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ANALYSIS OF LINEAR TIME VARYING SYSTEMS IN WAVELET DOMAIN

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ANALYSIS OF LINEAR TIME-VARYING SYSTEMS IN WAVELET DOMAIN

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Analysis of linear time-varying systems is one of the important problems in engineering. In many solution methods there are some approximations such that the system is modeled as time-invariant system and/or the solution is obtained in steady-state conditions without considering the transient response. The problem is to get a general analysis method for LTV systems is still continuing to investigate.

In this thesis, time-varying systems are analyzed in wavelet domain. The system equations are described in a higher order differential equation or state-space representation. To solve these equations, they are transferred to wavelet domain by forming algebraic matrix-vector relations using the wavelet transform coefficients. These relations are achieved by defining operator matrices concerned with addition-subtraction, multiplication, derivative and integral operators appear in system equations. Orthogonal and compact support wavelets provide a simple way to define these operator matrices. The operator matrices have been defined for orthogonal compactly supported wavelets. Linear time-varying system's dynamic equations are expressed as into algebraic matrix-vector relations by using the operator matrices. In application, firstly, system analysis in wavelet domain method is theoretically applied to differential equation and state-space representation of LTV systems and then some illustrative examples are given. Their results are discussed in conclusion.

This thesis consists of a literature survey on the linear time-varying systems and their solution techniques, the theoretical background of wavelets and wavelets transform,

the operational matrices of time domain operators and LTV system analysis with wavelets.

Key Words: Wavelet domain system analysis, Wavelet transform, Linear time-varying systems, Linear operators.

ÖZET

ZAMANLA DEĞİŞEN LİNEER SİSTEMLERİN DALGACIK ORTAMINDA ANALİZİ

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Zamanla değişen lineer sistemlerin çözümü mühendislik uygulamalarında karşılaşılan önemli problemlerden biridir. Birçok çözüm metodunda, zamanla değişmeyen sistem modellemesi ve/veya geçici rejim analizini dikkate almadan kalıcı çözümü bulmak için bazı yaklaşımlar bulunmaktadır. Zamanla değişen lineer sistemlerin tümü için uygulanabilecek genel bir çözüm metodu arayışı halen devam etmektedir.

Son zamanlarda bazı araştırmacılar dalgacık analizini zamanla değişen lineer sistemlerin çözümünü bulmak için belli sınırlamalarla uyguladılar Bu tezde, sistem denklemleri dalgacık katsayıları cinsinden dalgacık ortamına transfer edilir. Sistem denklemleri yüksek dereceli diferansiyel denklem ya da durum denklemleri olarak tanımlanır. Bu denklemleri dalgacık ortamında çözmek için cebirsel matris-vektör denklemlerine dönüştürülür. Bu ifadeler, sistem denklemlerinde bulunan ekleme-çıkarma, çarpma, türev ve integral operatörleri için tanımlanan operatör matrisleriyle ifade edilir. Dalgacıkların ortogonal ve lokal tanımlanabilirlik özelikleri, operatör matrislerinin basit bir şekilde tanımlanmasını sağlar. Operatör matrisleri ortogonal ve lokal tanımlı dalgacıklar için tanımlanır. Operatör matrisleri zamanla değişen doğrusal sistemlerin dinamik denklemlerini cebirsel matris-vektör ilişkisine dönüştürür. Uygulamada, ilk olarak, dalgacık ortamında sistem analiz metodu teorik olarak yapılan türetimlerle, zamanla değişen doğrusal sistemlerin diferansiyel ve durum denklemlerinde gösterilir.Daha sonra seçilen örnekler üzerinde pratik olarak uygulanır.Bu örneklerin çıktıları sonuç bölümünde tartışılır.

Bu çalışmada, zamanla değişen doğrusal sistemlerin literatür taraması ve çözüm teknikleri verilmektedir. Dalgacık ve dalgacık ortamı teorisinden yararlanılarak zamanla değişen doğrusal sistemlerin analizi yapılmaktadır. Bunun için sistem denklemlerinde zaman ortamında kullanılan operatörler dalgacık operator matrisleri olarak tanımlanıp; sistem denklemleri dalgacık ortamına transfer edilmektedir. Böylece bu denklemler çözülerek zamanla zamanla değişen doğrusal sistemlerin analizi dalgacık ortamında gerçekleştirilmektedir.

Anahtar Kelimeler: Dalgacık ortamında sistem analizi, dalgacık ortamı, Zamanla değişen lineer sistemler, lineer operatörler.

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LIST OF SYMBOLS/ABREVIATIONS*

A	Addition matrix
D	Derivative operator matrix
M	Multiplication matrix
P	Integral operator matrix
T	Linear operator
U	Wavelet coefficients input vector
Y	Wavelet coefficients output vector
$\psi(t)$	Mother wavelet
$h(t,t_0)$	Impulse response function
$\varphi(t)$	Scaling function
$\delta(\cdot)$	Knocker delta function
$\mathbf{\Phi(t,t_0)}$	State transition matrix
T_0	Fundamental period
$\frac{d^n}{dt^n}$	nth order time derivative
$L^2(R)$	Square integrable functions space
$\cup$	Union
$\cap$	Intersection
$\in$	Is an element of
$\notin$	Does not belong to
$\oplus$	Direct sum
$\perp$	Orthogonal
$\Leftrightarrow$	If and only if
$<.,.>$	Inner product
$\sum$	Sum

*First Latin, then Greek letters, both in alphabetical order.

CHAPTER 1

INTRODUCTION

In last two and a half decades, many researchers interest the wavelets. The early wavelets work was in the 1980's by Morlet, Grossmann, Meyer, Mallat, and others, but it was the paper by Ingrid Daubechies [1] in 1988 that caught the attention of the large number of the scientist who works on theoretical and applied sciences. Some of the application areas of the wavelets are applied mathematics, mathematical physics, numerical analysis, communication systems, signal and image processing-image and video coding, denoising, linear system analysis.

System analysis with wavelets is a new application area of wavelets. There are a few works applying the wavelets to get solution of LTV systems [2,3]. These works give the solution of first order differential equation defining the integral operator for Haar wavelet. Naturally this approach is suitable for state-space representation of LTV systems. Xiangqian L. and Lin Z**.** [2] applied the Haar integral operator to state-space representation of linear time-varying systems. As for C.H. Lee used this operator for transient analysis of linear systems [3].

Linear time-varying system analysis based on operator matrices is not a new analysis technique. This method is used by G. Tohumoglu [4] to analyze periodically time-varying linear systems. In this work, which is the main inspiration of us to do this thesis, operator matrices (didem, indem, modem, delay) for derivative, integral, multiplication and delay operation were introduced for Fourier series. The same matrices can be defined in wavelet domain in a similar way as in the spectral analysis method [4]

In this thesis, it is aimed to give a different solution method of linear time-varying systems in wavelet domain by doing steady-state analysis. A comparative discussion of this method with the other solutions methods such as spectral analysis method is given.

Having noted the importance and the applications of linear time-varying systems in this section, general properties and representations of linear time-varying systems and their solution techniques are introduced in the subsequent sections.

In chapter two, an introductory survey of wavelets and particularly, orthogonal wavelets and its properties will be given. Besides of theoretical aspects of wavelets, their applications will be mentioned. In chapter three, wavelet analysis of linear time-varying systems is explained theoretically. It will be introduced operational matrices such as derivative, integral operational matrices in wavelet domain for time-domain operators. Besides of these, multiplication and addition matrices will be defined, too.

Wavelet analysis method is suitable for computer programming; therefore, basically, MatLab functions for operational matrices are developed. Finally, in the last chapter; the results and conclusions are briefly summarized and some topics are proposed for further research.

1.1 IMPORTANCE OF LINEAR TIME-VARYING SYSTEMS

Time-varying systems take an important place in modern technology. They are widely used as in communication systems, power electronic circuits, electrical machinery and electronics.

In communication system, the communication channels are time varying due to movement of the source, receiver or scatters. Therefore, the channel is acting like time varying filters. Besides that, parametric amplifiers, parametric converters, time varying filters, switched capacitor networks, mixers and RF circuits are also different types of time-varying systems [5-8].

In power electronic circuits high power semiconductors devices such as thristors, diacs, triacs are used and these devices are either triggered externally or controlled by the response signals; in either case the controlling signal is periodic and these devices behave as periodically time-varying components. Because of time-varying nature of power systems, time-varying system analysis methods are used in the systems as, power system protection, power quality, power system transients, partial discharges, load forecasting, power system measurement.

In integrated circuits (IC) area, due to the heat generated by IC, circuit parameters are changing . The parameter variations need to be quantified in order to ensure a robust circuit.

1.2 LINEAR TIME-VARYING SYSTEMS

The system approach is a widely used in modeling electronic and mechanical systems. Linear systems are highly popular models due to their simplicity and convenience for mathematical analysis. Thus, many systems can be modeled as linear time-varying systems at least for a limited range of operation. Figure 1.1 describes the general notion of an input-output system in a block diagram. The input is u and the output is y to describe physical quantities and their relations.

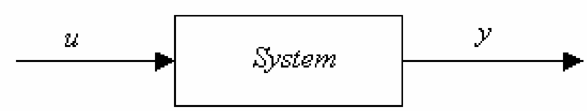

Figure 1.1 Input-Output System

To classify the system, let us define important qualifiers for a system.
A system is *linear* if it is satisfies the property of superposition, that is, for any couple of inputs and outputs $y_1 = f(u_1)$ and $y_2 = f(u_2)$, the equation $ay_1 + by_2 = af(u_1) + bf(u_2)$ must be satisfied for any couple of scalars a and b.

A system is *time-varying,* if a system parameters changes with time, otherwise it is called a time-*invariant system*. If a system satisfies linearity property and it has at least a time-varying component, it is called a linear time-varying system (LTVS), otherwise linear time-invariant system (LTIS). A small class of LTVS is called periodically time-varying system, whose components change periodically with time.

1.2.1 Linear Time-Varying Systems Representation

The relation between the input and the output of a time-varying system can be expressed in a variety of ways. This forms "characterization" (representation) of the system. Basically, the input-output relation of a linear time-varying system is described by differential equation as

$$a_n(t)\frac{d^n y}{dt^n}+...+a_1(t)\frac{dy}{dt}+a_0(t)y(t)=b_m(t)\frac{d^m u}{dt^m}+...b_1(t)\frac{du}{dt}+b_0(t)u(t) \qquad (1.1)$$

where $a_n(t)$ and $b_m(t)$ are known continuous functions of time. This equation is referred to as the fundamental equation of the system [9]. If there are more than one input and/or output in the system then, in general, we have more than one high order simultaneous differential equations containing multi-input, multi-output variables.

The classical differential equation solution techniques can be applied successfully to a small class of systems and corresponding basis functions can be found in [10]. This small class contains the systems, which are characterized by the following equations: Bessel equations, Weber equations, Hypergeometric equations, Airy equations and others.

The equation (1.1) defines a periodically time-varying linear system if the coefficients of functions $a_n(t)$ and $b_m(t)$ are periodic with the system's fundamental period T_0. For periodically time-varying systems, the periodicity makes it possible to apply some special techniques such as Floquet theory [11] and spectral analysis [4]. In spectral analysis fundamental differential equation of linear LTV system is expressed in terms of algebraic matrix-vector relation by defining operational matrices for derivative, integral, and any time-varying component behavior. The system equations are transferred to spectral domain. Thus, solution of the system equation can be easily obtained by using the matrix operations. The solution is computed in spectral domain in term of Fourier coefficients. Then it is carried to the time domain by applying inverse Fourier Transform. This method gives the steady-state analysis of periodically time-varying system. However the general analysis methods of LTV systems are still continuing to investigate.

Due to above mentioned difficulties of system representation by a single high order differential equation; state-space representation, has been developed. In modern system theory, it is preferred and found very convenient methods especially for computer simulations to use a set of N first order linear differential equations of the form Eq. (1.2a) together with the expression Eq. (1.2b) for the output.

$$x'(t) = A(t)x(t) + B(t)u(t) \tag{1.2a}$$
$$y(t) = C(t)x(t) + D(t)u(t) \tag{1.2b}$$

In these equations $x(t) \in R^n$, $u(t) \in R^n$, $y(t) \in R^m$ are the state, input and output respectively, at time $t \in R^+$; $A(t)$, $B(t)$, $C(t)$, $D(t)$ are matrices of order compatible with $x(t)$, $u(t)$ and $y(t)$, and their elements are known and they are piece-wise continues functions defined on R^+. It is well known that the state solution of Eq. (1.2a) is given by,

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \tag{1.3}$$

where $\Phi(t,t_0)$ is called the state transition matrix [10]. The $\Phi(t,t_0)$ is the key to the solution of Eq. (1.2a). Some solution techniques are given in [10] for different classes of linear system equations. The common one is commutative class. State-space representation of LTV system can be transformed into time-invariant representation through the commutative class by using transformation as,

$$x(t) = T(t)\bar{x}(t) \tag{1.4}$$

Here, $T(t)$ is the transformation matrix, which transforms the system representation into commutative or even a linear time-invariant system representation.

Although, It is concluded in [12], [13] that, the commutative property is not an inherent property of a dynamic system, but rather is just a system representation property it is difficult to find transformation matrix $T(t)$ in Eq. (1.4). Therefore it is not easy to get the solution of system if it is not in commutative class. Spectral analysis method [4] can be applied efficiently for this representation, if $A(t)$, $B(t)$, $C(t)$, $D(t)$ matrices are periodically time varying.

A LTV system is excited by an impulse function, that is the delta function, $\delta(t)$ and the system's response to the impulse function is called "impulse response" and denoted as $h(t,t_0)$. The system response $y(t)$ to the input $u(t)$ applied at the $t = t_0$ is given by the superposition integral

$$y(t) = \int_{t_0}^{t} h(t,\tau)u(\tau)d\tau \tag{1.5}$$

This superposition is expressed as *convolution* of input-output, that is

$$y(t) = h(t,t_0) * u(t) \tag{1.6}$$

However, a method for analytic expression of $h(t,t_0)$ is generally unknown and same difficulties mentioned in *differential equation* are valid for this representation.

Frequency domain approach for analysis of LTV is first developed by L.A. Zadeh [9]. Zadeh's approach is essentially an extension of the frequency analysis techniques commonly used in LTI systems. He defines a time-variable system function $H(s,t)$, for a variable linear network. This function possesses most of the fundamental properties of the transfer function of a fixed network. For this reason it is conveniently used to interpret the frequency domain behavior of systems and to realize the given frequency domain requirements in design problem. Further, once $H(s,t)$ has been determined, the response to any given input can be obtained by treating $H(s,t)$ as if it were the transfer function of a fixed network.

For a single-input, single-output time-varying linear system, which is initially, relaxed, the time-varying system function is defined by the relation

$$H(s,t) = \int_{-\infty}^{\infty} h(t,\tau) e^{-s(t-\tau)} d\tau \tag{1.7}$$

The response of linear system to any input $u(t)$, $t \geq t_0 \geq 0$, can be derived by

$$y(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} H(s,t) U(s) e^{-st} ds \tag{1.8}$$

where $U(s)$ is the Laplace transform of $u(t)$.

However, There are similar difficulties to determine $H(s,t)$ involved in solving the fundamental equation or the state equations of the system. To overcome some of difficulties the system equations transformed to spectral domain to use the spectral analysis techniques [4]. The spectral analysis method basically uses Fourier series expansion of variables in linear periodically time-varying systems.

CHAPTER 2

INTRODUCTION TO WAVELET TRANSFORM

In this chapter, it is introduced that the wavelets and the discrete wavelet transform from the classical viewpoint, based on the concept of multi-resolution analysis. It will not be given all deep details, but it will be limited to the topics that are the more relevant important parts for this thesis. For the details we refer to references [14-18].

A short history of wavelets and their practical applications are introduced in the following sections. It can be found that the brief explanation for what the wavelets are. Multiresolution analysis forms the backbone of the wavelet analysis, which is shortly given in section 2.5. In order to calculate the wavelet transform in an efficient way, the fast wavelet transform (FWT) is used that is given in the last part of this chapter.

2.1 HISTORY OF WAVELETS

The appearance of wavelets is a relatively recent development in mathematics. The name wavelet or *ondelette* was coined some ten years ago by French researchers, including Morlet, Arens, Fourgeau and Giard [19], Morlet [20], and Grossman and Morlet [21]. The existence of wavelet like functions has been known since the early part of the century (notable examples being what are now known as the Haar wavelet and the Littlewood- Paley wavelet) and many of the ideas now embodied in wavelets originated from work in subband coding in engineering, coherent states and renormalization group theory in physics and the study of Calderon-Zygmund operators in mathematics. However, it was not until recently that the unifying concepts necessary for a general understanding of wavelets were provided by researchers such as Mallat [22], Stromberg [23], Meyer[24], Daubechies [18], Battle [25] and Lemarié [26]. Since then, the growth of wavelet research in mathematics and their applications in different areas has been explosive with numerous authors contributing significantly.

2.2 WAVELET'S APPLICATIONS

There is an extensive wavelet literature within a variety of applications in several disciplines, e.g., Digital Signal and Image Processing in particular application, Medical and Biomedical Signal and Image Processing, fingerprint classification, remote sensing, target recognition, denoising etc, Numerical Solution to Partial Differential Equations, Seismic and Geophysical Signal Processing, The main focus in the literature has been on identification and classification methods from the analysis of measured signals, few works use wavelet transform as an analysis technique for the solution of voltages and currents which propagate throughout the system

2.3 WHAT ARE WAVELETS?

Real-world signals usually have the features that they are both limited in time and limited in frequency (band-limited). Time limited signals can be represented efficiently using a basis of block function. But block signals are not limited in frequency. Band-limited signals can be represented efficiently using a Fourier basis, but sines and cosines are not limited in time.

What we need is a compromise between the pure time-limited and band-limited basis that combines the best of both worlds: wavelets ("small wave").

The goal of most wavelet research is to create a set of basis functions and transform that will give an informative, efficient and useful description of a function or signal. Thus, a signal $f(t)$ decomposes into a basis of functions Ψ_i

$$f(t) = \sum_k a_i \Psi_i \qquad (2.1)$$

To have an efficient representation of the signal $f(t)$ using only a few coefficients a_i, it is very important to use a suitable family of functions Ψ_i. The functions Ψ_i should match the features of the data we want to represent.

If the signal is represented as a function of time, wavelets provide efficient localization in both time and frequency or scales. Hence, they can easily detect local features in a signal. Another central idea of wavelets is that of multiresolution analysis where the decomposition of a signal is in terms of the resolution of detail. Therefore the wavelet decomposition allows to analyzing a signal at different resolution levels (scales).

2.4 WAVELET TRANSFORM

The fundamental idea of the transform techniques is to transform signal from time domain to frequency domain. As it is known that the commonly used transform techniques in the signal compressions are the Fourier transform (FT), Short-time Fourier transform (STFT), Discrete Cosine transform (DCT) and Wavelets transform (WT). In the last two decades, the researchers interest the wavelets transform and its applications more precisely. The wavelets transform overcomes some resolution problems -which will be explained in the following paragraphs- arising in the Fourier and/or Short-time Fourier transform.

One of the commonly used transform techniques is the Fourier transform in signal frequency analysis [27]. The Fourier transform gives what frequency components exist in the signal, but it doesn't tell when these frequency components exist. Both of time and frequency information is not required when the signal is statinary whose frequency doesn't change with time. Therefore, FT is not suitable technique for nonstationary signals. However, the Fourier transform is suitably applied for windowed nonstationary signals which is called Short-time Fourier transform [28]. Once a window is chosen for the STFT, then the time-frequency resolution is fixed over entire time-frequency plane. This causes resolution problems. To overcome some resolution problems of the STFT, it is necessary that the new or modified transformation techniques. The wavelets transform (WT) emerged as a new transformation method.

Multiresolution analysis mainly deals with the problem of adjusting time and frequency resolution [29]. Considering Heisenberg inequality defined as the multiplication of time resolution $\Delta\tau$ and frequency resolution Δf is a fixed constant i.e. $\Delta\tau\Delta t \geq \frac{1}{4\pi}$, the time resolution decreases when frequency resolution increases or vice-versa, shortly, the time and scale resolutions are bounded (can not be arbitrarily small). In practical applications, high frequencies appear from time to time as short bursts, or spikes, however low frequencies are usually present during the entire duration of the signal. The WT are based on the multiresolution analysis which allows analyzing a signal at different resolution levels [30]. Wavelets can also be stretched or compressed to obtain low and high frequency components to be analyzed any signal at different resolutions. This provides multiresolution analysis in the frequency domain representation of signal.

The Continuous Wavelet Transform (CWT) of a signal $x(t) \in L^2(R)$ is defined as

$$CWT_x(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t)\, \psi * (\frac{t-\tau}{s})\, dt \tag{2.2}$$

$CWT(s,\tau)$ is a function of two variables, τ and s, the translation and scale parameters, respectively. $\psi(t)$ is the transforming function-usually a bandpass filter-called the mother wavelet. The wavelet function has oscillatory property. The large scales $(s >> 1)$ correspond to long basis functions, and will identify long-term trends in the signal to be analyzed. The small scales $(0 < s < 1)$ lead to short basis functions in order to define the short term behavior of the signal. Thus the scale parameter interpreted as inversely proportional with the frequency of the signal. The wavelet transform is a reversible transform and the reconstruction is possible if admissibility and regularity conditions are satisfied [29, 30].

For the discretization of wavelets transform, the scale and translation parameters are discritized as $s = s_0^{-j}$ and $\tau = k s_0^{-j} \tau_0$ with an index j and k and substituted into the mother wavelet to define discrete wavelet transform basis as

$$\psi_{j,k}(t) = s_0^{j/2} \psi(s_0^j t - k\tau_0) \tag{2.3}$$

The most convenient value is found to be "2" for s_0 and "1" for τ_0. This equation is carried into the Eq. (2.2) in order to define the Discrete Wavelet Transform (DWT). The multiresolution wavelet algorithm decomposes a signal x(t) by the help of scaling functions $\varphi(t)$ and wavelet functions $\psi(t)$. These functions together resolve the signal into its coarse and detail components. Thus, by using the multiresolution idea, the signal x(t) is defined in terms of scale and wavelet coefficients, $c(k)$ and $d(j,k)$, respectively. That is

$$x(t) = \sum_{k=-\infty}^{\infty} c(k)\varphi_k(t) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} d(j,k)\psi_{j,k}(t) \tag{2.4}$$

First summation gives a function that is low resolution or coarse approximation of $x(t)$, the second one represents the higher or finer resolution to give detail information of the signal.

The signal is split first via a two channel filter bank, then the lowpass version is split again using the same filter bank and so on. The filters used in multiresolution decomposition are called a *constant-Q* filter bank since the bandwidth at each channel, divided by its center frequency is constant. The wavelet functions and the scaling function, that represent any signal in Eq. (2.4) can be obtained by the process of the digital filtering and down sampling. The scale j (c_j) coefficients are filtered by two finite impulse response of lowpass and highpass digital filters with coefficients $h_0(n)$ and $h_1(n)$, respectively. After this operation, down sampling gives the next coarser j-1 scaling (c_{j-1}) and wavelet (d_{j-1}) coefficients. This process is illustrated in Figure 2.1, where $\downarrow 2$ denote a down sampling by 2.

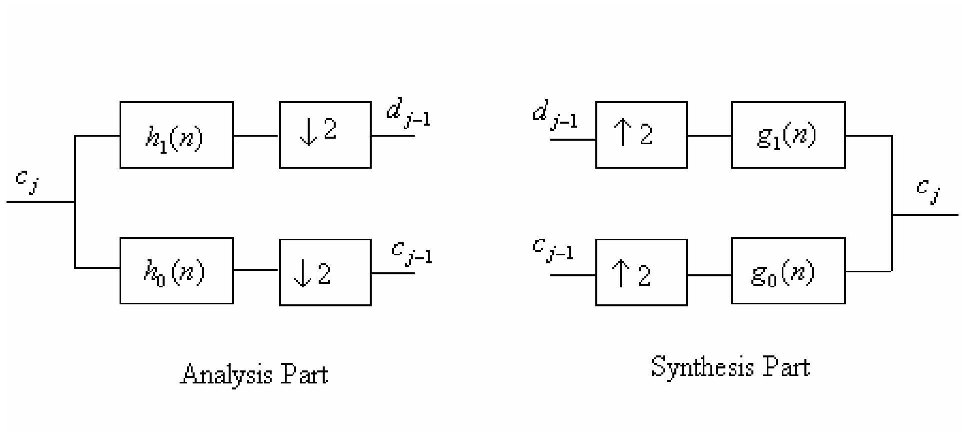

Figure 2.1. Subband Filtering Scheme for One Step Decomposition.

In Figure 2.1, the synthesis process is shown where in j-1 coarse scaling (c_{j-1}) and wavelet (d_{j-1}) coefficients are filtered and then upsampled to form the j-coarse scale value. The process denoted by $\uparrow 2$ is the upsampling by a factor 2. The filters used in analysis part $\mathrm{h_0(n)}$ lowpass and $\mathrm{h_1(n)}$ highpass filters and in synthesis part $\mathrm{g_0(n)}$ lowpass and $\mathrm{g_1(n)}$ highpass filters are Quadrature mirror filters [30, 31] and they have relations as

$$g_0(n) = h_0(\text{-}n) \;\; and \;\; g_1(n) = h_1(\text{-}n) \tag{2.5}$$

2. 5 DISCRETE WAVELETS AND MULTI-RESOLUTION ANALYSIS

The goal of multiresolution analysis is to developed representation of a function $f(t)$ at various levels of resolution. To achieve this, the given function is expanded in terms of basis functions $\varphi(t)$, which can be scaled to give multiresolution of the original function.

In order to develop a multiresolution analysis [32, 33] $L^2(R)$ is decomposed in nested sub-spaces V_j such that the closure of their unions is $L^2(R)$ and their intersection contains only the zero function.

$$\{0\} .... \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 .... L^2(R) \tag{2.6}$$

In the dyadic case, each subspace V_j is twice as large as V_{j-1}, and then a function $f(t)$ that belongs to one of these subspaces has the following properties:

$$f(t) \in V_j \Leftrightarrow \quad dilation \quad f(2t) \in V_{j+1} \tag{2.7}$$

$$f(t) \in V_j \Leftrightarrow \ translation \ \ f(t+1) \in V_j \tag{2.8}$$

If it can be found a function $\varphi(t) \in V_0$ such that its translation forms a Riesz basis for the space V_0, it is called scaling function or father function. That is,

$$V_0 = span\overline{\{\varphi_k(t),\ k \in Z\}} \quad where \quad \varphi_k(t) = \varphi(t-k) \tag{2.9}$$

Then $\varphi_{j,k}(t)$ form a basis for the space V_j, i.e,

$$V_j = span\overline{\{\varphi_{j,k}(t),\ k \in Z\}} \quad where \quad \varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k) \tag{2.10}$$

The nesting of the spans of $\varphi_{j,k}(t)$, denoted by V_j and shown in Eq. (2.6), is achieved by requiring that $\varphi(t) \in V_1$, which means that if $\varphi(t)$ is in V_0, it is also in V_1, the space spanned by $\varphi(2t)$. This means $\varphi(t)$ can be expressed in terms of a weighted sum of shifted $\varphi(2t)$ as

$$\varphi(t) = \sqrt{2}\sum_{k=0}^{L-1} h_0(k)\varphi(2t-k) \tag{2.11}$$

The coefficients h_0 are a sequence of real or perhaps complex numbers called the scaling function coefficients (or the scaling filter or the scaling vector) and $\sqrt{2}$ maintains the norm of the scaling function with the scale of two. The equation is referred to by different names to describe different interpretations or points of view. It is called the refinement equation, the multiresolution analysis equation, or dilation equation.

Now let us investigate the differences between subspaces V_{j-1} and V_j. It can be defined a new subspaces W_{j-1} such that it is the orthogonal complement of V_{j-1} in V_j

$$V_j = V_{j-1} \oplus W_{j-1}; \qquad V_{j-1} \perp W_{j-1} \tag{2.12}$$

where $\oplus$ represents a direct sum. W_j spaces are called as wavelet spaces. It follows that the spaces W_j are orthogonal and that

$$L^2(R) = \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots \tag{2.13}$$

Another way to describe the relation V_0 to the wavelet spaces is noting

$$W_{-\infty} \oplus \cdots \oplus W_{-1} = V_0 \tag{2.14}$$

which again shows that the scale of the scaling space can be chosen arbitrarily.

If it is introduced a function $\psi(t) \in W_0$ that obeys the properties Eq. (2.7) and (2.8), and its integer translates form a basis for the space W_0, it is called wavelet function or mother wavelet function.

$$W_0 = span\overline{\{\psi(t-k),\ k \in Z\}} \tag{2.15}$$

Then $\psi_{j,k}(t)$ is a Riesz basis for W_j, i.e,

$$W_j = span\overline{\{\psi_{j,k}(t),\ k \in Z\}} \quad where\ \ \psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) \tag{2.16}$$

Since these wavelets reside in the space spanned by the next narrower scaling function, $W_0 \subset V_1$, they can be represented by a weighted sum of shifted scaling function $\varphi(2t)$ as,

$$\psi(t) = \sqrt{2}\sum_{k=0}^{L-1} h_1(k)\varphi(2t-k) \tag{2.17}$$

$$where\ \ h_1(k) = (-1)^k h_0(L-k-1) \quad k = 0,1,....,L-1$$

Due to the multi-resolution analysis, the relations Eq.(2.11) and Eq.(2.17) are also valid between V_{j+1}, V_j and W_j for arbitrary j. The h_0 and h_1 are the filter coefficients that uniquely define the scaling function $\varphi(t)$ and the wavelet $\psi(t)$. The h_0 and h_1 are lowpass and highpass filter coefficients respectively.

The subspaces V_j are nested, and each of them can be split in two subspaces V_{j-1} and W_{j-1}. This means that V_j is a "coarse-resolution" representation of V_{j+1}, while W_j carries the "high-resolution" difference information between V_{j+1} and V_j. The Figure 2.2 pictorially shows the nesting of the scaling function spaces V_j for different scales j and how the wavelet spaces are the disjoint differences (expect for the zero element) or, the orthogonal complements.

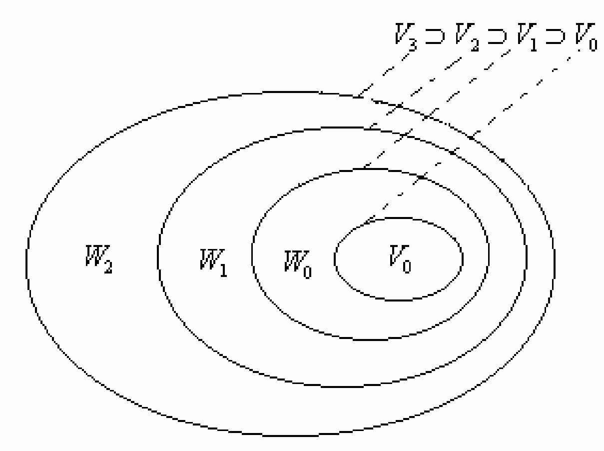

Figure 2.2: Scaling Function and Wavelet Vector Spaces

Now, by using $\varphi_k(t)$ and $\psi_{j,k}(t)$ functions it is possible to span the space $L^2(R)$. Any function $f(t) \in L^2(R)$ is written as a series expansion in terms of the scaling functions and wavelets.

$$f(t) = \sum_{k=-\infty}^{\infty} c(k)\varphi_k(t) + \sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty} d(j,k)\psi_{j,k}(t) \tag{2.18}$$

In this expansion, the first summation gives a function that is a low resolution or coarse approximation of $f(t)$. For each increasing index j in the second summation, a higher or finer resolution function is added, which adds increasing detail. This is somewhat analogous to a Fourier series where the higher frequency terms contain the detail of the signal.

It is stated that the subspaces V_j and W_j are orthogonal, which means, the basis functions $\varphi_{j,k}(t)$ and $\psi_{j,k}(t)$ are orthogonal to each other. Then

$$<\varphi_{j,k}(t), \psi_{j,k}(t)> = 0 \tag{2.19}$$

$$<\varphi_{j,k}(t), \varphi_{j,k'}(t)> = \delta_{k-k'} \tag{2.20}$$

$$<\psi_{j,k}(t), \psi_{j',k'}(t)> = \delta_{j-j'}\delta_{k-k'} \tag{2.21}$$

Therefore, the $c(k)$ and $d(j,k)$ coefficients in Eq. (2.18) can be found by taking inner products of $f(t)$ with the scaling and wavelet functions respectively,

$$c(k) = < f(t), \varphi_k(t) > \tag{2.22}$$

$$d(j,k) = < f(t), \psi_{j,k}(t) > \tag{2.23}$$

In Eq. (2.18) range of j extends to infinity but for a real application this wide range may not be required. For any practical signal that is bandlimited, there will be an upper scale $j = J$, above which the wavelet coefficients $d_j(k)$ are negligibly small [29]. Then high resolution description of $f(t)$ in terms of the scaling coefficients c_J is like that,

$$f(t)=\sum_{k=-\infty}^{\infty} c_J(k)\varphi_{J,k}(t) \tag{2.24}$$

Wavelets and scaling functions are compact support functions. These functions satisfy the following relations.

$$\psi_{j,k}(t)\,,\,\varphi_{j,k}(t)=\left\{\begin{matrix} \neq 0 & for \quad \frac{k}{2^j}\leq t<\frac{k+1}{2^j} \\ \\ 0 & otherwise \end{matrix}\right\} \tag{2.25}$$

and $$A_0=\int_{-\infty}^{\infty}\varphi(t)dt=1 \tag{2.26}$$

These relations give a chance to express the system equations in wavelet domain in a simple manner.

2.6 THE FAST WAVELET TRANSFORM

In many applications one never has to be deal with the scaling function or wavelets. Only the filter coefficients h_0 and h_1 in the defining Eq. (2.11), (2.17) and c_k , $d_{j,k}$ in the expansion Eq. (2.18) need to be considered.

In order to work directly with the wavelet transform coefficients, it is needed the relationship between the expansion coefficients at a lower scale level in terms of those at a higher scale. A function $f(t)\in V_{j+1}$ is expressible at scale $j+1$ with scaling functions.

$$f(t)=\sum_k c_{j+1}(k)\varphi_{j+1,k}(t) \tag{2.27}$$

At one scale lower resolution, wavelets are necessary for the detail not available at scale of $j+1$.

$$f(t) = \sum_k c_j(k)\varphi_{j,k}(t) + \sum_k d_j(k)\psi_{j,k}(t) \tag{2.28}$$

The relations between $c_{j+1}(k)$ and $c_j(l)$, $d_j(l)$ are defined in [29] as

$$c_j(k) = \sum_k h_0(m-2k)c_{j+1}(m) \tag{2.29}$$

$$d_j(k) = \sum_k h_1(m-2k)d_{j+1}(m) \tag{2.30}$$

In these equations, it is shown that the scale $j+1$ coefficients are " filtered " by two, lowpass and highpass FIR digital filters with coefficients $h_0(n)$ and $h_1(n)$, respectively.

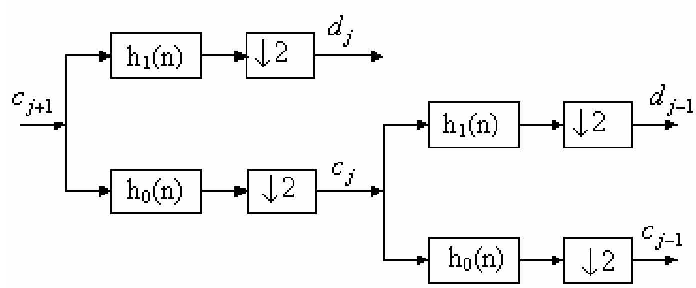

Figure 2.3: Two-Stage Two-Band Analysis Tree

This operation is known as the Fast Wavelet Transform (FWT). The FWT is implemented as two stage two-band Analysis Tree, which is shown in Figure 2.3.

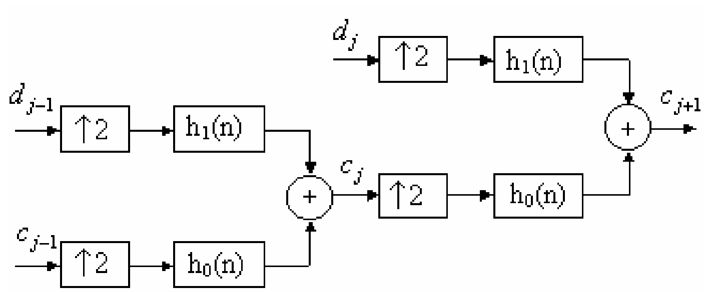

Figure 2.4: Two-Stage Two-Band Synthesis Tree

Similarly, the synthesis part is depicted in Figure 2.4. The c_{j+1} coefficients are obtained from the relation

$$c_{j+1}(k) = \sum_m c_j(m) h_0(k-2m) + \sum_m d_j(m) h_1(k-2m) \tag{2.31}$$

which is called the Inverse Wavelet Transform implementation. It is worth to note that the coefficients are upsampled before filtered.

CHAPTER 3

SYSTEM ANALYSIS

The general information about linear time-varying systems and their solution methods were given in chapter 1. But as mentioned before the analysis of LTVS is more complicated than the analysis of linear time-invariant systems. There are different analysis methods in time domain and frequency domain. Particularly, the spectral analysis method using Fourier domain is one of the well-known methods to find steady-state solution of periodically time-varying system [4]

In two and a half decades, the wavelet transform (WT) attracts the attention of scientists and its many different applications are seen in literature. The compact supportness and orthogonality properties of wavelets [1] make them a suitable tool for the analysis of LTVS. In literature, the solution of system equation in state-space representation is restrictively defined for integral operator using only Haar wavelets [2,3]. In this study, a general solution method in the wavelet domain for the analysis of linear time-varying systems is introduced

Dynamic equations of LTV systems can be converted to algebraic matrix-vector relations by operator matrices. It is possible and easy to define operator matrices for orthogonal wavelet series. Besides, compact support property gives us an opportunity to define a signal, which has local abrupt variation, in wavelet series expansion. Other orthogonal functions like Fourier or Bessel series are insensitive for this type of signals because of their global support.

In this chapter, the system equations will be expressed in wavelet domain as algebraic relations in terms of matrix-vector relations. The method computes the steady-state solution from the system equations whether in differential equation or state-space representation. In order to define time-variance of a signal in wavelet domain, firstly, wavelet modem matrix is formed. Then, operator matrices for different operators such as derivative, integral, multiplication and addition are presented. The method is illustratively applied on simple examples. The result is compared with the analytical solution.

3.1 OPERATOR ANALYSIS IN WAVELET DOMAIN

Let us first make a general observation about the representation of a linear operator T and wavelets. Suppose that $x(t)$ has the representation

$$x(t)=\sum_k x_J(k)\varphi_{j,k}(t) \tag{3.1}$$

Then, the operator effect on $x(t)$ is written as

$$Tx(t)=\sum_k x_J(k)T\varphi_{j,k}(t) \tag{3.2}$$

and, using the wavelet representation of the function $T\varphi_{j,k}(t)$, the equation becomes

$$\begin{aligned} Tx(t) &= \sum_k x_J(k)\sum_l < T\varphi_{j,k}(t),\varphi_{J,l}(t) > \varphi_{J,l}(t) \\ &= \sum_l (\sum_k < T\varphi_{j,k}(t),\varphi_{J,l}(t) > x_J(k))\varphi_{J,l}(t) \end{aligned} \tag{3.3}$$

In other words, the action of the operator T on the function $x(t)$ is directly translated into the action of the infinite matrix $A_T=\{< T\varphi_{J,k}(t),\varphi_{J,l}(t) >\}_{Jl,Jk}$ in the sequence $x_J(k)$. This representation of T as the matrix A_T is often referred to as the "standard representation " of T [34] The effect of the operator T on the orthogonal series is written in terms of the operator matrices. The main concept around these properties is the fact that the integral or derivative of an orthogonal series may be also expressed as an orthogonal series. All these properties provide a way to transform system dynamic equations into algebraic equations. The concept may be in fact related to similar property observed in Laplace transform. In recent years many papers have written to define the operator matrices in several domains. [4], [35-37] Generally, depends on the LTV system structure, the system equation may contain different linear operators such as derivative, integral, time delay.

Any linear operator on a signal or algebraic relations of signals can be transferred to matrix-vector relation by using the operator property of orthogonal series. In order to show this transformation approach, let us assume that the signals $x(t)$, and $y(t) \in L^2$ has a relation given as,

$$y(t) = T\{x(t)\} \tag{3.4}$$

where T is a linear operator. In wavelet domain, this relation can be expressed algebraically as

$$\mathbf{Y} = \mathbf{TX} \tag{3.5}$$

or explicitly

$$\begin{bmatrix} y_0 \\ y_{-1} \\ y_{+1} \\ y_{-2} \\ y_{+2} \\ \vdots \end{bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & \cdots \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & \\ & & \vdots & & & \end{bmatrix} \begin{bmatrix} x_0 \\ x_{-1} \\ x_{+1} \\ x_{-2} \\ x_{+2} \\ \vdots \end{bmatrix} \tag{3.6}$$

where $\mathbf{Y}$ and $\mathbf{X}$ are wavelet coefficient vectors of $y(t)$ and $x(t)$, respectively and T is the wavelet modem matrix corresponding to time operator. If operator T represents an nth order derivative or integral of $x(t)$, then the wavelet domain relation can be found recursively.

$$\mathbf{Y} = \mathbf{T}^n\mathbf{X} \tag{3.7}$$

where $\mathbf{T}^n$ operator matrix of nth order integral or derivative.

This theoretical approach will be shown on the specific operators to derive different operator matrices in the following subsections.

3.1.1 Wavelet Modem Matrix

In a time-varying network, if a signal $x(t)$ is multiplied by a scalar function $m(t)$, and the new signal $y(t)$ is calculated by the relation

$$y(t) = m(t)x(t) \tag{3.8}$$

Then, The relation is expressed in a wavelet expansion by using the Eq. (2.24)

$$\sum_n y_J(n)2^{J/2}\varphi(2^J t - n) = (\sum_l m_J(l)2^{J/2}\varphi(2^J t - l))(\sum_k x_J(k)2^{J/2}\varphi(2^J t - k)) \tag{3.9}$$

It is known that the scaling functions are compact support functions, thus the multiplications of scaling functions can be written as,

$$\varphi(2^J - k)\varphi(2^J - l) = 0 \qquad for \quad k \neq l$$
$$\varphi(2^J - k)\varphi(2^J - l) = \varphi^2(2^J - k) \qquad for \quad k = l$$

Thus the Eq. (3.9) is simply written as,

$$\sum_n y_J(n)2^{J/2}\varphi(2^J t - n) = \sum_k m_J(k)x_J(k)2^J\varphi^2(2^J t - k) \tag{3.10}$$

In this equation the wavelet coefficients $y_J(n)$ are obtained by inner product

$$y_J(n) =< m(t)x(t), \varphi_{j,n}(t) > \tag{3.11}$$

Hence

$$y_J(n) = \int_{-\infty}^{\infty} \sum_k m_J(k)x_J(k)2^J\varphi^2(2^J t - k)2^{J/2}\varphi(2^J t - n)$$

$$= \sum_k m_J(k) x_J(k) \int_{-\infty}^{\infty} 2^J \varphi^2(2^J t - k) 2^{J/2} \varphi(2^J t - n) dt \tag{3.12}$$

After the change of variable as $\tau = 2^J t - k$, it becomes

$$y_J(n) = \sum_k 2^{J/2} m_J(k) x_J(k) \int_{-\infty}^{\infty} \varphi^2(\tau) \varphi(\tau + k - n) d\tau \tag{3.13}$$

The compact supportness of scaling function $\varphi(t)$ provides the following expressions.

$$\int_{-\infty}^{\infty} \varphi^2(\tau) \varphi(\tau + k - n) d\tau = 0 \qquad for \quad k \neq n \tag{3.14}$$

$$\int_{-\infty}^{\infty} \varphi^2(\tau) \varphi(\tau + k - n) d\tau = \int_{-\infty}^{\infty} \varphi^3(\tau) d\tau = A \qquad for \quad k = n \tag{3.15}$$

It is clearly seen that in order to get $y_J(n)$ the value A must be computed. It is necessary some mathematical manipulation for A computation. The Eq. (3.15) is rewritten as

$$\int_{-\infty}^{\infty} \varphi(\tau) \varphi(\tau + m) \varphi(\tau + n) d\tau = A \delta(m) \delta(n) \tag{3.16}$$

where $\delta(\cdot)$ represents knocker delta function. Summing both sides over m, n gives

$$\sum_m \sum_n \int_{-\infty}^{\infty} \varphi(\tau) \varphi(\tau + m) \varphi(\tau + n) d\tau = A \tag{3.17}$$

after reordering, which is

$$\int_{-\infty}^{\infty} \varphi(\tau) \sum_m \varphi(\tau + m) \sum_m \varphi(\tau + n) d\tau = A \tag{3.18}$$

The relation given in [29] is

$$\sum_m \varphi(\tau + m) = \int \varphi(\tau) d\tau = A_0 \tag{3.19}$$

and using $\int \varphi(\tau) d\tau = 1$, therefore, it is easily seen that $A = A^3{}_0 = 1$

Hence, the wavelet coefficients $y_J(n)$ is computed as,

$$y_J(n) = 2^{J/2} m_J(n) x_J(n) \quad , \quad n = 0, \pm 1, \pm 2, \cdots \tag{3.20}$$

Therefore the time domain relation $y(t) = m(t)x(t)$ is transferred to wavelet domain by using the wavelet coefficients of $y(t)$, $m(t)$ and $x(t)$ signals, that is

$$\mathbf{Y = MX} \tag{3.21}$$

where $\mathbf{Y}$ and $\mathbf{X}$ are the wavelet coefficients vectors of $y(t)$ and $x(t)$, respectively, $\mathbf{M}$ is the wavelet modem matrix of $m(t)$.

$$\begin{bmatrix} y_0 \\ y_{-1} \\ y_{+1} \\ y_{-2} \\ y_{+2} \\ \vdots \end{bmatrix} = 2^{J/2} \begin{pmatrix} m_0 & & & & & 0 \\ & m_{-1} & & & & \\ & & m_{+1} & & & \\ & & & m_{-2} & & \\ & & & & m_{+2} & \\ 0 & & & & & \ddots \end{pmatrix} \begin{bmatrix} x_0 \\ x_{-1} \\ x_{+1} \\ x_{-2} \\ x_{+2} \\ \vdots \end{bmatrix} \tag{3.22}$$

Similarly, if $y(t) = m(t)n(t)x(t)$ relation, the corresponding wavelet domain relation becomes

$$\mathbf{Y = MNX} \tag{3.23}$$

It is seen that for individual multiplication functions, the wavelet modem matrices are multiplied.

3.1.2 Wavelet Derivative Matrix

Consider two signals $y(t)$ and $x(t) \in L^2(R)$. If the signals $y(t)$ is derivative of $x(t)$, i.e,

$$y(t) = \frac{d}{dt}(x(t)) \tag{3.24}$$

In order to find the corresponding equation in wavelet domain, the discrete wavelet expansions of the signals are carried into the above equation, such that

$$\sum_n y_J(n) 2^{J/2} \varphi(2^J t - n) = \frac{d}{dt} \sum_k x_J(k) 2^{J/2} \varphi(2^J t - k) \tag{3.25}$$

The wavelet coefficients $y_J(n)$ can be obtained by inner product of $x'(t)$ and $\varphi_{J,n}(t)$, where ` (prime) represents time derivative; that is

$$y_J(n) = \sum_k x_J(k) 2^{2J} \int_{-\infty}^{\infty} \varphi'(2^J t - k) \varphi(2^J t - n) dt \tag{3.26}$$

By change of variable $\tau = 2^J t - k$

$$y_J(n) = \sum_k x_J(k) 2^J \int_{-\infty}^{\infty} \varphi'(\tau) \varphi(\tau - (n-k)) d\tau \tag{3.27}$$

Let us call the integral part of the equation as r_i,

$$r_i = \int_{-\infty}^{\infty} \varphi'(\tau) \varphi(\tau - i) dt \tag{3.28}$$

Then the $y_J(n)$ becomes

$$y_J(n) = \sum_k x_J(k) 2^J r_{n-k} \quad , \quad n = 0, \pm 1, \pm 2, \cdots \tag{3.29}$$

which is the discrete wavelet coefficient relations of $y(t)$ and $x'(t)$. Therefore, the representation of $\frac{d}{dt}$ is completely determined from r_i. If the Eq. (3.29) is written in matrix-vector relation, it is obtained that

$$\mathbf{Y} = \mathbf{DX} \qquad \text{with} \qquad \mathbf{D} = 2^J \begin{pmatrix} r_0 & r_{+1} & r_{-1} & r_{+2} & r_{-2} & \\ r_{-1} & r_0 & r_{-2} & r_{+1} & r_{-3} & \\ r_{+1} & r_{+2} & r_0 & r_{+3} & r_{-1} & \cdots \\ r_{-2} & r_{-1} & r_{-3} & r_0 & r_{-4} & \\ r_{+2} & r_{+3} & r_{+1} & r_{+4} & r_0 & \\ & & \vdots & & & \end{pmatrix} \tag{3.30}$$

where, $\mathbf{Y}$ and $\mathbf{X}$ are the wavelet coefficients vectors. The matrix $\mathbf{D}$ is called the wavelet derivative matrix. Note that the derivative operator has higher order derivative such as n, then the matrix-vector relation in wavelet domain for nth order derivative is like that

$$\mathbf{Y} = \mathbf{D}^n \mathbf{X} \tag{3.31}$$

In order to compute the coefficients r_i, the condition

$$\sum_{i=1}^{L-2} i r_i = -1 \tag{3.32}$$

must be satisfied [34]. The relation for r_i coefficients are given as

$$r_i = 2[r_{2i} + \frac{1}{2} \sum_{k=1}^{L/2} a_{2k-1} (r_{2i-2k+1} + r_{2i+2k-1})] \tag{3.33}$$

It is stated in [34] that finite numbers of r_i are different from zero, that is

$r_i \neq 0 \quad for \quad -L+2 \leq i \leq L-2$ and $r_i = -r_{-i}$. The Eq.(3.33) can be solved iteratively with setting r_i coefficients initial values as,

$$r_i = [\cdots \quad 0 \quad 0.5 \quad 0 \quad -0.5 \quad 0 \quad \cdots]$$

The a_n coefficients are the autocorrelation coefficients of $H = \{h_k\}_{k=0}^{k=L-1}$ given in the form of

$$a_n = 2\sum_{i=0}^{L-1-n} h(i)h(i+n) \qquad n = 1,...L-1 \tag{3.34}$$

Thus the system of equations for r_i will be solved.

3.1.3 Wavelet Integral Matrix

Suppose that $x(t)$ and $y(t) \in L^2(R)$ and $y(t)$ is integral of $x(t)$,

$$y(t) = \int_{-\infty}^{t} x(t')dt' \tag{3.35}$$

This equation is expressed by using the discrete wavelet expansion of signals $y(t)$ and $x(t)$ as

$$\sum_n y_J(n)2^{J/2}\varphi(2^J t - n) = \int_{-\infty}^{t} \sum_k x_J(k)2^{J/2}\varphi(2^J t' - k)dt' \tag{3.36}$$

The wavelet coefficients of $y(t)$ are derived by inner product of $y(t)$ with $\varphi_{J,n}(t)$.

$$y_J(n) = \int_{-\infty}^{\infty} (\int_{-\infty}^{t} \sum_k x_J(k)2^{J/2}\varphi(2^J t' - k)dt')2^{J/2}\varphi(2^J t - n)dt \tag{3.37}$$

Since the $\varphi_{J,n}(t)$ is a compact support in $[\frac{n}{2^J},\frac{n+1}{2^J}]$, the boundaries of inner integral can be changed as

$$y_J(n)=\int_{-\infty}^{\infty}(\int_{-\infty}^{\frac{n+1}{2^J}}\sum_k x_J(k)2^{J/2}\varphi(2^J t'-k)dt')2^{J/2}\varphi(2^J t-n)dt \tag{3.38}$$

This equation is modified by change of variable as $u=2^J t-k$

$$y_J(n)=\int_{-\infty}^{\infty}(\int_{-\infty}^{n+1-k}\sum_k x_J(k)2^{-J/2}\varphi(u)du)2^{J/2}\varphi(2^J t-n)dt \tag{3.39}$$

In this relation $\varphi(u)$ is also compact support and it is nonzero in $[0,1]$. Thus the inner integral is nonzero for only $k\leq n$. Therefore the equation becomes

$$y_J(n)=\int_{-\infty}^{\infty}(\sum_k^n x_J(k)2^{-J/2}\int_0^1\varphi(u)du)2^{J/2}\varphi(2^J t-n)dt \tag{3.40}$$

Using the previously given relation $\int_{-\infty}^{\infty}\varphi(\tau)d\tau=1$, the equation is simply written as

$$y_J(n)=\sum_k^n x_J(k)\int_{-\infty}^{\infty}\varphi(2^J t-n)dt \tag{3.41}$$

In more compact form, by change of variable $\tau=2^J t-n$, it is easily obtained, that

$$y_J(n)=\sum_k^n x_J(k)2^{-J} \quad , \quad n=0,\pm1,\pm2,\cdots \tag{3.42}$$

If the integral expression $y(t)=\int_{-\infty}^{t}x(t')dt'$ is written in wavelet domain in matrix-vector form as,

$$\mathbf{Y = PX} \quad \text{with} \quad \mathbf{P} = 2^{-J}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 0 & \\ 1 & 1 & 1 & 1 & 1 & \\ & & \vdots & & & \end{pmatrix} \tag{3.43}$$

where $\mathbf{Y}$ and $\mathbf{X}$ are the wavelet coefficients vectors. The matrix $\mathbf{P}$ is called the wavelet integral matrix. The nth order integration is expressed in wavelet domain as

$$\mathbf{Y = P}^n\mathbf{X} \tag{3.44}$$

It is seen that the nth order integral corresponds to nth power of wavelet integral matrix.

3.1.4 Addition Matrix

It is previously spoken that the wavelet and scaling functions are compact-support functions. Using this property, the wavelet expansion of two additional functions is simply defined by the addition of wavelet and scaling coefficients, correspondingly. Let $a(t)$, $b(t)$ and $c(t)$ be $\in L^2(R)$ and by the relation $a(t) = b(t) + c(t)$. The discrete wavelet expansion of this relation is given,

$$\sum_k a_J(k)\varphi_{J,k}(t) = \sum_k b_J(k)\varphi_{J,k}(t) + \sum_k c_J(k)\varphi_{J,k}(t) \tag{3.45}$$

It is obviously seen that $a_J(k)$ wavelet coefficients are the addition of $b_J(k) + c_J(k)$ wavelet coefficients. Therefore the wavelet addition matrix is given as

$$\mathbf{A = B + C} \quad where \quad a_{i,j} = diag[b_{i,j} + c_{i,j}] \quad for \quad i, j = 0, \pm 1, \pm 2 \cdots \tag{3.46}$$

All operator matrices that are defined are computerized by writing functions and subprograms in MatLab. These functions and subprograms are given in Appendix A1

3.2 SOLUTION OF LTV SYSTEMS IN WAVELET DOMAIN

Consider the differential equation of linear time-varying system

$$a_n(t)\frac{d^n y}{dt^n}+\ldots+a_1(t)\frac{dy}{dt}+a_0(t)y(t)=b_m(t)\frac{d^m u}{dt^m}+\ldots b_1(t)\frac{du}{dt}+b_0(t)u(t)$$

This equation can be written in wavelet domain using the operator matrices. By using the previously defined wavelet derivative matrix and wavelet modem matrices, the wavelet domain expression for the LTV system differential equation is expressed as

$$[\mathbf{M}_{an}\mathbf{D}^n+\ldots+\mathbf{M}_{a1}\mathbf{D}+\mathbf{M}_{a0}]\mathbf{Y}=[\mathbf{M}_{bm}\mathbf{D}^m+\ldots+\mathbf{M}_{b1}\mathbf{D}+\mathbf{M}_{b0}]\mathbf{U} \qquad (3.47)$$

where $\mathbf{D}^i$ is the ith order wavelet derivative matrix, $\mathbf{M}_{ai}$, and $\mathbf{M}_{bi}$ are wavelet modem matrices of the functions $a_i(t)$, and $b_i(t)$ respectively. The wavelet coefficients vector $\mathbf{Y}$ represents the output function $y(t)$ and similarly $\mathbf{U}$ is the wavelet coefficients vector of input function $u(t)$. By assuming that the coefficient matrix of $\mathbf{Y}$ is nonsingular, the output vector can be computed from the relation

$$\mathbf{Y}=[\mathbf{M}_{an}\mathbf{D}^n+\ldots+\mathbf{M}_{a1}\mathbf{D}+\mathbf{M}_{a0}]^{-1}[\mathbf{M}_{bm}\mathbf{D}^m+\ldots+\mathbf{M}_{b1}\mathbf{D}+\mathbf{M}_{b0}]\mathbf{U} \qquad (3.48)$$

Theoretically, the dimensions of these matrices are infinite but for a practical application the system solution can be obtained by considering significant wavelet coefficients. Therefore the actual dimension of matrices depends on the number of wavelet coefficients considered in the solution.

Consider the state-space representation of a LTV system.

$$x'(t)=A(t)x(t)+B(t)u(t)$$
$$y(t)=C(t)x(t)+D(t)u(t)$$

The wavelet domain counterparts of these equations are

$$\mathbf{DX} = \mathbf{M_A X} + \mathbf{M_B U} \tag{3.49}$$

$$\mathbf{Y} = \mathbf{M_C X} + \mathbf{M_D U} \tag{3.50}$$

The state vector of LTV system is easily obtained as

$$\mathbf{X} = [\mathbf{D} - \mathbf{M_A}]^{-1}\mathbf{M_B U} \tag{3.51}$$

where it is assumed that matrix $[\mathbf{D} - \mathbf{M_A}]^{-1}$ is nonsingular.

By putting the state solution wavelet vector into the response equation, then the output of system is found as

$$\mathbf{Y} = [\mathbf{M_C}[\mathbf{D} - \mathbf{M_A}]^{-1}\mathbf{M_B} + \mathbf{M_D}]\mathbf{U} \tag{3.52}$$

It is seen that the output wavelet coefficient vector $\mathbf{Y}$ is defined in terms of algebraic matrix relations. The time domain expression of $y(t)$ can be computed from,

$$y(t) = \sum_{l} y_J(l) 2^{J/2} \varphi(2^J t - l) \tag{3.53}$$

Thus, all necessary fundamental derivations to analyze a linear time –varying system in wavelet domain is completed. All these theoretical derivations are applied on the following examples.

Example 1: Consider the linear time-varying RC circuit shown in Figure 3.1.

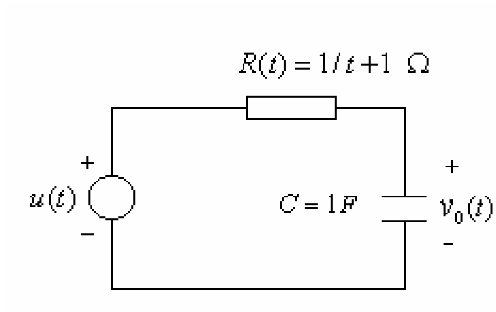

Figure 3.1: Time varying RC circuit.

The network equation is defined in differential equation form as

$$\frac{1}{t+1}\frac{dv_0}{dt} + v_0 = u(t) \quad \text{with the initial condition } v_0(0) = 0 \tag{3.54}$$

This first order time-varying differential equation is solved and the analytical solution is $v_0(t) = 1 - e^{-\frac{t^2}{2}-t}$. System differential equation is transferred to wavelet domain, and it is expressed as a matrix-vector relation in the form of

$$(\mathbf{M}_1\mathbf{D} + \mathbf{I})\mathbf{V} = \mathbf{U_1} \tag{3.55}$$

where $\mathbf{M}_1$ and $\mathbf{I}$ are the wavelet modem matrices of time function $1/t+1$ and constant coefficient 1, respectively. The $\mathbf{U_1}$ is the input wavelet coefficients vector of input function $u(t)$. The response vector $\mathbf{V}$ can be simply written as

$$\mathbf{V} = (\mathbf{M_1}\mathbf{D} + \mathbf{I})^{-1}\mathbf{U_1} \tag{3.56}$$

Particularly, in this example the wavelet matrix coefficients have been calculated using fourth order Daubechies (db4) wavelet and for $J = 2$ resolution value, given as

$$\mathbf{D}=\begin{bmatrix} 0 & 3.172 & -0.768 & \\ -3.172 & 0 & 3.172 & \cdots \\ 0.768 & -3.172 & 0 & \\ & \vdots & & \end{bmatrix},\ \mathbf{M}_1=\begin{bmatrix} 1 & & & 0 \\ & 0.8 & & \\ & & 0.67 & \\ 0 & & & \ddots \end{bmatrix},\mathbf{I}=\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{bmatrix},$$

For the input wavelet coefficient vector

$$\mathbf{U}_1=\begin{bmatrix}0.5 & 0.5 & 0.5 & \cdots\end{bmatrix}^T$$

The response wavelet coefficient vector is computed as

$$\mathbf{V}=\begin{bmatrix}0.083 & 0.19 & 0.28 & 0.35 & \cdots\end{bmatrix}^T$$

Then, the sampled response vector as capacitor voltage in steady state is obtained from Eq. (3.53) as

$$v(n)=\begin{bmatrix}0.17 & 0.37 & 0.55 & \cdots\end{bmatrix}^T$$

The analytical solution and wavelet domain solutions are depicted in Figure 3.2. In this figure the efficiency of the method is clearly observed. The error between analytical solution and the wavelet domain solution is very small.

In order to compute the total percentage error between the exact solution and the approximate solution, the error is defined by the relation

$$e_{total}=\frac{\sum_{n=1}^{N}\left|f_{exact}(n)-f_{approx}(n)\right|}{\sum_{n=1}^{N}\left|f_{exact}(n)\right|}100 \tag{3.57}$$

where N is the total number of sampling points. The sequences $f_{exact}(n)$ and $f_{approx}(n)$ are the exact and approximate solutions, respectively.

In this example, the total percentage error is computed 0.67 for the capacitor voltage with N=81 sample points.

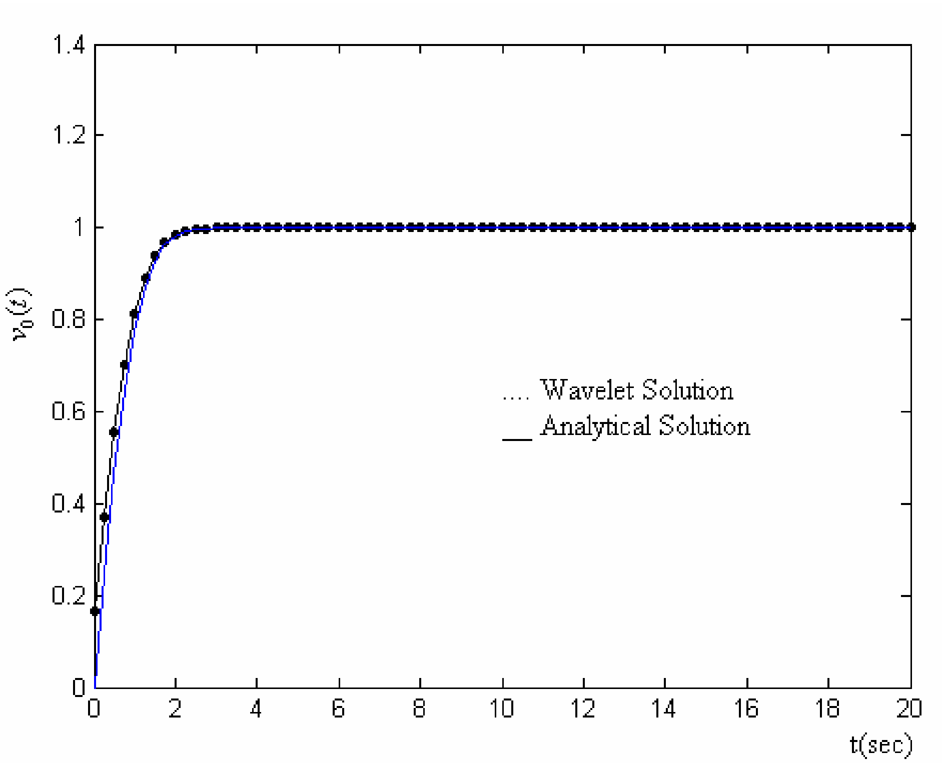

Figure 3.2:Analytical and Wavelet Domain Solutions of $v_0(t)$

Example 2: Consider the state-space representation of a LTV system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -2t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ with the initial condition } \mathbf{x(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.58}$$

The analytic solution of this system is expressed for $x_1(t)$ and $x_2(t)$

$$x_1(t) = \frac{1}{3} - \frac{1}{3}e^{-3t} \quad \text{and} \quad x_2(t) = \frac{1}{2} - \frac{1}{2}e^{-t^2} \tag{3.59}$$

The wavelet domain counterpart of Eq.(3.58) is as follows,

$$\begin{bmatrix} \mathbf{D} & 0 \\ 0 & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{-3} & 0 \\ 0 & \mathbf{M}_{-2t} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_t \end{bmatrix} \tag{3.60}$$

In this expression $\mathbf{U}_t$ and $\mathbf{U}_1$ are the wavelet coefficients vectors of $f(t) = t$ and constant 1, respectively. The wavelet coefficients vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ can be found as,

$$\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} [\mathbf{D} - \mathbf{M}_{-3}]^{-1} & \mathbf{0} \\ \mathbf{0} & [\mathbf{D} - \mathbf{M}_{-2t}]^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_t \end{bmatrix} \tag{3.61}$$

therefore

$$\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} [\mathbf{D} - \mathbf{M}_{-3}]^{-1} \mathbf{U}_1 \\ [\mathbf{D} - \mathbf{M}_{-2t}]^{-1} \mathbf{U}_t \end{bmatrix} \tag{3.62}$$

The wavelet matrix coefficients for $x_1(t)$ and $x_2(t)$ have been calculated using fourth order Daubechies (db4) wavelet and for $J = 2$ resolution value.

$$\mathbf{D} = \begin{bmatrix} 0 & 3.172 & -0.768 & & \\ -3.172 & 0 & 3.172 & \cdots & \\ 0.768 & -3.172 & 0 & & \\ & \vdots & & & \end{bmatrix}, \mathbf{M}_{-3} = \begin{bmatrix} -3 & & & 0 \\ & -3 & & \\ & & -3 & \\ 0 & & & \ddots \end{bmatrix}, \mathbf{U}_1 = [0.5 \quad 0.5 \quad 0.5 \quad \cdots]^T$$

$$\mathbf{M}_{2t} = \begin{bmatrix} 0 & & & 0 \\ & -0.5 & & \\ & & -1 & \\ 0 & & & \ddots \end{bmatrix} \qquad \mathbf{U}_t = [0 \quad 0.125 \quad 0.25 \quad \cdots]^T,$$

The wavelet coefficients vectors of $\mathbf{X}_1$ and $\mathbf{X}_2$ are computed as

$$\mathbf{X}_1 = [\ 0.07 \quad 0.12 \quad 0.15 \quad \cdots]^T$$
$$\mathbf{X}_2 = [-0.01 \quad 0.01 \quad 0.05 \quad \cdots]^T$$

Time domain sampled values of $x_1(t)$ and $x_1(t)$ are computed from Eq. (3.53) as,

$$x_1(n) = [\ 0.14 \quad 0.24 \quad 0.29 \quad \cdots]^T$$
$$x_2(n) = [-0.02 \quad 0.02 \quad 0.09 \quad \cdots]^T$$

The analytic and wavelet domain solution of $x_1(t)$ and $x_2(t)$ is shown in Figure 3.3. It is seen that there are minor differences between the exact and wavelet domain solutions of state variables. The total percentage error of $x_1(t)$ and $x_2(t)$ are 1.03 % and 0.16 %, respectively.

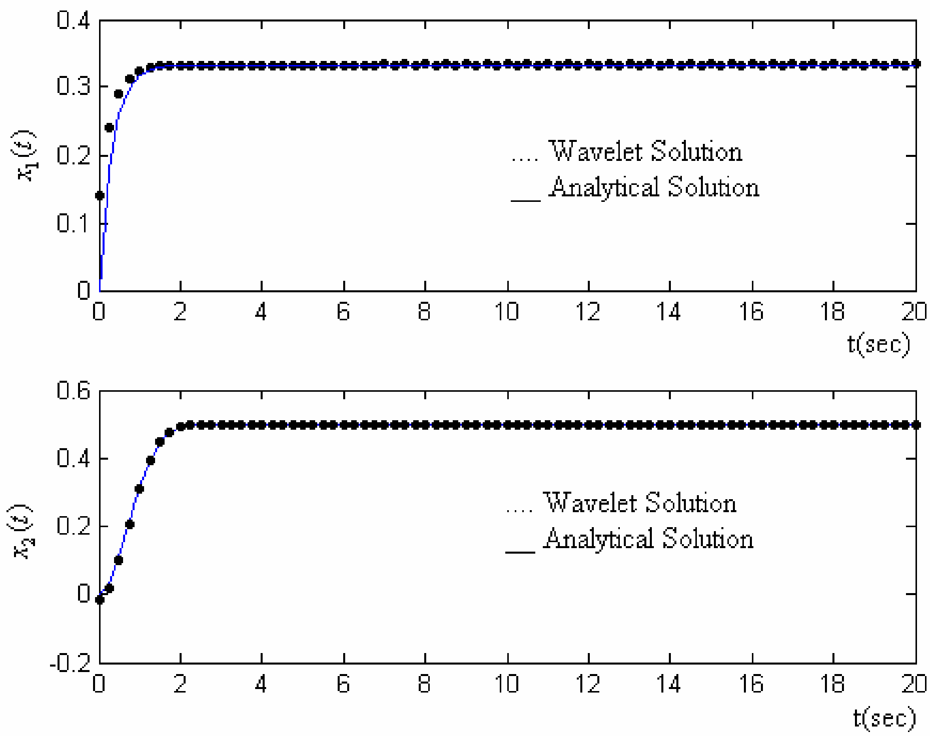

Figure 3.3 Analytical and Wavelet Domain Solutions of $x_1(t)$ and $x_2(t)$

The MatLab functions related with the example 1 and example 2 are given in Appendix A2

CHAPTER 4

RESULTS AND CONCLUSION

In this thesis we have introduced a new method to do analysis of LTV systems in wavelet domain. To solve system equations having the form differential equation or state equation, they are transferred to wavelet domain by forming algebraic matrix-vector relation using the wavelet transform coefficients.

In application of this new method, many examples have been solved and just two of them were put in this work, illustratively. The steady-state solutions of the chosen examples are compared by the exact solutions. It is seen that the error between the analytic solution and wavelet domain solutions is around 1% in total sampling points.

MatLab programs are prepared for all of the matrices related with the time domain operators such as wavelet modem matrix, wavelet derivative matrix, wavelet integral matrix, and wavelet addition matrix. The simulations for the studied examples are also used the MatLab toolbox together with the what we have contribute for them.

This thesis was presented in 11. Elektrik–Elektronik ve Bilgisayar Mühendisliği Ulusal Kongresi (İstanbul 2005) and the paper is published in the conference proceeding [38].

As a further study, the method can be modified in order to do the transient analysis of LTVS besides of steady-state analysis. In application, it is possible to derive all necessary operator matrices for a class of biorthogonal wavelets.

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APPENDIX A1

MATLAB FUNCTIONS

```
*******************************************************************
Mmatrix      Wavelet Modem Matrix
        M = Mmatrix(s1,J) constructs n dimensional wavelet modem matrix
        at resolution level J.
        s1 is the wavelet coefficients vector with related to signal m(t)
        n is the length of the vector s1
        n=length(s1)
*****************************************************************

function M = Mmatrix(s1,J)
n=length(s1);
for m = 1 : n
   M(m,m)=(2^(J/2))*s1(m);
end


*******************************************************************
Dmatrix        Wavelet Derivative Matrix
        D = Dmatrix(n,J,wname) constructs n dimensional wavelet derivative matrix
        at resolution level J with respect to a particular wavelet 'wname',
        'wname' is a string containing the wavelet name.
*****************************************************************

function D = Dmatrix(n,J,wname)

r = rcoef( wname );%rcoef('wname') computes the non-zero value of r
lr=length(r);
```

```
r = [wrev(-r) 0 r];
T = ( 2^J ) * convmtx(r',n+1);
for l = 1 : n + 1
   for k = 1 : n + 1
      D(l,k) = T(l+lr,k);
   end
end
```

```
*********************************************************************
rcoef   Recursive function of Wavelet Modem Matrix
      r=rcoef(wname) computes the entries of Wavelet Derivative Matrix
      recursively with respect to particular wavelet 'wname',
      wname' is a string containing the wavelet name.
*********************************************************************
```

```
function r=rcoef(wname)
ea = .000000000001;
s = 10;
a = autocorrelation( wname );
L = length(a) + 1;
n = L - 2;
rold = r_old( n );%Takes the initial values of r vector.
rnew = r_new( n );
N = length( rold );
while ( s > ea )
for i = 1 : N
   j = 2 * i;
   if ( j > N )
      u = 0;
   else
      u = rold( j );
   end
      temp = 0;
```

```
    for k = 1 : L / 2
        v = (2 * i) - (2 * k) + 1;
        z = (2 * i) + (2 * k) - 1;
        if ( ( v < -N ) | ( v > N ))
            v = 0;
        elseif ( ( v < 0 ) & ( v > -N ) )
            v = -rold( -v );
        else
            v = rold( v );
        end

        if ( ( z < -N ) | ( z > N ))
            z = 0;
        elseif ( ( z < 0 ) & ( z > -N ))
            z = -rold( -z );
        else
            z = rold( z );
        end
        temp = temp + a( 2 * k - 1 ) * ( v + z );
    end
    rnew( i ) = 2 * u + temp;
end
s = abs ( ( rnew( 1 ) - rold( 1 ) ) / rnew( 1 ) ) * 100;
rold = rnew;
end


******************************************************************
autcorrelation   Autocorrelation function of wavelets.
            a = autocorrelation( wname ) computes the autocorrelation coefficiensts of
            lo_d with respect to particular wavelet 'wname',
            [lo_d,hi_d,lo_r,hi_r]=wfilters(wname)
            'wname' is a string containing the wavelet name.
********************************************************************
```

```
function a = autocorrelation( wname )
[lo_d,hi_d,lo_r,hi_r]=wfilters(wname);
for n = 1:length(lo_d)-1
    t = 0;
    for i = 1:length(lo_d)-n
        t = t + lo_d( i ) * lo_d( i+n );
    end
    a(n) = 2 * t;
end
u = floor( length( a ) / 2 );
i = 1:u;
a( 2 * i ) = 0;



function rold = r_old(n)
rold = [-0.5 0];%Initial values of r vector.
for i = 1 :n-2
    rold = [rold 0];
end



function rnew = r_new(n)
rnew = 0;
for i = 1 : n-1
    rnew = [rnew 0];
end
*******************************************************************
Pmatrix Wavelet Integral Matrix
        P=Pmatrix(n,J) constructs n dimensional wavelet integral matrix
        at resolution level J.
*******************************************************************
function P=Pmatrix(n,J)
for k=1:n+1
```

```
   for l=1:k
      P(k,l)=2^(-J);
   end
end
```

****

Amatrix Wavelet Addition Matrix

A = Amatrix(s1,s2) constructs n dimensional wavelet addition matrix

s1 is the wavelet coefficients vector with related to signal a(t)

s2 is the wavelet coefficients vector with related to signal b(t)

n is the length of the vector s1

n=length(s1)

Length of s1 and s2 vectors must be same.

****

```
function A = Amatrix(s1,s2)
sum = s1 + s2;
lsum = length( sum )
for i = 1 : lsum
   for k = 1 : lsum
      if (i == k)
         A(i,k) = sum( i );
      else
         A(i,k) = 0;
      end
   end
end
```

APPENDIX A2

MatLab function for example 1

```
function [vww,va,D,M1,M2,U1,etp]=example1(t)

J=2;%Resolution level
n = 0 : 1 / (2^J) : t;
u = 0 : 1 / (2^J) : t + t * 0.2;
ln = length( n );
lu = length( u );

for i = 1 : lu;
    s1( i ) = 0.5 * ( 1 / ( u( i ) + 1 ) );
end

D = Dmatrix( lu-1, J, 'db4' );
M1 = Mmatrix( s1, 2 );
m = 1 : lu;
s2( m ) = 0.5 * 1;
M2 = Mmatrix( s2, J );
U1( m ) = 0.5 * 1;

vw_temp = inv( M1 * D + M2 ) * U1';%wavelet coefficients vector
vww = vw_temp( 1:ln );
vw =( 2^(J/2) )*vww;%time domain value of v(t)
va = 1 - 1 * exp( ( -n .* n ) / 2 - n );

% Total percentage error between analytical and wavelet domain solutions
num = 0;
```

```
den = 0;

for i = 1 : ln
   num = num + abs( va( i ) - vw( i ) );
   den = den + abs( va ( i ) );
end
etp = ( 100 * num ) / den;%Total percentage error
```

MatLab function for example 2

```
function [D,U1,M3,x1w,Ut,M2t,x2w,etp1,etp2]=example2(t)

J = 2;
n = 0 : 1 / ( 2^J ) : t;
ln = length( n );
n_temp = 0: 1 / ( 2^J ) : t + t * 0.2;
ln_temp = length( n_temp );

s1( 1 : ln_temp ) = ( 2^( -J/2 ) ) * 1;
U1 = s1';
s2 = -3 * s1;
M3 = Mmatrix( s2,J );
D = Dmatrix( ln_temp-1,J,'db4' );

x1w_temp = inv( D-M3 ) * U1;%wavelet coefficients vector of x1(t)
x1w = ( 2^(J/2) ) * x1w_temp( 1 : ln );% Time domain values of x1(t)
x1a = ( 1 / 3 ) - ( 1 / 3 ) * exp( -3 * n );%Analytical values of x1(t)

D = Dmatrix( ln_temp-1,J,'db4' );
s1 = ( 2^(-J/2) ) * n_temp;
Ut = s1';
s2 = -2 * ( n_temp ) * ( 2^( -J/2 ) );
M2t = Mmatrix( s2,J );
x2w_temp = inv( D-M2t ) * Ut;%wavelet coefficients vector of x2(t)
x2w = ( 2^(J/2) ) * x2w_temp( 1:ln );% Time domain values of x2(t)
```

```
x2a = 0.5 - 0.5 * exp( -( n .* n ) );%Analytical values of x2(t)

%Total percentage errors between Analytical and Wavelet Domain Solutions
num1 = 0;
num2 = 0;
den1 = 0;
den2 = 0;

for i = 1 : ln
    num1 = num1 + abs( x1w( i ) - x1a( i ) );
    den1 = den1 + abs( x1a( i ) );
    num2 = num2 + abs( x2w( i ) - x2a( i ) );
    den2 = den2 + abs( x2a( i ) );
end

etp1 = ( 100 * num1 ) / den1;%Total percentage error for x1(t)
etp2 = ( 100 * num2 ) / den2;%Total percentage error for x2(t)
```