some phenomena appear to be highly stochastic in the sense that the evolution of the system state appears to be governed by influences similar to those governing the role of dice or the decay of radioactive material. Other phenomena may be deterministic but the equations governing their behaviour are so complicated or so critically dependent on accurate observations of the state that accurate long-term observations are practically impossible.

In the mathematical point of view, general form of the non-linear equations are given by

$$P_1(x)\frac{d^n y}{dx^n} + P_2(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots\dots + P_n(x)y = f(x,y) \qquad (5.1)$$

The function f(x,y) is both function of x and y. We must face the fact that it is usually very difficult, if not impossible, to find a solution of a given differential equation. For example, consider equation of simple pendulum;

$$\frac{d^2\theta}{dt^2} + w^2 \sin\theta = 0 \qquad (5.2)$$

This is one of the most simple equation of classical mechanics. But to find an analytical solution we need to make an approximation. $(\sin\theta \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots)$

5.2. A Finite Population Model

In a finite world, no population can become infinite; limiting factors of one kind or another must come into play. One possible formula for the variation of t with N, is in the form

$$\frac{dN}{dt} = \sum_{n=0}^{m} \varepsilon_n N^n \tag{5.3}$$

with the initial condition $N(0) = N_0 \geq 0$ where ε_n is constant and m is an integer number depends on the properties of physical system. N is the size of population at time t, and $\frac{dN}{dt}$ is the population growth rate. If $n>1$ then the equation is nonlinear equation. Linear part of the equation can easily be solved and the result is

$$N = N_0 e^{\varepsilon_1 t} + \frac{\varepsilon_0}{\varepsilon_1}(1 - e^{\varepsilon_1 t}) \tag{5.4}$$

the equation (5.3) occurs in many different applications. In here let's consider this equation as population growth equation. If we put $\varepsilon_0 = 0, \varepsilon_1 = \varepsilon, \varepsilon_2 = -\sigma$ the equation takes form

$$\frac{dN}{dt} = \varepsilon N - \sigma N^2 \tag{5.5}$$

Under this assumption, The differential equation describing the growth of population. The nonlinear term σN^2 is recognised as death rate. The general solution of the equation may be found by solving it as a Bernoulli equation using the substitution $N^{-1}=Z$. From this we easily find the population at time t is

$$N = \frac{\varepsilon N_0}{\sigma N_0 + (\varepsilon - \sigma N_0)e^{-\varepsilon t}} \tag{5.6}$$

assuming N_0 is the number of individuals present when t=0. A t becomes infinite, the factor $e^{-\varepsilon t}$ approaches zero and N approaches the limiting value ε/σ. The graph of the N is given in figure (5.1).

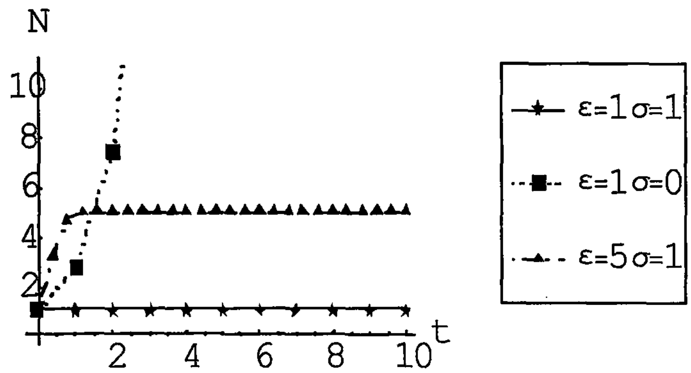


Figure 5.1:The plot of population growth equations with the different ε/σ values.

Although equation (5.5) was introduced as an approximate model for a biological population, the some kind of equation might be useful for other purposes, such as the prediction of stock inventories under restricted sources of supply and growing consumer demands or the estimation of economic trends in segments of the economy which are limited by finite resources.

5.3. A Method to Solve Nonlinear Heat Equation

Let us consider the heat flow in a medium. If there is heat flow in the medium the temperature will vary with time and coordinate. We base our analysis on the experimental laws and one can drive an equation which in the form,

$$a^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{k} f(x,t) \tag{5.7}$$

where a is a constant depends on the properties of medium. In a special case f(x,0)=0, implies that heat is neither generated nor absorbed in the body, then equations takes form

$$a^2\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \tag{5.8}$$

Equation (5.6) can be solved by using the method of separation of variables and the general solution of the equations is

$$u(x,t)=(A\cos\lambda x+B\sin\lambda x)e^{-\frac{\lambda^2 t}{a^2}} \tag{5.9}$$

The integral constant A, B and λ can be determined by using boundary conditions and Fourier transformation [30]. In here the solution of heat equation which consists of nonlinear term is investigated. In many physical systems the equation is given by

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u^3 \tag{5.10}$$

The last term u^3 is function of x and t. (for simplicity the parameter a is taken unity). It is obvious that the equation cannot be solved by using the method of separation of variables. In order to solve above equation we can substitute

$$u(x,t)=y(x)w(z) \tag{5.11}$$

after a few attempts one can find that y(x) and z should be

$$y(x)=x+k_1 \qquad z=\frac{1}{2}x^2+k_1x+3t \tag{5.12}$$

Substitution equations (5.9) and (5.10) into equation (5.8) we obtain

$$\frac{\partial^2 w}{\partial z^2}+w^3=0 \tag{5.13}$$

Equation (5.11) can be put in the form of Jacobi equation by changing the variable

$$\frac{\partial w}{\partial z} = v \text{, then } v\frac{dw}{dv} + w^3 = 0 \text{ and } v = \sqrt{2c - \frac{w^4}{2}} \qquad (5.14)$$

Combinations of equations (5.11) and (5.13) gives us

$$\frac{dw}{dz} = \sqrt{2c - \frac{w^4}{2}} \qquad (5.15)$$

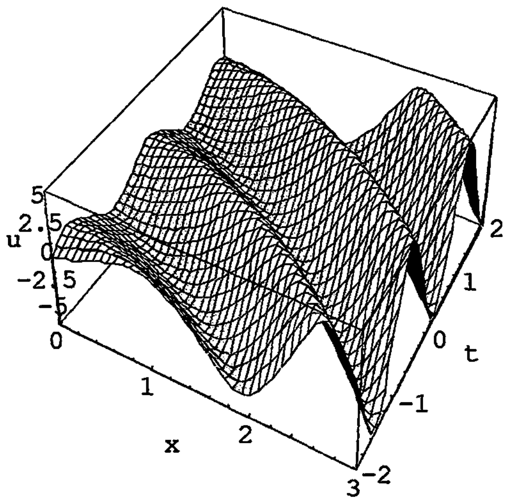
By using series method the above equation can be solved. In here the equation (5.14) can be solved by the help of MATHEMATİCA and the results is

$$w = -i\sqrt{2\sqrt{c}}\,\mathrm{JacobiSn}(ic^{\frac{1}{4}}(c_1 + z)) \qquad (5.16)$$

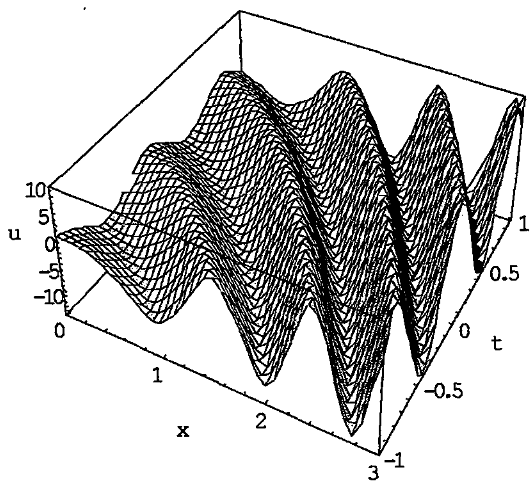
so that the general solution of the equation (5.8) is obtained,

$$u(x,t) = (x + k_1)w \qquad (5.17)$$

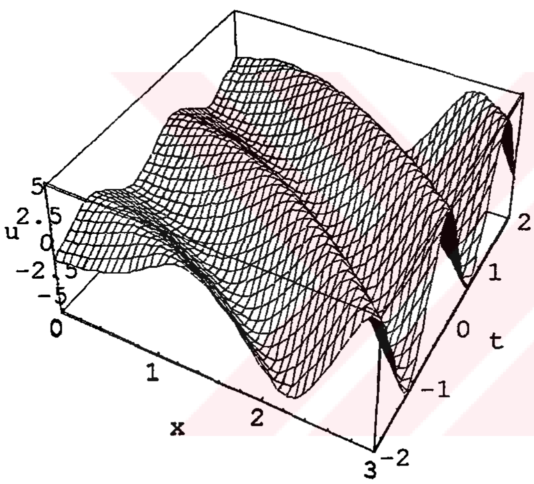
The integral constants c, c_1 and k_1 are determined by using boundary conditions. In here we investigated the effect of integral constants and the heat flow and the result are shown in figure 5.1



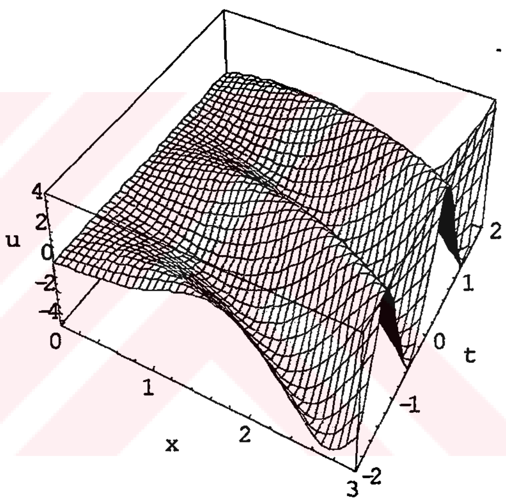
$c^{1/4}=1$, $c=1$, $k=1$



$c^{1/4}=2$, $c=1$, $k=1$



$c^{1/4}=1$, $c=0$, $k=1$



$c^{1/4}=1$, $c=1$, $k=0$

Figure 5.2. Shows the heat diffusivity against the displacement and time for different constants.

5.4 Lorenz Equations

The Lorenz Equation or Lorenz model is paradigm one for low dimensional chaos in dynamical system in synergetic and this model or is modifications are widely investigated in connection with modelling purposes in meteorology, hydrodynamics, laser physics, super conductivity, electronics, oil industry.

Meteorologist Edward Lorenz supposedly first discovered the chaos theory in 1960. He had created a weather model, which would predict weather with the use of twelve equations. One day Lorenz wanted to see a weather pattern he had already run through his model the day before. To save time, he entered the numbers from half way through the pattern into the model and let it run. When he later went back to the model it was drastically different from how it had been the day before. The reason for this in re entering the numbers, Edward had also cut off a few other numbers which it turn caused very big errors within the system. This ended up to be the beginning of research in the Chaos Theory.

Chaos theory attempts to explain the fact that complex and unpredictable result can and will occur in systems that are sensitive to their initial conditions. This alone does not mean that a system is chaotic, but every chaotic system has this property. Take for example a simple pendulum. For small oscillations the motion of the pendulum can be determined very easily as a sine functions. If this pendulum were started from a set of initial conditions many times, the behaviour would be roughly the same for each of trials.

Chaotic systems do not have this property. The smallest change in initial conditions could produce wildly different behaviour. They seem to move erratically, and any small perturbation could change the whole system drastically. However, these systems do have an underlying order.

The Lorenz Equations define three ordinary differential equations and determine the evaluation of the system. They describe the rate of change x, y, and z respectively. These equations attempt to model convection process in the atmosphere how air is warmed, then rises, is cooled, and falls again. Each of the variables x, y and z has a specific meaning in this system:

x- This variable is proportional to the speed of motion of the air due to convection.
y- This is measure of the temperature difference between the warm, rising air and the cool, falling air.
z- This is a measure of the vertical temperature difference as you move through the system from top to bottom.

There are also three constant in the equation set, which have large impact on the system. Sigma is the proportional to the Prandtl number, which is based upon the nature of the air involved. This usually takes a value of 10. Beta represents the size of the area by the modelling. It was originally set to 2.666. r is the systems Rayleigh number a parameter which dictates at which point convection will start in the system It is vital for changing from steady, stable convection to chaotic convection.

The Lorenz equations can be solved two methods. These are numerical methods and analytical methods.

5.4.1. Analytical Solution

a) A first integral of the Lorenz equations

b) Steady state solutions at constant time

c) Linearize the steady state solutions

d) x(t)=y(t) Asymptotic solution

e) By choosing appropriate parameters, σ, β, r values then solve the equations.

5.4.1.1 A first integral of the Lorenz equations

The Lorenz model of atmospheric circulation

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = rx - y - xz \qquad (5.18)$$
$$\frac{dz}{dt} = xy - \beta z(x - y)$$

admits for (σ, β, r)=(0,1/3,arbitrary) the first integral

$$[-\frac{3}{4}x^4 + \frac{4}{3}x(y - x) + (z - r + 1)x^2]e^{4t/3} = K, \qquad (5.19)$$

but can one go further , i.e. can one obtain more first integrals or even explicitly integrate? The answer is yes. By elimination of (y,z) one first builds the second order equation for x(t)

$$\frac{d^2x}{dt^2} = \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{x^3}{4} - \frac{K}{3x}e^{-4t/3} \tag{5.20}$$

For K=0 this equations admits the first integral

$$\frac{1}{x^2}\left(\frac{dx}{dt}\right)^2 + \frac{x^2}{4} = A^2 \tag{5.21}$$

and the general solution

$$x=(1/(2A))\cosh(t-t_0). \tag{5.22}$$

5.4.1.2.Steady State Solutions

A steady state of a system is a point in phase space from which the system will not change in time, once that state has been reached. In other words, it is a point,(x,y,z), such that the solution does not change, or where

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = 0, \qquad \frac{dz}{dt} = 0 \tag{5.23}$$

This point is usually referred to as a stationary point of the system. Now, set the time derivatives equal to zero in the Lorenz equations (5.18) , and solve the resulting system to show that there are three possible steady states, namely the points
(0,0,0), then

$$(\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1) \tag{5.24}$$

Remember that r is a positive real number, so that that there is only one stationary point when $0 \le r \le 1$, but all three stationary points are present when $r>1$

5.4.1.3. Linearization About the Steady States

The difficult part of doing any theoretical analysis of the Lorenz equations is that they are non-linear. So, why not approximate the non-linear problem by a linear one? For this one , we use the Taylor Series. There, we were approximating a function ,f(x), around a point by expanding the function in a Taylor series, and the first order Taylor approximation was simply a linear function in x. The approach we will take here is similar, but will get into Taylor series of functions of more than one variable: .f(x,y,z,....)

The basic idea is to replace the right hand side functions in equation (5.18) with a linear approximation about a stationary point, and then solve the resulting system of linear ODE's. Hopefully, we can then say something about the non-linear system at values of the solution close to the stationary point (remember that the Taylor series is only accurate close to the point we're expanding about).

So, let us first consider the stationary point(0,0,0) . If we linearize a function f(x,y,z) about (0,0,0) we obtain the approximation:

$$f(x,y,z) \approx f(0,0,0)+f_x(0,0,0).(x-0)+f_y(0,0,0).(y-0)+f_z(0,0,0).(z-0) \qquad (5.25)$$

If we apply this formula to the right hand side function for each of the ODE's in (5.18) then we obtain the following linearized system about(0,0,0) :

$$\begin{aligned} \frac{dx}{dt} &= \sigma x + \sigma y \\ \frac{dy}{dt} &= rx - y \\ \frac{dz}{dt} &= -\beta z \end{aligned} \qquad (5.26)$$

(note that each right hand side is now a linear function of x, y and z). It is helpful to write this matrix form as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5.27}$$

the reason for this being that the eigenvalues of the matrix give us valuable information about the solution to the linear system. In fact, it is a well-known result from the study of dynamical systems is that if the matrix in equation (5.27) has distinct eigenvalues , $\lambda_1, \lambda_2, \lambda_3$ and , then the solution to this equation is given by

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_{31} t} \tag{5.28}$$

and similarly for the other two solution components, y(t) and z(t) (the c_i's are constants that are determined by the initial conditions of the problem). This should not seem too surprising, if you think that the solution to the scalar equation is

$$\frac{dx}{dt} = \lambda \qquad x(t) = e^{\lambda t} \tag{5.29}$$

The eigenvalues of a matrix, A, are given by the roots of the characteristic equation, det(A-.λI). We determine the characteristic equation of the matrix in equation (5.29), and we find the eigenvalues of the matrix A are,

$$\begin{aligned} \lambda_1 &= \beta \\ \lambda_2 = \lambda_3 &= 1/2\left(\sigma - 1 \pm \sqrt{(\sigma-1)^2 + 4\sigma r}\right) \end{aligned} \tag{5.30}$$

When r>1, the same linearization process can be applied at the remaining two stationary points, which have eigenvalues that satisfy another characteristic equation:

$$\lambda^3 + (\sigma + \beta + 1)\lambda^2 + (r + \sigma)\beta\lambda + 2\sigma\beta(r - 1) = 0 \tag{5.31}$$

5.4.2. Numerical Solutions

The solutions of the Lorenz equations can be found numerical method. The initial value problem for a system of differential equations is to be solved by the well know Runge-Kutta method. A method of numerically integrating differential equations by using a trial step at the midpoint of an interval to cancel out lower-order error terms. The fourth-order formula is

$$\begin{aligned} K_1 &= hf(x_n, y_n) \\ K_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}K_1) \\ K_3 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}K_2) \\ K_4 &= hf(x_n + h, y_n + K_3) \\ y_{n+1} &= y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \end{aligned} \tag{5.32}$$

This method is reasonably simple and robust and is a good general candidate for numerical solution of differential equations when combined with an intelligent adaptive step-size routine. Runge-Kutta 4 method is vastly greater accuracy for the same amount of calculations.

To obtain the solution of equations numerical method are interpolated with the Mathematica and we examine the effect of the parameters and initial conditions.

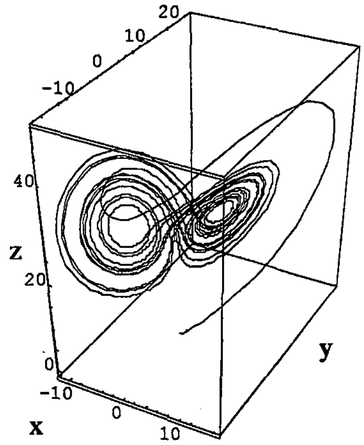


Figure 5.3: A plot of the solution to the Lorenz equations as an orbit in phase space. Parameters: σ =10, β =2.666, r=28; initial values: (x,y,z) =(0,1,0)

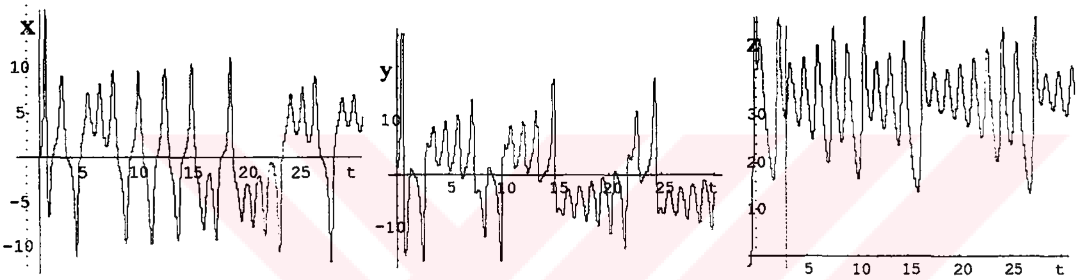


Figure 5.4. A plot of the solution to the Lorenz equations right to left x,y and z versus time. Parameters: σ =-6, β =-1, r=28; initial values: (x,y,z)=(0,1,0)

5.4.2.1 Effect of the Initial Conditions

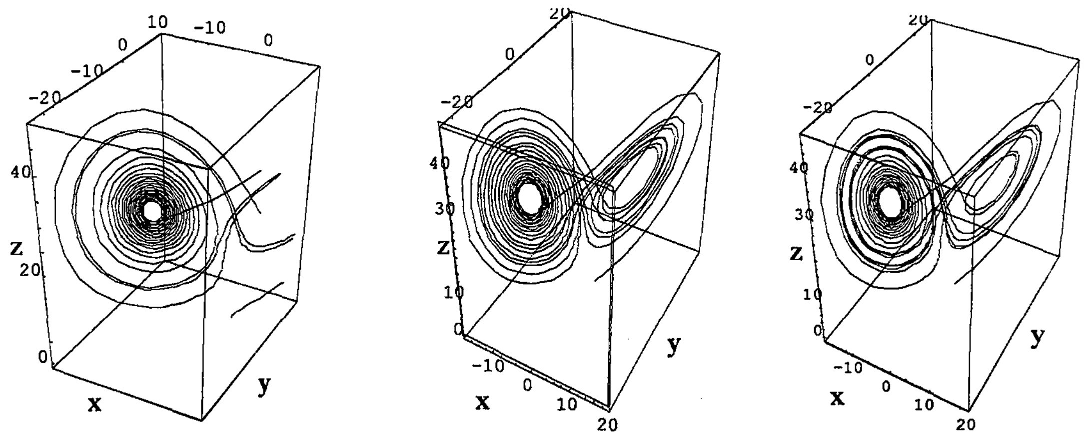


Figure 5.5. Shows the effect of the initial conditions

1. x=1, y=0, z=0 **2.** x=2, y=2, z=2 **3.** x=2, y=2z=2.002

5.4.2.2 Effect of the Parameters

Effect of the Sigma

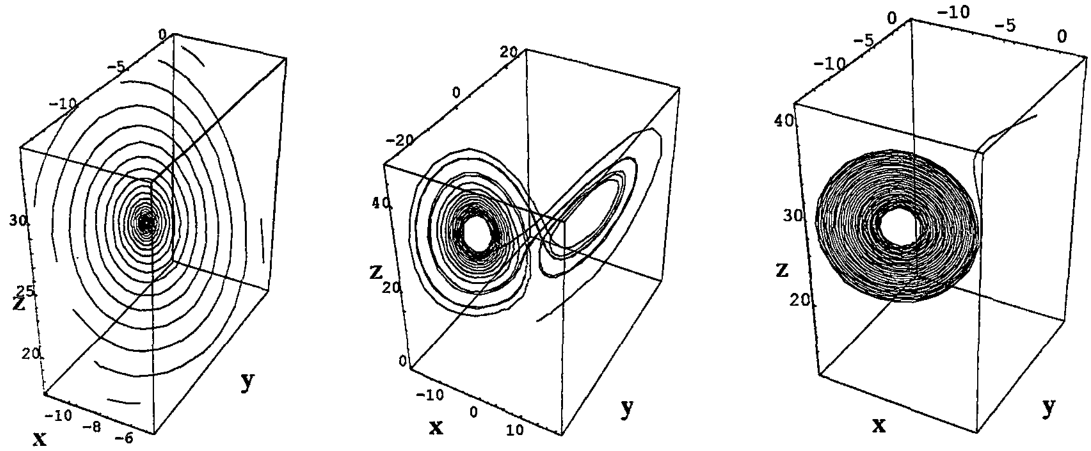


Figure 5.6. Shows the effect of the sigma.

1. $\sigma = 4, r = 28, \beta = 2.6$ **2.** $\sigma = 8, r = 28, \beta = 2.6$ **3.** $\sigma = 20, r = 28, \beta = 2.6$

Effect of r

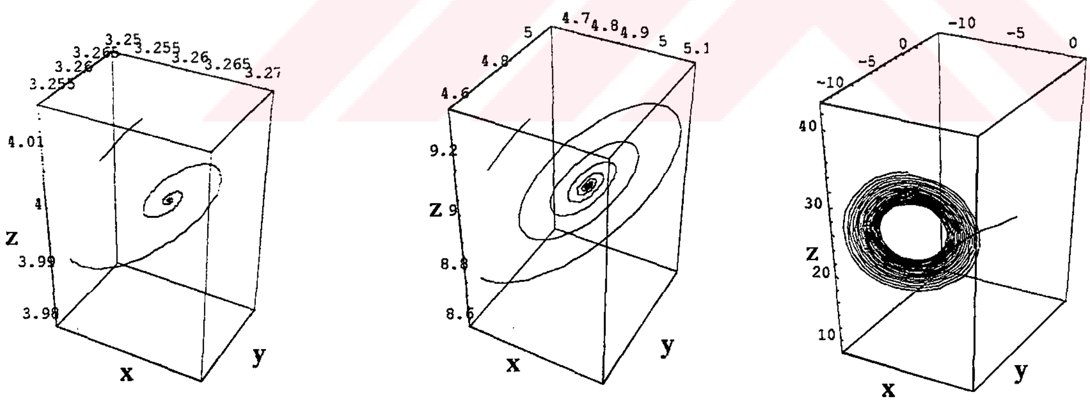


Figure 5.7. Shows the effect of the r.

1. $\sigma = 10, r = 5, \beta = 2.6$ **2.** $\sigma = 10, r = 10, \beta = 2.6$ **3.** $\sigma = 10, r = 25, \beta = 2.6$

Effect of Beta

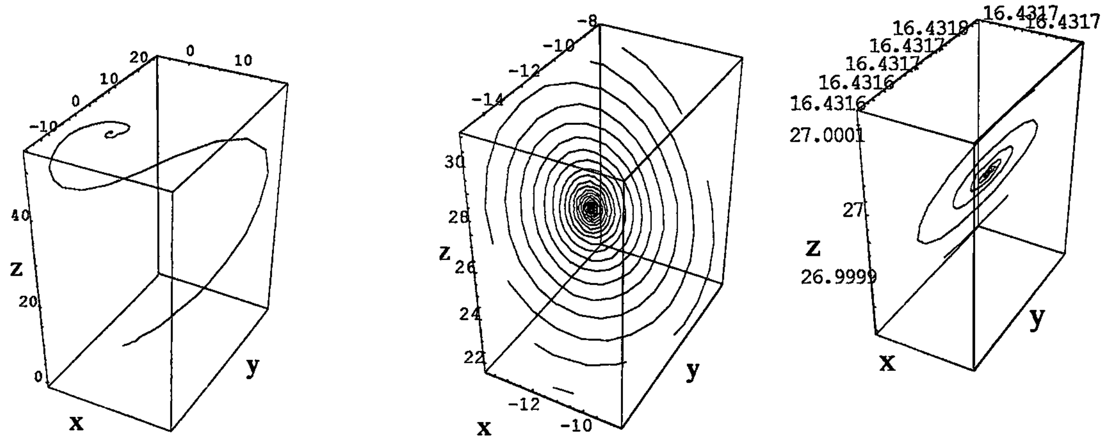


Figure 5.8. Shows the effect of the beta.

1. $\sigma = 10, r = 28, \beta = 0$ **2.** $\sigma = 10, r = 28, \beta = 5$ **3.** $\sigma = 10, r = 28, \beta = 10$

5.5. TOPOLOGICAL ANALYSIS OF CHAOTIC DYNAMICAL SYSTEMS

Topological ideas are present in almost all areas of today's mathematics. The subject of topology itself consists of several different branches, such as point set topology, algebraic topology and differential topology, which have relatively little in common. We shall trace the rise of topological concepts in a number of different situations.

The subject of this part is the analysis of data generated by a dynamical system operating in a chaotic regime. More specifically, this part describes how to extract, from chaotic data, topological invariant that determine the stretching and squeezing mechanisms responsible for generating these chaotic data.

A dynamical system consists of an abstract phase space or state space, whose coordinates describe the dynamical state at any instant; and a dynamical rule which specifies the immediate future trend of all state variables, given only the present values of those same state variables. Mathematically, a dynamical system is described by an initial value problem.

Dynamical systems are "deterministic" if there is a unique consequent to every state, and "stochastic" or random" if there is more than one consequent chosen from some probability distribution (the "perfect" coin toss has two consequent with equal probability for each initial state).

Every differential equation gives rise to a map, the time one map, defined by advancing the flow one unit of time. This map may or may not be useful. If the differential equation contains a term or terms periodic in time, then the time T map (where T is the period) is very useful--it is an example of a Poincaré section. The time T map in a system with periodic terms is also called a stroboscopic map, since we are effectively looking at the location in phase space with a stroboscope tuned to the period T. This map is useful because it permits us to dispense with time as a phase space coordinate: the remaining coordinates describe the state completely so long as we agree to consider the same instant within every period.

The original approach of the study of differential equations involved searches for exact analytic solutions. If they were not available, one attempted to use perturbation theory to approximate the solutions. While this approach is useful for determining explicit solutions, it is not useful for determining the general behaviour predicted by even simple nonlinear dynamical systems. Poincare realized the poverty of this approach over a century ago. His approach involved studying how an ensemble of nearby initial conditions. Poincare's approach to the study of differential equations evolved into the mathematical field we now call topology.

Topological tools are useful for the study of both conservative and dissipative dynamical systems. At present, they can be extended to "low" dimensional dissipative dynamical systems. Using these system to determine the stretching and squeezing mechanisms that build up strange attractors and to determine the properties of these strange attractors.

The dynamic systems that behave chaotically. Chaotic behaviour is defined by two properties:

1) Sensitive to initial conditions and
2) Recurrent non periodic behaviour

Sensitivity to initial conditions means that nearby points in phase space typically repel each other. That is, the distance between the points increases exponentially, at least for a sufficiently small time.

$$d(t) = d(0)e^{\lambda t} \tag{5.33}$$

Here d(t) is the distance separating two points at time t, d(0) is the initial distance separating them at t=0, t is sufficiently small, and the Lyapunov exponent λ is positive. To put it graphically, the two initial conditions are stretched apart.

Lyapunov exponents measure the rate at which nearby orbits converge or diverge. There are as many Lyapunov exponents as there are dimensions in the state space of the system, but the largest is usually the most important. Roughly speaking the (maximal) Lyapunov exponent is the time constant, lambda, in the expression for the distance between two nearby orbits, exp.(lambda * t). If lambda is negative, then the orbits converge in time, and the dynamical system is insensitive to initial conditions. However, if lambda is positive, then the distance between nearby orbits grows exponentially in time, and the system exhibits sensitive dependence on initial conditions.

There are basically two ways to compute Lyapunov exponents. In one way one chooses two nearby points, evolves them in time, measuring the growth rate of the distance between them. This is useful when one has a time series, but has the disadvantage that the growth rate is really not a local effect as the points separate. A better way is to measure the growth rate of tangent vectors to a given orbit.

If two nearby initial conditions diverged from each other exponentially in time for all times, they would eventually wind up at opposite ends of the universe. If motion in phase space is bounded, the two points will eventually reach a maximum separation and then begin to approach each other again. These exhibit, the two initial conditions are then squeezed together.

The stretching and squeezing mechanisms gives us the fractal structure of the attractor and determine the properties of these strange attractor. The strange attractor classify the three groups:

1) Metric invariant which include dimensions of various kinds and multifractal scaling functions
2) Dynamical invariant which include Lyapunov exponents.
3) Topological invariant which generally depend on the periodic orbits that exist in a strange attractor.

5.5.1. Analysis Chaotic Data Sets By Using Laser

In this section analysis the chaotic data sets generated by a laser. The use of lasers as a tested for generating deterministic chaotic signals has two major advantages over fluid systems, which had until that time been the principle source of chaotic data:

1) The time scales intrinsic to a laser (10^{-7} to 10^{-3} sec) are much shorter than the time scales for fluid experiments.
2) Reliable laser models exist in terms of a small number of ordinary differential equations.

We originally studied in detail in laser with modulated losses. A schematic of this laser shown in Fig(5.9). A Kerr cell is placed with in the cavity of a CO_2 gas laser. The electric field within the cavity is polarized by Brewster angle windows. The Kerr cell allows linearly polarized light to pass through it an electric field across the Kerr cell rotates the plane of polarization. As the polarization plane of the Kerr cell is rotated away from the polarization plane established by the Brewster angle windows, controllable losses are introduced into the cavity. If the Kerr cell is periodically modulated, output intensity is also modulated When the modulation amplitude is small, the output modulation is locked to the modulation of the Kerr cell. When the modulation amplitude is sufficiently large and the modulation frequency is comparable to the cavity-relaxation frequency, or one of its sub-harmonics, the laser-output intensity no longer remains locked to the signal driving the Kerr cell, and can even become chaotic.

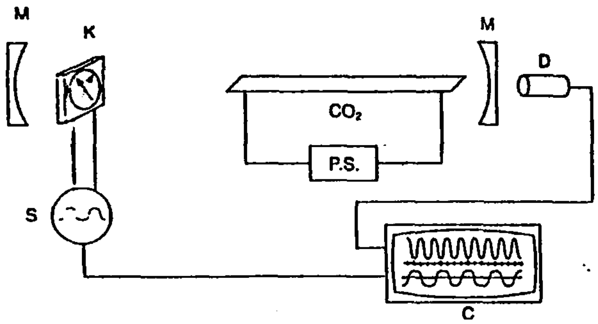


Fig. 5.9:Schematic representation of a laser with modulated losses. CO_2:laser tube containing CO_2 with Brewster windows; M: mirrors forming cavity; P.S.: power source; K: Kerr cell; S: signal generator; D: detector; C: oscilloscope and recorder. A variable electric field across the Kerr cell varies its polarization direction and modulates the electric field amplitude within the cavity.

The rate equations governing the laser intensity S and the population inversion N are

$$\begin{aligned} \frac{dS}{dt} &= -k_0 S[(1-N)+m\cos(wt)] \\ \frac{dN}{dt} &= -\gamma[(N-N_0)+(N_0-1)SN] \end{aligned} \tag{5.34}$$

Here m and w are modulation amplitude and angular frequency, respectively , of the Kerr cell; N_0 is the pump parameter, normalised to $N_0=1$ at the laser threshold ; and k_0 and γ are loss rates. In scaled form, this equation is

$$\begin{aligned} \frac{dU}{d\tau} &= [z - T\cos(\sigma\tau)]u \\ \frac{dz}{d\tau} &= (1-\varepsilon_1 z)-(1+\varepsilon_2 z)u \end{aligned} \tag{5.35}$$

where the scaled variables are,

$$u = S, z = k_0\kappa(N-1), t = \kappa\tau, T = k_0 m, \sigma = w\kappa, \varepsilon_1 = \kappa\gamma,$$
$$\varepsilon_2 = 1/\kappa k_0, \kappa^2 = 1/\gamma k_0 (N_0 - 1) \quad (5.36)$$

A bifurcation diagram for the laser, and the model (5.35), is shown in figure(5.10).The bifurcation diagram provides a nice summary for the transition between different types of motion that can occur as one parameter of the system is varied. This diagram is constructed by varying modulation amplitude T and keeping all other parameters. Before analysis this diagrams, explain the means of the bifurcation. Roughly speaking, a bifurcation is a qualitative change in an attractors structure as a control parameter is smoothly varied. For example, a simple equilibrium, or fixed point attractor, might give way to a periodic oscillation as the stress on a system increases. Similarly, a periodic attractor might become unstable and be replaced by a chaotic attractor. In Benard convection, to take a real world example, heat from the surface of the earth simply conducts its way to the top of the atmosphere until the rate of heat generation at the surface of the earth gets too high. At this point heat conduction breaks down and bodily motion of the air (wind!) sets in. The atmosphere develops pairs of convection cells, one rotating left and the other rotating right. So Bifurcation theory is a method for studying how solutions of a non-linear problem and their stability change as the parameters varies.

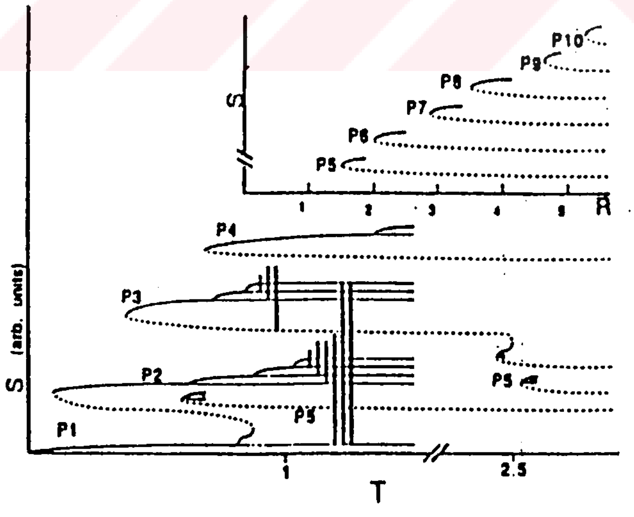


Fig.5.10:Bifurcation diagram for model (33) of the laser with modulated losses, with $\varepsilon_1 = 0.03, \varepsilon_2 = 0.009, \sigma = 1.5$. Stable periodic orbits (solid lines),regular saddles(dashed lines), and strange attractors are shown.

Now, this diagram shows that a period -one solution exists above the laser threshold (N_0>1) for T=0 and remains stable as T is increased until T~0.8. It becomes unstable at T~0.8, with a stable period -two orbit emerging from it in a period-doubling bifurcation. Contrary to what might be expected, this is not the early stage of a period doubling cascade, for the period -two orbit is annihilated at T~0.85 in an inverse saddle node bifurcation with a period-two regular saddle. This saddle node bifurcation destroys the basin of attraction of the period two orbit.

Subharmonics of period n are created in saddle-node bifurcation which is a pair of periodic orbits are created "out of nothing." One of the periodic orbits is always unstable (the saddle), while the other periodic orbit is always stable (the node). This bifurcation at increasing values of T and S (P2 at T~0.1, at T~0.3, P4 at T~0.7, P5 and higher shown in inset.) The evaluation of each sub-harmonics follows a standard scenario as T increases:

1) A saddle node bifurcation creates an unstable saddle and a node which is initially stable.
2) Each node becomes unstable and initiates a periodic doubling cascade as T increases.
3) Beyond accumulation there is a series of noisy orbits of period $nX2^k$ that undergo inverse period-halving bifurcation.

Higher sub-harmonics are generally created at larger values of T. They are created with smaller basin of attraction. Roughly speaking, the larger period orbits exists outside the smaller period orbits

The period doubling accumulation, inverse noisy period halving scenario described above is often interrupted by a crisis of one type or another.

Boundary Crisis: A regular saddle on a period -n branch in the boundary of a basin of attraction surrounding either the period-n node or one its periodic or noisy periodic granddaughter orbits collides with the attractor. The basin is annihilated or enlarged.

Internal Crisis: A flip saddle of period $nX2^k$ in the boundary of a basin surrounding a noisy period $nX2^{k+1}$ orbit collides with attractor to produce a noisy period - halving bifurcation.

External Crisis: A regular saddle of period -n' in the boundary of a period -n ($n' \neq n$) strange attractor collides with the attractor, thereby annihilating or enlarging the basin of attraction.

The bifurcation diagram shown in Fig(5.10). These included direct and inverse saddle node bifurcation, and boundary and external crises. As the laser operating parameters (k_0, γ, σ) change, the bifurcation diagram changes. The sub-harmonics orbits of period n created at increasing T values, there are orbits of period n do not appear to belong to that series of sub-harmonics. The clearest example is the period two orbit, which bifurcates from period one at T~0.8. Another is the period three orbit pair created in a saddle node bifurcation , which occurs at T~2.45.

These analyses shows, there are many coexisting basins of attraction, some containing a periodic attractor, others a strange attractor for using the laser.

CHAPTER 6

CONCLUSIONS

In this thesis, we have obtained eigenvalues and eigenvectors of the several different quantum mechanical problems by using Lie algebraic method. Firstly we have constructed specific differential realization of the SO(2,1) algebra. However, the choice of an appropriate realization of the generators is of greater significance, leading to particularly simple solution whenever the Hamiltonian can be expressed as a linear function of the generator. In particular, the generators were chosen as linear differential operators depending on two variables. This realization was inspired by the work of Alhassid on the potential group approach, in which the Hamiltonian is expressed in terms of the Casimir operator of the potential algebra. Noting that there are formal analogise between this approach and super symmetric quantum mechanics, we applied transformations to derive the Schrödinger equations for the orthogonal polynomial within the framework of the group theory. We have made solution of some orthogonal functions by using Lie algebraic method and have found eigenfunctions which depend on the Hermite polynomial of the Hamiltonian and our results have been compared with the Hermite polynomial. Then we have seen a good agreement with each other. These results are summarised in Table 3.1 and Table 3.3. The algebra derived from this procedure and showed the role of the potentials for a spectrum generating algebra. This study demonstrated the role of similarity transformations, and the connection between the Lie method and other procedures which lead to the diagonalization of very large matrices. Also we have studied a few complex potentials which are invariant under the combined symmetry PT and showed that even in all these cases the energy eigenvalues of the Schrödinger equations are real.

The present work can be generalised in various directions. Here we have considered two possible differential realization of the SO(2,1) generators. Other realization of this algebra are also known. The Hamiltonian will take different forms in different realizations of the generators.

In this thesis, also we have found the solution of the non-linear Heat equation and Lorenz equation. Analytical solution of these type equation are very difficult. For this reason, we have used Runge-Kutta 4 which is one of the method of numerical analysis. Then the resulting data are interpolated by the Mathematica. This results have been found in good agreement with the literature.

REFERENCES

[1] Joseph L., Mc Caurly, **Chaos Dynamics and Fractals,** Cambridge University Press, 1998. Olver P. J., **Application of Lie groups to Differential Equations** Springer, New York, 1986. Martin C. G., **Chaos in Classical and Quantum Mechanics,** Springer Verlag, 1990. Bhagovantom S. And Venkataroyudu T., **Theory of Groups and its Application to Physical Problems,** Academic Press 1996.

[2] Alhassid Y. Fyodorow, Y. V., **The Spectral Autocorrelation Functions in Weakly Open Chaotic Systems,** Preprint in JHEP, Chhao-dyn/9808003, 1998.

[3] Patric L. N., Chen L. Y., **Efficient Finite Difference Solutions to the Time Dependent Schrödinger Equation,** Journal of Comp. Phys. 130, pp. 266-268, 1997.

[4] David M. Fradkin, **Single-Boson Realizations of SO(3) and SO(2,1),** Journal of Physics A:Math. Gen. Volume 27, pp. 1261-1273, 12 October 1993.

[5] R. Dutt, A. Gangopadhyaya, C. Rasinariu and U. Sukhatme, **Coordinate Realizations of Deformed Lie Algebreas with Three Generators,** Physical Review A, Volume 60, Number 5, November 1999.

[6] N. Debergh, **The Relation between Polynomial Deformations of sl(2,R) and Quasi-Exact Solvability,** Journal of Physics A:Math. Gen. Volume 33, pp. 7109-7121, 3 August 2000.

[7] Brian G. W., **Classical Groups for Physicists,** John Willey and Sons, New York, 1974.

[8] James P. S., **Order parameters, Broken Symmetry and Topology,** Preprint in JHEP, cond-mat 9204009, 1992.

[9] M. Kibler and M. Daoud, **An Alternative basis for the Wigner-Racah Algebra of the Group SU(2),** JHEP, 9712034,17 Dec 1997.

[10] B L. Burrows, M. Cohen and Tova Feldman, **A Lie Algebraic Study of Some Schrödinger Equations,** Journal of Mathematical Physics, 35(11), November 1994.

[11] G. Levai, **Solvable Potentials Associated with SU(1,1) Algebreas: A Systematic Study,** Journal of Physics A: Math. Gen. Volume 27, pp. 3809-3828, 1994.

[12] R. A. Ionescu, A. Ludu and W. Scheid, **SU(1,1) algebraic Description of One-Dimensional Potentials within the R-Matrix Theory,** Journal of Physics A:Math. Gen. Volume 29, pp. 3669-3667, 29 January 1996.

[13] Fernando Casas, **Solution of Linear Partial Differential equations by Lie Algebraic Methods,** Journal of Computational and Applied Mathematics, Volume 76, pp. 159-170, 10 February 1996.

[14] M. P. Blencowe, H. F. Jones, and A. P. Korte, **Applying the Linear** δ **Expansion to** $i\phi^3$, JHEP, 9710173, 22 Oct 1997.

[15] C. Brif, A. Vourdas and A. Mann, **Analytical Representation Based on SU(1,1) Coherent States and their Applications.**

[16] C. Brif and Y. Ben-Aryeh, **SU(1,1) Intelligent States: Analytic Representation in the Unit Disk,** Journal of Physics A:Math. Gen. Volume 27, 8185 (1994)

[17] Krzysztof Kowalski and Jakup Rembielinski, **Groups and Non-linear Dynamical Systems,** JHEP,chao-dyn/9801020, 14 Jan 1998.

[18] Elman M. Shahverdiev, **A Generalized Method of Chaos Synchronization in Some Dynamical Systems,** Journal of Physical Society of Japan, Volume 67, Number 6, pp. 2-22 June 1998.

[19] V. Latora and A. Rapisarda, **Chaotic Dynamics and Superdiffusion in a Hamiltonian System with many Degrees of Freedom,** JHEP, cond-mat/9911317,19 Nov 1999.

[20] Konstantin B. Blyyuss, **Chaotic Behaviour of Non-linear waves and Solutions of perturbed Korteweg-de Vries Equation,** JHEP, patt-sol/99040001,9 Feb 2000.

[21] Sang-Yoon Kim, **Non-linear Dynamics of a Damped Magnetic Oscillator,** 5 Feb 1999

[22] Robert Gilmore, **Topological Analysis Of Chaotic Dynamical System,** Reviews of Modern Physics, Volume 70, Number 4, October 1998

[23] Carl M. Bender and Stefan Boettcher, **Real Spectra in Non-Hermition Hamiltonians Having PT Symmetry,** JHEP, physics/9712001, 6 May 1998.

[24] Avinash Khare, **PT Invariant Non-Hermition Potentials with Real QES Eigenvalues,** JHEP,quant-ph/0004019,6 Jul 2000.

[25] Miloslav Znojil, **New Set of Exactly Solvable Complex Potentials Giving the Real Energies,** JHEP,quant-ph/9912079,16 December 1999.

[26] Carl M. Bender and Stefan Boettcher, **Quasi-Exactly Solvable Quartic Potential,** JHEP,physics/9801007,8 Jan 1998.

[27] Miloslav Znojil and Geza Levai, **The Coulomb-Harmonic Oscillator Correspondence in PT Symmetric Quantum Mechanics,** Physics Let. A, Volume 271, pp.327-333, 20 Mar 2000.

[28] B. Bagchi and C. Quesne, **sl(2,C) as a Complex Lie Algebra and the Associated Non-Hermition Hamiltonians with Real Eigenvalues,** JHEP, math-ph/0008020, 10 August 2000.

[29] B. Bagchi F. Cannata and C. Quesne, **PT Symmetric Sextic Potential,** Phys. Let. A, Volume 269, pp. 79-82, 20 Mar 2000.

[30] C. Ray Wylie, Louis C. Barrett, **Advanced Engineering Mathematics,** McGraw-Hill Inc. New York 1995.

APPENDIX

A MATHEMATICA PROGRAM TO CONSTRUCT THE REALIZATION OF SO(2,1) ALGEBRA

The operators $J1_o$, $J1_+$ and $J1_-$ are the usual operators of SO (2, 1) Lie algebra and JC is Casimir operator. The operators are considered as the second order differential operators

$$m = 6;$$

$$J1_o = \text{FullSimplify}\left[\sum_{i=0}^{2} \Gamma_i[x]\, D[f[x], \{x, i\}] \text{ //. st1}\right]$$

$$J1_+ = \text{FullSimplify}\left[\sum_{i=0}^{2} \chi_i[x]\, D[f[x], \{x, i\}] \text{ //. st1}\right]$$

$$J1_- = \text{FullSimplify}\left[\sum_{i=0}^{2} A_i[x]\, D[f[x], \{x, i\}] \text{ //. st1}\right]$$

$$\begin{aligned} JC = \; & \text{Simplify}\Big[\sum_{i=0}^{m} \text{Coefficient}[2 * J1_o, f^{(i)}[x]] * D[J1_o, \{x, i\}] - \\ & \sum_{i=0}^{m} \text{Coefficient}[J1_+, f^{(i)}[x]] * D[J1_-, \{x, i\}] - \\ & \sum_{i=0}^{m} \text{Coefficient}[J1_-, f^{(i)}[x]] * D[J1_+, \{x, i\}]\Big]; \end{aligned}$$

The Casimir operator also should be in the second order differential equation form. Then the coeefficients of the higher order terms should be zero. In the following section the coefficients of the higher order terms are set zero

$$c4 = \text{Simplify}[\text{Coefficient}[JC, f^{(4)}[x]]]$$
$$c3 = \text{Simplify}[\text{Coefficient}[JC, f^{(3)}[x]]]$$
$$c2 = \text{Simplify}[\text{Coefficient}[JC, f^{(2)}[x]]]$$
$$c1 = \text{Simplify}[\text{Coefficient}[JC, f^{(1)}[x]]]$$
$$c0 = \text{Simplify}[\text{Coefficient}[JC, f^{(0)}[x]]]$$

```
FullSimplify[DSolve[{c3 == 0, c4 == 0}, {Γ1[x], Γ2[x]}, x]]
```

In the following section of the program the commutation relations are computed. JOM = $[J1_o, J1_-] + J1_-$, JPM = $[J1_+, J1_-] + 2\,J1_o$ and J0P = $[J1_o, J1_+] - J1_+$.

```
JOM =
 Simplify[Sum[Coefficient[J1o, f^(i)[x]] * D[J1-, {x, i}], {i, 0, m}] -
   Sum[Coefficient[J1-, f^(i)[x]] * D[J1o, {x, i}], {i, 0, m}] + J1-];
JPM =
 Simplify[Sum[Coefficient[J1+, f^(i)[x]] * D[J1-, {x, i}], {i, 0, m}] -
   Sum[Coefficient[J1-, f^(i)[x]] * D[J1+, {x, i}], {i, 0, m}] + 2 * J1o];
J0P =
 Simplify[Sum[Coefficient[J1o, f^(i)[x]] * D[J1+, {x, i}], {i, 0, m}] -
   Sum[Coefficient[J1+, f^(i)[x]] * D[J1o, {x, i}], {i, 0, m}] - J1+];
```

The operator commutations are computed in above section should be zero. By using this property one can obtain a general relation between arbitrary functions $\Gamma_i[x]$, $\Lambda_i[x]$ and $\chi_i[x]$. Note that the relation can be found under some assumptions given in text.

```
c4 = Coefficient[JOP, f^(4)[x]]
c3 = Coefficient[JOP, f^(3)[x]]
c2 = Coefficient[JOP, f^(2)[x]]
c1 = Simplify[Coefficient[JOP, f^(1)[x]]]
c0 = Simplify[Coefficient[JOP, f^(0)[x]]]
d4 = Coefficient[JPM, f^(4)[x]]
d3 = Coefficient[JPM, f^(3)[x]]
d2 = Coefficient[JPM, f^(2)[x]]
d1 = Simplify[Coefficient[JPM, f^(1)[x]]]
d0 = Simplify[Coefficient[JPM, f^(0)[x]]]
e4 = Coefficient[JOM, f^(4)[x]]
e3 = Coefficient[JOM, f^(3)[x]]
e2 = Coefficient[JOM, f^(2)
e1 = Simplify[Coefficient[JOM, f^(1)[
e0 = Simplify[Coefficient[JOM, f^(0)[x]]]
```

This part of the program should be modify by the user.

```
FullSimplify[DSolve[e0 == 0, χ0[x], x]]
FullSimplify[DSolve[e1 == 0, χ1[x], x]]
FullSimplify[DSolve[e2 == 0, χ02[x], x]]
```