MAPPINGS BETWEEN POTENTIALS

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ABSTRACT

MAPPINGS BETWEEN POTENTIALS

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In this work we scrutinize applications of the point canonical transformations (PCT), which have received a lot of attention in the literature recently. The method has been applied to exactly solvable central and non-central potentials together with quasi-exactly solvable potentials. The PCT technique has led to new insights in the studies of physical problems posed in this thesis. The applications allow something of a systematic approach enabling one to recognize the equivalence of superficially unrelated quantum mechanical problems.

Keywords: Point Canonical Transformation, quasi-exactly solvable potentials, exactly solvable central and non-central potentials.

ÖZET

POTANSİYELLER ARASINDAKİ DÖNÜŞÜMLER

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Bu çalışmada, yakın geçmişte literatürde çok sık görülen nokta kuralsal dönüşüm tekniklerinin uygulamaları gözden geçirildi. Metot, kesin ve tam çözümü olan merkezcil ve merkezcil olmayan potansiyellerle birlikte yarı çözümlü diğer potansiyellere uygulandı. Nokta kuralsal dönüşüm tekniği, bu tezde incelenen fizik problemlerine yeni bir bakış açısı getirdi. Yapılan uygulamalar sonucu görünüşte birbiriyle ilgisi bulunmayan kuantum mekaniksel problemlerin, sistematik bir yaklaşımla, aslında tek bir problemin farklı görünüşleri olduğu anlaşıldı.

Anahtar Kelimeler: Nokta kuralsal dönüşüm, yarı çözümlü potansiyeller, kesin çözümü olan merkezcil ve merkezcil olmayan potansiyeller.

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I dedicate this work to my family and my engaged Fatma ÜZGÜN; I wish to thank them for their support and encouragement during the preparation of this thesis work.

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CHAPTER I

INTRODUCTION

Various types of correspondence between the Kepler-Coulomb and the isotropic-oscillator systems have been extensively investigated since the influential works of Levi-Civita early this century [1]. Among the correspondences of interest are mappings that can be constructed between the radial equations of the quantum systems. This subject was initiated over 60 years ago in a paper by Schrödinger [2] addressing the solution of eigenvalue problems by factorization. Schrödinger discovered a connection between the radial equation of the three-dimensional quantum Coulomb problem and the radial equation of an N-dimensional quantum harmonic oscillator. Using a quadratic transformation in the radial coordinate, he showed that the mapping images all the states in the three-dimensional discrete Coulomb spectrum only for oscillators with $N=2$ or 4.

Schrödinger's idea was subsequently rediscovered or investigated by a number of authors, for a review see [3]. An extension relating the radial equations of the Coulomb system and the oscillator problem in the arbitrary dimensions was given in Ref. [4]. A more general mapping for arbitrary dimensions that involves a free parameter was presented in Ref. [5] along with the corresponding mappings to the supersymmetric partners of these systems. All these correspondences involve oscillators in even dimensions, and they incorporate constraints on the allowed range of angular momenta. It is possible in general to map all the states of the N-dimensional Coulomb system into half the states of an N'-dimensional oscillator, where N is greater than one and N' must be even. Recently, it has been also proposed that some restrictions on the dimensions or angular momenta can be removed with the introduction of suitable analytical deformations called quantum defects in one or both systems [6].

Solvable potential problems have played a dual role since the beginnings of quantum mechanics. First, they represented useful aids in modelling realistic physical problems, and second, they offered an interesting field of investigation in their own right. Related to this latter area, the concept of solvability has changed to some extent in recent years. Besides exactly solvable problems, for which the full bound-state energy spectrum and solutions could be given in general analytical form, quasi-exactly solvable (QES) and conditionally exactly solvable (CES) potential classes have also been identified recently, for a recent review see framework [7]. In the case of QES potentials only a finite number of eigenstates can be obtained exactly. While in the case of CES potentials analytical solutions are available only if some (or all) of the potential parameter are fine tuned to specific values.

The technique of changing the independent coordinate has always been a useful tool in the solution of the Schrödinger equation. For one thing, this allows something of a systematic approach, enabling one to recognize the equivalence of superficially unrelated quantum mechanical problems. An area where this can be interesting is one-body motion in central and noncentral potentials. The work presented in this thesis has addressed this old subject, and particular emphasis has been placed on the exactly and quasi-exactly solvable potential types and the mapping procedures between them, which are of great importance in applied physics and chemistry.

In 1971, Natanzon [8] wrote down (what he thought at that time to be) the most general solvable potentials i.e. for which the Schrödinger equation can be reduced to either the hypergeometric or confluent hypergeometric equation. It turns out that most of these potentials are exactly and quasi-exactly solvable potentials. One might ask if one can obtain these solutions from the explicitly solvable shape invariant ones, which is one of the goals of the present work. For investigating this problem posed here, we start with a Schrödinger equation, which is exactly solvable (for example one having a shape invariant potential), and see what happens to this equation under a point canonical transformation. We will show throughout the works presented in this thesis that in order for the Schrödinger equation to be mapped into another Schrödinger equation, there are severe restrictions on the nature of the coordinate transformations, and coordinate transformations which satisfy these restrictions give rise to new exactly/quasi-exactly solvable problems.

In this thesis, by the expertise we gain through the transformations between exactly and quasi-exactly solvable central potentials, we apply the point canonical transformation technique to also non central-potentials. Consequently, we show that non-central potentials having specific properties, which are discussed in this work, can also be mapped within each other. This investigation is very appealing and interesting because the related literature does not cover such applications.

At this stage it is noted that for benchmark calculations, which provide a testing ground for the results obtained via the mapping procedure used through the present work, one needs exact analytical expressions for the energy spectrum and wavefunctions of the system undertaken. For this we consider the powerful supersymmetric quantum mechanical framework [7], which is the application of supersymmetry to the usual quantum mechanics.

Supersymmetry is one of the most powerful idea that was invented by late 20th century physicists. In general terms, supersymmetry is the pairing of bosons and fermions in a unified, or symmetric, fashion. In quantum field theory, supersymmetry pairs bosonic and fermionic fields (into so-called supermultiplets) and allows transformations, which intermix the two. In string theory, the fundamental "strings" postulated to comprise all particles exhibit vibrational patterns, which are identified as "bosonic" or "fermionic". Strings that vibrate in both ways are called superstrings. Physicists have long striven to obtain a unified description of all basic interactions of nature, i.e.strong, electroweak, and gravitational interactions. Several ambitious attempts have been made until now, and it is now widely felt that supersymmetry (SUSY) is a necessary ingredient in any unifying approach [9]. Despite the beauty of the unifying properties of SUSY theories, there has so far been no experimental evidence of SUSY being realized in nature. In other words, it is fact that bosons and fermions are clearly distinguishable and SUSY is apparently broken in our present environment as we mentioned before, one of the important predictions of unbroken SUSY theories is the existence of SUSY partners of quarks, leptons and gauge bosons which have the same masses as their SUSY counterparts. The fact that no such particles however have been seen implies that SUSY must be spontaneously broken. Nevertheless, because the shape invariance condition is an integrability condition, using this condition and the hierarchy of Hamiltonians one can easily obtain the energy eigenvalues and eigenfunctions of any shape invariant potential (SIP) when SUSY is unbroken.

Nevertheless, the SUSY idea has led to new insights in the studies of nuclear physics, condensed matter physics, statistical physics and mathematical physics [10]. In particular, supersymmetric quantum mechanics (SUSYQM), which originally introduced in 1976 and re-discovered in 1981, nowadays attracts much attention .The algebra involved in SUSY is a graded Lie algebra which closes under a combination of commutation and anti-commutation relations.

Through the work carried out in this thesis, we have seen that using the ideas of SUSY and shape invariance, a number of potential problems can be solved algebraically. Most of these potentials are either one-dimensional or are central potentials which are again essentially one-dimensional but on the half line. It may be worthwhile to enquire if one can also algebraically solve some non-central but separable potential problems. As has been shown in one of the sections of the present work the answer to this question is yes. It turns out that the problem is algebraically solvable so long as the separated problems for each of the coordinates belong to the class of SIP.

The plan of the thesis is as follows. The first part of the thesis, Chapter II, is devoted to the transformations between exactly solvable shape invariant potentials, which consists of three sections. The first section of Chapter II discusses explicit point canonical transformations which map twelve types of shape invariant central potentials (which are known to be exactly solvable) into two potential classes. Hypergeometric and confluent hypergeometric functions give the eigenfunctions in these two classes respectively. The second section clarifies that inter-relations also between this two distinct potential family members are also possible with a suitable limiting procedure and redefinition of parameters within the frame of point canonical transformations. Furthermore, in the third section, we develop an algebraic framework to show that a similar mapping procedure exists between a class of non-central potentials. As an illustrative example, we discuss the inter-relation between the generalized Coulomb and oscillatory systems. The second part of the thesis, Chapter III, deals with the transformations between quasi-exactly solvable potentials. As an example, the relationship between a class of singular potentials in arbitrary dimensions is searched. The eigenvalues and eigenfunctions, together with those of the special cases of these potentials, are obtained in N-dimensional space. The explicit dependence of these potentials in higher-dimensional space is discussed within the frame of supersymmetric quantum mechanics, which has not been previously covered in the literature. Finally, concluding remarks, a brief summary of the whole work and an outlook are given in Chapter IV.

CHAPTER 2

MAPPINGS BETWEEN EXACTLY SOLVABLE SYSTEMS

2.1 Mapping of Shape Invariant Potentials Under Point Canonical Transformations

The application of supersymmetry to quantum mechanics [11] has received fresh interest in the problem of obtaining algebraic solutions of exactly solvable non-relativistic potentials. In an interesting paper, Gendenshtein [12] showed that whenever a parametric relation, the so-called 'shape invariance' condition, is satisfied by two supersymmetric partner potentials, the bound state spectra and eigenfunctions can be readily determined by purely algebraic means using factorability of the Hamiltonians. This generalization is in many respects equivalent to the earlier work of Schrödinger [2] and Infeld and Hull [13] . Using the concept of shape invariance, Dutt et al [14] have explicitly worked out the bound state spectra for eleven types of shape invariant potentials. Subsequently, using the operator formalism, Dabrowska et al [15] have shown an elegant way of writing eigenfunctions for all these problems. Recently, Barclay and Maxwell [16] have discussed one more type of shape invariant potentials. (This correspondence to the superpotential $W = A\tan(\alpha x) + B/A$, and is the trigonometric version of the Rosen-Morse potential.)

There also exist other solvable potentials, for which the factorization procedure is not applicable, since they are not shape invariant [17,18]. It has been shown by Cooper et al [18] that many such potentials [for example, the Natanzon potentials [8], can be generated by applying an operator transformation (f-transformation) to shape invariant potentials. This procedure neither preserves shape invariance nor, in general, transforms a potential into its supersymmetric partner potential. In fact, the general method of operator f-transformations yields new solvable potentials and does not depend on supersymmetry. Alternatively, the techniques of supersymmetric quantum mechanics can be used to generate multi-parameter families of solvable potentials, which are strictly isospectral to any given shape invariant potential [19]. The number of solvable families can be yet further enlarged by using the Abraham-Moses and Pursey procedures for deleting and inserting bound states [20] instead of the customary Darboux procedure used in supersymmetric quantum mechanics.

At this stage, it is quite natural to ask whether it is possible to inter-relate the twelve known types of shape invariant potentials among themselves via transformation analogues to the f-transformation. We find that this is indeed the case; the known types of shape invariant potentials can be grouped into two classes in the sense that the potentials in any class can all be mapped to a single potential in that class through point canonical transformations (PCT) [21]. PCT have been studied in the path integral approach to quantum mechanical problems [22,23]. Pak and Sokmen [24] and Inomata [25] suggested that PCT together with a path dependent time transformation (local time transformation) could reduce a few solvable potential kernels to the kernels of either the harmonic oscillator or Scarf potentials [23,26].However, in path integral calculations, the mathematical manoeuvring of steps becomes so complicated due to the combined transformations of space and time variables that the mapping of all shape invariant potentials has not yet been done.

In this chapter, we show that a much simpler approach consists of mapping through canonical transformation of coordinates, which interrelate the Hilbert spaces of various shape invariant potentials. A similar suggestion has been made recently by Junker [27]. The general method of transformation of the time-independent Schrödinger equation into a hypergeometric equation goes back to Manning [28].The method was further studied by other authors [29]. We re-establish the known result that the Coulomb and Morse potentials can be mapped into the three-dimensional harmonic oscillator. These types of potentials form one class. For these class I potentials, the eigenfunctions correspond to confluent hypergeometric functions, which can be written as Laguerre polynomials. Furthermore , using PCT, we show that potentials such as the Rosen-Morse (hyperbolic and trigonometric), Eckart, Pöschl-Teller (I and II),etc. can be mapped into the generalized Scarf potential, and they form a second class. For these class II potentials, the eigenfunctions correspond to hypergeometric functions, which can be written as associated Legendre functions.

After presenting the general formulation of PCT as applied to the Schrödinger equation, we illustrate the procedure with a simple example. We show the steps necessary to connect the energy eigenvalues and eigenfunctions of the hyperbolic Rosen-Morse potential with those of the generalized Scarf potential, which corresponds to the superpotential $W = -A\cot(\alpha x) + B\operatorname{cosec}(\alpha x),\ 0\langle \alpha x \langle \pi,\ A\rangle B$. For all other types of shape invariant potentials, we cite the appropriate transformations of coordinates and the energy eigenvalues and eigenfunctions in tables 1 and 2 (see appendix B). From the tables, one can easily find the sequence of transformations necessary to map any potential of a given class to another one belonging to the same class.

First we give the general PCT which transforms the time-independent Schrödinger equation for a given shape invariant potential $V(\alpha_i;x)$.

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(\alpha_i;x) - E(\alpha_i)\right]\psi(\alpha_i;x) = 0 \qquad (2.1.1)$$

to a corresponding one

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dz^2} + \widetilde{V}(\widetilde{\alpha}_i;z) - \widetilde{E}(\widetilde{\alpha}_i)\right]\widetilde{\psi}(\widetilde{\alpha}_i;z) = 0 \qquad (2.1.2)$$

for which $\widetilde{\psi}_n(\widetilde{\alpha}_i;z)$ and $\widetilde{E}_n(\widetilde{\alpha}_i)$ are assumed to be known for the shape invariant potential $\widetilde{V}(\widetilde{\alpha}_i;z)$ for each state labelled by the quantum number $n = 0,1,2,\ldots$.Here $\{\alpha_i\}$ and $\{\widetilde{\alpha}_i\}$ represent sets of parameters of the original (old) and transformed (new) potentials respectively.

Invoking a transformation of both the independent and dependent variables of the form

$$x = f(z) \qquad \psi(\alpha_i;x) = \upsilon(z)\widetilde{\psi}(\widetilde{\alpha}_i;z) \qquad (2.1.3)$$

equation (2.1.1) becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\tilde{\psi}}{dz^2}-\frac{\hbar^2}{m}\left\{\frac{\upsilon'}{\upsilon}-\frac{f''}{2f'}\right\}\frac{d\tilde{\psi}}{dz}$$

$$+\left[f'^2\{V(\alpha_i;f(z))-E(\alpha_i)\}+\frac{\hbar^2}{2m}\left(\frac{f''\upsilon'}{f'\upsilon}-\frac{\upsilon''}{\upsilon}\right)\right]\tilde{\psi}=0 \qquad (2.1.4)$$

(For the equation (2.1.4) see appendix A)

in which the prime denotes differentiation with respect to the variable z .To remove the first derivative term in (2.1.4) one requires

$$\upsilon(z)=C\sqrt{f'(z)} \qquad (2.1.5)$$

where C is a constant of integration. For a known transformation function f, one then finds the wavefunction of the original problem in terms of the known eigenfunctions

$$\psi(\alpha_i;f(z))=C\sqrt{f'(z)}\tilde{\psi}(\tilde{\alpha}_i;z) \qquad (2.1.6)$$

Once the desired eigenfunction is obtained in terms of the transformed variable, it may easily be expressed in terms of the true one by inverse transformation. Using (2.1.5) and comparing equations (2.1.2) and (2.1.4) term by term, we write

$$\tilde{V}(\tilde{\alpha}_i;z)-\tilde{E}(\tilde{\alpha}_i)=U(\alpha_i;z) \qquad (2.1.7)$$

where

$$U(\alpha_i;z)=f'^2\{V(\alpha_i;f(z))-E(\alpha_i)\}+\frac{\hbar^2}{4m}\left\{\frac{3}{2}\left(\frac{f''}{f'}\right)^2-\frac{f'''}{f'}\right\} \qquad (2.1.8)$$

The transformation function f has to be chosen such that the functional form of $U(\alpha_i;z)$ as given by (2.1.8) is identical to that of the known potential $\tilde{V}(\tilde{\alpha}_i;z)$ .The energy eigenvalues $E(\alpha_i)$ can then be determined from the known values of $\tilde{E}(\tilde{\alpha}_i)$ and the parameters obtained through inverse mapping. Our scheme is in many ways similar to that proposed in [27].

To see how the method works, we consider the Rosen-Morse potential [14].

$$V(A,B,\alpha;x) = A^2 + \frac{B^2}{A^2} + 2B\tanh(\alpha x) - A\left(A + \frac{\alpha\hbar}{\sqrt{2m}}\right)\sec h^2(\alpha x) \qquad (2.1.9)$$

Using the point canonical transformation

$$x \equiv f(z) = \frac{1}{\alpha}\tanh^{-1}(\cos(\alpha z)) \qquad (2.1.10)$$

one obtains from equation (2.1.8) and (2.1.10)

$$U(A,B,\alpha;z) = \left[A^2 + \frac{B^2}{A^2} - \frac{\hbar^2\alpha^2}{8m} - E\right]\operatorname{cosec}^2(\alpha z) + 2B\operatorname{cosec}(\alpha z)\cot(\alpha z)$$

$$-\left[A\left(A + \frac{\alpha\hbar}{\sqrt{2m}}\right) + \frac{\hbar^2\alpha^2}{8m}\right]. \qquad (2.1.11)$$

We now take the known potential to be the generalized Scarf potential [14]

$$\tilde{V}(\tilde{A},\tilde{B},\alpha;z) = -A^2 + \left(\tilde{A}^2 + \tilde{B}^2 - \frac{\tilde{A}\alpha\hbar}{\sqrt{2m}}\right)\operatorname{cosec}^2(\alpha z)$$

$$-\tilde{B}\left(2\tilde{A} - \frac{\alpha\hbar}{\sqrt{2m}}\right)\operatorname{cosec}(\alpha z)\cot(\alpha z) \qquad (2.1.12)$$

for which

$$\tilde{E}_n(\tilde{A},\tilde{B},\alpha) = \left(\tilde{A} + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 - \tilde{A}^2$$

$$\tilde{\psi}_n(\tilde{A},\tilde{B},\alpha;z) = [1-\cos(\alpha z)]^{\frac{\sqrt{2m}}{\hbar}\left(\frac{\tilde{A}-\tilde{B}}{2\alpha}\right)}[1+\cos(\alpha z)]^{\frac{\sqrt{2m}}{\hbar}\left(\frac{\tilde{A}+\tilde{B}}{2\alpha}\right)}$$

$$(2.1.13)$$

$$\times P_n^{\left[\frac{\sqrt{2m}}{\hbar}\left(\frac{\tilde{A}-\tilde{B}}{\alpha}\right)-\frac{1}{2},\ \frac{\sqrt{2m}}{\hbar}\left(\frac{\tilde{A}+\tilde{B}}{\alpha}\right)-\frac{1}{2}\right]}(\cos(\alpha z))$$

are known. Using equation (2.1.12), (2.1.13) and (2.1.14a) in (2.1.7) and comparing like terms we get

$$A^2+\frac{B^2}{A^2}-\frac{\hbar^2\alpha^2}{8m}-E=\tilde{A}^2+\tilde{B}^2-\frac{\tilde{A}\alpha\hbar}{\sqrt{2m}} \qquad (2.1.14a)$$

$$B=-\tilde{B}\left(\tilde{A}-\frac{\alpha\hbar}{2\sqrt{2m}}\right) \qquad (2.1.14b)$$

$$A\left(A+\frac{\alpha\hbar}{\sqrt{2m}}\right)+\frac{\hbar^2\alpha^2}{8m}=\tilde{A}^2+\tilde{E}=\left(\tilde{A}+\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 \qquad (2.1.14c)$$

From equations (2.1.14b) and (2.1.14c) one obtains

$$A=\tilde{A}-\frac{\hbar\alpha}{2\sqrt{2m}}+\frac{n\alpha\hbar}{\sqrt{2m}} \quad \text{and} \quad B=-\left(A-\frac{n\alpha\hbar}{\sqrt{2m}}\right)\tilde{B}=-\left(\tilde{A}-\frac{\hbar\alpha}{2\sqrt{2m}}\right)\tilde{B} \qquad (2.1.15)$$

Equation (2.1.14a) in conjunction with (2.1.15) gives the eigenvalues of the Rosen-Morse potential:

$$E_n(A,B,\alpha)=A^2-\left(A-\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2+B^2\left(\frac{1}{A^2}-\frac{1}{(A-n\alpha\hbar/\sqrt{2m})^2}\right) \qquad (2.1.16)$$

also from equations (2.1.6), (2.1.13) and (2.1.15); the unnormalized eigenfunction is obtained after inverse transformation of the variable

$$\psi_n(A,B,\alpha;x)=[1-\tanh(\alpha x)]^{p/2}[1+\tanh(\alpha x)]^{q/2}\,P_n^{(p,q)}(\tanh(\alpha x)) \qquad (2.1.17)$$

$$\begin{pmatrix}p\\q\end{pmatrix}=\frac{\sqrt{2m}}{\hbar\alpha}A-n\pm\frac{\sqrt{2m}}{\hbar\alpha}\frac{B}{(A-n\alpha\hbar/\sqrt{2m})}$$

The result given in equations (2.1.16) and (2.1.17) are the same as those obtained in [14] and [15] through operator techniques using the condition for shape invariance.

Similar mapping procedures can be followed starting from other types of shape invariant potentials. In Table 1 (see appendix B), we give the mapping functions for the Coulomb and Morse potentials, which may be expressed in terms of the three-dimensional harmonic oscillator. (the one-dimensional harmonic oscillator is the $\ell=0$ special case of the three-dimensional oscillator, and its Hermite polynomial eigenfunctions are easily expressed as confluent hypergeometric functions [30].) In Table 2 (see appendix B), we present results for all other known types of shape invariant potentials like the Eckart, Pöschl-Teller, etc, which can be mapped to the generalized Scarf potential. It is quite evident that the potentials in these two classes correspond to eigenfunctions which are represented by confluent hypergeometric and hypergeometric functions respectively. Finally, it should be mentioned that mappings via point canonical transformations can also be used to interrelate the reflection and transmission coefficients and S-matrices of various types of shape invariant potentials [31]

Let's apply this technique to transform from Morse to Coulomb potential. We firstly must equate both variables of the potentials.

$$y_M = \frac{2\sqrt{2m}Be^{-\alpha x}}{\alpha} \quad \text{(Variable of the Morse potential)},$$

where $\frac{2\sqrt{2}B}{\alpha} = \sqrt{m\omega}$, then above variable take a new form as,

$$y_M = m\omega e^{-\alpha x} \qquad (2.1.18)$$

and the other variable (Variable of the Coulomb potential);

$$y_C = m\omega z \qquad (2.1.19)$$

$$y_M = y_C$$

$$m\omega e^{-\alpha x} = m\omega z$$

$$x = f(z) = -\frac{1}{\alpha}\ln z \qquad (2.1.20)$$

Eq. (2.1.20) is our transformation function. After taking first and second derivative of the transformation function, we replace it into the Eq. (2.1.8). Then one obtains,

$$U(\alpha, z) = \frac{1}{z^2}\left[\frac{1}{\alpha^2}(A^2 - E) - \frac{\hbar^2}{8m}\right] - \frac{2B}{\alpha^2 z}\left(A + \frac{\alpha\hbar}{2\sqrt{2m}}\right) + \frac{B^2}{\alpha^2} \qquad (2.1.21)$$

Now, we take the known potential to be the Coulomb potential

$$\hat{V}(\alpha,z) = -\frac{e^2}{z} + \frac{\ell(\ell+1)\hbar^2}{2mz^2} + \frac{me^4}{2(\ell+1)^2\hbar^2} \tag{2.1.22}$$

$$\hat{E}(\alpha) = \frac{me^4}{2\hbar^2}\left(\frac{1}{(\ell+1)^2} - \frac{1}{(n+\ell+1)^2}\right) \tag{2.1.23}$$

$$\hat{\psi}_n = \left(\frac{2me^2 z}{\hbar(n+\ell+1)}\right)^{\ell+1} \exp\left(-\frac{1}{2}\frac{2me^2 z}{\hbar(n+\ell+1)}\right) \times L_m^{2\ell+1}\left(\frac{2me^2 z}{\hbar(n+\ell+1)}\right) \tag{2.1.24}$$

By using Eq. (2.1.7) we can write,

$$U(\alpha,z) = -\frac{e^2}{z} + \frac{\ell(\ell+1)\hbar^2}{2mz^2} + \frac{me^4}{2\hbar^2(n+\ell+1)^2} \tag{2.1.25}$$

Comparing (2.1.21) and (2.1.25),

$$-\frac{2B}{\alpha^2}\left(A + \frac{\alpha\hbar}{2\sqrt{2m}}\right) = -e^2 \tag{2.1.26a}$$

$$\frac{B^2}{\alpha^2} = \frac{me^4}{2\hbar^2(n+\ell+1)^2} \tag{2.1.26b}$$

$$\frac{1}{\alpha^2}(A^2 - E) - \frac{\hbar^2}{8m} = \frac{\ell(\ell+1)\hbar^2}{2m} \tag{2.1.26c}$$

From equations (2.1.26a) and (2.1.26b) one obtains

$$A = \frac{\alpha\hbar}{\sqrt{2m}}(n+\ell+1/2) \tag{2.1.27}$$

$$B = \frac{\alpha e^2\sqrt{2m}}{2\hbar(n+\ell+1)} \tag{2.1.28}$$

From (2.1.27), we can write

$$\ell = \frac{\sqrt{2m}A}{\alpha\hbar} - (n+1/2) \tag{2.1.29}$$

Conjunction of (2.1.26c) with (2.1.29) gives us the eigenvalues of the Morse potential:

$$E = A^2 - \left[A - \frac{n\alpha\hbar}{\sqrt{2m}}\right]^2 \tag{2.1.30}$$

Also from equations (2.1.6), (2.1.24), (2.1.28) and (2.1.29), the unnormalized eigenfunction is obtained:

$$\psi_n = y^{\left(\sqrt{2m}A/\alpha\hbar - n\right)} \exp(-\tfrac{1}{2}y) \times L_m^{\left(2\sqrt{2m}A/\alpha\hbar - 2n\right)}(y) \tag{2.1.31}$$

where $y = \dfrac{2\sqrt{2m}B}{\alpha} e^{-\alpha x}$.

2.2 Inter-Relations of Solvable Potentials

It is well known that the Natanzon potentials [32] are exactly solvable in non-relativistic quantum mechanics. These potentials are of two types corresponding to whether the Schrödinger equation can be reduced to either a hypergeometric or a confluent hypergeometric equation. Those that lead to a hypergeometric equation (confluent hypergeometric equation) will be called type-I (type-II) potentials. It has been shown [18,33,34] that the members within each class can be mapped into each other by point canonical transformations (PCT); however, members of these two different classes cannot be connected by a PCT. Since a hypergeometric differential equation reduces to a confluent hypergeometric one under appropriate limits, it is reasonable to expect that the potentials of the above mentioned two classes can also be connected by a similar procedure. The purpose of this note is to establish a connection between specifically chosen potentials in each class. A convenient choice is the so called shape invariant potentials [12,35] which form a distinguished class in the sense that their spectra can be determined entirely by an algebraic procedure, akin to that of the harmonic oscillator, without ever referring to the underlying differential equations. We provide a list of mappings that connect shape invariant type-I potentials to type-II potentials.

Before proceeding further, it is worth reviewing point canonical transformations in non-relativistic quantum mechanics. We consider a time-independent Schrödinger equation with a potential function $V(\alpha_i;x)$ that depends upon several parameters α_i (we will use $\hbar = 2m = 1$):

$$\left[-\frac{d^2}{dx^2}+V(\alpha_i;x)-E(\alpha_i)\right]\psi(\alpha_i;x)=0 \qquad (2.2.1)$$

Under a point canonical transformation, which replaces the independent variable x by z $(x=f(z))$ and transforms the wave function $[\psi(\alpha_i;x)=\upsilon(z)\tilde{\psi}(\alpha_i;z)]$, the Schrödinger equation transforms into:

$$-\frac{d^2\tilde{\psi}}{dz^2}-\left\{\frac{2\upsilon'}{\upsilon}-\frac{f''}{f'}\right\}\frac{d\tilde{\psi}}{dz}+\left\{f'^2[V(\alpha_i;f(z))-E(\alpha_i)]+\left(\frac{f''\upsilon'}{f'\upsilon}-\frac{\upsilon'}{\upsilon}\right)\right\}\tilde{\psi}=0$$

Requiring the first derivative term to be absent gives $\upsilon(z)=C\sqrt{f'(z)}$. This then leads to another Schrödinger equation with a new potential.

$$\left[-\frac{d^2}{dz^2}+\left\{f'^2[V(\alpha_i;f(z))-E(\alpha_i)]+\frac{1}{2}\left(\frac{3f''^2}{2f'^2}-\frac{f'''}{f'}\right)\right\}\right]\tilde{\psi}(\alpha_i;z)=0. \qquad (2.2.2)$$

In general this is an eigenvalue equation, unless $\{f'^2[V(\alpha_i;f(z))-E(\alpha_i)]\}$ has a term independent of z, which will act like the energy term for the new Hamiltonian. This condition constraints allowable choices for the function $f(z)$. For a general potential $V(\alpha_i;f(z))$, many choices for $f(z)$ are still possible that would give rise to Schrödinger type eigenvalue equations, and thus, if we have one solvable model, we can generate many others from it.

Ref. [33] contains a list of functions $f(z)$ that relate all shape invariant potentials of type-I (type-II) to the Scarf (harmonic oscillator) potential. In the following we will present two examples where suitable limits take one beyond class barriers, and connect type-I potentials to those of type-II. In particular we shall exhibit the limiting procedures that convert (a) the Scarf potential into the harmonic oscillator potential; and (b) the generalised Pöschl-Teller into either the Morse or the harmonic oscillator potentials. In Table 3 (see appendix C), we provide additional examples of limiting procedures and redefinition of parameters.

2.2.1 Scarf potential to harmonic oscillator:

The Scarf potential, given by

$$V_{Scarf}(x) = -A^2 + (A^2 + B^2 - A\alpha)\sec^2(\alpha x) - B(2A - \alpha)\tan(\alpha x)\sec(\alpha x) \qquad (2.2.3)$$

after shifting the $x \to \left(r - \frac{\pi}{2\alpha}\right)$, and redefinition of parameters

$$A \to \left(\frac{\omega}{\alpha} + \alpha\frac{(\ell+1)}{2}\right), B \to \left(\frac{\omega}{\alpha} - \alpha\frac{(\ell+1)}{2}\right)$$

(2.2.3) takes the new form

$$\mathrm{V(r)} = -\left(\frac{\omega^2}{\alpha^2} + \omega(\ell+1) + \frac{\alpha^2(\ell+1)^2}{4}\right) + \left(\frac{2\omega^2}{\alpha^2} - \omega\right)\left(\frac{1}{1+\cos\alpha r}\right)$$

$$+ \frac{\alpha^2 \ell(\ell+1)}{2}\left(\frac{1}{1-\cos\alpha r}\right) \qquad (2.2.4)$$

and then taking the limit $\alpha \to 0$ Scarf potential goes into the three-dimensional harmonic oscillator potential (HO)

$$V_{HO} = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+1)}{r^2} - \left(\ell + \frac{3}{2}\right)\omega \qquad (2.2.5)$$

Now we do same limiting procedure for the energy eigenvalue

$$E_{Scarf} = (A + n\alpha)^2 - A^2 = 2n\alpha A + n^2\alpha^2 \qquad (2.2.6)$$

$$E_{HO} = \lim_{\alpha \to 0} 2n\alpha\left[\frac{\omega}{\alpha} + \alpha\left(\frac{\ell+1}{2}\right)\right] + n^2\alpha^2$$

$$E_{HO} = 2n\omega \qquad (2.2.7)$$

2.2.2 Generalised Pöschl-Teller potential to Morse:

The generalised Pöschl-Teller potential (GPT)

$$V_{GPT} = A^2 + (A^2 + B^2 + A\alpha)\operatorname{cosech}^2(\alpha r + \beta))$$
$$- B(2A + \alpha)\operatorname{cosech}(\alpha r + \beta)\coth(\alpha r + \beta) \qquad (2.2.8)$$

can be converted into two shape invariant potentials of type-II by taking appropriate limits. One obtains the Morse potential when $B \to \frac{1}{2}Be^{\beta}$, and one takes the limit $\beta \to \infty$. Alternatively, one gets the three dimensional harmonic oscillator potential when [36]

$$r \to x,\ A \to \left(\frac{\omega}{\alpha} - \alpha\frac{(\ell+1)}{2}\right),\ B \to \left(\frac{\omega}{\alpha} + \alpha\frac{(\ell+1)}{2}\right),\ \alpha \to 0,\ \beta \to 0 \qquad (2.2.9)$$

Let's discuss how generalized Pöschl-Teller potential converted to the Morse potential. To do this,

$$\left.\begin{array}{l} A \to A \\ B \to \dfrac{Be^{\beta}}{2} \\ r \to x \\ \beta \to \infty \end{array}\right\} \qquad (2.2.10)$$

Using (2.2.10), equation (2.2.8) take the form,

$$V(x) = A^2 + \left(A^2 + \frac{1}{4}B^2e^{2\beta} + A\alpha\right)\frac{1}{\sinh^2(\alpha x + \beta)} - \frac{1}{2}Be^{\beta}(2A + \alpha)\frac{\cosh(\alpha x + \beta)}{\sinh^2(\alpha x + \beta)}$$

(2.2.11)

and then taking limit of the (2.2.11) we can easily obtain Morse potential

$$V_M(x) = A^2 + B^2e^{-2\alpha x} + 2B(A + \alpha/2)e^{-\alpha x} \qquad (2.2.12)$$

Because the eigenvalue of generalized Pöschl-Teller is independent of β, even we take it's limit it won't be affected and both eigenvalues of the potentials will be same. So,

$$E_{GPT} = E_M = A^2 - (A - n\alpha)^2 \tag{2.2.13}$$

To get the three-dimensional harmonic oscillator potential as we said above will use (2.2.9). Before taking limit our potential comes to form of

$$V(x) = \left(\frac{\omega^2}{\alpha^2} - \omega(\ell+1) + \frac{\alpha^2(\ell+1)^2}{4}\right) + \left(\frac{2\omega^2}{\alpha^2} + \omega\right)\left(-\frac{1}{1+\cosh(\alpha x+\beta)}\right)$$

$$+\frac{\alpha^2\ell(\ell+1)}{2}\left(-\frac{1}{1-\cosh(\alpha x+\beta)}\right) \tag{2.2.14}$$

Now let's take limit of (2.2.14)

$$V(x) = \frac{1}{4}\omega^2 x^2 + \frac{\ell(\ell+1)}{x^2} - \omega\left(\ell + \tfrac{3}{2}\right) \tag{2.2.15}$$

The same limiting procedure must be used for the eigenvalue of the generalized Pöschl-Teller potential to get that of three-dimensional harmonic oscillator.

$$E_{GPT} = A^2 - (A - n\alpha)^2 = 2n\alpha A - n^2\alpha^2 \tag{2.2.16}$$

$$E_{HO} = \lim_{\alpha \to 0} 2n\alpha\left(\frac{\omega}{\alpha} - \alpha\left(\frac{\ell+1}{2}\right)\right) - n^2\alpha$$

$$E_{HO} = 2n\omega \tag{2.2.17}$$

Here it is worth noting that we have given straightforward routes for going from type-I to type-II potentials. Type-I potentials give rise to hypergeometric differential equation, which has three regular singular points. Two of them merge in the limiting procedures stated above, and as expected one gets a confluent hypergeometric equation. The reverse procedure of going from type-II to type-I is not well defined. We also provide a figure with information on different limiting procedures and point canonical transformations that take type-I potentials among each other or reduce them to type-II potentials.

2.3 Mapping of Non-central Potentials Under Point Canonical Transformations

2.3.1 Introduction

In recent years, there has been considerable interest in studying exactly solvable quantum mechanical problems using algebraic approach [37]. In this respect, supersymmetric quantum mechanics (SUSYQM) [7] has been found to be an elegant and useful prescription for obtaining closed analytic expressions both for the energy eigenvalues and eigenfunctions for a large class of one-dimensional (or spherically symmetric three-dimensional) problems. An interesting feature of SUSYQM is that for a shape invariant system [7,13] the entire spectrum can be determined algebraically without ever referring to underlying differential equations.

As has been shown recently [38,39], the idea of supersymmetry and shape invariance can also be used to obtain exact solutions of a wide class of non-central but separable potentials in algebraic fashion. In these works, it emerges that the angular part, as well as the radial part, of the Laplacian of the Schrödinger equation can indeed be dealt with using the idea of shape invariance, hence the radial and the angular pieces of the Schrödinger equation can both be treated within the same framework.

It is well known that the Natanzon potentials [32] are exactly solvable in non-relativistic quantum mechanics. These potentials are known to group into two disjoint classes depending on whether the Schrödinger equation can be reduced to either a hypergeometric or a confluent hypergeometric equation. It has been shown that [18,33,34] the members within each class can be mapped into each other by point canonical transformations (PCT); however members of these two different classes cannot be connected by PCT. Nevertheless, it is reasonable to expect that the potentials of the above mentioned two classes can also be connected by a similar procedure since a hypergeometric differential equation reduces to a confluent hypergeometric one under appropriate limits. Gangopadhyaya and his co-workers have shown [40] that this is indeed the case by establishing a connection between the two classes with appropriate limiting procedures and redefinition of parameters, thereby inter-relating all known solvable central potentials.

At this stage, it is quite natural to ask whether it is also possible to inter-connect non-central potentials among themselves via canonical transformations of coordinates. To the best of our knowledge, the answer of this question or the feasibility of application of PCT to non-central potentials for mapping purposes has not been discussed earlier in the literature. In this respect such an attempt will be interesting. Through this chapter we will show that the problem posed is algebraically solvable and mappings are possible between non-central but separable potentials so long as the separated problems for each of the coordinates belong to the class of shape invariant potentials (SIP).

The whole development is very elegant, appealing, and yet rather simple, so that any student of quantum mechanics should be able to understand and appreciate it. Indeed, we strongly feel that the material presented here can be profitably included in future quantum mechanics courses and textbooks. Accordingly, we have kept this chapter at a pedagogical level and made it as self-contained as possible. In the following section, we review briefly PCT in non-relativistic quantum mechanics. Section 3 explains how the results for the known SIP may be used to inter-relate two super-integrable systems: the generalized Coulomb and oscillator systems. Some concluding remarks are given in the last section. Throughout the present work the natural units $\hbar = 2m = 1$ are used.

2.3.2. Operator transformation

We consider a time-independent Schrödinger equation with a shape invariant potential $V(\alpha_i;x)$ that may depend upon several parameters α_i

$$\left[-\frac{d^2}{dx^2}+V(\alpha_i;x)-E(\alpha_i)\right]\psi(\alpha_i;x)=0\,. \tag{2.3.1}$$

Under a point canonical transformation, which replaces the independent variable x by $z\,(x=f(z))$ and transforms the wavefunction $[\psi(\alpha_i;x)=v(z)\tilde{\psi}(\alpha_i;z)]$, the Schrödinger equation transforms into:

$$\left[-\frac{d^2}{dz^2}+\widetilde{V}(\widetilde{\alpha}_i;z)-E(\widetilde{\alpha}_i)\right]\widetilde{\psi}(\widetilde{\alpha}_i;z)=-\frac{d^2\widetilde{\psi}}{dz^2}-\left(\frac{2v'}{v}-\frac{f''}{f'}\right)\frac{d\widetilde{\psi}}{dz}$$

$$+\left\{f'^2\left[V(\alpha_i;f(z))-E(\alpha_i)\right]+\left(\frac{f''v'}{f'v}-\frac{v''}{v}\right)\right\}\widetilde{\psi}=0, \qquad (2.3.2)$$

in which $\widetilde{\alpha}_i$ represents sets of parameters of the transformed potentials, and the prime denotes differantion with respect to the variable z. To remove the first derivative term in (2.3.2) for the purpose of having a Schrödinger like equation, one requires $v(z)=c\sqrt{f'(z)}$ where c is a constant of integration. This then leads to another Schrödinger equation with a new potential,

$$\left[-\frac{d^2}{dz^2}+U(\alpha_i;z)\right]\widetilde{\psi}(\widetilde{\alpha}_i;z)=0, \qquad (2.3.3)$$

where

$$U(\alpha_i;z)=f'^2\left[V(\alpha_i;f(z))-E(\alpha_i)\right]+\frac{1}{2}\left(\frac{3f''^2}{2f'^2}-\frac{f'''}{f'}\right)=\widetilde{V}(\widetilde{\alpha}_i;z)-\widetilde{E}(\widetilde{\alpha}_i). \qquad (2.3.4)$$

In general, this is not an eigenvalue equation, unless $\left\{f'^2\left[V(\alpha_i;f(z))-E(\alpha_i)\right]\right\}$ has a term independent of z, which will act like the energy term for the new Hamiltonian. This condition constraints allowable choices for the function $f(z)$. For a general potential $V(\alpha_i;f(z))$, many choices for $f(z)$ are still possible that would give rise to Schrödinger type eigenvalue equations, and thus, if we have one solvable model, we can generate many others from it.

More precisely, the transformation function $f(z)$ has to be chosen such that the functional form of $U(\alpha_i;z)$ as given by (2.3.4) is identical to that of the well-known exactly solvable SIP. This is indeed the case if

$$f'^2\left[V(\alpha_i;f(z))-E(\alpha_i)\right]=V_{SIP}(\alpha_i;z)-\frac{1}{2}\left(\frac{3f''^2}{2f'^2}-\frac{f'''}{f'}\right), \qquad (2.3.5)$$

where $V_{SIP}(\alpha_i; z)$ is a member of the shape invariant potential family. Ref. [33] contains a list of functions $f(z)$, hence one can easily find the sequence of transformations necessary to map any shape invariant potential of a given class to another one belonging to the same class.

We note that PCT have been studied in the path integral approach to quantum mechanical problems [22-25]. However, in path integral calculations, the mathematical manoeuvring of steps becomes so complicated due to the combined transformations of space and time variables that the mapping of all SIP has not yet been done. In this respect, the operator transformation introduced by Ref. [33], and reviewed above, is a much simpler approach consists of mapping through canonical transformations of coordinates which inter-relate the Hilbert spaces of various SIP.

2.3.3 Inter-relations of solvable non-central potentials

Non-central potentials are normally not discussed in most textbooks on quantum mechanics. This is presumably because most of them are not analytically solvable. However, it is worth noting that there is a class of non-central potentials in three-dimensions for which the Schrödinger equation separable. This section deals with the link between two systems, so-called super-integrable systems, involving a generalized form of such potentials: a system known in quantum chemistry as the Hartmann system and a system of potential use in quantum chemistry and nuclear physics. Both systems correspond to ring-shaped potentials. They admit two maximally super integrable systems as the limiting cases: the Coulomb-Kepler system and the isotropic harmonic oscillator system in three-dimensions. Three-dimensional potentials that are singular along curves have received a great deal of attention in recent years. In particular, the Coulombic ring-shaped potential [41] reviewed in quantum chemistry by Hartmann and co-workers [42], and the oscillatory ring-shaped potential [43], systematically studied by Quesne [44], have been investigated from a quantum mechanical viewpoint by using various approaches. As ring-shaped systems they may play an important role in all situations where axial symmetry is relevant. For example, the Coulomb-Kepler system is of interest for ring-shaped molecules like cyclic polyenes [42]. Further, the harmonic oscillator system is of potential use in the study of (super-) deformed nuclei.

2.3.3.1. The generalized Coulomb system

For the sake of clarity, and to demonstrate the simplicity of the present approach, we review the algebraic framework to solve exactly the generalized Coulomb system studied in a recent work [39]. This will make clear that how the results for the known SIP may be used to algebraically obtain in a closed form the eigenvalues for a non-central but separable potential. In addition, as the initial potential to be mapped is the generalized Coulomb system here, this review would also be very useful in understanding further the mapping of the generalized Coulomb system to the oscillatory system having a non-central potential discussed in the next section.

The Coulombic ring-shaped, or Hartmann, potential (energy) is

$$V=-Z\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}+\frac{1}{2}Q\frac{1}{x_1^2+x_2^2} \quad Z>0 \quad Q>0, \qquad (2.3.6)$$

where $Z=\eta\sigma^2$ and $Q=q\eta^2\sigma^2$ in the notation of Kibler and Negadi [41] and of Hartmann [42]. Such an $O(2)$ invariant potential reduces to an attractive Coulomb potential in the limiting case $Q=0$ and this will prove useful for checking purposes. Clearly, Eq. (2.3.6) is a special case of the potential (in spherical coordinates)

$$V_{GC}(r,\theta)=\frac{A}{r}+\frac{B}{r^2\sin^2\theta}+C\frac{\cos\theta}{r^2\sin^2\theta}\ , \qquad (2.3.7)$$

introduced by Makarov *et al.* [45]. The importance of the potential in (2.3.7) lies on the fact that compound Coulomb plus Aharanov-Bohm potential [46] and Hartmann ring-shaped potential, originally proposed as model for the benzene molecule are mathematically linked to this potential. In fact the energy spectrum for these two potentials can be obtained directly [39] by considering these as a special case of the general non-central potential in (2.3.7).

The Schrödinger equation in spherical polar coordinates for a particle in the presence of a potential $V(r,\theta)$ can be reduced to the two ordinary differential equations,

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(E-\frac{A}{r}-\frac{\ell(\ell+1)}{r^2}\right)R=0, \qquad (2.3.8)$$

$$\frac{d^2P}{d\theta^2}+\cot\theta\frac{dP}{d\theta}+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}-\frac{(B+C\cos\theta)}{\sin^2\theta}\right]P=0, \qquad (2.3.9)$$

(see appendix D)

if the corresponding total wave function can be written as $\Psi(r,\theta,\vartheta)=R(r)P(\theta)e^{im\vartheta}$. It is not difficult to see that Eq. (2.3.8) is the same we obtain in solving the problem of an electron in a Coulomb-like field. Bearing in mind the discussion given in the previous section and using the transformation $\theta\to z$ through a mapping function $\theta=f(z)$, one obtains

$$\frac{d^2P}{dz^2}+\left[-\frac{f''}{f'}+f'\cot f\right]\frac{dP}{dz}+f'^2\left[\ell(\ell+1)-\frac{m^2}{\sin^2 f}-\frac{(B+C\cos f)}{\sin^2 f}\right]P=0 \qquad (2.3.10)$$

(see appendix D)

which seems a Schrödinger-like equation if $\frac{f''}{f'}=f'\cot f$ that leads to

$$\theta\equiv f=2\tan^{-1}(e^z)\ ,\ \sin\theta=\operatorname{sec}hz\ \ ,\ \ \cos\theta=-\tanh z. \qquad (2.3.11)$$

Eq. (2.3.9) now reads

$$\frac{d^2P}{dz^2}+\left[\ell(\ell+1)\operatorname{sec}h^2z+C\tanh z\right]P=(m^2+B)P, \qquad (2.3.12)$$

which can be rearranged as

$$-\frac{d^2P}{dz^2}+\left[\lambda^2\tanh^2z-C\tanh z\right]P=\left[\lambda^2-(m^2+B)\right]P, \qquad (2.3.13)$$

where $\lambda^2 = \ell(\ell+1)$. The full potential in (2.3.13) has the form of the Rosen-Morse-II potential, which is well known to be, shape invariant. More specifically, the potential

$$V_{R-M}(z) = a_0(a_0+1)\tanh^2 z + 2b_0 \tanh z, \qquad (b_0 \prec a_0{}^2) \qquad (2.3.14)$$

has energy eigenvalues [42]

$$E_n = a_0(a_0+1) - (a_0-n)^2 - \frac{b_0{}^2}{(a_0-n)^2} \qquad n = 0,1,2,... \qquad (2.3.15)$$

For our case $E_n = \lambda^2 - (m^2 + B)$ and using the corresponding constants $a_0 = \ell$, $b_0 = -\frac{C}{2}$ we obtain

$$\ell = n + \left[\frac{\left(m^2+B\right) + \sqrt{\left(m^2+B\right)^2 - C^2}}{2}\right]^{1/2}. \qquad (2.3.16)$$

The energy eigenvalues obtained from (2.3.8) for the Coulomb potential ($A = -Ze^2$) are

$$E_N = \frac{-Z^2e^4}{4[N+\ell+1]^2} \qquad N = 0,1,2,... \qquad (2.3.17)$$

Therefore our final eigenvalues for a bound electron in a Coulomb potential as well a combination of a non-central potential given by (2.3.9) are

$$E_N = \frac{-Z^2e^4}{4\left\{N+n+1+\left[\frac{\left(m^2+B\right)+\sqrt{\left(m^2+B\right)^2-C^2}}{2}\right]^{1/2}\right\}^2}. \qquad (2.3.18)$$

2.3.3.2 Mappings between the two distinct systems

Now, we are ready to illustrate the mapping procedure by starting off with the system described in (2.3.7). We show here the steps necessary to relate the generalized Coulomb system discussed above to a system having a generalized oscillatory potential, so called the generalized Aharonov-Bohm plus oscillator systems,

$$V_{GHO}(r,\theta)=\widetilde{A}r^2+\frac{\widetilde{B}}{(r\cos\theta)^2}+\frac{\widetilde{C}}{(r\sin\theta)^2}, \tag{2.3.19}$$

where $\widetilde{A},\widetilde{B},\widetilde{C}$ being strictly positive constants. The potential above is of the V_3 in the V_1-V_4 classification by Makarov and collaborators [45]. The limiting case $\widetilde{B}=0,\ \ \widetilde{C}=0$ corresponds to an isotropic harmonic oscillator and will serve for testing results to be obtained. In case when $\widetilde{B}=0$ we get the well-known ring-shape oscillator potential which was investigated in many studies.

The strategy followed is to start the transformation with appropriate Schrödinger equations, which must be exactly solvable having a shape invariant potential, explicitly these are Eqs. (2.3.8) and (2.3.13), and to see what happens to these equations under a point canonical transformation. In order for the Schrödinger equation to be mapped into another Schrödinger equation, there are severe restrictions on the nature of the coordinate transformation. Coordinate transformations, which satisfy these restrictions, give rise to new solvable problems. When the relationship between coordinates is implicit, then the new solutions are only implicitly determined, while if the relationship is explicit then the newly found solvable potentials are also shape invariant which will be the case in the present work.

As the first step, we proceed with the transformation of the well-known system in (2.3.8) that corresponds to the central portion of the non-central potential in (2.3.7). For convenience, we will call our initial coordinates r and our final coordinates z. Using the point canonical transformation $r=f(z)=z^2$, one readily obtains from Eq. (2.3.4)

$$U_1 = -4Ze^2 + \frac{16\ell(\ell+1)+3}{4z^2} - (4E_N)z^2, \tag{2.3.20}$$

where E_N is the energy eigenvalue in (2.3.17) corresponding the initial shape invariant Coulomb potential. As the angular momentum barrier term in (2.3.20) can algebraically be expressed in the form

$$\frac{16\ell(\ell+1)+3}{4z^2} = \frac{\tilde{\ell}(\tilde{\ell}+1)}{z^2} \quad , \quad \tilde{\ell} = 2\ell + \frac{1}{2}, \tag{2.3.21}$$

the consideration of Eq. (2.3.20) together with the right hand side of Eq. (2.3.4) yields

$$U_1 = \frac{w^2 z^2}{4} + \frac{\tilde{\ell}(\tilde{\ell}+1)}{z^2} - 2w(N+\ell+1) = \tilde{V}_1 - \tilde{E}_1 , \tag{2.3.22}$$

where the relation between the original and transformed potential parameters is $w = 2Ze^2/(N+\ell+1)$. It is obvious that the transformed new potential, which represents the first term in (2.3.19) where $\tilde{A} = w^2/4$, is the shape invariant isotropic harmonic oscillator potential

$$\tilde{V}_1 = \tilde{A}z^2 + \frac{\tilde{\ell}(\tilde{\ell}+1)}{z^2}, \tag{2.3.23}$$

and the corresponding energy eigenvalues are

$$\tilde{E}_1 = w\left(2N + \tilde{\ell} + \frac{3}{2}\right). \tag{2.3.24}$$

The next step is to transform the system in (2.3.13), which is identical to (2.3.9) involving the non-central part of the potential in (2.3.7). This mapping will enable us to see clearly the final form of the transformed (new) non-central potential, and to derive an explicit expression (like Eq. (2.3.16) for $\tilde{\ell}$ appeared in (2.3.24) in terms of the transformed potential parameters of the non-central portion.

Following a similar algebraic treatment as above, and using a proper transformation function

$$z \equiv (\tilde{z}) = \tanh^{-1}(\cosh 2\tilde{z}), \qquad (2.3.25)$$

where z and $\tilde{z}$ denote the initial and final coordinates respectively, one can readily transform the radial Schrödinger equation in (2.3.13) and obtain from Eqs. (2.3.25) and (2.3.4)

$$U_2 = (2\ell+1)^2 - \frac{\left(m^2+B+C-\frac{1}{4}\right)}{\cosh^2\tilde{z}} + \frac{\left(m^2+B-C-\frac{1}{4}\right)}{\sinh^2\tilde{z}} = \tilde{V}_2 - \tilde{E}_2 . \qquad (2.3.26)$$

In the above equation, the full potential $\tilde{V}(\tilde{z})$ resembles the shape invariant Pöschl-Teller II type potential

$$V_{PT}(\tilde{z}) = -\tilde{a}_0(\tilde{a}_0+1)\mathrm{sec}h^2\tilde{z} + \tilde{b}_0(\tilde{b}_0-1)\mathrm{cosec}h^2\tilde{z}, \qquad (2.3.27)$$

with the eigenenergies

$$E_{PT} = -\left(\tilde{a}_0 - \tilde{b}_0 - 2n\right)^2 . \qquad (2.3.28)$$

Comparing the similar terms in Eqs. (2.3.26-2.3.28), and bearing in mind the relation $\tilde{\ell} = 2\ell + 1/2$ from (2.3.21), one finds

$$\tilde{a}_0 = -\frac{1}{2} + \sqrt{m^2+B+C} \quad , \quad \tilde{b}_0 = \frac{1}{2} \pm \sqrt{m^2+B-C} \quad ,$$

$$\tilde{a}_0 - \tilde{b}_0 - 2n = \tilde{\ell} + \frac{1}{2} \quad \Rightarrow \quad \tilde{\ell} = 2n + \frac{1}{2} \pm \sqrt{m^2+B-C} + \sqrt{m^2+B+C} \qquad (2.3.29)$$

Thus, Eq. (2.3.24) reads

$$\tilde{E} = w\left(2N + 2n + 2 \pm \sqrt{m^2+B-C} + \sqrt{m^2+B+C}\right), \qquad (2.3.30)$$

which are the eigenvalues of the transformed non-central potential in (2.3.19) where

$$\widetilde{B} = m^2 + B - C - \frac{1}{4} \quad , \quad \widetilde{C} = \widetilde{\ell} + \frac{1}{2} - m^2 \ . \tag{2.3.31}$$

The final form of the transformed potential, Eq. (2.3.19), can easily be seen by the use of inverse transformation of the potential in (2.3.26) via the mapping functions in Eq.(2.3.11). Eq. (2.3.30), which agrees with Eq. (2.3.3) of Ref. [47] and Eq. (2.3.27) of Ref. [39], shows that for each quantum number (N, n) we have two levels for $B \neq 0$ case and one level for $B = 0$ case. The two parts of the energy spectrum for the $\pm$ signs correspond to odd (for +) and even (for -) solutions. In other words, the eigenvalues of the generalized oscillator do not restrict to the eigenvalues of the ring shape oscillator.

We finally remark that although here we have only focussed on eigenvalues and spherical polar coordinates, generalization of the technique used to describe eigenfunctions of the present systems in analytical form (using the discussion in section 2 and the analytical forms of the unnormalized wave functions in Refs. [33,39], and to other non-central potentials in any orthogonal curvilinear coordinate system is quite straightforward.

2.3.4 Concluding Remarks

In this chapter the Schrödinger equation with a class of non-central but separable potentials has been studied and we have shown that such potentials can be easily inter-related among themselves within the framework of point canonical coordinate transformations, as the corresponding eigenvalues may be written down in a closed form algebraically using the well known results for the shape invariant potentials. Although the literature covered similar problems, to our knowledge an investigation such as the one we have discussed in this chapter was missing. It is quite obvious that similar mapping procedures can be followed starting from other types of shape invariant potentials, which may lead to solve analytically other complicated systems involving different kind of non-central potentials. With the above considerations the researchers hope to stimulate further examples of applications of the present method in important problems of physics.

CHAPTER 3

MAPPINGS BETWEEN QUASI-EXACTLY SOLVABLE SYSTEMS

3.1 Introduction

Singular potentials have attracted much attention in recent years for a variety of reasons, two of them being that (i) the ordinary perturbation theory fails badly for such potentials, and (ii) in physics, one often encounters phenomenological potentials that are strongly singular at the origin such as certain type of nucleon-nucleon potentials, singular models of fields in zero dimensions, etc. Thus a study of such potentials is of interest, both from the fundamental and applied point of view.

One of the challenging problems in non-relativistic quantum mechanics is to find exact solutions to the Schrödinger equation for potentials that can be used in different fields of physics. Recently, several authors obtained exact solutions for the fourth-order inverse-power potential

$$V_1(r)=\frac{A_1}{r}+\frac{A_2}{r^2}+\frac{A_3}{r^3}+\frac{A_4}{r^4} \qquad (3.1.1)$$

using analytical methods [48-50]. These methods yield exact solutions for a single state only for a potential of type (1) with restrictions on the coupling constants. The interest is mainly due to the wide applicability of these type inverse-power potentials. Some areas of interest are ion-atom scattering [51], several interactions between the atoms [52], low-energy physics [53], interatomic interactions in molecular physics [54] and solid-state physics [55].

The advent of supersymmetry has had a significant impact on theoretical physics in a number of distinct disciplines. One subfield that has been receiving much attention is supersymmetric quantum mechanics [10] in which the Hamiltonians of distinct systems are related by a supersymmetry algebra. In this work, we are concerned with, via supersymmetric quantum mechanics, clarifying the relationship between two distinct systems having an interaction potential of type (1) and interacting through

$$V_2(r) = B_1 r^2 + \frac{B_2}{r^2} + \frac{B_3}{r^4} + \frac{B_4}{r^6} \tag{3.1.2}$$

singular even-power potentials, which have been widely used in a variety of fields, e.g. see [53,56]. In recent years, the higher order anharmonic potentials have drawn more attentions of physicists and mathematicians in order to partly understand a newly discovered phenomena such as the structural phase transitions [57], the polaron formation in solids [58], the concept of false vacuo in field theory [59], fibre optics [60], and molecular physics [61]. In addition, some 60 years ago Michels *et al.* [62] proposed the idea of simulating the effect of pressure on an atom by enclosing it in a impenetrable spherical box. Since that time there have been a large number of publications, for an overview see [63], dealing with studies on quantum systems enclosed in boxes, which involve an interaction potential that is a special case ($B_2 = 0$) of (2). This field has received added impetus in recent years because of the fabrication of semiconductor quantum dots [64].

The main motivation behind this work is to reveal the existence of a link between potentials of type (1) and (2) in $N-$dimensional space, and between their special cases such as a Mie-type potential (or Kratzer) [65] and pseudoharmonic-like (or Goldman-Krivchenkov) potential [66] in higher dimensions, which to our knowledge has never been appeared in the literature. On the other hand, with the advent of growth technique for the realization of the semiconductor quantum wells, the quantum mechanics of low-dimensional systems has become a major research field. The work presented in this chapter would also be helpful to the literature in this respect as the results can readily be extended to lower dimensions as well.

3.2 The Schrödinger Equation In *N*-Dimensional Space

It is well known that the general framework of the non-relativistic quantum mechanics is by now well understood and its predictions have been carefully proved against observations. Physics is permanently developing in a tight interplay with mathematics. It is of importance to know therefore whether some familiar problems are a particular case of a more general scheme or to search if a map between the radial equations of two different systems exists. It is hence worthwhile to study the Schrödinger equation in the arbitrary dimensional spaces which has attracted much more attention to many authors. Many efforts have in particular been produced in the literature over several decades to study the stationary Schrödinger equation in various dimensions with a central potential containing negative powers of the radial coordinates [67, and the references therein].

The radial Schrödinger equation for a spherically symmetric potential in *N*-dimensional space (we shall use through this chapter the natural units such that $\hbar = m = 1$)

$$-\frac{1}{2}\left[\frac{d^2R}{dr^2}+\frac{N-1}{r}\frac{dR}{dr}\right]+\frac{\ell(\ell+N-2)}{2r^2}R=\left[E-V(r)\right]R \tag{3.2.1}$$

is transformed to

$$-\frac{d^2\psi}{dr^2}+\left[\frac{(M-1)(M-3)}{4r^2}+2V(r)\right]\psi=2E\psi \tag{3.2.2}$$

where ψ, the reduced radial wave function, is defined by,

$$\psi(r)=r^{(N-1)/2}\,R(r) \tag{3.2.3}$$

and

$$M=N+2\ell \tag{3.2.4}$$

(see appendix E)

Eq. (3.2.2) can also be written as

$$-\frac{1}{2}\frac{d^2\psi}{dr^2}+\left[\frac{\Lambda(\Lambda+1)}{2r^2}+V(r)\right]\psi=E\psi \tag{3.2.5}$$

where $\Lambda=(M-3)/2$. We see that the radial Schrödinger equation in N dimensions has the same form as the three-dimensional one. Consequently, given that the potential has the same form in any dimension, the solution in three dimensions can be used to obtain the solution in any dimension simply by using the substitution $\ell \to \Lambda$. It should be noted that N and ℓ enter into expressions (3.2.2) and (3.2.50) in the form of the combinations $N+2\ell$. Hence, the solutions for a particular central potential $V(r)$ are the same as long as $M(=N+2\ell)$ remains unaltered. Therefore the $s-$wave eigensolutions $(\psi_{\ell=0})$ and eigenvalues in four-dimensional space are identical to the p-wave solutions $(\psi_\ell=1)$ in two-dimensions.

The technique of changing the independent coordinate has always been useful tool in the solution of the Schrödinger equation. For instance, this allows something of a systematic approach enabling to recognize the equivalence of superficially unrelated quantum mechanical problems. Many recent papers have adressed this old subject. In the light of these works we proceed by substituting $r=\alpha\rho^2/2$ and $R=F(\rho)/\rho^\lambda$, λ an integer, suggested by the known transformations between Coulomb and harmonic oscillator problems [2,3,5,68,69] and used to show the relation between the perturbed Coulomb problem and the sextic anharmonic oscillator in arbitrary dimensions [70,71], we transform Eq. (3.2.1) to another Schrödinger-like equation in $N'=2N-2-2\lambda$ dimensional space with angular momentum $L=2\ell+\lambda$,

$$-\frac{1}{2}\left[\frac{d^2F}{d\rho^2}+\frac{N'-1}{\rho}\frac{dF}{d\rho}\right]+\frac{L(L+N'-2)}{2\rho^2}F=\left[\hat{E}-\hat{V}(\rho)\right]F \tag{3.2.6}$$

where

$$\hat{E}-\hat{V}(\rho)=E\alpha^2\rho^2-\alpha^2\rho^2V(\alpha\rho^2/2) \tag{3.2.7}$$

(see appendix E)

and α is a parameter to be adjusted properly. Note that leaving re-scaling constant α arbitrary for now gives us an additional degree of freedom. When we discuss bound state eigenvalues later, we can use this to allow the values of the potential coefficients to be completely independent of each other. Thus, by this transformation, in general, the N-dimensional radial wave Schrödinger equation with angular momentum ℓ can be transformed to a $N' = 2N - 2 - 2\lambda$ dimensional equation with angular momentum $L = 2\ell + \lambda$. If we choose $\alpha^2 = 1/|E|$, with E corresponding the eigenvalue for the inverse power potential of Eq. (3.1.1), then Eq. (3.2.6) corresponds to the Schrödinger equation of a singular even-power potential

$$\hat{V}(\rho) = \rho^2 + \frac{4A_2}{\rho^2} + \frac{8A_3}{\rho^4}|E|^{1/2} + \frac{16A_4}{\rho^6}|E| \tag{3.2.8}$$

with eigenvalue

$$\hat{E} = \frac{-2A_1}{|E|^{1/2}} \tag{3.2.9}$$

Thus, the system given by Eq. (3.1.1) in N-dimensional space is reduced to another system defined by Eq. (3.1.2) in $N' = 2N - 2 - 2\lambda$ dimensional space. In other words, by changing the independent variable in the radial Schrödinger equation, we have been able to demonstrate a close equivalence between singular potentials of type (1) and (2). Note that when $N = 3$ and $\lambda = 0$ one finds $N' = 4$, and when $\lambda = 1$ we get $N' = 2$. It is also easy to see that $N' + 2L$ does not depend on λ, which leads to map two distinct problems in three- and four-dimensional space [71].

3.3 Mappings between Two Distinct Systems

A. Quasi-Exactly Solvable Case

Since Eq. (3.2.2) for the reduced radial wave $\psi(r)$ in the N-dimensional space has the structure of the one-dimensional Schrödinger equation for a spherically symmetric potential $V(r)$, we may define the supersymmetric partner potentials [10]

$$V_{\pm}(r) = W^2(r) \pm W'(r) \tag{3.3.1}$$

which has a zero-energy solution, and the corresponding eigenfunction is given by

$$\psi_{n=0}(r) \propto \exp\left[\pm \int^r W(r)\,dr\right] \tag{3.3.2}$$

In constructing these potentials one should be careful about the behaviour of the wave function $\psi(r)$ near $r = 0$ and $r \to \infty$. It may be mentioned that $\psi(r)$ behaves like $r^{(M-1)/2}$ near $r = 0$ and it should be normalizable. For the inverse power potential of Eq. (3.1.1) we set

$$W(r) = \frac{-a}{r^2} + \frac{c}{r} - b, \quad b, c \succ 0 \tag{3.3.3}$$

and identify $V_+(r)$ with the effective potential so that

$$V_+(r) = \left(\frac{2A_4}{r^4} + \frac{2A_3}{r^3} + \frac{2A_2}{r^2} + \frac{2A_1}{r}\right) + \frac{(M-1)(M-3)}{4r^2} - 2E_{n=0} \tag{3.3.4}$$

and substituting Eq. (3.3.3) into Eq. (3.3.1) we obtain

$$V_+(r) = \frac{a^2}{r^4} + \frac{2a(1-c)}{r^3} + \frac{c(c-1)+2ab}{r^2} - \frac{2bc}{r} + b^2 \tag{3.3.5}$$

and the relations between the parameters satisfy the supersymmetric constraints

$$a = \pm\sqrt{2A_4} \quad ; \quad c = 1 - \frac{A_3}{\pm\sqrt{2A_4}} \tag{3.3.6}$$

The potential (1) admits the exact solutions

$$\psi_{n=0}(r) = N_0\, r^c \exp\left(\frac{a}{r} - br\right) \tag{3.3.7}$$

where N_0 is the normalization constant, with the physically acceptable eigenvalues

$$E_{n=0} = -\frac{b^2}{2} = -\frac{1}{16A_4}\left[\frac{A_3}{\sqrt{2A_4}}\left(1+\frac{A_3}{\sqrt{2A_4}}\right)-\frac{1}{4}(M-1)(M-3)-2A_2\right]^2 \quad (3.3.8)$$

in the case of $a \prec 0$ and under the constraints

$$A_1 = -\left(1+\frac{A_3}{\sqrt{2A_4}}\right)\sqrt{-2E_{n=0}} \quad (3.3.9)$$

The results obtained agree with those in Refs. [49,50,67] for three-dimensions. Note that in order to retain the well-behaved solution at $r \to 0$ and at $r \to \infty$ we have chosen $a = -\sqrt{2A_4}$.

The expressions obtained above can easily be extended to the lower dimensions. For example, one can readily check that our two-dimensional solutions ($N = 2$, $\ell \to \ell - 1/2$) for the inverse power potential considered are in excellent agreement with the literature [67]. The ground state solutions in arbitrary dimensions for the Coulomb ($A_2 = A_3 = A_4 = 0$), and for a Kratzer ($A_3 = A_4 = 0$) [65], and for an inverse-power ($A_3 = 0$) [48,49] potentials can also be found from the above prescriptions.

For the singular even-power anharmonic oscillator potential of Eq. (3.1.2), we set

$$W(r) = \mu r + \frac{\delta}{r} + \frac{\eta}{r^3} \quad , \qquad \delta \succ 0 \quad (3.3.10)$$

which leads to

$$\psi_{n=0}(r) = C_0 r^{\delta} \exp\left(\frac{\mu r^2}{2} - \frac{\eta}{2r^2}\right) \quad (3.3.11)$$

with C_0 being the corresponding normalization constant, and identify $V_+(r)$ with the effective potential so that

$$V_+(r)=\left(\frac{2B_4}{r^6}+\frac{2B_3}{r^4}+\frac{2B_2}{r^2}+2B_1r^2\right)+\frac{(M-1)(M-3)}{4r^2}-2\widetilde{E}_{n=0}$$

$$=W^2(r)+W'(r)=\frac{\eta^2}{r^6}+\frac{(2\delta-3)}{r^4}\eta+\frac{\delta(\delta-1)+2\eta\mu}{r^2}+\mu^2r^2+\mu(2\delta+1) \quad (3.3.12)$$

and the relations between the potential parameters satisfy the supersymmetric constraints

$$\eta=\pm\sqrt{2B_4} \quad ; \quad \delta=\frac{3}{2}+\frac{B_3}{\eta} \quad ; \quad \mu=\mp\sqrt{2B_1} \quad (3.3.13)$$

As we are dealing with a confined particle system, the negative values for η and μ would of course be the right choice to ensure the well-behaved nature of the wave function behaviour at the origin and at infinity. Hence, physically meaningful ground state energy eigenvalues for the potential of interest are

$$\widetilde{E}_{n=0}=-\frac{\mu}{2}(2\delta+1)=\sqrt{\frac{B_1}{2}}\left\{2+\sqrt{1-16\sqrt{B_1B_4}+8B_2+(M-1)(M-3)}\right\} \quad (3.3.14)$$

At this point we should report that our results reproduce those obtained by [63,72,73] when potential (2) (in case $B_2=0$) is confined to an impenetrable spherical box in 2- and 3-dimensions. It is also not difficult to see that if one takes $\eta=0$ in Eq. (3.3.12), then Eq. (3.3.14) becomes the exact energy spectra of $N-$dimensional harmonic oscillator. Further, one easily check that in case $B_4=B_3=0$, the above energy expression correctly reproduce the eigenvalues of the pseudo-type potential in 3-dimension [74] which is the subject of the next section.

Finally, we wish to discuss briefly the explicit mapping between the singular potentials given by Eqs. (3.1.1) and (3.1.2). If one consider the transformed anharmonic oscillator potential of Eq. (3.2.8) and repeat the above mathematical procedure carried out through Eqs. (3.3.10-3.3.14), then the corresponding eigenvalue equation reads

$$\hat{E}_{n=0}=-2\mu\left(1+\frac{A_3}{\sqrt{2A_4}}\right) \quad (3.3.15)$$

Using the physically acceptable definition of A_1 in Eq. (3.3.9), the above equation can be rearranged as

$$\hat{E}_{n=0} = -\frac{2A_1}{|E_{n=0}|} \tag{3.3.16}$$

where $E_{n=0}$ has been described in Eq. (3.3.8). This brief discussion shows explicitly the relation between the two singular potentials in higher dimensions and verifies Eq. (3.2.9).

B. Exactly Solvable Case

Kasap [74] and his co-workers used supersymmetric quantum mechanics to find exact results for the special cases of the singular potentials of (3.1.1) and (3.1.2), more precisely the solutions of the Kratzer and pseudoharmonic potentials in three dimensions. Their results can be easily generalized to $N-$dimensions by the substitution $\ell \to \Lambda = (M-3)/2$ as indicated in section II. This extension to arbitrary dimensions helps us in constructing the map between these two distinct systems.

The study of anharmonic oscillators has raised a considerable amount of interest because of its various applications especially in molecular physics. The Morse potential is commonly used for the anharmonic oscillator. However, its wave function does not vanish at the origin, but those for Mie-type and pseudoharmonic potentials do. The Mie-type potential possesses the general features of the true interaction energy, inter-atomic and inter-molecular, and dynamical properties of solids [75]. On the other hand, the pseudoharmonic potential may be used for the energy spectrum of linear and non-linear systems [66]. The Mie-type and pseudo-harmonic potentials are two special kinds of analytically solvable singular-power potentials as they have the property of shape-invariance.

Starting with the general form of the Mie-type potential

$$V(r) = D_0\left[\frac{p}{q-p}\left(\frac{\sigma}{r}\right)^q - \frac{q}{q-p}\left(\frac{\sigma}{r}\right)^p\right] \tag{3.3.17}$$

where D_0 is the interaction energy between two atoms in a molecular system at $r=\sigma$, and $q \succ p$ is always satisfied. If we take $q=2p$ and $p=1$, we arrive at a special case of the potential in Eq. (3.3.17), which is exactly solvable

$$V(r)=\frac{A}{r^2}-\frac{B}{r} \tag{3.3.18}$$

where $A=D_0\sigma^2$ and $B=2D_0\sigma$. The above potential, the so-called Kratzer potential, includes the terms, which give the representation of both the steep repulsive branch and the long-range attraction. A single minimum occurs at $r=\sigma$ where the energy is $-D_0$. Considerable interest has recently been shown in this potential as a model to describe inter-nucleon vibration [76] and, in applications this Mie type potential offers one of the most important exactly solvable models of atomic and molecular physics and quantum chemistry [77].

We set the superpotential for the Kratzer effective potential

$$W(r)=\frac{B/2}{\beta+\left(\beta^2+C\right)^{1/2}}-\frac{\beta+\left(\beta^2+C\right)^{1/2}}{r} \tag{3.3.19}$$

where

$$C=\frac{\Lambda(\Lambda+1)}{2}+A \quad , \quad \Lambda=\ell+\frac{1}{2}(N-3) \quad , \quad \beta=\frac{1}{2\sqrt{2}} \tag{3.3.20}$$

and obtained the exact spectrum in $N-$dimensional space as

$$E_n=-\left(\frac{B/2\beta}{2n+1+\left[(2\Lambda+1)^2+A/\beta^2\right]^{1/2}}\right)^2 \quad , \qquad n=0,1,2,\ldots \tag{3.3.21}$$

and from Eq. (3.3.2) the exact unnormalized ground state wavefunction can be expressed as

$$\psi_{n=0}(r)=r^{1/2\left\{1+\left[(2\Lambda+1)^2+A/\beta^2\right]^{1/2}\right\}}\times\exp\left(-\frac{Br/4\beta^2}{1+\left[(2\Lambda+1)^2+A/\beta^2\right]^{1/2}}\right) \tag{3.3.22}$$

The excited state wavefunctions can be easily determined from the usual approach in supersymmetric quantum mechanics [10] and the normalization coefficients for each quantum state wave function can be analytically worked out using the explicit recurrence relation given in a recent work [78].

As a second application, we consider the general form of the pseudoharmonic potential

$$\widetilde{V}(r)=V_0\left(\frac{r}{r_0}-\frac{r_0}{r}\right)^2=\widetilde{B}r^2+\frac{\widetilde{A}}{r^2}-2V_0 \tag{3.3.23}$$

which can be used to calculate the vibrational energies of diatomic molecules with the equilibrium bond length r_0 and force constant $k=8V_0/r_0^{\,2}$, and set the corresponding superpotential as

$$W(r)=\sqrt{\widetilde{B}}r-\frac{\beta+(\beta^2+\widetilde{C})^{1/2}}{r} \tag{3.3.24}$$

where $\widetilde{B}=V_0/r_0^{\,2}$, $\widetilde{C}=[\Lambda(\Lambda+1)+2\widetilde{A}]/2$, $\widetilde{A}=V_0 r_0^{\,2}$. The exact full spectrum of the potential in arbitrary dimensions is

$$\widetilde{E}_n=2\beta\sqrt{\widetilde{B}}\left\{4n+2+\left[(2\Lambda+1)^2+\widetilde{A}/\beta^2\right]^{1/2}\right\}-2V_0 \tag{3.3.25}$$

and the unnormalized exact ground state wave function is

$$\psi_{n=0}(r)=r^{1/2\left\{1+\left[(2\Lambda+1)^2+\widetilde{A}/\beta^2\right]^{1/2}\right\}}\times\exp\left(-\frac{\sqrt{\widetilde{B}}r^2}{4\beta}\right) \tag{3.3.26}$$

Using the discussion in section II, one can transform the Kratzer potential in Eq. (3.3.18) to its dual potential- shifted (by $2V_0$) pseudoharmonic-like potential in Eq. (3.3.23) with some restrictions in potential parameters. In the light of Eqs. (3.2.7-3.2.9), the transformed potential reads

$$\hat{V}(\rho) = \rho^2 + \frac{4A}{\rho^2} \tag{3.3.27}$$

which is in the form of the Goldman-Krivchenkov potential. Here $A(= D_0\sigma^2)$ is the Kratzer potential parameter and, considering Eqs. (3.3.23) through Eq. (3.3.25), constraints on the potential parameters are such that $\widetilde{B} = 1$ and $\widetilde{A} = 4A$. In this case corresponding eigenvalues are

$$\hat{E}_{n'} = \frac{2B}{|E_n|^{1/2}} = 4\beta\left\{1 + 2n' + \left[1 + 4\Lambda'(\Lambda'+1) + \frac{A}{\beta^2}\right]\right\} \quad , \quad \Lambda' = L + \frac{1}{2}(N'-3) \tag{3.3.28}$$

where $B\,(= 2D_0\sigma)$ and E_n are the coupling parameter and the eigenenergy values (Eq. (3.3.21)), respectively, of the Kratzer potential.

The ensuing relationships among the dimensionalities and quantum numbers of the two distinct systems considered here in this section are:

$$N' = 2N - 2 - 2\lambda \quad , \quad L = 2\ell + \lambda \quad , \quad n' = 2n - 2 + \lambda \tag{3.3.29}$$

Clearly, the mapping parameter λ must be an integer if n', L, n and ℓ are integers. It is worthwhile to discuss briefly the physics behind this transformation in the light of the comprehensive work of Kostelecky *et al.* [2,3,5,68,69]. We note that it is a general feature of this map that the spectrum of the $N-$dimensional problem involving Kratzer potential is related to the half the spectrum of the $N-$dimensional problem involving Goldman-Krivchenkov potential for any even integer N'. However, the quantities in Eq. (3.3.29) have parameter spaces that are further restricted by the properties chosen for the map. For instance, suppose we wish to map all states corresponding the $N-$dimensional Kratzer potential into that corresponding Goldman-Krivchenkov potential. Since on physical grounds we know that $N' \geq 2$, $n' \geq 0$, $L \geq 0$, we must impose $N \geq 2 + \lambda$, $n \geq 1 - \lambda/2$, $\ell \geq -\lambda/2$. This yields the bound $-2\ell \leq \lambda \leq N - 2$. Further requiring $n \geq 1$, $\ell \geq 0$ restricts the bound to $0 \leq \lambda \leq N - 2$. We conclude that all states of the $N-$dimensional Kratzer problem can be mapped into the appropriate Goldman-Krivchenkov problem, except for $N = 1$.

As an example, consider the three-dimensional Kratzer problem. Assuming we wish to map all its states into those of its dual-the Goldman-Krivchenkov potential, we must impose $0 \le \lambda \le 1$. First, take $\lambda = 0$. Then, the $s-$orbitals in Kratzer potential ($n \ge 1, \ell = 0$) are related to the ($n' = 2n - 2 \ge 0, L = 0$) states of the four-dimensional Goldman-Krivchenkov problem. Similarly, the $p-$states ($n \ge 2, \ell = 1$) correspond to the ($n' = 2n - 2 \ge 0, L = 0$) same problem. Next, suppose $\lambda = 1$. The states corresponding the potential in Eq. (3.3.18) are then mapped into the odd-integer states of the two-dimensional oscillator problem of Eq. (3.3.27). The $s-$orbitals of Kratzer potential ($n \ge 1, \ell = 0$) map into the ($n' = 2n - 1 \ge 1, L = 1$) anharmonic oscillator states corresponding Goldman-Krivchenkov potential, while the Kratzer $p-$orbitals ($n \ge 2, \ell = 1$) map into the ($n' = 2n - 1 \ge 3, L = 1$) oscillator states of Goldman-Krivchenkov problem. As a rule, in both cases ($\lambda = 0,1$), the lowest-lying states of Goldman-Krivchenkov potential are excluded, one by one, with each higher value of ℓ.

As a final remark, a student of introductory quantum mechanics often learns that the Schrődinger equation is exactly solvable (for all angular momenta) for two central potentials in equations (3.3.18) and (3.3.27), and for also their special cases ($A = 0$) the Coulomb and harmonic oscillator problems. Less frequently, the student made aware of the relation between these two problems, which are linked by a simple change of the independent variable discussed in detail through this chapter. Under this transformation, energies and coupling constants trade places, and orbital angular momenta are re-scaled. Thus, we have in this section shown that there is really only one quantum mechanical problem, not two involving the Kratzer and Goldman-Krivchenkov potentials, which can be exactly solved for all orbital angular momenta.

3.4 Conclusion

The main aim of this work has been to establish a very general connection between a class of singular potentials in higher dimensional space through the application of a suitable transformation. Although much work had been done in the literature on similar problems, an investigation as the one we have discussed in this chapter was missing to our knowledge. In addition, it is shown that the supersymmetric quantum mechanics yields exact solutions for a single state only for the quasi-exactly solvable potentials such as the ones given in equations (3.1.1) and (3.1.2) with some restrictions on the potential parameters in N – dimensional space, unlike the shape invariant exactly solvable potentials. We have also shown how to obtain exact solutions to such problems in any dimension by applying an adequate transformation to previously known three-dimensional results. This simple and intuitive method discussed through this chapter is easy to be generalized.

CHAPTER 4

SUMMARY AND CONCLUSION

Through the thesis work, explicit point canonical transformations (PCT), which map exactly solvable shape invariant potentials into two potential classes have been discussed. The eigenfunctions in these two classes have been given by hypergeometric and confluent hypergeometric functions respectively. The members within each class then have been mapped to each other, and the members of two different classes have also been connected by suitable transformations. An algebraic structure have been developed to demonstrate that similar mapping procedure can also be used to transform exactly solvable non-central potentials to each other.

We have, in addition, shown that supersymmetric quantum mechanics, as an alternative treatment to PCT, is also a useful tool for obtaining algebraic solutions of exactly solvable and quasi-exactly solvable non-relativistic potentials. Finally, to obtain a general connection between a class of singular potentials in higher dimensional space we have employed both treatments, supersymmetric and PCT techniques, within the same framework and obtained the exact results.

Although, the literature is rich with such applications, to the best of our knowledge, the works carried out through in particular section (2.3) and Chapter 3 were missing in the literature. Hence, we believe that the works presented in this thesis would in this respect illuminate some unresolved questions in the related areas of atomic physics and chemistry.

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APPENDICES

APPENDIX A

To understand how equation (2.1.4) in the first section of chapter 2 is formed let us use some mathematics. It is given,

$$x = f(z) \qquad \text{and} \qquad \psi(\alpha, x) = \upsilon(z)\tilde{\psi}(\alpha, z) \tag{A.1}$$

then

$$dx = f'dz \quad \text{and} \quad \frac{dz}{dx} = \frac{1}{f'} \Rightarrow \frac{d}{dx} = \frac{dz}{dx}\frac{d}{dz} = \frac{1}{f'}\frac{d}{dz}$$

so,

$$\frac{d^2}{dx^2} = -\frac{f''}{f'^3}\frac{d}{dz} + \frac{1}{f'^2}\frac{d^2}{dz^2} \tag{A.2}$$

and

$$\frac{d^2\psi(\alpha,x)}{dx^2} = -\frac{f''}{f'^3}\frac{d(\upsilon(z)\tilde{\psi}(\alpha,z))}{dz} + \frac{1}{f'^2}\frac{d^2(\upsilon(z)\tilde{\psi}(\alpha,z))}{dz^2}$$

$$\frac{d^2\psi(\alpha,x)}{dx^2} = -\frac{f''}{f'^3}\left[\upsilon'\tilde{\psi} + \upsilon\frac{d\tilde{\psi}}{dz}\right] + \frac{1}{f'^2}\frac{d}{dz}\left[\upsilon'\tilde{\psi} + \upsilon\frac{d\tilde{\psi}}{dz}\right]$$

$$\frac{d^2\psi(\alpha,x)}{dx^2} = \frac{\upsilon}{f'^2}\frac{d^2\tilde{\psi}(\alpha,z)}{dz^2} + \left(\frac{2\upsilon'}{f'^2} - \frac{f''\upsilon}{f'^3}\right)\frac{d\tilde{\psi}}{dz} + \left(\frac{\upsilon''}{f'^2} - \frac{f''\upsilon'}{f'^3}\right)\tilde{\psi}(\alpha,z) \tag{A.3}$$

If we replace equation (A.1) and (A.3) into (2.1.1) in the first section of chapter 2 we get,

$$-\frac{\hbar^2}{2m}\frac{d^2\tilde{\psi}}{dz^2} - \frac{\hbar^2}{m}\left\{\frac{\upsilon'}{\upsilon} - \frac{f''}{2f'}\right\}\frac{d\tilde{\psi}}{dz}$$

$$+\left[f'^2\{V(\alpha_i; f(z)) - E(\alpha_i)\} + \frac{\hbar^2}{2m}\left\{\frac{f''\upsilon'}{f'\upsilon} - \frac{\upsilon''}{\upsilon}\right\}\right]\tilde{\psi} = 0 \tag{A.4}$$

And to see how we obtain Equation (2.1.5), one needs to remove the term involving first derivative in Eq. (2.1.4) which leads to Eq.(2.1.5) in chapter 2 as shown below,

$$\frac{\upsilon'}{\upsilon}-\frac{f''}{2f'}=0 \Rightarrow \upsilon=C\sqrt{f'(z)} \tag{A.5}$$

APPENDIX B

Table 1: Interrelation of energy eigenvalues and eigenfunctions obtained by mapping via PCT among shape invariant potentials of class I, i.e. those whose eigenfunctions correspond to confluent hypergeometric functions

Name of Potential	Superpotential W	Potential V	Trans-formation $f(z)$	Relation among Parameters	Eigenvalues $E_n^{(-)}$	Variable y	Radial part of Wavefunction ψ_n
Harmonic Oscillator	$\sqrt{\frac{m}{2}}\tilde{\omega}z - \frac{(\tilde{\ell}+1)\hbar}{\sqrt{2m}z}$ $0 \leq z \langle \infty$	$\frac{1}{2}m\tilde{\omega}^2 z^2 + \frac{\tilde{\ell}(\tilde{\ell}+1)\hbar^2}{2mz^2}$ $-(\tilde{\ell}+\frac{3}{2})\hbar\tilde{\omega}$	—	—	$2n\hbar\tilde{\omega}$	$y = m\tilde{\omega}z^2$	$\exp(-\frac{1}{2}y)y^{(\tilde{\ell}+1)/2}$
Coulomb	$\sqrt{\frac{m}{2}}\frac{e^2}{(\ell+1)\hbar}$ $-\frac{(\ell+1)\hbar}{\sqrt{2m}x}$ $(0 \leq x \leq \infty)$	$-\frac{e^2}{x} + \frac{\ell(\ell+1)\hbar^2}{2mx^2}$ $+\frac{me^4}{2(\ell+1)^2\hbar^2}$	z^2	$\tilde{\omega} = \frac{2e^2}{\hbar(n+\ell+1)}$ $\tilde{\ell} = 2\ell + \frac{1}{2}$	$\frac{me^4}{2\hbar^2}\left(\frac{1}{(\ell+1)^2} - \frac{1}{(n+\ell+1)^2}\right)$	$y = \frac{2me^2 x}{\hbar(n+\ell+1)}$	$y^{\ell+1}\exp(-\frac{1}{2}y)$ $\times L_m^{2\ell+1}(y)$
Morse	$A - Be^{-\alpha x}$ $(-\infty \langle x \langle \infty)$	$A^2 + B^2 e^{-2\alpha x}$ $-2B\left(A + \frac{\alpha\hbar}{2\sqrt{2m}}\right)e^{-\alpha x}$	$-\frac{2}{\alpha}\ln z$	$\tilde{\omega} = \frac{2\sqrt{2}B}{\alpha\sqrt{m}}$ $\tilde{\ell} = \left(\frac{2\sqrt{2m}A}{\hbar\alpha} - 2n - \frac{1}{2}\right)$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$y = \frac{2\sqrt{2m}B}{\alpha}e^{-\alpha x}$	$y^{(\sqrt{2m}A/\hbar\alpha - n)}e^{(-\frac{1}{2}y)}$ $\times L_m^{(2\sqrt{2m}A/\hbar\alpha - 2n)}(y)$

Table 2: Interrelation of energy eigenvalues and eigenfunctions obtained by mapping via PCT among shape invariant potentials of class II, i.e. those eigenfunctions correspond to hypergeometric functions.

Name of Potential	Superpotential W	Potential V	Transformation $f(z)$	Relation among Parameters	Eigenvalues $E_n^{(-)}$	Variable y	Radial part of Wavefunction ψ_n
Scarf	$-\tilde{A}\cot\alpha z$ $+B\operatorname{cosec}\alpha z$ $(0\le\alpha z\le\pi;$ $\tilde{A}\rangle B)$	$-\tilde{A}^2+(\tilde{A}^2+\tilde{B}^2-\frac{\tilde{A}\alpha\hbar}{\sqrt{2m}})$ $\times\operatorname{cosec}^2\alpha z$ $-\tilde{B}(2\tilde{A}-\frac{\alpha\hbar}{\sqrt{2m}})\cot\alpha z$ $\times\operatorname{cosec}\alpha z$	——	——	$\left(\tilde{A}+\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$ $-\tilde{A}^2$	$y=\cos\alpha z$ $s=\frac{\sqrt{2m}}{\hbar}\frac{\tilde{A}}{\alpha}$ $\lambda=\frac{\sqrt{2m}}{\hbar}\frac{\tilde{B}}{\alpha}$	$(1-y)^{(s-\lambda)/2}$ $\times(1+y)^{(s+\lambda)/2}$ $\times\rho_n^{(s-\lambda-\frac{1}{2},s+\lambda-\frac{1}{2})}(y)$
Scarf	$A\tanh\alpha x$ $+B\operatorname{sec}h\alpha x$ $(-\infty\langle x\langle\infty)$	$\tilde{A}^2$ $+(\tilde{B}^2-\tilde{A}^2-\frac{\tilde{A}\alpha\hbar}{\sqrt{2m}})\operatorname{sec}h^2\alpha x$ $+B\left(2A+\frac{\alpha\hbar}{\sqrt{2m}}\right)\operatorname{sec}h\alpha x$ $\times\tanh\alpha x$	$\frac{\sinh^{-1}(-i\cos\alpha z)}{\alpha}$	$\tilde{A}=-A$ $\tilde{B}=iB$	A^2 $-\left(A-\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$y=\sinh\alpha x$ $s=\frac{\sqrt{2m}A}{\hbar\alpha}$ $\lambda=\frac{\sqrt{2m}B}{\hbar\alpha}$	$(1+y^2)^{-s/2}$ $\times\exp(-\lambda\tan^{-1}y)$ $\times\rho_n^{(-s-i\lambda-\frac{1}{2},-s+i\lambda-\frac{1}{2})}(iy)$

Table 2 (Continued)

Name of Potential	Superpotential W	Potential V	Transformation $f(z)$	Relation among parameters	Eigenvalues $E_n^{(-)}$	Variable y	Radial part of Wavefunction ψ_n
Rosen-Morse-I	$A\tan\alpha x+\frac{B}{A}$ $(-\infty\langle x\langle\infty)$	$-A^2+\frac{B^2}{A^2}$ $+2B\tan\alpha x$ $+A\left(A-\frac{\alpha\hbar}{\sqrt{2m}}\right)$ $\times\sec^2\alpha x$	$\frac{\cos^{-1}(\mathrm{cosec}\,\alpha z)}{\alpha}$	$\tilde{A}=-A$ $-(n-\frac{1}{2})\frac{\hbar\alpha}{\sqrt{2m}}$ $\tilde{B}=-\frac{iB}{\left(A+\frac{n\alpha\hbar}{\sqrt{2m}}\right)}$	$\left(A+\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$ $-A^2+\frac{B^2}{A^2}$ $-\frac{B^2}{\left(A+\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2}$	$y=\tan\alpha x$ $s=\frac{\sqrt{2mA}}{\hbar\alpha}$ $\lambda=\frac{\sqrt{2mB}}{\hbar\alpha}$ $a=\frac{\sqrt{2m\lambda}}{\hbar\alpha(s+n)}$	$(1+y^2)^{-(s+n)/2}$ $\times\exp(-a\tan^{-1}y)$ $\times\rho_n^{(-s-n+ia,s-n-ia)}(-iy)$
Rosen-Morse-II	$A\tanh\alpha x+\frac{B}{A}$ $(-\infty\langle x\langle\infty)$	$A^2+\frac{B^2}{A^2}$ $+2B\tan\alpha x$ $-A\left(A+\frac{\alpha\hbar}{\sqrt{2m}}\right)$ $\times\mathrm{sec}h^2\alpha x$	$\frac{\tanh^{-1}(\cos\alpha z)}{\alpha}$	$\tilde{A}=A$ $-(n-\frac{1}{2})\frac{\hbar\alpha}{\sqrt{2m}}$ $\tilde{B}=-\frac{B}{\left(A-\frac{n\alpha\hbar}{\sqrt{2m}}\right)}$	$-\left(A-\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$ $+A^2+\frac{B^2}{A^2}$ $-\frac{B^2}{\left(A-\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2}$	$y=\tanh\alpha x$ $s=\frac{\sqrt{2mA}}{\hbar\alpha}$ $\lambda=\frac{\sqrt{2mB}}{\hbar\alpha}$ $a=\frac{\sqrt{2m\lambda}}{\hbar\alpha(s-n)}$	$(1-y)^{(s-n+a)/2}$ $\times(1+y)^{(s-n-a)/2}$ $\times\rho_n^{(s-n+a,s-n-a)}(y)$

Table 2 (continued)

Name of Potential	Superpotential W	Potential V	Transformation $f(z)$	Relation among parameters	Eigenvalues $E_n^{(-)}$	Variable y	Radial part of Wavefunction ψ_n
Rosen-Morse-II	$A\coth\alpha x$ $-B\operatorname{cosech}\alpha x$ $(0 \le x \langle \infty$; $A \langle B)$	$A^2 + (B^2 + A^2 + \frac{A\alpha\hbar}{\sqrt{2m}}) \times \operatorname{cosech}^2\alpha x - B\left(2A + \frac{\alpha\hbar}{\sqrt{2m}}\right) \times \coth\alpha x \operatorname{cosech}\alpha x$	$\frac{\cosh^{-1}(\cos\alpha z)}{\alpha}$	$\tilde{A} = -A$ $\tilde{B} = -B$	$A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$y = \cosh\alpha x$ $s = \frac{\sqrt{2m}A}{\hbar\alpha}$ $\lambda = \frac{\sqrt{2m}B}{\hbar\alpha}$	$(y-1)^{(\lambda-s)/2} \times (1+y)^{-(s+\lambda)/2} \times \rho_n^{(\lambda-s-\frac{1}{2}, -\lambda-s-\frac{1}{2})}(y)$
Eckart	$-A\coth\alpha x + \frac{B}{A}$ $(0 \le x \langle \infty$; $B \rangle A^2)$	$A^2 + \frac{B^2}{A^2} - 2B\coth\alpha x + A\left(A - \frac{\alpha\hbar}{\sqrt{2m}}\right) \times \operatorname{cosech}^2\alpha x$	$\frac{\coth^{-1}(\cos\alpha z)}{\alpha}$	$\tilde{A} = -A - (n - \frac{1}{2})\frac{\hbar\alpha}{\sqrt{2m}}$ $\tilde{B} = -\frac{B}{\left(A + \frac{n\alpha\hbar}{\sqrt{2m}}\right)}$	$A^2 + \frac{B^2}{A^2} - \left(A + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 - \frac{B^2}{\left(A + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2}$	$y = \coth\alpha x$ $s = \frac{\sqrt{2m}A}{\hbar\alpha}$ $\lambda = \frac{\sqrt{2m}B}{\hbar\alpha}$ $a = \frac{\sqrt{2m}\lambda}{\hbar(s-n)}$	$(y-1)^{-(s+n-a)/2} \times (1+y)^{-(s+n+a)/2} \times \rho_n^{(-s+a-n, -s-a-n)}(y)$

Table 2 (continued)

Name of Potential	Superpotential W	Potential V	Transformation $f(z)$	Relation among parameters	Eigenvalues $E_n^{(-)}$	Variable y	Radial part of Wavefunction ψ_n
Pöschl-Teller I	$A\tan\alpha x$ $-B\cot\alpha x$ $(0 \le \alpha x \le \frac{\pi}{2})$	$-(A+B)^2$ $+A\left(A-\frac{\alpha\hbar}{\sqrt{2m}}\right)\sec^2\alpha x$ $+B\left(B-\frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{cosec}^2\alpha x$	$z/2$	$\tilde{A}=\left(\frac{A+B}{2}\right)$ $\tilde{B}=\left(\frac{A-B}{2}\right)$	$\left(A+B+\frac{2n\alpha\hbar}{\sqrt{2m}}\right)^2$ $-(A+B)^2$	$y=1-2\sin^2\alpha x$ $s=\frac{A}{\alpha}$ $\lambda=\frac{B}{\alpha}$	$(1-y)^{\lambda/2}$ $\times(1+y)^{s/2}$ $\times\rho_n^{(\lambda-\frac{1}{2},s-\frac{1}{2})}(y)$
Pöschl-Teller II	$A\tanh\alpha x$ $-B\coth\alpha x$ $(0 \le x \langle \infty$; $B \langle A)$	$(A-B)^2$ $-A\left(A+\frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}^2\alpha x$ $+B\left(B-\frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{cosech}^2\alpha x$	$\frac{\sinh^{-1}(i\sin\frac{\alpha z}{2})}{\alpha}$	$\tilde{A}=\left(\frac{B-A}{2}\right)$ $\tilde{B}=\left(\frac{A+B}{2}\right)$	$(A-B)^2$ $-\left(A-B-\frac{2n\alpha\hbar}{\sqrt{2m}}\right)^2$	$y=1+2\sinh^2\alpha x$ $s=\frac{A}{\alpha}$ $\lambda=\frac{B}{\alpha}$	$(1-y)^{\lambda/2}$ $\times(1+y)^{-s/2}$ $\times\rho_n^{(\lambda-\frac{1}{2},-s-\frac{1}{2})}(y)$

APPENDIX C

Table 3: Limiting procedures and redefinition of parameters that relate type-I to type-II potentials.

Type-I Potential	**Type-II Potential**	**Limits & Redef. of Parameters**
Generalized Pöschl-teller $V(r)=A^2+\frac{(A^2+B^2+A\alpha)}{\sinh^2(\alpha r+\beta)}$ $-\frac{B(2a+\alpha)\coth(\alpha r+\beta)}{\sinh(\alpha r+\beta)}$ $-\beta\langle\alpha r\langle\infty$ $E_n=A^2-(A-n\alpha)^2$ $A\langle B$	**Harmonic Oscillator** $V(r)=\frac{1}{4}\omega^2r^2+\frac{\ell(\ell+1)}{r^2}$ $-\left(\ell+\frac{3}{2}\right)\omega$ $0\langle r\langle\infty\,,\quad E_n=2n\omega$	$A\rightarrow\left[\frac{\omega}{\alpha}-\alpha\left(\frac{\ell+1}{2}\right)\right]$ $B\rightarrow\left[\frac{\omega}{\alpha}+\alpha\left(\frac{\ell+1}{2}\right)\right]$ $\alpha\rightarrow0,\quad\beta\rightarrow0$
	Morse Potential $V(x)=A^2+B^2e^{-2\alpha x}$ $-2B(A+\frac{\alpha}{2})e^{-\alpha x}$ $-\infty\langle x\langle\infty$ $E_n=A^2-(A-n\alpha)^2$	$A\rightarrow A$ $B\rightarrow\frac{Be^{\beta}}{2}$ $r\rightarrow x$ $\beta\rightarrow\infty$
Scarf $V(x)=-A^2+\frac{(A^2+B^2-A\alpha)}{\cos(\alpha x)}$ $-\frac{B(2A-\alpha)\tan(\alpha x)}{\cos(\alpha x)}$ $-\frac{\pi}{2\alpha}\langle x\langle\frac{\pi}{2\alpha},\quad A\rangle B,$ $E_n=(A+n\alpha)^2-A^2$	**Harmonic oscillator** $V(r)=\frac{1}{4}\omega^2r^2+\frac{\ell(\ell+1)}{r^2}$ $-\left(\ell+\frac{3}{2}\right)\omega$ $0\langle r\langle\infty$ $E_n=2n\omega$	$A\rightarrow\left[\frac{\omega}{\alpha}+\alpha\left(\frac{\ell+1}{2}\right)\right]$ $B\rightarrow\left[\frac{\omega}{\alpha}-\alpha\left(\frac{\ell+1}{2}\right)\right]$ $x\rightarrow r+\frac{\pi}{2\alpha}$ $\alpha\rightarrow0$
Scarf (Hyperbolic) $V(x)=A^2+\frac{(-A^2+B^2-A\alpha)}{\cosh^2(\alpha x+\beta)}$ $+\frac{B(2A+\alpha)\tanh(\alpha x+\beta)}{\cosh(\alpha x+\beta)}$ $-\infty\langle x\langle\infty\,,\quad A\rangle0$ $E_n=-(A-n\alpha)^2+A^2$	**Morse Potential** $V(x)=A^2+B^2e^{-2\alpha x}$ $-2B(A+\frac{\alpha}{2})e^{-\alpha x}$ $-\infty\langle x\langle\infty$ $E_n=A^2-(A-n\alpha)^2$	$A\rightarrow A$ $B\rightarrow-\frac{Be^{\beta}}{2}$ $\beta\rightarrow\infty$

Type-I Potential	Type-II Potential	Limits & Redef. of Parameters
Eckart $V(r)=A^2+\frac{B^2}{A^2}-2B\coth\alpha r$ $+A(A-\alpha)\operatorname{cosech}^2(\alpha r)$ $0\langle r\langle\infty,\quad B\rangle A^2,\quad A\rangle 0$ $E_n=-(A+n\alpha)^2+A^2$ $+\frac{B^2}{A^2}-\frac{B^2}{(A+n\alpha)^2}$	**Coulomb** $V(r)=-\frac{e^2}{r}+\frac{\ell(\ell+1)}{r^2}$ $+\frac{e^4}{4(\ell+1)^2}$ $0\langle r\langle\infty$ $E_n=\frac{e^4}{4}(\frac{1}{(\ell+1)^2}$ $-\frac{1}{(n+\ell+1)^2})$	$A\to\alpha(\ell+1)$ $B\to\frac{\alpha}{2}e^2$ $\alpha\to 0$
Rosen-Morse I $V(r)=-A^2+\frac{B^2}{A^2}+2B\tan\alpha x$ $+A(A-\alpha)\sec^2(\alpha x)$ $-\frac{\pi}{2\alpha}\langle x\langle\frac{\pi}{2\alpha}$ $E_n=(A+n\alpha)^2-A^2$ $+\frac{B^2}{A^2}-\frac{B^2}{(A+n\alpha)^2}$	**Coulomb** $V(r)=-\frac{e^2}{r}+\frac{\ell(\ell+1)}{r^2}$ $+\frac{e^4}{4(\ell+1)^2}$ $0\langle r\langle\infty$ $E_n=\frac{e^4}{4}(\frac{1}{(\ell+1)^2}$ $-\frac{1}{(n+\ell+1)^2})$	$A\to\alpha(\ell+1)$ $B\to\frac{\alpha}{2}e^2$ $x\to r-\frac{\pi}{2\alpha}$ $\alpha\to 0$

APPENDIX D

The general schrödinger equation in spherical coordinates is,

$$-\nabla^2\psi + V\psi = E\psi \tag{D.1}$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{D.2}$$

$$V(r,\theta) = \frac{A}{r} + \frac{B}{r^2\sin^2\theta} + C\frac{\cos\theta}{r^2\sin^2\theta} \tag{D.3}$$

Therefore, (D.1) becomes

$$-\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi$$

$$+\left(\frac{A}{r} + \frac{B}{r^2\sin^2\theta} + C\frac{\cos\theta}{r^2\sin^2\theta}\right)\psi = E\psi \tag{D.4}$$

Defining $\psi(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi)$ and using seperation of variables we obtain

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + Er^2 - Ar + \frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} - \frac{(B + C\cos\theta)}{\sin^2\theta} = 0 \tag{D.5}$$

setting

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + Er^2 - Ar = \Lambda \tag{D.6a}$$

and

$$\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}-\frac{(B+C\cos\theta)}{\sin^2\theta}=-\Lambda \tag{D.6b}$$

and having in mind $\Lambda=\ell(\ell+1)$, Eq. (D.6a) reduces to

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+(E-A/r)R=\frac{\ell(\ell+1)R}{r^2}$$

Rearranging

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(E-\frac{A}{r}-\frac{\ell(\ell+1)}{r^2}\right)R=0 \tag{D.7}$$

and considering the new form of (D.6b)

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dY}{d\theta}\right)+\frac{d^2Y}{d\phi^2}-(B+C\cos\theta)Y+\ell(\ell+1)\sin^2\theta Y=0$$

As $Y(\theta,\phi)=P(\theta)\Phi(\phi)$ then ,

$$\frac{\sin\theta}{p}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\ell(\ell+1)\sin^2\theta-(B+C\cos\theta)=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \tag{D.8}$$

Now we equate both sides of (D.8) to m^2 then

$$\Phi=Fe^{\pm im\phi} \tag{D.9}$$

and

$$\frac{d^2P}{d\theta^2}+\cot\theta\frac{dP}{d\theta}+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}-\frac{(B+C\cos\theta)}{\sin^2\theta}\right]P \tag{D.10}$$

Using transformation $\theta\rightarrow z$ through a mapping $\theta=f(z)$,

$$d\theta=f'dz \text{ and } \frac{dz}{d\theta}=\frac{1}{f'}$$

$$\frac{d}{d\theta}=\frac{dz}{d\theta}\frac{d}{dz}=\frac{1}{f'}\frac{d}{dz}$$

so,

$$\frac{dP}{d\theta}=\frac{1}{f'}\frac{dP}{dz} \tag{D.11a}$$

and

$$\frac{d^2}{d\theta^2}=\frac{d}{d\theta}\left(\frac{d}{d\theta}\right)=\frac{1}{f'}\frac{d}{dz}\left(\frac{1}{f'}\frac{d}{dz}\right)=-\frac{1}{f'^3}\frac{d}{dz}+\frac{1}{f'^2}\frac{d^2}{dz^2}.$$

Finally,

$$\frac{d^2P}{d\theta^2}=-\frac{1}{f'^3}\frac{dP}{dz}+\frac{1}{f'^2}\frac{d^2P}{dz^2}, \tag{D.11b}$$

if one replaces (D.11a) and (D.11b) into (D.10) then one arrives at

$$\frac{d^2P}{dz^2}+\left[-\frac{f''}{f'}+f'\cot f\right]\frac{dP}{dz}+f'^2\left[\ell(\ell+1)-\frac{m^2}{\sin^2 f}-\frac{(B+C\cos f)}{\sin^2 f}\right]P=0 \tag{D.12}$$

APPENDIX E

The radial Schrödinger equation for a spherically symmetric potential $V(r)$ in N -dimensional space ;

$$-\frac{1}{2}\left[\frac{d^2R}{dr^2}+\frac{N-1}{r}\frac{dR}{dr}\right]+\frac{\ell(\ell+N-2)}{2r^2}R=[E-V(r)]R \tag{E.1}$$

is transformed to another form using some transformation terms,

$$X(r)=r^{\frac{(N-1)}{2}}R(r) \tag{E.2}$$

which leads to,

$$\frac{dR(r)}{dr}=\frac{d}{dr}\left[r^{(1-N)/2}X(r)\right]=r^{(1-N)/2}\frac{dX(r)}{dr}+X(r)\frac{(1-N)}{2}r^{-(N+1)/2} \tag{E.3}$$

$$\frac{d^2R(r)}{dr^2}=(1-N)r^{-(N+1)/2}\frac{dX(r)}{dr}+r^{(1-N)/2}\frac{d^2X(r)}{dr^2}+\frac{(N-1)(N+1)}{4}r^{-(N+3)/2}X(r) \tag{E.4}$$

If we insert Eq.(E.3) and (E.4) into Eq.(E.1);

$$-\frac{1}{2}\left[\frac{d^2X(r)}{dr^2}-\frac{(N-1)(N-3)}{4r^2}X(r)\right]+\frac{\ell(\ell+N-2)}{2r^2}X(r)=(E-V(r))X(r)$$

$$-\frac{d^2X(r)}{dr^2}+\frac{N^2+4\ell N+4\ell^2-4(N+2\ell)+3}{4r^2}X(r)+2V(r)X(r)=2EX(r) \tag{E.5}$$

As, $M=N+2\ell$ then;

$$-\frac{d^2X(r)}{dr^2}+\left[\frac{(M-1)(M-3)}{4r^2}+2V(r)\right]X(r)=2EX(r) \tag{E.6}$$

Now we substitute $r=\dfrac{\alpha\rho^2}{2}$ and $R=\dfrac{F(\rho)}{\rho}$ then transform Eq. (E.1) to another Schrödinger-type equation in $(N'=2N-4)$-dimensional space with angular momentum $L=2\ell+1$,

$$-\frac{1}{2}\left[\frac{d^2R}{dr^2}+\frac{N-1}{r}\frac{dR}{dr}\right]+\frac{\ell(\ell+N-2)}{2r^2}R=\left[E-V(r)\right]R \qquad \text{(E.1)}$$

As $r=\dfrac{\alpha\rho^2}{2}$, the derivative terms,

$$\frac{d}{dr}=\frac{d\rho}{dr}\frac{d}{d\rho}=\left(\frac{1}{\alpha\rho}\right)\frac{d}{d\rho} \qquad \text{(E.7)}$$

$$\frac{d^2}{dr^2}=-\frac{1}{\alpha^2\rho^3}\frac{d}{d\rho}+\frac{1}{\alpha^2\rho^2}\frac{d^2}{d\rho^2} \qquad \text{(E.8)}$$

As our aim is to find $\dfrac{d^2R}{dr^2}$ and $\dfrac{dR}{dr}$ terms, then considering $R=\dfrac{F(\rho)}{\rho}$, we find

$$\frac{dR}{dr}=-\frac{1}{\alpha\rho^3}F(\rho)+\frac{1}{\alpha\rho^2}\frac{dF(\rho)}{d\rho} \qquad \text{(E.9)}$$

and

$$\frac{d^2R}{dr^2}=\frac{1}{\alpha^2\rho}\left[\frac{3F(\rho)}{\rho^4}-\frac{3}{\rho^3}\frac{dF(\rho)}{d\rho}+\frac{1}{\rho^2}\frac{d^2F(\rho)}{d\rho^2}\right] \qquad \text{(E.10)}$$

If we replace Eq.(E.9) and (E.10) into Eq.(E.1) we get

$$-\frac{1}{2\alpha^2\rho^3}\frac{d^2F(\rho)}{d\rho^2}-\left[\frac{2N-4-1}{2\alpha^2\rho^4}\right]\frac{dF(\rho)}{d\rho}$$

$$+\left[\frac{(2N-4)-1+2\ell(2\ell+2N-4)}{2\alpha^2\rho^5}\right]F(\rho)=\left[E-V(\frac{\alpha\rho^2}{2})\right]\frac{F(\rho)}{\rho} \qquad \text{(E.11)}$$

Substituting $L=2\ell+1$, and $N'=2N-4$,

$$-\frac{1}{2\alpha^2\rho^3}\frac{d^2F(\rho)}{d\rho^2}-\left(\frac{N'-1}{2\alpha^2\rho^4}\right)\frac{dF(\rho)}{d\rho}+\left[\frac{L(L+N'-2)}{2\alpha^2\rho^5}\right]F(\rho)=$$

$$\left[E-V(\frac{\alpha\rho^2}{2})\right]\frac{F(\rho)}{\rho} \qquad \text{(E.12)}$$

and multiplying both sides of Eq.(E.12) with the term $\alpha^2\rho^2$ then,

$$-\frac{1}{2}\left[\frac{d^2F(\rho)}{d\rho^2}+\frac{(N'-1)}{\rho}\frac{dF(\rho)}{d\rho}\right]+\frac{L(L+N'-2)}{2\rho^2}F(\rho)=\left[\hat{E}-\hat{V}(\rho)\right]F(\rho) \qquad \text{(E.13)}$$

where

$$\hat{E}-\hat{V}(\rho)=E\alpha^2\rho^2-\alpha^2\rho^2V(\frac{\alpha\rho^2}{2}) \qquad \text{(E.14)}$$

which is Eq.(3.9) in chapter 3.

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