A COMPUTATIONAL TOOL FOR DETERMINING OPTIMUM SHAPES OF LINEARLY ELASTIC ARCHES

M. Sc. Thesis

in

Civil Engineering
University of Gaziantep

By

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January 2003

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ABSTRACT

A COMPUTATIONAL TOOL FOR DETERMINING OPTIMUM SHAPES OF LINEARLY ELASTIC ARCHES

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M.S. in Civil Engineering
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January 2003, 51 pages

This thesis deals with the development of reliable and efficient computational tools to analyze and find optimum shapes of linear elastic arch structures in static situations. The finite element method is used to determine the stress and displacements based on variable thickness *C(0)* continuity Timoshenko-Hencky beam theory. An automated analysis and optimization procedure is adopted which integrates finite element analysis, parametric cubic spline geometry definition, automatic mesh generation, sensitivity analysis and mathematical programming methods.

Key words: Arch structures, static analysis, shape optimization, finite element method.

ÖZ

LİNEER ELASTİK KEMER YAPILARIN OPTİMUM ŞEKİLLERİNİN BULUNMASI İÇİN BİLGİSAYARLI HESAP TEKNİĞİ

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Tez Yöneticisi: Doç. Dr. Mustafa ÖZAKÇA
Ocak 2003, 51 sayfa

Bu tez, lineer elastik kemer yapıların statik analizi ve optimum tasarımı için güvenli ve verimli bir bilgisayar programı geliştirilmesini kapsamaktadır. Değişken kalınlıklı, izoparametrik ve *C(0)* sürekliliğinin sağlandığı Timoshenko-Hencky kiriş teorisine bağlı sonlu elemanlar metodu kullanılarak deplasmanlar, gerilmeler ve şekil değiştirme enerjileri hesaplanmıştır. Bu çalışmada sonlu elemanlar metodu, kübik eğrilerle geometri tanımlama, ağ üretimi, hassasiyet analizi ve matematik programlama metotlarını birleştiren bir entegre analiz ve optimizasyon tekniği kullanılmaktadır.

Anahtar kelimeler: Kemer yapılar, statik analiz, şekil optimizasyonu, sonlu elemanlar metodu.

ACKNOWLEDGEMENTS

I would like to express my sincere and gratitude thanks to my supervisor, Assoc. Prof. Dr. Mustafa Özakça, for his guidance, advice, encouragement and suggestions during the preparation of this thesis.

And I would like to thank Res. Ass. Nildem Tayşi and Res. Ass. M. Tolga Göğüş for great help.

And also my special thanks are reserved for my family for their encouragement, understanding, patience and great help.

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Abbreviations:

CAD	Computer Aided Design
DOF	Degrees Of Freedom
FD	Finite Difference
FE	Finite Element
MMA	Method of Moving Asymptotes
MR	Mindlin-Reissner
SA	Semi-Analytical
SE	Strain Energy

Latin Symbols:

Scalar

A	area
$C(0)$	order of continuity
d	displacement
E	Young's modulus
$F(\mathbf{s})$	objective function to be minimized
G	modulus of rigidity
$g_j(\mathbf{s})$	inequality constraint function
g_u, g_v, g_w	distributed loadings associated with u, v and w displacement
g_ϕ, g_ψ	distributed loadings associated with ϕ and ψ displacement
$h_k(\mathbf{s})$	equality constraint function
I	total virtual work

I^e	total work from element *e*
J	Jacobian
ℓ	the arc length parameter of the curve
L	span length
M_l, M_y, M_{ly}	stress resultant due to bending
N_i	shape function associated with node *i*
N_l, N_y, N_{ly}	membrane stress resultant
n_{dv}	number of design variables
Q_l, Q_y	shear forces
r,z	radial and vertical coordinate
R	radius of curvature
s	design variables
t	thickness
t_i	thickness at node *i*
u_ℓ, v_ℓ, w_ℓ	displacement components in ℓ, y and *n*-direction
u, v, w	global displacement parameters
$\|\mathbf{W}\|^2$	total strain energy
$\|\mathbf{W}\|_b^2$	strain energy due to bending
$\|\mathbf{W}\|_m^2$	strain energy due to membrane
x_i, y_i, z_i	typical cartesian coordinates of node *i*
x, y, z	global cartesian coordinates

Vector

$\mathbf{d}$	vector of unknown displacements (eigenvector)
$\mathbf{d}_i^p$	vector of nodal degrees of freedom
$\mathbf{d}_i^e$	displacement (eigenvector) vector associated with element *e* and node *i*
$\mathbf{f}$	vector of structural load
$\mathbf{f}_i^e$	nodal force vector associated with node *i*

$\mathbf{g}$	vector of distributed loading
$\mathbf{u}$	displacement field vector

Matrix

$\mathbf{B}$	strain-displacement matrix
$\mathbf{B}_{mi}^{e}$	membrane strain-displacement matrix for element *e* and node *i*
$\mathbf{B}_{bi}^{e}$	bending strain-displacement matrix for element *e* and node *i*
$\mathbf{B}_{si}^{e}$	shear strain-displacement matrix for element *e* and node *i*
$\mathbf{B}_{mi}$	membrane strain-displacement matrix for node *i*
$\mathbf{B}_{bi}$	bending strain-displacement matrix for node *i*
$\mathbf{B}_{si}$	shear strain-displacement matrix for node *i*
$\overline{\mathbf{B}}_{mi}, \overline{\mathbf{B}}_{bi}, \overline{\mathbf{B}}_{si}$	transformed strain-displacement matrices
$\mathbf{B}_{pi}$	in-plane strain displacement matrix
$\mathbf{B}_{si}$	shear strain displacement matrix
$\mathbf{D}$	matrix of rigidities
$\mathbf{D}_m$, $\mathbf{D}_b$, $\mathbf{D}_s$	matrices of membrane, bending and shear rigidities
$\mathbf{D}_p$, $\mathbf{D}_s$	in-plane and transverse elasticity matrix
$\mathbf{J}$	Jacobian matrix
$\mathbf{K}_{ij}^{e}$	stiffness matrix associated with element *e* and node *i* and *j*
$\mathbf{K}$	symmetric, banded stiffness matrix
$\mathbf{K}_{mij}^{e}$	membrane stiffness matrix for element *e* and node *i* and *j*
$\mathbf{K}_{bij}^{e}$	bending stiffness matrix for element *e* and node *i* and *j*
$\mathbf{K}_{sij}^{e}$	shear stiffness matrix for element *e* and node *i* and *j*
$\mathbf{M}$	global mass matrix
$\mathbf{M}_{ij}^{e}$	mass matrix associated with element *e* and node *i* and *j*
$\mathbf{N}$	shape function matrix
$\mathbf{T}$	transformation matrix

Greek Symbols:

α	angle between local and global axes
β	arch opening angle
δ_t	mesh density
Δs_k	small perturbation of design variables s_k
$\varepsilon_m, \varepsilon_b, \varepsilon_s$	membrane, bending and transverse shear strain vectors
$\varepsilon_\ell, \varepsilon_y$	strain in ℓ direction and longitudinal strain
$\gamma_{\ell y}, \gamma_{yn}$	shear strain
κ	shear modification factor
κ_ℓ	curvature in the ℓ-direction
κ_y	longitudinal curvature
$\kappa_{\ell y}$	twisting curvature
ν	Poisson's ratio
ξ, η, ζ	isoparametric element natural (curvilinear) coordinates
σ	stress component
$\sigma_m^p, \sigma_b^p, \sigma_s^p$	membrane, bending and shear stress resultant vectors
ρ	mass density
$\partial \ell$	partial differential of ℓ

CHAPTER 1

INTRODUCTION

1.1 Motivation and Principal Objectives

The ultimate motivation of the thesis is static analysis and Structural Shape Optimization (SSO) of arch structures using reliable, versatile and efficient computational tools. The specific objectives of this work can be summarized as:

- Linear elastic analysis of arch structures based on Mindlin-Reissner (MR) theory and integrating locking-free finite element (FE) analysis, cubic splines to define the arch geometry and automatic mesh generation are presented. The accuracy and relative performance of a family of linear, quadratic and cubic finite elements are examined by comparing with other numerical solutions.

- Static optimization: in which the structure is subjected to static external loads. Here, the aim of the SSO is generally to obtain the best geometric shape for the arc structures; so that it can carry the imposed loads safely and economically. The type of optimization considered is minimization of strain energy and weight by constraining the volume of the arch material and stresses.

1.2 Arch Structures

Arches are one of the earliest structures used by man in the history. Arch structures provide esthetics and strength through their geometry. Arches carry pure compression in its primary element. Structures of this type, which have the configuration of an inverted cable, have a pure compressive thrust along the rib of the arch under the specific loading condition for which it was designed. Variations in this loading will introduce some bending in the arch, but compression will remain the dominant mode of action. Secondary members, which attach from portions of the structure to the arch, are compression members when they attach from above the arch or tension members when they attach from below the arch [1,2].

Arches provide economical solutions for crossing large spans. They are frequently used in bridge structures, dome roofs, and for openings in masonry walls. They can be built with materials of high compressive strength and low tensile strength, such as concrete and masonry.

Note that the definition of an arch is a structural one, not geometrical. Here again the relation structure-shape comes into play. Not every structure that looks like an arch is one. For a structure to be an arch certain conditions have to be satisfied which link the geometry with the flow of internal forces. False arches abound in our built environment [3].

What makes an arch an arch is:

- support conditions enabling horizontal reactions;
- correspondence of the shape with the load.

Here the structure-shape or force–form relation finds its utmost expression. Keeping 'true to form' will avoid the arch-arc confusion, at least on the semantic/syntactic level

1.3 Literature Survey

Masonry arch structures are often encountered in old historical building such as carrying elements for the vertical loading.

Nowadays, steel and concrete have replaced stones. These are monolithic structures, so they do not have a problem with shear forces. Usage of steel and concrete in structures decreased the number of arch structures.

But after the improvements in technology the usage of arch structures in long spans such as bridges, mosques are again increased.

1.3.1 Static analysis

The question of the choice of the proper finite elements to the analysis of the curved structures is a subject of numerous papers and many authors have devoted their efforts to it. Yamada and Ezawa [4] presented stiffness matrix of the curved finite elements for the analysis of circular arches. Litewka and Rakowski [5] derived exact stiffness matrix for a curved thick beam with constant curvature including the flexural, shear and axial effects. The stiffness matrix of finite ring elements under the in-plane forces using polynomial displacement functions was derived by Meck [6].

Finite element analysis of the Timoshenko beam problem has been frequently used as a starting point for a better understanding of the much more complex problem of constructing accurate finite element approximations for the Reissener-Mindlin plate problem. Cheng and Han [7] discussed some finite elements methods for Timoshenko beam, circular arch and Reissener-Mindlin plate problems. They used to avoid locking phenomenon, the reduced integration technique and added a bubble function space to increase the solution accuracy.

Lee and Sin [8] presented the formulation of a curved beam element with 3 nodes for curvature to eliminate the shear/membrane locking phenomenon. The element was based on curvature so that it might represent the bending energy fully, and the shear/membrane strain energy was incorporated into the formulation by the equilibrium equations. They introduced to deal with general boundary conditions, a transformation matrix between nodal curvature and nodal displacement.

In most of the papers, curved beam elements with polynomial shape functions are used. Pandian and Chandra [9] adopted the cubic polynomials as the shape functions to the displacement of thin elements. They obtained improvement of results by the application of reduced integration and adoption of a least-square fit. Stolarski and Chiang [10] applied the combination of linear, quadratic and cubic shape functions for the components of the displacement.

The elements with other types of curvature are also the subject of some papers, e.g. Marquis and Wang [11] derived the stiffness matrix for a parabolic beam element in which the flexural, axial and shear deformation effects are taken into account. Friedman and Kosmatka [12] developed an accurate two-node finite element for curved shear deformable beams. The element was demonstrated to converge to the results obtained from a shear deformable straight beam when the beam becomes shallower.

Kikuchi [13] presented a mathematical analysis of some finite element models for thin circular arches. He obtained error estimates for some compatible and mixed finite elements. In order to analyze numerical behaviors of the finite element solutions for thin arches, he employed the theory of mixed finite elements with perturbations as well as the techniques of asymptotic expansion.

In recent years the application of finite elements to problems of large deflection and stability analysis has been of considerable interest. Chakraborty and Majumdar [14] derived a high precision finite element for large deflection and post-buckling analysis of thin arches of arbitrary shapes. They developed a simple general arch element suitable for the non-linear analysis of thin arches. The representation of arbitrary geometry was based on a computer-aided design technique which uses vector valued nodal handles and cubic Hermitian interpolation polynomials.

Chucheepsakul and Huang [15] presented a finite element solution of a simply supported beam with variable arc length under a point load. They used finite element method and Newton-Raphson iteration process to solve this highly nonlinear problem. They gave numerical results from the finite element method and compared favorably with elliptic-integral and shooting-optimization methods.

Detailed treatment of the static finite element analysis of arches with arbitrarily cross-sections can be found in many text books [16-19].

1.3.2 Structural shape optimization

Analytical methods for solving shape optimization problems have been used for a long time. Perhaps the best known early work is the work by Michell (1904). During the last 20 years, optimal structural design has received considerable attention. The development of powerful computers and the implementation of efficient general algorithms has stimulated renewed interest in this field, which is as old as structural engineering. One of the first treatments of the problem of selecting an optimum shape of a structure is by Zienkiewicz and Campbell [20]. They used the finite element method with node coordinates as design variables to find an optimum shape.

Ding [21] reviewed numerical and analytical methods for shape optimization of structures. Several steps in the shape optimization process, such as model description, selection of objective function and shape variables, representation of boundary shape, finite element mesh generation and refinement, sensitivity analysis and solution methods, were reviewed in detail. Tadjbakhsh [22] developed an algorithm which determines the optimum profile of an arch considering the stability constraints.

In general optimization techniques used in structural engineering design can be categorized into four distinct approaches: (1) Optimality Criteria (OC) methods; (2) Genetic Algorithm (GA); (3) Evolution Strategies (ES); and (4) Mathematical Programming (MP) methods.

The OC methods are developed from indirectly applied the Kuhn-Tucker conditions of non-linear mathematical programming combined with Lagrangian multipliers. The Kuhn-Tucker conditions provide the necessary requirements for an optimum solution and the Langrangian multipliers are used to include the associated constraints.

OC methods are based on continuous design variables. For the case where discrete variables are desired using OC methods a two-step procedure is typically used. First, the optimization problem is solved using continuous variables. Second, a set of discrete values is estimated by matching the values obtained from the continuous solution. OC methods use a single cross-sectional property of a structural member as the design variable. All other cross-sectional properties are expressed as a function of the selected design variable.

No and Aguinagalde [23] used optimality criterion to optimize structures modeled by means of the Finite Element Method. Uzman and Daloğlu [24] used optimality criteria method to develop an optimum design of arch structures with uniform and/or varying cross section subjected to displacement, stresses and minimum depth constraints. The optimality criteria method is employed to obtain its solution which was reported to be quite effective in solving nonlinear optimum design problems by Saka and Hayalioğlu [25,26] and Saka and Ülker [27].

Genetic Algorithm (GA) is a strategy that models the mechanisms of genetic evolution which that operates on a population of design variable set defining a potential solution is called a string. Each string is made up of a series of characters, typically binary numbers, representing the values of the discrete design variables for a particular solution. The fitness of each string is a measurement of performance of the design variables as defined by the objective function and the constraints. GA basically consists of three parts:

(1) coding and decoding variables into strings;

(2) evaluating the fitness of each solution string; and

(3) applying genetic operators to generate the next generation of solution strings.

The fitness of each string is evaluated by performing some type of system analysis to compute a value of the objective function. If the solution violates constraints, the value of the objective function is penalized.

The core characteristics of a GA are based on the principles of survival of the fittest and adaptation and the ability to deal with discrete optimum design problems and do not require derivatives of functions, unlike classical optimization.

The advantages of applying a GA to optimized design of structures include discrete design variables, open format for constraint statements and multiple load cases [28]. A GA does not require an explicit relationship between the objective function and the constraints. Instead, the value of the objective function for a set of design variables is adjusted to reflect any violation of the constraints. Peng and Fairfield [29] presented an integrated design optimization combining the mechanism method with genetic algorithms.

The Evolutionary Structural Optimization (ESO) method is based on the concept of slowly removing the inefficient material and or gradually shifting the material from the strongest part of the structure to the weakest part until the structure evolves towards the desired optimum. And also ESO are the application of combinatorial optimization methods based on probabilistic searching. These algorithms have some selection process based on fitness of individuals and some recombination operators. Both ESOs and GAs imitate biological evolution in nature and combine the concept of artificial survival of the fittest with evolutionary operators to form a robust search mechanism

ESO method for shape and layout optimization has been proposed by Xie and Steven [30] which is based concept of gradually removing redundant elements to achieve optimal design. It is widely recognized that combinatorial optimization techniques are in general more robust and present a better global behavior than Mathematical Programming (MP) methods [31].

Mathematical Programming can be subdivided into linear programming and non-linear programming. The major characteristic of linear programming is that the objective function(s) and the associated constraints are expressed as a linear combination of the design variables. To apply linear programming techniques to structural optimization, the relationship between the objective function and the constraints to the design variables have to be linearized. However, when a linear relationship is used to model a non-linear structural response, errors are inevitable.

Mathematical programming techniques were first used in design of frames by Moses [32]. Many authors have since then investigated the shape optimization problem.

Majid and Saka [33] and Topping [34] employed mathematical programming techniques in the shape optimization of rigidly jointed frames. In some of these works, the minimum weight was replaced by minimum cost. It was demonstrated in all these works that the shape of a frame can successfully be treated as a design parameter.

1.4 About Computer Program

The adaptivity and shape optimization group (ADOPT) at the University College of Swansea were developing a series of basic analysis and shape optimization programs for straight and curved planform prismatic plates and shells [35]. These programs have been run on VAX mainframes and require NAG subroutine libraries. In this thesis, these programs are modified to assist structural engineer to design structurally efficient forms and provide considerable insight into the structural behavior of arches. The intention is to deal with arch structure idealizations problem involving static analysis. The following improvements are done in original programs:

- Existing plate programs are modified and some new subroutines are written for analysis and optimization of arch structures.
- Program is re-written in FORTRAN 90 using double precision and run on personal computers.
- Program is simplified and written in modular form. The modules of program are mesh generation, static analysis, sensitivity and mathematical programming.
- Static analysis and optimization are integrated in one program.
- New linear equation solver so-called *"skyline method"* which is possible to use more efficient storage scheme, is implemented.
- A mathematical programming method, DOT is integrated to the program.
- The capability of the present program is increased such as program can handle different loading, objective function and constraints.
- In chapters 2-4, the case studies are carried out related with arch structures.

1.5 Layout of the Thesis

The layout of thesis is now described:

- Chapter 2 is devoted to the static analysis of arch structures. The basic theory and FE formulation is first presented. Several numerical examples are studied.
- Chapter 3 deals with various aspects of the optimization process including the definition and selection of the design variables, the sensitivity analysis and the optimization algorithm adopted.
- Chapter 4 presents several examples which demonstrate the optimal shapes and thickness distributions obtained for arch structure optimization problems.
- Finally in Chapter 5, some brief conclusion are presented together with some suggestions for future work.

CHAPTER 2

STATIC ANALYSIS

2.1 Introduction

The finite element method is now widely employed in analysis of curved structural members such as arches and shells. Modern matrix methods have two main methods. The first one flexibility method is based on deflection per unit force and the second one stiffness method is force per unit deflection. These methods are based on two formulations.

- Euler – Bernoulli formulations are based on the assumption that normals to the midsurface remain straight and normal to the midsurface after deformation. Only direct stresses in the spanning directions contribute directly to the strain energy. This theory is suitable for thin beam. See Figure 2.1

- Mindlin-Reissner (or Timoshenko-Hencky) formulations on the other hand are based on the assumption that normals to the midsurface remain straight but not necessarily normal to the midsurface after deformation. Deflections are small compared with the beam thickness. This theory is suitable for thick beam. See Figure 2.2.

The selection of finite elements is quite delicate especially when the members are thin and flexible. We often experience that the deflections calculated by certain finite element models are unbelievably smaller than the exact ones.

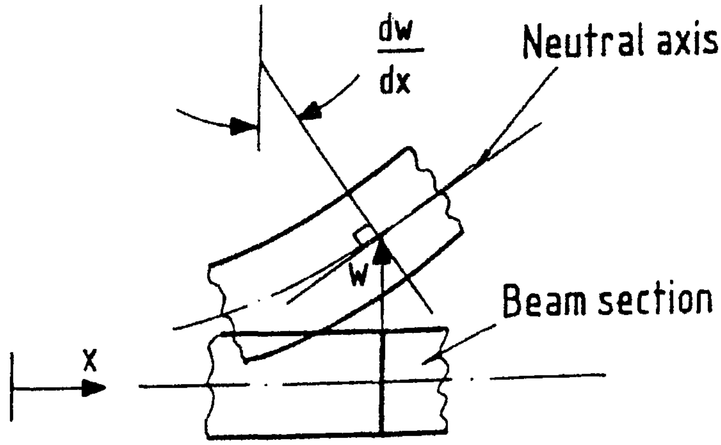


Figure 2.1 Euler – Bernoulli formulations [18]

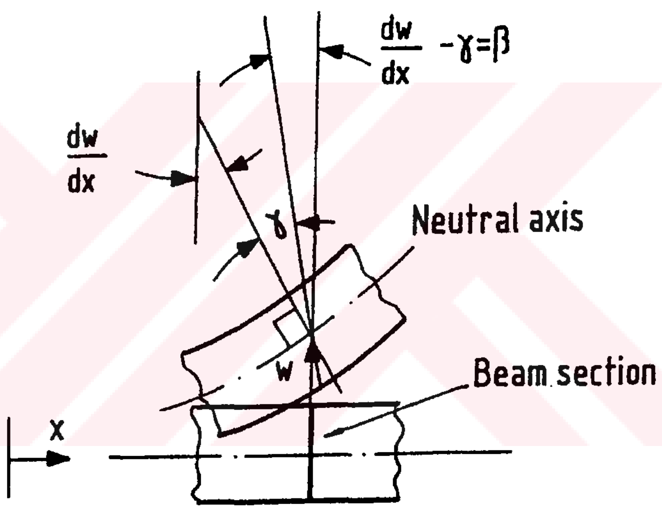


Figure 2.2 Mindlin-Reissner formulations [18]

In this chapter, we consider the basic formulation and associated finite elements based on Timoshenko (Mindlin-Reissner) beam models for the static analysis of arches. The arches are essentially curved Timoshenko beams and the elements used are based on the work of Potts and Day [36] and extended Hinton and co workers.

2.2 Finite Element Formulation of Arch Structures

Consider the MR curved shell element shown in Figure 2.3. The displacement components u_ℓ and w_ℓ, expressed in terms of axes which are tangential and normal, may be written in terms of global displacements u and w as

$$u_\ell = u\cos\alpha + w\sin\alpha$$
$$w_\ell = -u\sin\alpha + w\cos\alpha \tag{2.1}$$

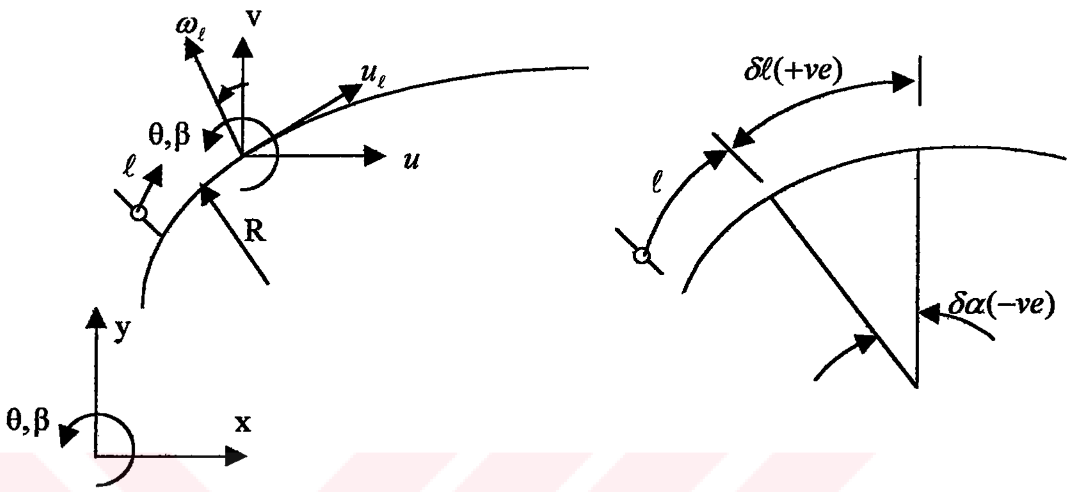


Figure 2.3 Definition of curved Mindlin-Reissner finite elements

where α is shown in Figure 2.3. The radius of curvature R may be obtained from the expression

$$\frac{d\alpha}{d\ell} = -\frac{1}{R} \tag{2.2}$$

The total potential energy for a typical MR is given as

$$\Pi(u_\ell, w_\ell, \theta) = 1/2\int\left([\varepsilon_m]^T D_m\varepsilon_m + [\varepsilon_b]^T D_b\varepsilon_b + [\varepsilon_s]^T D_s\varepsilon_s\right)d\ell - \int w_\ell q\,d\ell - (M\overline{\theta} + N\overline{u}_\ell + \overline{w}_\ell) \tag{2.3}$$

where the axial strain is given by the expression

$$\varepsilon_m = \frac{du_\ell}{d\ell} + \frac{w_\ell}{R} \tag{2.4}$$

or re-writing interms of the global displacements

$$\varepsilon_m = \frac{du}{d\ell}\cos\alpha + \frac{dw}{d\ell}\sin\alpha \tag{2.5}$$

The bending strain or curvature may be written as

$$\varepsilon_b = -\frac{d\theta}{d\ell} \tag{2.6}$$

and the shear strain is given as

$$\varepsilon_s = \frac{dw_\ell}{d\ell} - \theta - \frac{u_\ell}{R} \tag{2.7}$$

or

$$\varepsilon_s = -\theta - \frac{du}{d\ell}\sin\alpha + \frac{dw}{d\ell}\cos\alpha \tag{2.8}$$

Also, note that the axial, flexural and shear rigidities have the form

$$D_m = EA; \qquad D_b = EI; \qquad D_s = \kappa GA \tag{2.9}$$

where E is the elastic modulus, A is the cross-sectional area, I is the moment of inertia, G is the shear modulus and κ is the shear modification factor and is taken as $5/6$ for an arch of rectangular cross-section.

The loading in (2.3) consists of a distributed pressure loading q, as well as couples M, axial forces N or lateral forces Q applied along a circumferential circle at $\ell = \overline{\ell}$. Note that $\overline{u}_\ell$, $\overline{w}_\ell$ and $\overline{\theta}$ are the corresponding displacement and rotation value at $\ell = \overline{\ell}$.t

2.2.1 Finite element idealization

Using n-noded, $C(0)$ line elements, the global displacement parameters u, and θ may be interpolated, within each beam element by the expressions

$$u = \sum_{i=1}^{n} N_i u_i; \qquad w = \sum_{i=1}^{n} N_i w_i; \qquad \theta = \sum_{i=1}^{n} N_i \theta_i \tag{2.10}$$

where u_i, w_i and θ_i are typical nodal displacement degrees of freedom and $N_i(\xi)$ is the shape function associated with node i which, for 2-noded linear elements, have the form

$$N_1 = 1/2(1-\xi); \qquad N_2 = 1/2(1+\xi) \tag{2.11}$$

for 3-noded, quadratic elements

$$N_1 = \xi/2(\xi+1); \qquad N_2 = (1-\xi^2); \qquad N_3 = \xi/2(\xi+1) \tag{2.12}$$

and for 4-noded, cubic elements

$$N_1 = 9/16(1/9-\xi^2)(\xi-1); \qquad N_2 = 27/16(1-\xi^2)(1/\xi-\xi)$$

$$N_3 = 26/16(1-\xi^2)(1/\xi+\xi); \qquad N_4 = -9/16(1/9-\xi^2)(\xi+1) \tag{2.13}$$

These elements are essentially isoparametric so that

$$x = \sum_{i=1}^{n} N_i x_i; \qquad y = \sum_{i=1}^{n} N_i y_i; \qquad t = \sum_{i=1}^{n} N_i t_i \tag{2.14}$$

where x_i, y_i and t_i are typical coordinates and thickness of node i respectively. Note also that the Jacobian

$$J = \frac{d\ell}{d\xi} = \left[\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2\right]^{1/2}; \qquad d\ell = J d\xi \tag{2.15}$$

where

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i; \qquad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} y_i. \tag{2.16}$$

Also note that

$$\sin\alpha = \frac{dy}{d\xi}\frac{1}{J}; \qquad \cos\alpha = \frac{dx}{d\xi}\frac{1}{J} \tag{2.17}$$

and

$$\frac{dN_i}{d\ell} = \frac{dN_i}{d\xi}\frac{1}{J}. \tag{2.18}$$

The axial strains ε_m may then be expressed as

$$\varepsilon_m = \sum_{i=1}^{n} \mathbf{B}^e_{mi}\mathbf{d}^e_i \tag{2.19}$$

where

$$\mathbf{B}^e_{mi} = [(\partial N_i/\partial\ell)\cos\alpha \quad (\partial N_i/\partial\ell)\sin\alpha \quad 0] \tag{2.20}$$

and

$$\mathbf{d}^e_i = [u_, , w_i, \theta_i]^T. \tag{2.21}$$

The flexural strain or curvatures ε_b can be written as

$$\varepsilon_b = \sum_{i=1}^{n} \mathbf{B}^e_{bi}\mathbf{d}^e_i \tag{2.22}$$

where

$$\mathbf{B}^e_{bi} = [(0 \quad 0 \quad -dN_i/d\ell] \tag{2.23}$$

and the shear strain ε_s is approximated as

$$\varepsilon_s = \sum_{i=1}^{n} \mathbf{B}_{si}^e \mathbf{d}_i^e \tag{2.24}$$

where

$$\mathbf{B}_{si}^e = \left[-(dN_i/d\ell)\sin\alpha \quad (dN_i/d\ell)\cos\alpha \quad -N_i \right] \tag{2.25}$$

Thus, neglecting circumferential line loads and couples, the contribution to the total potential from element *e* may be expressed as

$$\Pi^e = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2} [\mathbf{d}_i^e]^T \mathbf{K}_{ij}^e \mathbf{d}_j^e - \sum_{i=1}^{n} [\mathbf{d}_i^e]^T \mathbf{f}_i^e \tag{2.26}$$

where the submatrix of the strip stiffness matrix $[\mathbf{K}_{ij}^e]$ linking nodes *i* and *j* has the form

$$\mathbf{K}_{ij}^e = \int_{-1}^{+1} \{\mathbf{B}_{mi}^T D_m \mathbf{B}_{mj} + \mathbf{B}_{bi}^T D_b \mathbf{B}_{bj} + \mathbf{B}_{si}^T D_s \mathbf{B}_{sj}\} J d\xi \tag{2.27}$$

and the consistent nodal force vector associated with node *i* is written as

$$\mathbf{f}_i^e = \begin{bmatrix} \int_{-1}^{+1} N_i q \sin\alpha J d\xi \\ \int_{-1}^{+1} N_i q \cos\alpha J d\xi \\ 0 \end{bmatrix} \tag{2.28}$$

2.2.2 Strain energy evaluation

The Strain Energy (SE) of the FE solution $\|\hat{\mathrm{W}}\|^2$ for the beam is computed as the sum of the membrane, bending and shear SEs

$$\|\hat{\mathrm{W}}\|^2 = \|\hat{\mathrm{W}}\|_b^2 + \|\hat{\mathrm{W}}\|_m^2 + \|\hat{\mathrm{W}}\|_s^2$$

$$\|\hat{\mathrm{W}}\|_b^2 \approx \int_\Omega [\hat{\sigma}_b]^{\mathrm{T}} \mathrm{D}_b^{-1} \hat{\sigma}_b d\Omega$$

$$\|\hat{\mathrm{W}}\|_m^2 \approx \int_\Omega [\hat{\sigma}_m]^{\mathrm{T}} \mathrm{D}_m^{-1} \hat{\sigma}_m d\Omega$$

$$\|\hat{\mathrm{W}}\|_s^2 \approx \int_\Omega [\hat{\sigma}_s]^{\mathrm{T}} \mathrm{D}_s^{-1} \hat{\sigma}_s d\Omega \tag{2.29}$$

in which $\hat{\sigma}_b = [M_\ell]^T$ contains the FE bending moments, $\hat{\sigma}_m = [N_\ell]$ contains the FE membrane forces and $\hat{\sigma}_s = [Q_\ell]$ is the FE shear force.

2.3 Examples of Static Analysis

2.3.1 Cantilever beam

Problem definition: The first example is a cantilever beam with a square cross-section as shown in Figure 2.4. The following material properties are used: elastic modulus $E = 200\ GPa$ and Poisson's ratio $\nu = 0.3$.

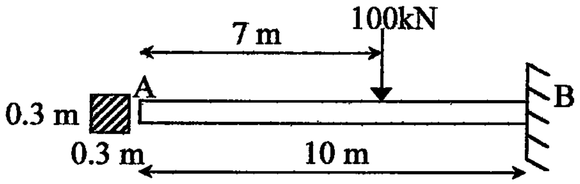


Figure 2.4 Cantilever beam

Discussion of results: The results obtained using a fine mesh of 4-noded cubic elements are also compared with exact solution. Rotation at point A and reactions at point B are respectively $3.33x10^{-3}, -300.0\,kNm, 100\,kN$ are obtained from exact solution and FE solution.

2.3.2 Portal frame

Problem definition: The next example is portal frame with a square $0.3 \times 0.3\ m$ cross-section, which is simple supported as shown in Figure 2.5.

Discussion of results: Used material properties are elastic modulus $E = 200\ GPa$ and Poisson's ratio $\nu = 0.3$. In the analysis 2-noded linear Timoshenko elements is used with 173 DOF. The results are tabulated in Table 2.1 and compared with non-commercial program CMEFRAME. Remarkably good agreement is obtained.

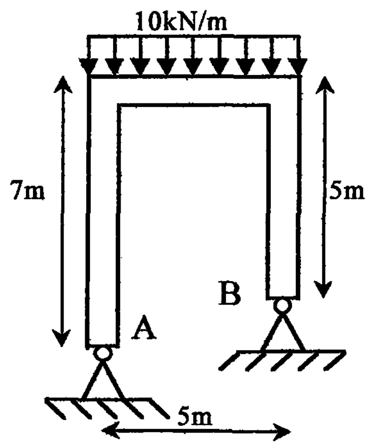


Figure 2.5 Portal frame

Table 2.1 Comparisons of member forces of portal frame

points	reactions	present study	CMEFRAME
A	horizontal (*kN*)	+1.843	+1.850
	vertical (*kN*)	+25.737	+25.740
	rotation (*rad.*)	$+1.668\times10^{-4}$	$+1.676\times10^{-4}$
B	horizontal (*kN*)	-1.843	-1.850
	vertical (*kN*)	+24.262	+24.260

2.3.3 Arches of uniform cross-section

Problem definition: This example involves analysis of square arches under two different load condition, which have been studied by Roark [37]. An arch of uniform cross-section with opening angle 120° is considered. The arch, which is radius of curvature of R = 10 m and 0.3×0.3 m cross-section, has the following material properties: elastic modulus $E = 200$ GPa, Poisson's ratio $\nu = 0.3$.

In the analysis two different loading cases is considered;

a) Point load at the crown
b) Uniformly normal loading

shown in Figure 2.6(a)-(b).

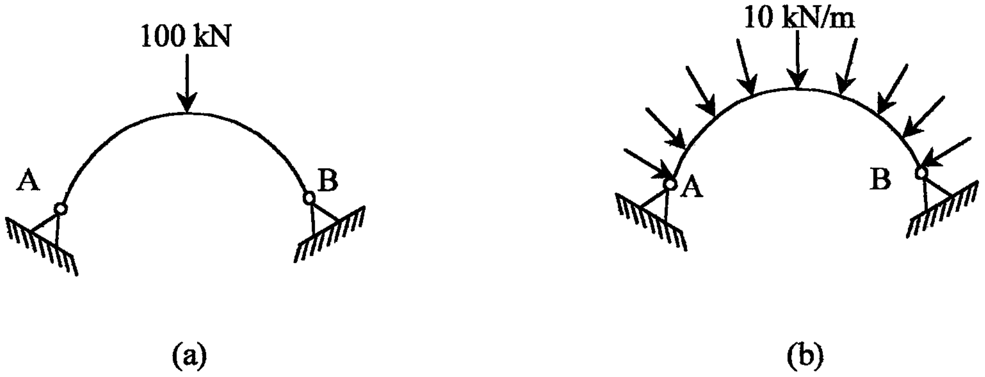


Figure 2.6 Loading conditions of uniform cross-section arch

Discussion of results: The results are tabulated in Table 2.2 and compared with results obtained from Roark [37]. Close agreement between results can be observed.

Table 2.2 Comparisons of member forces of uniform cross-section arch

		load case a		load case b	
points	reactions	present study	Roark [37]	present study	Roark [37]
A	horizontal (*kN*)	63.67	63.06	49.95	49.96
	rotation (*rad.*)	0.001	0.001	0.121×10^{-4}	0.111×10^{-4}

2.3.4 Arches of uniform cross-section

Problem definition: The next example involve a series of arches with rectangle cross-sections, which have been studied by Litewka and Rakowski [ ]. An arch of uniform cross-section with opening angle ω = 120° is considered. The arch, which is radius of curvature of R = 4 m and 0.6×0.4 m cross-section, has the following material properties: elastic modulus $E = 30$ GPa, Poisson's ratio $\nu = 0.17$.

The analysis are carried out for two different fixity conditions and three different loading cases;

i. Clamped-clamped arch

a) Vertical point load at the crown,

b) Horizontal point load at the crown,

c) Moment at the crown.

ii. Simple support arch

d) Vertical point load at the crown,

e) Horizontal point load at the crown,

f) Moment at the crown.

shown in Figure 2.7.

Discussion of results: The results of the normalized displacements: u_c/l, w_c/l and θ_c/ω obtained using a fine mesh of 4-noded elements are also compared with solution obtained by Litewka and Rakowski [5], in Table 2.3 and Table 2.4.

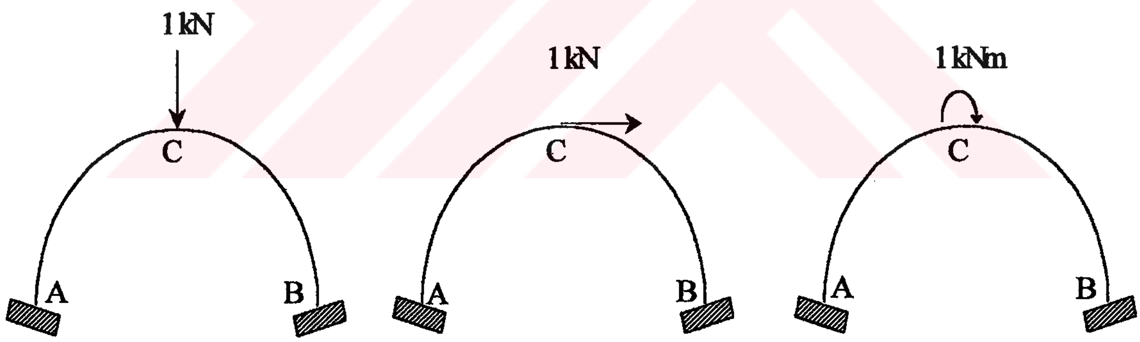


Figure 2.7 Loading conditions of uniform cross-section clamped arch

Table 2.3 Comparisons of clamped-clamped arch results

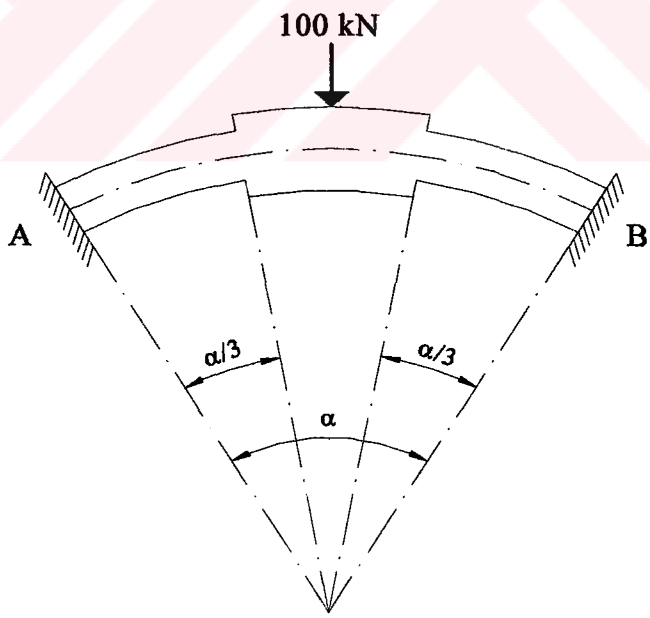
	u_c/l		w_c/l		θ_c/ω	
	present	Ref [5]	present	Ref [5]	present	Ref [5]
a	6.192×10^{-10}	0.000	2.516×10^{-7}	2.488×10^{-7}	2.164×10^{-8}	0.000
b	1.236×10^{-7}	1.252×10^{-7}	6.192×10^{-10}	0.000	3.621×10^{-7}	3.796×10^{-7}
c	9.095×10^{-8}	9.490×10^{-8}	5.436×10^{-9}	0.000	1.079×10^{-6}	1.082×10^{-6}

Table 2.4 Comparisons of simple support arch results

	u_c/l		w_c/l		θ_c/ω	
	present	Ref. [5]	present	Ref. [5]	present	Ref. [5]
a	9.243 x10^{-9}	0.000	2.799 x10^{-7}	3.047x10^{-7}	3.741 x10^{-8}	0.000
b	2.765 x10^{-7}	2.884 x10^{-7}	9.240 x10^{-9}	0.000	7.770 x10^{-7}	8.064 x10^{-7}
c	1.952 x10^{-7}	2.016 x10^{-7}	9.396 x10^{-9}	0.000	1.362 x10^{-6}	1.361 x10^{-6}

2.3.5 Arches of non-uniform cross-section

Problem definition: The last example is an arch of non-uniform cross-section with opening angle 40°. The arch, which is radius of curvature of R = 100 *m*, 0.8×1.0 *m* and 1.0×1.0 *m* cross-sections and elastic modulus E = 2.85056 *GPa*. Shown in Figure 2.8.



.**Figure 2.8** Geometry of non-uniform arch

Discussion of results: In the analysis 4-noded linear elements are used with 51 DOF. The results are tabulated in Table 2.5 and compared with results obtained from SAP(2000). Again, excellent agreement in the results is obtained.

Table 2.5 Comparisons of member forces of non-uniform arch

points	reactions	present study	SAP 2000
A	horizontal (*kN*)	+24.655	+24.657
	vertical (*kN*)	+50.000	+50.000
B	horizontal (*kN*)	-24.655	-24.657
	vertical (*kN*)	+50.000	+50.000
C	vertical displacement (*cm*)	-8.413	-8.411

CHAPTER3

OPTIMIZATION ALGORITHM

3.1 Introduction

Arch structures provide economical solutions for crossing large spans. Compared with beams smaller cross sections can be used in arches as the axial forces are dominant. Further economy can be achieved in design of these structures by using structural optimization. In structural design it is necessary to obtain an appropriate geometric shape for the structure so that it can carry the imposed loads safely and economically. The optimisation process uses the results of the finite element analysis to predict the optimum shape. In Structural Shape Optimization (SSO), the original mesh is gradually changed as the iterative optimization method proceeds and the structural geometry changes. SSO tools can be developed by the efficient integration of structural shape definition procedures, automatic mesh generation, structural analysis and mathematical programming methods.

3.2 Mathematical Definition of Optimization Problem

The optimization problem can be stated in the formal mathematical language of nonlinear programming as:

$$\text{minimize} : F(\mathbf{s}) \tag{3.1}$$

subject to

$$g_j(\mathbf{s}) \leq 0$$

$$h_k(\mathbf{s}) = 0$$

$$s_i^l \leq s_i \leq s_i^u$$

in which $F(\mathbf{s})$ is the objective function to be minimized, $g_j(\mathbf{s})$ is behavioral constraint, $h_k(\mathbf{s})$ is an equality constraint and $s_i^l \leq s_i \leq s_i^u$ are geometric constraints. The subscripts j, k and i denote the number of behavioral constraints, equality constraints and design variables respectively. The terms s_i^l and s_i^u refer to the specified lower and upper bounds on the design variables.

Table 3.1 summarizes the commonly used design variables, objective functions and constraints functions in structural shape optimization. Note that, in general, the functions F, g_j and h_k may all be nonlinear implicit functions of the design variables s.

Table 3.1 Design variables, objective functions and constraints used in structural shape optimization

Design variables s

- Coordinates of key points s_k
- Thickness at key points t_k

Objective functions $F(\mathbf{s})$

- Weight
- Strain energy
- Error energy
- Stress leveling

Constraint functions $g(\mathbf{s})$

- Stress constraint
- Displacement constraint
- Volume constraint

In static situation minimization of strain energy, weight, error energy or stress leveling subject to stress, displacement and/or weight constraints. In addition explicit geometrical constraints are imposed on the design variables to avoid impractical geometries. For example, a minimum element thickness is defined to avoid zero or negative element thickness values. It is worth mentioning here that the objective function and the constraint hull may be non-convex and there fore local optima may exist.

3.3 Structural Shape Optimization Algorithm

The basic algorithm for structural shape optimization is shown in Figure 3.1.

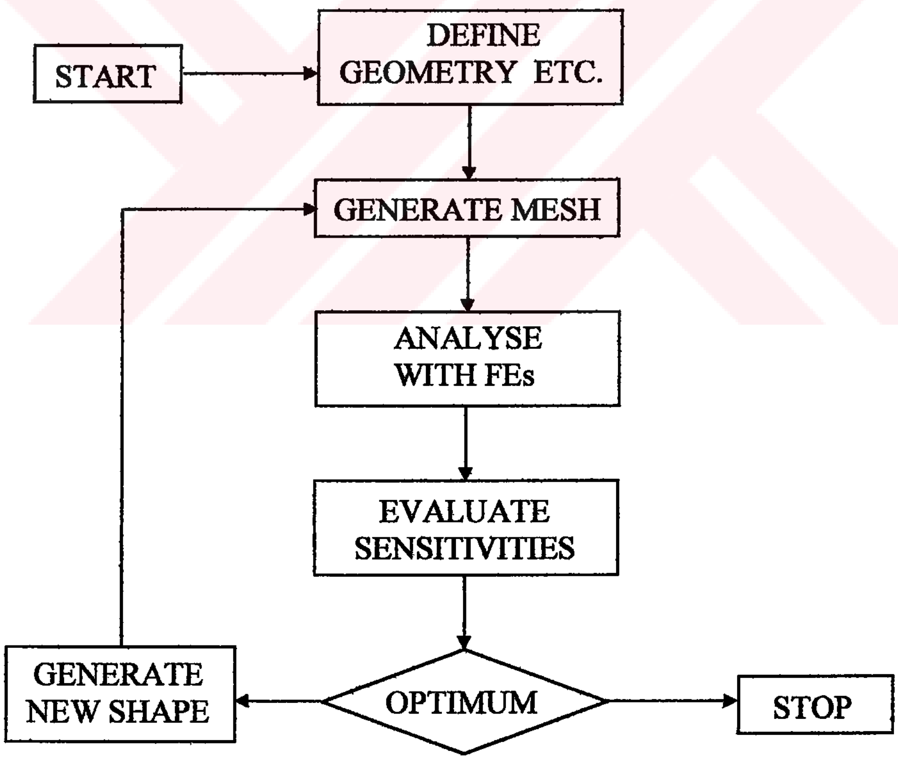


Figure 3.1 Basic algorithm to structural shape optimization

If we present an overview of a typical structural shape optimization procedure which is based on the following algorithm:

1. Define the optimization problem including the objective function, constraints and design variables. The objective function and constraint functions are highly nonlinear with respect to shape variables in most cases.
2. The initial structural shape of the arch is defined using cubic spline segments with the coordinates specified at the certain key points. The thickness distribution may also be defined using cubic splines with thickness values specified at the key points. Design variables $s^{(1)} = [s_1^{(1)}, s_2^{(1)}, \ldots, s_n^{(1)}]^T$ may include the coordinates and thickness of some specific points. (The superscript denotes the design number in other words the optimization iteration number.)
3. Main problem in the shape optimization is the geometry modeling and finite element mesh generation. This may be achieved with an automatic mesh generator for a prescribed mesh density.
4. In the FE analysis the arch is modeled using linear, quadratic or cubic, MR, curved, variable thickness elements. Carry out a FE analysis of current design $s^{(c)}$ and evaluate the objective function and constraints. Details of the FE formulation is given in Chapter 2.
5. In static situation the sensitivities of strain energy, displacement and stress resultant of the current design to small changes in the design variables are evaluated. Methods for evaluating the sensitivities may be semi analytical or based on finite differences.
6. Modify the current design and evaluate the design changes $\Delta s^{(c)}$ using a suitable optimization algorithm, such as sequential quadratic programming (SQP).
7. Check the new design changes $\Delta s^{(c)}$. If the design changes $\Delta s^{(c)}$ are non-zero then update the design vector to

$$s^{(c+1)} = s^{(c)} + \Delta s^{(c)} \tag{3.2}$$

and new values of the design variables are sent to mesh generator and the whole sequence of operations is repeated. Otherwise stop.

3.4 Geometry Definition

3.4.1 Structural shape definition

The way used to describe the shape of the structure is the key element in the process of obtaining the optimum shape. The definition and control of the geometric model of the structure is a complex task. Detail information is given in ref [38].

Three methods are used:
(1) boundary nodes are used for shape representation;
(2) boundary shape is described by piecewise polynomials;
(3) boundary shape is described by spline or spline blending functions.
Spline, as a shape representation, can eliminate the problem derived by using high order polynomials to describe the boundary. Because spline functions are composed of low order polynomial pieces which are combined to maximize smoothness [21].

The geometric representation of arch structure is shown in Figure 3.2. which is formed by either a single or an assembly of segments. Each segment is a cubic spline curve passing through certain key points. Some key points are common to different segments at their points of intersection. To represent a straight line the analysts have to provide a minimum of two key points lying on the segment and three points to represent a curved line.

Another important aspect in shape optimization is the number of key points used to define the shape of the structure. For arch structure, the more key points used the better the representation of geometry. However, it should be noted that in SSO procedures increasing the number of key points leads to an increase in the number of design variables and is likely to lead to greater computational expense.

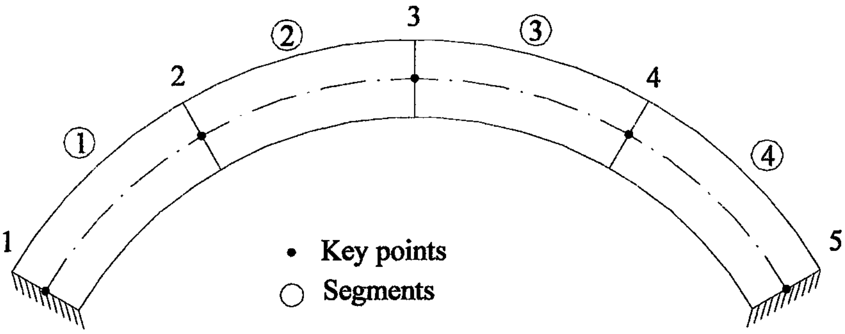


Figure 3.2 Geometric representation of arch structure

3.4.2 Structural thickness definition

The thickness of the arch is specified at some or all of the key points of the structure and then interpolated using cubic splines or lower order functions. Only uniform thickness variations in structural shape optimization over constrain the optimization process and does not give the greatest opportunity for objective function improvement.

3.4.3 Selection of constraint points

The objective is to minimize the weight of the arch by reducing the depth of the cross section. However, there may be some limitations on the displacements to consider while the optimum depth of the cross section is searched. Hence, there is an objective such as the minimization of the weight of the arch to find the optimum values of the depths, and there are some restrictions like the upper bounds of the displacements and stresses, and also the lower bound of the depths. In SSO procedures where remeshing is performed at every iteration the function cannot be associated with the nodes since their number and position do not remain constant. Therefore, apart from being used to represent the shape and thickness, the key points are also used as stress sampling points the verify whether the stress constraint has been satisfied or not.

Although this approach is satisfactory in most cases, it can be dangerous, since the maximum value of the stress may not occur at a key point. To avoid this potential problem, the points where the maximum stress occur are also taken as constraint points in addition to the key points. This approach has been found to be reliable.

3.5 Mesh Generation

The second main problem in the shape optimization is the finite element mesh generation. A fixed finite element model, like in the sizing optimization, is no longer valid to assure the accuracy of the structural analysis as the shape of the boundary changes, since the accuracy of various portions of the finite element mesh will change [21].

One general way of handling this problem is to take advantage of the automated mesh generation in the computer program of the shape optimization. Mesh generation should be robust, versatile and efficient. Here, we use a mesh generator which incorporates a remeshing facility to allow for the possibility of refinement. It also allows for a significant variation in mesh spacing throughout the region of interest. The mesh generator can generate meshes of two, three and four noded elements. To control the spatial distribution of element sizes or mesh density throughout the domain, it is convenient to specify the mesh density at a sequence of points in the structure. Figure 3.3 shows a mesh of arch structure.

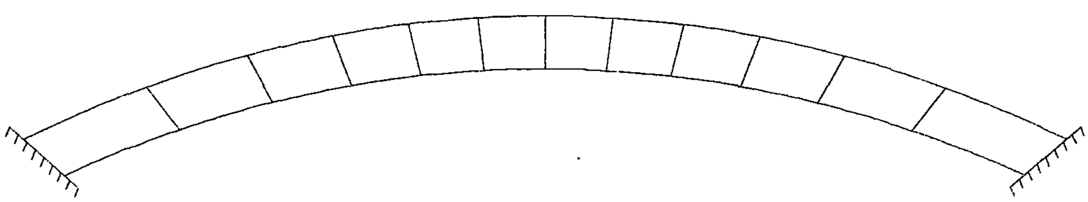


Figure 3.3 Mesh representation of arch structure

The mesh density is a piecewise linear function of the values of mesh size δ at some points along the midsurface of the structure. At the initial stages of the analysis, mesh density values given at the two end points of each segment will be sufficient if only a uniform or a linearly varying mesh density is required [39].

3.6 Structural Analysis

In the FE analysis the arch is modeled using linear, quadratic or cubic, MR, curved, variable thickness elements. These belong to a family introduced by Potts and Day [36] and subsequently extended by Hinton et al [35]. They are well tried and tested good performers and to avoid locking or overstiff behaviour for shear stiff arches, reduced integration of the stiffness matrices with a one, two- and three-point Gauss-Legendre rule is adopted for the linear, quadratic and cubic elements respectively. Theory and implementation of FE method for static analyses are given in Chapters 2 respectively.

3.7 Sensitivity Analysis

Design sensitivity analysis, that is, the calculation of quantitative information on how the response of a structure is affected by changes in the variables that define its shape, is a fundamental requirement for the shape optimization. We calculate the sensitivities of items such as strain energy. Methods for evaluating the sensitivities may be purely analytical or can be based on finite differences in which case the choice of the step size may be crucial. Alternatively, we may use semi-analytical methods which are partly analytical and partly based on finite differences.

Sensitivity analysis consists of the systematic calculation of the derivatives of the response of the FE model with respect to parameters characterizing the model i.e. the design variables which may be length, thickness or shape. FE structural analysis programs are used to calculate the response quantities such as displacements, stresses etc. The first partial derivatives of the structural response quantities with respect to the shape (or other) variables provide the essential information required to couple mathematical programming methods and structural analysis procedures. The sensitivities provide the mathematical programming algorithm with search directions for optimum solutions.

In the present study, both the finite difference and semi analytical methods are used to calculate sensitivities. The finite difference method uses a difference formula to numerically approximate the derivatives. The semi-analytical method which was originally proposed by Zienkiewicz and Campbell [20] is quite popular in shape optimization and it combines the analytical and finite difference methods. The derivatives of some quantities are evaluated using finite difference whereas for the others the analytical method is adopted. These two methods are accurate, computationally efficient and sensitive to roundoff and truncation errors associated with step size.

In the FE displacement approach the equilibrium equations

$$\mathbf{K}\mathbf{d} = \mathbf{f} \tag{3.3}$$

Differentiating (3.3) with respect to design variable s_k we have

$$\mathbf{K}\frac{\partial \mathbf{d}}{\partial s_k} + \frac{\partial \mathbf{K}}{\partial s_k}\mathbf{d} = \frac{\partial \mathbf{f}}{\partial s_k} \tag{3.4}$$

On rearranging equation (3.4) we get

$$\mathbf{K}\frac{\partial \mathbf{d}}{\partial s_k} = \mathbf{f}_\mathbf{k}^* \tag{3.5}$$

where

$$\mathbf{f}_\mathbf{k}^* = (\frac{\partial \mathbf{f}}{\partial s_k} - \frac{\partial \mathbf{K}}{\partial s_k}\mathbf{d}) \tag{3.6}$$

In general, if we have several design variables s_k we get

$$\mathbf{K}(\frac{\partial \mathbf{d}}{\partial s_1}, \frac{\partial \mathbf{d}}{\partial s_2}, \ldots, \frac{\partial \mathbf{d}}{\partial s_n}) = (\mathbf{f}_1^*, \mathbf{f}_2^*, \ldots, \mathbf{f}_n^*) \tag{3.7}$$

Thus, to obtain the derivatives of the displacement with respect to each design variable we should calculate $\partial \mathbf{K} / \partial s_k$ and $\partial \mathbf{f} / \partial s_k$. These quantities can be computed either analytically, semi - analytically or using a global finite difference method.

Semi – analytical approach can be used in which the derivatives $\partial \mathbf{K} / \partial s_k$ and $\partial \mathbf{f} / \partial s_k$ are approximated by a forward finite difference scheme [38]. In this case we store the factored **K** matrix during the solution of (3.3) and calculate the pseudo- load matrix $\mathbf{f}_k^*$ i.e. we compute the terms $\partial \mathbf{f} / \partial s_k$ and $\partial \mathbf{K} / \partial s_k$. These derivatives are computed by recalculating the new $\mathbf{K}(s_k + \Delta s_k)$, and $\mathbf{f}(s_k + \Delta s_k)$ for a small perturbation Δs_k of the design variables and by applying the forward finite difference scheme so that

$$\frac{\partial \mathbf{K}}{\partial s_k} \approx \frac{\mathbf{K}(s_k + \Delta s_k) - \mathbf{K}(s_k)}{\Delta s_k} \tag{3.8}$$

$$\frac{\partial \mathbf{f}}{\partial s_k} \approx \frac{\mathbf{f}(s_k + \Delta s_k) - \mathbf{f}(s_k)}{\Delta s_k} \tag{3.9}$$

Derivative of displacements and stress resultants: To get $\partial \mathbf{d} / \partial s_k$ and $\partial \sigma / \partial s_k$ global finite difference method is used and following expressions may be written

$$\frac{\partial \mathbf{d}}{\partial s_k} \approx \frac{\mathbf{d}(s_k + \Delta s_k) - \mathbf{d}(s_k)}{\Delta s_k} \tag{3.10}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial s_k} \approx \frac{\boldsymbol{\sigma}(s_k + \Delta s_k) - \boldsymbol{\sigma}(s_k)}{\Delta s_k} \tag{3.11}$$

where Δs_i is step size, $\mathbf{d}(s_k + \Delta s_k)$ is evaluated by solving

$$\mathbf{K}(s_k + \Delta s_k)\mathbf{d}(s_k + \Delta s_k) = \mathbf{f}(s_k + \Delta s_k) \tag{3.12}$$

and $\sigma(s_k + \Delta s_k)$ is found from

$$\sigma(s_k + \Delta s_k) = \mathbf{D}(s_k + \Delta s_k)\mathbf{B}(s_k + \Delta s_k)\mathbf{d}(s_k + \Delta s_k) \tag{3.13}$$

Derivative of volume: The volume derivative is calculated using a forward finite difference approximation

$$\frac{\partial V}{\partial s_k} \approx \frac{V(s_k + \Delta s_k) - V(s_k)}{\Delta s_k} \tag{3.14}$$

where the volume V of the whole structure (or cross-sectional area of the structure may also be used) can be calculated by adding the volumes of numerically integrated FEs.

3.8 Mathematical Programming

Various numerical algorithms have been applied to solve the shape optimization of structures. They can be divided into the following categories:

- method of moving asymptotes (MMA),
- sequential quadratic programming (SQP),
- sequential linear programming (SLP),
- penalty function methods,
- feasible direction methods.

In the present work we use SQP which is considered to be quite powerful for a wide class problems.

3.8.1 Sequential quadratic programming

The SQP, the principal concept is to use a variable metric algorithm to create an approximation to the Lagrangian function. This function is used in a quadratic programming subproblem to find a search direction in design space which will drive the design to the satisfaction of the Kuhn-Tucker necessary conditions for optimality. A one dimensional search is then performed to achieve the design improvements [38].

The optimization process usually begins with a proposed design s_k^0, provided as input. The design is then typically updated by modifying s_k as

$$s_k^q = s_k^{q-1} + \alpha r_k^q \tag{3.15}$$

where q is the iteration number. The vector $\mathbf{r} = [r_1, r_2, \ldots.]^T$ is the search direction and α is a scalar move parameter.

No effort has been made to study the mathematical programming methods used for SSO procedures and algorithm is used here essentially as a 'black box'.

CHAPTER 4

OPTIMIZATION EXAMPLES

4.1 Introduction

In this chapter a series of static examples will be considered which illustrate the algorithm for shape and thickness optimization of arch structures. The design variables are used to define the shape or the thickness variation or both.

Only linear elastic behavior is considered and the optimized shape and thickness distributions are not checked for buckling under the given set of loads. Although some of the optimal shapes of the structures obtained may look impractical, they could serve as a guide to designing practical shapes and as an educational tool.

4.2 Strain Energy Minimization of Arch Structure Under the Point Load

Problem Definition: The geometry of the structure is shown in Figure 4.1. The arch has a radius of 10 *m* and opening angle 120°. The following material properties are assumed: elastic modulus $E = 200$ *GPa,* Poisson's ratio $\nu = 0.3$.

The shape of uniform arch is defined using two segments and five key points. Strain energy minimization with a constraint that the total material volume of structure is limited to 1.9 m^3. A total of two shape and three thickness design variables as shown in Figure 4.2 are considered. Use is made of design variables linking procedures to maintain symmetry of the structure.

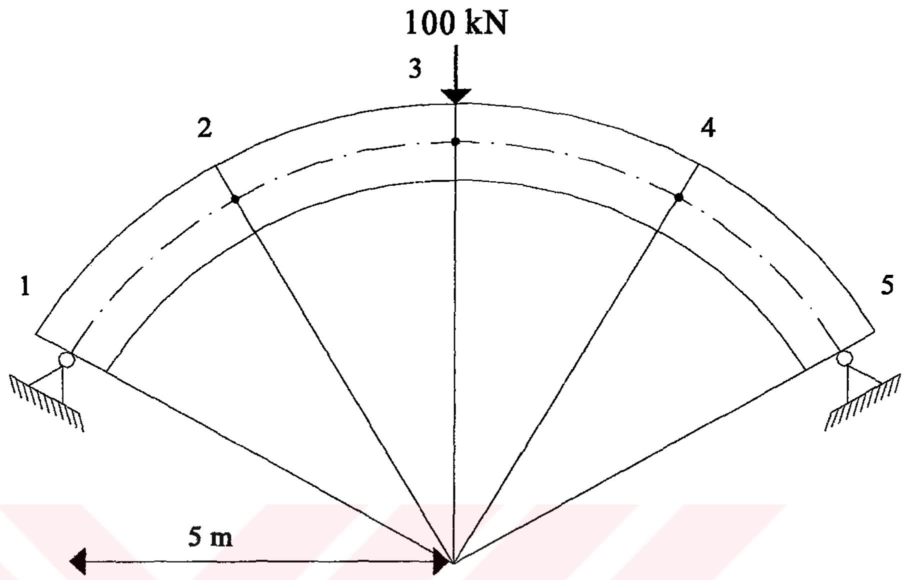


Figure 4.1 Geometry of uniform arch

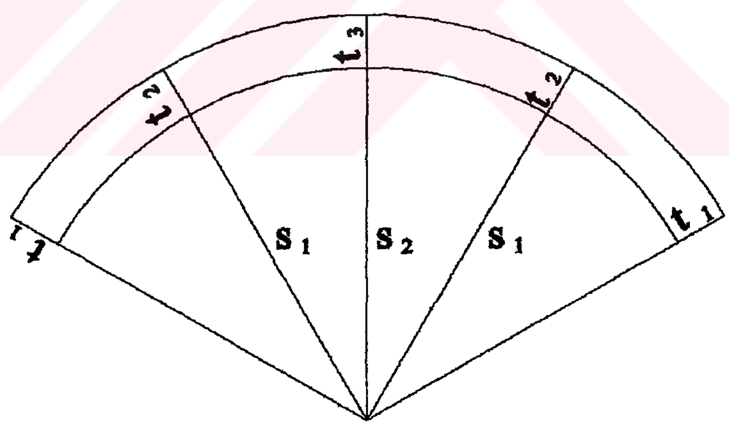


Figure 4.2 Location of design variables

Optimization is carried out for four types of design variables.

i-) only shape design variables s_1,s_2

ii-) only thickness design variables t_1,t_2,t_3

iii-) both shape and thickness design variables are considered together

iv-) only thickness design variables t_1,t_2,t_3 after type *i*

Discussion of results: The analysis is carried out using cubic elements and 53 DOF. Table 4.1-4.4 presents initial and optimal design variables together with their bounds, Table 4.5 presents initial, optimal values and percent improvements of total strain energy and Table 4.6 presents % contributions to total strain energy. We used thickness design variables after shape optimization so highest improvement is obtained in type *iv*.

Table 4.1 Uniform arch: values of the design variables (Type *i*-only shape)

design variables				opt. design variables
type	min.	initial	max.	
s_1	5.0	10.0	15.0	8.99
s_2	5.0	10.0	15.0	10.78

Table 4.2 Uniform arch: values of the design variables (Type *ii*-only thickness)

design variables				opt. design variables
type	min.	initial	max.	
t_1	0.1	0.3	0.8	0.20
t_2	0.1	0.3	0.8	0.23
t_3	0.1	0.3	0.8	0.64

Table 4.3 Uniform arch: values of the design variables (Type *iii*-both thickness and shape)

design variables				opt. design variables
type	min.	initial	max.	
s_1	5.0	10.0	15.0	9.50
s_2	5.0	10.0	15.0	10.22
t_1	0.1	0.3	0.8	0.10
t_2	0.1	0.3	0.8	0.25
t_3	0.1	0.3	0.8	0.7

Table 4.4 Uniform arch: values of the design variables
(Type *iv*-only thickness after type *i*)

design variables				opt. design variables
type	min.	initial	max.	
t_1	0.1	0.3	0.8	0.31
t_2	0.1	0.3	0.8	0.32
t_3	0.1	0.3	0.8	0.31

Table 4.5 Initial-optimal values and % improvement of total strain energy

	strain energy ($\times 10^{-4}$)		
type	initial	optimum	% decrease
i	3639.027	93.711	97.4
ii	3639.027	2282.044	37.3
iii	3639.027	924.429	74.6
iv	3639.027	89.420	97.6

Table 4.6 % Contributions to total strain energy

% contributions to strain energy				
case	shape	membrane	bending	shear
i	initial	1.911	97.470	0.619
	optima	99.923	0.077	0.000
ii	Initial	1.911	97.470	0.619
	optima	2.819	96.381	0.800
iii	initial	1.911	97.470	0.619
	optima	9.970	89.258	0.771
iv	initial	99.923	0.077	0.000
	optima	99.925	0.075	0.000

4.3 Strain Energy Minimization of Beam

Problem Definition: The beam which is of length 10 *m* and 0.3×0.3 *m* cross-section, has the following material properties: elastic modulus E = 200 *GPa,* Poisson's ratio $\nu = 0.3$.

The shape of the beam is defined using two segments and eleven key points. A total of five shape design variables as shown in Figure 4.3 are considered.

In the analysis two different loading case;

i-) Point load at the crown
ii-) Uniformly normal loading
shown in Figure 4.4.

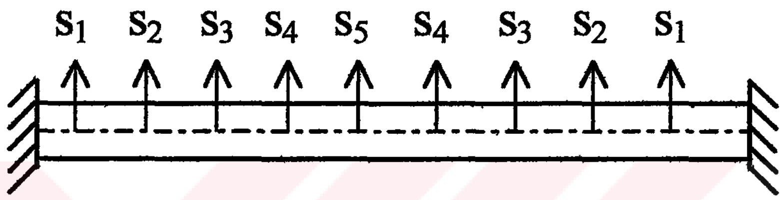


Figure 4.3 Location of design variables

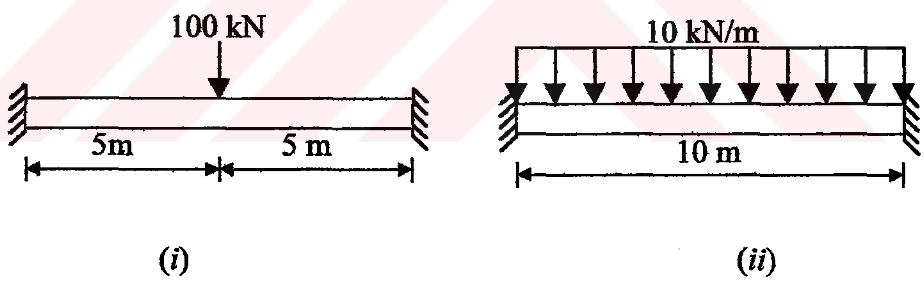


Figure 4.4 Geometry of beams

Discussion of results: Table 4.7 and 4.8 presents initial and optimal design variables together with their bounds. Table 4.9 presents initial, optimal values and percent improvements of total strain energy and Table 4.10 presents %contributions to total strain energy. As you can see from optimum shape of the beam is becoming an arch structure, because of structural advantages of arch structure. The optimum shape of structures are shown in Figure 4.5 and 4.6.

Table 4.7 Beam: values of the design variables (case *i*)

design variables				opt. design variables
type	min.	initial	max.	
s_1	0.01	0.0	1.5	0.2266
s_2	0.01	0.0	1.5	0.5910
s_3	0.01	0.0	1.5	0.9088
s_4	0.01	0.0	1.5	1.2733
s_5	0.01	0.0	1.5	1.5000

Table 4.8 Beam: values of the design variables (case *ii*)

design variables				opt. design variables
type	min.	initial	max.	
s_1	0.01	0.0	1.5	0.5107
s_2	0.01	0.0	1.5	1.0010
s_3	0.01	0.0	1.5	1.3010
s_4	0.01	0.0	1.5	1.5000
s_5	0.01	0.0	1.5	1.5000

Table 4.9 Initial-optimal values and % improvement of total strain energy

	strain energy ($\times 10^{-4}$)		
case	initial	optimum	% decrease
i	3901.358	157.341	95.9
ii	1043.251	44.033	95.8

Table 4.10 % Contributions to total strain energy

% contributions to strain energy				
case	shape	membrane	bending	shear
i	initial	0.000	98.889	1.111
	optima	89.279	9.289	1.432
ii	initial	0.000	98.615	1.385
	optima	91.858	7.145	0.997

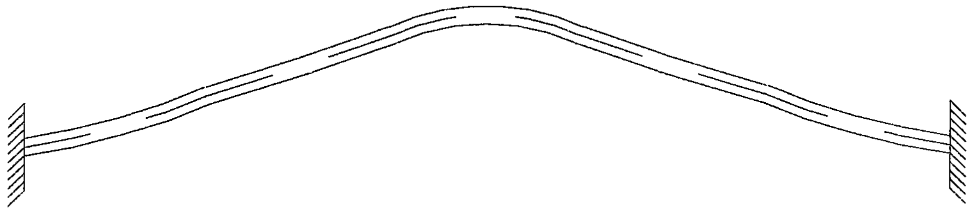


Figure 4.5 Optimum shape of case *i*

Figure 4.6 Optimum shape of case *ii*

4.4 Volume Minimization of Arch Structure Under the Point Load

Problem Definition: The next example involve two arches with rectangle cross-sections, which have been studied by Uzman and Daloğlu [24].The depth of arch is taken as 5 *m*. The modulus of elasticity of the material E = 20700 kN/cm^2 , the allowable stress σ_{all} = 20.25 kN/cm^2 and Poisson's ratio ν = 0.3. Only a displacement constraint, the vertical displacement at the crown, is limited to 0.5 *cm*.

The shape of uniform arch is defined using two segments and five key points. The geometry of the structure and the location of three thickness design variables are shown in Figure 4.7. Use is made of design variables linking procedures to maintain symmetry of the structure.

In the analysis two different fixity conditions;

i-) Clamped-clamped arch

ii-) Simple support arch

Discussion of results: Table 4.11 and Table4.12 presents initial and optimal design variables together with their bounds and Table 4.13 presents comparisons of initial, optimal values and percent improvements of volume. The results of optimizations compare very well with those obtained by Uzman and Daloğlu [24]. For case(*i*) the problem of volume minimization 36.9 percent decrease and for case(*ii*) 20.4 percent decrease is obtained.

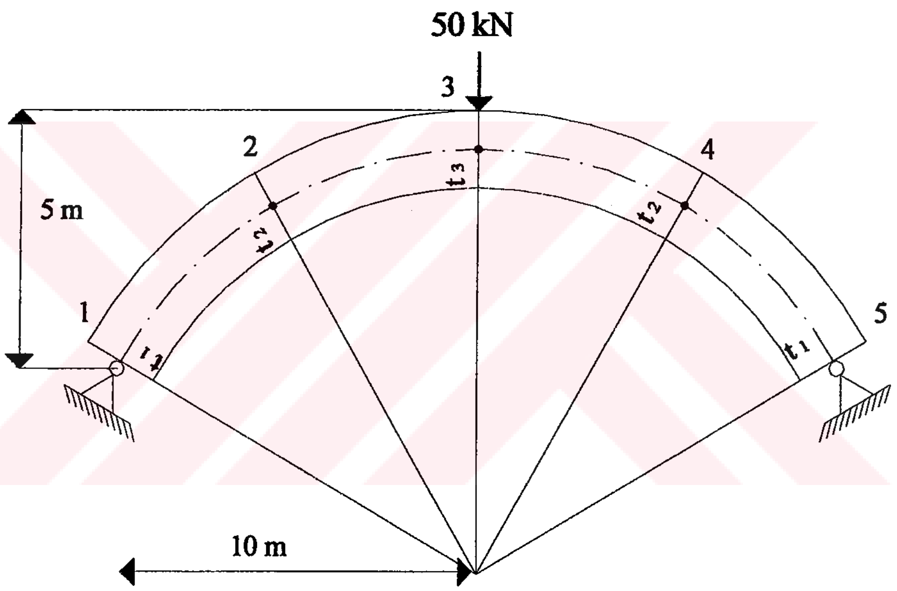


Figure 4.7 Geometry of arch and location of design variables

Table 4.11 Arch: values of the design variables (case *i*)

design variables				opt. design variables
type	min.	Initial	max.	
t_1	0.08	0.35	0.7	0.08
t_2	0.08	0.35	0.7	0.25
t_3	0.08	0.35	0.7	0.31

Table 4.12 Arch: values of the design variables (case *ii*)

design variables				opt. design variables
type	min.	initial	max.	
t_1	0.08	0.35	0.7	0.16
t_2	0.08	0.35	0.7	0.31
t_3	0.08	0.35	0.7	0.31

Table 4.13 Comparisons of initial and optimum values of volume and % improvement

	present study			Uzman and Daloğlu [24]		
	volume			volume		
	initial	optimum	% decrease	initial	optimum	% decrease
i	0.800	0.505	36.9	0.800	0.562	29.8
ii	0.800	0.637	20.4	0.800	0.667	16.6

4.5 Strain Energy Minimization of Non-Uniform Cross Section Arch

Problem definition: The last example is an arch of continuously varying cross-section with opening angle 60°. The arch, which is radius of curvature of $R = 100$ *m* and elastic modulus $E = 0.14431$ *GPa.* Its beginning and ending cross-section values are 0.6×1.0 *m* and 1.4×1.0 *m* respectively. Shown in Figure 4.8.

The shape of uniform arch is defined using two segments and five key points. Strain energy minimization with a constraint that the total material volume of structure is limited to 105 m^3. A total of two shape and five thickness design variables as shown in Figure 4.9 are considered.

Optimization is carried out for three types of design variables.

i-) only shape design variables s_1,s_2

ii-) only thickness design variables t_1,t_2,t_3

iii-) both shape and thickness design variables are considered together

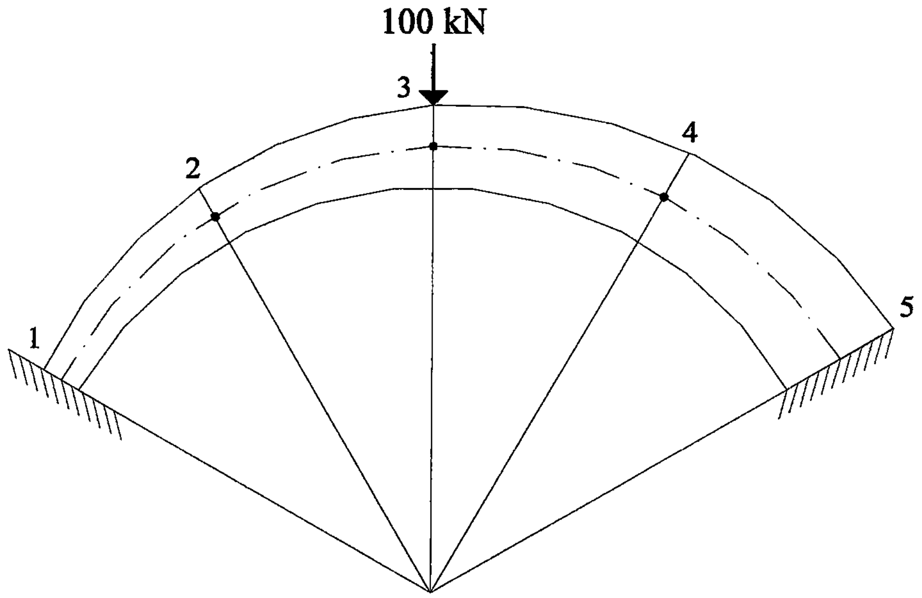


Figure 4.8 Geometry of non-uniform arch

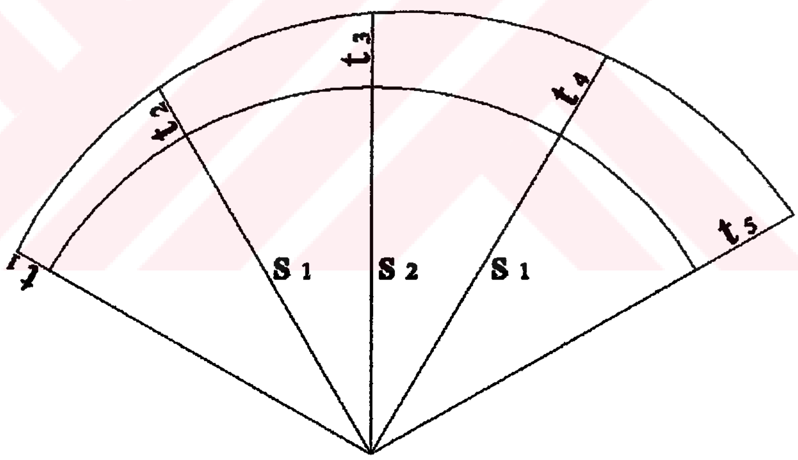


Figure 4.9 Location of design variables

Discussion of results: The analysis is carried out using cubic elements and 195 DOF. Table 4.14-4.16 presents initial and optimal design variables together with their bounds, Table 4.17 presents initial, optimal values and percent improvements of total strain energy and Table 4.18 presents %contributions to total strain energy. Highest improvement is obtained in type *i* because membrane energy increased highest value in type *i*..

Table 4.14 Non-uniform arch: values of the design variables (Type *i*-only shape)

design variables				opt. design variables
type	min.	initial	max.	
s_1	50.0	100.0	150.0	97.38
s_2	50.0	100.0	150.0	102.62

Table 4.15 Non-uniform arch: values of the design variables (Type *ii*-only thickness)

design variables				opt. design variables
type	min.	initial	max.	
t_1	0.2	0.6	2.0	0.93
t_2	0.2	0.8	2.0	0.99
t_3	0.2	1.0	2.0	1.24
t_4	0.2	1.2	2.0	0.96
t_5	0.2	1.4	2.0	1.11

Table 4.16 Non-uniform arch: values of the design variables (Type *iii*-both thickness and shape)

design variables				opt. design variables
type	min.	initial	max.	
s_1	50.0	100.0	150.0	96.63
s_2	50.0	100.0	150.0	103.37
t_1	0.2	0.6	2.0	0.69
t_2	0.2	0.8	2.0	0.88
t_3	0.2	1.0	2.0	1.07
t_4	0.2	1.2	2.0	1.15
t_5	0.2	1.4	2.0	1.32

Table 4.17 Initial-optimal values and % improvement of total strain energy

	strain energy ($\times 10^{-4}$)		
type	initial	optimum	% decrease
i	443.407	27.538	93.8
ii	443.407	288.969	34.8
iii	443.407	73.99	83.3

Table 4.18 % Contributions to total strain energy

% contributions to strain energy				
case	shape	membrane	bending	shear
i	initial optima	4.881 74.951	94.864 24.980	0.256 0.068
ii	initial optima	4.881 7.159	94.864 92.486	0.256 0.354
iii	initial optima	4.881 24.356	94.864 75.430	0.256 0.243

CHAPTER 5

CONCLUSION AND FURTHER WORK

5.1 Conclusion

In the present work computational tools have been developed for geometric modeling, automatic mesh generation, analysis and shape optimization of arch structures. During the research work, an efficient, reliable, robust and flexible computer program has been developed based on the previous work [35]. Several examples have been studied and used to test and to demonstrate the capabilities offered by these computational tools. Based on the above studies the following general conclusions can be drawn.

5.1.1 Structural analysis

- FEs of the types presented in the present work, which can perform well in *thick*, *thin* and *variable thickness* cases have proved to be most appropriate for the analysis and optimization of arch structures due to their inexpensiveness, accuracy and reliability.
- The results obtained using FE analysis tools generally compare well with those obtained from other sources based on alternative formulations. The results illustrate that the FE methods presented here can be used with confidence for the static analysis of arch structures which may have rectangular cross-sections.

5.1.2 Structural optimization

- Definition of the shape variables is crucial, i.e. the parameterization of the optimization model has to have as few degrees of freedom as possible to simplify the optimization task and as many degrees of freedom as necessary so that the problem is not over-constrained. The optimum solution obtained is only the optimum for this particular problem definition; in only very rare cases will it be the global optimum.
- The application of SSO in conjunction with finite element analysis is an efficient an effective method, in particular for problems with a great number of design variables and a reasonable number of design constraints.
- It is important to identify the objective of a particular problem clearly and then use the correct function to solve the problem.
- Constraints are important since they guide the optimizer to the optimum and restrict the design space to a useful feasible domain.
- Shape optimization with a strain energy minimization as the objective seems to be a mathematically better behaved problem than those defined using volume/weight minimization as objective function.
- The more accurate the information given to the optimizer, the faster the convergence achieved. FE solutions in combination with the semi-analytical sensitivity method deliver more accurate function and derivative values.
- In certain special cases, we have demonstrated that the SE and volume can be improved by as much as 97 and 37 percent respectively using shape optimization.
- The introduction of thickness as well as shape variation leads to a further improvement in the objective function of the optimal structures.
- Some of the optimum shapes presented in this thesis are not practical and are included to show how shape and thickness changes can substantially alter the values of objective function. However, manufacturing constraints etc. can be imposed to produce more practical optimum designs.
- The tools developed in the present work are useful creative design aids for structural engineers. They also offer potential as educational aids.

5.2 Further Work

The following possibilities exist for extending the various aspect of the present work.

5.2.1 Structural analysis

- The finite element analysis code could extend to handle the buckling analysis of arch structures
- The FEs developed and used in this thesis are inexpensive and accurate. However, it would be a desirable feature to include error estimation and adaptive analysis procedures. An error estimation and adaptivity technique could be investigated for dynamic problems.

5.2.2 Structural optimization

- The stability characteristics, post buckling and general nonlinear behavior of the optimized structures could be investigated so that their safety is assured.
- The optimization of structures constructed from composite materials could be considered.
- In SSO of structures analyzed by FE method, the use of other objective functions such as minimization of error energy could be investigated. Further, multi-objective problems should be considered in which there are several objective functions to be minimized and/or maximized. In static problems, multiple load cases should be considered.
- The user friendliness of the SSO algorithm could be investigated by developing a modern user interface. It would also be of interest to integrate the present SSO algorithm with a standard CAD system.

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