APPLICATIONS OF A NOVEL APPROACH TO RELATIVISTIC AND NON-RELATIVISTIC PROBLEMS IN PHYSICS

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ABSTRACT

APPLICATIONS OF A NOVEL APPROACH TO RELATIVISTIC AND NON-RELATIVISTIC PROBLEMS IN PHYSICS

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We have developed an algebraic approach for the treatment of time-independent Schrödinger equation with constant/non-constant masses within the frame of non-relativistic quantum theory. The model developed then has been successfully applied in various fields of physics involving exactly/approximately solvable potentials. The first part of the thesis work is devoted to the presentation of these applications, such as the application results for non-central potentials, applicability of the scheme for scattering theory and careful analysis of the application results for quantum systems with position-dependent masses in arbitrary dimensions.

After gaining confidence from the successful applications of this novel formalism to non-relativistic systems, we have extended our investigations by studying the applicability of the model also for the relativistic considerations in the light of Klein-Gordon and Dirac equations involving only bound quantum states. These considerations require naturally the split of the relativistic equations into two parts, unlike the other models in the literature, which provide a clear visualization of the relativistic contributions, in an explicit manner, to the solution in the non-relativistic limit.

Key words: Schrödinger Equation, Dirac Equation, Klein-Gordon Equation, Exactly-solvable Systems, Non-central Potentials, Position-dependent Mass.

ÖZET

RÖLATİVİSTİK VE RÖLATİVİSTİK OLMAYAN FİZİK PROBLEMLERİNE YENİ BİR YAKLAŞIMIN UYGULAMALARI

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Rölativistik olmayan kuantum mekaniği çerçevesinde sabit/değişken kütleli zamandan bağımsız Schrödinger dalga denkleminin çözümü için analitik bir metod geliştirildi. Bu metod, tam ve yaklaşık olarak çözülebilen potansiyel içeren farklı fizik problemlerine başarıyla uygulandı. Tez çalışmasının ilk kısmında, bahsedilen uygulama sonuçları tartışıldı. Bu uygulamalardan bazıları söz konusu modelin kullanımı ile merkezcil olmayan potansiyel içeren Schrödinger denkleminin çözümü, modelin quantum saçılma teorisine uygulanabilirliği ve keyfi boyutta değişken kütle içeren Schrödinger denkleminin yeni model çerçevesin- de dikkatle incelenmesidir.

Modelimizin rölativistik olmayan sistemlere başarılı uygulamalarından edindiğimiz güvenle; Klein-Gordon ve Dirac denklemlerini göz önünde bulundurarak, geliştirilen analitik çözüm tekniğinin rölativistik problemlere de uygulanabilirliği araştırıldı. Literatürde mevcut diğer modellerden farklı olarak; yeni uygulama neticesinde rölativistik denklemler, doğal ve belirgin olarak iki ayrı denkleme ayrılır ki bu sonuç rölativistik düzeltmelerin tam ve kesin olarak gözlenebilirliğini sağlar.

Anahtar kelimeler: Schrödinger Denklemi, Dirac Denklemi, Klein-Gordon Denklemi, Tam Çözülebilir Sistemler, Merkezcil olmayan Potansiyeller, Değişken Kütleler.

I dedicate this work to my family
especially to my daughter
Nursena

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CHAPTER 1

INTRODUCTION

An exact solution of physical systems has a great importance. Especially, in the case of Schrödinger equation there is only a few selected problems that can be exactly solvable. At this point perturbation theory (PT), which comprises mathematical methods that are used to find an approximate solution to a problem, has a vital role. PT is applicable if the problem at hand can be formulated by adding a small term to the mathematical description of the exactly solvable problem. In this manner, there are lot's of problems for example magnetic interaction (spin-orbit coupling), the electrostatic repulsion of electrons and the influence of external fields. The inter-connection between exactly-solvable systems and PT has formed one of the purpose of this thesis to reveal a new and more effective method to deal with analytical solution of quantum mechanical problems.

Actually, a number of models have been developed to treat perturbative systems. For instance, performing explicit calculations in non-relativistic quantum mechanics using the familiar Rayleigh-Schrödinger perturbation expansion is rendered difficult by the presence of summations over all intermediate unperturbed eigenstates. Alternative perturbation procedures have been proposed to avoid this difficulty, notably the logarithmic perturbation theory (LPT) [1-4] and the Dalgarno-Lewis technique [5-8]. The virtue of LPT is its avoidance of the cumbersome summation over states for second- and higher-order corrections in Rayleigh-Schrödinger perturbation theory. Unfortunately, it has problems of its own in calculating corrections to excited states, owing to presence of nodes in the wave functions. Various schemes have been proposed to circumvent these singularities [4, 9, 10].

Such is the status of LPT after over 20 years of active development. Meanwhile, supersymmetric quantum mechanics (SSQM) [11, 12] has been developed immensely since the first models were introduced [13, 14]. Several approximation methods using SSQM formalism have been proposed, including the supersym-

metric perturbation theory (SSPT) of Cooper and Roy [15]. Recently, Lee [16] has shown that SSPT and LPT are entirely equivalent and fortuitously, each turns out to resolve difficulties encountered in the other. Namely, LPT formulas for energy corrections obviate tedious procedures in the SSQM method, while the use of SSQM partner potentials with virtually identical bound state spectra solves difficulties with excited states encountered in LPT. Although the iterative procedure in SSPT may not actually reduce the calculational workload, it does cast the calculations into a physically-motivated, visualizable framework.

In this thesis, starting from the first principles, we have developed a more economical scheme which yields simple but closed perturbation theory formulae leading to the Riccati equation from which one can actually obtain all the perturbation corrections to both energy level shifts and wave functions for all states, unlike the other models mentioned above. In the application of the present method to the n^{th} excited state, one requires knowledge of the unperturbed eigenfunction but no knowledge of the other eigenvalues or eigenfunctions is necessary. The procedure introduced here does not involve either tedious explicit factoring out of the zeros of unperturbed eigenfunction [1, 2] or introduction of ghost states [4] as were the cases encountered for applying LPT to excited states. The present model also offers explicit expressions for the energy corrections, which are absent in SSPT, and provides a clean route to the excited states, which are cumbersome to analyze in both LPT and SSPT. Therefore, our results can be thought of as a generalization of logarithmic and supersymmetric based perturbation theories. This is one of the vital points in the present work.

In the following Chapter we have introduced the model and discussed briefly the physics behind the formulation. In Chapter 3 the model applied to non-relativistic problems. Section 3.1 involves an application leading to closed solutions which provide a search the reason behind exact solvability of some non-central but separable potentials. To the best of our knowledge, this question has not been discussed in the literature. To show the effectiveness of the formalism, we also have extended it to the scattering domain in Section 3.2 and compared the results obtained with those in the literature. In Section 3.3 we have considered effective mass Hamiltonians with spatially varying mass within the frame of the model underlined and discussed the interrelation between the exact solvability and ambiguity parameters in such Hamiltonians. This is one of the ongoing debates in the literature. Finally, Section 3.4 involves a careful extension of one-dimensional calculations to higher dimensions.

Chapter 4 illustrates further applications of the formalism to some other interesting problems in the relativistic region. For instance, Section 4.1 involves powerful applications of the model of interest via the Klein-Gordon equation in-

volving different interactions. These applications clarify the appearance of the relativistic equation in the non-relativistic limit for the potentials considered and shed a light to discuss explicitly relativistic contributions. In the similar manner, we have analyzed Dirac equation in Section 4.2, and suggested a refined and improved model to solve relativistic equations in terms of orthogonal polynomials. This more flexible and elegant prescription works for all exactly and quasi-exactly solvable potential family. In addition, now the excited state wave functions can be easily expressed simultaneously with the ground-state solution, unlike the previous model used through the thesis work. This is a considerable improvement in the formalism presented. We finally note that, the compact scheme of this remarkable new formalism is also applicable for the all non-relativistic problems with constant and position-dependent masses.

CHAPTER 2

FORMALISM

An exact solution of the Schrödinger equation exists only for a few idealized problems; hence sometimes it has to be solved using an approximation method such as the perturbation theories (PT) available in the literature [1-10, 15, 16], which constitute one of the most powerful tools available in the study of quantum mechanics in the atoms and molecules. PT are applied to those cases in which the real system can be described by a small change in an exactly solvable idealized system. In this form we can describe a great number of problems encountered especially in atomic physics, in which the nucleus provides the strong central potential for the electrons; further interactions of less strength are described by the perturbation. But in spite of widespread application of this theory, its basic analytical properties are poorly understood. Within this context, the main objective in this Chapter is to use the spirit of the perturbation theories to be able to introduce a novel and more flexible formalism to treat all the potentials in non-relativistic quantum theory, which would also clarify the vague points in PT.

Let us first start with basic definitions in the supersymmetric quantum theory (SUSYQM), which is well known in the literature [11]. The goal in SSQM is to solve the Riccati equation,

$$W^2(r) - \frac{\hbar}{\sqrt{2m}} W'(r) = V(r) - E_0 \ , \tag{2.1}$$

where $V(r)$ is the potential of interest and E_0 is the corresponding ground state energy. If we find $W(r)$, the so called superpotential, we have of course found the ground state wave function via,

$$\psi_0(r) = N \ \exp\left[-\frac{\sqrt{2m}}{\hbar} \int^r W(z) dz \right] \ , \tag{2.2}$$

where N is the normalization constant. If $V(r)$ is a shape invariant potential, we can in fact obtain the entire spectrum of bound state energies and wave functions

via ladder operators [11]. Through the present work, this basic ingredient of SSQM given by (2.1) and (2.2) will be later extended and used for the treatment of excited states.

Now, suppose that we are interested in a potential for which we do not know $W(r)$ exactly, and the corresponding Hamiltonian is not factorizable but almost factorizable. More specifically, we assume that $V(r)$ differs by a small amount from a potential $V_0(r)$ plus angular momentum barrier if any, for which one solves the Riccati equation explicitly. For the consideration of spherically symmetric potentials, the corresponding Schrödinger equation for the radial wave function has the form

$$\frac{\hbar^2}{2m}\frac{\psi_n''(r)}{\psi_n(r)} = [V(r) - E_n] \quad , \quad V(r) = \left[V_0(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right] + \Delta V(r), \tag{2.3}$$

where ΔV is a perturbing potential. Let us write the wave function ψ_n as

$$\psi_n(r) = \chi_n(r)\phi_n(r) \ , \tag{2.4}$$

in which χ_n is the known normalized eigenfunction of the unperturbed Schrödinger equation whereas ϕ_n is a moderating function corresponding to the perturbing potential. Substituting (2.4) into (2.3) yields

$$\frac{\hbar^2}{2m}\left(\frac{\chi_n''}{\chi_n} + \frac{\phi_n''}{\phi_n} + 2\frac{\chi_n'}{\chi_n}\frac{\phi_n'}{\phi_n}\right) = V - E_n \ . \tag{2.5}$$

Instead of setting the functions χ_n and ϕ_n, we will set their logarithmic derivatives using the spirit of Eqs. (2.1) and (2.2);

$$W_n = -\frac{\hbar}{\sqrt{2m}}\frac{\chi_n'}{\chi_n} \quad , \quad \Delta W_n = -\frac{\hbar}{\sqrt{2m}}\frac{\phi_n'}{\phi_n} \tag{2.6}$$

which leads to

$$\frac{\hbar^2}{2m}\frac{\chi_n''}{\chi_n} = W_n^2 - \frac{\hbar}{\sqrt{2m}}W_n' = \left[V_0(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right] - \epsilon_n \ , \tag{2.7}$$

where ϵ_n is the eigenvalue of the unperturbed and exactly solvable potential, and

$$\frac{\hbar^2}{2m}\left(\frac{\phi_n''}{\phi_n} + 2\frac{\chi_n'}{\chi_n}\frac{\phi_n'}{\phi_n}\right) = \Delta W_n^2 - \frac{\hbar}{\sqrt{2m}}\Delta W_n' + 2W_n\Delta W_n = \Delta V(r) - \Delta\epsilon_n \ , \tag{2.8}$$

in which $\Delta\epsilon_n$ is the eigenvalue for the perturbed potential, and $E_n = \epsilon_n + \Delta\epsilon_n$. Then, Eq. (2.5), and subsequently Eq. (2.3), reduces to

$$(W_n + \Delta W_n)^2 - \frac{\hbar}{\sqrt{2m}}(W_n + \Delta W_n)' = V - E_n \ , \tag{2.9}$$

which is similar to Eq. (2.1). In principle as one knows explicitly the solution of Eq. (2.7), namely the whole spectrum and corresponding eigenfunctions of the

unperturbed interaction potential, the aim here is to solve only Eq. (2.8), which is the main result of this study, leading to the solution of Eqs. (2.3) and (2.9).

Eq. (2.8) is a closed analytical form in comparing to lengthy expressions in perturbation theories existed in the literature, in particular for the excited states. In this respect, the present formulation has a more general form than the available perturbation theories. Though this point will be clear below and also in the next Chapters through the applications, it would be convenient at this stage to clarify how Eq. (2.8) involves in a compact form the standard perturbation theory expressions.

For the perturbation technique, we have initially assumed that we could split the given potential in two parts, Eq. (2.3). The main part corresponds to a shape invariant potential, Eq. (2.7), for which the superpotential is known analytically and the remaining part is treated as a perturbation, Eq. (2.8). If necessary, one can expand the functions related to the perturbation in terms of the perturbation parameter λ,

$$\begin{aligned} \Delta V(r;\lambda) &= \sum_{k=1}^{\infty} \lambda^k \Delta V_k(r) \ , \\ \Delta W_n(r;\lambda) &= \sum_{k=1}^{\infty} \lambda^k \Delta W_{nk}(r) \ , \\ \Delta \epsilon_n(\lambda) &= \sum_{k=1}^{\infty} \lambda^k \Delta \epsilon_{nk} \ , \end{aligned} \tag{2.10}$$

where λ will eventually be set equal to one. Substitution of the above expansion into Eq. (2.8) by equating terms with the same power of λ on both sides yields up to $O(\lambda^3)$

$$2W_n \Delta W_{n1} - \frac{\hbar}{\sqrt{2m}} \Delta W'_{n1} = \Delta V_1 - \Delta \epsilon_{n1} \ , \tag{2.11}$$

$$\Delta W_{n1}^2 + 2W_n \Delta W_{n2} - \frac{\hbar}{\sqrt{2m}} \Delta W'_{n2} = \Delta V_2 - \Delta \epsilon_{n2} \ , \tag{2.12}$$

$$2\left(W_n \Delta W_{n3} + \Delta W_{n1} \Delta W_{n2}\right) - \frac{\hbar}{\sqrt{2m}} \Delta W'_{n3} = \Delta V_3 - \Delta \epsilon_{n3} \ , \tag{2.13}$$

which are exactly SSPT expressions appeared in [15, 16] but for the case $n = 0$. Eq. (2.8) and its expansion, Eqs. (2.11-2.13), give a flexibility for the easy calculations of the perturbative corrections to energy and wave functions for the n^{th} state of interest through an appropriately chosen perturbed superpotential, unlike the other perturbation theories. We will show in the next Chapters through the applications that this feature of the present model leads to a simple framework in obtaining the corrections to all states without using complicated and tedious mathematical procedures. It is noted that through the thesis work the natural units ($\hbar = 2m = 1$) will be used.

CHAPTER 3

APPLICATIONS TO NON-RELATIVISTIC PROBLEMS

3.1 Systematic Search of Exactly Solvable Non-Central Potentials

Gaining confidence from the recent applications [17-19] of the model presented in the previous Chapter, we aim in this section to illustrate the idea of this basic but powerful technique can also be readily used to search exact solvability of non-central potentials, which would clarify the systematic behind such algebraic treatments. To our knowledge, such work does not exist in the literature.

Analytically solvable potentials are important for a number of reasons such as providing model problems to analyze, to start perturbation theory expansions from, or to provide complete sets of basis functions for solving real problems. In this respect, using the ideas of supersymmetry and shape invariance [11], many authors [20-23] obtained the solutions of a wide class of non-central potentials in a closed form. Additionally, in a recent work [24] similar techniques have been used to determine the spectrum of a vibrational molecular system and using some well-known shape invariant potentials the authors have obtained energy levels of triatomic molecules for 12 classes of non-central but separable potentials. Nevertheless, the answer of the natural question that why some of the non-central potentials can be solved exactly has not been discussed in the literature, which is the task of this section.

For the consideration of exactly solvable non-central potentials,

$$U(r,\theta,\varphi) = U_1(r) + \frac{U_2(\theta)}{r^2} + \frac{U_3(\varphi)}{r^2\sin^2(\theta)}, \tag{3.1}$$

the time-independent Schrödinger equation reads

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[E - U_1 - \frac{\ell(\ell+1)}{r^2}\right]R = 0, \tag{3.2}$$

$$\frac{d^2P}{d\theta^2}+\cot\theta\frac{dP}{d\theta}+\left[\ell(\ell+1)-\frac{m^2}{\sin^2(\theta)}-U_2\right]P=0 \tag{3.3}$$

$$\frac{d^2\phi}{d\varphi^2}+(m^2-U_3)\phi=0 \tag{3.4}$$

where the total wave function is $\Psi(r,\theta,\varphi)=R(r)P(\theta)\phi(\varphi)$ and $\ell=0,1,2,...,n-1$ together with $m=0,\pm1,...,\pm\ell$ are respectively the orbital and azimuthal quantum numbers. Here, the crucial point is that each part of the physically total interaction potential, namely U_1, U_2 and U_3, should be analytically solvable. As Eqs. (3.2) and (3.4), having an exactly solvable potential, were well discussed in the literature based in particular on the supersymmetric quantum theory [11], we apply the presented technique to only Eq. (3.3) to discuss the systematic behind such equations. However, one should bear in mind that the same procedure also can be employed easily in Eqs. (3.2) and (3.4), if necessary.

To proceed we use a mapping function $\theta=f(z)$ which transforms Eq. (3.3) into

$$\frac{d^2P}{dz^2}+\left(-\frac{f''}{f'}+f'\cot f\right)\frac{dP}{dz}+f'^2\left[\ell(\ell+1)-\frac{m^2}{\sin^2 f}-U_2(f)\right]P=0. \tag{3.5}$$

The aim here is to have a Schrödinger-like equation, therefore the second term above is removed with the choice of $\theta\equiv f=2\tan^{-1}(e^z)$ which yields $\sin\theta=\sec hz,\ \cos\theta=-\tanh z$, leading to

$$-\frac{d^2P}{dz^2}+[U_2(z)-\ell(\ell+1)\sec h^2z]P=-m^2P \tag{3.6}$$

Now, the question is which forms of U_2 reproduce analytical solutions. To answer this question one needs to use the discussion given by Eqs. (2.3) through (2.9). As the whole interaction potential is,

$$V(z)=V_0(z)+\Delta V(z)=-\ell(\ell+1)\sec h^2z+U_2(z), \tag{3.7}$$

in case the angular part of the potential $U_2=0$ in (3.6), the remain piece leads to the well-known shape invariant exactly solvable potential [11]. It can be readily solved by the supersymmetric quantum theory,

$$W_{n=0}(z)=\ell\tanh z,\quad \varepsilon_n=-(\ell-n)^2,\quad n=0,1,2,... \tag{3.8}$$

where $W_{n=0}$ and ε_n denote respectively the superpotential and energy eigenvalue for the unperturbed potential, $V_0=-\ell(\ell+1)\sec h^2z$. Note that the corresponding wave functions reproduce standard properties of the spherical harmonics [21].

At this stage, with the consideration of Eq. (2.8)

$$\Delta W^2_{n=0}(z)-\Delta W'_{n=0}(z)+2(\ell\tanh z)\Delta W_{n=0}(z)=\Delta V(z)-\Delta\varepsilon_{n=0}, \tag{3.9}$$

one arrives at

$$\Delta W_{n=0} = \frac{b}{\ell} \quad , \quad \Delta\varepsilon_{n=0} = -\frac{b^2}{\ell^2} \quad , \quad \Delta V(z) = 2b\tanh z, \tag{3.10}$$

where b is a constant. It makes clear that the full interaction potential has now the form of the Rosen-Morse II potential. A brief study of all shape invariant exactly solvable potentials [11] in one dimensional space, together with the results obtained in [22], clarify the physically meaningful choice of b as to be $b = -\frac{\gamma}{2}$ with γ be another constant relating to the system of interest. For the completeness, one should now take consider the whole superpotential,

$$W_{n=0}^{total}(z) = W_{n=0}(z) + \Delta W_{n=0}(z) = \ell\tanh z - \frac{\gamma}{2\ell} \quad , \tag{3.11}$$

which will reproduce the whole spectrum. Proceeding within the framework of supersymmetric quantum mechanics, the energy spectrum for the total potential $V = -\ell(\ell+1)\sec h^2 z - \gamma\tanh z$ is given in the form

$$E_n = -(\ell - n)^2 - \frac{\gamma^2}{4(\ell - n)^2} = \varepsilon_n + \Delta\varepsilon_n, \tag{3.12}$$

which is the proof of the stated theory. To see the exactly solvable form of the angular potential U_2, we use inverse mapping $\sec hz = \sin\theta, \tanh z = -\cos\theta$ and bearing in mind Eqs. (3.1) and (3.5), then arrives at

$$U_2(\theta) = \frac{\gamma\cos\theta}{r^2\sin^2(\theta)} \quad . \tag{3.13}$$

From the mathematical point of view, $r^2\sin^2\theta$ in the calculations of exactly solvable forms of U_2 comes naturally, see Eqs. (3.1) and (3.5). Hence one can generalize the above potential involving a constant, if necessary, related to the physical system considered. In this case, such potentials are given as

$$U_2(\theta) = \frac{\beta + \gamma\cos\theta}{r^2\sin^2(\theta)} \quad , \tag{3.14}$$

in which β appears, from Eq. (3.5), only on RHS of Eq. (3.6) as a piece of energy value. Considering Eqs. (3.6) and (3.12) for the presence of β, we obtain

$$\ell = n + \left[\frac{(m^2+\beta) + \sqrt{(m^2+\beta^2) - \gamma^2}}{2}\right]^{\frac{1}{2}} . \tag{3.15}$$

This ℓ- value is used in the energy expression given for the central potential $U_1(r)$ to introduce the complete spectrum of analytically solvable non-central potentials. The present result agrees with Eq. (2.10) of Ref. [22], where the generalized Coulomb potential was discussed.

Clearly, Eq. (3.9) is the most significant equation of this section, which defines exactly the possible forms of $U_2(\theta)$ yielding analytical solutions. The main point here of course is to be able to find an analytical expression for ΔW and subsequently ΔV for the definition of the total θ-dependent perturbing potential, which requires exactly the satisfaction of Eq. (3.9) leading to a closed form for the complete spectrum.

One should however note that although we have focussed here on the eigenvalues, calculation of the corresponding eigenfunctions within the same model is quite straightforward, see Eq. (2.2). In addition, we believe that this generalization would considerably extend the list of exactly solvable non-central potentials for which the solution can be obtained algebraically in a simple and elegant manner as discussed here.

A similar study is conducted here for the shape invariant Pösch-Teller II type potential [11], within the frame of Eqs. (3.6) through (3.9). This choice requires

$$\Delta W_{n=0}(z) = -\alpha \coth z, \tag{3.16}$$

which, from (3.9), reproduces

$$\Delta V(z) = U_2(z) = \alpha(\alpha-1)\csc h^2 z \quad , \quad \Delta\varepsilon_{n=0} = -\alpha(\alpha-2\ell). \tag{3.17}$$

Thus, the full angle dependent potential in (3.6) turns into $V = -\ell(\ell+1)\sec h^2 z + \alpha(\alpha-1)\csc h^2 z$ having a complete spectrum in the form of $E_n = -(\ell-\alpha-2n)^2 = \varepsilon_n + \Delta\varepsilon_n$. Using the same algebraic procedure as before, one obtains the related exactly solvable non-central potential as

$$U_2(\theta) = \frac{\delta}{r^2\sin^2(\theta)} + \frac{c}{r^2\cos^2(\theta)} \quad , \tag{3.18}$$

in which δ and $c = \alpha(\alpha-1)$ are constants. As explained in the previous example, δ appears as a piece of energy eigenvalue on RHS of Eq. (3.6), therefore

$$m^2 + \delta = (\ell - \alpha - 2n)^2 \tag{3.19}$$

from which one defines the ℓ - value as

$$\ell = 2n + \left(\frac{1}{2} \pm \sqrt{\frac{1}{4}+c}\right) + (m^2+\delta)^{\frac{1}{2}}, \tag{3.20}$$

that is the same result with compared to (26) of [22], and also agrees with the related references in [22]. If $U_1(r)$ is taken as the harmonic oscillator potential, then the whole non-central potential with $U_3(\varphi) = 0$ corresponds to the generalized oscillatory potential. To find the full spectrum for such a potential, Eq. (3.20) is invoked to the energy spectrum of $U_1(r)$ [22].

In this section, we have attempted to explore the effectiveness of the recently developed formalism through which we have made successfully the complete mathematical analysis of the reason behind exact solvability of some Schrödinger equations with a class of non-central but separable potentials, for which the complete spectrum and eigenfunctions can be written down algebraically using the well known results for the shape invariant potentials. Generalization of our technique to other non-central potentials is quite straightforward and use of the present model may also be useful for solving other quantum mechanical complicated systems analytically.

The work presented here was published in [25].

3.2 An Approach To Potential Scattering

In the studies [17-19] and [25, 26] published recently, a time-independent novel perturbation theory presented in the previous Chapter has been developed in the bound state domain, which is non-perturbative, self-consistent and systematically improvable, and used to treat successfully significant problems in different fields of physics. With the confidence gained from these applications, we aim through the present work to show that similar techniques can also be used in the continuum, which would be useful in particular for calculations in nuclear physics.

It is well known that there are many scattering problems in which the interaction between the projectile and the target decomposes naturally into two parts ($V = V_0 + \Delta V$) as done in the earlier application in this thesis work. This division is especially useful if the scattering wave function under the action one part can be obtained exactly (V_0), while the effect of the other (ΔV) can be treated in some approximation as in the present formalism.

For simplicity, we here confine ourselves to $s-$wave scattering from a potential which is assumed that vanishes beyond a finite radius R. The associated total wave function behaves at large distances

$$\Psi(r) = \frac{1}{k}\sin(kr + \delta), \quad r \geq R, \tag{3.21}$$

where δ is the $s-$wave phase shift.

Our present treatment of scattering has concerned itself primarily with determining how the solutions of the free Schrödinger equation are affected by the presence of the interaction. Within the framework of the present formalism we suppose that the solutions of Eq.(2.7) are known, or are easily found, to give the corresponding phase shift δ_0. Considering the expansion $\delta = \delta_0 + \lambda\delta_1 + \lambda^2\delta_2 + ...$,

as in Eq.(2.10), we aim here to derive explicitly solvable and easily accessible expressions for the phase shift contributions at successive perturbation orders.

3.2.1 First-Order Phase Shift Correction

Keeping in mind Eq.(3.21) and considering the discussion in previous section, at the first perturbation order one has

$$(W + \lambda \Delta W_1) = -k \cot(kr + \delta_0 + \lambda \delta_1), \quad W_n = -\frac{\chi'}{\chi} = -k \cot(kr + \delta_0), \quad (3.22)$$

from where the superpotential relating to the perturbing interaction

$$\Delta W_1(r) = \frac{k \delta_1}{\sin^2(kr + \delta_0)} \ , \quad (3.23)$$

is obtained assuming that $\sin \lambda \delta_1 \cong \lambda \delta_1$ and $\cos \lambda \delta_1 \cong 1$.

In the second step, one needs to employ Eq. (2.11) to arrive at another expression for ΔW_1. Rearranging the terms, $\Delta W_1' - 2W \Delta W_1 = (\Delta \varepsilon_1 - \Delta V_1)$ and multiply both sides by the integrating factor $\exp(-2 \int_0^r W(z) dz)$, which is the square of the unperturbed wave function $\chi^2(r)$ through Eq.(2.6), one obtain

$$\frac{d}{dr} \left[\chi^2(r) \Delta W_1(r) \right] = \chi^2(r) (\Delta \varepsilon_1 - \Delta V_1). \quad (3.24)$$

The integration, and the remove of $\Delta \varepsilon_1$ term due to the consideration of elastic scattering process here, yields

$$\Delta W_1(r) = -\frac{1}{\chi^2(r)} \int_0^r \chi^2(z) \Delta V_1(z) dz. \quad (3.25)$$

As $\chi = \frac{1}{k} \sin(kr + \delta_0)$ in the asymptotic region, comparison of Eqs.(3.23) and (3.25) reproduces the first-order change in the phase shift

$$\delta_1 = -k \int_0^\infty \chi^2(r) \Delta V_1(r) dr. \quad (3.26)$$

If necessary, the corresponding change in the wave function can easily be obtained by the substitution of Eq.(3.25) into (2.6), $\phi_1 = \exp(-\int \Delta W_1)$. For the reliability of the present expression obtained, Eq (3.26), one may compare it with that reproduced by other methods. For example, in the limiting case where the unperturbed potential vanishes, the unperturbed $s-$wave function is reduced to a plane wave $\chi(r) = \sin(kr)/k$, and the first-order change in the phase shift becomes

$$\delta_1 = -\frac{1}{k} \int_0^\infty \sin^2(kr) \Delta V_1(r) dr \quad (3.27)$$

which is just the first Born approximation for the phase shift [27]. In addition, the well known expression for $s-$wave scattering amplitude by the two-potential formula in scattering theory [27],

$$f_1 = -e^{2i\delta_0} \int_0^\infty \chi^2(r) \Delta V_1(r) dr \tag{3.28}$$

where the phase factor in front of the integration arises because of the standing wave boundary conditions, justifies once more our result since $f_1 = -e^{2i\delta_0}\delta_1/k$ and, equating this to the above equation leads immediately to Eq.(3.26).

The present result has a widespread applicability, which may also be used in the treatment of scattering length problems. At low-energy limit, the phase shift is related to the scattering length $\delta_{k\to 0} \to -ka$ where $a = a_0 + \lambda a_1 + \lambda^2 a_2 + ...$ may be expanded in a perturbation series similar to the phase shift. Outside the range of the potential, the unperturbed wave function behaves as $\chi \to (r - a_0)$. Thus, the first correction to the scattering length is

$$a_1 = \lim_{r\to\infty} \left[\int_0^r (z - a_0)^2 \Delta V_1(z) dz \right] \tag{3.29}$$

which can be calculated for a given ΔV_1. The scattering length has an important physical significance. In the low-energy limit only the $s-$wave makes a nonzero contribution to the cross section, so that the angular distribution of the scattering is spherically symmetric and the total cross section is $4\pi(a_0 + \lambda a_1 + ...)^2$. This is also exactly the result obtained in most textbooks for the low-energy scattering of a hard sphere of radius. Thus the scattering length is the effective radius of the target at zero energy.

As a last example, consider the case of the angular momentum barrier as the unperturbed potential $V_0 = \ell(\ell+1)/r^2$ that produces $[rj_\ell(kr)]$ with a phase shift $\delta_0 = -\ell\pi/2$. For a trivial perturbation let us choose $\Delta V_1 = \lambda/r^2$, due to which the angular momentum is slightly perturbed $\overline{\ell} \approx \ell + \lambda/(2\ell+1) + O(\lambda^2)$. Therefore the phase shift correction at first-order is $\delta_1 = -\pi/2(2\ell+1)$. Again, this exact result confirms the reliability of Eq. (3.26).

3.2.2 Second-Order Phase Shift Correction

To solve Eq.(2.12) which is our second-order perturbation equation, for ΔW_2 we mimic the preceding calculation. The integration factor is the same. In fact, examining Eqs.(2.11) and (2.12), the only difference is that the quantity $\Delta V_1 - \Delta\varepsilon_1$ is replaced by $\Delta V_2 - \Delta W_1^2 - \Delta\varepsilon_2$. As $\Delta\varepsilon_2$ term is zero due to the process of interest, ΔW_2 is thus

$$\Delta W_2(r) = -\frac{1}{\chi^2(r)} \int_0^r \chi^2(z) \left[\Delta W_1^2(z) - \Delta V_2(z) \right] dz. \tag{3.30}$$

Bearing in mind that $\chi = \frac{1}{k}\sin(kr+\delta_0)$ for the region $r \geq R$, the second-order expansion in the superpotential similar to Eq.(3.22) provides another expression for ΔW_2 which is

$$\Delta W_2(r) = k\delta_1^2 \frac{\cot(kr+\delta_0)}{\sin^2(kr+\delta_0)} + \frac{k\delta_2}{\sin^2(kr+\delta_0)} \tag{3.31}$$

Comparison of Eqs.(3.30) and (3.31), together with the substitution of (3.23) in (3.30), leads to an auxiliary function for the second order phase shift correction,

$$\delta_2(r) = -\frac{1}{k}\int_0^r \Delta V_2(z)\sin^2(kz+\delta_0)dz + k\delta_1^2 \int_0^r \frac{dz}{\sin^2(kz+\delta_0)} - \delta_1^2\cot(kr+\delta_0), \tag{3.32}$$

where a singularity appears in the second integral at $z = 0$. This problem can be circumvented by replacing the lower limit of the integral with R. Assuming $\Delta V = \Delta V_1$ as in realistic problems of nuclear physics, which means that $\Delta V_2 = 0$, the $r-$dependent phase shift correction in the second-order is given in the form of

$$\delta_2(r) = \delta_1^2\cot(kR+\delta_0) - 2\delta_1^2\cot(kr+\delta_0). \tag{3.33}$$

As an alternative treatment, which leads to a concrete comparison, one can go back to Eq.(3.30) and split $\chi^2\Delta W_1^2$ term in two parts as $(\chi^2\Delta W_1)(\Delta W_1)$ allowing to invoke Eq.(3.25). In this case the comparison of the result with the expansion in (3.31) gives

$$\delta_2 = -k\int_0^\infty \chi^2(r)\Delta V_1(r)dr \int_R^r \frac{dz}{\chi^2(z)}\left[\int_z^R \chi^2(y)\Delta V_1(y)dy - \frac{\delta_1}{k}\right] + \delta_1^2\cot(kR+\delta_0) \tag{3.34}$$

which is in agreement with the work in [28]. In addition, the use of (3.26) in (3.33) transforms it into Eq. (3.34). Furthermore, the reader is reminded that the second Born approximation for the phase shift can be most easily derived using the variable phase equation approach [29],

$$\delta_2 = 2k^2\int_0^\infty \chi^2(r)\Delta V_1(r)\cot(kr)dr \int_0^r \chi^2(y)\Delta V_1(y)dy \tag{3.35}$$

which, in the light of Eq. (3.24), is the same result as we find from Eq (3.34), by putting $\delta_0 = 0$. Higher order terms can also be evaluated in the same manner.

The new model introduced in Chapter 2 for time-independent perturbation theory has been successfully extended from the bound state region to the scattering domain. For the clarification, the work has been carried out with the consideration of $s-$wave scattering only. However, generalization of the formalism to higher partial waves in the scattering domain does not cause any problem. The inclusion of the centrifugal barrier contribution in the effective potential for instance leads to the replacement of the $s-$wave phase shift with $\delta_\ell - \ell\pi/2$ due

to the related wave function $\chi(r) = \sin(kr + \delta_\ell - \ell\pi/2)/k$ in the asymptotic region, supposing both the unperturbed and perturbed potentials vanish at a large $r > R_1$ which means that in the region $R_1 < r \leq R$ there is then only the centrifugal barrier contribution. This inclusion requires simply to repeat the present calculations for the replacement in the phase shift.

It should be stressed that, anything that can be achieved from the present formalism must also be obtainable from the works [28, 30] in the literature. For instance, considering the bound state region, Bender's formalism [30] can be simplified by introducing the auxiliary function $F_N(r)$ such that the whole wave function $\Psi_N(r) = \chi(r)F_N(r)$ where denotes the perturbation order. The first-order correction can then be written as $\frac{d}{dr}\left[\chi^2\frac{dF}{dr}\right] = (\Delta V_1 - \Delta\varepsilon_1)\chi^2$ which corresponds exactly to the present treatment by Eq.(3.24) when we identify $\Delta W_1 = dF/dr$. The higher order calculations can be linked to ours in the similar manner. Whereas, the works of Milward and Wilkin [28] may be related to the present formalism in both domain, the bound and scattering region by making a relation between their probability density distributions/derivatives and our ΔW functions, such as $\Delta W_0 = -P_0'/2P_0$ at the zeroth order, $\Delta W_1 = (-P_1/2P_0)'$ at the first order and $\Delta W_2 = (-P_2/2P_0)'$ at the second order *etc.* Nevertheless, the present technique provides a clean and explicit route for the calculations without tedious and cumbersome integrals.

The energy variation of the scattering wave function and phase shift can also be studied by perturbing in the energy. We wish to stress that all these effects depend purely upon the perturbation and the unperturbed wave function; explicit knowledge of the unperturbed potential is not necessary.

The work presented in this section appeared in [31].

3.3 Remarks on Exact Solvability of Quantum Systems with Non-Constant Mass

The study of quantum mechanical systems with position dependent mass has raised some important conceptional questions, such as the ordering ambiguity of the momentum and mass operators in the kinetic energy term, the boundary conditions at abrupt interfaces characterized by discontinuities in the mass function, *etc.* Therefore, the form of the effective mass Hamiltonian has been a controversial subject in the literature. In recent years there has been a growing interest in the study of such systems due to the applications in condensed matter physics and other areas involving quantum many body problem. These applications have stimulated a lot of work in the literature regarding the development of

techniques for the treatment of such systems, for a recent review, see [32-37] and the related references therein. In all these works the main concern is in obtaining the energy spectra and/or wave functions for quantum systems with spatially dependent effective mass. Moreover, exact solvability requirements result in constraints on the potential functions for the given mass distribution. Though there has been a large consensus in favor of BenDaniel and Duke Hamiltonian (BDD) [38] proposed in the literature as an appropriate one, the question of the exact form of the kinetic energy operator is still an open problem for such systems.

Within this context, the present section involves an alternative scheme to obtain unambiguously the Schrödinger equation with non-constant particle mass, which makes clear the relationship between the exact solvability of the Schrödinger equation and the ordering ambiguity. The model explored here restricts naturally the possible choices of ordering and provide us a clear comparison between the solutions of different but physically plausible effective Hamiltonians clarifying the physics behind ambiguity.

To achieve our goal defined above, the present non-perturbative formalism is employed here. In this unified model, the BDD Hamiltonian is considered as an unperturbed term while modifications due to other effective Hamiltonians are treated as an additional potential in the same framework. This realization is of prime significance in the calculation of physical processes, which so far did not receive adequate attention.

There are several ways to define the kinetic energy operator when the mass is variable. Since the momentum and mass operators no longer commute, the generalization of the Hamiltonian is not trivial and this kind of physical problem is intrinsically ambiguous. Starting with the von Roos effective mass kinetic energy operator [39], which has the advantage of an inbuilt Hermicity,

$$H_{\nu R} = \frac{1}{4}[m^{\alpha}(\hat{z})\hat{p}m^{\beta}(\hat{z})\hat{p}m^{\gamma}(\hat{z}) + m^{\gamma}(\hat{z})\hat{p}m^{\beta}(\hat{z})\hat{p}m^{\alpha}(\hat{z})] + V(\hat{z}), \tag{3.36}$$

where $\alpha+\beta+\gamma = -1$. By the correspondence in wave mechanics $\hat{p} \to -i\hbar\frac{d}{dz}$, $\hat{z} \to z$ and on setting

$$m(z) = m_0 M(z), \qquad \hbar = 2m_0 = 1, \tag{3.37}$$

where $M(z)$ is the dimensionless form of the mass function, the effective mass equation can be written in a differential form,

$$-\frac{d}{dz}\left[\frac{1}{M(z)}\frac{d\Psi(z)}{dz}\right] + V^{eff}(z)\Psi(z) = E\Psi(z), \tag{3.38}$$

Here, $V^{eff}(z)$ is termed the effective potential energy whose algebraic form de-

pends on the Hamiltonian employed

$$V^{eff}(z) = V_0(z) + U_{\alpha\gamma}(z) = V_0(z) - (\frac{\alpha+\gamma}{2})\frac{M''}{M^2} + (\alpha\gamma+\alpha+\gamma)\frac{M'^2}{M^3} \quad , \quad (3.39)$$

in which the first and second derivatives of $M(z)$ with respect to z are denoted by M' and M'', respectively. The effective potential is the sum of the real potential profile $V_0(z)$ and the modification $U_{\alpha\gamma}(z)$ emerged from the location dependence of the effective mass. A different Hamiltonian leads to a different modification term. Some of them are the ones of BDD [38] ($\alpha = \gamma = 0$), Bastard [40] ($\alpha = -1$), Zhu-Kroemer (ZK) [41] ($\alpha = \gamma = -\frac{1}{2}$) and Li-Kuhn [42] ($\beta = \gamma = -\frac{1}{2}$).

Considering the supersymmetric treatment of effective mass Hamiltonians by Plastino and his co-workers [43]

$$A\Psi = \frac{1}{\sqrt{M}}\frac{d\Psi}{dz} + W\Psi, \qquad A^+\Psi = -\frac{d}{dz}\left(\frac{\Psi}{\sqrt{M}}\right) + W\Psi, \quad (3.40)$$

where A and A^+ are linear operators and $W(z)$ is a superpotential, the supersymmetric Hamiltonians are expressed as

$$H_1 = A^+A = -\frac{1}{M}\frac{d^2}{dz^2} - \left(\frac{1}{M}\right)'\frac{d}{dz} + W^2 - \left(\frac{W}{\sqrt{M}}\right)', \quad (3.41)$$

and

$$H_2 = AA^+ = H_1 + \frac{2W'}{\sqrt{M}} - \left(\frac{1}{\sqrt{M}}\right)\left(\frac{1}{\sqrt{M}}\right)''. \quad (3.42)$$

From which, supersymmetric partner potentials are

$$V_1^{SUSY} = W^2 - \left(\frac{W}{\sqrt{M}}\right)', \quad V_2^{SUSY} = V_1^{SUSY} + \frac{2W'}{\sqrt{M}} - \left(\frac{1}{\sqrt{M}}\right)\left(\frac{1}{\sqrt{M}}\right)''. \quad (3.43)$$

At this stage, we use the spirit of our non-perturbative approach expressing the total wave function as a product,

$$\Psi(z) = \Phi(z)\Theta(z). \quad (3.44)$$

In the above equation, Φ denotes the wave function corresponding to the unperturbed piece of the effective potential in Eq. (3.39) while Θ is the moderating function due to the modified term $U_{\alpha\gamma}$ therein.

The use of (3.44) in (3.38) yields

$$\frac{1}{M}\left(\frac{\Phi''}{\Phi} + \frac{\Theta''}{\Theta} + 2\frac{\Phi'}{\Phi}\frac{\Theta'}{\Theta}\right) - \frac{M'}{M^2}\left(\frac{\Phi'}{\Phi} + \frac{\Theta'}{\Theta}\right) = V_{eff} - E, \quad (3.45)$$

which reduces to the usual Schrödinger equation with a constant mass when $M \to 1$. With the consideration of (3.41), where the superpotential now can be given as

$$W(z) = W_0(z) + \Delta W(z), \quad (3.46)$$

with W_0 and ΔW being superpotentials corresponding to the unperturbed potential (V_0) and modification term ($U_{\alpha\gamma}$) respectively, Eq. (3.45) is transformed into a couple of equation,

$$W_0^2 - \left(\frac{W_0}{\sqrt{M}}\right)' = V_0 - E_0, \quad W_0 = -\frac{1}{\sqrt{M}}\frac{\Phi'}{\Phi}, \tag{3.47}$$

$$\Delta W^2 - \left(\frac{\Delta W}{\sqrt{M}}\right)' + 2W_0\Delta W = U_{\alpha\gamma} - \Delta E, \quad \Delta W = -\frac{1}{\sqrt{M}}\frac{\Theta'}{\Theta}. \tag{3.48}$$

In the above equations, $E = E_0 + \Delta E$ due to $V_{eff} = V_0 + U_{\alpha\gamma}$. Therefore one can easily see the contributions, if any, to the energy and wave function due to the use of effective Hamiltonians other than BDD which represents the unperturbed Hamiltonian in the present scenario since it has no modification term, see (3.39).

We are familiar with (3.47) as a standard supersymmetric treatment of the Schrödinger equation for the exact solutions. However, Eq. (3.48) is new and is the most significant piece of the work presented in this section. Because it is a non-perturbative approach by Riccati equation, which reproduces the whole corrections coming from $U_{\alpha\gamma}$ if, of course, Eq. (3.48) is exactly solvable.

To proceed we remind a general consensus [36] that the resolution of the ordering ambiguity in this problem could come from a scheme that starts with the relativistic Dirac equation with spatially varying mass then taking the non-relativistic limit. This is due to the fact that the Dirac equation is inherently free from the ordering ambiguity and that taking the non-relativistic limit is a well defined procedure. Bearing in mind this point we propose a correct choice of ΔW as

$$\Delta W = \left(\frac{\alpha+\gamma}{2}\right)\frac{M'}{M^{3/2}}, \tag{3.49}$$

which directs us to find correct ordering parameter(s) leading to the physically plausible effective Hamiltonian(s). Through Eq. (3.48), the parameters get decoupled in a natural way and the ambiguity in the choice of proper kinetic energy operator disappears. Substituting (3.49) into (3.48), we obtain

$$\Delta W^2 - \left(\frac{\Delta W}{\sqrt{M}}\right)' = U_{\alpha\gamma}, \quad \Delta E = -2W_0\Delta W, \tag{3.50}$$

if either $\alpha = \gamma = 0$ which yields the BDD Hamiltonian or $\alpha = \gamma = -\frac{1}{2}$ corresponding to the ZK Hamiltonian. It is stressed that the results are independent of any choice of $M(z)$ and in case $\alpha = \gamma = 0$ Eq. (3.48) vanishes. This restriction is in agreement with the discussion in Ref. [44] and also with the work of Bagchi *et all* [34].

Though the present formalism has a wide spread applicability, for clarity we now simply consider the two examples which were investigated in Ref. [43].

This consideration will shed a light in understanding the interrelation between the BDD and ZK effective Hamiltonians bearing in mind the results presented in [43] for the systems of interest.

The simplest case of the shape invariance integrability condition [11], leading to exactly solvable potentials, corresponds a uniform energy shift ε between partner potentials,

$$V_2^{SUSY}(z,\varepsilon) - V_1^{SUSY}(z,\varepsilon) = \varepsilon = 2E_0 \tag{3.51}$$

since ΔE term appearing in the partners due to $U_{\alpha\gamma}$ cancels each other. The replacement of (3.43) into (3.51) gives

$$\frac{2\left(W_0' + \Delta W'\right)}{\sqrt{M}} - \left(\frac{1}{\sqrt{M}}\right)\left(\frac{1}{\sqrt{M}}\right)'' = \varepsilon, \tag{3.52}$$

from which one finds the superpotentials leading to the hamiltonian with V_0,

$$W_0(z) = -\frac{1}{2}\left(\frac{1}{\sqrt{M}}\right)' + \frac{\varepsilon}{2}\int^z \sqrt{M(y)}dy, \tag{3.53}$$

since $\Delta W = -(M'/2M^{3/2}) = \left(1/\sqrt{M}\right)'$. To finalize the full treatment, one needs the total superpotential, $W = W_0 + \Delta W$ from which the results in [43], Eq. (35) and the subsequent equations, can easily be reproduced.

From this short discussion, it is obvious that (i) there will be no contribution to E_0 due to the modification term. For this reason total energies in both system having a constant mass and position dependent mass are equal. (ii) From (3.48), the contribution of $U_{\alpha\gamma}$ to the unperturbed wave function is (for the ground state)

$$\Theta_{n=0}(z) = exp\left(-\int^z \sqrt{M(y)}\Delta W(y)dy\right) = m^{1/2}. \tag{3.54}$$

Thus, going back to (3.44) along with Eqs. (3.47) and (3.53), the full unnormalized ground state wave function is expressed as

$$\Psi_{n=0}(z) = \left[m^{-1/4}(z)\Phi(\bar{z})\right] m^{1/2}(z) = m^{1/4}(z)\Phi(\bar{z}), \tag{3.55}$$

where $\bar{z} = \int^z \sqrt{M(y)}dy$, which supports the reliability of the present formalism [32]. The excited state wave functions can be determined [11] in algebraic fashion by successive application of the linear operators in (3.40) upon the ground state wave function. (iii) The both choice, namely the BDD and ZK Hamiltonians are represented with a unique superpotential leading to exactly equivalent wave functions for shape invariant potentials. (iv) From (3.43), as $\alpha = \gamma = -\frac{1}{2}$, one gets

$$V_2^{SUSY} = \left(V_1^{SUSY} + U_{\alpha\gamma}\right) + \frac{2W'}{\sqrt{M}}, \tag{3.56}$$

pointing a duality between BDD and ZK schemes, which reveals the suggestions in [32, 34].

Let us proceed with another example in Ref. [43] where the superpotential leads to a Morse-like spectra,

$$W(z, A) = A + f(z), \tag{3.57}$$

in which, within the frame of the present formalism, $f(z) = f_0(z) + \Delta f(z)$ that turns the form of (3.57) into

$$W(z, A) = [A + f_0(z)] + \Delta f(z) = W_0 + \Delta W(z) \tag{3.58}$$

From the shape invariance condition $V_2^{SUSY}(z, A) = V_1^{SUSY}(z, A-\lambda)+R(A)$ used in the supersymmetric quantum theory [11], where A is the potential parameter and R involving both parameter, A and λ , leads to the ground state energy of the system. In the light of the work carried out in [43], the substitution of (3.58) in (3.43) within the frame of shape invariance condition above produces

$$\frac{2\left(f_0' + \Delta f'\right)}{\sqrt{M}} - \frac{1}{\sqrt{M}}\left(\frac{1}{\sqrt{M}}\right)'' = \lambda\left(\frac{1}{\sqrt{M}}\right)' - 2\lambda\left(f_0 + \Delta f\right). \tag{3.59}$$

Remembering $\Delta W = \Delta f = \left(\frac{1}{\sqrt{M}}\right)'$ for $\alpha = \gamma = -\frac{1}{2}$, the above equation is rearranged as

$$f_0'(z) + b_1(z) f_0(z) = b_2(z), \tag{3.60}$$

where

$$b_1 = \lambda\sqrt{M}, \quad b_2 = -\left[\frac{\lambda}{2}\sqrt{M}\left(\frac{1}{\sqrt{M}}\right)' + \frac{1}{2}\left(\frac{1}{\sqrt{M}}\right)''\right]. \tag{3.61}$$

From (3.59) it is clear that Δf term affects only b_2, since when $\Delta f \to 0 \;\; b_2 \to -b_2$. The solution of differential equation in (3.60) gives

$$f_0(z) = \left\{C + \int^z b_2(y)dy \; exp\left[\int^x b_1(t)dt \;\right]\right\} \times exp\left[-\int^z b_1(y)dy \;\right], \tag{3.62}$$

where C is an integration constant. Employing the mass function used in [43], $M = [(\alpha + z^2)/(1 + z^2)]^2$, we obtain

$$W(z) = W_0 + \Delta W =$$

$$= \left(A + C exp\left[-\lambda\left\{z + (\alpha - 1)\arctan x\right\}\right] - \frac{z(\alpha-1)}{(\alpha + z^2)^2}\right) + 2\frac{z(\alpha-1)}{(\alpha + z^2)^2}, \tag{3.63}$$

that is Eq. (53) in [43]. From (3.47), the corresponding potential function, energy and wave function can be expressed as in [43], which are out of interest

in this section. Generalization of the above discussion to a formalism which is applicable to all spatially varying masses, yields

$$W(z) = W_0 + \Delta W = \left\{ A + Cexp\left[-\int^z b_1(y)dy \right] - \left(\frac{1}{2\sqrt{M}} \right)' \right\} + \left(\frac{1}{\sqrt{M}} \right)' \tag{3.64}$$

Plastino and co-workers [43] studied this problem in case $\alpha = \gamma = 0$ considering only the BDD Hamiltonian and arrived at Eq. (53) in their work, which addresses (3.63) in our work. This means that BDD and ZK effective Hamiltonians in fact reproduce same results employing an identical superpotential, which once more supports the realization introduced by (3.56) that they are their supersymmetric partners.

In this section we have discussed the problem of solvability and ordering ambiguity in quantum mechanics for the systems with a position dependent mass. The present scheme restricts the possible choices of ordering. Proceeding with this consideration it has been observed that the only physically allowable BDD and ZK Hamiltonians for exactly solvable systems are in fact their supersymmetric partners that reproduce identical results in their independent considerations due to use of an identical superpotential. We hope that this observation would make a contribution to the ongoing debate in the literature regarding the isospectral effective mass Hamiltonians.

The work presented in this section was published in [45].

3.4 $N-$Dimensional Schrödinger Equations with Position-Dependent Mass

Additionally to the successful applications illustrated in the previous sections, and our other published works in [17-19, 25, 26, 31, 45] dealing with various fields of physics, we will in this section extend further our calculations. In the previous section, we have investigated the relation between the solutions of physically acceptable effective mass Hamiltonians proposed in the literature for the treatment of one dimensional problems. Using our expertise taken from these applications, we aim here to tackle the more difficult problem of generating exact solutions for position-dependent mass Schrödinger equations (PDMSE) in $N-$dimension, as the most of the related works in the literature have been devoted to one-dimensional systems [46-48].

The concept of PDMSE is known to play an important role in different branches of physics. This formalism has been extensively used in nuclei, quantum liquids, ^3He and metal clusters. Another area wherein the such concepts

provide very useful tool is the study of electronic properties of many condensed-matter systems, such as semiconductors and quantum dots. In particular, recent progress in crystal-growth techniques for producing non-uniform semiconductor specimens, wherein the carrier effective mass depends on position, has considerably enhanced the interest in the theoretical description of semiconductor heterostructures. It has also recently been signalled in the rapidly growing field of PT-symmetric or more generally pseudo-Hermitian quantum mechanics. For an excellent recent review, leading to the related references, the reader is referred to [46-49].

Tracking down solvable potentials in PDMSE has always aroused interest. Apart from being useful in understanding of many physical phenomena, the importance of searching for them also stems from the fact that they very often provide a good starting point for undertaking perturbative calculations of more complex systems.

As is well known, the general form of radial PDMSE with Hermitian Hamiltonians in one-dimension gives rise to

$$-\frac{d}{dz}\left[\frac{1}{M(z)}\frac{d\Phi(z)}{dz}\right]+V^{eff}(z)\Phi(z)=\lambda\Phi(z) \quad , \tag{3.65}$$

where the effective potential

$$V^{eff}(z)=V_0(z)+U_{\alpha\gamma}(z)=V_0(z)-\frac{(\alpha+\gamma)}{2}\frac{M''}{M^2}+(\alpha\gamma+\alpha+\gamma)\frac{M'^2}{M^3} \quad , \tag{3.66}$$

depends on the mass term and ambiguity parameters. Here a prime denotes derivative with respect to the variable, $M(z)$ is the dimensionless form of the mass function $m(z)=m_0M(z)$ and we have set $\hbar=2m_0=1$. The effective potential is the sum of the real potential profile $V_0(z)$ and the modification $U_{\alpha\gamma}(z)$ emerged from the location dependence of the effective mass. A different Hamiltonian leads to a different modification term. Some of them are the ones proposed by BenDaniel-Duke [38] ($\alpha=\gamma=0$), Bastard [40] ($\alpha=-1$), Zhu-Kroemer [41] ($\alpha=\gamma=-1/2$) and Li-Kuhn [42] ($\gamma=-1/2,\alpha=0$).

Considering the works in [45, 50], the radial piece of PDMSE in arbitrary dimensions for spherically symmetric potentials and mass functions reads

$$\left\{\frac{d^2}{dr^2}+\frac{M'}{M}\left(\frac{N-1}{2r}-\frac{d}{dr}\right)-\frac{L(L+N+2)+(N-1)(N-3)/4}{r^2}+M\left[E-V_{eff}(r)\right]\right\}\Psi(r) \quad , \tag{3.67}$$

where we assume that $\Psi(r)=F(r)G(r)$ which leads to

$$\frac{1}{M}\left(\frac{F''}{F}+\frac{G''}{G}+2\frac{F'}{F}\frac{G'}{G}\right)-\frac{M'}{M^2}\left(\frac{F'}{F}+\frac{G'}{G}\right)=U_{eff}-E. \tag{3.68}$$

The effective potential in higher dimensions ($N > 1$) now is transformed to the form

$$U_{eff}(r) = V_0(r) + U_{\alpha\gamma}(r) - \frac{M'}{M^2}\frac{(N-1)}{2r} + \frac{L(L+N-2)+(N-1)(N-3)/4}{Mr^2} \quad , \tag{3.69}$$

in which L is the angular momentum. As the one-dimensional calculations require $N = 1$ and $L = 0$, Eq.(3.69) reduces in this case to $U_{eff}(r) = V_{eff}(r) = V_0 + U_{\alpha\gamma}$ as in Ref. [45] discussed also in the previous section, which provides us a reliable testing ground.

Keeping in mind the spirit of the technique used simply in the previous section, we split Eq. (3.68) in two parts deviating from the treatments in [46-49]

$$W^2(r) - \left[\frac{W(r)}{\sqrt{M}}\right]' = V_0(r) - \varepsilon, \quad W = -\frac{F'}{\sqrt{M}F} \quad , \tag{3.70}$$

where ε is the corresponding energy of the required quantum state F_n ($n = 0, 1, 2, ...$) for V_0 which is assumed in this model as an exactly solvable mass-dependent potential, and

$$\Delta W^2(r) - \left[\frac{\Delta W(r)}{\sqrt{M}}\right]' + 2W(r)\Delta W(r) = \Delta V(r) - \Delta E, \qquad \Delta W(r) = -\frac{G'}{\sqrt{M}G} \quad , \tag{3.71}$$

where

$$\Delta V(r) = U\alpha\gamma(r) - \frac{M'}{M^2}\frac{(N-1)}{2r} + \frac{L(L+N-2)+(N-1)(N-3)/4}{Mr^2} \tag{3.72}$$

Note that the total energy appearing in (3.68) is $E = \varepsilon + \Delta E$ and, in one-dimension the modification term ΔV becomes $U_{\alpha\gamma}$ as in Ref. [45]. This clarifies that the corrections due to the higher dimensions arise because of the second and third term on RHS of Eq. (3.72).

From the present theoretical consideration, Eq. (3.70) has an algebraic solution leading to closed analytical expressions for the wave functions and energy eigenvalues, hence one needs to solve Eq. (3.71) exactly. To proceed further, with the consideration of relativistic Dirac equations having no ambiguity parameters, we confidently choose

$$\Delta W(r) = \frac{(\alpha+\gamma)}{2}\frac{M'}{M^{3/2}} - \frac{(N+2L-1)}{2\sqrt{M}r} \quad , \tag{3.73}$$

in which the second term disappears for $N = 1$ as in [45]. Within the frame of Eq. (3.71), this choice leads us

$$W(r)\Delta W(r) = \frac{M'}{2rM^2}\left[\frac{(\alpha+\gamma)(N-1)}{2} + (\alpha+\gamma+1)L\right] - \frac{\Delta E}{2} \quad , \tag{3.74}$$

that is the backbone of the present section.

From the definition of the effective potential in Eq. (3.66), we also note that the use of Eqs. (3.71) and (3.72) naturally restricts the choice of some ambiguity parameters yielding different physically acceptable effective mass Hamiltonians, allowing only $\alpha = \gamma = 0$ (Ben-Daniel Duke Hamiltonian) and $\alpha = \gamma = -1/2$ (Zhu-Kroemer Hamiltonian) cases. This observation clarifies that the unperturbed part (V_0) of the effective potential in (3.70) should correspond to the case $\alpha = \gamma = 0$, having well known solutions in one dimension, while $\alpha = \gamma = -1/2$ is used to calculate $U_{\alpha\gamma}$ in (3.72). Obviously, all the corrections coming from the higher dimensions to the energy and well-behaved wave function terms can be systematically calculated for a given M with the consideration of Eqs. (3.71-3.74) in the light of corresponding W in (3.70).

Recently, some researches have been devoted to the analysis of the classification of quantum systems with position-dependent mass regarding their exact solvability. On a similar basis, Plastino and his co-workers [43] applied an approach within the supersymmetric quantum mechanical framework, for the case $\alpha = \gamma = 0$, to such systems and succeeded to show that some one-dimensional systems with non-constant mass have a supersymmetric partner with the same effective mass. They were also able to solve exactly some particular cases by constructing the superpotential $[W(r)]$ from the form of the effective mass $[M(r)]$ and generalize the concept of the shape invariance for these systems.

For illustration, the superpotential expressions given by [43] for the systems having harmonic oscillator and Morse-like spectra can be easily used in Eq. (3.70) to serve explicit expressions for the corrections to the one-dimensional solutions obtained by considering the Ben-Daniel-Duke effective Hamiltonian in their [43] calculations. This simple investigation enables us testing our results, because all the corrections should disappear in case $N = 1$ and $\alpha = \gamma = 0$ leading to the expressions in [43]. For clarity, this section involves only the application on the harmonic oscillator system. However, the generalization of the present model yielding self-consistent calculations, reproducing $W(r)$ term within the model for any system of interest, will be discussed later.

According to Ref. [43], $W(r)$ term in Eqs. (3.70) and (3.74) is

$$W(r) = \frac{\omega}{2}\int^{r}\sqrt{M(z)}dz + \frac{1}{2}\left(\frac{1}{\sqrt{M}}\right)', \quad \omega = 2\varepsilon_{n=0} \quad , \tag{3.75}$$

for the system having harmonic oscillator spectra. Hence, use of Eqs. (3.71) through (3.74) gives

$$\begin{aligned}\Delta E = {} & \tfrac{M'}{rM^2}\left[\tfrac{(\alpha+\gamma)(N-1)}{2} + L(\alpha+\gamma+1)\right] + \tfrac{(N+2L-1)\omega}{2r\sqrt{M}}\int^{r}\sqrt{M(z)}dz \\ & + (\tfrac{1}{\sqrt{M}})'\left\{(\alpha+\gamma)\left[\omega\int^{r}\sqrt{M(z)}dz + \left(\tfrac{1}{\sqrt{M}}\right)'\right] + \tfrac{(N+2L-1)}{2r\sqrt{M}}\right\} \quad ,\end{aligned} \tag{3.76}$$

which is the explicit form of the energy corrections for a given smooth mass. Clearly, it can be seen that for a constant mass $M \to 1$, Eq. (3.76) reduces to $(N + 2L - 1)\omega/2$ for arbitrary dimensions [50] while in one dimension it goes to zero for a non-constant mass in case $\alpha = \gamma = 0$ [43]. Furthermore, from Eqs. (3.71) and (3.73), the modification term for the corresponding wave function is

$$G(r) = \exp\left(-\int^r \sqrt{M(z)}\Delta W(z)dz\right) = r^{(N+2L-1)/2}M^{-(\alpha+\gamma)/2}. \qquad (3.77)$$

As Eq. (3.70) is analytically solvable having a closed expression for $W(r)$ given by Eq. (3.75) reproducing explicit expressions for ε and F, the corresponding total energy and wave function can easily be calculated through $E = \varepsilon + \Delta E$ and $\Psi = FG$ for the system of interest with a location dependent mass. At this stage it is also noted that the formalism suggested here seems superior to the usual treatment in supersymmetric quantum theory that in principle start with the ground state and builds up excited state wave functions by the use of some linear operators ($A^{\pm}$) whereas there is no such restriction in the present theory providing flexible investigations.

Although the procedure used in the formalism seems reasonable, the use of other works as in the previous section for an appropriate $W(r)$ term to solve Eq. (3.70) may be seen as a drawback of the model. To remove this seeming deficiency, we propose here a unified treatment within the model considering the recent work in [51].

Many of the special functions $H(g)$ of mathematics represent solutions to differential equations of the form

$$\frac{d^2H(g)}{dg^2} + Q(g)\frac{dH(g)}{dg} + R(g)H(g) = 0 \quad , \qquad (3.78)$$

where the functions $Q(g)$ and $R(g)$ are well defined for any particular function [52]. Since in this section we are interested in bound state wave functions, we should restrict ourselves to polynomial solutions of Eq. (3.78). Bearing in mind Eq. (3.78), the substitution of $\Phi(z) = H[g(z)]f(z)$ in Eq. (3.65) leads to the second-order differential equation

$$\frac{1}{M}\left(\frac{f''}{f} + \frac{H''g'^2}{H} + \frac{g''H'}{H} + 2\frac{H'g'f'}{Hf}\right) - \frac{M'}{M^2}\left(\frac{f'}{f} + \frac{H'g'}{H}\right) = V_{eff} - \lambda \quad , \qquad (3.79)$$

in which primes denote derivatives with respect to g and z for the functions $H(g)$, $g(z)$ and $f(z)$ respectively. With the confidence gained by the similarity between Eqs. (3.79) and (3.68), one can safely use the present treatment splitting Eq. (3.79) in two pieces

$$W^2(z) - \left[\frac{W(z)}{\sqrt{M}}\right]' = V_0(z) - \varepsilon, \qquad W = -\frac{f'}{\sqrt{M}f} \quad , \qquad (3.80)$$

and

$$\Delta W^2(z) - \left[\frac{\Delta W(z)}{\sqrt{M}}\right]' + 2W(z)\Delta W(z) = \Delta V(z) - \Delta E, \quad \Delta W = -\frac{H'g'}{\sqrt{M}H} \quad , \tag{3.81}$$

which is similar to Eqs. (3.70) and (3.71), where $\lambda = \varepsilon + \Delta E$ and $V_{eff} = V_0 + \Delta V$.

After all, it can be clearly seen that Eq. (3.80) is the one required for obtaining an explicit expression for W term used in Eq. (3.70) corresponding to an exactly solvable system considered in one-dimension ($\alpha = \gamma = 0$). However, to proceed further, the functions f and g should be solved as H, Q and R are known in principle. Now, equating like terms between the resulting expression in (3.78) and (3.79) gives

$$Q[g(z)] = \frac{1}{g'}\left(\frac{g''}{g'} + \frac{2f'}{f} - \frac{M'}{M}\right), \quad R[g(z)] = \frac{1}{g'^2}\left[\frac{f''}{f} - \frac{M'}{M}\frac{f'}{f} + M(E-V)\right] \quad , \tag{3.82}$$

where, from the definition of Q,

$$f(z) \approx (\frac{M}{g'})^{1/2} \exp\left[\frac{1}{2}\int^{g(z)} Q(g)dg\right]. \tag{3.83}$$

Consideration of Eqs. (3.79) through (3.82) suggests a novel prescription

$$\Delta V(z) - \Delta E = -\frac{g'^2}{M}R[g(z)] \quad , \tag{3.84}$$

which, for plausible M and R functions, provides a reliable expression for $g(z)$. It is remarked that in the constant mass case $M \to 1$ this procedure reduces to the well known formalism which has been thoroughly investigated [53-55] that, together with [51], justify our new proposal in solving PDMSE. The more detailed investigation of this treatment will be discussed elsewhere.

In this section, a general method has been presented to address the question of corrections to the solution in one-dimension for a large class of $N-$ dimensional and exactly solvable PDMSE. We have also described how to extend the method to the case where the necessary function $W(r)$ in (3.70) generating algebraically solvable potentials in one dimension are present, which initiates calculations in the model leading to explicit expressions for the modifications due to both the use of physically plausible Zhu-Kroemer effective Hamiltonian ($\alpha = \gamma = -1/2$) and higher dimensional treatments. The main results are consistent with the other related works in the literature, which allow a non-perturbative treatment of these issues.

Although, for clarity we have illustrated an application of the method for an easily accessible case of interest, it can be readily employed in various typical situations. In view of the importance in calculating such corrections in physics,

we believe that the present model would serve as a useful toolbox to treat even more realistic situations which now occur in experimental observations with the advent of the quantum technology.

The work discussed here appeared in [56].

CHAPTER 4

APPLICATIONS TO RELATIVISTIC PROBLEMS

After successful applications [17-19, 25, 26, 31, 45] of the recently introduced new model to some non-relativistic problems in various fields of physics which are discussed in detail in the previous Chapter, we consider here the application of the model to some other interesting problems in the relativistic region as well. These applications will involve the well known Klein- Gordon (K-G) and Dirac equations.

These two relativistic equations have a lot of attraction in the last few years [57-62]. In addition to these works, we approach to the same problem in a different perspective to improve the framework of the model used through the thesis study. These alternative treatment will provide safely the separation of the relativistic and non-relativistic contributions in an explicit algebraic form, which has not been discussed earlier in the literature as in the present consideration. Thus, the ground-state and excited-state wave functions can be generated systematically in a unique frame in terms of orthogonal polynomials. This is a considerable refinement of the previously used model.

4.1 Practicing with K-G Equation

4.1.1 A Search on Klein-Gordon Equation

Due to the significance of exactly solvable relativistic equations for the systems under the influence of strong potentials in the vast area of physics, a considerable increasing interest in the study of the Klein-Gordon and Dirac equations has appeared in the literature. However, to our knowledge the relation between the strengths of the vector and scaler potentials and the relativistic corrections coming to the non-relativistic solutions has not been fully explored, although the literature involves many valuable applications on this matter.

Within this context, and using the spirit of the works in [17-19, 25, 26, 31, 45], the work presented in this Chapter deals with the K-G equation which is carefully decomposed here in two pieces to see unambiguously the behavior of the spinless particle in the non-relativistic domain and the modifications brought by the relativistic effects. Before proceeding, one needs to visualize the possible non-relativistic limit of this equation through a crude approximation, which will provide a better understanding in building the formalism presented here.

We remind that in the presence of vector and scalar potentials the (1+1)-dimensional time-independent K-G equation for a spinless particle of rest mass m reads

$$-(\hbar c)^2\Psi_n'' + (mc^2 + V_S)^2\Psi_n = (E_n - V_V)^2\Psi_n \quad , \quad n = 0, 1, 2, ... \tag{4.1}$$

where E is the relativistic energy of the particle, c is the velocity of the light and $\hbar$ is the Planck constant. The vector and scalar potentials are given by $V_V(r)$ and $V_S(r)$, respectively. In the non-relativistic approximation (potential energies small compared to mc^2 and $E \cong mc^2$), Eq. (4.1) becomes

$$-\frac{\hbar^2}{2m}\Psi_n'' + (V_S + V_V)\Psi_n \approx (E_n - mc^2)\Psi_n \ . \tag{4.2}$$

Eq. (4.2) shows that Ψ obeys the Schrödinger equation with binding energy equal to $E - mc^2$, and without distinguishing the contributions of vector and scalar potentials.

Now, bearing in mind this form of the K-G equation in the non-relativistic domain and our earlier treatments in the previous sections, we suggest that the full relativistic wave function in (4.1) may be expressed as $\Psi = \chi\phi$ where χ denotes the behavior of the wave function in the non-relativistic region and ϕ is the modification function due to the relativistic effects. This consideration transforms Eq. (4.1) into a couple of equation

$$\frac{\chi_n''}{\chi_n} = 2(mV_S + E_nV_V) - \varepsilon_n \ , \tag{4.3}$$

$$\frac{\phi_n''}{\phi_n} + 2\frac{\chi'}{\chi}\frac{\phi_n'}{\phi_n} = (V_S^2 - V_V^2) - \Delta\varepsilon_n \ , \tag{4.4}$$

where again the natural units $\hbar = c = 1$ is employed as in the previous sections. This would provide for a clear comparison with the other works in the literature. In the above equations, ε and $\Delta\varepsilon$ represent the binding energy within the non-relativistic limit and the modification term because of the relativistic consideration (if any), respectively. Note that $E^2 - m^2 = \varepsilon + \Delta\varepsilon$ and the relativistic corrections are involved within the frame of Eq. (4.4) in a non-perturbative way. This simple but more flexible presentation of the K-G formalism is compatible

with the crude approximation, Eq. (4.2) used for revealing the appearance of K-G equation in the non-relativistic limit, and also confirms the nice discussion in Ref. [58] about the possible misinterpretation in the related literature regarding the relativistic extensions of the given potentials which behave in a similar manner at the non-relativistic domain. Additionally, it is noticeable that relativistic contributions in case $V_S = \pm V_V$ disappear whereas they can be calculated explicitly through (4.4), which will be discussed below by the examples. Eq. (4.3), yields the free particle solution if $V_S = -V_V$ because $E \approx m$ in the limit where this equation is valid while (4.3) reproduces Schrödinger like non-relativistic solutions for the case $V_S = V_V$, which overall justify the reliability of the formalism when the ongoing discussions considered in the literature, e.g. [58, 60].

For practical calculations, Eqs. (4.3) and (4.4) are expressed by the Riccati equation,

$$W_n^2 - \frac{W_n'}{\sqrt{2m}} = 2(mV_S + E_n V_V) - \varepsilon_n, \quad W_n = -\frac{1}{\sqrt{2m}}\frac{\chi_n'}{\chi_n} \ , \tag{4.5}$$

$$\Delta W_n^2 - \frac{\Delta W_n'}{\sqrt{2m}} + 2W_n \Delta W_n = (V_S^2 - V_V^2) - \Delta\varepsilon_n, \quad \Delta W_n = -\frac{1}{\sqrt{2m}}\frac{\phi_n'}{\phi_n} \ . \tag{4.6}$$

It is worth to note that if the whole potential $(2mV_S + 2mV_V + V_S^2 - V_V^2)$ is an exactly solvable then the above equations reduce to a simple form within the framework of the usual supersymmetric quantum theory [11] where a unified treatment like Eq.(4.5) is employed with $W_n^{SUSY} = W_n + \Delta W_n$. However, if Eq. (4.6) has no analytical solution one cannot use W_n^{SUSY} concept in dealing with such problems. To overcome this drawback of the formalism, the elegant reliable technique leading to approximate solutions of (4.6) has been introduced in Chapter 2 and its applications discussed in Chapter 3 and their published forms appeared in Ref. [17-19, 25, 26, 31, 45] for any state of interest. Therefore, the standard treatment of the supersymmetric quantum mechanics may be seen as a particular case of the present scheme.

As an illustrative example, we start with the well known Hulthén potential which is frequently used in the literature to justify theoretical models introduced. Considering the related works [47, 62], the scaler and vector potentials are chosen as

$$V_S(r) = -\frac{S_0}{e^{\alpha r} - 1}, \quad V_V(r) = -\frac{V_0}{e^{\alpha r} - 1} \ , \tag{4.7}$$

which, in the light of Eq. (4.5), restricts us to define

$$W_{n=0} = -\frac{\alpha/\sqrt{2m}}{e^{\alpha r} - 1} + A \ , \tag{4.8}$$

leading to $A = \frac{\sqrt{m/2}}{\alpha}\left(U_0 - \frac{\alpha^2}{2m}\right)$ where $U_0 = 2(mS_0 + E_{n=0}V_0)$. The correspond-

ing non-relativistic energy and unnormalized wave function in the ground state

$$\varepsilon_{n=0} = -A^2 = -\frac{(2mU_0 - \alpha^2)^2}{8m\alpha^2}, \quad \chi_{n=0} = e^{-\sqrt{2m}\int W_{n=0}dz} = (1 - e^{-\alpha r})e^{-\sqrt{2m}Ar} \ , \tag{4.9}$$

which are in agreement with the work [63] performed in the non-relativistic frame. From Eq.(4.4), bound states requirements such that $V_S > V_V$ and $E^2 - m^2 = \varepsilon + \Delta\varepsilon < 0$ subsequently $m > E$ are satisfied. In this case, with the consideration of (4.6), we set ΔW

$$\Delta W = -\frac{\delta(\alpha/\sqrt{2m})}{e^{\alpha r} - 1} + B \ , \tag{4.10}$$

from where $B = -\sqrt{m/2}\delta U_0/\alpha(\delta + 1)$. It is stressed that for $\delta \to 0$, the relativistic effects due to strong interactions die away because $\Delta W \to 0$, together with $\Delta V \to 0$ and $\Delta\varepsilon \to 0$. From equations (4.6) and (4.10), in case $\delta > 0$, the relativistic contributions to the non-relativistic solutions are

$$\Delta\varepsilon_{n=0} = -B(B + 2A) = \frac{\delta U_0}{2\alpha^2(\delta + 1)^2}\left[mU_0(\delta + 2) - \alpha^2(\delta + 1)\right] \ ,$$

$$\phi_{n=0} = e^{-\sqrt{2m}\int \Delta W_{n=0}dz} = (1 - e^{-\alpha r})^{\delta} e^{-\sqrt{2m}Br} \quad . \tag{4.11}$$

Thus, the full solutions corresponding the total potential $-U_0/(1 - e^{\alpha r}) + (V_{S_0}^2 - V_{V_0}^2)/(1 - e^{\alpha r})^2$ are

$$E_{n=0}^2 - m^2 = \varepsilon_{n=0} + \Delta\varepsilon_{n=0} = -\frac{1}{8m\alpha^2}\left[\frac{2mU_0}{\delta + 1} - \alpha^2\right]^2 \ ,$$

$$\Psi_{n=0} = \chi_{n=0}\phi_{n=0} = (1 - e^{-\alpha r})^{\delta+1} e^{-\left[\frac{mU_0}{\alpha(\delta+1)} + \frac{\alpha}{2}\right]r} \ . \tag{4.12}$$

The results agree with [62]. The justification of the scenario used in terms of the findings above can also be easily observed if one starts directly from the K-G equation and use the introduced form of $W_{n=0}^{SUSY}$ in a Riccati equation similar to (4.5) but for the whole potential. However, such a treatment is not so practical due to the screening of the relativistic contributions in the calculated results.

Though we have considered only the ground state solutions here, the extension of the prescription used to the excited states does not cause any problem if the potential in (4.5) has an algebraic solution. For the clarification of this point, the reader is referred to [31]. It is importantly stressed that neither the Hulthén potential [63] nor the effective Hulthén like potential appeared here are shape invariant [11], unlike the wrong consideration in the recent analysis of the same problem [47]. Therefore, an algebraic expression for the whole spectrum of the total potential is not available.

Furthermore, as the use of (4.10) in (4.6) reproduces $\delta(\delta+1)\frac{\alpha^2}{2m} = S_0^2 - V_0^2$, one can safely express the parameter δ, related to the relativistic contributions through strong interactions in case the scaler potential is larger than the vector potential, in the explicit form

$$\delta = -\frac{1}{2} + \sqrt{\frac{2m}{\alpha^2}(S_0^2 - V_0^2) + \frac{1}{4}}, \tag{4.13}$$

which supports the earlier work in [62] and physically interesting discussion therein regarding the relation between the reasonable solutions and the strengths of vector/tensor potentials through the parameter δ.

As the second illustration, we focus on the recently investigated [59] mixed perturbed Coulomb like scalar and vector potentials,

$$V_S(r) = \frac{S_0}{r} + S_1 r + S_2 r^2, \quad V_V(r) = \frac{V_0}{r} + V_1 r + V_2 r^2. \tag{4.14}$$

Although this problem has been well discussed in the literature with the consideration of exact solvability depending on the potential parameters, there is an alternative case stayed behind the study in [59], which is one of the subject of this section. Secondly, and more significantly, the theoretical consideration here proposes a scheme for a systematic treatment of the relativistic effects if the corresponding equation, Eq. (4.4) or Eq. (4.6), is not analytically solvable, whereas the work in [59] lacks of such flexibility.

By the use of Eq. (4.5) and considering the whole discussion in this section, one finds the corresponding solution in a closed algebraic form for the potential, $2\left[(mS_0 + E_n V_0)/r + (mS_1 + E_n V_1)r + (mS_2 + E_n V_2)r^2\right]$, in the non-relativistic region where $V_S = \pm V_V$, with the choice

$$W_{n=0} = \sqrt{\frac{m}{2}}a - \frac{1}{\sqrt{2m}r} + \sqrt{c}r, \tag{4.15}$$

in which $a = -2(mS_0 + E_n V_0)$ and $c = 2(mS_2 + E_n V_2)$. This choice, with the natural restriction on the potential parameters such that

$$mS_1 + E_n V_1 = -(mS_0 + E_n V_0)\sqrt{4m(mS_2 + E_n V_2)}, \tag{4.16}$$

reveals the binding energy at the non-relativistic limit as

$$\varepsilon_n = -(b^2/4c) + \sqrt{c}(2n+3)/\sqrt{2m} = -2m(mS_0 + E_n V_0)^2 + (2n+3)\sqrt{(S_2 + E_n V_2/m)} \tag{4.17}$$

where $b = 2(mS_1 + E_n V_1)$. The wave function in this domain can readily be calculated in the light of (4.5). A detailed study of a similar problem in arbitrary dimensions can be found in [17]. The reader is also referred to the related references therein for the complicated relationship between the potential parameters and the radial quantum number ($n = 0, 1, 2...$).

In spite of the shape invariance character of the potential in the non-relativistic limit discussed above, having a closed algebraic form for the whole spectrum, the inclusion of the relativistic effects ($V_S^2 - V_V^2 > 0$) turns the total potential into the quasi-exactly solvable case [50], which is indeed interesting from the physical point of view. In the contrary, it is reminded that for instance the usual exponential potential has no analytical solution at the non-relativistic domain but the relativistic contribution transforms it an exactly solvable Morse like potential. At this point however, we suggest an alternative scenario for the approximate calculation of relativistic contributions for any quantum state, as long as Eq. (4.5) is analytically solvable as in the present case. Namely, if one expands Eq. (4.6), up to e.g. third order then obtains, as discussed in Section 3.2

$$2W_n\Delta W_{n1} - \frac{\Delta W'_{n1}}{\sqrt{2m}} = \Delta V_1 - \Delta\varepsilon_{n1}, \tag{4.18}$$

$$\Delta W_{n1}^2 + 2W_n\Delta W_{n2} - \frac{\Delta W'_{n2}}{\sqrt{2m}} = \Delta V_2 - \Delta\varepsilon_{n2}, \tag{4.19}$$

$$2(W_n\Delta W_{n3} + \Delta W_{n1}\Delta W_{n2}) - \frac{\Delta W'_{n3}}{\sqrt{2m}} = \Delta V_3 - \Delta\varepsilon_{n3}, \tag{4.20}$$

keeping in mind that

$$\Delta V(r;\lambda) = \sum_{k=1}^{\infty}\lambda^k\Delta V_k(r),\ \ \Delta W_n(r;\lambda) = \sum_{k=1}^{\infty}\lambda^k\Delta W_{nk}(r),\ \ \Delta\varepsilon_n(\lambda) = \sum_{k=1}^{\infty}\lambda^k\Delta\varepsilon_{nk} \tag{4.21}$$

where λ and k denote the perturbation parameter and perturbation order, respectively. It should be remarked that as the system is algebraically solvable in the non-relativistic domain, which means that the corresponding wave functions for the all states are known explicitly, one can easily define $W_n = -\chi'_n/\sqrt{2m}\chi_n$ to be used through Eqs. (4.18-4.20).

To proceed further, considering the perturbation potential shifted by $2(S_0S_1 - V_0V_1)$ because of the relativistic effects,

$$\Delta V = V_S^2 - V_V^2 = \frac{(S_0^2 - V_0^2)}{r^2} + 2(S_0S_2 - V_0V_2)r + (S_1^2 - V_1^2)r^2 + 2(S_1S_2 - V_1V_2)r^3 + (S_2^2 - V_2^2)r^4, \tag{4.22}$$

one needs to chose proper ΔW_{nk} values to satisfy equations at successive perturbation orders such as (4.18-4.20) which lead to the approximate energy $\Delta\varepsilon_n = \sum_k \Delta\varepsilon_{nk}$ and wave function $\phi_n = e^{-\sqrt{2m}\int\sum_k \Delta W_{nk}dz}$ values to obtain the modified relativistic extension, $E_n^2 - m^2 = \varepsilon_n + \Delta\varepsilon_n$ and $\Psi_n = \chi_n\phi_n$, of the results in the non-relativistic domain. Though, this formalism does not seem so practical, in particular for the system under consideration due to the quite complicated relationship between the potential parameters in higher quantum states, it could be easy for the other physical systems, see the work [31], and may work efficiently.

The work presented in this section was published in [64].

4.1.2 Bound State Solutions of Klein-Gordon Equation with the Kratzer Potential

The solution of Kratzer potential within the frame of non-relativistic physics is well known in the literature, see for instance [50]. By means of the increasing interest to the exact solutions for the relativistic equations, the solution of Kratzer potential is recently investigated [61] in the light of K-G equation, but only for the consideration of equal scalar and vector potentials leading to approximate energy solutions. In this section, which is based on the discussion presented in the previous section, we consider the general case where the scalar potential is unequal to the vector potential, bearing in mind the existence of bound states. The results obtained are compared to those in [61] to clarify the importance of the present formalism revealing that the consideration of mixed equal potentials such as [61] does not in usual reproduce the relativistic effects. In fact, such calculations give solely an idea about the appearance of K-G equations in the non-relativistic border.

Now, let us focus on the scalar and vector potentials in the form

$$V_S = \frac{A_1}{r^2} - \frac{B_1}{r}, \quad V_V = \frac{A_2}{r^2} - \frac{B_2}{r}, \tag{4.23}$$

which, in the light of Eq. (4.5), restricts us to define

$$W_{n=0} = -\frac{c+1}{r} + \frac{k}{2(c+1)}, \quad c > 0,\ k > 0 \tag{4.24}$$

where $k = 2mB_1 + 2EB_2$ and

$$c = \frac{(A_1B_1 - A_2B_2)}{\sqrt{A_1^2 - A_2^2}} \tag{4.25}$$

or

$$c(c+1) = 2mA_1 + 2EA_2$$

$$c = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2(mA_1 + EA_2)}. \tag{4.26}$$

Although we have two definitions of c, (4.25) and (4.26), we will use only the physically acceptable one which is (4.26). Because (4.25), which has no physical meaning, doesn't reproduce physically acceptable c-values when compared to the works of Castro [60] in case $A_1 = A_2 = 0$. Thus, the corresponding full non-relativistic energy spectrum and unnormalized wave function in the ground state are in the form of

$$\varepsilon_n = -\frac{k^2}{4(n+c+1)^2}, \quad \chi_{n=0} = r^{c+1}e^{-\frac{kr}{2(c+1)}}, \tag{4.27}$$

that is in agreement with those in [50] that was performed in the non-relativistic frame.

With the consideration of (4.6), we set ΔW as

$$\Delta W = -\frac{a}{r^2}\ , \quad a > 0 \tag{4.28}$$

where $a = \sqrt{A_1^2 - A_2^2}$. The procedure until here shows us that $A_1 > A_2$, $|B_1| < |B_2|$, and thus the requirement for bound states such that $V_S > V_V$ and $E^2 - m^2 = \varepsilon + \Delta\varepsilon < 0$, subsequently $m > E$ are satisfied. For the case $V_V = \pm V_S$, a vanishes, consequently $\Delta W \to 0$, together with $\Delta V \to 0$. From equation (4.6) and (4.26), in case $a > 0$, the relativistic contributions to the non-relativistic solutions are

$$\Delta\varepsilon_n = 0, \quad \phi_{n=0} = e^{-\sqrt{2m}\int \Delta W_{n=0} dz} = e^{-\frac{a}{r}}. \tag{4.29}$$

Hence, the full solutions corresponding the total potential $2mV_S + 2E_nV_V + V_S^2 - V_V^2$ are

$$E_n^2 - m^2 = \varepsilon_n + \Delta\varepsilon_n = -\frac{k^2}{4(n+c+1)^2} = -\frac{4(mB_1 + E_nB_2)^2}{\left[2n+1+\sqrt{1+8(mA_1+E_nA_2)}\right]^2} < 0, \tag{4.30}$$

$$\psi_{n=0} = \chi_{n=0}\phi_{n=0} = r^{c+1}\exp\left(-\frac{a}{r} - \frac{kr}{2(c+1)}\right). \tag{4.31}$$

Though the energy correction is zero in this specifically chosen example, however this is not the case in general for other problems [64]. It is stressed at this point that one can directly solve infact the K-G equation, without use of a separation procedure as in the present scheme, employing the total form of $W_{n=0}^{SUSY}$ above in a Riccati equation similar to (4.5) in connection with the whole potential. However, such a treatment is not so practical due to the screening of the relativistic contributions in the calculation results.

To test the reliability of (4.30) let $A_1 = A_2 = 0$, then

$$E_n = m\frac{-\frac{B_1B_2}{(n+1)^2} \pm \sqrt{1 - \frac{B_1^2 - B_2^2}{(n+1)^2}}}{1 + \frac{B_2^2}{(n+1)^2}}\ , \tag{4.32}$$

that overlaps with result in [60].

Since we know the solution of the problem, we will start to analyze some special cases.

(1) In the case of a pure scalar potential, $V_V = 0$, where we have $A_2 = B_2 = 0$ and $V = V_S = \frac{A_1}{r^2} - \frac{B_1}{r} > 0$,

$$E_n = \pm m\sqrt{1 - \frac{4B_1^2}{\left[2n+1+\sqrt{1+8mA_1}\right]^2}}\ , \tag{4.33}$$

so that the bound state energy levels for particles and antiparticles are symmetric about $E_{n=0}$. If $A_1 = 0$ then Eq. (4.33) reduces to the solution in [60].

(2) In the pure vector potential case, $V_S = 0$, we have only a potential term $V = V_V = \frac{A_2}{r^2} - \frac{B_2}{r}$; $A_1 = B_1 = 0$. Then we have bound state energy

$$E_n^2 - m^2 = -\frac{4E_n^2 B_2^2}{\left[2n + 1 + \sqrt{1 + 8E_n A_2}\right]^2} \ , \tag{4.34}$$

the expression of E_n is so complicated in order to to observe its physical meaning. Because of this we use power series to have an approximate energy solution

$$E_n = m\left[1 - \frac{B_2^2}{2(n + 1 + 2mA_2)^2}\right]. \tag{4.35}$$

In this circumstances, the energy spectrum consists of energy levels either for particle $V_V > 0$ or for antiparticles $V_V < 0$. To compare with the study in Ref. [60] we choose $A_2 = 0$, then (4.30) will have same bound state energy

$$E_n = \pm\frac{m}{\sqrt{1 + \frac{B_2^2}{(n+1)^2}}}. \tag{4.36}$$

(3) For $V_V = \pm V_S$, it is note that relativistic contributions disappear in this case. So we are dealing only with the non-relativistic limit solutions. There are two conditions which first one is **(i)** $V_V = V_S$, it requires the equality of parameters of both potentials, $A_2 = A_1$ and $B_2 = B_1$, $c(c+1) = 2mA_1 + 2EA_2 = 2(m + E)A_1$ with $k = 2(m + E)B_1$. In the light of this point and together with Eq. (4.30) one obtains

$$E_n^2 - m^2 = -\frac{4(m + E_n)^2 B_1^2}{\left[2n + 1 + \sqrt{1 + 8(m + E_n)A_1}\right]^2} \ , \tag{4.37}$$

to avoid from a complicated expression for E_n , we set $\sqrt{m - E_n} = \alpha$ and then expand (4.37) as a power series of α. Leaving out the α^2 and higher terms because of their negligible small values as compared to α, we find $\alpha = \frac{2\sqrt{2m}B_1}{\left[2n+1+\sqrt{1+16mA_1}\right]}$, and then from $m - E_n = \alpha^2$

$$E_n = m - \frac{8mB_1^2}{\left[2n + 1 + \sqrt{1 + 16mA_1}\right]^2}. \tag{4.38}$$

This result agrees with the circumstance in Ref. [61]. The second test is done by Castro [60]; when we assume $A_1 = 0$ in (4.37) we get

$$E_n = m\left[\frac{(n + 1)^2 - B_1^2}{(n + 1)^2 + B_1^2}\right], \tag{4.39}$$

that is same with the case in [60]. Energy levels obtained in (4.38) and (4.39) correspond to bound states of particles. In this case there are no energy levels

for antiparticles. The second condition for the third case is **(ii)** $V_V = -V_S$ where $A_2 = -A_1$ and $B_2 = -B_1$ then the energy spectrum

$$E_n^2 - m^2 = -\frac{4(m - E_n)^2 B_1^2}{\left[2n + 1 + \sqrt{1 + 8(m - E_n)A_1}\right]^2}\ , \tag{4.40}$$

as in the previous case we are using power series expansion about the $\sqrt{m - E_n} = \alpha$ one obtains

$$E_n = -m\left(\frac{2(n+1)^2}{B_1^2} - 1\right), \tag{4.41}$$

which for $A_1 = 0$ Eq. (4.40) reduces to Castro's [60] related situation

$$E_n = -m\left[\frac{(n+1)^2 - B_1^2}{(n+1)^2 + B_1^2}\right] \tag{4.42}$$

in contrast to previous case , now energy levels in (4.41) and (4.42) correspond to antiparticles and so there is no any energy spectrum for the particles.

The present systematic study obviously recovers a number of earlier results for many different potentials in a natural unified way and also leads to new findings. The idea put forward in this section would be used to explore a great number of relativistic systems and can be also extended to the case of the Dirac equation that is discussed in detail in the following section.

The work discussed here appeared in [65]

4.2 Practicing with Dirac Equation

In the previous section we have attempted to understand the nature of K-G equation within the formalism used through the thesis work. Now, we will apply the same model to the Dirac equation involving spin 1/2 cases, and attempt to improve it in order to remove deficiencies in the model

4.2.1 Non-relativistic Limit

Dirac equation for scalar and vector potentials is given by [66] ($\hbar = c = 1$)

$$H\Psi = \{\alpha.\mathbf{p} + \beta(m + V_S) + V_V\}\Psi \tag{4.43}$$

where $\mathbf{p}$ is the momentum operator, E and m are the energy and rest mass of the particle, V_S and V_V are scalar and vector potentials respectively.

To separate angular part of Eqn. (4.43) from the radial part one follows

$$\Psi_{jm}^{\ell} = \begin{bmatrix} \frac{iG_{\ell j}}{r}\varphi_{jm}^{\ell} \\ \frac{F_{\ell j}}{r}\frac{\sigma.\mathbf{r}}{r}\varphi_{jm}^{\ell} \end{bmatrix} \tag{4.44}$$

where σ represents the Pauli spin matrices [67] and

$$G_{\ell j} = \left\{ \begin{array}{ll} G_j^+ & j = \ell + 1/2 \\ G_j^- & j = \ell - 1/2 \end{array} \right\} \quad F_{\ell j} = \left\{ \begin{array}{ll} F_j^+ & j = \ell + 1/2 \\ F_j^- & j = \ell - 1/2 \end{array} \right\}$$

$$\varphi_{jm}^{\ell} = \left\{ \begin{array}{ll} \varphi_{jm}^+ & j = \ell + 1/2 \\ \varphi_{jm}^- & j = \ell - 1/2 \end{array} \right\}.$$

Then using the relations

$$\sigma.\mathbf{p}\frac{f(r)}{r}\varphi_{jm}^{\ell} = -\frac{i}{r}(\frac{df}{dr} + \frac{kf}{r})\frac{\sigma.\mathbf{r}}{r^2}\varphi_{jm}^{\ell} \tag{4.45}$$

and

$$\sigma.\mathbf{p}\frac{\sigma.\mathbf{r}}{r^2}\frac{f(r)}{r}\varphi_{jm}^{\ell} = -\frac{i}{r}(\frac{df}{dr} - \frac{kf}{r})\varphi_{jm}^{\ell} \tag{4.46}$$

where

$$k = \left\{ \begin{array}{ll} -(\ell+1) = -(j+1/2) & j = \ell + 1/2 \\ +\ell = +(j+1/2) & j = \ell - 1/2 \end{array} \right\},$$

the radial equations are

$$-\frac{dF(r)}{dr} + \frac{k}{r}F(r) = (E - m - V_S - V_V)G(r) \tag{4.47}$$

$$\frac{dG(r)}{dr} + \frac{k}{r}G(r) = (E + m + V_S - V_V)F(r). \tag{4.48}$$

From (4.47) and (4.48), omitting the derivatives of V_S and V_V, we obtain

$$\left\{ -\frac{d^2}{dr^2} + \frac{k(k+1)}{r^2} + (V_S^2 - V_V^2) + (2mV_S + 2EV_V) \right\} G = (E^2 - m^2)G. \tag{4.49}$$

Eqn. (4.49) is similar to (4.1) with an additional barrier like term which shows the spin effect of the Dirac equation. When $k = -1$, which means $\ell = 0$, this spin effect term disappears, then Eqn. (4.49) reduces to K-G equation.

The full relativistic spin-up wave function G may be expressed as $G = \chi\phi$, considering our previous treatments, where χ denotes the behavior of the wave function in the non-relativistic region and ϕ is the modification function due to relativistic effect. This substitution transform Eq. (4.49) into a couple of equation expressed by the Riccati equation.

$$W_n^2 - \frac{W_n'}{\sqrt{2m}} = 2(mV_S + E_nV_V) + \frac{k(k+1)}{r^2} - \varepsilon_n, \quad W_n = -\frac{1}{\sqrt{2m}}\frac{\chi_n'}{\chi_n}\ , \tag{4.50}$$

$$\Delta W_n^2 - \frac{\Delta W_n'}{\sqrt{2m}} + 2W_n\Delta W_n = (V_S^2 - V_V^2) - \Delta\varepsilon_n, \quad \Delta W_n = -\frac{1}{\sqrt{2m}}\frac{\phi_n'}{\phi_n}\ . \tag{4.51}$$

With the consideration of the scalar and vector potentials in the form

$$V_S = \frac{A_1}{r^2} - \frac{B_1}{r}, \quad V_V = \frac{A_2}{r^2} - \frac{B_2}{r}, \tag{4.52}$$

for which one needs to define

$$W_{n=0} = -\frac{c+1}{r} + \frac{b}{2(c+1)}, \quad c > 0,\ b > 0, \tag{4.53}$$

and following the similar procedure in section (4.1.2) we find

$$\varepsilon_n = -\frac{b^2}{4(n+c+1)^2}, \quad \chi_{n=0} = r^{c+1}e^{-\frac{br}{2(c+1)}}, \tag{4.54}$$

where $b = 2mB_1 + 2EB_2$ and $c = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2(mA_1 + EA_2) + k(k+1)}$ and assuming

$$\Delta W = -\frac{a}{r^2}\ , \quad a > 0 \tag{4.55}$$

thus the relativistic contributions to the non-relativistic solutions are

$$\Delta\varepsilon_n = 0, \quad \phi_{n=0} = e^{-\sqrt{2m}\int \Delta W_{n=0}dz} = e^{-\frac{a}{r}}, \tag{4.56}$$

where $a = \sqrt{A_1^2 - A_2^2}$. So, the full solutions corresponding the total potential $(2mV_S + 2E_nV_V + V_S^2 - V_V^2 + \frac{k(k+1)}{r^2})$ are

$$E_n^2 - m^2 = \varepsilon_n + \Delta\varepsilon_n = -\frac{b^2}{4(n+c+1)^2} = -\frac{4(mB_1 + E_nB_2)^2}{\left[2n+1+\sqrt{1+8(mA_1+E_nA_2)+4k(k+1)}\right]^2} < 0, \tag{4.57}$$

$$G_{\ell j}(r) = \chi_{n=0}\phi_{n=0} = r^{c+1}\exp\left(-\frac{a}{r} - \frac{br}{2(c+1)}\right). \tag{4.58}$$

There is no energy correction, unlike the case in Section 4.1.2, nevertheless we have now a modification term for the wave function. However, we can not say this is the general case for other problems.

For a comparison, if

$$V_S = V_V = \frac{1}{2}\left(\frac{\hat{A}}{r^2} - \frac{\hat{B}}{r}\right) \tag{4.59}$$

which causes to disappearance of relativistic effects, hence the non-relativistic energy expression will approximately be

$$E_n = m - \frac{2m\hat{B}^2}{\left[2n+1+\sqrt{(2k-1)^2+8m\hat{A}}\right]^2} \tag{4.60}$$

which overlaps with those in [61].

Although we have analyzed the results only for one of the spinors of the particle, such investigation can easily be repeated for the other spinor that is denoted by $F(r)$, following the same procedure.

Finally, we note that in case $k = -1$, all the results obtained here are transformed to those given by the K-G equation for $\ell = 0$ case represented in section (4.1.2). This justifies the reliability of the present results.

4.2.2 A Search on Dirac Equation via Orthogonal Polynomials

Exact solutions of systems in physics has a great importance. To provide such solutions the method carried out in the earlier sections needs to use the linear operators which has a great algebraic difficulty. To overcome this drawback in the formalism, we propose here to use orthogonal polynomials.

For a particle in a field, Dirac equation is transformed to Eq. (4.49), that is given as

$$\left\{-\frac{d^2}{dr^2}+\frac{k(k+1)}{r^2}+(V_S^2-V_V^2)+(2mV_S+2\varepsilon V_V)\right\}G=(\varepsilon^2-m^2)G\ , \quad (4.61)$$

where ε and m are the energy and rest mass of the particle, V_S and V_V are the scalar and vector potentials respectively. This Schrödinger-like equation can be defined as

$$\frac{G''(r)}{G(r)}=V(r)-E\ , \quad (4.62)$$

where $V(r)=\frac{k(k+1)}{r^2}+(V_S^2-V_V^2)+(2mV_S+2\varepsilon V_V)$ and $E=\varepsilon^2-m^2$. As is well known, the solution of (4.62) generally takes the form for exactly solvable cases

$$G(r)=f(r)F[s(r)]. \quad (4.63)$$

The substitution of (4.63) into (4.62) yields the second-order differential equation

$$\left(\frac{f''}{f}+\frac{F''s'^2}{F}+\frac{s''F'}{F}+\frac{2F's'f'}{Ff}\right)=V-E\ , \quad (4.64)$$

and rearranging (4.64) for a more useful form, one gets

$$F''+\left(\frac{s''}{s'^2}+2\frac{f'}{s'f}\right)F'+\left(\frac{f''}{s'^2f}+\frac{E-V}{s'^2}\right)F=0\ . \quad (4.65)$$

Eq. (4.65) is in the form of the most familiar second-order differential equations to the hypergeometric type [52],

$$F''(s)+\frac{\tau(s)}{\sigma(s)}F'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}F(s)=0\ , \quad (4.66)$$

where σ and $\tilde{\sigma}$ are at most second degree polynomials, and τ is a first degree polynomial. The form of $\frac{\tau(s)}{\sigma(s)}$ and $\frac{\tilde{\sigma}(s)}{\sigma^2(s)}$ is well defined for any special function $F(g)$ [52, 54]. From (4.66) it follows that

$$\frac{s''}{s'^2}+2\frac{f'}{s'f}=\frac{\tau(s)}{\sigma(s)}\ , \qquad \frac{f''}{fs'^2}+\frac{E-V}{s'^2}=\frac{\tilde{\sigma}}{\sigma^2}\ . \quad (4.67)$$

From our earlier works, the energy and potential terms in (4.67) can be decomposed in two pieces, which provides a clear understanding for the individual

contributions of the F and f terms to the whole of the solutions, such that $E - V = (E_f + E_F) - (V_f + V_F)$. Therefore, the second equality in (4.67) is transformed to a couple of equation

$$\frac{f''}{f} = V_f - E_f \ , \qquad -\frac{\tilde{\sigma}}{\sigma^2}s'^2 = V_F - E_F \ , \tag{4.68}$$

where f can be expressed in an explicit form, due to the first part in (4.67)

$$f(r) = (s')^{-1/2} \exp\left[\frac{1}{2}\int^{s(r)} \frac{\tau(s)}{\sigma(s)} ds\right] . \tag{4.69}$$

Since the corresponding σ, $\tilde{\sigma}$ and τ terms are well known for a given polynomial (F), the transformation function (s) in (4.68), and afterwards f in (4.69), are easily defined. So, from (4.63), the corresponding total wave function is readily obtained for the whole spectrum.

The potential and total energy terms for the Dirac equation in this case

$$\frac{f''}{f} = V_f - E_f, \qquad V_f = 2mV_S + 2\varepsilon V_V + \frac{k(k+1)}{r^2} \ , \tag{4.70}$$

$$-\frac{\Lambda}{\sigma}s'^2 = V_F - E_F, \qquad V_F = V_S^2 - V_V^2 \ , \tag{4.71}$$

and

$$E_f + E_F = \varepsilon^2 - m^2 \ . \tag{4.72}$$

To understand how efficiently this method works, some physically possible potentials are solved in the following sections to obtain their eigenvalues and eigenfunctions, within the frame of the present formalism.

Dirac oscillator

We first choose the vector and scalar potentials having equal magnitudes to get an exact solutions for this system. Otherwise the system considered becomes quasi-exactly solvable, hence we set $V_V = V_S = ar^2$. In this respect, right-hand sides of equations (4.70) and (4.71) gives

$$V_f = 2a(m+\varepsilon)r^2 + \frac{k(k+1)}{r^2} \ , \qquad V_F = 0 \ , \tag{4.73}$$

where, for spin-up case, $k = -(\ell+1)$ with ℓ being the angular momentum quantum number.

Considering the generalized Laguerre polynomials $L_n^\alpha(s)$ related to confluent hypergeometric functions, one sees that [52]

$$\sigma = s \ , \quad \tau = \alpha + 1 - s \ , \quad \Lambda = n \ . \tag{4.74}$$

From Eq. (4.68), and keeping in mind that the right-hand side of (4.68) should involve the three dimensional harmonic oscillator, obviously one realizes that $s = \frac{1}{2}\omega r^2$. Then, substituting (4.74) with the $s-$definition it is not hard to find that

$$f = Cr^{\alpha+1/2}e^{\frac{-\omega r^2}{4}} \ , \tag{4.75}$$

where $C = \frac{1}{\sqrt{2}}(\frac{\omega}{2})^{\frac{\alpha}{2}}$. This makes possible to predict V_f and E_f as

$$\frac{f''}{f} = V_f - E_f \ , \qquad E_f = (\alpha+1)\omega \ ,$$

$$V_f = \frac{1}{4}\omega^2 r^2 + \frac{(\alpha-1/2)(\alpha+1/2)}{r^2} \ , \tag{4.76}$$

where $\alpha = -(k+1/2) = \ell + \frac{1}{2}$. To find also V_F and E_F, we should concentrate on $-\frac{\Lambda}{\sigma}s'^2 = V_F - E_F$. After some simple algebra we find

$$V_F = 0 \ , \qquad E_F = 2n\omega \ . \tag{4.77}$$

Thus, the full energy spectrum and wave functions are given as

$$E = E_f + E_F = (\alpha + 1 + 2n)\omega = (2n + \ell + 3/2)\omega \ ,$$

$$\Psi = fF = Cs^{\frac{(\ell+1)}{2}}e^{-\frac{s}{2}}L_n^{(\ell+\frac{1}{2})}(s) \ . \tag{4.78}$$

Consequently, the relativistic energy of the Dirac oscillator reads

$$\varepsilon_n^2 = m^2 + (2n + \ell + 3/2)\omega. \tag{4.79}$$

The results obtained are in agreement with those in [54] which considers only the non-relativistic case, and also, for proper parameters these results are agree well with the study of Alhaidari [70].

Hence, it is importantly noted that, choosing the equal magnitudes for vector and tensor potentials we have obtained the appearance of the Dirac equation in the non-relativistic limit, removing the relativistic corrections. We also remark that the present algebraic treatments have been performed for spin-up case. Obviously, following similar procedure, one can easily repeat the same calculations for the spin-down case where now $k = +\ell$.

To find the Dirac equation for antiparticles we can use Eqs. (4.47) and (4.48) that can be written as

$$-F''(r) + \frac{k(k-1)}{r^2}F + (V_S^2 - V_V^2)F = (\varepsilon^2 - m^2)F \ , \tag{4.80}$$

repeating the same procedure just discussed, we obtain the same results as in Eqs. (4.76) and (4.77), but now $k = \alpha + \frac{1}{2} = -(\ell+1)$. This small difference

changes the energy spectrum and the behavior of wave function that the total energy and eigenfunction in Eq. (4.78) reads

$$E = E_f + E_F = (\alpha + 1 + 2n)\omega = (2n - \ell - 1/2)\omega \ ,$$

$$\Psi = fF = Ds^{\frac{-(\ell+1)}{2}} e^{-\frac{s}{2}} L_n^{-(\ell+\frac{3}{2})}(s). \tag{4.81}$$

In this case the relativistic energy for antiparticles becomes

$$\varepsilon_n^2 = m^2 - (2n - \ell - 1/2)\omega \ . \tag{4.82}$$

Dirac-Coulomb Problem

The relativistic hydrogen atom is also an exactly solvable system within the frame of Dirac equation where the piece of potentials are now properly defined as

$$\frac{f''}{f} = V_f - E_f, \qquad V_f = V_S^2 - V_V^2 \ , \tag{4.83}$$

$$-\frac{\tilde{\sigma}}{\sigma^2} s'^2 = V_F - E_F, \qquad V_F = 2mV_S + 2\varepsilon V_V + \frac{k(k+1)}{r^2} \ . \tag{4.84}$$

For again the equal vector and scalar potentials; $V_V = V_S = -\frac{b}{r}$ one gets

$$V_f = 0 \ , \qquad V_F = -\frac{2(m+\varepsilon)b}{r} + \frac{k(k+1)}{r^2} \ . \tag{4.85}$$

In order to apply the present orthogonal polynomial technique, we choose the most suitable generalized Laguerre polynomial $[F = e^{-s/2} s^{\frac{\alpha+1}{2}} L_n^{\alpha}(s)]$ where

$$\sigma = 1 \ , \quad \tau = 0 \ , \quad \tilde{\sigma} = \frac{2n + \alpha + 1}{2s} + \frac{1 - \alpha^2}{4s^2} - \frac{1}{4}, \tag{4.86}$$

leading to $s = ar$, and to be in convenience with [54], we set $a = \frac{e^2}{n+\ell+1}$ that reduces $s = \frac{e^2}{n+\ell+1} r$. Then, Eq. (4.68) reads

$$f = \frac{(n + \ell + 1)^{\frac{1}{2}}}{e} = \text{constant} = C. \tag{4.87}$$

This means that $\frac{f''}{f} = V_f - E_f = 0$, while

$$V_F = -\frac{2(m+\varepsilon)b}{r} + \frac{\ell(\ell+1)}{r^2} \ , \qquad E_F = -\frac{e^4}{4(n+\ell+1)^2} \ , \tag{4.88}$$

where $k = -(\ell + 1)$ and $\alpha = 2\ell + 1$. Thus for this system, the full energy spectrum and wave functions are

$$E = E_f + E_F = E_F = -\frac{e^4}{4(n+\ell+1)^2} \ , \tag{4.89}$$

$$\Psi = fF = Ae^{-s/2}s^{\frac{\alpha+1}{2}}L_n^{\alpha}(s) = Ce^{-s/2}s^{\ell+1}L_n^{2\ell+1}(s). \tag{4.90}$$

Therefore, the relativistic energy for the Dirac-coulomb problem is found as

$$\varepsilon_n^2 = m^2 - \frac{e^4}{4(n+\ell+1)^2}. \tag{4.91}$$

The results are in agreement with [54] and, with a suitable parameters, overlap with those of [70]. Repeating the same calculations for antiparticles we get total energy spectrum and eigenfunctions as

$$E = E_f + E_F = E_F = -\frac{e^4}{4(n-\ell-1)^2}\ , \tag{4.92}$$

$$\Psi = fF = Be^{-s/2}s^{\frac{\alpha+1}{2}}L_n^{\alpha}(s) = Be^{-s/2}s^{-(\ell+1)}L_n^{-2\ell-3}(s). \tag{4.93}$$

where $k = \frac{\alpha+1}{2} = -(\ell+1)$ in this case. And the corresponding relativistic energy now becomes

$$\varepsilon_n^2 = m^2 - \frac{e^4}{4(n-\ell-1)^2}. \tag{4.94}$$

Dirac-Morse Problem

Another exactly solvable system in Dirac equation is the Morse potential which describes the interaction of atoms in the diatomic molecules. For this problem the pieces of the potentials are described as in the form of

$$\frac{f''}{f} = V_f - E_f, \qquad V_f = V_S^2 - V_V^2\ , \tag{4.95}$$

$$-\frac{\tilde{\sigma}}{\sigma^2}s'^2 = V_F - E_F, \qquad V_F = 2mV_S + 2\varepsilon V_V\ , \tag{4.96}$$

where $V_S = -De^{-ar} + [\sqrt{A^2+m^2} - m]$; $V_V = -Ce^{-ar}$ then, one gets

$$V_f = (D^2-C^2)e^{-2ar} - 2D[\sqrt{A^2+m^2}-m]e^{-ar} + A^2 - 2m[\sqrt{A^2+m^2}-m]\ , \tag{4.97}$$

and

$$V_F = -2(mD+\varepsilon C)e^{-ar} + 2m[\sqrt{A^2+m^2} - m]\ . \tag{4.98}$$

The sum of these two potentials is the well known Morse potential

$$V_{tot} = (D^2 - C^2)e^{-2ar} - 2\left(D\sqrt{A^2+m^2} + \varepsilon C\right)e^{-ar} + A^2. \tag{4.99}$$

Choosing the generalized Laguerre polynomials for this problem,

$$\sigma = s\ \ ,\ \ \tau = \alpha + 1 - s\ \ ,\ \ \Lambda = n\ , \tag{4.100}$$

one, from Eq. (4.68), gets $s = \frac{2B}{a}e^{-ax}$ with

$$V_F = -2naBe^{-ax}, \quad E_F = 0. \tag{4.101}$$

Substituting the definition of s into (4.69) we obtain the function f

$$f = \frac{1}{\sqrt{\alpha}} s^{\alpha/2} e^{-\frac{s}{2}} \ , \tag{4.102}$$

and also substituting f into (4.68) the other piece of potential and energy are extracted as

$$V_f = B^2 e^{-2ax} - (1+\alpha)aBe^{-ax} \ , \quad E_f = -\frac{a^2\alpha^2}{4} \ . \tag{4.103}$$

Thus, the total potential

$$V = V_f + V_F = B^2 e^{-2ax} - aB(2n+1+\alpha)e^{-ax} \ , \tag{4.104}$$

and the full energy spectrum, together with wave functions read,

$$E = E_f + E_F = -(A-na)^2 \ , \quad \Psi = fF = \frac{1}{\sqrt{a}} s^{\alpha/2} e^{-\frac{s}{2}} L_n^{\alpha}(s) \ , \tag{4.105}$$

where $A = \frac{a}{2}(2n+\alpha)$, and the relativistic energy is

$$\varepsilon_n^2 = m^2 - (A-na)^2 \ . \tag{4.106}$$

These results overlaps with those in [70] and [71].

Dirac- Rosen-Morse-I Type Potential

For another application, one may set

$$V_S = V_V = (A\tanh(ar) + B)^2, \tag{4.107}$$

which requires $V_f = V_S^2 - V_V^2 = 0$, and using (3.71) one defines

$$V_f = -2(m+\varepsilon)\left[A^2 \sec h^2(ar) - 2AB\tanh(ar)\right] + 2(m+\varepsilon)(A^2+B^2). \tag{4.108}$$

Now, we must decide which type of Jacobi polynomials should be used to obtain the exact solution for the potential in (4.108). We follow that $F = (1-s)^{(\alpha+1)/2}(1+s)^{(\beta+1)/2}P_n^{\alpha,\beta}(s)$

$$\sigma = 1 \ , \ \tau = 0 \ , \ \tilde{\sigma} = \frac{1-\alpha^2}{4(1-s^2)} + \frac{1-\beta^2}{4(1+s^2)} + \frac{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)}{2(1-s^2)} \ , \tag{4.109}$$

and to define s-function,

$$-\frac{\tilde{\sigma}}{\sigma^2}s'^2 = V_F - E_F = -\frac{(1-\alpha^2)s'^2}{4(1-s^2)} - \frac{(1-\beta^2)s'^2}{4(1+s^2)} - \frac{\left[2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)\right]s'^2}{2(1-s^2)} \ . \tag{4.110}$$

Since we have to get a constant (E) on the left-hand side, there must be at least one term on the right-hand side, from which a constant arises. In the most general case this must be one of the terms containing the parameters n, α, and β of the Jacobi polynomials. Thus, from $\frac{s'^2}{1-s^2}{}^2 = a^2$ we found $s = \tanh ar$, where a is a constant. Due to this, we find

$$V_F = -a^2\left[\frac{(\alpha+\beta)(\alpha+\beta+2)}{4} - q\right]\sec h^2(ar) + \frac{a^2(\alpha^2-\beta^2)}{2}\tanh(ar)\ , \quad (4.111)$$

$$E_F = a^2\left[1 - \frac{(\alpha^2+\beta^2)}{2}\right]\ , \quad (4.112)$$

where $q = n(n+\alpha+\beta+1)$. From the values of s, τ and σ one can now define the other part of the wave function f, considering (4.69),

$$f = \frac{1}{\sqrt{a}}(1-s)^{-1/2}(1+s)^{-1/2}\ . \quad (4.113)$$

Remember that the left-hand side of (4.68), with substitution of f, gives us

$$V_f = 0\ ,\quad E_f = -a^2\ . \quad (4.114)$$

In sum,

$$V_{tot} = -a^2\left[\frac{(\alpha+\beta)(\alpha+\beta+2)}{4} - q\right]\sec h(ar) + \frac{a^2(\alpha^2-\beta^2)}{2}\tanh(ar)\ , \quad (4.115)$$

$$E = E_f + E_F = \varepsilon^2 - m^2 = -(C-na)^2 - \frac{D^2}{(C-na)^2}\ , \quad (4.116)$$

$$\Psi = fF = (1-s)^{\alpha/2}(1+s)^{\beta/2}P_n^{\alpha,\beta}(s)\ , \quad (4.117)$$

where to be in convenience with the literature [54] we set $\alpha = -\gamma + n + \frac{\lambda}{\gamma-n}$, $\beta = -\gamma + n - \frac{\lambda}{\gamma-n}$ while $C = \gamma a$ and $D = \lambda a$. Finally, we obtain the relativistic energy of the system as

$$\varepsilon_n^2 = m^2 - (C-na)^2 - \frac{D^2}{(C-na)^2}\ . \quad (4.118)$$

The results are also in agreement with [72] for suitable parameters.

Dirac- Eckart-I Type Potential

It is well known that the one-dimensional form of this potential is exactly solvable and shape invariant in non-relativistic quantum mechanics as in the others discussed above. In order to obtain an exactly solvable analytical solution for such a potential, in also for relativistic case, we again consider the special case for the scalar and vector potentials having equal magnitudes, $V_S(r) = V_V(r)$ which can be introduced to be

$$V_S = V_V = (-A\coth(ar) + B)^2\ , \quad (4.119)$$

providing that $V_f = V_S^2 - V_V^2 = 0$, and

$$V_F = 2(m+\varepsilon)A^2\csc(h)^2(ar) - 4(m+\varepsilon)AB\coth(ar) + 2(m+\varepsilon)(A^2+B^2)\ . \quad (4.120)$$

To finalize, we choose the best suitable Jacobi polynomials $[F = (1-s)^{(\alpha+1)/2}(1+s)^{(\beta+1)/2}P_n^{\alpha,\beta}(s)]$

$$\sigma = 1\ ,\ \tau = 0\ ,\ \tilde{\sigma} = \frac{1-\alpha^2}{4(1-s^2)} + \frac{1-\beta^2}{4(1+s^2)} + \frac{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)}{2(1-s^2)}\ , \quad (4.121)$$

leading to $\frac{s'^2}{1-s^2}^2 = a^2$. One solution of this equation is $s = \coth(ar)$. Then, it yields

$$V_F = a^2\left[q - \frac{(\alpha+\beta)(\alpha+\beta+2)}{4}\right]\csc h^2(ar) - \frac{a^2(\beta^2-\alpha^2)}{2}\coth(ar)\ , \quad (4.122)$$

$$E_F = -(C+na)^2 - \frac{D^2}{(C+na)^2}\ , \quad (4.123)$$

where $q = n(n+\alpha+\beta+1)$, $\alpha = -(\gamma+n) + \frac{\lambda}{\gamma+n}$, $\beta = -(\gamma+n) - \frac{\lambda}{\gamma+n}$ while $C = \gamma a$ and $D = \lambda a^2$. Now, we realize that

$$f = \frac{1}{\sqrt{a}}(1-s)^{-1/2}(1+s)^{-1/2}\ . \quad (4.124)$$

Replacing f into (4.68) gives us

$$V_f = 0\ ,\quad E_f = -a^2\ . \quad (4.125)$$

Thus, the total potential term the and full spectrum solutions are explicitly given as

$$V_{tot} = a^2\left[q - \frac{(\alpha+\beta)(\alpha+\beta+2)}{4}\right]\csc h^2(ar) - \frac{a^2(\beta^2-\alpha^2)}{2}\coth(ar)\ . \quad (4.126)$$

$$E = E_f + E_F = \varepsilon^2 - m^2 = -(C+na)^2 - \frac{D^2}{(C+na)^2}\ , \quad (4.127)$$

$$\Psi = fF = (1-s)^{\alpha/2}(1+s)^{\beta/2}P_n^{\alpha,\beta}(s)\ , \quad (4.128)$$

The relativistic energy of the system therefore

$$\varepsilon_n^2 = m^2 - (C+na)^2 - \frac{D^2}{(C+na)^2}\ . \quad (4.129)$$

The results found in this section are in agreement with those of [72] for appropriate parameters. All results obtained are tabulated in Table 1.

Throughout the study in this last section, we have realized that some physically meaningful systems can be solved exactly if the the strengths of the vector and scalar potentials are equal to each other, except the Dirac-Morse problem. Otherwise, the systems of interest becomes quasi-exactly solvable and one may need in this case to propose a more refined model to solve uniquely the Schrödinger, Klein-Gordon and Dirac equations in terms of orthogonal polynomials.

	Oscillator	*Coulomb*	*Morse*
V_S	ar^2	$-b/r$	$-Ae^{-ar}+(\sqrt{B^2+m^2}-m)$
V_V	ar^2	$-b/r$	$-Ce^{-ar}$
s	$\frac{1}{2}\omega r^2$	$(e^2/(n+\ell+1))r$	$(2D/a)e^{-ax}$; $D^2=A^2-C^2$
σ	s	1	s
τ	$\alpha+1-s$	0	$\alpha+1-s$
$\tilde{\sigma}$	ns	$\frac{2n+\alpha+1}{2s}+\frac{1-\alpha^2}{4s^2}-\frac{1}{4}$	ns
ε	$(m^2+(2n+\ell+3/2)\omega)^{\frac{1}{2}}$	$\left(m^2-\frac{e^4}{4(n+\ell+1)^2}\right)^{\frac{1}{2}}$	$\left(m^2+B^2-\frac{a^2\alpha^2}{4}\right)^{\frac{1}{2}}$
Ψ	$s^{\frac{(2\alpha+1)}{4}}e^{-\frac{s}{2}}L_n^{(\alpha)}(s)$	$s^{\frac{\alpha+1}{2}}e^{-s/2}L_n^{\alpha}(s)$	$s^{\alpha/2}e^{-\frac{s}{2}}L_n^{\alpha}(s)$
α	$\ell+1/2$	$2\ell+1$	$\frac{2(A\sqrt{B^2+m^2}+\varepsilon C)}{a\sqrt{A^2-C^2}}-1-2n$

	Rosen − Morse	*Eckart*
V_S	$(A\tanh(ar)+B)^2$	$(-A\coth(ar)+B)^2$
V_V	$(A\tanh(ar)+B)^2$	$(-A\coth(ar)+B)^2$
s	$\tanh ar$	$\coth ar$
σ	1	1
τ	0	0
$\tilde{\sigma}$	$\frac{1-\alpha^2}{4(1-s^2)}+\frac{1-\beta^2}{4(1+s^2)}+\frac{c_n}{(1-s^2)}$	$\frac{1-\alpha^2}{4(1-s^2)}+\frac{1-\beta^2}{4(1+s^2)}+\frac{c_n}{(1-s^2)}$
ε	$\left(m^2+\eta-\frac{a^2(\alpha^2+\beta^2)}{2}\right)^{\frac{1}{2}}$	$\left(m^2+\zeta-\frac{a^2(\alpha^2+\beta^2)}{2}\right)^{\frac{1}{2}}$
Ψ	$(1-s)^{\alpha/2}(1+s)^{\beta/2}P_n^{\alpha,\beta}(s)$	$(s-1)^{\alpha/2}(1+s)^{\beta/2}P_n^{\alpha,\beta}(s)$
α	$\gamma-n+\frac{\lambda}{\gamma-n}$	$-\gamma-n+\frac{\lambda}{\gamma+n}$
β	$\gamma-n-\frac{\lambda}{\gamma-n}$	$-\gamma-n-\frac{\lambda}{\gamma+n}$
c_n	$n(n+\alpha+\beta+1)+\frac{1}{2}(\alpha+1)(\beta+1)$	$n(n+\alpha+\beta+1)+\frac{1}{2}(\alpha+1)(\beta+1)$

Table 4.1: Relativistic energy and unnormalized eigenfunctions of the five potentials deduced within the present Dirac formalism discussed in Section 4.2.2. In the treatment of Rosen-Morse and Eckart potentials, the notation carried out in [73] is used.

CHAPTER 5

CONCLUSION

No single approximation method available in the literature is ideal for every problem. SSPT and LPT based theories avoid the Rayleigh-Schrödinger summation, but it can lead to nasty integrals and more effort in particular for excited states. The method is valuable when the integrals can be done exactly or by a reliable numerical procedure. Otherwise, the Rayleigh-Schrödinger summation, even when it does not give an exact answer, starts not to look so bad after all. This was the motivation behind the work introduced in this thesis. The present perturbation model appears in this respect to be superior for the excited states and provides a quick route to the calculation of all corrections within the frame of the perturbation theory, which considerably simplify one's calculational workload.

The power and elegancy of the unified model developed in this thesis which is, in a sense, complete are illustrated via applications to non-relativistic and relativistic problems. Firstly, we have attempted to explore the effectiveness of the formalism introduced through which we have made successfully the complete mathematical analysis of the reason behind exact solvability of some Schrödinger equations with a class of non-central but separable potentials, for which the complete spectrum and eigenfunctions can be written down algebraically using the well known results for the shape invariant potentials.

With the second example, the theory put forward here has been successfully extended from the bound state region to the scattering domain. For the clarification, the work has been carried out with the consideration of $s-$wave scattering only, considering phase shifts at low orders. However, generalization of the formalism to higher partial waves in the scattering domain does not cause any problem. The energy variation of the scattering wave function and phase shift can of course be studied if required by perturbing in the energy. We wish to stress that all these effects depend purely upon the perturbation and the unperturbed wave function; explicit knowledge of the unperturbed potential is not

necessary.

The third illustration in the first part of the thesis, as another application in non-relativistic domain, has discussed the problem of exact solvability and ordering ambiguity in quantum mechanics for the systems with a position-dependent mass. The present scheme has restricted the possible choices of ordering. Proceeding with this consideration it has been observed that the only physically allowable BDD and ZK Hamiltonians only for exactly solvable potentials are in fact their supersymmetric partners that reproduce identical results in their independent considerations due to use of an identical superpotential. We hope that this observation would make a contribution to the ongoing debate in the literature regarding the isospectral effective mass Hamiltonians.

As a last application of the model in the non-relativistic region, a general method has been presented to address the question of corrections to the solution in one-dimension for a large class of $N-$ dimensional and exactly solvable PDMSE. We have also described how to extend the method from the case where the required function $W(r)$ in (3.70) generates algebraically solvable potentials in one dimension. The procedure suggested in there initiates calculations in the model leading to explicit expressions for the modifications due to both the use of physically plausible Zhu-Kroemer effective Hamiltonian ($\alpha = \gamma = -1/2$) and higher dimensional treatments. The main results are consistent with the other related works in the literature, which allow a non-perturbative treatment of these issues.

Although, we have illustrated some simple but interesting applications of the method for non-relativistic systems, through Chapter 3, it can be readily employed in various other typical situations. In view of the importance in calculating such corrections in physics, we believe that the present prescription would serve as a useful toolbox to treat even more realistic situations which now occur in experimental observations with the advent of the quantum technology.

In the case of relativistic systems, Klein-Gordon and Dirac equations have been analyzed carefully. These applications have justified once more, additional to our previous non-relativistic considerations, the success and power of the model employed in the calculations. During the discussions in Chapter 4, we have suggested that a relativistic problem may be treated as modifications plus the solution in non-relativistic limit. For this purpose, the system of interest has divided in two parts; first part constitutes the exactly solvable portion and the second part corresponds to the corrections to this non-relativistic part. Applications of the technique to different potentials in the relativistic region confirm the reliability of the unified treatment leading to accurate and explicit solutions.

In the final section of Chapter 4, we have also attempted to improve the

model used through the thesis. The refined new model by passes the difficulties in calculating excited state wave functions, unlike our previous model where one needs to use linear operators which lead to tedious and cumbersome calculations. The improved model, however, provides simply the entire spectrum wave functions in a unique frame in terms of special functions. This refinement in the model can also easily be used for Schrödinger equation with constant/non-constant masses.

Hence, the solutions (in terms of orthogonal polynomials) of Dirac equation with analytically solvable potentials have been investigated in the last section of the thesis within the refined model by transforming the relativistic equation into a Schrödinger like one. Earlier results have been discussed and certain solutions of a large class of potentials have been given in this section. The work presented there involves significant idea of connecting the methods used in the analysis of exactly solvable potentials in non-relativistic quantum mechanics with the solution procedure of Dirac equation. A straightforward generalization would of course be the application of the scheme to other relativistic equations for integer spin cases. Beyond its intrinsic importance as a new solution for a fundamental equation in physics, we also expect the present simple method would find a widespread application in the study of different quantum mechanical and nuclear scattering systems. Along this line, the works are in progress.

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FOREIGN LANGUAGES

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PUBLICATIONS

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