

SUPERSYMMETRIC QUANTUM
MECHANICS AND ITS APPLICATIONS
IN PHYSICS

130908

Ph. D Thesis

in

GAZİANTEP ÜNİVERSİTESİ
MÜHÜRÜ

Engineering Physics
University of Gaziantep

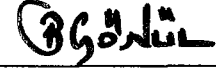
By

Okan ÖZER

June 2003

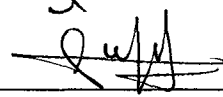
130908

Approval of the Graduate School of Natural and Applied Sciences.



Prof. Dr Bülent GÖNÜL
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.



Prof. Dr Ömer F. BAKKALOĞLU
Head of Department

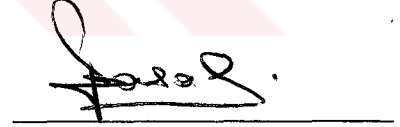
This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.



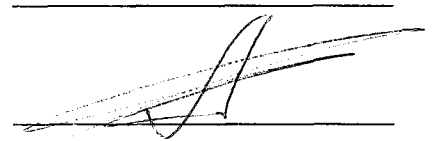
Prof. Dr Bülent GÖNÜL
Supervisor

Examining Committee Members:

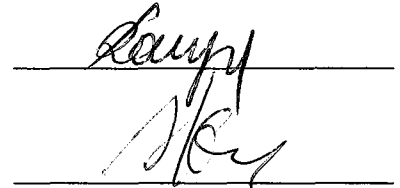
Prof. Dr Ergün KASAP



Prof. Dr Bülent GÖNÜL



Prof. Dr Asker Ali ABİYEYEV



Assoc. Prof. Dr Ramazan KOÇ

Assist. Prof. Dr Hümbet AHMADOV

ABSTRACT**SUPERSYMMETRIC QUANTUM MECHANICS AND ITS APPLICATIONS
IN PHYSICS**

ÖZER, Okan

Ph. D in Engineering Physics

Supervisor: Prof. Dr Bülent GÖNÜL

June 2003, 95 pages

Theoretical formulation of the supersymmetric quantum mechanics is reviewed and its applications to some physical problems are studied. The supersymmetric quantum mechanics (SUSYQM) is based on the factorization of the Schrödinger equation, leading to generalized operators and partner potentials. The Hamiltonian hierarchy and the factorization in the method are briefly discussed. It is also combined with the perturbation theory which is used for the systems that are not exactly solvable. The method is applied to the Hulthén potential and an expression for the energy levels which gives satisfactory values for the non-zero angular momentum states of the potential is obtained. It is shown that a very general connection can be established between a class of singular potentials in N -dimensional space through the application of a suitable transformation by SUSYQM. Using this way, a connection between screened Coulomb and anharmonic oscillator potentials is obtained. A general mapping procedure is described for the transformation of a differential equation with a position-dependent mass to the Schrödinger equation with a constant mass under canonical transformations. In the frame of SUSYQM, it is shown that the Schrödinger equation in its new form may have solutions if the original potential is solvable and shape-invariant one. Finally, It is demonstrated how the SUSYQM method can also be employed for the construction of n -parameter family of potentials which possess localized positive energy state(s) in the continuum.

Key words: Supersymmetric Quantum Mechanics, Shape invariant potentials, Hamiltonian hierarchy, Supersymmetric perturbation theory, Hulthén potential, N -dimensional space, Position-dependent mass, Bound states in continuum.

ÖZ

SÜPERSİMETRİK KUANTUM MEKANİĞİ VE FİZİKTEKİ
UYGULAMALARI

ÖZER, Okan

Doktora Tezi, Fizik Mühendisliği Bölümü

Tez Yöneticisi: Prof. Dr Bülent GÖNÜL

Haziran 2003, 95 sayfa

Süpersimetrik kuantum mekaniğinin teorik altyapısı gözden geçirilerek bazı fiziksel problemlere uygulamaları çalışıldı. Metod, türetilmiş operatörler ve eş potansiyelleri içeren, Schrödinger dalga denkleminin çarpanlara ayrılması üzerine kuruludur. Modelin özünde bulunan Hamilton silsilesi ve çarpanlara ayrılma olayı kısaca tartışıldı. Tam çözülemeyen sistemler için kullanılan pertürbasyon teorisi süpersimetrik kuantum mekaniği çerçevesinde incelendi. Metod, Hulthén potansiyeline uygulandı ve potansiyelin $\ell > 0$ açısal momentum değerleri için tatmin edici enerji seviye değerleri veren bir ifade elde edildi. Süpersimetrik kuantum mekaniği tarafından uygun bir dönüşüm uygulamasıyla N -boyutlu uzayda, bir grup tekil potansiyel arasında çok genel bir bağlantı kurulabileceği gösterildi. Bu teknik kullanılarak, perdelenmiş Coulomb potansiyeli ve tam harmonik olmayan salınımlı sistemler arasında bir bağ elde edildi. Kuralsal dönüşümler altında, pozisyona bağlı kütle ifadeli bir diferansiyel denklemin sabit kütleli Schrödinger dalga denkleminin dönüşümü için genel bir eşleme işlemi tanımlandı. Yeni formundaki Schrödinger dalga denkleminin, eğer ana potansiyel tam çözülebilir ve şekil değişmezliği özelliğine sahipse süpersimetrik kuantum mekaniği çerçevesi içinde tam çözümleri olabileceği gösterildi. Serbest bölge içinde, lokalize ancak pozitif enerji seviyelerine sahip n -parametrelili potansiyel ailelerinin oluşturulması için metodun nasıl uygulanacağı gösterildi.

Anahtar kelimeler: Süpersimetrik kuantum mekaniği, Şekil değişmezli potansiyeller, Hamilton hiyerarşisi, Süpersimetrik pertürbasyon teorisi, Hulthén potansiyeli, N -boyutlu uzay, Pozisyona bağlı kütle, Serbest bölge içinde bağlı sistemler.

*To My Dear Wife ASLIHAN
and
To My Dear Parents and Brother*

**20. YÜZ YIL
T.C. YATIRIM MENKUL DEĞERLER A.Ş.**

ACKNOWLEDGEMENTS

I would like to express my thanks to my dear supervisor, Prof. Dr Bülent Gönül, not only for his guidance, encouragement, and suggestions but also for his really great and warm friendship in the construction of such a work. There is no doubt that our many discussions have always been constructive and instructive.

I am indebted to my colleagues in the department of Engineering Physics. In particular, I would like to thank to Assoc. Prof. Dr Ramazan Koç and Res. Assist. Eser Olğar for their guidance to me through the use of the software *Mathematica* which made my calculations simple, and to Assist. Prof. Dr Andrew Beddall for \LaTeX that saved considerably my time in the preparation of the thesis. I also would like to thank to my dear friend Murat Oduncuoğlu for his reassuring conversations at teatime.

My special thanks go to my mother, father and brother, who were a prop to me during this work by their faithful prays.

To my wife, Aslihan, I offer my sincere thanks for her unshakable faith in me and her willingness to endure with me; thank you Aslı - for everything.

TABLE OF CONTENTS

CHAPTER	
1 INTRODUCTION	1
2 BASIC FORMALISM	7
2.1 Factorization of the Hamiltonian	7
2.2 Shape-invariance	11
3 SUPERSYMMETRIC PERTURBATION THEORY	14
3.1 Formalism of SSPT	14
3.1.1 For ground-state	16
3.1.2 For excited states	18
3.2 Example: Perturbed harmonic oscillator	19
3.2.1 Ground-state corrections	20
3.2.2 Excited states corrections	21
3.3 Conclusion	23
4 HAMILTONIAN HIERARCHY AND THE HULTHÉN POTENTIAL	24
4.1 Supersymmetric solution of the Hulthén potential	25
4.2 Conclusion	29
5 SUPERSYMMETRY AND THE RELATIONSHIP BETWEEN A CLASS OF SINGULAR POTENTIALS IN ARBITRARY DIMENSIONS	30
5.1 The Schrödinger equation in N -dimensional space	31
5.2 Mappings between two distinct systems	33
5.2.1 Quasi-exactly solvable case	33
5.2.2 Exactly solvable case	36
5.3 Conclusion	39
6 ON THE RELATIONSHIP BETWEEN THE SCREENING COULOMB AND ANHARMONIC OSCILLATOR POTENTIALS IN ARBITRARY DIMENSIONS	40
6.1 The Schrödinger equation in N -dimensional space	42
6.2 Mappings between the two distinct systems	42
6.3 Results and Discussion	46
6.4 Concluding Remarks	47

7	QUANTUM SYSTEMS WITH POSITION-DEPENDENT MASS	49
7.1	Supersymmetric Approach	49
7.1.1	Shape-Invariant Potentials with Position-Dependent Mass	50
7.1.1.1	Harmonic-oscillator-like Potential	51
7.1.1.2	Morse-like Potential	52
7.1.2	Conclusion	54
7.2	Schrödinger Equation with Position-Dependent Mass Under Point Canonical Transformations	54
7.2.1	Transformation of Schrödinger Equation	55
7.2.1.1	Application with Harmonic Oscillator Potential	57
7.2.1.2	Application with Morse Potential	58
7.2.2	Conclusion	58
7.3	Application to Quantum Well Lasers	59
7.3.1	A Physical Background on the Effective Mass	60
7.3.2	Pösch-Teller Type Potential	61
7.3.3	Effective-Mass Hamiltonians	64
7.3.4	Applications	65
7.3.4.1	For $m(x) = m_0 \exp(\lambda x)$	66
7.3.4.2	For $m(x) = m_0 \left(\frac{a+q^2}{1+q^2}\right)^2$	67
7.3.5	Discussion and Conclusion	68
8	BOUND STATES IN CONTINUUM GENERATED BY SUPERSYMMETRIC QUANTUM MECHANICS	72
8.1	Theoretical considerations	73
8.2	The One-Parameter Family of Potentials	75
8.2.1	The Free Particle on the half-line	75
8.2.2	Coulomb Potential	77
8.3	The Two-Parameter Family of Potentials	81
8.3.1	Coulomb Potential	81
8.4	BIC of complex potentials generated by SUSYQM	81
8.4.1	Flat Potential	83
8.5	Conclusion	84
9	GENERAL CONCLUSION	86
	REFERENCES	88
	PUBLICATIONS	94
	VITA	95

LIST OF TABLES

Table

4.1	Energy eigenvalues of the Hulthén potential as a function of screening parameter for various states in atomic units.	28
6.1	The first four eigenvalues of the screening Coulomb potential in Eq. (6.6) as a function of the screening parameter in atomic units.	48
6.2	Ground-state eigenvalues of the anharmonic potential in Eq. (6.13)	48
7.1	Single-band effective mass Hamiltonians, Eq. (7.83).	65

LIST OF FIGURES

Figure

8.1	Example of potential $\hat{V}(r)$ and the associated BIC wave function $\hat{\psi}_0(r)$ (dashed) in the one-parameter family for $k = 1.0$ and $\lambda = 0.5$	76
8.2	Example of potential $\hat{V}(r)$ and the associated BIC wave function $\hat{\psi}_0(r)$ (dashed) in the one-parameter family for $k = 1.0$ and $\lambda = 5.0$	76
8.3	The BIC potential (solid) derived from the repulsive Coulomb potential (dotted) for $Z = 6, k = 1.0$ and $\lambda = 1.0$. It is explicitly seen again how the BIC potential oscillates around the original Coulomb potential.	79
8.4	The wave functions (unnormalized) corresponding to the BIC potential (solid) and the repulsive Coulomb potential (dotted) for $Z = 6, k = 1.0$ and $\lambda = 1.0$	79
8.5	The BIC potential (solid) derived from the attractive Coulomb potential (dotted) for $Z = -2, k = 1.0$ and $\lambda = 0.5$. It is explicitly seen how the BIC potential oscillates around the original Coulomb potential.	80
8.6	The wave functions (unnormalized) corresponding to the BIC potential (solid) and the attractive Coulomb potential (dotted) for $Z = -2, k = 1.0$ and $\lambda = 0.5$	80
8.7	Isospectral two-parameter family of BIC potential derived from the attractive Coulomb potential (dotted) for $Z = -2, \lambda_1 = -1.1$ (solid) and $\lambda_1 = -1.001$ (dashed curve) for fixed $\lambda_0 = -1.1$	82
8.8	Isospectral two-parameter family of BIC potential derived from the attractive Coulomb potential (dotted) for $Z = -2, \lambda_0 = -1.1$ (solid) and $\lambda_0 = -1.01$ (dashed curve) for fixed $\lambda_1 = -1.001$	82
8.9	The complex potential generated from initial potential ($y = 0$ line) using the parameter values $k = 1, C_r = 1, C_i = 1$. Real and Imaginary parts are shown by solid line and dashed line, respectively.	84

CHAPTER 1

INTRODUCTION

Physicist have long striven to obtain an unified description of all basic interactions of nature, i.e. strong, electro-weak, and gravitational interactions. Several ambitious attempts have been made in the last two decades, and it is now widely felt that supersymmetry (SUSY) is a necessary ingredient in any unifying approach [1]. SUSY relates bosonic and fermionic degrees of freedom. The algebra involved in SUSY is a graded Lie algebra which closes under a combination of commutation and anti-commutation relations.

Supersymmetry is not an independent theory or model by itself, but rather a concept incorporated into other theories via some framework-specific mechanism. In field or string theory, this incorporation actually adds new content to the physical predictions of the theories. If supersymmetry is real, every known fermion should have a "*partner*" boson, and vice-versa. However, no two particles we know today are supersymmetric partners of each other. So there must be an entire set of new particles we have never seen before. It is not seen the "*superpartners*" because they have very large masses out of the reach of present-day accelerators. But, if supersymmetry is really a "*symmetry*" in nature, we expect partner particles at least to have equal masses. Thus theorists postulate that supersymmetry is "*broken*" in our nature. How or why this is so is not really known, and a large body of research in the last two decades has been devoted to studying mechanism which might break supersymmetry.

The SUSY idea has led to new insights in the studies of nuclear physics, condensed matter physics, statistical physics and mathematical physics [2]. In particular, supersymmetric quantum mechanics (SUSYQM), which originally introduced by Nicolai in 1976 [3] and re-discovered in 1981 by Witten [4], nowadays attracts much attention [5].

As a one-particle, non-relativistic, quantum mechanical theory, SUSYQM can be understood entirely in terms of familiar old quantum mechanics. It recast traditional methods of solving problems in quantum mechanics in a novel, more elegant framework. The starting point in the quantum mechanics is to solve the

Schrödinger equation which reads in one-dimension

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi(x) \quad (1.1)$$

The quantity in brackets is called "*Hamiltonian*". The normal program is as follows: plug in the potential energy function $V(x)$ that describes our system of interest and solve the differential Eq. (1.1) for the energies E and wave functions $\psi(x)$ which are "*normalizable*". This limitation restricts the possible values of E to a discrete set of values, generally shown by E_n .

The SUSYQM tells us that we should consider two (or more) potentials $V(x)$ simultaneously in solving that equation. Applying an operator to the initial equation (1.1), we transform it into a new Schrödinger equation, with a new interaction potential. The supersymmetric formalism implies that all the solutions of this new equation are expressed by algebraic expressions in terms of the solutions of the initial equation. Moreover, the new potential is expressed in terms of the initial potential and of one particular solution of the initial equation, called *factorization solution*. Different factorization solutions provide different transformed potentials, but the algebraic structure of the method does not depend on the particular choice of factorization solution. The trick is to pick potentials $V(x)$ which yield *exactly* the same spectrum of energies E_n . Such potentials are called *partner potentials* in SUSYQM. In summary, it can be said that supersymmetric transformations, via operator technique, allow the construction of new Schrödinger equations, starting from a given one, and that everything known about the initial equation is also known about the new ones. This supersymmetric transformations leads us to two types of applications:

- Supersymmetric transformations can add, remove or modify bound states by acting on the zeroes of the wavefunctions;
- they can modify cross sections by acting on the phase of the wavefunction.

The remove (suppression) of bound states and phase shift calculations have already been studied in our earlier works [6, 7], and other transformations such as the adding procedure is one of our works discussed in this thesis.

Solving the *partner potentials* sometimes depend on the nature of the problem. Since the appearance of quantum mechanics, there has been continual interest in models for which the corresponding Schrödinger equation is exactly solvable. Solvable potential problems have played a dual role since the beginnings of quantum mechanics. First, they represented useful aids in modeling

realistic physical problems, and second, they offered an interesting field of investigation in their own right. Related to this latter area, the concept of solvability has changed to some extent in recent years. With regards to solvability of the Schrödinger equation there are three interesting classes of the potentials;

The first class is the exactly solvable potentials allowing to obtain in explicit form all energy levels and corresponding wave functions. The hydrogen atom and harmonic oscillator are the best-known examples of this type. We discuss the solution of Hulthén potential within the frame of SUSYQM and give exact solutions for $\ell = 0$ angular momentum quantum states. For $\ell > 0$ momenta, we develop a semi-algebraic technique to solve more accurately its corresponding eigenvalues using the supersymmetric formulae and the first-order perturbation theory. The results obtained for all states are compared with other numerical calculations and seen well agreement between them. In addition, the work presented here leads us to understand explicitly the usual concept of shape invariance and to suggest one extension of this concept, assuming the possibility of constructing a general expression for all potentials of the super-family. In connection with this, an interesting question arises that if the extended shape invariance is a necessary condition to the potential to be exactly solvable. The answer of this question is discussed in a great detail, and we find that the Hulthén potential for $\ell = 0$ case is not shape invariant but exactly solvable. However, the super family of the Hulthén potential is shape invariant.

The second class is the so-called quasi-exactly solvable (QES) potentials for which a finite number of eigenstates of corresponding Hamiltonian can be found exactly in explicit form. The first examples of QES potentials were given in [8]. Subsequently several methods for generating QES potentials were worked out and as a result many QES potentials were found [9]. Three different methods that are based respectively on the polynomial ansatz for wave functions, the point canonical transformation, and the SUSYQM are described in the work of Gangopadhyaya and his co-workers [9]. Recently, an anti-isospectral transformation called also as duality transformation was introduced [10]. This transformation relates the energy levels and wave functions of two QES potentials. In [11] a new QES potential was discovered using this anti-isospectral transformation. In the following, two examples for the QES potentials have been investigated and the relation between the eigenvalues of these potentials has been obtained in the N -dimensional space using the SUSYQM.

The third class is the conditionally-exactly solvable (CES) potentials for which the eigenvalue problem for the corresponding Hamiltonian is exactly solvable only when the parameters of the potential obey certain conditions. Such a class of potentials was first considered in [12]. It is interesting to note that in [13]

it was demonstrated the equivalence of the condition required for the potential obtained in [12] to be a CES potential with the condition that this potential can be put in an explicitly supersymmetric form. Recently, new examples of CES potentials have been discovered [14].

Physics is permanently developing in a tight interplay with Mathematics. It is of importance to know therefore whether some familiar problems are a particular case of a more general scheme or to search if a map between the radial equations of two different systems exists. It is hence worthwhile to study the Schrödinger equation in the arbitrary dimensional spaces which has attracted much more attention to Physics. Such an investigation is the one of the objectives of the present thesis.

In addition to all, the study of quantum systems with position-dependent effective masses has taken the subject of much activity in recent years. The Schrödinger equation with nonconstant mass provides an interesting and useful model for the description of many physical problems. The effective-mass approximation is an important and widely used tool for the determination of the electronic properties of semiconductors and quantum dots [15]. Interest in this kind of approach is growing nowadays due to recent progress in crystal-growth techniques for the production of nonuniform semiconductor specimens. Much work has been done in recent years on this study and some exactly soluble models with smooth potentials and mass steps have been discovered [16]. The concept of effective mass is also relevant in connection with the energy density functional (EDF) approach to the quantum many-body problem. The EDF formalism has provided reasonable theoretical predictions of many experimental properties for several quantum many-body systems. Within the EDF approach, the nonlocal terms of the associated potential can be often interpreted as a position dependence on an appropriate effective mass. This formalism has been extensively used in many applications [17].

Since the momentum and the mass operators no longer commute in case of spatially varying mass, a question concerning the correct form of the kinetic energy operator of the generalized Hamiltonian has arisen. This problem of ordering ambiguity is a long standing one in quantum mechanics. The application of supersymmetry ideas to nonrelativistic quantum mechanics has provided an understanding of analytically solvable Hamiltonians, as well as a set of powerful approximate schemes for dealing with problems admitting of no exact solution [5]. The concept of shape invariance has played a fundamental role in these developments, since it allowed both for (i) a unified treatment of all the already known textbook cases of potentials admitting analytical solutions and (ii) a systematic procedure for generating novel exactly solvable systems. We have through the

thesis work considered the application of the supersymmetric approach to quantum systems with position-dependent mass and to extend the concept of shape invariance to the nonconstant mass scenario. Alternatively, the more powerful 'point canonical transformation' method has also been applied to the Schrödinger equation with position dependent mass and the results obtained are checked out within the framework of the SUSYQM.

Finally, we discuss the general problem of using SUSY transformations to find out the families of potentials which are phase equivalent to initial one with a less bound state. The answer of the following interesting question leads us to families of *isospectral* potentials, that have bound states in the continuum: Since there are partner potentials, it is what the various forms of these partners to initial one are. When the answer is given by SUSYQM, it is seen that there may exist n-parameter family of potentials having bound states in the continuum. Starting from the Schrödinger equation for a potential whose ground-state wave function is known, this method permits one to generate families of new potentials, which may look quite different from the original one, but have exactly the same spectrum. To generate new potentials with bound states in the continuum, SUSYQM formalism is extended for obtaining isospectral potentials and applied to potentials with a continuum of scattering states. It is seen that, while the wave functions in the continuum of the original potential are non-normalizable, the wave functions generated by SUSYQM are normalizable and thus representing a bound state. One can construct one-parameter or two-parameter families of supersymmetric partner potentials with one or two bound states in the continuum.

The plan of the thesis is as follows. After giving some introductory remarks about the basic formalism of the method, we introduce the *perturbation theory* in the frame of SUSY. We deal with the Hamiltonian hierarchy problem of the Hulthén potential in Chapter 4 within the frame of the SUSYQM and find that the associated supersymmetric partner potentials simulate the effect of the centrifugal barrier. Incorporating the supersymmetric solutions and using the first-order perturbation theory, we obtain an expression for the energy levels of the Hulthén potential which gives satisfactory values for the non-zero angular momentum states.

In Chapter 5, we will obtain in N -dimensional space the eigenvalues of the potentials $V_1(r) = \frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3} + \frac{A_4}{r^4}$, $V_2(r) = B_1 r^2 + \frac{B_2}{r^2} + \frac{B_3}{r^4} + \frac{B_4}{r^6}$, and of the special cases of these potentials such as the Kratzer and Goldman-Krivchenkov potentials. The explicit dependence of these potentials in higher-dimensional space is discussed, which have not been previously linked.

A mapping is obtained relating radial screened Coulomb systems to radial

anharmonic oscillators in N -dimensional space in Chapter 6. We have shown, using the formalism of SUSYQM, that the exact solutions of these potentials exist when the parameters satisfy certain constraints.

The solutions to the Schrödinger equation describing a particle characterized by a position-dependent effective mass has been considered in Chapter 7.1 since the study on such systems has been the subject of much activities in recent years. After giving some introductory remarks, we have first applied the SUSYQM to quantum systems with position-dependent mass and extend the concept of shape invariance to the nonconstant mass scenario. We introduce an alternative but more powerful transformation technique in Chapter 7.2 for the systems of interest, which describes the mapping procedure between two distinct systems having constant and spatially varying masses.

With experiences gained through the works presented in Chapter 7.1 and Chapter 7.2, we proceed in Chapter 7.3 with a physical application involving Pösch-Teller type potential. That potential has a great importance in quantum well lasers since it possesses an interesting property that a suitable choice of its parameters delivers unity transmission of free states in quantum well profiles. We also observe that the distinct effective mass Hamiltonians proposed in the literature in fact describe exactly equivalent systems having identical spectra and wave functions as far as exact solvability is concerned. This observation clarifies the Hamiltonian dependence of the band-offset ratio for quantum wells.

In the last Chapter of the thesis, we introduce an interesting case in which some quantum mechanical systems have been constructed of local potentials with bound states embedded in the continuum range. In the beginning of the Century, it was discovered that a class of potentials may give isolated levels embedded in the continuum of positive energy states. Since some scalar potentials that tend to zero at infinite distance possess one or several discrete *positive* bound states for the Schrödinger single-particle wave equation, how the method of SUSYQM can be used to generate families of such potentials has been considered.

CHAPTER 2

BASIC FORMALISM

The fundamental results of supersymmetric quantum mechanics (SUSYQM) can be derived in a number of ways, differing only in detail but not in content. We take as starting point the factorization of the Schrödinger Hamiltonian, a second-order differential operator, into two first-derivative factors. We first try to guess the form that such operators must take. As we work to make the pieces of the puzzle fit together, the basic equations and framework of SUSYQM will pop right out.

2.1 Factorization of the Hamiltonian

We begin with the time-independent Schrödinger equation -introduced by Eq. (1.1)- describing a particle of mass m moving in a one-dimensional potential $V(x)$. Focus on its ground state wave function

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi_0^+(x)}{dx^2} + V^+(x) \psi_0^+(x) = E_0^+ \psi_0^+(x) \quad (2.1)$$

Why the superscripts? The '+' distinguishes this system from a 'partner' system to be constructed shortly. Why the ground state? Well, the reason will be clear soon.

Let H^+ be the Hamiltonian operator for this system:

$$H^+ = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^+(x) . \quad (2.2)$$

It will be very convenient to make the right-hand side of (2.1) be zero, as follows:

$$(H^+ - E_0^+) \psi_0^+(x) = 0 . \quad (2.3)$$

This has the effect of scaling the potential by the constant value E_0^+ . Such scaling does not change the physics. In particular, the wave functions are unaffected. Postulate that the operator on the left-hand side can be factored as

$$H^+ - E_0^+ = A^+ A^- . \quad (2.4)$$

and it is guessed that these operators are in the form

$$\begin{aligned} A^+ &= -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \\ A^- &= +\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \end{aligned} \quad (2.5)$$

where $W(x)$ is some function, called *superpotential* in SUSYQM, whose exact form will be determined soon.

To examine what exactly does A^+A^- operators do to a function $f(x)$, apply it directly to that function:

$$\begin{aligned} (A^+A^-)f(x) &= \left[-\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right] \left[\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right] f(x) \\ &= \left[-\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right] \left[\frac{\hbar}{\sqrt{2m}} \frac{df(x)}{dx} + W(x)f(x) \right] \\ &= -\frac{\hbar^2}{2m} \frac{d^2 f(x)}{dx^2} - \left[\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} W(x)f(x) \right] \\ &\quad + W(x) \frac{\hbar}{\sqrt{2m}} \frac{df(x)}{dx} + W^2(x)f(x) \\ &= \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}} W'(x) + W^2(x) \right] f(x). \end{aligned}$$

For this result, it is required

$$W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x) = V^+(x) - E_0^+. \quad (2.6)$$

If we can find a function W satisfying this equation, then we will have successfully factorized the Hamiltonian in (2.4).

There is a clever way to solve for W . Suppose it were the case that A^- annihilated the ground state wave function ψ_0^+ , that means, $A^-\psi_0^+ = 0$. Then $A^+A^-\psi_0^+$ would also be zero, and Eq. (2.3) would be satisfied. So we look for a W which forces this to be happened. It is

$$A^-\psi_0^+(x) = \frac{\hbar}{\sqrt{2m}} \frac{d\psi_0^+(x)}{dx} + W(x)\psi_0^+(x) = 0, \quad (2.7)$$

and giving that

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \psi_0^+(x). \quad (2.8)$$

And also from this equation, it is obvious that

$$\psi_0^+(x) = N \exp \left[-\frac{\sqrt{2m}}{\hbar} \int^x W(y) dy \right], \quad (2.9)$$

where N is the normalization constant. Eq. (2.8) says that we can find $W(x)$ explicitly if we know the ground state wave function. To do this, we are to solve

the Schrödinger equation completely. We find that we need to know the solutions already. But Eq. (2.9) says something different: It says that if we can find W independently of any knowledge of ψ_0^+ then Eq. (2.9) tells us the ground state wave function. And this we could do by solving the differential equation (2.6) for W and E_0^+ . In essence, we have transformed the Schrödinger equation into a completely different, but equivalent one. In more general terms, we have found a first-order, nonlinear differential equation equivalent to our original second-order, linear differential equation. Such first-order nonlinear equivalents are classified under the general name of *Riccati equations*.

In order to obtain a partner potential and to factorize the Schrödinger equation, the only thing we do is to reverse the order of the factors A^\pm and get something interesting. Reversing the order of factors leads to a new Hamiltonian such as

$$A^- A^+ = H^- - E_0^+ , \quad (2.10)$$

where $H^- = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^-(x)$ and it is found that

$$\begin{aligned} (A^- A^+) f(x) &= \left[\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right] \left[-\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right] f(x) \\ &= \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}} W'(x) + W^2(x) \right] f(x) , \end{aligned}$$

for this result, it is required that the potential of this new Hamiltonian H^- must satisfy

$$W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x) = V^-(x) - E_0^+ . \quad (2.11)$$

From that equation, it is seen that there is a relation between partner potentials and it can be found as following: If we rewrite that equation with a little difference,

$$V^-(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x) + \frac{\hbar}{\sqrt{2m}} W'(x) - \frac{\hbar}{\sqrt{2m}} W'(x) + E_0^+ , \quad (2.12)$$

and using another fact,

$$V^+(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x) + E_0^+ , \quad (2.13)$$

one obtains

$$V^-(x) = V^+(x) + 2 \frac{\hbar}{\sqrt{2m}} W'(x) . \quad (2.14)$$

When the superpotential term is replaced in terms of wave function, we get finally,

$$V^-(x) = V^+(x) - \frac{\hbar^2}{m} \frac{d^2}{dx^2} \ln \psi_0^+ . \quad (2.15)$$

We can easily find interesting connections between these two systems described by H^+ and H^- . First, consider an eigenfunction of H^+ satisfying

$$H^+\psi_n^+ = E_n^+\psi_n^+, \quad (2.16)$$

where E_n^+ is the n^{th} energy of H^+ . In terms of A^\pm , we get

$$(A^+A^- + E_0^+)\psi_n^+ = E_n^+\psi_n^+. \quad (2.17)$$

Applying A^- to the left of both equations, we obtain

$$A^-A^+A^-\psi_n^+ + E_0^+A^-\psi_n^+ = E_n^+A^-\psi_n^+, \quad (2.18)$$

then we can group terms

$$(A^-A^+ + E_0^+)(A^-\psi_n^+) = E_n^+(A^-\psi_n^+), \quad (2.19)$$

which we notice Eq. (2.10) to get

$$H^-(A^-\psi_n^+) = E_n^+(A^-\psi_n^+). \quad (2.20)$$

This is the interesting thing that says $A^-\psi_n^+$ is a solution to the Schrödinger equation for H^- , with energy E_n^+ . It is obvious that we can go the other way around. Starting with an eigenfunction ψ_m^- of H^- with energy E_m^- leads us to obtain

$$(A^-A^+ + E_0^+)\psi_m^- = E_m^-\psi_m^-. \quad (2.21)$$

Applying A^+ to the left of both sides and grouping terms appropriately we get

$$(A^+A^- + E_0^+)(A^+\psi_m^-) = E_m^-(A^+\psi_m^-), \quad (2.22)$$

or shortly

$$H^+(A^+\psi_m^-) = E_m^-(A^+\psi_m^-), \quad (2.23)$$

so $A^+\psi_m^-$ is an eigenfunction of H^+ , with an energy E_m^- .

In summary, A^- applied to any eigenfunction of H^+ gives an eigenfunction of H^- , with the same energy eigenvalue. A^+ applied to any eigenfunction of H^- gives an eigenfunction of H^+ , again with the same eigenvalue. This guarantees that the two systems in fact have *identical* energy spectra, almost. Notice that A^- kills the ground state wave function ψ_0^+ of H^+ , by construction, so H^+ has no corresponding eigenstate at the same energy. The lowest state of H^- must then correspond to the first 'excited' state of H^+ , or $E_0^- = E_1^+$. In general,

$$E_n^- = E_{n+1}^+. \quad (2.24)$$

From the standard quantum mechanics, the spectrum of a Hamiltonian in one dimension is *nondegenerate*. That is, to any one energy corresponds only one wave function. So when we have found an eigenfunction of H^- with energy E_n^- , we know it is the n^{th} eigenstate of H^- . So we can say

$$\begin{aligned} A^- \psi_n^+ &= a \psi_{n-1}^- \\ A^+ \psi_m^- &= b \psi_{m+1}^+ , \end{aligned} \quad (2.25)$$

where a and b are normalization constants to be determined. These constants are necessary we want all the wave functions here to be normalizable ones. To find out these constants, we remember some 'rules' of quantum mechanics: A 'normalized' eigenvector $|\psi\rangle$ satisfies not only $\langle\psi|\psi^*\rangle < \infty$, but also $\langle\psi|\psi^*\rangle = 1$ where ψ^* is the complex conjugate of ψ . Consider the norm of $A^-|\psi_n^+\rangle = a|\psi_{n-1}^-\rangle$. Using adjointness of A^\pm on the one hand, we get

$$\begin{aligned} \langle\psi_n^+|A^+A^-|\psi_n^+\rangle &= \langle\psi_n^+|(H^+ - E_0^+)|\psi_n^+\rangle \\ &= (E_n^+ - E_0^+) \langle\psi_n^+|\psi_n^+\rangle \\ &= E_n^+ - E_0^+ \end{aligned}$$

since $|\psi_n^+\rangle$ is assumed to be normalized eigenfunction of H^+ . On the other hand,

$$\begin{aligned} \langle\psi_n^+|A^+A^-|\psi_n^+\rangle &= (A^-|\psi_n^+\rangle)^\dagger (A^-|\psi_n^+\rangle) \\ &= a^* |\psi_{n-1}^-\rangle^\dagger a |\psi_{n-1}^-\rangle \\ &= a^* a \langle\psi_{n-1}^-|\psi_{n-1}^-\rangle \\ &= |a|^2 \end{aligned}$$

since $|\psi_{n-1}^-\rangle$ is also supposed to be normalized. Comparing these two results and choosing a to be real, $a = \pm\sqrt{E_n^+ - E_0^+}$. A perfectly analogous argument shows that $b = \pm\sqrt{E_m^- - E_0^+}$. Thus the operator equations become

$$\begin{aligned} A^- \psi_n^+ &= \pm\sqrt{E_n^+ - E_0^+} \psi_{n-1}^- , \\ A^+ \psi_m^- &= \pm\sqrt{E_m^- - E_0^+} \psi_{m+1}^+ . \end{aligned}$$

These give the complete relations between the wave functions of the partner systems.

2.2 Shape-invariance

As it is well known, there are a number of analytically solvable potentials in nonrelativistic quantum mechanics for which all the energy eigenvalues and eigenfunctions are explicitly obtained. Some examples may be the Coulomb,

harmonic oscillator, Eckart, Morse and Pösch-Teller potentials. These potentials are solvable since they have the property of "shape-invariance" as pointed out by Gendenshtein [18].

What one means by shape invariant potentials is given as following. If the pair of supersymmetric partner potentials $V^\pm(x)$ defined by Eqs. (2.6) and (2.11) are similar in shape, differing only in the (x -independent) parameters that appear in them, then they are said to be shape invariant. More specifically, if $V^+(x; a_0)$ is any potential, its supersymmetric partner $V^-(x; a_0)$ must satisfy the requirement of shape-invariance

$$V^-(x; a_1) = V^+(x; a_2) + R(a_1) \quad (2.26)$$

where a_0 is a set of parameters, a_1 is a function of a_0 , [$a_1 = f(a_0)$], and the remainder $R(a_1)$ is independent of x .

The eigenstates of shape invariant potentials can now be easily obtained. To do this, it is constructed a series of Hamiltonians H^s , $s = 0, 1, 2, \dots$ where $H^{(0)} \equiv H^+$, $H^{(1)} \equiv H^-$.

$$H^{(s)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^+(x; a_s) + \sum_{k=1}^{s+1} R(a_k) \quad (2.27)$$

where $a_s = f^s(a_0)$, i. e., the function f applied s times. If it is compared the spectrum of $H^{(s)}$ with that of $H^{(s+1)}$, in view of Eqs. (2.26) and (2.27), then one gets

$$\begin{aligned} H^{(s+1)} &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^+(x; a_{s+1}) + \sum_{k=1}^{s+1} R(a_k) \\ &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^-(x; a_s) + \sum_{k=1}^s R(a_k) . \end{aligned} \quad (2.28)$$

Comparing Eqs. (2.27) (2.28), it is seen that H^s and H^{s+1} are supersymmetric partner Hamiltonians and hence have identical bound-state energy spectra except for the lowest level of H^s whose energy is

$$E_0^{(s)} = \sum_{k=1}^s R(a_k) . \quad (2.29)$$

This follows from Eq. (2.27) and the fact that $E_0^+ = 0$. On going back from H^s to H^{s-1} , one would reach $H^{(1)} (= H^-)$ and $H^{(0)} (= H^+)$, whose ground-state energy is zero and its n th energy level being coincident with the ground state of Hamiltonian H^n ($n = 1, 2, 3, \dots$). Hence, the complete energy spectrum of H^+ is given by

$$E_n^{(+)} = \sum_{k=1}^n R(a_k) \quad , \quad E_0^{(+)} = 0 . \quad (2.30)$$

Summarizing up, we can give the fundamental equations that is heavily used in SUSYQM and also in coming Chapters:

$$A^\pm = \mp \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x)$$

$$H^\pm = A^\pm A^\mp = -\frac{\hbar}{\sqrt{2m}} \frac{d^2}{dx^2} + V^\pm(x)$$

$$V^\pm(x) = W^2(x) \mp \frac{\hbar}{\sqrt{2m}} W'(x) + E^\pm$$

$$W(x) = \pm \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \psi_0^\mp$$

$$\psi_0^\pm(x) = N^\pm \exp \left[\mp \frac{\sqrt{2m}}{\hbar} \int^x W(y) dy \right]$$

$$E_n^\mp = E_{n\pm 1}^\pm$$

$$A^\mp \psi_n^\pm = \sqrt{E_n^\pm - E_0^\mp} \psi_{n\mp 1}^\mp$$

CHAPTER 3

SUPERSYMMETRIC PERTURBATION THEORY

In this Chapter, we examine briefly the *perturbation theory*, a method to solve approximately systems difficult to solve analytically, in the framework of SUSYQM. The *supersymmetric perturbation theory* (SSPT) is entirely equivalent to a method in standard quantum mechanics, called *logarithmic perturbation theory* that is not an aim of this thesis. This equivalence allows the tools of each theory to resolve difficulties or inefficiencies in the other. The demonstration of these two methods are studied explicitly in Ref. [19].

In this thesis we present the supersymmetric version of perturbation theory, in theory, and proof the theory with the simplest example, perturbed harmonic oscillator.

3.1 Formalism of SSPT

The goal in SUSYQM is to solve the Riccati equation as we mention before. Let's start with:

$$W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) = V(x) - E_0, \quad (3.1)$$

where $V(x)$ is the potential we are interested in and E_0 is its ground state energy. If we find $W(x)$, we have of course found the ground-state wave function, via.

$$\psi_0(x) = N \exp \left[-\frac{\sqrt{2m}}{\hbar} \int_0^x W(y) dy \right], \quad (3.2)$$

where N is the normalization constant. If, as illustrated earlier, $V(x)$ is a shape-invariant potential, we can in fact find the entire spectrum of bound-state energies and wave functions via ladder operators.

Suppose, now, that we are interested in a potential $V(x)$ for which we don't know $W(x)$ exactly. So, our Hamiltonian is not factorizable, but 'almost factorizable'. Specially, we assume that V differs from a potential V_0 , for which we have solved the Riccati equation, by a small amount, controlled by a parameter λ . We assume that everything in sight is expandable in successive orders of λ . We than try to solve the Riccati equation up to a given order in λ .

We are thus attempting to solve the following equation:

$$W^2(x; \lambda) - \frac{\hbar}{\sqrt{2m}} W'(x; \lambda) = V(x; \lambda) - E_0(\lambda) . \quad (3.3)$$

We postulate we can expand each function in terms of λ :

$$V(x; \lambda) = \sum_{n=0}^{\infty} \lambda^n V_n(x) \quad (3.4)$$

$$E_0(\lambda) = \sum_{n=0}^{\infty} \lambda^n E_0^n \quad (3.5)$$

$$W(x; \lambda) = \sum_{n=0}^{\infty} \lambda^n W_n(x) \quad (3.6)$$

Since we begin a problem given a specific $V(x; \lambda)$, we know that the $V_n(x)$ are given by

$$V_n(x) = \frac{1}{n!} \left. \frac{\partial^n V}{\partial \lambda^n} \right|_{\lambda=0} , \quad (3.7)$$

using the Taylor's series expansion of V about $\lambda = 0$. The E_0^n and W_n , for which we must solve, are the n^{th} order corrections to the ground-state energy and superpotential. So we can calculate:

$$W^2(x; \lambda) = W_0^2(x) + \lambda(2W_0W_1) + \lambda^2(W_1^2 + 2W_0W_2) + \dots \quad (3.8)$$

$$W'(x; \lambda) = W_0'(x) + \lambda W_1'(x) + \lambda^2 W_2'(x) + \dots \quad (3.9)$$

We plug these into the Riccati equation for $V(x)$ and equate terms of equal order in λ . In zeroth-order, we have

$$W_0^2(x) - \frac{\hbar}{\sqrt{2m}} W_0'(x) = V_0(x) - E_0^0 . \quad (3.10)$$

This is the equation for which we are supposed already to know the solution. The first-order terms become:

$$2W_0(x)W_1(x) - \frac{\hbar}{\sqrt{2m}} W_1'(x) = V_1(x) - E_0^1 . \quad (3.11)$$

To second-order:

$$W_1^2(x) + 2W_0(x)W_2(x) - \frac{\hbar}{\sqrt{2m}} W_2'(x) = V_2(x) - E_0^2 . \quad (3.12)$$

We note something very convenient about these results. The equation to be solved at each order is linear and first-order in the variable for which we want to solve. All other terms in the equation involve terms from lower-order equations. Thus, if we solve the equation for one order, we then have enough information to solve the next order equation explicitly. The only difficulty at each step is to find to correct value for the constant E_0^n . Shortly we will see how to do this.

Let's solve the first-order equation. Rearranging the terms,

$$W_1' - \frac{2\sqrt{2m}}{\hbar} W_0 W_1 = \frac{\sqrt{2m}}{\hbar} (E_0^1 - V_1). \quad (3.13)$$

Multiply both sides by the 'integrating factor' $\exp(-\frac{2\sqrt{2m}}{\hbar} \int_0^x W_0(y) dy)$:

$$\exp\left(-\frac{2\sqrt{2m}}{\hbar} \int_0^x W_0 dy\right) W_1' - \exp\left(-\frac{2\sqrt{2m}}{\hbar} \int_0^x W_0 dy\right) \frac{2\sqrt{2m}}{\hbar} W_0 W_1 = \frac{\sqrt{2m}}{\hbar} \exp\left(-\frac{2\sqrt{2m}}{\hbar} \int_0^x W_0 dy\right) (E_0^1 - V_1)$$

The point of that exercise was to turn the left-hand side into the derivative of a product:

$$\frac{d}{dx} \left[\exp\left(-\frac{2\sqrt{2m}}{\hbar} \int_0^x W_0(y) dy\right) W_1(x) \right] = \frac{\sqrt{2m}}{\hbar} \exp\left(-\frac{2\sqrt{2m}}{\hbar} \int_0^x W_0(y) dy\right) (E_0^1 - V_1(x)).$$

Notice that we now have

$$\frac{d}{dx} \left[\frac{1}{N^2} \psi_0^0(x)^2 W_1(x) \right] = \frac{\sqrt{2m}}{\hbar} \frac{1}{N^2} \psi_0^0(x)^2 (E_0^1 - V_1(x)), \quad (3.14)$$

using (3.2) in zeroth-order. Cancel the factors of N and integrate.

$$\psi_0^0(y)^2 W_1(y) \Big|_{-\infty}^x = \frac{\sqrt{2m}}{\hbar} \int_{-\infty}^x \psi_0^0(y)^2 (E_0^1 - V_1(y)) dy. \quad (3.15)$$

Now, $\psi_0^0(x)^2$ vanishes for $x \rightarrow -\infty$, so we have an expression for W_1 ;

$$W_1(x) = \frac{\sqrt{2m}}{\hbar} \frac{1}{\psi_0^0(x)^2} \int_{-\infty}^x \psi_0^0(y)^2 (E_0^1 - V_1(y)) dy. \quad (3.16)$$

To solve the equation for W_2 , we mimic the preceding calculation. The integrating factor is the same. In fact, examining Eq. (3.11) and Eq. (3.12), the only difference is that the quantity $V_1 - E_0^1$ is replaced by $V_2 - W_1^2 - E_0^2$. W_2 is thus

$$W_2(x) = \frac{\sqrt{2m}}{\hbar} \frac{1}{\psi_0^0(x)^2} \int_{-\infty}^x \psi_0^0(y)^2 (E_0^2 + W_1^2(y) - V_2(y)) dy. \quad (3.17)$$

According to these formulas, we can calculate W_1 and W_2 explicitly only when we know what the energy corrections E_0^1 and E_0^2 are. We will now see how this is done.

3.1.1 For ground-state

Whatever the perturbation procedure tells us that the corrections to the superpotential and energies must be, we know that these corrections must leave the wave functions normalizable. There are two procedures for imposing this condition and thereby obtaining the correct energy corrections. The procedures are equivalent, although one is slightly quicker than the other.

First, the quick method. We return to Eq. (3.15), and take $x \rightarrow \infty$. Then the left-hand side vanishes entirely. On the right-hand side, we notice that

$$\int_{-\infty}^{\infty} E_0^1 \psi_0^0(y)^2 dy = E_0^1, \quad (3.18)$$

since ψ_0^0 is normalized, so we conclude

$$E_0^1 = \int_{-\infty}^{\infty} \psi_0^0(x)^2 V_1(x) dx. \quad (3.19)$$

For the second-order correction E_0^2 , we need only replace $V_1(x)$ with $V_2(x) - W_1^2(x)$:

$$E_0^2 = \int_{-\infty}^{\infty} \psi_0^0(xy)^2 (V_2(x) - W_1^2(x)) dx. \quad (3.20)$$

This is the highest-order energy correction considered here. We look again at the formula (3.2) for the ground-state wave function:

$$\psi_0(x) = N \exp \left[-\frac{\sqrt{2m}}{\hbar} \int_0^x W(y) dy \right]. \quad (3.21)$$

To first-order in λ ,

$$\begin{aligned} \psi_0(x) &= N \exp \left[-\frac{\sqrt{2m}}{\hbar} \int_0^x (W_0(y) + \lambda W_1(y)) dy \right] \\ &= N \exp \left[-\frac{\sqrt{2m}}{\hbar} \int_0^x W_0(y) dy \right] \exp \left[-\lambda \frac{\sqrt{2m}}{\hbar} \int_0^x W_1(y) dy \right]. \end{aligned}$$

Expand the second exponential to first-order in λ , and get:

$$\psi_0(x) = N \exp \left[-\frac{\sqrt{2m}}{\hbar} \int_0^x W_0(y) dy \right] \left[1 - \lambda \frac{\sqrt{2m}}{\hbar} \int_0^x W_1(y) dy \right]. \quad (3.22)$$

Observe that the first factor is $\psi_0^0(x)$, and use Eq. (3.16) to write this as

$$\psi_0(x) = \psi_0^0(x) \left[1 - \lambda \frac{2m}{\hbar^2} \int_{y=0}^x \frac{1}{\psi_0^0(y)^2} \int_{z=-\infty}^y \psi_0^0(z)^2 (E_0^1 - V_1(z)) dz dy \right]. \quad (3.23)$$

As $x \rightarrow \infty$, we want to keep $\psi_0(x)$ close to zero. Looking at the integrand of the y integral, we see that for large values of y , $\frac{1}{\psi_0^0(y)^2}$ will blow up. To compensate, the z integral must die quickly as $y \rightarrow \infty$. At the very least, this requires

$$\int_{-\infty}^{\infty} \psi_0^0(z)^2 (E_0^1 - V_1(z)) dz = 0. \quad (3.24)$$

But this, we see, is precisely the condition we used to obtain Eq. (3.19). We have already derived this condition based on earlier considerations, and didn't need to examine the explicit formula for the wave function ψ_0 in terms of the superpotential W .

3.1.2 For excited states

To obtain corrections to excited state energies, we take the standard SUSYQM route. Each excited state of the potential we are interested in corresponds to the ground state of one of the partner potentials in the hierarchy. Our task is to figure out exactly how to notate the formulas that exploit this relationship.

We begin with the Riccati equation for V^+ :

$$W(x)^2 - \frac{\hbar}{\sqrt{2m}}W'(x) = V^+(x) - E_0^+ . \quad (3.25)$$

We then solve for W and E_0 perturbatively. We now form the partner potential of V^+ in the usual way:

$$W(x)^2 + \frac{\hbar}{\sqrt{2m}}W'(x) = V^-(x) - E_0^+ . \quad (3.26)$$

The bound state spectra of the partner systems obey the SUSYQM relation:

$$E_n^+ = E_{n-1}^- , \quad n \geq 1 . \quad (3.27)$$

If we can get an approximate solution for the ground-state of V^- to $O(\lambda^n)$, then, we have in fact found the first excited state E_1^+ of V^+ to $O(\lambda^n)$. And to find the ground state energy of V^- , we can use the machinery of the previous section.

To use that machinery, we must solve a new Riccati equation for V^- :

$$U(x)^2 - \frac{\hbar}{\sqrt{2m}}U'(x) = V^-(x) - E_0^- . \quad (3.28)$$

That is, we find a superpotential¹ U which generates a pair of partners; the *first* of which is V^- . To use formulas from the previous section we expand everything in orders of λ . So V^- is

$$V^-(x; \lambda) = V_0^-(x) + \lambda V_1^-(x) + \lambda^2 V_2^-(x) + \dots . \quad (3.29)$$

Using $W(x)^2 + \frac{\hbar}{\sqrt{2m}}W'(x) = V^-(x) - E_0^+$, we have

$$V_0^- = W_0^2 + \frac{\hbar}{\sqrt{2m}}W_0' + E_0^{+0} \quad (3.30)$$

$$V_1^- = 2W_0W_1 + \frac{\hbar}{\sqrt{2m}}W_1' + E_0^{+1} \quad (3.31)$$

$$V_2^- = W_1^2 + 2W_0W_2 + \frac{\hbar}{\sqrt{2m}}W_2' + E_0^{+2} \quad (3.32)$$

The lowest-order equations² for U and E_0^- are then

$$U_0^2 - \frac{\hbar}{\sqrt{2m}}U_0' = V_0^- - E_0^{-0} \quad (3.33)$$

¹ to avoid confusion of W_i , we have used different term U_i .

² Expressions like E_0^{+2} don't mean E_0 to the power of +2, but the second-order correction to E_0 .

$$2U_0U_1 - \frac{\hbar}{\sqrt{2m}}U_1' = V_1^- - E_0^{-1} \quad (3.34)$$

$$U_1^2 + 2U_0U_2 - \frac{\hbar}{\sqrt{2m}}U_2' = V_2^- - E_0^{-2} \quad (3.35)$$

We expect to be able to solve for U_0 and E_0^{-0} easily, at least for unperturbed potentials which are shape-invariant. The form of these equations is identical to what we had in previous section, so the formulas transfer exactly. For the superpotential corrections,

$$U_1(x) = \frac{\sqrt{2m}}{\hbar} \frac{1}{\psi_0^{-0}(x)^2} \int_{-\infty}^x \psi_0^{-0}(y)^2 (E_0^{-1} - V_1^-(y)) dy, \quad (3.36)$$

and

$$U_2(x) = \frac{\sqrt{2m}}{\hbar} \frac{1}{\psi_0^{-0}(x)^2} \int_{-\infty}^x \psi_0^{-0}(y)^2 (E_0^{-2} + U_1(y)^2 - V_2^-(y)) dy. \quad (3.37)$$

For the energies,

$$E_1^{+1} = E_0^{-1} = \int_{-\infty}^{\infty} \psi_0^{-0}(x)^2 V_1^-(x) dx, \quad (3.38)$$

and

$$E_1^{+2} = E_0^{-2} = \int_{-\infty}^{\infty} \psi_0^{-0}(x)^2 (V_2^-(x) - U_1(x)^2) dx. \quad (3.39)$$

The left-hand equalities show explicitly the whole point of this procedure: find the corrections to the first excited state of V^+ by working with the *ground* state of V^- . Also, in these formulas, recall that V_1^- and V_2^- are given by Eq. (3.32) and Eq. (3.33).

This process can clearly be repeated to get higher excited energies of V^+ . For example, after we complete the calculations just described, we generate the next partner for V^- . We find its (approximate) ground state energy. Then we have the first excited energy of V^- , and hence the second excited energy of V^+ . Perhaps this procedure gets tedious for the fifth excited state of V^+ . But, SUSYQM doesn't necessarily simplify one's calculational workload. It instead fits the calculations into a *conceptually* simplified framework.

3.2 Example: Perturbed harmonic oscillator

We give the simplest perturbation example we think of. Consider a harmonic oscillator, whose spring constant increases slightly, from k to $(1 + \lambda)k$. Since $\omega = \sqrt{k/m}$, $V_0(x) = \frac{1}{2}m\omega^2x^2$ gets perturbed to $V(x) = \frac{1}{2}m(1 + \lambda)\omega^2x^2$. Thus the $O(\lambda)$ term of the perturbation is

$$V_1(x) = \frac{1}{2}m\omega^2x^2 \quad (3.40)$$

Now, the perturbed potential is nothing special. It can of course be solved exactly- it's just another harmonic oscillator. But that is precisely its merit (that will be a test for the quality of the approximation, too).

3.2.1 Ground-state corrections

The unperturbed problem has as its solution:

$$W_0(x) = \sqrt{\frac{m}{2}} \omega x , \quad (3.41)$$

$$E_0^0 = \frac{\hbar\omega}{2} . \quad (3.42)$$

The ground-state wave function of the unperturbed SHO is

$$\psi_0^0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) . \quad (3.43)$$

We plug this and the perturbation Eq. perturbv1 into the formula Eq. (3.19) for the first-order energy correction, and obtain

$$E_0^1 = \frac{m\omega^2}{2} \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx , \quad (3.44)$$

which is easily evaluated:

$$E_0^1 = \frac{\hbar\omega}{4} . \quad (3.45)$$

The first-order superpotential correction Eq. (3.16) becomes

$$W_1(x) = \frac{\sqrt{2m}}{\hbar} e^{m\omega x^2/\hbar} \int_{-\infty}^x e^{-m\omega y^2/\hbar} \left(\frac{\hbar\omega}{4} - \frac{m\omega^2 y^2}{2}\right) dy , \quad (3.46)$$

which reduces to

$$W_1(x) = \sqrt{\frac{m}{2}} \frac{\omega x}{2} . \quad (3.47)$$

Now we can calculate the second-order energy correction using Eq. (3.20):

$$E_0^2 = \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar} \left(-\frac{m\omega^2 x^2}{8}\right) dx = -\frac{\hbar\omega}{16} . \quad (3.48)$$

Although we don't need it further calculations, let's find the second-order superpotential correction W_2 , just for the record. By Eq. (3.17), we have

$$W_2(x) = \frac{\sqrt{2m}}{\hbar} e^{m\omega x^2/\hbar} \int_{-\infty}^x e^{-m\omega y^2/\hbar} \left(-\frac{\hbar\omega}{16} + \frac{m\omega^2 y^2}{8}\right) dy , \quad (3.49)$$

which simplifies to

$$W_2(x) = -\sqrt{\frac{m}{2}} \frac{\omega x}{8} . \quad (3.50)$$

We have found terms up to second-order, then our complete results are now:

$$W(x; \lambda) = \sqrt{\frac{m}{2}} \left(1 + \frac{\lambda}{2} - \frac{\lambda^2}{8}\right) \omega x , \quad (3.51)$$

$$E_0(\lambda) = \frac{\hbar\omega}{2} \left(1 + \frac{\lambda}{2} - \frac{\lambda^2}{8}\right) . \quad (3.52)$$

The expansions in λ look like they are converging to

$$1 + \frac{\lambda}{2} - \frac{\lambda^2}{8} + \dots \rightarrow \sqrt{1 + \lambda}. \quad (3.53)$$

Well, of course! The answers, after all, are found by just taking $\omega \rightarrow \omega\sqrt{1 + \lambda}$ in the unperturbed formulas:

$$W_0(x) = \sqrt{\frac{m}{2}}\omega x \rightarrow W(x) = \sqrt{\frac{m}{2}}\omega x\sqrt{1 + \lambda}, \quad (3.54)$$

$$E_0^0 = \frac{\hbar\omega}{2} \rightarrow E_0 = \frac{\hbar\omega}{2}\sqrt{1 + \lambda}. \quad (3.55)$$

SSPT, then, has passed the test for the ground-state energies. Now to the excited states.

3.2.2 Excited states corrections

Let's use the equations of previous section to calculate the first-excited energy of the perturbed harmonic oscillator to first order. Using Eq. (3.30), the partner potential V^- to zeroth order is

$$V_0^-(x) = \frac{m\omega^2 x^2}{2} + \frac{\hbar}{\sqrt{2m}}\sqrt{\frac{m}{2}}\omega + \frac{\hbar\omega}{2} = \frac{m\omega^2 x^2}{2} + \hbar\omega. \quad (3.56)$$

That's nothing new, of course. We already know that the partner of the harmonic oscillator is the harmonic oscillator itself, raised by an amount $\hbar\omega$. The first-order term of V^- is given by Eq. (3.31):

$$\begin{aligned} V_1^-(x) &= 2 \left(\sqrt{\frac{m}{2}}\omega x \right) \left(\sqrt{\frac{m}{2}}\frac{\omega x}{2} \right) + \frac{\hbar}{\sqrt{2m}}\sqrt{\frac{m}{2}}\frac{\omega}{2} + \frac{\hbar\omega}{4} \\ &= \frac{m\omega^2 x^2}{2} + \frac{\hbar\omega}{2}, \end{aligned} \quad (3.57)$$

while the second-order term is given by Eq. (3.32):

$$V_2^-(x) = \frac{m\omega^2 x^2}{8} + 2\sqrt{\frac{m}{2}}\omega x \left(-\sqrt{\frac{m}{2}}\frac{\omega x}{8} \right) + \frac{\hbar}{\sqrt{2m}} \left(-\sqrt{\frac{m}{2}}\frac{\omega}{8} \right) - \frac{\hbar\omega}{16} \quad (3.58)$$

$$= -\frac{\hbar\omega}{8}. \quad (3.59)$$

Thus, the partner potential of our perturbed SHO, accurate to second order in λ , is

$$V^-(x; \lambda) = \frac{m\omega^2 x^2}{2}(1 + \lambda) + \left(1 + \frac{\lambda}{2} - \frac{\lambda^2}{8} \right) \hbar\omega. \quad (3.60)$$

To zeroth-order, the ground-state energy of this potential is just

$$E_0^{-0} = \frac{\hbar\omega}{2} + \hbar\omega = \frac{3\hbar\omega}{2}, \quad (3.61)$$

which we recognize to be the first excited energy of the original unperturbed SHO. Since the ground-state wave function for V^- is the same as the one for V^+ , Eq. (3.38) gives simply for the first-order correction to E_0^- :

$$\begin{aligned} E_0^{-1} &= \int_{-\infty}^{\infty} \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega x^2/\hbar} \left(\frac{m\omega^2 x^2}{2} + \frac{\hbar\omega}{2} \right) dx \\ &= \frac{\hbar\omega}{4} + \frac{\hbar\omega}{2} = \frac{3\hbar\omega}{4}. \end{aligned} \quad (3.62)$$

Now, to get the second-order correction, we need to know the new superpotential U to first order. But it is quite clear that this will be the same as the superpotential W , given by Eq. (3.51), for the original potential V^+ . In particular, the first-order term is

$$U_1(x) = \sqrt{\frac{m}{2}} \frac{\omega x}{2}, \quad (3.63)$$

so Eq. (3.39) gives for the second-order energy correction

$$\begin{aligned} E_0^{-2} &= \int_{-\infty}^{\infty} \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega x^2/\hbar} \left(-\frac{\hbar\omega}{8} - \frac{m\omega^2 x^2}{8} \right) dx \\ &= -\frac{\hbar\omega}{8} - \frac{\hbar\omega}{16} = -\frac{3\hbar\omega}{16}. \end{aligned} \quad (3.64)$$

The ground-state energy of V^- , and the first excited state of V^+ , is thus

$$E_1^+ = E_0^- = \frac{3\hbar\omega}{2} \left(1 + \frac{\lambda}{2} - \frac{\lambda^2}{8} \right), \quad (3.65)$$

which again agrees with the exact result to second order.

The generalization to higher excited states is quite easy. First renotate V^+ as $V^{(0)}$ and V^- as $V^{(1)}$. From Eq. (3.60) and our earlier study of the SHO, we are led to expect the n^{th} partner potential in the hierarchy to be

$$V^{(n)}(x; \lambda) = \frac{m\omega^2 x^2}{2}(1 + \lambda) + \left(1 + \frac{\lambda}{2} - \frac{\lambda^2}{8} \right) n \hbar\omega \quad (3.66)$$

to second order. Each potential shares the same ground-state wave function.

The ground-state energy in zeroth-order is just

$$\begin{aligned} E_0^{(n)0} &= \frac{\hbar\omega}{2} + n \hbar\omega \\ &= \left(n + \frac{1}{2} \right) \hbar\omega. \end{aligned} \quad (3.67)$$

The first-order correction is

$$\begin{aligned} E_0^{(n)1} &= \int_{-\infty}^{\infty} \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega x^2/\hbar} \left(\frac{m\omega^2 x^2}{2} + \frac{n \hbar\omega}{2} \right) dx \\ &= \frac{\hbar\omega}{4} + \frac{n \hbar\omega}{2} \\ &= \left(n + \frac{1}{2} \right) \frac{\hbar\omega}{2}, \end{aligned} \quad (3.68)$$

while the second-order correction is

$$\begin{aligned}
 E_0^{(n)2} &= \int_{-\infty}^{\infty} \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega x^2/\hbar} \left(-\frac{n\hbar\omega}{8} - \frac{m\omega^2 x^2}{8} \right) dx \\
 &= \frac{\hbar\omega}{16} - \frac{n\hbar\omega}{8} \\
 &= -\left(n + \frac{1}{2}\right) \frac{\hbar\omega}{8}.
 \end{aligned} \tag{3.69}$$

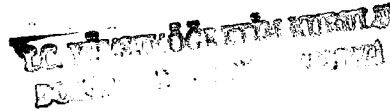
Then, $E_0^{(n)}$ (which is of course the same thing as $E_n^{(0)}$) becomes

$$E_n^{(0)} = E_0^{(n)} = \left(n + \frac{1}{2}\right) \hbar\omega \left(1 + \frac{\lambda}{2} - \frac{\lambda^2}{8}\right). \tag{3.70}$$

This is the n^{th} excited energy of the original perturbed harmonic oscillator to second order.

3.3 Conclusion

In fact, there are more systems to be studied in the framework of SSPT. We have just given the theory with the simplest example for a quantum system. This example illustrates how SSPT allows us (as in the SUSYQM) to use the formulas for the ground state even for the excited state energies, by moving to a partner potential whose ground state coincides with the excited state we are interested in. An important point to reiterate about the preceding calculations is this: To find the energy of the excited states of V^+ , all we had to do was perform calculations using the *ground-state* wave function of V^- (or of V^n , later in the hierarchy). It is also obvious that the solutions become complex as the corrections to higher order increases.



CHAPTER 4

HAMILTONIAN HIERARCHY AND THE HULTHÉN POTENTIAL

The Hulthén potential [20, 21] is one of the important short-range potentials in physics. The potential is given by

$$V(r) = -\frac{Ze^2\delta e^{-\delta r}}{(1 - e^{-\delta r})} \quad (4.1)$$

where Z is a constant and δ is the screening parameter. If the potential is used for atoms, the Z is identified with the atomic number. This potential is a special case of the Eckart potential [22] which has been widely used in several branches of physics and its bound-state and scattering properties have been investigated by a variety of techniques (see e.g., [23] and references therein).

Unfortunately, the radial Schrödinger equation for the Hulthén potential can be solved analytically only for the states with zero angular momentum [20, 21, 24, 25]. For $\ell \neq 0$, a number of methods have been employed to evaluate bound-state energies numerically [23], [26]-[36]. In connection with this, we present in this study a method within the frame of SUSYQM using an effective Hulthén potential for non-zero angular momentum states, which can be solved analytically.

In SUSYQM (for a recent review see [5]) one often deals with hierarchy problem. Within the context of the SUSYQM one can generate a Hamiltonian hierarchy, the adjacent members of which are supersymmetric partners in that they share the same eigenvalue spectrum except for the missing ground state. In the case of Coulomb potential $V_c(r)$, the Hamiltonian hierarchy corresponds to the addition of an appropriate centrifugal potential and the so-called accidental degeneracy is recovered as a natural consequence [37]. In this study we shall examine the implication of the Hamiltonian hierarchy for the Hulthén potential. At small values of the radial coordinate r , the Hulthén potential behaves like a Coulomb potential whereas for large values of r it decreases exponentially so that its capacity for bound state is smaller than $V_c(r)$. In contrast to the Hulthén potential, the Coulomb problem is analytically solvable for all energies and all

angular momenta. Because of the similarity and points of contrast mentioned above between Coulomb and Hulthén potentials, it may be of considerable interest to generate the supersymmetric partners of the Hulthén potential and study their eigenfunctions and eigenvalues. In the following we outline the basic idea of the SUSYQM and set up the formalism for dealing with the Hulthén problem.

4.1 Supersymmetric solution of the Hulthén potential

The supersymmetric formalism has already been used to study some aspects of the Hulthén potential [34]-[36]. Here the exact analytical solution for this potential is re-obtained for $\ell = 0$ state in the light of the works described in Refs. [34]-[36], [38] to show the consistency of the method and see how partners of the Hulthén potential simulate the effect of the centrifugal barrier, which makes clear the physics behind partner Hamiltonians in connection with the states having $\ell \neq 0$ angular momenta.

In the approach followed here the first step taken is to look for an effective potential similar to the original Hulthén potential. Inspired by SUSYQM, we propose an ansatz for the superpotential,

$$W_{\ell+1}(r) = -\frac{\hbar}{\sqrt{2m}} \frac{(\ell+1)\delta e^{-\delta r}}{(1-e^{-\delta r})} + \sqrt{\frac{m}{2}} \frac{e^2}{\hbar} \left[\frac{1}{(\ell+1)} - \frac{(\ell+1)\beta}{2} \right], \quad (4.2)$$

where $(\ell+1)$ denotes the partner number with $\ell = 0, 1, 2, \dots$, and $\beta = \frac{\hbar^2 \delta}{me^2}$ which is a dimensionless quantity. This kind of superpotential choice leads to the $(\ell+1)$ -th member of the Hamiltonian hierarchy:

$$V_{\ell+1}(r) - E_{\ell+1}^{n=0} = W_{(\ell+1)}^2(r) - \frac{\hbar}{\sqrt{2m}} \frac{d}{dr} W_{(\ell+1)}(r), \quad (4.3)$$

$$V_{\ell+1}(r) = \frac{\hbar^2}{2m} \frac{\ell(\ell+1)\delta^2 e^{-2\delta r}}{(1-e^{-\delta r})^2} - e^2 \frac{\delta e^{-\delta r}}{(1-e^{-\delta r})} \left[1 - \ell(\ell+1) \frac{\beta}{2} \right]. \quad (4.4)$$

We introduce an expression for the bound-state energies of the above potential, considering the shape invariance requirement [18],

$$E_{\ell+1}^n = -\frac{me^4}{2\hbar^2} \left[\frac{1}{(n+\ell+1)} - \frac{(n+\ell+1)(\beta)}{2} \right]^2; \quad n = 0, 1, 2, \dots \quad (4.5)$$

and the corresponding ground-state eigenfunctions are

$$\begin{aligned} \Psi_{(\ell+1)}^{n=0}(r) &= N \exp \left(-\frac{\sqrt{2m}}{\hbar} \int^r W_{(\ell+1)}(r') dr' \right) \\ &= N (1-e^{-\delta r})^{\ell+1} \exp \left\{ -\frac{me^2}{\hbar^2} \left[\frac{1}{(\ell+1)} - \frac{(\ell+1)\beta}{2} \right] r \right\}. \end{aligned} \quad (4.6)$$

It is reminded that for a number of purposes it is convenient to have the wavefunction in such a compact analytical form. The first eigenfunction corresponds

to the minimum energy for each ℓ . In terms of the hierarchy of Hamiltonians, we present here the lowest state wavefunctions for each member. The excited state wavefunctions can be determined [5, 39] from the usual approach in SUSYQM.

For $\ell = 0$ the potential in Eq. (4.4) leads to the usual Hulthén potential which has an interesting property such that when the angular momentum is zero, it is not shape invariant in the sense expressed in [18]. However, it is still possible to construct a general form of the potentials in the shape invariant super-family of Hamiltonians as seen in Eq. (4.4) where the first member corresponds to the Hulthén potential which can be solved exactly in analytic form. One can easily verify that the energy eigenvalues and eigenfunctions for $\ell = 0$ case of Eqs. (4.5) and (4.6) are the same given in Refs. [24, 25]. This supports the suggestion [36], [40]-[42] that the Gedenshtein's condition of shape invariance is sufficient but not a necessary condition in the construction of exactly solvable but non-shape invariant potentials.

Eq. (4.4) can be rearranged as

$$V_H^{eff}(r) = V_{\ell+1}(r) = -e^2 \frac{\delta e^{-\delta r}}{(1 - e^{-\delta r})} + \frac{\ell(\ell+1)\hbar^2 \delta^2}{2m(1 - e^{-\delta r})^2} e^{-\delta r}, \quad (4.7)$$

which is known in literature as the approximate Hulthén effective potential introduced by Greene and Aldrich [43] in their method to generate pseudo-Hulthén wavefunctions for $\ell \neq 0$ states. For small δr , Eq. (4.7) is a good approximation to the realistic Hulthén effective potential, and unlike the original case the radial Schrödinger equation for this potential is solvable analytically through Eqs. (4.5) and (4.6). In addition, the partner potentials in Eq. (4.7) gives the necessary repulsive core due to angular momentum. For instance, for small r the second term in Eq. (4.7) behaves as a p -wave centrifugal barrier for the second member of the super-family. Since we know that the centrifugal potential is effective only in this region (i.e., small r), eigensolution of the potential for $\ell = 1$ in Eq. (4.6) can be regarded as the approximate p -wave solution for the Hulthén potential. Clearly, one can get other supersymmetric partners and their solutions in an explicit form for $\ell \neq 0$ states. The present simple and elegant method is a clear cut of the iteration technique introduced by Laha *et al.* [34, 35]

For the sake of completeness, it is of interest to note that for small values of δ , the potential in Eq. (4.7) closely approximates the effective Coulomb potential rather well,

$$V_H^{eff}(r, \delta \cong 0) \rightarrow V_C^{eff}(r) = -\frac{e^2}{r} + \frac{\ell(\ell+1)\hbar^2}{2mr^2}, \quad (4.8)$$

and the corresponding energy eigenvalue for the potential of Eq. (4.8), together with its ground state wavefunction for $\ell \neq 0$ states, obtained easily via Eqs. (4.5) and (4.6) overlap with those, e.g. in Ref. [39]. This makes clear the work

of Lam and Varshni [24] in which they showed that if one uses as trial functions eigenvectors of the Hulthén potential rather than those of the simple Coulomb potential, excellent results for the energies of the states of the screened Coulomb potential can be obtained with simple variational wave functions containing only one parameter.

An important quantity of interest for the Hulthén potential (and for other similar screened potentials) is the critical screening parameter δ_c , which is that value of δ for which the binding energy of the level in question becomes zero. Using Eq. (4.5), in atomic units,

$$\delta_c = \frac{2}{(n + \ell + 1)^2}, \quad (4.9)$$

which works well for all n values in case $\ell = 0$ when compared to those in Table III of Ref. [23], but fails for non-zero angular momentum states. Consequently, the eigenenergies obtained via Eq. (4.5) for $\ell \neq 0$ states deviates from the accurate values obtained by numerical techniques and presented in Table I of Ref. [23]. This may be understood as follow. If Eq. (4.7) is written in the form

$$V_H^{eff}(r) = -e^2 \frac{\delta e^{-\delta r}}{(1 - e^{-\delta r})} + \frac{\ell(\ell + 1)(\hbar^2)}{2mr^2} + \left[\frac{\ell(\ell + 1)\hbar^2\delta^2}{2m(1 - e^{-\delta r})^2} e^{-\delta r} - \frac{\ell(\ell + 1)\hbar^2}{2mr^2} \right], \quad (4.10)$$

the exact energy eigenvalues for the realistic effective Hulthén potential may be given as

$$E_H^{n\ell} = -\frac{me^4}{2\hbar^2} \left[\frac{1}{(n + \ell + 1)} - \frac{(n + \ell + 1)\beta}{2} \right]^2 + \Delta E. \quad (4.11)$$

where ΔE is the contribution, which does not appear in Eq. (4.9), due to the last term in Eq. (4.10). The clear interpretation of Eqs. (4.10) and (4.11) is that the potential barrier term prevents us to build the super-family as in the $\ell = 0$ case, since the potential-the first two terms in Eq. (4.10)-is not exactly solvable hence the supersymmetry is broken for $\ell \neq 0$ due to the potential barrier term. It is easy however to verify that for small values of δ , ΔE goes to zero while Eq. (4.10) becomes an expression for the effective Coulomb potential in which case the accidental degeneracy is recovered as a natural consequence.

The usefulness of the Hulthén potential would be enhanced if one obtains an analytical expression for the exact energies of the non-zero angular momentum states. The work along this line is in progress in the frame of broken supersymmetry. Further, in the light of the supersymmetric solutions discussed in this study we suggest here, as an alternative to other various methods [23], [26]-[33] investigating the bound-state properties of the Hulthén potential, an elegant approach for the calculation of the whole energy spectrum of the potential using

Table 4.1: Energy eigenvalues of the Hulthén potential as a function of screening parameter for various states in atomic units.

State	δ	Present Calculations	Numerical Integration (Ref. [23])	Variational (Ref. [23])	Lai and Lin (Ref. [26])	Patil (Ref. [28])	'Iang and Chan (Ref. [31])	Matthys and De Meyer (Ref. [33])
2p	0.025	0.1127605	0.1127605	0.1127605		0.11276		0.1127604
	0.050	0.1010425	0.1010425	0.1010425	0.101043	0.10104	0.1010424	0.1010425
	0.075	0.0898478	0.0898478	0.0898478		0.08985		
	0.100	0.0791794	0.0791794	0.0791794	0.079179	0.07918	0.0791794	0.0791794
	0.150	0.0594415	0.0594415	0.0594415		0.059445		0.0594415
	0.200	0.0418884	0.0418880	0.0418880	0.041886	0.041895	0.0418857	0.0418860
	0.250	0.0266060		0.0266111				
	0.300	0.0137596	0.0137900	0.0137878	0.013790			0.0137900
	0.350	0.0036146	0.0037931	0.0037734	0.003779	0.038375		
3p	0.025	0.0437068	0.0437069	0.0437069	0.043707	0.0437085		0.0437071
	0.050	0.0331632	0.0331645	0.0331645	0.033165	0.033185	0.03316518	0.0331650
	0.075	0.0239331	0.0239397	0.0239397		0.0240165		
	0.100	0.0160326	0.0160537	0.0160537	0.016054	0.01622	0.01606772	0.0160537
	0.150	0.0043599	0.0044663	0.0044660	0.004466	0.046995		0.0044664
3d	0.025	0.0436030	0.0436030	0.0436030	0.043603	0.0436025		0.0436030
	0.050	0.0327532	0.0327532	0.0327532	0.032753	0.032745	0.0327532	0.0327532
	0.075	0.0230306	0.0230307	0.0230307		0.02299		
	0.100	0.00144832	0.0144842	0.0144832	0.014484	0.01439	0.0144842	0.0144842
	0.150	0.0132820	0.0013966	0.0013894	0.001391	0.0013755		0.0013965
4p	0.025	0.0199480	0.0199489	0.0199489	0.019949	0.01995		0.0199490
	0.050	0.0110430	0.0110582	0.0110582	0.011058	0.011075	0.0110725	0.0110583
	0.075	0.0045385	0.0046219	0.0046219	0.004622	0.0046585		0.0046224
	0.100	0.0004434	0.0007550	0.0007532	0.000754	0.000752		
4d	0.025	0.0198460	0.0198462	0.0198462	0.019846	0.019845		0.0198462
	0.050	0.0106609	0.0106674	0.0106674	0.010667	0.010668	0.0106690	0.0106674
	0.075	0.0037916	0.0038345	0.0038344	0.003834	0.003875		0.0038346
4f	0.025	0.0196911	0.0196911	0.0196911	0.019691	0.01969		0.0196911
	0.050	0.0100618	0.0100620	0.0100620	0.010062	0.010045	0.0100620	0.0100619
	0.075	0.0025468	0.0025563	0.0025557	0.002556	0.002557		0.0025563
5p	0.025	0.0094011	0.0094036				0.0094087	
	0.050	0.0026056	0.0026490					
5d	0.025	0.0092977	0.0093037				0.0093050	
	0.050	0.0022044	0.0023131					
5f	0.025	0.0091507	0.0091521				0.0091523	
	0.050	0.0017421	0.0017835					
5g	0.025	0.0089465	0.0089465				0.0089465	
	0.050	0.0010664	0.0010159					
6p	0.025	0.0041493	0.0041548					
6d	0.025	0.0040452	0.0040606					
6f	0.025	0.0038901	0.0039168					
6g	0.025	0.0036943	0.0037201					

the first-order perturbation theory,

$$E_H^{n\ell} = E_{\ell+1}^n + \frac{\ell(\ell+1)\hbar^2}{2m} \int_0^\infty [\psi_{\ell+1}^n(r)]^2 \left(\frac{1}{r^2} - \frac{\delta^2}{(1-e^{-\delta r})^2} e^{-\delta r} \right) dr \quad (4.12)$$

which gives satisfactory values when compared (see Table 4.1) with the results obtained by the various methods for the eigenenergies of $\ell \neq 0$ levels. The accuracy of the present calculations may be improved incorporating higher-order perturbations for in particular large values of the screening parameter.

4.2 Conclusion

We have obtained the exact analytical eigenfunctions and eigenvalues for the Hulthén potential within the framework of SUSYQM for the case $\ell = 0$. The approach consists of making an ansatz in the superpotential which satisfies the Riccati equation by an effective potential. For $\ell = 0$ the effective potential obtained is identical to the Hulthén potential. However, for $\ell \neq 0$ the effective supersymmetric potential has a slightly different structure than the Hulthén potential. This deviation has led us to introduce a simple expression that yields reasonable results for the non-zero angular momentum state energies. We stress that even though the problem has been attacked by different methods, our simple and elegant methodology [44] is powerful because it provides an insight into the relation between theoretical partner Hamiltonians in the frame of SUSYQM and physical states of the system considered.

CHAPTER 5

SUPERSYMMETRY AND THE RELATIONSHIP BETWEEN A CLASS OF SINGULAR POTENTIALS IN ARBITRARY DIMENSIONS

Singular potentials have attracted much attention in recent years for a variety of reasons, two of them being that (i) the ordinary perturbation theory fails badly for such potentials, and (ii) in physics, one often encounters phenomenological potentials that are strongly singular at the origin such as certain type of nucleon-nucleon potentials, singular models of fields in zero dimensions, etc. Thus a study of such potentials is of interest, both from the fundamental and applied point of view.

As we mentioned in the previous chapters, one of the challenging problems in non-relativistic quantum mechanics is to find exact solutions to the Schrödinger equation for potentials that can be used in different field of physics. Recently, several authors obtained exact solutions for the fourth-order inverse-power potential

$$V_1(r) = \frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3} + \frac{A_4}{r^4} \quad (5.1)$$

using analytical methods [45]-[47]. These methods yield exact solutions for a single state only for a potential of type (5.1) with restrictions on the coupling constants. The interest is mainly due to the wide applicability of these type inverse-power potentials. Some areas of interest are ion-atom scattering [48], several interactions between the atoms [49], low-energy physics [50], interatomic interactions in molecular physics [51] and solid-state physics [52].

The advent of supersymmetry has had a significant impact on theoretical physics in a number of distinct disciplines. One subfield that has been receiving much attention is the SUSYQM [2, 5] in which the Hamiltonians of distinct systems are related by a supersymmetry algebra. In this Chapter, we are concerned with, via SUSYQM, clarifying the relationship between two distinct systems having an interaction potential of type (5.1) and interacting through

$$V_2(r) = B_1 r^2 + \frac{B_2}{r^2} + \frac{B_3}{r^4} + \frac{B_4}{r^6} \quad (5.2)$$

singular even-power potentials which have been widely used in a variety of fields, e.g. see [50, 53, 54]. In recent years, the higher order anharmonic potentials have drawn more attentions of physicists and mathematicians in order to partly understand a newly discovered phenomena such as the structural phase transitions [55], the polaron formation in solids [56], the concept of false vacuo in field theory [57], fiber optics [58], and molecular physics [59]. In addition, some 60 years ago Michels *et al.* [60] proposed the idea of simulating the effect of pressure on an atom by enclosing it in a impenetrable spherical box. Since that time there have been a large number of publications, for an overview see [61], dealing with studies on quantum systems enclosed in boxes, which involve an interaction potential that is a special case ($B_2 = 0$) of (5.2). This field has received added impetus in recent years because of the fabrication of semiconductor quantum dots [62].

The main motivation behind this work is to reveal the existence of a link between potentials of type (5.1) and (5.2) in N -dimensional space, and between their special cases such as a Mie-type potential (or Kratzer) [63] and pseudoharmonic-like (or Goldman-Krivchenkov) potential [64] in higher dimensions, which to our knowledge has never been appeared in the literature. On the other hand, with the advent of growth technique for the realization of the semiconductor quantum wells, the quantum mechanics of low-dimensional systems has become a major research field. The work presented in this study would also be helpful to the literature in this respect as the results can readily be extended to lower dimensions as well.

5.1 The Schrödinger equation in N -dimensional space

It is well known that the general framework of the non-relativistic quantum mechanics is by now well understood and its predictions have been carefully proved against observations. Physics is permanently developing in a tight interplay with mathematics. It is of importance to know therefore whether some familiar problems are a particular case of a more general scheme or to search if a map between the radial equations of two different systems exists. It is hence worthwhile to study the Schrödinger equation in the arbitrary dimensional spaces which has attracted much more attention to many authors. Many efforts have in particular been produced in the literature over several decades to study the stationary Schrödinger equation in various dimensions with a central potential containing negative powers of the radial coordinates [65, 66], and the references therein.

The radial Schrödinger equation for a spherically symmetric potential in N -dimensional space (we shall use through this Chapter the natural units such

that $\hbar = m = 1$)

$$-\frac{1}{2} \left[\frac{d^2 R}{dr^2} + \frac{N-1}{r} \frac{dR}{dr} \right] + \frac{\ell(\ell + N - 2)}{2r^2} R = [E - V(r)]R \quad (5.3)$$

is transformed to

$$-\frac{d^2 \Psi}{dr^2} + \left[\frac{(M-1)(M-3)}{4r^2} + 2V(r) \right] \Psi = 2E\Psi \quad (5.4)$$

where Ψ , the reduced radial wave function, is defined by

$$\Psi(r) = r^{(N-1)/2} R(r) \quad (5.5)$$

and

$$M = N + 2\ell \quad (5.6)$$

Eq.(5.4) can also be written as

$$-\frac{1}{2} \frac{d^2 \Psi}{dr^2} + \left[\frac{\Lambda(\Lambda + 1)}{2r^2} + V(r) \right] \Psi = E\Psi \quad (5.7)$$

where $\Lambda = (M - 3)/2$. We see that the radial Schrödinger equation in N -dimensions has the same form as the three-dimensional one. Consequently, given that the potential has the same form in any dimension, the solution in three dimensions can be used to obtain the solution in any dimension simply by using the substitution $\ell \rightarrow \Lambda$. It should be noted that N and ℓ enter into expressions (5.4) and (5.7) in the form of the combinations $N + 2\ell$. Hence, the solutions for a particular central potential $V(r)$ are the same as long as $M (= N + 2\ell)$ remains unaltered. Therefore the s-wave eigensolutions ($\Psi_{\ell=0}$) and eigenvalues in four-dimensional space are identical to the p-wave solutions ($\Psi_{\ell=1}$) in two-dimensions.

The technique of changing the independent coordinate has always been useful tool in the solution of the Schrödinger equation. For instance, this allows something of a systematic approach enabling to recognize the equivalence of superficially unrelated quantum mechanical problems. Many recent papers have addressed this old subject. In the light of these works we proceed by substituting $r = \alpha\rho^2/2$ and $R = F(\rho)/\rho^\lambda$, λ an integer, suggested by the known transformations between Coulomb and harmonic oscillator problems [67, 68, 69] and used to show the relation between the perturbed Coulomb problem and the sextic anharmonic oscillator in arbitrary dimensions [70, 71], we transform Eq. (5.3) to another Schrödinger-like equation in $N' = 2N - 2 - 2\lambda$ dimensional space with angular momentum $L = 2\ell + \lambda$,

$$-\frac{1}{2} \left[\frac{d^2 F}{d\rho^2} + \frac{N' - 1}{\rho} \frac{dF}{d\rho} \right] + \frac{L(L + N' - 2)}{2\rho^2} F = [\hat{E} - \hat{V}(\rho)]F \quad (5.8)$$

where

$$\hat{E} - \hat{V}(\rho) = E\alpha^2\rho^2 - \alpha^2\rho^2V(\alpha\rho^2/2) \quad (5.9)$$

and α is a parameter to be adjusted properly. Note that leaving re-scaling constant α arbitrary for now gives us an additional degree of freedom. When we discuss bound state eigenvalues later, we can use this to allow the values of the potential coefficients to be completely independent of each other. Thus, by this transformation, in general, the N -dimensional radial wave Schrödinger equation with angular momentum ℓ can be transformed to a $N' = 2N - 2 - 2\lambda$ dimensional equation with angular momentum $L = 2\ell + \lambda$. If we choose $\alpha^2 = 1/|E|$, with E corresponding the eigenvalue for the inverse power potential of Eq. (5.1), then Eq. (5.8) corresponds to the Schrödinger equation of an singular even-power potential

$$\hat{V}(\rho) = \rho^2 + \frac{4A_2}{\rho^2} + \frac{8A_3}{\rho^4}|E|^{1/2} + \frac{16A_4}{\rho^6}|E| \quad (5.10)$$

with eigenvalue

$$\hat{E} = \frac{-2A_1}{|E|^{1/2}} \quad (5.11)$$

Thus, the system given by Eq. (5.1) in N -dimensional space is reduced to another system defined by Eq. (5.2) in $N' = 2N - 2 - 2\lambda$ dimensional space. In other words, by changing the independent variable in the radial Schrödinger equation, we have been able to demonstrate a close equivalence between singular potentials of type (5.1) and (5.2). Note that when $N = 3$ and $\lambda = 0$ one finds $N' = 4$, and when $\lambda = 1$ we get $N' = 2$. It is also easy to see that $N' + 2L$ does not depend on λ , which leads to map two distinct problems in three- and four-dimensional space [71].

5.2 Mappings between two distinct systems

5.2.1 Quasi-exactly solvable case

Since Eq. (5.4) for the reduced radial wave $\Psi(r)$ in the N -dimensional space has the structure of the one-dimensional Schrödinger equation for a spherically symmetric potential $V(r)$, we may define the supersymmetric partner potentials [5]

$$V_{\mp}(r) = W^2(r) \pm W'(r) + E_0^{\mp} \quad (5.12)$$

which has a zero-energy solution, and the corresponding eigenfunction is given by

$$\Psi_{n=0}^{\pm}(r) \propto \exp[\mp \int^r W(r) dr] \quad (5.13)$$

In constructing these potentials one should be careful about the behaviour of the wave function $\Psi(r)$ near $r = 0$ and $r \rightarrow \infty$. It may be mentioned that

$\Psi(r)$ behaves like $r^{(M-1)/2}$ near $r = 0$ and it should be normalizable. For the inverse power potential of Eq. (5.1) we set

$$W(r) = \frac{-a}{r^2} + \frac{c}{r} - b \quad , \quad b, c > 0 \quad (5.14)$$

and identify $V_-(r)$ with the effective potential so that

$$V_-(r) = \left(\frac{2A_4}{r^4} + \frac{2A_3}{r^3} + \frac{2A_2}{r^2} + \frac{2A_1}{r} \right) + \frac{(M-1)(M-3)}{4r^2} - 2E_{n=0} \quad (5.15)$$

and substituting Eq. (5.14) into Eq. (5.12) we obtain

$$V_-(r) = \frac{a^2}{r^4} + \frac{2a(1-c)}{r^3} + \frac{c(c-1) + 2ab}{r^2} - \frac{2bc}{r} + b^2 \quad (5.16)$$

and the relations between the parameters satisfy the supersymmetric constraints

$$a = \pm \sqrt{2A_4} \quad ; \quad c = 1 - \frac{A_3}{\pm \sqrt{2A_4}} \quad (5.17)$$

The potential (5.1) admits the exact solutions

$$\Psi_{n=0}(r) = N_0 r^c \exp\left(\frac{a}{r} - br\right) \quad (5.18)$$

where N_0 is the normalization constant, with the physically acceptable eigenvalues

$$\begin{aligned} E_{n=0} &= -\frac{b^2}{2} \\ &= -\frac{1}{16A_4} \left[\frac{A_3}{\sqrt{2A_4}} \left(1 + \frac{A_3}{\sqrt{2A_4}}\right) - \frac{1}{4}(M-1)(M-3) - 2A_2 \right]^2 \end{aligned} \quad (5.19)$$

in the case of $a < 0$ and under the constraints

$$A_1 = -\left(1 + \frac{A_3}{\sqrt{2A_4}}\right) \sqrt{-2E_{n=0}} \quad (5.20)$$

The results obtained agree with those in Refs. [46, 47, 65, 66] for three-dimensions. Note that in order to retain the well-behaved solution at $r \rightarrow 0$ and at $r \rightarrow \infty$ we have chosen $a = -\sqrt{2A_4}$.

The expressions obtained above can easily be extended to the lower dimensions. For example, one can readily check that our two-dimensional solutions ($N = 2, \ell \rightarrow \ell - 1/2$) for the inverse power potential considered are in excellent agreement with the literature [65, 66]. The ground state solutions in arbitrary dimensions for the Coulomb ($A_2 = A_3 = A_4 = 0$), and for a the Kratzer ($A_3 = A_4 = 0$) [63], and for an inverse-power $A_3 = 0$ potentials can also be found from the above prescriptions.

For the singular even-power anharmonic oscillator potential of Eq. (5.2), we set

$$W(r) = \mu r + \frac{\delta}{r} + \frac{\eta}{r^3} \quad , \quad \delta > 0 \quad (5.21)$$

which leads to

$$\Psi_{n=0}(r) = C_0 r^\delta \exp\left(\frac{\mu r^2}{2} - \frac{\eta}{2r^2}\right) \quad (5.22)$$

with C_0 being the corresponding normalization constant, and identify $V_-(r)$ with the effective potential so that

$$\begin{aligned} V_-(r) &= \left(\frac{2B_4}{r^6} + \frac{2B_3}{r^4} + \frac{2B_2}{r^2} + 2B_1 r^2\right) + \frac{(M-1)(M-3)}{4r^2} - 2\tilde{E}_{n=0} \\ &= W^2(r) + W'(r) \\ &= \frac{\eta^2}{r^6} + \frac{\eta(2\delta-3)}{r^4} + \frac{\delta(\delta-1) + 2\eta\mu}{r^2} + \mu^2 r^2 + \mu(2\delta+1) \end{aligned} \quad (5.23)$$

and the relations between the potential parameters satisfy the supersymmetric constraints

$$\eta = \pm\sqrt{2B_4} \ ; \ \delta = \frac{3}{2} + \frac{B_3}{\eta} \ ; \ \mu = \pm\sqrt{2B_1} \quad (5.24)$$

As we are dealing with a confined particle system, the positive values for η and the negative values for μ would of course be the right choice to ensure the well behaved nature of the wave function behaviour at the origin and at infinity. Hence, physically meaningful ground state energy eigenvalues for the potential of interest are

$$\begin{aligned} \tilde{E}_{n=0} &= -\frac{\mu}{2}(2\delta+1) \\ &= \sqrt{\frac{B_1}{2}} \left\{ 2 + \sqrt{1 - 16\sqrt{B_1 B_4} + 8B_2 + (M-1)(M-3)} \right\} \end{aligned} \quad (5.25)$$

At this point we should report that our results reproduce those obtained by [61, 72, 73] when potential (5.2) (in case $B_2 = 0$) is confined to an impenetrable spherical box in 2- and 3-dimensions. It is also not difficult to see that if one takes $\eta = 0$ in Eq. (5.23), then Eq. (5.25) becomes the exact energy spectra of N -dimensional harmonic oscillator. Further, one easily check that in case $B_4 = B_3 = 0$, the above energy expression correctly reproduce the eigenvalues of the pseudo-type potential in 3-dimension [74] which is the subject of the next section.

Finally, we wish to discuss briefly the explicit mapping between the singular potentials given by Eqs. (5.1) and (5.2). If one consider the transformed anharmonic oscillator potential of Eq. (5.10) and repeat the above mathematical procedure carried out through Eqs. (5.21-5.25), then the corresponding eigenvalue equation reads

$$\hat{E}_{n=0} = -2\mu\left(1 + \frac{A_3}{\sqrt{2A_4}}\right) \quad (5.26)$$

Using the physically acceptable definition of A_1 in Eq. (5.20), the above equation can be rearranged as

$$\hat{E}_{n=0} = -\frac{2A_1}{|E_{n=0}|^{1/2}} \quad (5.27)$$

where $E_{n=0}$ has been described in Eq. (5.19). This brief discussion shows explicitly the relation between the two singular potentials in higher dimensions and verifies Eq. (5.11).

5.2.2 Exactly solvable case

Kasap [74] and his co-workers used th SUSYQM to find exact results for the special cases of the singular potentials of (5.1) and (5.2), more precisely the solutions of the Kratzer and pseudoharmonic potentials in three dimensions. Their results can be easily generalized to N -dimensions by the substitution $\ell \rightarrow \Lambda = (M-3)/2$ as indicated in this section. This extension to arbitrary dimensions helps us in constructing the map between these two distinct systems.

The study of anharmonic oscillators has raised a considerable amount of interest because of its various applications especially in molecular physics. The Morse potential is commonly used for anharmonic oscillator. However, its wave function does not vanish at the origin, but those for Mie-type and pseudoharmonic potentials do. The Mie-type potential possesses the general features of the true interaction energy, inter-atomic and inter-molecular, and dynamical properties of solids [54]. On the other hand, the pseudoharmonic potential may be used for the energy spectrum of linear and non-linear systems [64]. The Mie-type and pseudo-harmonic potentials are two special kinds of analytically solvable singular-power potentials as they have the property of shape-invariance.

Starting with the general form of the Mie-type potential

$$V(r) = D_0 \left[\frac{p}{q-p} \left(\frac{\sigma}{r}\right)^q - \frac{q}{q-p} \left(\frac{\sigma}{r}\right)^p \right] \quad (5.28)$$

where D_0 is the interaction energy between two atoms in a molecular system at $r = \sigma$, and $q > p$ is always satisfied. If we take $q = 2p$ and $p = 1$, we arrive at a special case of the potential in Eq. (5.28), which is exactly solvable

$$V(r) = \frac{A}{r^2} - \frac{B}{r} \quad (5.29)$$

where $A = D_0\sigma^2$ and $B = 2D_0\sigma$. The above potential, the so-called Kratzer potential, includes the terms which give the representation of both the steep repulsive branch and the long-range attraction. A single minimum occurs at $r = \sigma$ where the energy is $-D_0$. Considerable interest has recently been shown in this potential as a model to describe inter-nucleon vibration [75] and, in applications this Mie type potential offers one of the most important exactly solvable models of atomic and molecular physics and quantum chemistry [76].

We set the superpotential for the Kratzer effective potential

$$W(r) = \frac{B/2}{\beta + (\beta^2 + C)^{1/2}} - \frac{\beta + (\beta^2 + C)^{1/2}}{r} \quad (5.30)$$

where

$$C = \frac{\Lambda(\Lambda + 1)}{2} + A, \quad \Lambda = \ell + \frac{1}{2}(N - 3), \quad \beta = \frac{1}{2\sqrt{2}} \quad (5.31)$$

and obtained the exact spectrum in N -dimensional space as

$$E_n = \left(\frac{B/2\beta}{2n + 1 + [(2\Lambda + 1)^2 + A/\beta^2]^{1/2}} \right)^2, \quad n = 0, 1, 2, \dots \quad (5.32)$$

and from Eq. (5.13) the exact unnormalized ground state wavefunction can be expressed as

$$\Psi_{n=0}(r) = r^{1/2\{1+[(2\Lambda+1)^2+A/\beta^2]^{1/2}\}} \times \exp\left(-\frac{Br/4\beta^2}{1+[(2\Lambda+1)^2+A/\beta^2]^{1/2}}\right) \quad (5.33)$$

The excited state wavefunctions can be easily determined from the usual approach in SUSYQM [5] and the normalization coefficients for each quantum state wave function can be analytically worked out using the explicit recurrence relation given in a recent work [13, 14, 77].

As a second application, we consider the general form of the pseudoharmonic potential

$$\tilde{V}(r) = V_0\left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2 = \tilde{B}r^2 + \frac{\tilde{A}}{r^2} - 2V_0 \quad (5.34)$$

which can be used to calculate the vibrational energies of diatomic molecules with the equilibrium bond length r_0 and force constant $k = 8V_0/r_0^2$, and set the corresponding superpotential as

$$W(r) = \sqrt{\tilde{B}}r - \frac{\beta + (\beta^2 + \tilde{C})^{1/2}}{r} \quad (5.35)$$

where $\tilde{B} = V_0/r_0^2$, $\tilde{C} = [\Lambda(\Lambda + 1) + 2\tilde{A}]/2$, $\tilde{A} = V_0r_0^2$. The exact full spectrum of the potential in arbitrary dimensions is

$$\tilde{E}_n = 2\beta\sqrt{\tilde{B}}\{4n + 2 + [(2\Lambda + 1)^2 + \tilde{A}/\beta^2]^{1/2}\} - 2V_0 \quad (5.36)$$

and the unnormalized exact ground state wave function is

$$\Psi_{n=0}(r) = r^{1/2\{1+[(2\Lambda+1)^2+\tilde{A}/\beta^2]^{1/2}\}} \times \exp\left(-\frac{\sqrt{\tilde{B}}r^2}{4\beta}\right) \quad (5.37)$$

Using the discussion in section 5.1, one can transform the Kratzer potential in Eq. (5.29) to its dual potential- shifted (by $2V_0$) pseudoharmonic-like potential in Eq. (5.34) with some restrictions in potential parameters. In the light of Eqs. (5.9-5.11), the transformed potential reads

$$\hat{V}(\rho) = \rho^2 + \frac{4A}{\rho^2} \quad (5.38)$$

which is in the form of the Goldman-Krivchenkov potential. Here $A(= D_0\sigma^2)$ is the Kratzer potential parameter and, considering Eqs. (5.34) through Eq. (5.36), constraints on the potential parameters are such that $\tilde{B} = 1$ and $\tilde{A} = 4A$. In this case corresponding eigenvalues are

$$\begin{aligned}\hat{E}_{n'} &= \frac{2B}{|E_n|^{1/2}} = 4\beta\left\{1 + 2n' + \left[1 + 4\Lambda'(\Lambda' + 1) + \frac{A}{\beta^2}\right]\right\}, \\ \Lambda' &= L + \frac{1}{2}(N' - 3)\end{aligned}\quad (5.39)$$

where $B(= 2D_0\sigma)$ and E_n are the coupling parameter and the eigenenergy values (Eq. 5.32), respectively, of the Kratzer potential.

The ensuing relationships among the dimensions and quantum numbers of the two distinct systems considered here in this section are :

$$N' = 2N - 2 - 2\lambda, \quad L = 2\ell + \lambda, \quad n' = 2n - 2 + \lambda \quad (5.40)$$

Clearly, the mapping parameter λ must be an integer if n' , L , n and ℓ are integers. It is worthwhile to discuss briefly the physics behind this transformation in the light of the comprehensive work of Kostelecky *et al.* [68, 69]. We note that it is a general feature of this map that the spectrum of the N -dimensional problem involving Kratzer potential is related to the half the spectrum of the N' -dimensional problem involving Goldman-Krivchenkov potential for any even integer N' . However, the quantities in Eq. (5.40) have parameter spaces that are further restricted by the properties chosen for the map. For instance, suppose we wish to map all states corresponding the N -dimensional Kratzer potential into those corresponding Goldman-Krivchenkov potential. Since on physical grounds we know that $N' \geq 2$, $n' \geq 0$, $L \geq 0$, we must impose $N \geq 2 + \lambda$, $n \geq 1 - \lambda/2$, $\ell \geq -\lambda/2$. This yields the bound $-2\ell \leq \lambda \leq N - 2$. Further requiring $n \geq 1$, $\ell \geq 0$ restricts the bound to $0 \leq \lambda \leq N - 2$. We conclude that all states of the N -dimensional Kratzer problem can be mapped into the appropriate Goldman-Krivchenkov problem, except for $N = 1$.

As an example, consider the three-dimensional Kratzer problem. Assuming we wish to map all its states into those of its dual-the Goldman-Krivchenkov potential, we must impose $0 \leq \lambda \leq 1$. First, take $\lambda = 0$. Then, the s -orbitals in Kratzer potential ($n \geq 1, \ell = 0$) are related to the ($n' = 2n - 2 \geq 0, L = 0$) states of the four-dimensional Goldman-Krivchenkov problem. Similarly, the p -states ($n \geq 2, \ell = 1$) correspond to the ($n' = 2n - 2 \geq 0, L = 0$) same problem. Next, suppose $\lambda = 1$. The states corresponding the potential in Eq. (5.29) are then mapped into the odd-integer states of the two-dimensional oscillator problem of Eq. (5.38). The s -orbitals of Kratzer potential ($n \geq 1, \ell = 0$) map into the ($n' = 2n - 1 \geq 1, L = 1$) anharmonic oscillator states corresponding Goldman-Krivchenkov potential, while the Kratzer p -orbitals ($n \geq 2, \ell = 1$) map into the

($n' = 2n - 1 \geq 3, L = 1$) oscillator states of Goldman-Krivchenkov problem. As a rule, in both cases ($\lambda = 0, 1$), the lowest-lying states of Goldman-Krivchenkov potential are excluded, one by one, with each higher value of ℓ .

As a final remark, a student of introductory quantum mechanics often learns that the Schrödinger equation is exactly solvable (for all angular momenta) for two central potentials in Eqs. (5.29) and (5.38), and for also their special cases ($A = 0$) the Coulomb and harmonic oscillator problems. Less frequently, the student made aware of the relation between these two problems, which are linked by a simple change of the independent variable discussed in detail through the Chapter. Under this transformation, energies and coupling constants trade places, and orbital angular momenta are rescaled. Thus, we have in this section shown that there is really only one quantum mechanical problem, not two involving the Kratzer and Goldman-Krivchenkov potentials, which can be exactly solved for all orbital angular momenta.

5.3 Conclusion

The main aim of this work has been to establish a very general connection between a class of singular potentials in higher dimensional space through the application of a suitable transformation. Although much work had been done in the literature on similar problems, an investigation as the one we have discussed was missing to our knowledge [78]. In addition, it is shown that the SUSYQM yields exact solutions for a single state only for the quasi-exactly solvable potentials such as the ones given in Eqs. (5.1) and (5.2) with some restrictions on the potential parameters in N -dimensional space, unlike the shape invariant exactly solvable potentials. We have also shown how to obtain exact solutions to such problems in any dimension by applying an adequate transformation to previously known three-dimensional results. This simple and intuitive method discussed through this Chapter is easy to be generalized. The application of this method to other potentials involving non-central ones are in progress.

CHAPTER 6

ON THE RELATIONSHIP BETWEEN THE SCREENING
COULOMB AND ANHARMONIC OSCILLATOR POTENTIALS
IN ARBITRARY DIMENSIONS

Although modern computational facilities permit the construction of solutions for any well-behaved potentials to any degree of accuracy, there remains continuing interest in exact solutions for a wider range of potentials. In connection with this, the technique of changing the independent coordinate has always been a useful tool in the solution of the Schrödinger equation. For one thing, this allows something of a systematic approach, enabling one to recognize the equivalence of superficially unrelated quantum mechanical problems. For example, solvable Natanzon [79] potentials in non-relativistic quantum mechanics are known to group into two disjoint classes depending on whether the Schrödinger equation can be reduced to a hypergeometric or a confluent hypergeometric equation. All the potentials within each class are connected via point canonical transformations. Gangopadhyaya and his co-workers [80] established a connection also between the two classes with appropriate limiting procedures and redefinition of parameters, thereby inter-relating all known solvable potentials. In order for the Schrödinger equation to be mapped into another Schrödinger equation, there are severe restrictions on the nature of the coordinate transformation. Coordinate transformations which satisfy these restrictions give rise to new solvable problems. When the relationship between coordinates is implicit, then the new solution are only implicitly determined, while if the relationship is explicit then the newly found solvable potentials are also shape invariant. In a more specific special application of these ideas, Kostelecky *et al.* [68] were able to relate, using an explicit coordinate transformation, the Coulomb problem in N -dimensions with the N -dimensional harmonic oscillator. Other explicit applications of the coordinate transformation idea can be found in the review articles of Haymaker and Rau [81].

Many recent papers, [70], [71] and the references therein, have addressed this subject of coordinate transformation placing a particular emphasis on QES power-law potentials, which is also the subject of the present work in some extent.

The generalization to higher dimensions of one-dimensional QES potentials was discussed in a recent paper [82] and a few specific N -dimensional solutions were listed. In that work, applying a suitable transformation, these potentials were shown to be related to the isotropic oscillator plus Coulomb potential system, some normalized isolated solutions for this system were obtained.

The importance in the study of QES potentials, apart from intrinsic academic interest, rests on the possibility of using their solutions to test the quality of numerical methods and in the possible existence of real physical systems that they could represent. For instance, anharmonic oscillators and screening Coulomb (or Yukawa) potentials represent simplified models of many situations found in different field of physics. These problems have been studied for years and a general solution has not yet been found.

The problem of quantum anharmonic oscillators has been the subject of much discussion for decades, both from an analytical and a numerical point of view, because of its important applications in quantum-field theory [83], molecular physics [84], and solid-state and statistical physics [85, 86]. Various methods have been used successfully for the one-dimensional anharmonic oscillators with various types of anharmonicities. Relatively less attention has been given to the anharmonic oscillators in higher-dimensional space because of the presence of angular-momentum states that make the problem more complicated. The recent works [70, 71] have shown that there are many interesting features of the anharmonic oscillators and the perturbed Coulomb problems in higher-dimensional space, and discussed the explicit dependence of these two potentials.

Using the spirit of the works in Refs. [70, 71], we show the mappings between screened Coulomb potentials and anharmonic oscillator potentials in N -dimensional space, which have not been previously linked. The connection between these potentials are also checked numerically by the use of the Lagrange-mesh calculation technique [87, 88]. Next we study the N -dimensional screened Coulomb problem and higher order anharmonic oscillators within the framework of SUSYQM [5] and have shown that SUSYQM yields exact solutions for a single state only for such quasi-exactly solvable potentials in higher dimensions with some constraints on the coupling constants. These constraints differ from each eigenvalue, and hence various solutions do not correspond to the same potential and are not orthogonal. We have not found these solutions discussed in the literature when this study has been performed.

6.1 The Schrödinger equation in N -dimensional space

The radial Schrödinger equation for a spherically symmetric potential in N -dimensional space, as discussed in the earlier Chapter,

$$-\frac{1}{2} \left[\frac{d^2 R}{dr^2} + \frac{N-1}{r} \frac{dR}{dr} \right] + \frac{\ell(\ell+N-2)}{2r^2} R = [E - V(r)]R \quad (6.1)$$

is transformed to

$$-\frac{d^2 \Psi}{dr^2} + \left[\frac{(M-1)(M-3)}{4r^2} + 2V(r) \right] \Psi = 2E\Psi \quad (6.2)$$

where $\Psi(r) = r^{(N-1)/2} R(r)$, the reduced radial wave function and $M = N + 2\ell$. Note that the solutions for a particular central potential are the same as long as M remains unaltered. For instance, the s -wave eigensolutions and eigenvalues in four-dimensional space are identical to the p -wave solutions in two-dimensions.

We substitute $r = \alpha\rho^2/2$ and $R = F(\rho)/\rho^\lambda$ and transform Eq.(6.1) to another Schrödinger-like equation in $N' = 2N - 2 - 2\lambda$ dimensional space with angular momentum $L = 2\ell + \lambda$,

$$-\frac{1}{2} \left[\frac{d^2 F}{d\rho^2} + \frac{N'-1}{\rho} \frac{dF}{d\rho} \right] + \frac{L(L+N'-2)}{2\rho^2} F = [\hat{E} - \hat{V}(\rho)]F \quad (6.3)$$

where

$$\hat{E} - \hat{V}(\rho) = E\alpha^2\rho^2 - \alpha^2\rho^2V(\alpha\rho^2/2) \quad (6.4)$$

and α is a parameter to be adjusted properly. Note that leaving rescaling constant α arbitrary for now gives us an additional degree of freedom. When we discuss bound state eigenvalues later, we can use this to allow the values of the potential coefficients to be completely independent of each other. Thus, by this transformation, the N -dimensional radial wave Schrödinger equation with angular momentum ℓ can be transformed to a $N' = 2N - 2 - 2\lambda$ dimensional equation with angular momentum $L = 2\ell + \lambda$. The significant of the mapping parameter λ will be discussed in the following section.

6.2 Mappings between the two distinct systems

A student of introductory quantum mechanics often learns that the non-relativistic Schrödinger equation can be solved numerically for all angular momenta for the screened Coulomb and anharmonic oscillator problems. Less frequently, the student is made aware of the relation between these two problems, which are linked by a simple change of the independent variable discussed in detail through this section. Under this transformation, energies and coupling

constants trade places, and orbital angular momenta are rescaled. Thus, we have shown here that there is really only one quantum mechanical, not two.

The static screened Coulomb potential

$$V_{SC}(r) = -e^2 \frac{\exp(-\delta r)}{r} \quad (6.5)$$

where δ is a screening parameter, is known to describe adequately the effective interaction in many-body environment of a variety of fields such as atomic, nuclear, solid-state and plasma physics. In nuclear physics it goes under the name of the Yukawa potential (with e^2 replaced by another coupling constant), and in plasma physics it is commonly known as the Debye-Hückel potential. Eq. (6.5) also describes the potential of an impurity in a metal and in a semiconductor. Since the Schrödinger equation for such potentials does not admit exact analytic solutions, various approximate methods, [24, 89] and the references therein, both analytic and numerical, have been developed.

For the purpose of clarity and concreteness in mapping the screened Coulomb systems into the anharmonic oscillator systems, we consider a screened Coulomb potential with low screening parameter such that

$$\begin{aligned} V_{SC}(r) &= -e^2 \frac{\exp(-\delta r)}{r} \\ &\cong -\frac{e^2}{r} + e^2\delta - \frac{e^2\delta^2}{2}r + \frac{e^2\delta^3}{6}r^2 - \frac{e^2\delta^4}{24}r^3 + \frac{e^2\delta^5}{120}r^4 \\ &= \frac{A_1}{r} + A_2 + A_3r + A_4r^2 + A_5r^3 + A_6r^4 \end{aligned} \quad (6.6)$$

and neglect the other terms in this expansion. Using the formalism of SUSYQM [5] we set the superpotential

$$W(r) = \frac{a_1}{r} + a_2 + a_3r + a_4r^2, \quad a_2 < 0, \quad a_4 < 0 \quad (6.7)$$

for the potential in (6.6) and identify $V_-(r)$ supersymmetric partner potential with the corresponding effective potential so that

$$\begin{aligned} V_-(r) &= W^2(r) + W'(r) \\ &= \frac{2a_1a_2}{r} + [a_2^2 + a_3(2a_1 + 1)] + 2(a_1a_4 + a_4 + a_2a_3)r + (2a_2a_4 + a_3^2)r^2 \\ &\quad + 2a_3a_4r^3 + a_4^2r^4 + \frac{a_1(a_1 - 1)}{r^2} \\ &= \left(\frac{2A_1}{r} + 2A_2 + 2A_3r + 2A_4r^2 + 2A_5r^3 + 2A_6r^4 \right) \\ &\quad + \frac{(M-1)(M-3)}{4r^2} - 2E_{n=0} \end{aligned} \quad (6.8)$$

where $n = 0, 1, 2, \dots$ is the radial quantum number and its connection to the principal quantum number n_p , which denotes the energy levels, is known as

$n_p = n + \ell + 1$. The relations between the parameters in (6.8) satisfy the supersymmetric constraints

$$a_1 = \frac{M-2}{2}, \quad a_2 = \frac{2A_1}{M-1}, \quad a_3 = -\frac{A_5}{\sqrt{2A_6}}, \quad a_4 = -\sqrt{2A_6} \quad (6.9)$$

Note that in order to retain the well-behaved solution at $r \rightarrow \infty$ we have chosen the negative sign in a_4 . The potential in (6.6) admits the exact solutions

$$\Psi_{n=0}(r) = N_0 r^{a_1} \exp\left(a_2 r + \frac{a_3}{2} r^2 + \frac{a_4}{3} r^3\right) \quad (6.10)$$

where N_0 is the normalization constant, with the physically acceptable eigenvalues

$$E_{n=0} = A_2 - \frac{1}{2} \left[\frac{4A_1^2}{(M-1)^2} - \frac{A_5}{\sqrt{2A_6}} M \right] \quad (6.11)$$

under the constraints

$$A_1 = -(M-1) \frac{8A_6 A_4 - 2A_5^2}{16A_6 \sqrt{2A_6}} \quad (6.12)$$

The results obtained agree with those in Ref. [76].

If we choose $\alpha^2 = 1/|E_{n=0}|$ in Eq. (6.4), the screened Coulomb problem in (6.6) with the corresponding eigenvalue in Eq. (6.11) can be transformed to an anharmonic oscillator problem such that

$$\hat{V}(\rho) = \left(1 + \frac{A_2}{|E_{n=0}|}\right) \rho^2 + \frac{A_3}{2|E_{n=0}|^{3/2}} \rho^4 + \frac{A_4}{4|E_{n=0}|^2} \rho^6 + \frac{A_5}{8|E_{n=0}|^{5/2}} \rho^8 + \frac{A_6}{16|E_{n=0}|^3} \rho^{10} \quad (6.13)$$

with the eigenvalue

$$\hat{E}_{n=0} = \frac{-2A_1}{|E_{n=0}|^{1/2}} \quad (6.14)$$

Thus, the system given by Eq. (6.6) in N -dimensional space is reduced to another system defined by Eq. (6.13) in $N' = 2N - 2 - 2\lambda$ dimensional space. In other words, by changing the independent variable in the radial Schrödinger equation, we have been able to demonstrate a close equivalence between the screened Coulomb potential and anharmonic oscillator potentials.

For almost two decades, the study of higher order anharmonic potentials have been much more desirable to physicists and mathematicians in understanding a few newly discovered phenomena such as structural phase transitions [55], polaron formation in solids [56], and the concept of false vacuum in the field theory [57]. Unfortunately, in these anharmonic potentials, not much work has been carried out on the potential like the one defined by (6.13) except the works in Refs. [66], [90] and [91]. Our investigation in N -dimensional space, beyond showing an explicit connection between two distinct systems involving potentials

of type (6.6) and (6.13), would also be helpful to the literature regarding the solutions of such potentials in arbitrary dimensions due to the recent wide interest in the lower-dimensional field theory.

For the anharmonic oscillator potential in (6.13), we set the corresponding superpotential

$$W(r) = ar^5 + br^3 + \frac{c}{r} + dr, \quad a < 0, \quad d < 0 \quad (6.15)$$

which leads to

$$\Psi_{n=0}(r) = C_0 r^c \exp\left(\frac{a}{6}r^6 + \frac{b}{4}r^4 + \frac{d}{2}r^2\right) \quad (6.16)$$

with C_0 being the corresponding normalization constant, and identifying $V_-(r)$ with the effective potential we arrive at an expression for the physically meaningful ground state eigenvalues of the anharmonic oscillator potential in arbitrary dimensions,

$$\hat{E}_{n=0} = -\frac{d}{2}(2c+1) = \frac{8A_6A_6 - 2A_5^2}{16A_6\sqrt{2A_6}} \frac{M'}{|E_{n=0}|^{1/2}} \quad (6.17)$$

where $M' = N' + 2L$, and the relations between the potential parameters satisfy the supersymmetric constraints

$$\begin{aligned} a &= \pm \sqrt{\frac{A_6}{8}} \frac{1}{|E_{n=0}|^{3/2}}, \quad b = \frac{A_5}{8a} \frac{1}{|E_{n=0}|^{5/2}}, \\ c &= \frac{M' - 1}{2}, \quad d = \frac{1}{2a} \left(\frac{A_4}{2|E_{n=0}|^2} - b^2 \right). \end{aligned} \quad (6.18)$$

As we are dealing with a confined particle system, the negative values for a and d would of course be the right choice to ensure the well behaved nature of the wave function behaviour at the origin and at infinity. Our results are in agreement with the literature existing for three-dimensions [90, 91] (in case $N' = 3$) and for two-dimensions [66] (in case $N' = 2, L \rightarrow L - 1/2$).

To show explicitly the physics behind this transformation, we return back to Eq. (6.8) and consider the relation between the parameters

$$A_4 = a_2 a_4 + \frac{a_3^2}{2} \Rightarrow 8A_6A_4 - 2A_5^2 = -\frac{16A_1A_6\sqrt{2A_6}}{M-1} \quad (6.19)$$

and the substitution of Eq. (6.19) into Eq.(6.17) leads to

$$\hat{E}_{n=0} = -\frac{M'}{M-1} \frac{A_1}{|E_{n=0}|^{1/2}} \quad (6.20)$$

To be in consistent with Eq. (6.14) we must impose $0 \leq \lambda \leq 1$ as an integer, such that

$$\frac{M'}{M-1} = \frac{2(N-1-\lambda) + 2(2\ell + \lambda)}{N + 2\ell - 1} = 2 \quad (6.21)$$

It is a general feature of this map that, in both cases ($\lambda = 0, 1$), the spectrum of the N -dimensional screened Coulomb problem is related to half the spectrum of the N' -dimension anharmonic oscillator for any even integer N' . The reader is referred to [68, 69] for a comprehensive discussion of similar conformal mappings in the language of physics.

It is worthwhile at this stage to note that recently Chaudhuri and Mondal [70] studied the relations between anharmonic oscillators and perturbed Coulomb potentials in higher dimensions but their results correspond only to the case when $\lambda = 1$, in this case the three-dimensional perturbed Coulomb problem and the four-dimensional anharmonic oscillator cannot be related. However, by introducing an extra degree of freedom for the mapping parameter ($\lambda = 0$) through our equations, we can reproduce the well-known results found in the literature in three-dimensions. With this exact correspondence we can check Eq. (6.14), using exact results for the screened Coulomb potential, and calculated numerical results for the anharmonic oscillator potential.

6.3 Results and Discussion

In this section numerical applications of the transformation presented in the previous section are given. Calculations to check the validity of the equations developed for the screened Coulomb and anharmonic potentials are also given here. Table 6.1 displays the exact eigenvalues of the screened Coulomb potential in three-, and five-dimensions obtained using the Lagrange-mesh calculation technique [87, 88] for selected values of the potential parameters. Highly accurate Lagrange-mesh calculation results agree well with the best existing numerical and theoretical values obtained in three-dimensions [24, 89]. Due to the constraint in the potential parameter A_1 expressed in Eq. (6.12), we are not able to show in the same table the corresponding exact energy values which can be calculated by Eq. (6.11). For the work of interest in this study we set $A_1 = -1$, consequently the adequate δ -values satisfying the condition in Eq. (6.12) fall outside the scope of the presented work which has been performed for only low screening parameters.

Further, our calculation results shown in Table 6.1 make clear that the eigenvalues of the five-dimensional screened Coulomb problem with any angular momentum quantum number ℓ , for a particular δ value, are equal to the same system with $\ell+1$ in three-dimensions, due to $M = N+2\ell$ which remains unaltered for these states. The tabulated results support the work of Imbo and Sukhatme [92] in which they formulated SUSYQM for spherically symmetric potentials in N spatial dimensions and showed that the supersymmetric partner of a given

potential can be effectively treated as being $N + 2$ ($\ell \rightarrow \ell + 1$) dimensions. This fact was exploited in their calculations using the shifted $1/N$ expansion.

It is also noted that for very small values of the screening parameter, the screening Coulomb potential reduces to the Coulomb potential that is shape invariant having supersymmetric character. Therefore, the related supersymmetric partner potentials, such as V_ℓ and $V_{\ell+1}$, are expected to have the same spectra except the ground-state energy. This can easily be seen in Table 6.1 for the case of $\delta = 0.001$ in both arbitrary dimensions. For instance, the supersymmetric partner of the s -orbital ($\ell = 0$) spectrum of hydrogen is the p -orbital ($\ell = 1$) spectrum of the same system.

Finally, the exact calculated eigenvalues, by the use of Eq. (6.14), for the anharmonic oscillator in four-dimensions from the known exact results for the screened Coulomb problem in three-dimensions are displayed in Table 6.2. These eigenvalues are checked by the Lagrange-mesh calculations and tabulated in the same table. The agreement is in general very good.

6.4 Concluding Remarks

The mapping problems in arbitrary dimensions have been the subject of several papers and have served to illustrate various aspects of quantum mechanics of considerable pedagogical value. As the objective of this presentation we have highlighted a different facet of these studies and established a very general connection between the screened Coulomb and anharmonic oscillator potentials in higher dimensional space through the application of a suitable transformation, the purpose being to emphasize the pedagogical value residing in this interrelationship between two of the most practical applications of quantum mechanics. Although much work had been done in the literature on similar problems, an investigation as the one we have discussed was missing to our knowledge.

As a concluding remark, we note that accurate solutions of the Schrödinger equation for the screening Coulomb (or Yukawa) potential is needed in electron scattering from neutral atoms as well as in nuclear structure calculation. However, for high- Z neutral atoms, this potential is very narrow and deep, for which the wave function of the trapped electron is sharply peaked near the origin and the numerical solution (especially the wave function) is both difficult and inaccurate. Since the screened Coulomb potential is not shape invariant, exact analytic solutions are not possible. In this situation a reliable solution, which is the one proposed in a simple form through the present work considering various types of correspondence between anharmonic and deformed Coulomb potentials, is of particular importance.

Table 6.1: The first four eigenvalues of the screening Coulomb potential in Eq. (6.6) as a function of the screening parameter in atomic units.

In three-dimensional space					
δ	ℓ	n=0	n=1	n=2	n=3
0.001	0	-0.499 000	-0.124 003	-0.054 562	-0.030 262
	1	-0.124 002	-0.054 561	-0.030 261	-0.019 018
	2	-0.054 561	-0.030 260	-0.019 017	-0.012 914
0.005	0	-0.495 019	-0.120 074	-0.050 720	-0.026 537
	1	-0.120 062	-0.050 708	-0.026 526	-0.015 428
	2	-0.050 684	-0.026 503	-0.015 406	-0.009 474
0.010	0	-0.490 075	-0.115 293	-0.046 199	-0.022 356
	1	-0.115 245	-0.046 153	-0.022 313	-0.011 622
	2	-0.046 061	-0.022 228	-0.011 543	-0.006 070
0.020	0	-0.480 296	-0.106 148	-0.038 020	-0.015 377
	1	-0.105 963	-0.037 852	-0.015 232	-0.005 891
	2	-0.037 515	-0.014 939	-0.005 653	-0.001 521
0.025	0	-0.475 461	-0.101 776	-0.034 329	-0.012 495
	1	-0.101 492	-0.034 079	-0.012 287	-0.003 770
	2	-0.033 573	-0.011 865	-0.003 458	0.000 253

In five-dimensional space					
δ	ℓ	n=0	n=1	n=2	n=3
0.001	0	-0.124 002	-0.054 561	-0.030 261	-0.019 018
	1	-0.054 561	-0.030 260	-0.019 017	-0.012 914
	2	-0.030 259	-0.019 016	-0.012 912	-0.009 237
0.005	0	-0.120 062	-0.050 708	-0.026 526	-0.015 428
	1	-0.050 684	-0.026 503	-0.015 406	-0.009 474
	2	-0.026 468	-0.015 373	-0.009 443	-0.005 961
0.010	0	-0.115 245	-0.046 153	-0.022 313	-0.011 622
	1	-0.046 061	-0.022 228	-0.011 543	-0.006 070
	2	-0.022 099	-0.011 425	-0.005 965	-0.002 980
0.020	0	-0.105 963	-0.037 852	-0.015 232	-0.005 891
	1	-0.037 515	-0.014 939	-0.005 653	-0.001 521
	2	-0.014 491	-0.005 286	-0.001 263	0.000 885
0.025	0	-0.101 492	-0.034 079	-0.012 287	-0.003 770
	1	-0.033 573	-0.011 865	-0.003 458	0.000 253
	2	-0.011 216	-0.002 974	0.000 524	0.003 087

Table 6.2: Ground-state eigenvalues of the anharmonic potential in Eq. (6.13)

In four-dimensional space					
δ	ℓ	L	$ E_{n=0} $ (taken from Table 6.1)	$\hat{E}_{n=0}$ Lagrange-mesh calculations	$\hat{E}_{n=0}$ Exact value (Eq. 6.14)
0.001	0	0	0.499 000	2.831 259	2.831 259
	1	2	0.124 002	5.679 579	5.679 573
	2	4	0.054 561	8.562 285	8.562 268
0.005	0	0	0.495 019	2.842 624	2.842 622
	1	2	0.120 062	5.772 014	5.772 012
	2	4	0.050 684	8.883 704	8.883 714
0.010	0	0	0.490 075	2.856 927	2.856 924
	1	2	0.115 245	5.891 401	5.891 406
	2	4	0.046 061	9.318 882	9.318 871
0.020	0	0	0.480 296	2.885 862	2.885 862
	1	2	0.105 963	6.144 014	6.144 024
	2	4	0.037 515	10.325 883	10.325 891
0.025	0	0	0.475 461	2.900 499	2.900 498
	1	2	0.101 492	6.277 884	6.277 896
	2	4	0.033 573	10.915 282	10.915 281

CHAPTER 7

QUANTUM SYSTEMS WITH POSITION-DEPENDENT MASS

7.1 Supersymmetric Approach

The one-dimensional time-independent Schrödinger equation associated with a particle endowed with a position-dependent effective mass is given in its most general form by [16, 93]

$$-\left(\frac{\hbar^2}{2m(x)}\right) \frac{d^2\psi(x)}{dx^2} - \left[\frac{d}{dx} \left(\frac{\hbar^2}{2m(x)}\right)\right] \frac{d\psi(x)}{dx} + V(x)\psi(x) = E\psi(x) \quad (7.1)$$

where $m(x)$ is the particle's effective mass, $V(x)$ denotes the potential, $\psi(x)$ is the particle's wave function, and E is the eigenenergy.

We now consider the expressions of the supersymmetric formalism dealing with Eq. (7.1). From now on $m = m(x)$ denotes a function of x . Proceeding as in the case of constant mass, we introduce (i) a superpotential $W(x)$ and (ii) the associated pair of operators A^+ and A^- defined by

$$A^+\psi = -\frac{\hbar^2}{\sqrt{2m}} \frac{d\psi}{dx} + W\psi, \quad A^-\psi = +\frac{d}{dx} \left(\frac{\hbar\psi}{\sqrt{2m}}\right) + W\psi. \quad (7.2)$$

Notice that, due to the position dependence of the mass, d/dx and $\hbar/\sqrt{2m}$ do not commute anymore. The associated supersymmetric Hamiltonians can be obtained as following. The Hamiltonian H_1 corresponding to a particle with a position-dependent mass moving in the one dimensional potential is

$$H_1 = A^+A^- = -\left(\frac{\hbar^2}{2m}\right) \frac{d^2}{dx^2} - \left(\frac{\hbar^2}{2m}\right)' \frac{d}{dx} - \left(\frac{\hbar W}{\sqrt{2m}}\right)' + W^2. \quad (7.3)$$

where *prime* denotes the first derivative with respect to x .

Note that first two terms in above equation is the kinetic energy operator, T and the last two terms correspond to potential energy operator, V . The one dimensional potential definition from the above equation is obtained as

$$V_1 = W^2 - \left(\frac{\hbar W}{\sqrt{2m}}\right)'. \quad (7.4)$$

The associated supersymmetric partner Hamiltonian is now

$$\begin{aligned}
 H_2 &= A^- A^+ \\
 &= - \left(\frac{\hbar^2}{2m} \right) \frac{d^2}{dx^2} - \left(\frac{\hbar^2}{2m} \right)' \frac{d}{dx} - \left(\frac{\hbar W}{\sqrt{2m}} \right)' \\
 &\quad + W^2 + \frac{2\hbar W'}{\sqrt{2m}} - \left(\frac{\hbar}{\sqrt{2m}} \right) \left(\frac{\hbar}{\sqrt{2m}} \right)'' . \quad (7.5)
 \end{aligned}$$

where *primes* denotes the derivative with respect to x .

The supersymmetric partner potential of V_1 is found as

$$V_2 = W^2 - \left(\frac{\hbar W}{\sqrt{2m}} \right)' + \frac{2\hbar W'}{\sqrt{2m}} - \left(\frac{\hbar}{\sqrt{2m}} \right) \left(\frac{\hbar}{\sqrt{2m}} \right)'' . \quad (7.6)$$

The partner potential V_2 depends on not only the potential V_1 but also the form of the effective mass $m(x)$.

As a summary, the supersymmetric formalism allows for the construction of pairs of partner Hamiltonians H_1 and H_2 for systems with a position-dependent mass which have the same spectra (with the exception of the ground state of H_1).

7.1.1 Shape-Invariant Potentials with Position-Dependent Mass

The concept of shape invariance constitutes the basis of a powerful and elegant generalization of the well-known procedure for solving potentials using the operators. A potential $V(x, a)$ depending on a set of parameters a is said to be shape invariant when it is related to its supersymmetric partner as

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1). \quad (7.7)$$

The Equation (7.7) says that the partner potentials V_1 and V_2 have the same form, but characterized by different values of the set of parameters a . As in the usual SUSYQM, there is a relation between the parameters a_1 and a_2 ,

$$a_2 = f(a_1), \quad (7.8)$$

and again $R(a_1)$ is an x -independent potential shift, which determines the eigenenergies of the system for given potential and mass definition.

The eigenenergies of a shape-invariant potential are then given by

$$E_n = \sum_{i=1}^n R(a_i). \quad (7.9)$$

where

$$a_{i+1} = f(a_i), \quad i = 1, 2, \dots, n-1. \quad (7.10)$$

All the usual exactly solvable potentials, as well as many discovered ones, belong to the class of shape-invariant potentials [5]. On the basis of these supersymmetric formalism for systems with nonconstant mass, two examples of that shape-invariant integrability condition are going to be considered for a mass, m .

7.1.1.1 Harmonic-oscillator-like Potential

The simplest case of the shape invariance condition for the partner potentials is the differing only by a uniform energy shift, say β . So, the Eq. (7.7) can be written as

$$V_2(x; \beta) = V_1(x; \beta) + \beta \quad (7.11)$$

Starting with that point and replacing the expressions (7.4) and (7.6) (and setting $\hbar = 1$, after that step) for the partner potentials into the (7.11), one obtains

$$\frac{2W'}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}}\right) \left(\frac{1}{\sqrt{2m}}\right)'' = \beta = \text{const.} \quad (7.12)$$

Solving for the superpotential term, $W(x)$, one gets

$$W(x) = \frac{1}{2} \left(\frac{1}{\sqrt{2m}}\right)' + \frac{1}{2}\beta \int^x \sqrt{2m(z)} dz. \quad (7.13)$$

The above equation indicates that for each effective mass, $m(x)$, it is *possible* to obtain a superpotential $W(x)$ whose associated pair of partner potentials complies with the condition (7.11). So, let the definition of x -dependent mass be given by

$$m(x) = m_0 \left(\frac{\alpha + x^2}{1 + x^2}\right)^2. \quad (7.14)$$

For the mathematical simplicity, set $m_0 = \text{unity}$. This effective mass verifies the conditions

$$m(0) = \alpha^2, \quad m(\infty) = \lim_{x \rightarrow \infty} m(x) = 1. \quad (7.15)$$

Replacing Eq. (7.14) in Eq. (7.13), we obtain the superpotential as

$$W(x) = \frac{\beta x}{\sqrt{2}} + \frac{(\alpha - 1)}{\sqrt{2}} \left[\beta \arctan(x) + \frac{x}{(\alpha + x^2)^2} \right]. \quad (7.16)$$

Since the superpotential term is obtained explicitly, this leads us to define the shape-invariant potential function given in Eq. (7.4). The first-partner potential is obtained as

$$V(x) = \frac{\beta^2}{2} [x + (\alpha - 1) \arctan(x)]^2 + \frac{(\alpha - 1)}{2(\alpha + x^2)^4} [3x^4 + (4 - 2\alpha)x^2 - \alpha] - \frac{\beta}{2}. \quad (7.17)$$

In this case the potential function is characterized by one single parameter β . Note that the potential term obtained in Eq. (7.17) reduces directly to harmonic

oscillator potential when $\alpha = 1$ which means $m(x) \rightarrow m_0$. The function R , which is x -independent term, appearing in Eq. (7.7) reduces to the identity function, that is, $R(\beta) = \beta$. The general expression for the energy levels, as given in Eq. (7.9), of such a shape-invariant potential leads to the harmonic-oscillator-like spectra

$$E_n = n \beta. \quad (7.18)$$

The different values of parameter α determines the smoothness or the sharpness of the partner potential of $V(x)$. In spite of the different aspects exhibited by the above potential terms for such values, they all share *the same energy spectra*.

7.1.1.2 Morse-like Potential

The supersymmetric solution to Morse potential for constant mass condition has been given in Table I of the Ref. [39]. In this Section, the position-dependent mass solution for the same potential is going to be considered, in the framework of SUSYQM. The superpotential will be considered as following,

$$W(x; A) = A + g(x), \quad (7.19)$$

Here, A is the parameter characterizing the potential function $V(x; A)$ ($A \equiv a_1$). Since the shape-invariance condition is depending on a set of parameters for the partner potentials, a_2 should be defined in terms of a_1 in supersymmetry with reference equation (7.7). The assumption is given in Ref. [94] as

$$a_2 = a_1 - \gamma. \quad (7.20)$$

Thus, the shape-invariance condition, given in Eq. (7.7), adopts the form

$$V_2(x; A) = V_1(x; A - \gamma) + R(A). \quad (7.21)$$

Substituting the expression (7.19) into the Eqs. (7.4) and (7.6) (by setting $\hbar = 1$) and replacing the results into the Eq. (7.21), one gets a differential equation that the superpotential W must verify if $V_1(x; A)$ is to be shape invariant. This is

$$\begin{aligned} W^2(x; A) - \left(\frac{W(x; A)}{\sqrt{2m}} \right)' + \frac{2W'(x; A)}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}} \right) \left(\frac{1}{\sqrt{2m}} \right)'' \\ = - \left(\frac{W(x; A - \gamma)}{\sqrt{2m}} \right)' + W^2(x; A - \gamma) + R(A). \end{aligned} \quad (7.22)$$

From this differential equation, it is to verify that $R(A)$ is

$$R(A) = 2A\gamma - \gamma^2 = A^2 - (A - \gamma)^2, \quad (7.23)$$

so that the condition (7.22) can be solvable for $g(x)$,

$$\frac{dg(x)}{dx} = -\gamma\sqrt{2m} g(x) + \frac{\gamma}{2}\sqrt{2m} \left(\frac{1}{\sqrt{2m}}\right)' + \frac{1}{2} \left(\frac{1}{\sqrt{2m}}\right)'' . \quad (7.24)$$

Solving this differential equation for $g(x)$ leads us to define the superpotential $W(x; A)$ with respect to the position-dependent mass definition. The general expression of $g(x)$ is found as following

$$g(x) = \left[B + \int^x b_2(s) \exp \left\{ - \int^s b_1(t) dt \right\} ds \right] \times \exp \left\{ \int^x b_1(s) ds \right\}, \quad (7.25)$$

where B is an integration constant and the functions $b_{1,2}(x)$ are given as, respectively,

$$b_1(x) = -\gamma\sqrt{2m}, \text{ and } b_2(x) = \frac{\gamma}{2}\sqrt{2m} \left(\frac{1}{\sqrt{2m}}\right)' + \left(\frac{1}{\sqrt{2m}}\right)'' . \quad (7.26)$$

Now, it can be obtained that the energy eigenvalues simply by replacement of Eq. (7.23) into the general expression, obtained previously in Eq. (7.9), for the energy levels of shape-invariant potentials. The definition of parameter a_k is

$$\begin{aligned} a_k &= A - (k-1)\gamma, \\ R(a_k) &= a_k^2 - a_{k+1}^2 = 2A\gamma - (2k-1)\gamma^2, \quad (k=1, 2, \dots) \end{aligned} \quad (7.27)$$

and finally

$$E_n = A^2 - (A - n\gamma)^2 . \quad (7.28)$$

After that point, the same expression, given in Eq. (7.14), for the position-dependent mass will be considered for the Morse-like potential, as discussed previously in harmonic-oscillator-like potential example. Using that mass form and replacing it into the integral term given in Eq. (7.25), the superpotential of that position-dependent mass is obtained as

$$W(x) = A + B \exp\{-\sqrt{2}\gamma[x + (\alpha - 1) \arctan(x)]\} + \frac{x(\alpha - 1)}{\sqrt{2}(\alpha + x^2)^2} . \quad (7.29)$$

Since we obtain the superpotential term, the corresponding potential function can be obtained by replacing $W(x)$ into the Eq. (7.4)

$$\begin{aligned} V(x) &= (A + B \exp\{-\sqrt{2}\gamma[x + (\alpha - 1) \arctan(x)]\})^2 \\ &\quad + B\gamma \exp\{-\sqrt{2}\gamma[x + (\alpha - 1) \arctan(x)]\} \\ &\quad + \frac{(\alpha - 1)[3x^4 + (4 - 2\alpha)x^2 - \alpha]}{2(\alpha + x^2)^4} . \end{aligned} \quad (7.30)$$

The function obtained in Eq. (7.30) is a Morse-like potential¹ and has the same energy spectra with Morse potential though the mass is a function of position. Note that if the parameter α is equal to "1", which means that $m(x) \rightarrow m_0$ see Eq. (7.14), then the above potential reduces to the conventional Morse potential with constant mass.

¹ in Eq. (7.30), γ corresponds to $1/\sqrt{2}$ according to the work in [39].

7.1.2 Conclusion

We have reviewed and shown that the quantum system corresponding to a particle endowed with a position-dependent mass $m(x)$ moving in a potential $V_1(x)$ admits of a supersymmetric partner system with the same effective mass and a supersymmetric partner potential $V_2(x)$. This pair of partner potentials can be expressed in terms of (i) the effective mass $m(x)$ and (ii) an appropriate superpotential $W(x)$. Both potentials $V_{1,2}(x)$ exhibit the same energy spectra except for the ground state of $V_1(x)$, which has no associated state of $V_2(x)$, as in the usual SUSYQM with constant mass condition.

Adopting an appropriate ansatz for the superpotential $W(x)$, it is possible to obtain potential functions $V_1(x)$ that are related to their supersymmetric partners $V_2(x)$ by the shape-invariance integrability condition. The eigenenergies (and also eigenfunctions) of these potentials can be found in algebraic fashion. As a simple illustration of the above procedure, we have re-proceeded how to find, for a given effective mass $m(x)$, a shape-invariant potential whose partner potential is equal to a shifted version of the original one. This is the most simple form of shape invariance. The concomitant operators comply with the well-known commutator algebra of the constant mass harmonic oscillator, allowing for an algebraic determination of the energy eigenvalues and eigenfunctions. We also considered a slightly more complicated shape-invariant condition leading to systems with nonconstant mass that exhibit a Morse-like spectra. Again, the energy eigenvalues are algebraically determined.

7.2 Schrödinger Equation with Position-Dependent Mass Under Point Canonical Transformations

In Ref. [95], it has been shown that the point canonical transformations, so called "*f-transformations*", can be applied to solve the one-dimensional, non-relativistic Schrödinger equation with constant mass. The main purpose there was to show that the known types of shape invariant potentials could be grouped into two classes in the sense that the potentials in any class could all be mapped to a single potential in that class under *f*-transformation. The study gives explicit point canonical transformations which map twelve types of shape invariant potentials (which are known to be exactly solvable) into two potential classes. The procedure used there was to invoke the appropriate transformation of both the dependent and independent variables in the Schrödinger equation with constant mass. The resultant new equation was solved and it was shown that the all observables agree with those in Ref. [39].

Here, through the present Section, we generalize this transformation tech-

nique for the mappings between two different systems having constant and non-constant masses.

7.2.1 Transformation of Schrödinger Equation

As a starting point, we rewrite the Schrödinger equation with effective mass,

$$-\left(\frac{\hbar^2}{2m(x)}\right) \frac{d^2\psi(x)}{dx^2} - \left[\frac{d}{dx} \left(\frac{\hbar^2}{2m(x)}\right)\right] \frac{d\psi(x)}{dx} + V(x)\psi(x) = E\psi(x), \quad (7.31)$$

and transform both the dependent and independent variables as given in Ref. [95] but with a little difference: We will also transform the position-dependent mass definition with respect to independent variable, x .

$$x \rightarrow f(z), \quad \psi(x) \rightarrow \nu(z)\phi(z), \quad \text{and} \quad m(x) \rightarrow m(f(z)) = \bar{m}(z). \quad (7.32)$$

From now, "prime" sign indicates the derivation of functions with respect to "z". When the differential terms are transformed, one gets

$$\frac{d}{dx} = \frac{1}{f'} \frac{d}{dz}, \quad \frac{dm(x)}{dx} = \frac{1}{f'} \frac{d\bar{m}(z)}{dz}. \quad (7.33)$$

And the wave function transformations gives

$$\frac{d\psi}{dx} = \frac{1}{f'} (\nu' \phi + \nu \phi'), \quad (7.34)$$

$$\frac{d}{dx} \left(\frac{d\psi}{dx}\right) = -\frac{f''}{f'^3} (\nu' \phi + \nu \phi') + \frac{1}{f'^2} (\nu'' \phi + 2\nu' \phi' + \nu \phi''). \quad (7.35)$$

Inserting all these terms into the Eq. (7.31), and multiplying the result by $\left(\frac{f'^2}{\nu}\right) \left(\frac{\bar{m}}{m_0}\right)$, one gets

$$-\frac{\hbar^2}{2m_0} \phi'' - \frac{\hbar^2}{2m_0} \left(\frac{2\nu'}{\nu} - \frac{f''}{f'} - \frac{\bar{m}'}{\bar{m}}\right) \phi' - \frac{\hbar^2}{2m_0} \left(\frac{\nu''}{\nu} - \frac{f''\nu'}{f'\nu} - \frac{\bar{m}'\nu'}{\bar{m}\nu}\right) \phi + \frac{\bar{m}}{m_0} f'^2 [V(f(z)) - E] \phi = 0. \quad (7.36)$$

where m_0 denotes the constant mass.

Since we try to obtain Schrödinger equation with constant mass form, it is clear in Eq. (7.36) that the second term should be equal to zero. Then, one arrives at

$$\nu = \sqrt{f' \bar{m}}. \quad (7.37)$$

Additionally, if $f'^2 = \sqrt{\frac{m_0}{\bar{m}}}$ is chosen in Eq. (7.36), one gets

$$\nu = C(m_0 \bar{m})^{1/4}, \quad (7.38)$$

where C is the integral constant and given as $C = 1/\sqrt{m_0}$ since the wave function $\psi(x)$ is square integrable, i.e. $\langle \psi(x) | \psi(x) \rangle = 1$. This kind of appointment of f' leads us to define the new coordinate system z in terms of x . Remembering the condition

$$\frac{dz}{dx} = \frac{1}{f'(z)}, \quad (7.39)$$

we then can write

$$\frac{dz}{dx} = \sqrt{\frac{\bar{m}(z)}{m_0}}, \quad (7.40)$$

and finally we obtain

$$z = \frac{1}{\sqrt{m_0}} \int^x dt \sqrt{\bar{m}(t)}. \quad (7.41)$$

This expression leads us to express the shape-invariant potential in term of the new coordinate variable depending on the x -dependent mass definition.

The third term -which is in terms of f , ν and \bar{m} - now can be found in the transformed form of position-dependent mass equation. It is expressed as

$$\frac{\nu''}{\nu} - \frac{f''\nu'}{f'\nu} - \frac{\bar{m}'\nu'}{\bar{m}\nu} = \frac{4\bar{m}\bar{m}'' - 5\bar{m}'^2}{16\bar{m}^2}. \quad (7.42)$$

Turning back to Eq. (7.36) with these terms, we reach to the transformed form of Schrödinger equation with position-dependent mass in z coordinate system. It is written as

$$-\frac{\hbar^2}{2m_0}\phi'' + \frac{\hbar^2}{2m_0} \left(\frac{5\bar{m}'^2 - 4\bar{m}\bar{m}''}{16\bar{m}^2} \right) \phi + [V(f(z)) - E]\phi = 0. \quad (7.43)$$

Since $m(x) = \bar{m}(z) = m(f(z))$, one can write

$$\bar{m}'(z) = \frac{d\bar{m}(z)}{dz} = \frac{dx}{dz} \frac{d}{dx} [\bar{m}(z)] = \sqrt{\frac{m_0}{\bar{m}(z)}} m'(x) \Big|_{z=x}, \quad (7.44)$$

where $\frac{dx}{dz} = f'(x) = \sqrt{\frac{m_0}{m(x)}}$. And for $\bar{m}''(z)$, it is found that

$$\bar{m}''(z) = m_0 \left(\frac{m''(x)}{m(x)} - \frac{1}{2} \frac{m'^2(x)}{m^2(x)} \right). \quad (7.45)$$

Finally, replacing all these terms into the Eq. (7.43) one reaches to the mapped Schrödinger equation with an effective mass

$$-\frac{\hbar^2}{2m_0} \frac{d^2\phi}{dz^2} + \left\{ \frac{\hbar^2}{32m^3(z)} [7m'^2(x) - 4m(x)m''(x)] \right\}_{x=z} \phi + [V(f(z)) - E]\phi = 0. \quad (7.46)$$

For a known transformation function $f(z)$, one then finds the wavefunction $\psi(x)$ of the original problem in terms of the product of known function $\nu(z)$ and eigenfunction $\phi(z)$ without any change in the energy eigenvalue. Once the

desired eigenfunction, $\phi(z)$ is obtained in terms of the transformed variable z , it may easily be expressed in terms of the corresponding physical one by inverse transformation.

Eq. (7.46) has exactly the same form as given in Ref. [96] although there is some missing terms and typographical mistakes in that work. Now, to have exactly solvable system, the potential term $V(f(z))$ should be given, to include the shape-invariant potential function, as

$$V(f(z)) = V_{SIP}(f(z)) - V_{mass}(x) \quad (7.47)$$

where $V_{SIP}(f(z))$ represents the solvable "shape-invariant potential" term in z coordinate and

$$V_{mass}(x) = \frac{7m'^2(x) - 4m(x)m''(x)}{32m^3(x)}. \quad (7.48)$$

Finally, we have reached a Schrödinger equation transformed by the "*f-transformation*" method into a new form including position-dependent mass definition. Now, we are going to apply this procedure to potentials mentioned in the framework of SUSYQM in the previous Section.

7.2.1.1 Application with Harmonic Oscillator Potential

The harmonic oscillator potential is given in atomic units (setting $\hbar = m_0 = 1$ for simplicity) as

$$V(x) = \frac{\beta^2}{2}x^2 \quad (7.49)$$

and the energy states of this potential is known as

$$E_n = n\beta. \quad (7.50)$$

As an example, the same position-dependent effective mass given in Eq. (7.14) will be considered. For this mass definition, the integral in Eq. (7.41) will be in the form

$$z = \int^x dx \sqrt{\left(\frac{\alpha + x^2}{1 + x^2}\right)^2}, \quad (7.51)$$

and the transformed coordinate becomes as

$$z = x + (\alpha - 1) \arctan(x). \quad (7.52)$$

So that, the harmonic oscillator potential can be written in its new form as $V_{SIP}(f(z))$

$$V_{SIP}(x) = \frac{\beta^2}{2}[x + (\alpha - 1) \arctan(x)]^2. \quad (7.53)$$

The $V_{mass}(x)$ term takes the form

$$V_{mass}(x) = \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[-3x^4 - (4 - 2\alpha)x^2 + \alpha], \quad (7.54)$$

since the x -dependent mass function is given as in Eq. (7.14). Finally, the $V(f(z))$ in Eq. (7.47) can be written as

$$V(x) = \frac{\beta^2}{2}[x + (\alpha - 1) \arctan(x)]^2 + \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 + (4 - 2\alpha)x^2 - \alpha]. \quad (7.55)$$

The Eq. (7.55) is exactly the same with Eq. (7.17) which is obtained in the framework of supersymmetry. The eigenvalues of this harmonic-like potential, Eq. (7.55), is exactly given as in Eq. (7.50).

7.2.1.2 Application with Morse Potential

Now, we will apply the method to the Morse potential with the same x -dependent mass function. From [39], using the atomic scale, the potential can be given as

$$V_{Morse}(x) = A^2 + B^2 e^{-2x} - 2B \left(A + \frac{1}{2\sqrt{2}} \right) e^{-x}, \quad (7.56)$$

and the energy values of the this potential is

$$E_n = A^2 - \left(A - \frac{n}{\sqrt{2}} \right). \quad (7.57)$$

for the constant mass condition. Using the same procedure done in the previous Section, we can obtain the $V_{SIP}(f(z))$ term for Morse potential as

$$V_{SIP}(x) = A^2 + B^2 e^{-2[x+(\alpha-1)\arctan(x)]} - 2B \left(A + \frac{1}{2\sqrt{2}} \right) e^{-[x+(\alpha-1)\arctan(x)]}, \quad (7.58)$$

and the $V_{mass}(x)$ term is same as given in Eq. (7.54). So that, $V(f(z))$ is written as

$$V(x) = A^2 + B^2 e^{-2[x+(\alpha-1)\arctan(x)]} - 2B \left(A + \frac{1}{2\sqrt{2}} \right) e^{-[x+(\alpha-1)\arctan(x)]} + \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 + (4 - 2\alpha)x^2 - \alpha]. \quad (7.59)$$

As it can be seen that the Eq. (7.59) is exactly in agreement with the Eq. (7.30) which is obtained in the framework of supersymmetry. The eigenvalues of Eq. (7.59) is expressed as in Eq. (7.57).

7.2.2 Conclusion

In this Section, a general mapping procedure has been described for the transformation of a differential equation with position-dependent mass to the Schrödinger equation with a constant mass under canonical transformations.

The Schrödinger equation in its new form has shown that the solution can exist if the original potential is solvable and shape-invariant one. Starting with Ref. [95] for the "point canonical transformation", we have shown that the method of [95] can exactly be applied to Schrödinger equation with x -dependent mass function. In fact, the results we obtained [97] are in agreement with the solutions of [39] for the potentials mentioned above.

It is also noted that the transformation technique gives excellent harmony with that of the SUSYQM studied in Section 7.1. In each method, the extra term emerging from the position-dependent mass function has the same form.

It is shown that the SUSYQM can be safely applied to the quantum systems with a position-dependent effective mass. As an illustrative example, the two well known potentials in physics, harmonic oscillator and Morse potentials, have been considered which provide benchmark calculations. We have dealt with a transformation method in which the dependent and independent variables are transformed into a new form to solve the Schrödinger equation with an effective mass. In the next Section, both methods the supersymmetric approach and mapping procedure have been combined to solve the Pösch-Teller type potential with a given mass definition and shown that the energy eigenvalues of the Schrödinger equation with a position-dependent mass is as in the form of constant mass scenario.

7.3 Application to Quantum Well Lasers

In this Section, we will apply the methods presented in the previous Sections to Pösch-Teller type potential which can safely be applied to quantum well lasers for analytical discussion.

The Pösch-Teller type potential has a great importance in Semiconductor Physics. Among various potential shapes, this potential with a constant mass condition has been attracted some attention recently since it has an interesting property that a suitable choice of its parameters delivers unity transmission of free states in quantum well profiles. Unfortunately, the idealized constant-mass Pösch-Teller Hamiltonian is not realizable in the common, in quantum wells, because the the effective mass therein necessarily varies together with position.

To observe the effects for a given x -dependent mass definition in the solutions, we first apply the "*f-transformation*" method to the potential undertaken and then supersymmetric techniques are manipulated to solve the new potential's energy spectra in the following Sections, which will establish a connection between the two methods experienced in the previous Sections.

7.3.1 A Physical Background on the Effective Mass

In the study of quantum systems, the Schrödinger equation with nonconstant mass provides an interesting and useful model for the description of many physical problems. The most extensive use of such an equation is in the physics of semiconductor nanostructures. This field has arisen due to the impressive development of sophisticated technologies of semiconductor growth, like molecular beam epitaxy, which made it possible to grow ultra-thin semiconductor structures, with very prominent quantum effects [15]. The motion of electrons (and also *holes*) in these structures may often be described by the envelope function effective-mass Schrödinger equation, where the material composition- (i.e., the position-) dependent effective mass of carriers replaces the constant particle mass in the conventional Schrödinger equation. The most popular of these structures is the semiconductor quantum wells, and the Schrödinger equation here is effectively one-dimensional. When the Schrödinger equation with effective mass is written, it is seen that the momentum and the mass operators is no longer commute in such a case. So, a question concerning the correct form of the kinetic energy operator of the generalized Hamiltonian has arisen. In fact, there are many possible permutations to represent the kinetic energy operators which frequently used, Hermitian ones, found in the literature are the BenDaniel and Duke Hamiltonian, the Bastard Hamiltonian, and the Zhu and Kroemer Hamiltonian, see Ref. [98] and references therein. A recent study about these Hamiltonians have been carried out [99] and seen that they can be treated as the supersymmetric partner of each other in the framework of supersymmetry.

To discuss the physical background of the mass functions used in the previous Sections, Eq. (7.14), we consider here a physical example to quantum well structures in which the carriers motion is position-dependent, $Al_zGa_{1-z}As$ where z is the composition fraction of Al , quantum well composition may be given. For $Al_zGa_{1-z}As$ with $0 \leq z \leq 0.45$, the band-gap energy, the effective mass of electron in the conduction band, and the effective masses of heavy and light holes in the valence band is taken the following relations in the Semiconductor Physics:

$$E_g(Al_zGa_{1-z}As) = \delta + vz , \quad (7.60)$$

$$m_e(Al_zGa_{1-z}As) = \kappa_e + \mu_e z , \quad (7.61)$$

$$m_{hh}(Al_zGa_{1-z}As) = \kappa_{hh} + \mu_{hh} z , \quad (7.62)$$

and

$$m_{lh}(Al_zGa_{1-z}As) = \kappa_{lh} + \mu_{lh} z , \quad (7.63)$$

where $\delta = 1.424$, $v = 1.247$, $\kappa_e = 0.076m_0$, $\mu_e = 0.083m_0$, $\kappa_{hh} = 0.35m_0$,

$\mu_{hh} = 0.5m_0$, $\kappa_{lh} = 0.8m_0$, and $\mu_{lh} = 0.10m_0$. The free electron mass is denoted by m_0 .

In the band gaps, the electron effective mass, m_e , is given in the form of x -dependent function

$$m_e(Al_xGa_{1-x}As) = \kappa_e + function(x) . \quad (7.64)$$

where the second term in Eq. (7.64) corresponds to Eq. (7.14). To get exactly solvable system, we have shifted the physical mass definition in Eq. (7.64) by κ_e through the earlier Sections. In quantum well studies, the conduction-band-offset ratio is denoted by Q_C and its relation to the conduction-band-offset energy is $V_C^0 = Q_C \Delta E_g$, where ΔE_g is the band-gap difference between $Al_xGa_{1-x}As$ and $GaAs$, and given as $\Delta E_g = vz$. V_C^0 is related to the potential depth parameter (A, B) used in the related equations in earlier Sections. The reader is referred to Ref. [98] for a detail discussion of this subject and the related equations for the light holes, heavy holes in the valence band.

In summary, the position-dependent mass definition should be given since the motion of carriers in the semiconductor structures have great importance to investigate the band structure and to calculate the "gain efficiency" in quantum well lasers. Such an expression leads us to define many possible Hamiltonians whether they are *Hermitian* or *not Hermitian*. In the concept of the following Subsection we have started with one of them called the BenDaniel and Duke Hamiltonian [100].

7.3.2 Pösch-Teller Type Potential

The Pösch-Teller type potential [39] is given in atomic units as

$$V_{PT}(x) = A^2 - A \left(A + \frac{\gamma}{\sqrt{2}} \right) \text{sech}^2(\gamma x) , \quad (7.65)$$

and the energy eigenvalues of this potential is given by

$$E_n = A^2 - A \left(A - \frac{n \gamma}{\sqrt{2}} \right)^2 , \quad (7.66)$$

for constant mass condition ($m_0 = 1$).

Now, using the same position-dependent mass definition given in Eq. (7.14), we find the transformed coordinate z as found in Eq. (7.52) which is

$$z = x + (\alpha - 1) \arctan(x) , \quad (7.67)$$

and inserting this into the Eq. (7.65) we find the $V_{SIP}(x)$, which is solvable and shape-invariant, in its new form as

$$V_{SIP}(x) = A^2 - A \left(A + \frac{\gamma}{\sqrt{2}} \right) \text{sech}^2\{\gamma [x + (\alpha - 1) \arctan(x)]\} , \quad (7.68)$$

and the extra term emerging from the transformation of Schrödinger equation with effective mass is same as given in Eq. (7.54) in Section 7.2. It is

$$V_{mass}(x) = \frac{(\alpha - 1)}{2(\alpha + x^2)^4} [-3x^4 - (4 - 2\alpha)x^2 + \alpha]. \quad (7.69)$$

So, the new potential term $V(x)$ after these transformations is written as

$$V(x) = A^2 - A \left(A + \frac{\gamma}{\sqrt{2}} \right) \operatorname{sech}^2\{\gamma [x + (\alpha - 1) \arctan(x)]\} + \frac{(\alpha - 1)}{2(\alpha + x^2)^4} [-3x^4 - (4 - 2\alpha)x^2 + \alpha]. \quad (7.70)$$

Note that again, the potential $V(x)$ directly reduces to the usual Pösch-Teller type potential when $\alpha = 1$.

After that point, we will turn back to SUSYQM. As given in Section 7.1, supersymmetric partner potentials can be determined if the superpotential term, $W(x; a_1)$, is known. The superpotential term can be found out by solving the differential equation (called as "*Riccati Equation*") given in Eq. (7.4) for the first partner of supersymmetric potential. So that, the transformed potential term in equation (7.70) corresponds to " $V_1(x)$ " from which the " $W(x; a_1)$ " can be determined. The differential equation which should be solved for $W(x; a_1)$ is written as

$$V(x) = V_1(x) = W(x; a_1)^2 - \left(\frac{W(x; a_1)}{\sqrt{2} m(x)} \right)', \quad (7.71)$$

where *prime* indicates the derivative with respect to x . Since we know the special solution of that Riccati Equation for a constant mass condition for $W(x)$, which is

$$W(x) = A \tanh(\gamma x). \quad (7.72)$$

Using Eq. (7.72) and replacing x with $x + (\alpha - 1) \arctan(x)$, we can find the general solution of Eq. (7.71) in transformed form.

After the solution of Eq. (7.71) using the same definition of (7.14), one finds that the superpotential should be in such form

$$W(x; A) = A \tanh\{\gamma [x + (\alpha - 1) \arctan(x)]\} + \frac{x(\alpha - 1)}{\sqrt{2}(x^2 + \alpha)^2}. \quad (7.73)$$

Then the first partner potential $V_1(x; A)$ by using Eq. (7.71) for $W(x; A)$, given in Eq. (7.73), is found as

$$V_1(x; A) = A^2 - A \left(A + \frac{\gamma}{\sqrt{2}} \right) \operatorname{sech}^2\{\gamma [x + (\alpha - 1) \arctan(x)]\} + \frac{(\alpha - 1)}{2(\alpha + x^2)^4} [3x^4 + (4 - 2\alpha)x^2 - \alpha], \quad (7.74)$$

and the second partner supersymmetric potential, $V_2(x; A)$, is found as

$$V_2(x; A) = A^2 + A \left(A - \frac{\gamma}{\sqrt{2}} \right) \operatorname{sech}^2 \{ \gamma [x + (\alpha - 1) \arctan(x)] \} \\ + \frac{(\alpha - 1)}{2(\alpha + x^2)^4} [3x^4 + (4 - 2\alpha)x^2 - \alpha] . \quad (7.75)$$

Since the usual SUSYQM connects a relation between partner potentials by shifting parameter, we can find that parameter using the Eq. (7.7) given in Section 7.1. It is seen that in Eq. (7.74) if we write $V_1(x; A \rightarrow A - \frac{\gamma}{\sqrt{2}})$ and proceed the Eq. (7.7), one finds that

$$V_2(x; A) - V_1(x; A - \frac{\gamma}{\sqrt{2}}) = A^2 - \left(A - \frac{\gamma}{\sqrt{2}} \right)^2 , \quad (7.76)$$

and it is seen that

$$a_1 = A \quad , \quad \text{and} \quad a_2 = A - \frac{\gamma}{\sqrt{2}} . \quad (7.77)$$

Since

$$E_n^+ = \sum_{k=1}^{n-1} R_k = \sum_{k=1}^n (a_k^2 - a_{k+1}^2) \quad (7.78)$$

then it is found that the energy spectra of the potential $V_1(x; A)$ which is the supersymmetric partner of $V(x)$ is

$$E_n^+ = A^2 - \left(A - \frac{n\gamma}{\sqrt{2}} \right)^2 \quad (7.79)$$

that is exactly in agree with the result of Ref. [39]. Since we are looking for the energy states of the Pösch-Teller potential, the exact energy levels of this potential is found from Eq. (7.79) by subtracting the extra term A^2 . So, the energy states is given as

$$E_n = - \left(A - \frac{n\gamma}{\sqrt{2}} \right)^2 . \quad (7.80)$$

Because the mass and the momentum operators are not commutable, there are different possible permutations to represent the kinetic energy operator in the Schrödinger equation as we mentioned above. In the following Sections, we have introduced a generalized treatment procedure for effective-mass Hamiltonians proposed in the literature and involved applications for two different but physically meaningful mass considerations.

7.3.3 Effective-Mass Hamiltonians

The single-band effective-mass approach to a quantum-well problem requires that the envelope function ψ satisfy the effective-mass equation

$$\hat{H}\psi = E\psi \quad (7.81)$$

where E is the energy eigenvalue and \hat{H} is the Hamiltonian operator consisting of kinetic energy operator (\hat{T}) and the potential energy operator (\hat{V}),

$$\hat{H} = \hat{T}(\hat{x}) + \hat{V}(\hat{x}) \quad (7.82)$$

Due to the compositional variation in a quantum well as a function of location, the kinetic energy and the potential energy are expressed as position-dependent operators in Eq. (7.82). The kinetic energy operator can be considered to be composed of four elements: $1/\sqrt{m(\hat{x})}$, $1/\sqrt{m(\hat{x})}$, \hat{p} , and \hat{p} , where \hat{x} and \hat{p} are position and momentum operators, respectively. Because the reason given above, there are different possible expressions for the kinetic energy operator, and also for the Hamiltonian operator. All of these single-band effective-mass Hamiltonians are special cases of a general form of the Hamiltonian introduced by von Roos [101],

$$H_{vR} = \frac{1}{4} \left[m^\alpha(\hat{x}) \hat{p} m^\beta(\hat{x}) \hat{p} m^\gamma(\hat{x}) + m^\gamma(\hat{x}) \hat{p} m^\beta(\hat{x}) \hat{p} m^\alpha(\hat{x}) \right] + V(\hat{x}), \quad (7.83)$$

where $\alpha + \beta + \gamma = -1$.

By the correspondence in wave mechanics $\hat{p} \rightarrow -i\hbar \frac{d}{dx}$ and $\hat{x} \rightarrow x$, the effective mass equation, Eq. (7.81), together with any possible Hamiltonian proposed in the literature as the special cases of Eq. (7.83) can be written in a differential form,

$$-\frac{\hbar^2}{2} \frac{d}{dx} \left[\frac{1}{m(x)} \frac{d\psi(x)}{dx} \right] + V^{eff}(x)\psi(x) = E\psi(x), \quad (7.84)$$

where $V^{eff}(x)$ is termed the effective potential energy whose algebraic form depends on the Hamiltonian employed (see Table 7.1),

$$\begin{aligned} V^{eff}(x) &= V(x) + U_{\alpha\gamma}(x) \\ &= V(x) - \frac{\hbar^2}{4m^3(x)} \left[(\alpha + \gamma) m(x)m''(x) - 2(\alpha\gamma + \alpha + \gamma) m'^2(x) \right], \end{aligned} \quad (7.85)$$

in which the first and second derivatives of $m(x)$ with respect to x are denoted by m' and m'' , respectively. The effective potential is the sum of the real potential profile $V(x)$ and the modification $U_{\alpha\gamma}(x)$ emerged from the location dependence of the effective mass. A different Hamiltonian leads to a different modification term.

Table 7.1: Single-band effective mass Hamiltonians, Eq. (7.83).

Hamiltonian	α	β	γ
Ref. [100]	0	-1	0
Ref. [102, 103]	-1	0	0
Ref. [104]	-1/2	0	-1/2
Ref. [105]	0	-1/2	-1/2

7.3.4 Applications

A square quantum well subject no external field is usually modeled by a step profile which discontinues at the heterojunctions. The commonly used interface conditions at the heterojunctions are: (i) the continuity of the envelope function and its first derivative, or (ii) the continuity of the envelope function and its first derivative divided by the effective mass. Within each flat region of the square quantum well, the mass is a constant, and the effective potentials, Eq. (7.85), are identical because the derivatives of the mass with respect to the position vanish. Thus envelope functions within each flat region are independent of the Hamiltonian used for the analysis. If the same interface conditions at the discontinue points between two adjacent regions are essentially imposed for the distinct effective Hamiltonians, the eigensolutions will be exactly identical. As a result, the effective-mass Hamiltonians in (7.85) are expected to produce the same transition energy, namely, the band-offset ratio will be independent of the Hamiltonian if the heterojunction is modeled by a step function with essentially imposed interface conditions. This point will be discussed in the next Section within the frame of our results.

However, the discontinuity of the square quantum-well model implies an infinite external electric field at the heterojunctions, and this is not physically possible. In reality, the potential changes over a few monolayers for a perfect microscopic interface. In this Section, the square quantum well is modelled by more realistic smoothed profiles, which leads to different effective potential descriptions. This modified realistic potential profiles remove the discontinuity of the sharp square quantum well. Considering the realistic quantum well applications in the works [105, 106], we test our models introduced in the previous

Sections employing the two different but physically meaningful effective mass variations discussed in the following.

7.3.4.1 For $m(x) = m_0 \exp(\lambda x)$

To demonstrate the simplicity of the models used we consider here a particular case, as a first illustrative example, for which one have exact solutions. That is a particle with exponentially increasing or decaying, $|\lambda| \propto 1/L$ with L being the quantum-well width, in the presence of a potential with similar behaviour that will be defined for each Hamiltonian in this Section.

Starting with Eqs. (7.46-7.48), within the framework of the coordinate transformation technique and taking the harmonic oscillator potential as a solvable potential, $V_{SIP}(x) = Bx^2$, and bearing in mind that $z = (2/\lambda) \exp(\lambda x/2)$ from Eq. (7.41), we obtain a variety of exactly solvable effective potential descriptions appearing in the original effective mass equation, Eq. (7.84). These are, corresponding to distinct effective mass Hamiltonians presented by Eqs. (7.83) and (7.85),

$$V_{BDD}^{eff}(x) = V_0 \exp(\lambda x) - \frac{3\hbar^2 \lambda^2}{32m_0} \exp(-\lambda x) \quad , \quad \alpha = \gamma = 0 \quad , \quad \beta = -1 \quad , \quad (7.86)$$

for the BenDaniel-Duke Hamiltonian [100],

$$V_{G-W}^{eff}(x) = V_0 \exp(\lambda x) + \frac{5\hbar^2 \lambda^2}{32m_0} \exp(-\lambda x) \quad , \quad \beta = \gamma = 0 \quad , \quad \alpha = -1 \quad , \quad (7.87)$$

for the Gora and Williams (or the Bastard) Hamiltonian [102, 103], and

$$V_{Z-K}^{eff}(x) = V_0 \exp(\lambda x) + \frac{\hbar^2 \lambda^2}{32m_0} \exp(-\lambda x) \quad , \quad \alpha = \gamma = -\frac{1}{2} \quad , \quad \beta = 0 \quad , \quad (7.88)$$

for the Zhu and Kroemer Hamiltonian [104]. We also obtain exactly the same result as in (7.88) for the Li-Kuhn Hamiltonian [105] ($\beta = \gamma = -1/2$, $\alpha = 0$). In the above equations, $V_0 = 4B/\lambda^2 = \hbar^2 \lambda^2 / 32m_0$. From Eqs. (7.84) and (7.46), it is clear that though the appearance and behaviour of the potentials in (7.86)-(7.88) are dissimilar, they have identical spectra, $E_n = (n+1/2)\hbar\omega$, where $\omega = \sqrt{2B/m_0}$. This result supports the similar discussion presented in Sec. 3 of a recently published paper [107]. Moreover, from Eqs. (7.32) and (7.37), these Hamiltonians have the same wavefunctions, $\psi_n(x) = N_n [\exp(\lambda x)]^{1/4} \psi_n(z)$ where $\psi_n(z) = \left\{ \frac{1}{2^n n!} \sqrt{\frac{\eta}{\pi}} \right\}^{1/2} H_n(\sqrt{\eta} z) \exp\left(-\frac{\eta z^2}{2}\right)$ with $\eta = \sqrt{2Bm_0}/\hbar$. As a result, the distinct effective Hamiltonians considered here are not only isospectral but also describe identically equivalent systems as far as exact solvability is concerned. This significant result is also confirmed below with use of the supersymmetric approach described in the Section 7.1 as an alternative treatment.

For the supersymmetric considerations, in the light of Eq. (35) of [94] we propose a superpotential

$$W(x) = \frac{\sqrt{2m_0}\delta}{\hbar\lambda} \exp(\lambda x/2) - \frac{\hbar\lambda}{4\sqrt{2m_0}} \exp(-\lambda x/2), \quad (7.89)$$

where $\delta = \hbar\omega$, which yields the supersymmetric partner potentials in Eqs. (7.4) and (7.6) in the form

$$\begin{aligned} V_1(x) &= V_0 \exp(\lambda x) - \frac{3\hbar^2\lambda^2}{32m_0} \exp(-\lambda x) - \frac{\delta}{2}, \\ V_2(x) &= V_0 \exp(\lambda x) - \frac{3\hbar^2\lambda^2}{32m_0} \exp(-\lambda x) + \frac{\delta}{2}, \end{aligned} \quad (7.90)$$

which is the simplest case of the shape invariance integrability condition given by (7.7) due to the partner potentials in (7.90) differing only by a uniform energy shift by δ . Note that the first partner has the same shape as in (7.86) corresponding to the BenDaniel-Duke effective potential. Considering the shifting term $\delta/2$, together with Eq. (7.7), one can easily find the corresponding energy spectrum, $E_n = (n+1/2)\hbar\omega$, which overlaps with the one found through the transformation technique. For the other exactly solvable shape invariant effective potentials we use a simple expression,

$$V_{SIP}^{eff}(x) = V_{BDD}^{eff}(x) - U_{\alpha\gamma}(x) = V_1(x) - U_{\alpha\gamma}(x), \quad (7.91)$$

in which $U_{\alpha\gamma}(x)$ is the modification term in Eq. (7.85),

$$U_{\alpha\gamma}^{G-W}(x) = -\frac{\hbar^2\lambda^2}{4m_0} \exp(-\lambda x), \quad U_{\alpha\gamma}^{Z-K}(x) = U_{\alpha\gamma}^{L-K} = -\frac{\hbar^2\lambda^2}{8m_0} \exp(-\lambda x). \quad (7.92)$$

The substitution of $V_1(x)$ in Eq. (7.90) into Eq. (7.91) leads us to arrive at Eqs. (7.87) and (7.88), which confirms the reliability of the coordinate transformation technique developed in the previous Sections. Once more it is clear that the potentials in Eqs. (7.86)-(7.88) will have the same energy spectra and wavefunctions as a unique superpotential is used, Eq. (7.89), for the generation of these analytically solvable potentials within the frame of supersymmetric quantum theory. For the relation between the superpotential and wavefunction in case of the supersymmetric considerations the reader is referred to [94, 108].

7.3.4.2 For $m(x) = m_0 \left(\frac{a+q^2}{1+q^2}\right)^2$

This physically convenient mass variation, in which a is a positive constant and $q(= \bar{\lambda}x)$ involves the variable with a positive width parameter $\bar{\lambda} \propto 1/L$, considered here to convince the reader for that the models introduced here works for all smoothly varying masses, unlike the recent models [16, 109]. Among

various potential shapes (quantum well profiles), Pöschl-Teller potential with a constant electron mass [110, 111] has great importance. Unfortunately, this idealized potential is not realizable in the common, graded ternary alloy based quantum wells, because of the effective mass therein necessarily varies, together with the potential. In this respect, the result shown in this Section and its related discussion given in the next Section would be helpful in designing realistic ternary alloy based structures with properties equivalent to those of idealized Pöschl-Teller potential.

From Eq. (7.41), the relation between the transformed coordinate and the old one is $z = x + (1/\bar{\lambda})(a-1)\tan^{-1}q$, and considering the Pöschl-Teller potential as one possible choice from analytically solvable potential family and working within the framework of the coordinate transformation method, one arrives at a class of exactly solvable potentials for the use in Eq. (7.84) belonging to different effective mass Hamiltonians,

$$\begin{aligned} V_{BDD}^{eff}(x) &= V_{PT}(z) + \frac{(a-1)[3q^4 + q^2(4-2a) - a]\hbar^2\bar{\lambda}^2}{2m_0(q^2+a)^4}, \\ V_{Z-K}^{eff}(x) &= V_{PT}(z) - \frac{(a-1)[3q^4 + 2q^2 - a]\hbar^2\bar{\lambda}^2}{2m_0(q^2+a)^4}, \\ V_{L-K}^{eff}(x) &= V_{PT}(z) + \frac{(a-1)^2q^2\hbar^2\bar{\lambda}^2}{2m_0(q^2+a)^4}, \\ V_{G-W}^{eff}(x) &= V_{PT}(z) - \frac{(a-1)[3q^4 + q^2(6-4a) - a]\hbar^2\bar{\lambda}^2}{2m_0(q^2+a)^4}, \end{aligned} \quad (7.93)$$

where $V_{PT}(z) = -A\left(A + \frac{\lambda\hbar}{\sqrt{2m_0}}\right) \text{sech}^2\left\{\bar{\lambda}\left[x + \frac{(a-1)\tan^{-1}q}{\bar{\lambda}}\right]\right\}$. Note that for $a \rightarrow 1$, $m(x) \rightarrow m_0$, all the above effective potentials reduce to the conventional Pöschl-Teller potential, $V^{eff}(x) \rightarrow V_{PT}(x)$, due to $z \rightarrow x$. From Eqs. (7.84), (7.37) and (7.46), the bound state energy spectra and wavefunctions corresponding to the potentials in (7.93) are $E_n = -\left(A - \frac{n\lambda\hbar}{\sqrt{2m_0}}\right)$ and $\psi_n(x) = N_n\sqrt{\frac{a+q^2}{1+q^2}}\psi(z)$ with $\psi(z)$ being the well described wavefunction [25] for the solution of the usual Schrödinger equation with a constant-mass Pöschl-Teller potential.

All these results are fully confirmed with use of the supersymmetric expressions presented in the Section 7.3.2 by adopting an ansatz for the superpotential,

$$W(x) = A \tanh\left\{\bar{\lambda}\left[x + \frac{(a-1)\tanh^{-1}q}{\bar{\lambda}}\right]\right\} + \frac{(a-1)q\hbar\bar{\lambda}}{\sqrt{2m_0}(a+q^2)^2}. \quad (7.94)$$

7.3.5 Discussion and Conclusion

The applications given in the previous Sections make clear the band-offset ratio dependence on the effective mass Hamiltonians, which is significant for

quantum well applications. The conduction-band-offset ratio, which is the ratio of the conduction-band offset to the total band gap of the heterojunction, has been investigated in various quantum wells because of its fundamental importance and application. The ratio has been measured by spectroscopic and electrical methods. From Duggan's and Kroemer's review articles [112] about the experimental and theoretical works, it can be seen that spectroscopic techniques are preferred over electrical ones in exploring the band-offset ratio via a quantum well, and that researchers using spectroscopy usually try to match their data with the theoretical results to determine the band offset. To demonstrate simply the band-offset ratio variation due to the choice of the Hamiltonian, we will focus here on transition energies between levels in conduction and valence bands. Following the works [105, 113] and considering only single-band effective-mass equations for the electron and the hole, Eq. (7.84), one finds the transition energy in the form of $E_T = E_e + E_h + E_G$ where E_e, E_h are the eigenenergies in the conduction and valence bands, respectively, and E_G is the band-gap energy. From the previous Sections, it is obvious that the effective Hamiltonians undertaken will yield the same transition energies between identical transition levels due to the identical values for E_e and E_h . Hence, the band-offset ratio for the BenDaniel-Duke Hamiltonian and for the others can be found by solving $E_T^{others}(Q^{others}) = E_T^{BDD}(Q^{BDD})$, with Q being the band-offset ratio. Recall the relation between the conduction band-offset ratio and the conduction-band-offset energy, $V_C^0 = Q\Delta E_g$, where ΔE_g denotes the band-gap difference between binary and ternary materials. Having in mind that the band-offset ratio of a quantum well determines the barrier height of the conduction band and valence bands, V_C^0 corresponds to the depth of the effective potentials discussed through the Chapter. This leads to the realization of the band-offset ratio dependence on the effective-mass Hamiltonian due to the underlying differences between the strengths of the potentials obtained. As a result of this, in the interpretation of a given spectrum, the Hamiltonian employed in the analysis cannot be regarded independently of the band-offset ratio utilized, unlike the case of a simple square quantum well consideration. Therefore, an attempt to determine the band-offset ratio from experimental data by matching calculated transition energies with spectral peaks would involve large inaccuracy.

There is another interesting point behind the present results. A systematic procedure based on the inverse spectral theory and supersymmetric transformations has been recently proposed [111] for optimized design of semiconductor quantum well structures via tailoring the quantum well potential, which enables shifting bound states in a quantum well and makes the search for the best desired energy spectrum and potential shape. By varying the free parameters appearing

in the procedure one can then design a convenient optimized structure. However, in these notable works the idealized constant-mass Pöschl-Teller potential, which allows one to set analytically the spacing between states, considered since a direct implementation of a more realistic position-dependent effective mass related to the position-dependent potential in their theoretical considerations is not trivial. The optimization of continuously graded structures thus require more sophisticated techniques. In this respect, we believe that the applications given in the previous Sections, in particular the one involved the Pöschl-Teller potential, give a lot of material for experimenting in the optimized quantum well laser design. For instance, though we have explicitly shown that the energy spectra of the realistic ternary alloy based structures with a carrier having a spatially varying mass are equivalent to those of the constant-mass Pöschl-Teller Potential, the maximization of the gain may be accomplished via changing the quantum well profile which is in turn changes the wave functions. Hence the consideration of our results within supersymmetric transformations, as an alternative to the recent procedure proposed, in particular the significant difference between the wavefunctions by ν shown in (7.32), relating the solutions of the Schrödinger equations with constant and location dependent masses, and use of the more realistic effective potentials in (7.93) in tailoring process instead of the standard Pöschl-Teller potential, should reproduce more reliable results in order to help for the best design of such structures.

Moreover, Plastino and his co-workers [114] recently studied some simple one-dimensional quantum mechanical systems characterized by a piecewise flat potential and mass to illustrate the influence of a non-constant mass on the density of the bound state energy levels. With the consideration of a finite potential well they showed that the number of bound states is less than those of the constant mass situation when the effective mass inside the well is lower than that of outside (m_0), and the opposite behaviour occurs when the effective mass inside the well is larger than the mass outside. However, our applications in Section 7.3.4 do not confirm their work. This contradiction may raise a further discussion on the reliability of the present results, which can be clarified as follows. In the two different variable mass definitions, $m(x)$, for the quantum well used in the previous Section, $m(x) > m_0$ for the case $\lambda > 0$ and $a > 1$ while $m(x) < m_0$ in case $\lambda < 0$ and $a < 1$. The consideration of $m(x) > m_0$ case leads to single potential wells, sharper than that of the standard potential corresponding to constant-mass potentials, whereas $m(x) < m_0$ case give rise to bistable double well potentials, like the related illustrations in [94]. In spite of the different aspects exhibited by the effective potentials defined in Section 7.3.4, we have clearly shown that they share the same energy spectra regardless of λ and a

values, together with the constant-mass potentials ($\lambda = 0, a = 1$). Consequently, one expects that the density of bound states for the systems we interest should not depend on the variation of a carrier mass.

As a conclusion, we have discussed the problem of solvability and ordering ambiguity in quantum mechanics as the form of the effective-mass Hamiltonian has been a controversial subject due to the location dependence of the effective mass. It was shown through particular examples that the exact solvability depends not only on the form of the potential, but also on the spatial dependence on the mass. Within the framework of the two different theoretical treatments, the effective-mass Schrödinger equation has been transformed to a constant-mass Schrödinger equation and we have clarified that the Schrödinger equations with different masses and potentials can be exactly isospectral. We have also shown that though the potential energy of the BenDaniel-Duke Hamiltonian differs from the effective potentials of other Hamiltonians proposed in the literature by a term caused by the mass dependence on location, the exact analytical solutions to the effective-mass equations do not change with the Hamiltonian. As far as we know, this feature was not perceived until now. The discussion given behind the results obtained may be of interest, e.g., in the design and optimization of semiconductor quantum wells. In addition to their practical applications, we believe that the study of quantum mechanical systems with a position-dependent mass within the framework of the present models will raise many interesting conceptual problems of fundamental nature. In particular, the methods used in this study may be extended to find applications in also the study of quasi- and conditionally-exactly solvable systems with non-constant masses.

CHAPTER 8

BOUND STATES IN CONTINUUM GENERATED BY SUPERSYMMETRIC QUANTUM MECHANICS

The solutions of the time-independent Schrödinger equation for a single particle are often discussed by considering piecewise constant potentials. The simplest such example for a local scalar potential is the square potential well. As every potential function of bounded support, the solution in the external region is not square integrable if the particle energy is positive. From this, it is deduced that these potentials possess only (discrete) negative bound states. On the other hand, there is a continuum of positive energy states that are not square integrable. In 1929, von Neumann and Wigner [115] realized that it was possible to construct potentials which have quantum mechanical bound states embedded in the classical energy continuum (BIC's). This is implicitly extended to local potentials in general in the literature [116, 117]. So, it is known that some potentials exist that tend to zero at infinity and yet possess square-integrable eigenfunctions with positive energy.

Further developments by many authors [118, 119, 120] have produced more examples and a better understanding of the kind of potential that can have such bound states. Recently, extensive work has been devoted to generating isospectral potentials by the method of SUSYQM [4, 121]. Starting from the Schrödinger equation for a potential, whose ground-state wave function is known, this method permits one to generate families of new potentials, which may look quite different from the original one, but have exactly the same spectrum. To generate new potentials with bound states in the continuum, SUSYQM formalism is extended for obtaining isospectral potentials and applied to potentials with a continuum of scattering states. It is seen that, while the wave functions in the continuum of the original potential are non-normalizable, the wave functions generated by SUSYQM are normalizable and thus representing a bound state. One can construct one-parameter or two-parameter families of supersymmetric partner potentials with one or two bound states in the continuum.

8.1 Theoretical considerations

The one-dimensional time-independent Schrödinger equation associated with a particle (with a constant-mass) is given by

$$-\frac{d^2}{dr^2}\psi_0(r) + V(r)\psi_0(r) = E\psi_0(r) , \quad (8.1)$$

where $V(r)$ is a spherically symmetric potential. In the SUSYQM, this equation is expressed in terms of the operators (as we did before) defined as

$$H_0\psi_0(r) = A_0^+ A_0^- \psi_0(r) = E\psi_0(r) , \quad (8.2)$$

where

$$A_0^+ = -\frac{d}{dr} + W(r) , \quad A_0^- = +\frac{d}{dr} + W(r) , \quad (8.3)$$

in which $W(r)$ is defined by *superpotential* and given as $W(r) = -\frac{d}{dr} \ln \psi_0(r)$. Then, the supersymmetric partner Hamiltonian of Eq. (8.2) can be given by

$$H_1 = A_0^- A_0^+ = -\frac{d^2}{dr^2} + V_1(r) \quad (8.4)$$

where the supersymmetric partner potential $V_1(r)$ is

$$V_1(r) = W^2(r) + W'(r) = V_0(r) - 2\frac{d^2}{dr^2} \ln \psi_0(r) , \quad (8.5)$$

where prime denotes differentiation with respect to r . If the potential $V_0(r)$ has eigenfunctions $\psi_0(r)$ at energies E_n , then the supersymmetric partner potential $V_1(r)$ has the same energy eigenvalues as $V_0(r)$ with eigenfunctions $A_0^- \psi_n(r)$, except that there is no ground state at $E = 0$ since $A_0^- \psi_0(r) = 0$. This is the standard procedure for deleting the ground state and obtaining $V_1(r)$. The procedure can be repeated 'upward', producing potentials $V_2(r), V_3(r), \dots$, with ground states $\psi_1(r), \psi_2(r), \dots$, at energies E_2, E_2, \dots until the top potential $V_n(r)$ holds no bound state. Although the potential V_1 does not have an eigenenergy E_0 , the function $1/\psi_0(r)$ satisfies the Schrödinger equation with potential $V_1(r)$ and energy E_0 . The other linearly independent solution is $\int_{-\infty}^r 1/\psi_0^2(z) dz$. So that the most general solution of the Schrödinger equation for the potential $V_1(r)$ at energy E_0 is

$$\Phi_0(r; \lambda_0) = \frac{I_0(r) + \lambda_0}{\psi_0(r)} \quad (8.6)$$

where

$$I_0(r) = \int_{-\infty}^r \psi_0^2(z) dz . \quad (8.7)$$

Starting with a potential $V_1(r)$, we can again use the standard SUSYQM procedure to add the state E_0 by using the general solution $\Phi_0(r; \lambda_0)$ and get

$$\hat{V}_0(r; \lambda_0) = V_1(r) - 2\frac{d^2}{dr^2} \ln \Phi_0(r; \lambda_0) , \quad (8.8)$$

since the function $1/\Phi_0(r; \lambda_0)$ can now be normalizable ground state wave function of $\hat{V}_0(r; \lambda_0)$ by providing that λ_0 does not lie in the interval $-1 \leq \lambda_0 \leq 0$. Therefore, we can replace $V_1(r)$ in the Eq. (8.8) and find a one-parameter family of potential $\hat{V}_0(r; \lambda_0)$ isospectral to $V_0(r)$ for $\lambda_0 > 0$ or $\lambda_0 < -1$:

$$\begin{aligned}\hat{V}_0(r; \lambda_0) &= V_0(r) - 2 \frac{d^2}{dr^2} \ln(\psi_0(r)\Phi_0(r; \lambda_0)) \\ &= V_0(r) - 2 \frac{d^2}{dr^2} \ln(I_0(r) + \lambda_0) .\end{aligned}\quad (8.9)$$

Note that this expression contains the potential $V_0(r)$ and as $\lambda_0 \rightarrow \pm\infty$, $\hat{V}_0(r; \lambda_0) \rightarrow V_0(r)$. The normalizable wave function corresponding to $\hat{V}_0(r; \lambda_0)$ is found as

$$\hat{\psi}_0(r; \lambda_0) = 1/\Phi_0(r; \lambda_0) = \frac{\psi_0(r)}{I_0(r) + \lambda_0} .\quad (8.10)$$

In order to produce a two-parameter family of isospectral potentials, we go from $V_0(r)$ to $V_1(r)$ and from $V_1(r)$ to $V_2(r)$ by deleting two lowest states of $V_0(r)$ and then we re-add these two states E_0 and E_1 by the procedure as described above.

The most general solutions of the Schrödinger equation for the potential $V_2(r)$ are given by $\Phi_1(r; \lambda_1) = (I_1(r) + \lambda_1)/\psi_1(r)$ at E_1 and $A_1^+ \Phi_0(r; \lambda_0)$ at E_0 . Then, as we did before, we find an isospectral one-parameter family of $V_1(r)$

$$\hat{V}_1(r; \lambda_1) = V_1(r) - 2 \frac{d^2}{dr^2} \ln(I_1(r) + \lambda_1) ,\quad (8.11)$$

where

$$I_1(r) = \int_{-\infty}^r \psi_1^2(z) dz .\quad (8.12)$$

and the solutions of the Schrödinger equation for potentials $V_2(r)$ and $\hat{V}_1(r; \lambda_1)$ are related by a new SUSY operator

$$\hat{A}_1^-(\lambda_1) = \frac{d}{dr} + \frac{d}{dr} \ln(\Phi_1(r; \lambda_1)) ,\quad (8.13)$$

therefore, the solution $\Phi_0(r; \lambda_0, \lambda_1)$ at E_0 for $\hat{V}_1(r; \lambda_1)$ is given as

$$\Phi_0(r; \lambda_0, \lambda_1) = \hat{A}_1^-(\lambda_1) A_1^+ \Phi_0(r; \lambda_0) .\quad (8.14)$$

Since $1/\Phi_0(r; \lambda_0, \lambda_1)$ is normalizable function, it is the ground state at E_0 of a new potential, which results in a two-parameter family of isospectral systems $\hat{V}_0(r; \lambda_0, \lambda_1)$:

$$\begin{aligned}\hat{V}_0(r; \lambda_0, \lambda_1) &= V_0(r) - 2 \frac{d^2}{dr^2} \ln[\psi_0(r)\psi_1(r)\Phi_1(r; \lambda_1)\Phi_0(r; \lambda_0, \lambda_1)] \\ &= V_0(r) - 2 \frac{d^2}{dr^2} \ln[\psi_0(r) (I_1(r) + \lambda_1) \Phi_0(r; \lambda_0, \lambda_1)] .\end{aligned}\quad (8.15)$$

for $\lambda_i > 0$ or $\lambda_i < -1$, $i = 0, 1, \dots, n$.

In all previous works, $\psi_0(r)$ was taken to be the nodeless, normalizable ground state wave function of the starting potential $V(r)$. However, the equations above can be generalized to the cases where $\psi_0(r)$ is any solution of Eq. (8.1) with arbitrary energy E . So that it can be taken to be a scattering solution at a positive energy $E = k^2$ of the potential $V(r)$ which vanishes at $r = \infty$. Taking $\psi_0(r = 0) = 0$ satisfies one of the required boundary conditions, but clearly $\psi_0(r)$ oscillates as $r \rightarrow \infty$ and has an amplitude which does not decrease. Consequently, the integral $I_0(r)$ in Eq. (8.7) grows like r at large r and $\hat{\psi}_0(r; \lambda_0)$ is now square integrable for different λ_i values, while the original wave function $\psi_0(r)$ was not square integrable. It is seen that all the potentials $\hat{V}(r, \lambda)$ have a BIC with energy E_0 . Note also Eq. (8.10) that $\hat{\psi}_0(r; \lambda_0)$ has the same zeros as the original $\psi_0(r)$. At these zero points, partner potentials are equal.

8.2 The One-Parameter Family of Potentials

The development of the n-parameter family of isospectral potentials in SUSYQM to generate potentials with n-bound state in the continuum has been presented in the previous section. Now, it is given to explicitly illustrate how one can apply the above procedure to obtain a potential possessing one bound state in the continuum.

8.2.1 The Free Particle on the half-line

The free particle, $V(r) \equiv 0$, solution for the half-line ($0 \leq r < \infty$) is considered as an example here. For the potential given, the wave function which vanishes at $r = 0$ is defined as

$$\psi_0(r) = \sin(kr) , \quad (8.16)$$

for the corresponding energy $E_0 = k^2 > 0$. So that, the integral term in Eq. (8.7) becomes as

$$I_0(r) = \frac{2kr - \sin(2kr)}{4k} . \quad (8.17)$$

It is observed that $I_0(r)$ does not vanish as $r \rightarrow \infty$, and this causes the partner wave function to be square integrable. Then, using Eq. (8.10) we get

$$\hat{\psi}_0(r, \lambda) = \frac{4k \sin(kr)}{2kr - \sin(2kr) + 4k\lambda} . \quad (8.18)$$

The family member of potential $\hat{V}(r, \lambda)$, from Eq. (8.8), is given by

$$\hat{V}(r, \lambda) = \frac{32 k^2 \sin^4(kr)}{(2kr - \sin(2kr) + 4k\lambda)^2} - \frac{8 k^2 \sin(2kr)}{2kr - \sin(2kr) + 4k\lambda} , \quad (8.19)$$

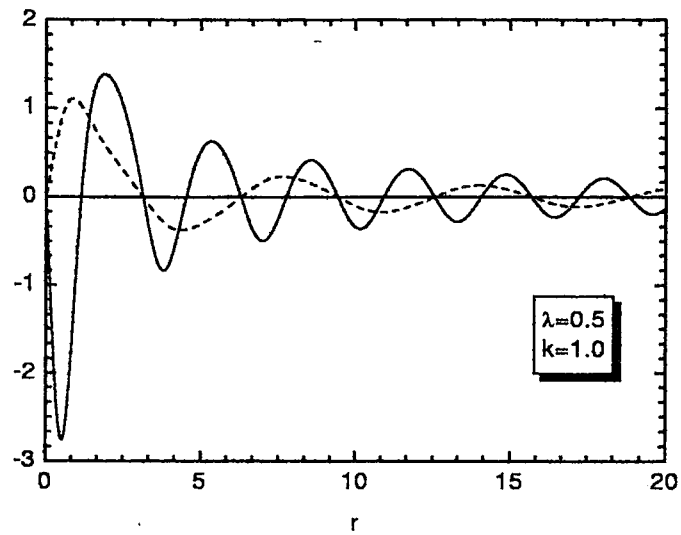


Figure 8.1: Example of potential $\hat{V}(r)$ and the associated BIC wave function $\hat{\psi}_0(r)$ (dashed) in the one-parameter family for $k = 1.0$ and $\lambda = 0.5$.

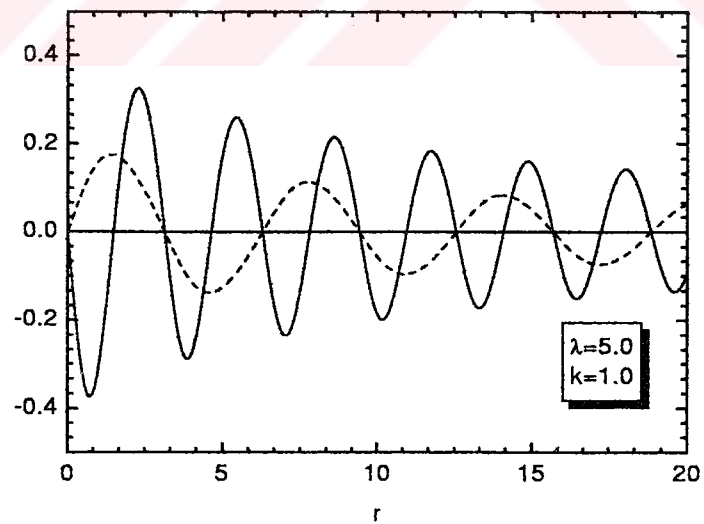


Figure 8.2: Example of potential $\hat{V}(r)$ and the associated BIC wave function $\hat{\psi}_0(r)$ (dashed) in the one-parameter family for $k = 1.0$ and $\lambda = 5.0$.

For the special values of the parameters k and λ , the potential \hat{V} and its wave functions are shown in Figs. 8.1 and 8.2.

The original potential has become oscillatory potential in the asymptotic region and the new wave function at $E_0 = k^2$ also oscillates and it has become normalizable. As $\psi_0(r)$ appears in the numerator of $\hat{V}(r, \lambda)$, every node of $\hat{\psi}_0(r, \lambda)$ is associated with a node of $\hat{V}(r, \lambda)$. The value of the eigenenergy E_0 clearly is above the asymptotic value, zero, of the potential. Evidently, the many oscillations of this potential, none of them able to hold a bound state, conspire in such a way as to keep the particle trapped.

The parameter λ which appears in the denominator of Eq. (8.18) plays the role of a damping distance; its magnitude indicates the value of r at which the integral $I_0(r)$ becomes a significant damping factor, both for the new potential and the new wave function. The parameter λ must be restricted to values greater than zero in order to avoid infinities in $\hat{V}(r, \lambda)$ and in the wave functions. It is clear that in the limit $\lambda \rightarrow \infty$, $\hat{V}(r, \lambda)$ becomes identical to $V(r)$.

8.2.2 Coulomb Potential

Starting from the expression of Coulomb potential $V(r) = \frac{Z}{r}$, for either repulsive or negative values of Z , we can construct the one-parameter family of isospectral potentials possessing a normalizable positive energy wave function. the unbound, $\ell = 0$ wave function satisfying the Schrödinger equation can be written in standard form

$$\psi_0'' + \left(1 - 2\frac{\tilde{\eta}}{\rho}\right) \psi_0, \quad (8.20)$$

where $\rho = \sqrt{Er}$ and $\tilde{\eta} = \frac{Z}{2\sqrt{E}}$. Useful expressions for the solutions in the regions near and far from the origin are available in the literature [122]. For the repulsive Coulomb potentials, the positive energy solution of Eq. (8.20) can be written as the real function

$$\psi_0(\rho) = C_0(\tilde{\eta}) e^{-i\rho} M(1 - i\tilde{\eta}, 2, 2i\rho) \quad (8.21)$$

where

$$C_0(\tilde{\eta}) = (e^{-\pi\tilde{\eta}/2}) |\Gamma(1 + i\tilde{\eta})| \quad (8.22)$$

and $M(a, b, z)$ is Kummer's function involving confluent hypergeometric solutions which in the asymptotic limit approach sine waves phase shifted by a logarithmic term. Using tabulated expressions for the Coulomb wave functions [122] and doing the integral I_0 , one obtains the BIC wave function for representative values for λ . Here, we show the corresponding one-parameter BIC partner potential, given by Eq. (8.9), to the repulsive Coulomb potential. Figure 8.3 shows the BIC partner to repulsive Coulomb potential for specific values of $Z = 6, \lambda = 1$, and

$k = 1$ while Fig. 8.4 shows the corresponding wave functions of corresponding potentials. It is seen explicitly that the potential that holds a bound state of positive energy shows an oscillatory behaviour about the original potential as observed in the previous example. Since the oscillation vanishes whenever ψ_0 vanishes, one has $\hat{V} = V$ at each node of ψ_0 . We have also shown the BIC partner potential for the attractive Coulomb potential for special values of $\lambda = 0.25, k = 1$, and $Z = -2$. While the Fig. 8.5 shows the BIC partner potential to the attractive Coulomb potential for the values $V = -2/r$ at $E_0 = k^2$, Fig. 8.6 shows the unnormalized wave function of the unbound state in the continuum of that potential. For comparison, the original Coulomb wave potentials wave functions are also given by dotted lines in all figures.



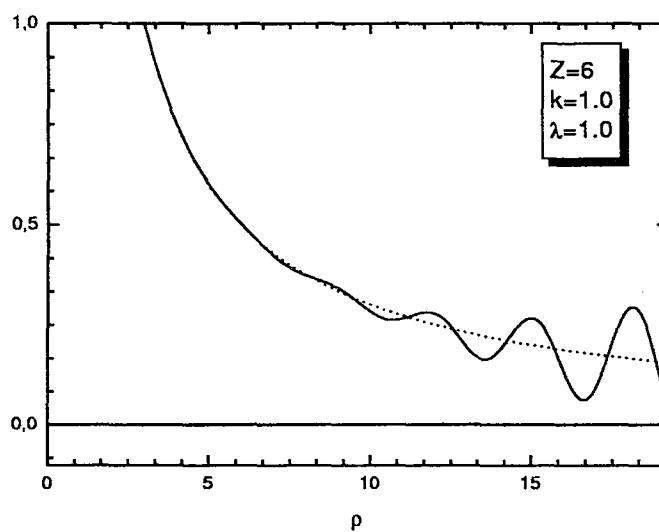


Figure 8.3: The BIC potential (solid) derived from the repulsive Coulomb potential (dotted) for $Z = 6$, $k = 1.0$ and $\lambda = 1.0$. It is explicitly seen again how the BIC potential oscillates around the original Coulomb potential.

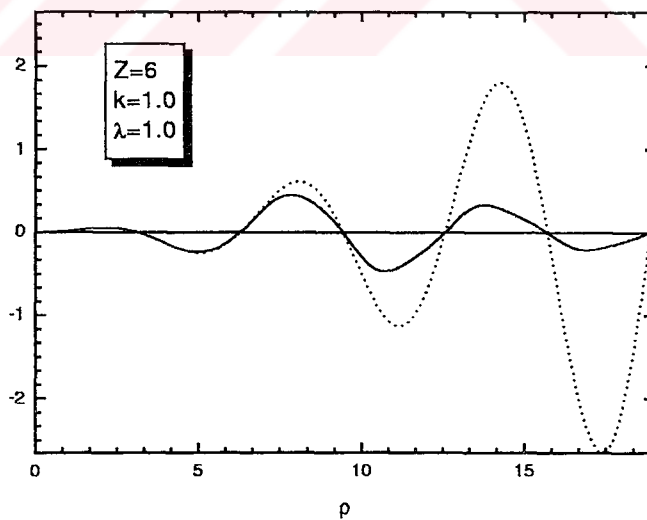


Figure 8.4: The wave functions (unnormalized) corresponding to the BIC potential (solid) and the repulsive Coulomb potential (dotted) for $Z = 6$, $k = 1.0$ and $\lambda = 1.0$.

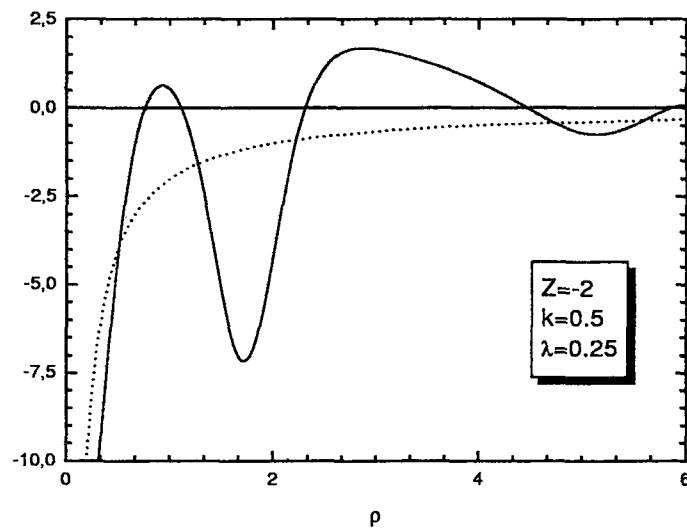


Figure 8.5: The BIC potential (solid) derived from the attractive Coulomb potential (dotted) for $Z = -2$, $k = 1.0$ and $\lambda = 0.5$. It is explicitly seen how the BIC potential oscillates around the original Coulomb potential.

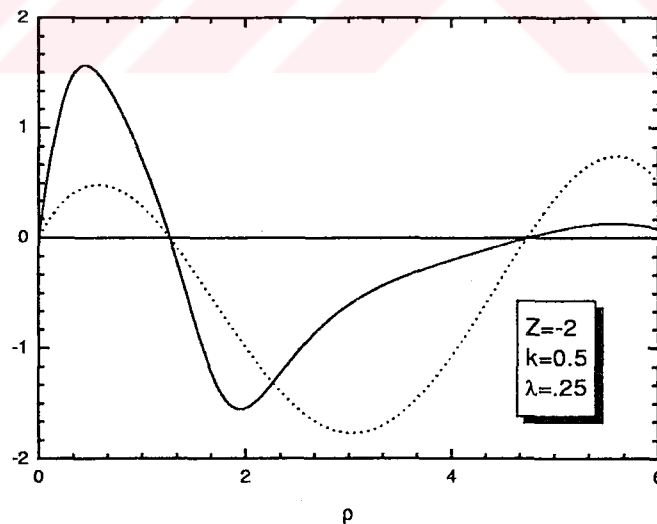


Figure 8.6: The wave functions (unnormalized) corresponding to the BIC potential (solid) and the attractive Coulomb potential (dotted) for $Z = -2$, $k = 1.0$ and $\lambda = 0.5$.

8.3 The Two-Parameter Family of Potentials

8.3.1 Coulomb Potential

As an example for the two-parameter family of potentials we will consider again the attractive Coulomb potential for the value of $Z = -2$ but for bound states. The s -wave effective potential for the Coulomb is $V(r) = -2/r$. The energy eigenstates for that potential is found as $E_0 = -1$ and $E_1 = -1/4$ and the wave functions are $\psi_0(r) = 2re^{-r}$ and $\psi_1(r) = r^2e^{-r/2}/\sqrt{24}$. The corresponding procedure is the same as given in the previous sections. Using the related equations, (8.7) and (8.12), we get $I_0(r) = -e^{2r}(1 + 2r + 2r^2)$ and $I_1(r) = -e^{-r}(1 + r + \frac{1}{2}r^2 + \frac{1}{6}r^3 + \frac{1}{24}r^4)$. So we can construct the two-parameter family $\hat{V}_0(r; \lambda_0, \lambda_1)$ and we have plotted this potential in Fig. 8.7 by keeping λ_0 fixed at a value -1.1 , and varying λ_1 for different values. It is seen obviously that the shallower well with bound state at $E_1 = -\frac{1}{4}$ moves to $r = \infty$ as $\lambda_1 \rightarrow -1$. From Fig. 8.8, it is also explicitly seen that this is true for E_0 as $\lambda_0 \rightarrow -1$ at fixed values of $\lambda_1 = -1.1$.

8.4 BIC of complex potentials generated by SUSYQM

As we have shown in previous sections, SUSYQM is a method that can be used to get bound states in continuum. Application of this technique to a real potential leads to bound states only on the half-line. In order to get bound states on the full line a complex potential generated by SUSYQM is introduced [123]. The Schrödinger equation and its general solution is the starting point in order to generate complex potentials isospectral with the initial one. Consider the Schrödinger equation,

$$H\psi = E\psi \quad (8.23)$$

where ψ is any solution of that equation. Having a particular solution $\psi(x)$, the general solution for energy E is given by

$$\tilde{\psi}(x) = \psi(x) + C\psi(x) \int_{x_0}^x \frac{dt}{|\psi(t)|^2}. \quad (8.24)$$

It is important to note that C may take complex values and x_0 is real and arbitrary, hence the superpotential can be a complex function. If it is chosen to be real, it leads to a real W and standard notation of SUSYQM. By taking C to be complex and following the procedure in the section 8.1, by just taking that $\tilde{\psi}$ is complex, we get the general results of generation one-parameter complex potentials having bound state in the continuum. The resultant complex potential can be given as

$$\hat{V}_0(\lambda; x) = V_0 - 2 \frac{d^2}{dx^2} \ln(I(x) + \lambda) \quad (8.25)$$

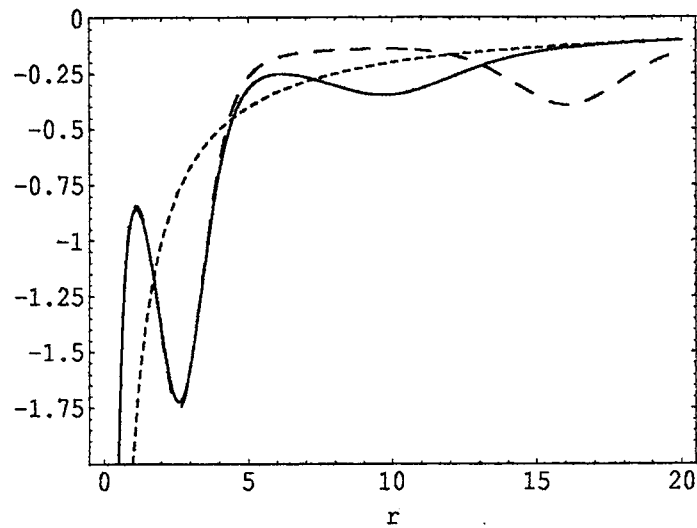


Figure 8.7: Isospectral two-parameter family of BIC potential derived from the attractive Coulomb potential (dotted) for $Z = -2$, $\lambda_1 = -1.1$ (solid) and $\lambda_1 = -1.001$ (dashed curve) for fixed $\lambda_0 = -1.1$.

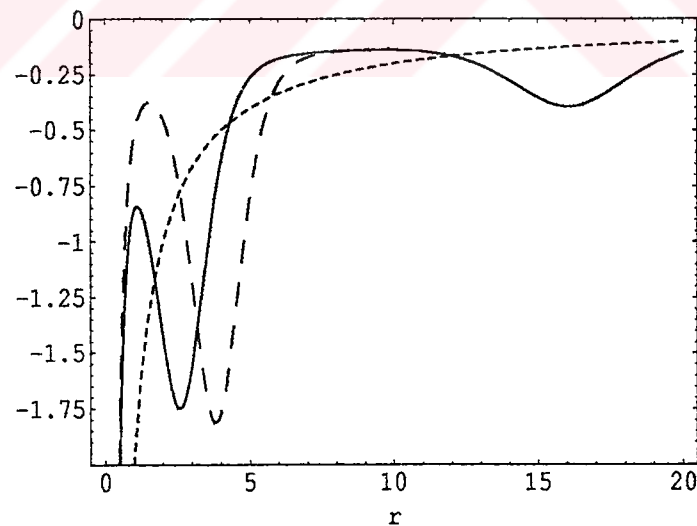


Figure 8.8: Isospectral two-parameter family of BIC potential derived from the attractive Coulomb potential (dotted) for $Z = -2$, $\lambda_0 = -1.1$ (solid) and $\lambda_0 = -1.01$ (dashed curve) for fixed $\lambda_1 = -1.001$.

where

$$I(x) = \int_{x_0}^x \tilde{\psi}(z)^2 dz . \quad (8.26)$$

The unnormalized eigenfunction of the partner Hamiltonian for energy E can be given by, as we found before,

$$\hat{\psi}(\lambda; x) = C \frac{\tilde{\psi}(x)}{I(x) + \lambda} . \quad (8.27)$$

Depending on the character of parameters C and λ , there are some possible cases: If these are both real, the wave functions can be normalized only if $C = 0$, which is the standard SUSYQM case and gives a real partner potential. If the constant $C = C_r + iC_i$ is real ($C_i = 0$, with nonzero C_r), then for any λ one can find a coordinate x where the denominator $\lambda + I(x)$ in the above equations equals zero, so the wave functions are not normalizable. If C is truly complex and λ is real, the requirement for the normalizability of wave functions imposes some constraints upon these parameters, as we will show below for a specific potential.

8.4.1 Flat Potential

The solution of the Schrödinger equation for the flat potential for an arbitrary energy, $E > 0$, leads one to the general solution of the form

$$\tilde{\psi}(x) = \cos(kx) + C \sin(kx) , \quad (8.28)$$

where $k = \sqrt{E}$. We now employ SUSYQM to find the complex potential that will have a bound state at this energy. Using the related equations, we have obtained above, we can give this partner as

$$\hat{V}_0(\lambda; x) = -2 \left(\frac{2k(\cos(kx) + C \sin(kx))(-\sin(kx) + C \cos(kx))}{\lambda + I(x)} - \frac{(\cos(kx) + C \sin(kx))^4}{(\lambda + I(x))^2} \right) , \quad (8.29)$$

and the corresponding wave function for this potential at this energy reads

$$\hat{\psi}(\lambda; x) = \frac{\cos(kx) + C \sin(kx)}{\lambda + \frac{x}{2} + \frac{\sin(2kx)}{4k} + \frac{C}{2k}(1 - \cos(2kx)) + C^2 \left(\frac{x}{2} - \frac{\sin(2kx)}{4k} \right)} . \quad (8.30)$$

In general case with $C_i \neq 0$, singularities do not appear provided

$$\lambda \neq -\frac{x_0}{2} - \frac{\sin(2kx_0)}{4k} - \frac{C_r^2 + C_i^2}{4kC_r} [1 - \cos(2kx_0)] \quad (8.31)$$

is satisfied. We show the complex partner potential generated from the flat potential by using SUSYQM transformations in Fig. 8.9

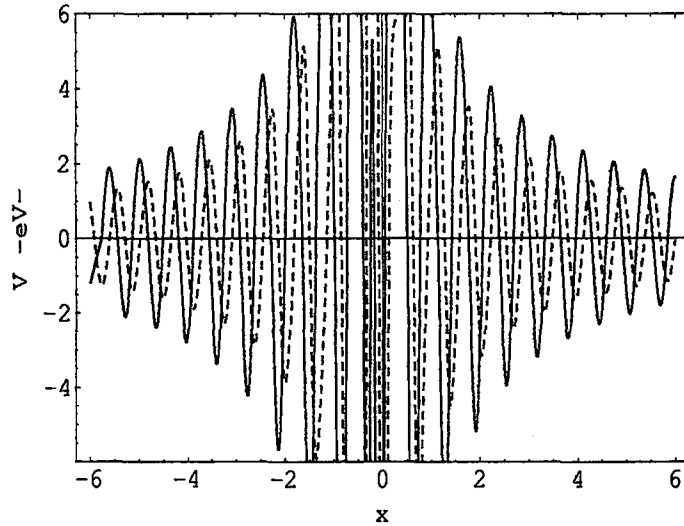


Figure 8.9: The complex potential generated from initial potential ($y = 0$ line) using the parameter values $k = 1, C_r = 1, C_i = 1$. Real and Imaginary parts are shown by solid line and dashed line, respectively.

8.5 Conclusion

It has been demonstrated how the SUSYQM method can be applied for the construction of BIC's. Starting from a potential $V(r)$, It is able to show how to generate a one or two-parameter family of potentials which possess localized positive energy state(s) in the continuum. The only requirement that the potential $V(r)$ must satisfy in order that it have such a continuum is that it approaches a *constant* as $r \rightarrow \infty$.

It has been illustrated the one-parameter method for an analytically solvable case: $V(r) = 0$, the free particle where the solution of Schrödinger equation with the potential $V(r)$ is oscillatory at large r , which can take to be form of $\sin(kr)$. Therefore the integral in the denominator of the new wave function, $\hat{\psi}_0$, expression makes it to be normalizable as r gets large values. Thus the procedure for constructing a BIC from an initial potential $V(r)$ is valid for potentials which approach a constant as $r \rightarrow \infty$. As a second example, we have given the repulsive and attractive Coulomb potentials as a BIC potential, separately. For both potentials, it is seen that the BIC potential oscillates around the initial one and approaches a constant (vanishing) as $r \rightarrow \infty$ like in the free case.

In the two-parameter case, we used attractive Coulomb potential to illustrate how the BIC potential oscillates and how a bound state moves towards continuum range with appropriate values of parameters.

Finally, we have investigated the supersymmetric generated complex potentials and the localized bound states in continuum of these kind of potentials

with complete real spectrum. As an example we have considered the free potential and shown the complex BIC potential. The parameters of the complex potential have to be chosen so that the normalizability of the wave function is satisfied.



CHAPTER 9

GENERAL CONCLUSION

The application of supersymmetry ideas to non-relativistic quantum mechanics has provided a deeper understanding of analytically solvable Hamiltonians, as well as a set of powerful approximate schemes for dealing with problems admitting of no exact solution as discussed through the present work. The concept of shape invariance plays a fundamental role in these developments, since it allows both for (i) a unified treatment of all the already known textbook cases of potentials admitting analytical solutions and (ii) a systematic procedure for generating novel exactly solvable systems as in the case of effective Hulthén like potential which is shape invariant but the first member of this super family is the not shape invariant that is the Hulthén potential for $\ell = 0$.

We have established a very general connection between a class of singular potentials in higher dimensional space through the application of a suitable transformation. We have also shown that the SUSYQM yields exact solutions for a single state only for the quasi-exactly solvable potentials with some restrictions on the potential parameters in N -dimensional space, unlike the shape invariant exactly solvable potentials. We have given how to obtain exact solutions to such problems in any dimension by applying an adequate transformation to previously known three-dimensional results. This simple and intuitive method discussed through this study is easy to be generalized. Additionally, we have again given a very general connection between two potentials in higher dimensional space by using SUSYQM transformations. They are linked by a simple change of the independent variable as discussed and it shows that there is really one quantum mechanical problem, but not two. The successful connection of the most practical applications of such quantum mechanical potentials, like screened Coulomb and anharmonic, is of great importance since the initial potential (screened Coulomb) is not shape invariant. So that our solution leads one to use its partner solution in N -dimensions.

As being another study, the application of the supersymmetric approach to quantum systems with position dependent mass unlike the case of constant mass has been employed through the works presented in this thesis. The Schrödinger

equation with non-constant mass provides an interesting and useful model for the description of many physical problems. The most extensive use of such an equation is in the physics of semiconductor nanostructures and the most popular of these structures is the semiconductor quantum well (QW) lasers. As an application of SUSYQM, to describe a method for the optimized design of QW structures, in respect to maximizing the stimulated gain in such lasers may be worth to study on as a future work for one interested in this area by using the results of related study in this thesis. It relies on applying SUSYQM to an initial Hamiltonian, in order to both map one bound state below the spectral range of the initial Hamiltonian (adding procedure), and to generate a parameter-controlled family of isospectral Hamiltonians with the desired energy spectrum, which has been briefly discussed through thesis. By varying the control parameter, one changes the potential shape and thus the gain calculations.

And finally, we have demonstrated how the SUSYQM method can be applied for the construction of potentials having bound states in the continuum. We have been able to show how to generate n -parameter family of potentials which possess a localized positive energy state, starting from a potential $V(r)$ which has no any energy states in the continuum. The only requirement is seen that the initial potential must vanish in the asymptotic region. It has been illustrated the n -parameter method for analytically solvable potentials. Even there is no any force acting on the particle or it is disappearing, SUSYQM method shows us that there is an oscillating potential vanishing in the asymptotic region. Thus we have shown that the procedure for constructing a BIC from an initial potential can be done by appropriate transformations of SUSYQM. Moreover, we have also investigated the supersymmetric generated complex potentials and the bound states in continuum of these kind of potentials with complete real spectrum.

LIST OF REFERENCES

- [1] J. Wess and B. Zumino, Nucl. Phys. B **70**, 39 (1974); M. F. Sohnius, Phys. Rep. **128**, 39 (1985).
- [2] V. A. Kostelecky and D. K. Campbell, *Supersymmetry in Physics*, North Holland, 1985.
- [3] H. Nicolai, J. Phys. A **9**, 1497 (1976).
- [4] E. Witten, Nucl. Phys. B **202**, 253 (1982); E. Witten, Nucl. Phys. B **188**, 513 (1985).
- [5] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251**, 267 (1995), and references therein.
- [6] O. Özer, *Phase-Equivalent Supersymmetric Deep and Shallow Potentials*, MSc. Thesis, July 1999.
- [7] B. Gönül, O. Özer, and M. Yilmaz, Eur. Phys. J. A **9**, 19 (2000).
- [8] V. Singh, S.N. Biswas, K. Dutta, Phys. Rev. D **18**, 1901 (1978); G.P. Flessas, Phys. Lett A **72**, 289 (1979); M. Razavy, Am. J. Phys. **48**, 285 (1980); A. Khare, Phys.Lett A **83**, 237 (1981).
- [9] A. V. Turbiner, A. G. Ushveridze, Phys. Lett. A **126**, 181 (1987); A. Turbiner, Sov. Phys. JETP **67**, 230 (1988); A. Turbiner, Commun. Math. Phys. **118**, 467 (1988); M. A. Shifman, Int. Jour. Mod. Phys. A **4**, 2897 (1989); P. Roy, Y. P. Varshni, Mod. Phys. Lett. A **6**, 1257 (1991); A. V. Turbiner, Phys. Lett B **276**, 95 (1992); M. Tater and A. V. Turbiner, J. Phys. A **26**, 697 (1993); A. Gangopadhyaya *et al.*, Phys. Lett. A **208**, 261 (1995); V. V. Ulyanov *et al.*, Fiz. Nizk. Temp. **23**, 110 (1997).
- [10] A. Krajewska, A. Ushveridze and Z. Walczak, Mod. Phys. Lett A **12**, 1225 (1997).
- [11] A. Khare, B. P. Mandal, J. Math. Phys. **39**, 3476 (1998).
- [12] A. De Souza Dutra, Phys. Rev. A **47**, R2435 (1993).
- [13] N. Nag, R. Roychoudhury, Y.P. Varshni, Phy. Rev. A **49**, 5098 (1994).
- [14] R. Dutt et.al., J. Phys A **28**, L107 (1995); G. Junker and P. Roy, Phys. Lett A **232**, 155 (1997); G. Junker, P. Roy, preprint quant-ph/9803024.

- [15] G. Bastard, *Wave Mechanics Applied to Semiconductor Heterostructures* (Les Editions de Physique, Les Ulis, France, 1992).
- [16] L. Dekar, L. Chetouani, and T. F. Hammann, *J. Math. Phys.* **39**, 2551 (1998); L. Dekar, L. Chetouani, and T. F. Hammann, *Phys. Rev. A* **59**, 107 (1999).
- [17] P. Ring and P. Schuck, *The Nuclear Many Body Problem* (Springer-Verlag, New York, 1980); F. Arias de Saavedra, J. Boronat, A. Polls, and A. Fabrocini, *Phys. Rev. B* **50**, 4248 (1994); M. Barranco, M. Pi, S. M. Gatica, E. S. Hernandez, and J. Navarro, *Phys. Rev. B* **56**, 8997 (1997).
- [18] L. Gendenshtein, *JETP Letters* **38**, 356 (1983).
- [19] Christopher Lee, *Phys. Lett. A* **267**, 101 (2000).
- [20] L. Hulthén, *Ark. Mat. Astron. Fys.* **28A**, 5 (1942).
- [21] L. Hulthén, *Ark. Mat. Astron. Fys.* **29B**, 1 (1942).
- [22] C. Eckart, *Phys. Rev.* **35**, 1303 (1930).
- [23] Y. P. Varshni, *Phys. Rev. A* **41**, 4682 (1990).
- [24] C. S. Lam and Y. P. Varshni, *Phys. Rev. A* **4**, 1875 (1971).
- [25] S. Flügge, *Practical Quantum Mechanics*, Springer, Berlin, 1974.
- [26] C. S. Lai and W. C. Lin, *Phys. Lett. A* **78**, 335 (1980).
- [27] R. Dutt, U. Mukherji, *Phys. Lett. A* **90**, 395 (1982).
- [28] S. H. Patil, *J. Phys. A* **17**, 575 (1984).
- [29] V. S. Popov, V.M. Weinberg, *Phys. Lett. A* **107**, 371 (1985).
- [30] B. Roy, R. Roychoudhury, *J. Phys. A* **20**, 3051 (1987).
- [31] A. Z. Tang, F. T. Chan, *Phys. Rev. A* **35**, 911 (1987).
- [32] C. H. Lai, *J. Math. Phys.* **28**, 1801 (1987).
- [33] P. Matthys, H. De Meyer, *Phys. Rev. A* **38**, 1168 (1988).
- [34] U. Laha, C. Bhattacharyya, K. Roy, B. Talukdar, *Phys. Rev. C* **38**, 558 (1988).
- [35] B. Talukdar, U. Das, C. Bhattacharyya, P. K. Bera, *J. Phys. A* **25**, 4073 (1992).
- [36] E. D. Filho, R. M. Ricotta, *Mod. Phys. Lett. A* **10**, 1613 (1995).
- [37] C. V. Sukumar, *J. Phys. A* **18**, L57 (1985).
- [38] E. Drigo, R. M. Ricotta, hep-th/0002015.

- [39] R. Dutt, A. Khare, U. Sukhatme, Am. J. Phys., **56**, 163 (1988).
- [40] F. Cooper, J.N. Ginocchio, A. Khare, Phys. Rev. D **36**, 2458 (1987).
- [41] C. X. Chuan, J. Phys. A **24**, L1165 (1991).
- [42] E. Drigo, R. M. Ricotta, hep-th/9607068.
- [43] R. L. Greene, C. Aldrich, Phys. Rev. A **14**, 2363 (1976).
- [44] B. Gönül, O. Özer, Y. Cançelik, and M. Koçak, Phys. Lett. A **275**, 238 (2000).
- [45] A. O. Barut, M. Berrondo and G. Garcia-Calderon, J. Math. Phys. **21**, 1851 (1980).
- [46] S. Özçelik and M. Şimşek, Phys. Lett. A **152**, 145 (1991); S. Özçelik, Tr. J. of Physics **20**, 1233 (1996).
- [47] E. Papp, Phys. Lett. A **157**, 192 (1991).
- [48] G. F. Gribakun and V. V. Flambaum, Phys. Rev. A **48**, 546 (1993); R. Szymtkowski, J. Phys. A **28**, 7333 (1995).
- [49] E. Vogt and G. H. Wannier, Phys. Rev. **95**, 1190 (1954); D. R. Bates and I. Esterman, *Advances in Atomic and Molecular Physics* (Academic Press, New York, 1970).
- [50] A. O. Barut, J. Math Phys. **21**, 568 (1980).
- [51] M. Kaplus and R. N. Porter, *Atoms and Molecular Physics* (Cambridge, 1970); R. J. Le Roy and R. B. Bernstein, J. Chem Phys. **52**, 3869 (1970); R. J. Le Roy and W. Lam, Chem. Phys. Lett. **71**, 544 (1980).
- [52] A. M. Sherry and M. Kumar, J. Chem. Solids **52**, 1145 (1991).
- [53] B. H. Bransden and C. J. Joachain, *Physics of Atoms and Molecules* (Longman, London, 1983).
- [54] G. C. Maitland, M. Rigby, E. B. Smith and W. A. Wakeham, *Intermolecular forces* (Oxford Univ. Press, Oxford, 1987).
- [55] A. Khare and S. N. Behra, Pramana J. Phys. **14**, 327 (1980).
- [56] D. Emin and T. Holstein, Phys. Rev. Lett. **36**, 323 (1976); Phys. Today **35**, 34 (1982).
- [57] S. Coleman, *Aspects of Symmetry, selected Erice Lectures* (Cambridge Univ. Press, Cambridge, 1988).
- [58] H. Hashimoto, Int. J. Electron, **46**, 125 (1979); Opt. Commun., **32**, 383 (1980).
- [59] M. S. Child, J. Phys. A **31**, 657 (1998).

- [60] A. Michels, J. de Boer, and A. Bijl, *Physica*, **4**, 981 (1937).
- [61] Y. P. Varshni, *Can. J. Phys.* **75**, 907 (1997).
- [62] M. A. Reed, *Sci. Am.* **268**, 118 (1993); D. S. Chuu, C. M. Hsiao, and W. N. Mei, *Phys. Rev. B* **46**, 3898 (1992).
- [63] G. Mie, *Ann. Physik* **11**, 657 (1903); A. Kratzer, *Z. Phys.* **3**, 289 (1920).
- [64] I. I. Goldman and V. D. Krivchenkov, *Problems in Quantum Mechanics* (Pergamon Press, New York, 1961); L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory* (Oxford, Pergamon, 1977).
- [65] S. Dong, Z. Ma and G. Esposito, *Found. Phys. Lett.* **12**, 465 (1999); quant-ph/9902081.
- [66] S. Dong and Z. Ma, quant-ph/9901037.
- [67] E. Schrödinger, *Proc. R. Irish Acad. Sec. A* **46**, 183 (1941); D. Bergmann and Y. Frishman, *J. Math. Phys.* **6**, 1855 (1965); A.K. Grant and J. L. Rosner, *Am. J. Phys.* **62**, 310 (1994).
- [68] V. A. Kostelecky, M. M. Nieto, and D. R. Truax, *Phys. Rev. D* **32**, 2627 (1985).
- [69] V. A. Kostelecky and N. Russell, *J. Math. Phys.* **37**, 2166 (1996).
- [70] R. N. Chaudhuri and M. Mondal, *Phys Rev. A* **52**, 1850 (1995).
- [71] D. A. Morales and Z. Parra-Mejias, *Can. J. Phys.* **77**, 863 (1999).
- [72] O. Mustafa, math-ph/0101030; S. Dong, X. Hou and Z. Ma, quant-ph/9808037.
- [73] S. Dong, *Int. J. Theor. Phys.* **39**, 1119 (2000); *J. Phys. A* **31**, 9855 (1998).
- [74] E. Kasap, B. Gönül and M. Şimşek, *Chem. Phys. Lett.* **172**, 499 (1990).
- [75] D. Secrets, *J. Chem. Phys.* **89**, 1017 (1988); J. Morales *et al.*, *J. Math. Chem.* **21**, 273 (1997).
- [76] M. Znojil, *J. Math. Chem.* **26**, 157 (1999); R. L. Hall and N. Saad, *J. of Chem. Phys.* **109**, 2983 (1998).
- [77] C. Jia, X. Wang, X. Yao and P. Chen, *J. Phys. A* **31**, 4763 (1998).
- [78] B. Gönül, O. Özer, M. Koçak, D. Tutcu and Y. Cançelik, *J. Phys. A* **34**, 8271 (2001).
- [79] G. A. Natanzon, *Theor. Math. Phys.* **38**, 146 (1979).
- [80] A. Gangopadhyaya, P. K. Panigrahi and U. P. Sukhatme, *Helv. Phys. Acta* **67**, 363 (1994).
- [81] R. Haymaker and A. R. P. Rau, *Am. J. Phys.* **54**, 928 (1986).

- [82] H. A. Mavromatis, *J. Math. Phys.* **39**, 2592 (1998).
- [83] C. Itzykson, and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
- [84] C. Reid, *J. Mol. Spectrosc.* **36**, 183 (1970).
- [85] C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1986).
- [86] R. K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1986).
- [87] D. Baye, *J. Phys. B* **28**, 4399 (1995).
- [88] M. Hesse and D. Baye, *J. Phys. B* **32**, 5605 (1999).
- [89] R. Dutt, A. Ray and P. P. Ray, *Phys. Lett. A* **83**, 65 (1981).
- [90] R. S. Kaushal, *Ann. Phys. (N.Y)* **206**, 90 (1991).
- [91] R. S. Kaushal and D. Parashar, *Phys. Lett. A* **170**, 335 (1992).
- [92] T. D. Imbo and U. P. Sukhatme, *Phys. Rev. Lett.* **54**, 2184 (1985).
- [93] J.-M. Lévy-Leblond, *Phys. Rev. A*, **52**, 1845 (1995).
- [94] A. R. Plastino, A. Rigo, M. Casas, F. Gracias and A. Plastino, *Phys. Rev. A* **60**, 4318 (1999).
- [95] R. De, R. Dutt and U. Sukhatme, *J. Phys. A* **25**, L843 (1992).
- [96] B. Roy and P. Roy, preprint quant-ph/0106028.
- [97] Bülent Gönül, Okan Özer, Beşire Gönül and Fatma Üzgün, *Mod. Phys. Lett. A* **17**, 2453 (2002).
- [98] T. Li and K. J. Kuhn, *Phys. Rev. B* **47**, 760 (1994).
- [99] Beşire Gönül, Bülent Gönül, Dilek Tutcu and Okan Özer, *Mod. Phys. Lett. A* **17**, 2057 (2002).
- [100] D. J. BenDaniel and C. B. Duke, *Phys. Rev.* **152**, 683 (1966).
- [101] O. von Roos, *Phys. Rev. B* **27**, 7547 (1983).
- [102] T. Gora, F. Williams, *Phys. Rev.* **177**, 1179 (1969).
- [103] G. Bastard, *Phys. Rev. B* **24**, 5693 (1981).
- [104] Q. G. Zhu, H. Kroemer, *Phys. Rev. B* **27**, 3519 (1983).
- [105] T. Li, K. J. Kuhn, *Phys. Rev. B* **47**, 12760 (1993).
- [106] S. Datta, *Quantum Phenomena, Modular Series on Solid State Devices*, Vol. 8 (Addison-Wesley, Reading, Ma, 1989), Chaps. 1 and 6; S. Adachi, *J. Appl. Phys.* **58**, R1 (1985); P. Lawaetz, *Phys. Rev. B* **4**, 3460 (1971).
- [107] B. Roy and P. Roy, *J. Phys. A* **35**, 3961 (2002).

- [108] V. Milanović and Z. Iković, *J. Phys. A* **32**, 7001 (1999).
- [109] A. De Souza Dutra, C. A. S. Almeida, *Phys. Lett. A* **275**, 25 (2000).
- [110] S. B. Haley, *Am. J. Phys.* **65**, 237 (1997); B. Y. Tong, *Solid State Commun.*, **104**, 679 (1997); B. Y. Tong, N. Kiruischeva, *Phys. Lett. A* **29**, 49 (1997).
- [111] G. Todorović, V. Milanović, Z. Ikonić and D. Indjin, *Phys. Lett. A* **279**, 268 (2001); S. Tomić, V. Milanović, Z. Ikonić, *IEEE J. Quant. Elect.*, **37**, 1337 (2001); S. Tomić, V. Milanović, M. Tadić and Z. Ikonić, *Solid State Comm.*, **113**, 221 (2000); S. Tomić, V. Milanović, Z. Ikonić, *Optics Comm.*, **143**, 214 (1997).
- [112] G. Duggan, *J. Vac. Sci. Tech. B* **3**, 1224 (1985); H. Kroemer, *Surf. Sci.* **174**, 299 (1986).
- [113] A. Chomette, B. Deveaud, M. Baudet, P. Auvray and A. Regreny, *J. Appl. Phys.* **59**, 3835 (1986).
- [114] A. R. Plastino, A. Puente, M. Casas, F. Garcias and A. Plastino, *Revista Mexicana De Fisica*. **46**, 78 (2000).
- [115] J. von Neumann and E. Wigner, *Phys. Z.* **30**, 465 (1929).
- [116] L. Landau and E. Lifchitz, *Quantum Mechanics* (Pergamon, New York, 1965).
- [117] A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1961), Vol. 1.
- [118] F. H. Stillinger and D. R. Herrick, *Phys. Rev. A* **11**, 446 (1975).
- [119] N. Meyer-Vernet, *Am. J. Phys.* **50**, 354 (1982).
- [120] M. Robnik, *J. Phys. A* **12**, 1175 (1979).
- [121] A. Khare and U. Sukhatme, *J. Phys. A* **22**, 2847 (1989).
- [122] M. Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions* (U. S. GPO, Washington, D. C., 1964).
- [123] V. Milanović, Z. Ikonić, *Phys. Lett A* **293**, 29 (2002).

PUBLICATIONS

- [1] B. Gönül, O. Özer, and M. Yılmaz, Eur. Phys. J. A **9**, 19 (2000).
- [2] B. Gönül, O. Özer, Y. Cançelik, and M. Koçak, Phys. Lett. A **275**, 238 (2000).
- [3] B. Gönül, O. Özer, M. Koçak, D. Tutcu and Y. Cançelik, J. Phys. A **34**, 8271 (2001).
- [4] Beşire Gönül, Bülent Gönül, Dilek Tutcu and Okan Özer, Mod. Phys. Lett. A **17**, 2057 (2002).
- [5] Bülent Gönül, Okan Özer, Beşire Gönül and Fatma Üzgün, Mod. Phys. Lett. A **17**, 2453 (2002).
- [6] Bülent Gönül, Okan Özer, Mehmet Koçak, Unified treatment of screening Coulomb and anharmonic oscillator potentials in arbitrary dimensions, Submitted to the *Physica Scripta*.

VITA

The author was born in Aydın, in December, 1972. He graduated from İzmir Eşrefpaşa Elementary, Secondary and High Schools in İzmir, in 1984, 1987, and 1990, respectively. He received his B. S. degree in Engineering Physics Department from University of Gaziantep in July 1996. He worked as a Technical Service Manager in a GSM Company in İzmir from 1996 to 1997. Then he has been accepted as a Research Assistant in the Department of Engineering Physics at the University of Gaziantep. He has completed his M. Sc. degree, in 1999, under the topic of "Phase-Equivalent Supersymmetric Deep and Shallow Potentials". He has started his Ph. D. program in 1999 and his main areas of interest are Supersymmetric Quantum Mechanics and its applications.

He married Aslıhan in October 2002.