Employing Meta-heuristics and Fuzzy Ranking Functions for Direct Solution of Fuzzy Mathematical Programs

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Dedicated to my sweet daughter Nisa…

ABSTRACT

EMPLOYING META-HEURISTICS AND FUZZY RANKING FUNCTIONS FOR DIRECT SOLUTION OF FUZZY MATHEMATICAL PROGRAMS

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Primary objective of this study is to present how fuzzy mathematical programming models can be solved by employing metaheuristic algorithms and ranking methods for fuzzy numbers without requiring a transformation into a crisp model. Up to date various solution approaches are proposed to solve different fuzzy mathematical programming models. The main difficulty in fuzzy mathematical programming is to solve fuzzy models using the existing solution algorithms. In the existing approaches to overcome this problem the crisp equivalent of the fuzzy models are obtained. In this study, a direct solution method is proposed to solve fuzzy mathematical programming problems with different fuzzy parameters and fuzzy mathematical programming problems with fuzzy decision variables. In the proposed direct solution method, ranking methods for fuzzy numbers and metaheuristic algorithms are used. The effectiveness of the proposed direct solution method is proved with different examples.

Keywords: fuzzy mathematical programming, fuzzy decision variables, ranking methods for fuzzy numbers, classification of fuzzy mathematical programming models

ÖZET

BULANIK MATEMATİKSEL PROGRAMLARIN DOĞRUDAN ÇÖZÜMÜNDE META-SEZGİSELLERİN VE BULANIK SIRALAMA FONKSİYONLARININ UYGULANMASI

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Bu çalışmada, bulanık matematiksel programlama modellerinin bulanık olmayan (kesin) modele dönüştürülmesine ihtiyaç duyulmadan bulanık sayı karşılaştırma yöntemleri ve meta-sezgisel algoritmalar kullanılarak çözülmesi amaçlanmıştır. Bugüne kadar, bulanık matematiksel programlama modellerini çözmek için çeşitli çözüm yaklaşımları önerilmiştir. Bulanık matematiksel programlamadaki esas zorluk bulanık modelin mevcut çözüm algoritmaları ile çözülememesidir. Önerilen yaklaşımlarda bu problemi aşmak için bulanık model kesin modele dönüştürülmeye çalışılmaktadır. Bu çalışmada, değişik parametrelerin bulanık sayı olarak tanımlandığı bulanık matematiksel programlama problemleri ve karar değişkenlerinin bulanık sayı olarak tanımlandığı bulanık matematiksel programlama problemlerini doğrudan çözmek için bir çözüm yöntemi önerilmiştir. Önerilen doğrudan çözüm yönteminde, bulanık sayı karşılaştırma metotları ve meta-sezgisel algoritmalar kullanılmıştır. Önerilen çözüm yönteminin geçerliliği değişik örneklerle gösterilmiştir.

Anahtar Kelimeler: bulanık matematiksel programlama, bulanık karar değişkenleri, bulanık sayı karşılaştırma yöntemleri, bulanık matematiksel programlama modellerinin sınıflandırılması

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CHAPTER 1

INTRODUCTION

1.1. Introduction

This chapter briefly introduces the context of the research and describes the fuzzy set theory, fuzzy numbers and fuzzy arithmetic, and fuzzy mathematical programming. Definition of fuzzy set and basic concepts of fuzzy set are explained in section 1.2. Fuzzy numbers and fuzzy arithmetic are described in section 1.3. Fuzzy mathematical programming is explained in section 1.4. Finally, the objective of the thesis and the methodology used in the solution procedure are explained and the organization of the thesis is presented.

1.2. Fuzzy Set Theory

Fuzzy set theory provides a means for representing uncertainties. Fuzzy set theory is a marvelous tool for modeling the kind of uncertainty associated with vagueness, with imprecision, and/or with a lack of information regarding a particular element of the problem at hand (Ross, 2004).

In mathematics, grouping concepts are stated by sets. In classical set theory, a set encloses a member or does not enclose. But, for some events grouping is not so precise and clear. For example, ‘tall people’, ‘fat people’ or ‘cold day’ groupings and categories do not have sharp boundaries. In these sets, membership is defined by degrees or grades. These sets are known as fuzzy sets and the underlying theory is called fuzzy set theory. Fuzzy sets are used when sharp boundaries cannot be defined.

1.2.1. Definition of the fuzzy set

In crisp sets the transition for an element in the universe between membership and nonmembership in a given set is well-defined. For an element in a universe that contains fuzzy sets, this transition can be gradual. This transition among various degrees of membership can be thought of as conforming to the fact that the boundaries of the fuzzy set are vague and ambiguous (Ross, 2004, pp.34).

A fuzzy set is a class of objects with a continuum of grades of membership (Zadeh, 1965).

If X is the universe whose generic element is denoted by x, a *fuzzy set* A in X is a function $A : X \rightarrow [0, 1]$ (Bector and Chandra, 2005, pp.21).

For the function A, generally, μ_A is used and the fuzzy set A is characterized by its membership function $\mu_A : X \rightarrow [0, 1]$. The value $\mu_A(x)$ at x represents the grade of membership of x in A, the degree to which x belongs to fuzzy set A (Bector and Chandra, 2005, pp.21; Zadeh, 1965).

An important difference between a fuzzy set and a crisp set is; a crisp set has a unique membership function but a fuzzy set can have an infinite number of membership functions to represent it. Another important difference is; elements in a fuzzy set, because their membership need not be complete (can be gradual), can also be members of other fuzzy sets on the same universe (Ross, 2004).

The membership functions can be represented with very different shapes of graphs. It cannot be said a particular shape is much suitable. It turns out, however, that many applications are not very sensitive to variations in shape. So, generally, a simple shape, such as the triangular or trapezoidal shape is used to define membership functions (Klir and Yuan, 1995).

1.2.2. Basic concepts of fuzzy sets

1.2.2.1. The α-cut and the strong α-cut

Given a fuzzy set $\widetilde{A}$ defined on X and any number $\alpha \in [0,1]$, the α-cut, A_α and its variant the strong α-cut, $A_{\alpha+}$, are the crisp sets

$$A_\alpha = \{x | \mu_{\widetilde{A}}(x) \geq \alpha\} \tag{1.1}$$

$$A_{\alpha+} = \{x | \mu_{\widetilde{A}}(x) > \alpha\}. \tag{1.2}$$

The α-cut (or the strong α-cut) of a fuzzy set $\widetilde{A}$ is the crisp set A_α (or the crisp set $A_{\alpha+}$) that contains all the elements of the universal set X whose membership grades in $\widetilde{A}$ are greater than or equal to (or only greater than) the specified value of α (Klir and Yuan, 1995). Any particular fuzzy set $\widetilde{A}$ can be transformed into an infinite number of α-cut sets, because there are an infinite number of values α on the interval [0,1] (Ross, 2005).

1.2.2.2. The support of a fuzzy set

The *support* of a fuzzy set $\widetilde{A}$ within a universal set X is the crisp set that contains all the elements of X that have nonzero membership grades in $\widetilde{A}$.

$$S(A) = \{x | \mu_{\widetilde{A}}(x) > 0\} \tag{1.3}$$

The support of $\widetilde{A}$ is exactly the same as the strong α-cut of $\widetilde{A}$ for α=0 (Klir and Yuan, 1995; Bector and Chandra, 2005).

1.2.2.3. The height of a fuzzy set

The *height*, h($\widetilde{A}$), of a fuzzy set $\widetilde{A}$ is the maximum membership value obtained by any element in that set.

$$h(\widetilde{A}) = \sup_{x \in X} u_{\widetilde{A}}(x) \tag{1.4}$$

If h($\widetilde{A}$) = 1, then the fuzzy set $\widetilde{A}$ is called a *normal* fuzzy set, if h($\widetilde{A}$) < 1 then it is called a *subnormal* fuzzy set (Klir and Yuan, 1995; Bector and Chandra, 2005).

1.2.2.4. Empty fuzzy set

A fuzzy set $\widetilde{A}$ is *empty* if and only if its membership function is identically zero, $\mu_{\widetilde{A}}(x) = 0$ for all $x \in X$ (Zadeh, 1965, Bector and Chandra, 2005).

1.2.2.5. Subset of a fuzzy set

A fuzzy set $\widetilde{A}$ is a *subset* of a fuzzy set $\widetilde{B}$ or $\widetilde{A}$ is contained in $\widetilde{B}$ if $\mu_{\widetilde{A}}(x) \le \mu_{\widetilde{B}}(x)$ for all $x \in X$ (Bector and Chandra, 2005).

1.2.2.6. Equality of fuzzy sets

Two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ are *equal* if and only if $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x)$ for all $x \in X$ (Zadeh, 1965).

1.2.2.7. Standard complement

The *standard complement* of a fuzzy set $\widetilde{A}$ is another fuzzy set, $\widetilde{A}'$, whose membership function is defined as $\mu_{\widetilde{A}'}(x) = 1 - \mu_{\widetilde{A}}(x)$ for all $x \in X$ (Bector and Chandra, 2005).

1.2.2.8. Union of fuzzy sets

The *union* of two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ is a fuzzy set $\widetilde{C}$, written as $\widetilde{C} = \widetilde{A} \cup \widetilde{B}$, whose membership function is defined as,

$$\mu_{\widetilde{C}}(x) = \max(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)) \qquad \text{for all } x \in X, \tag{1.5}$$

or

$$\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x) \qquad \text{for all } x \in X. \tag{1.6}$$

The union of $\tilde{A}$ and $\tilde{B}$ is the smallest fuzzy set containing both $\tilde{A}$ and $\tilde{B}$ (Zadeh, 1965).

1.2.2.9. Intersection of fuzzy sets

The *intersection* of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$, written as $\tilde{C} = \tilde{A} \cap \tilde{B}$, whose membership function is defined as,

$$\mu_{\tilde{C}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \qquad \text{for all } x \in X, \tag{1.7}$$

or

$$\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x) \qquad \text{for all } x \in X. \tag{1.8}$$

The intersection of $\tilde{A}$ and $\tilde{B}$ is the largest fuzzy set which is contained in both $\tilde{A}$ and $\tilde{B}$ (Zadeh, 1965).

1.2.2.10. Convexity of fuzzy sets

A fuzzy set $\tilde{A}$ in R is a *convex* fuzzy set if its α-cuts A_α are convex sets for all α ∈ (0,1]. A fuzzy set $\tilde{A}$ in R is a convex fuzzy set if and only if for all $x_1, x_2 \in$ R and $0 \le \lambda \le 1$,

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)). \tag{1.9}$$

The convexity of a fuzzy set does not mean that its membership function is a convex function. In fact, membership functions of convex fuzzy sets are functions that are quasi-concave and not convex (Bector and Chandra, 2005).

1.2.2.11. Bounded fuzzy set

A fuzzy set $\widetilde{A}$ in R is a *bounded* fuzzy set if its α-cuts A_α are bounded sets for all α ∈ (0,1] (Bector and Chandra, 2005).

1.2.2.12. The properties of fuzzy sets

Fuzzy sets follow the same properties as crisp sets. The following properties of crisp sets hold for fuzzy sets (Ross, 2004, Bector and Chandra, 2005).

$\widetilde{A} \cup \widetilde{B} = \widetilde{B} \cup \widetilde{A}$

$\widetilde{A} \cap \widetilde{B} = \widetilde{B} \cap \widetilde{A}$ (commutativity)

$\widetilde{A} \cup (\widetilde{B} \cup \widetilde{C}) = (\widetilde{A} \cup \widetilde{B}) \cup \widetilde{C}$

$\widetilde{A} \cap (\widetilde{B} \cap \widetilde{C}) = (\widetilde{A} \cap \widetilde{B}) \cap \widetilde{C}$ (associativity)

$\widetilde{A} \cup (\widetilde{B} \cap \widetilde{C}) = (\widetilde{A} \cup \widetilde{B}) \cap (\widetilde{A} \cup \widetilde{C})$

$\widetilde{A} \cap (\widetilde{B} \cup \widetilde{C}) = (\widetilde{A} \cap \widetilde{B}) \cup (\widetilde{A} \cap \widetilde{C})$ (distributivity)

$\widetilde{A} \cup \varnothing = \widetilde{A}$ *and* $\widetilde{A} \cap X = \widetilde{A}$

$\widetilde{A} \cap \varnothing = \widetilde{A}$ *and* $\widetilde{A} \cup X = \widetilde{A}$ (identity)

If $\widetilde{A} \subseteq \widetilde{B}$ and $\widetilde{B} \subseteq \widetilde{C}$, then $\widetilde{A} \subseteq \widetilde{C}$. (transitivity)

$(\widetilde{A}')' = \widetilde{A}$ (involution)

$(\widetilde{A} \cup \widetilde{B})' = \widetilde{A}' \cap \widetilde{B}'$

$(\widetilde{A} \cap \widetilde{B})' = \widetilde{A}' \cup \widetilde{B}'$ (De Morgan's law)

The following two properties of crisp sets do not hold for fuzzy sets.

$\widetilde{A} \cap \widetilde{A}' = \varnothing$ (law of contradiction)

$\widetilde{A} \cup \widetilde{A}' = X$ (law of excluded middle)

These two properties do not hold for fuzzy sets because of the characteristics of fuzzy sets, fuzzy sets can overlap and a set and its complement can also overlap.

1.3. Fuzzy Numbers and Fuzzy Arithmetic

1.3.1. Fuzzy Numbers

There are various types of fuzzy sets and one special type of them is fuzzy sets that are defined on the set R of real numbers. Membership functions of these sets have the form $\widetilde{A} : R \to [0,1]$ and can be viewed as fuzzy numbers or fuzzy intervals (Klir and Yuan, 1995).

The term fuzzy number is used to handle imprecise numerical quantities, such as "numbers that are close to a given real number", "about a given real number", "several", etc. (Chen and Hwang, 1992).

A fuzzy set $\widetilde{A}$ in R is called a fuzzy number if it satisfies the following conditions

(*i*) $\widetilde{A}$ is normal,

(*ii*) A_α is a closed interval for every $\alpha \in (0, 1]$,

(*iii*) The support of $\widetilde{A}$ is bounded (Bector and Chandra, 2005).

The fuzzy set $\widetilde{A}$ must be normal, because the fuzzy set $\widetilde{A}$ is defined as "real numbers close to *r*" so the membership grade of *r* in this set must be equal to 1. The support of a fuzzy number must be bounded and all α-cuts of $\widetilde{A}$ (for $\alpha \neq 0$) must be closed intervals to define meaningful arithmetic operations on fuzzy numbers in terms of standard arithmetic operations on closed interval. In addition to these, because of all α-cuts of a fuzzy number must be closed intervals for all $\alpha \in (0, 1]$, every fuzzy number is a convex fuzzy set (Klir and Yuan, 1995).

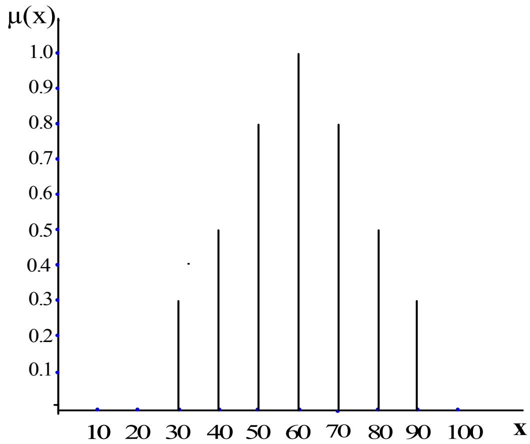
The membership function of a fuzzy number $\widetilde{A}$ denotes the grade of truth that $\widetilde{A}$ takes a specific number *r* (Chen and Hwang, 1992). The membership functions of fuzzy numbers may be, in general, piecewise defined functions and the following theorem shows this characterization (Klir and Yuan, 1995).

Let $\widetilde{A}$ be a fuzzy set in R, then $\widetilde{A}$ is a fuzzy number if and only if there exists a closed interval $[a, b] \neq \phi$ such that

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, & x \in [a,b], \\ l(x), & x \in (-\infty, a), \\ r(x), & x \in (b, \infty), \end{cases} \tag{1.10}$$

where (*i*) l is a function $(-\infty,a)$ to [0,1] that is increasing, continuous from the right and $l(x) = 0$ for $x \in(-\infty, w_1)$, $w_1 < a$ and (*ii*) r is a function from (b,∞) to [0,1] is decreasing continuous from the left and r(x) = 0 for $x \in (w_2,\infty)$, $w_2 > b$ (Bector and Chandra, 2005).

A fuzzy number can be represented in discrete or continuous form (Chen and Hwang, 1992).



x	30	40	50	60	70	80	90
μ(x)	0.3	0.5	0.8	1.0	0.8	0.5	0.3

Figure 1.1. A discrete fuzzy number (Chen and Hwang, 1992)

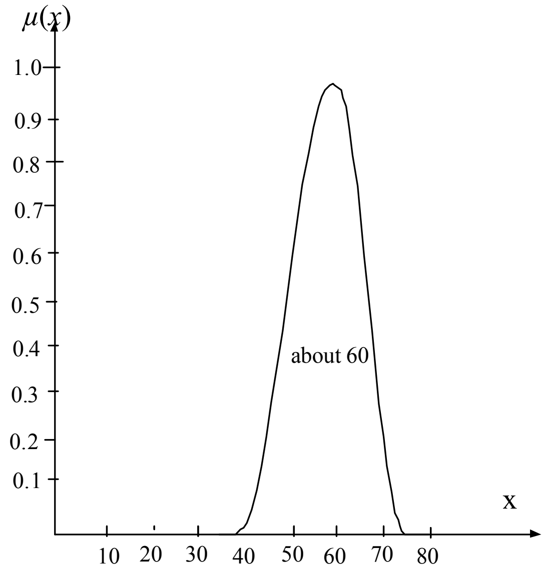


Figure 1.2. A continuous fuzzy number (Chen and Hwang, 1992)

1.3.2. Fuzzy arithmetic

Arithmetic of fuzzy numbers can be taken as a generalization of the interval arithmetic. The difference is, in interval arithmetic there is one (constant) level only, in fuzzy arithmetic there are several levels in [0,1]. All α-cuts of a fuzzy number are considered as an interval (Bector and Chandra, 2005). But, the approach based on the interval arithmetic is not the only approach for arithmetic of fuzzy numbers. There two different but equivalent approaches; the first approach is to use interval arithmetic on the α-cuts of fuzzy numbers. The second approach is based on the extension principle of Zadeh, by which operations on real numbers are extended to operations on fuzzy numbers (Klir and Yuan, 1995, Bector and Chandra, 2005).

1.3.2.1. Fuzzy arithmetic based on α-cuts

Let $\widetilde{A}$ and $\widetilde{B}$ are fuzzy numbers, A_α and B_α are α-cuts of these fuzzy numbers and * denotes any of the four basic arithmetic operations. Then a fuzzy set $\widetilde{A} * \widetilde{B}$ on R can be defined as

$$\widetilde{A} * \widetilde{B} = \bigcup_{\alpha} \alpha (A * B)_\alpha \qquad (1.11)$$

and

$$(A * B)_\alpha = A_\alpha * B_\alpha, \quad \alpha \in (0,1]. \tag{1.12}$$

$\widetilde{A}$ and $\widetilde{B}$ are fuzzy numbers, A_α, B_α and $(A*B)_\alpha$ are closed intervals for each $\alpha \in (0,1]$, so, $\widetilde{A} * \widetilde{B}$ will be a fuzzy number, not just a general fuzzy set (Klir and Yuan, 1995, Bector and Chandra, 2005).

If $\widetilde{A}$ and $\widetilde{B}$ are fuzzy numbers, the α level sets A_α and B_α can be written as $A_\alpha = [a_\alpha^L, a_\alpha^R]$ and $B_\alpha = [b_\alpha^L, b_\alpha^R]$. For a given $\alpha \in (0,1]$, the basic arithmetic operations can be computed by applying the interval arithmetic on the closed intervals A_α and B_α as follows (Bector and Chandra, 2005).

$$A_\alpha(+)B_\alpha = [a_\alpha^L + b_\alpha^L, a_\alpha^R + b_\alpha^R] \tag{1.13}$$

$$A_\alpha(-)B_\alpha = [a_\alpha^L - b_\alpha^R, a_\alpha^R - b_\alpha^L] \tag{1.14}$$

$$A_\alpha(\cdot)B_\alpha = [a_\alpha^L b_\alpha^L, a_\alpha^R b_\alpha^R] \tag{1.15}$$

$$A_\alpha(\div)B_\alpha = [\frac{a_\alpha^L}{b_\alpha^R}, \frac{a_\alpha^R}{b_\alpha^L}], \quad 0 \notin [b_\alpha^L, b_\alpha^R] \tag{1.16}$$

The multiplication of a fuzzy number by a real number k > 0 can be defined as follows.

$$(k \cdot A)_\alpha = k \cdot A_\alpha = [ka_\alpha^L, ka_\alpha^R] \tag{1.17}$$

1.3.2.2. Fuzzy arithmetic based on extension principle

Let $\widetilde{A}$ and $\widetilde{B}$ are fuzzy numbers and * denotes any of the four basic arithmetic operations. Then using Zadeh's extension principle a fuzzy number $\widetilde{A} * \widetilde{B}$ is defined as (Bector and Chandra, 2005)

$$\mu_{\widetilde{A}*\widetilde{B}}(z) = \sup_{z=x*y} \min(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(y)) \quad \text{for all } z \in R. \tag{1.18}$$

The basic arithmetic operations can be computed by applying the extension principle as follows.

$$\mu_{\tilde{A}(+)\tilde{B}}(z) = \sup_{z=x+y} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \tag{1.19}$$

$$\mu_{\tilde{A}(-)\tilde{B}}(z) = \sup_{z=x-y} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \tag{1.20}$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(z) = \sup_{z=x\cdot y} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \tag{1.21}$$

$$\mu_{\tilde{A}(\div)\tilde{B}}(z) = \sup_{z=x/y} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \tag{1.22}$$

1.3.2.3. Special types of fuzzy numbers

The set of fuzzy number is rather large and their arithmetic is in general computationally expensive, so, special types of fuzzy numbers are generally used in the literature and real life.

1.3.2.3.1. L-R fuzzy number

A fuzzy number $\tilde{A}$ is called an L-R fuzzy number if its membership function $\mu_{\tilde{A}} : R \to [0,1]$ has the following form

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\dfrac{x-a}{\alpha}\right), & (a-\alpha) \le x < a,\ \alpha > 0, \\ 1 & , a \le x \le b, \\ R\left(\dfrac{x-b}{\beta}\right), & b < x \le (b+\beta),\ \beta > 0, \\ 0 & , otherwise \end{cases} \tag{1.23}$$

where L and R are piecewise continuous functions, L is increasing, R is decreasing and $L(x) = L(-x)$, $R(x) = R(-x)$, $L(0)= R(0)=1$. The L-R fuzzy number $\tilde{A}$ can be written as $\tilde{A} = (a, b, \alpha, \beta)_{LR}$. L and R are called as the left and right reference functions, a and b are starting and end points of the flat interval, α is the left spread and β is the right spread (Bector and Chandra, 2005).

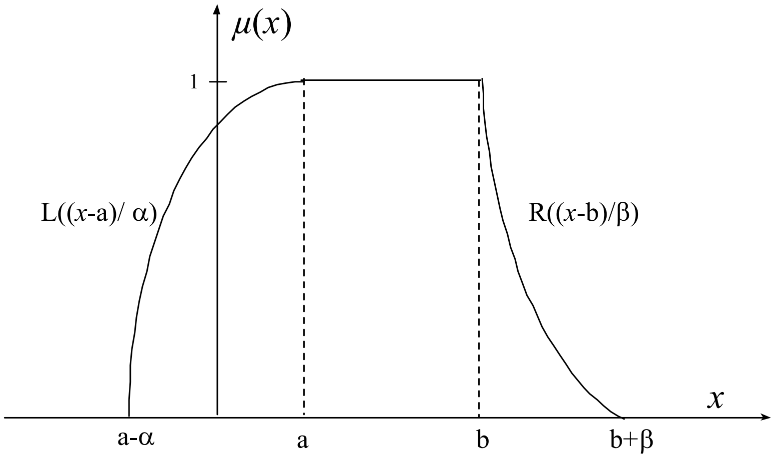


Figure 1.3. L-R fuzzy number (Bector and Chandra, 2005)

If $\tilde{A} = (a_1, b_1, \alpha, \beta)_{LR}$ and $\tilde{B} = (a_2, b_2, \gamma, \delta)_{LR}$ are two L-R fuzzy numbers the arithmetic operations of these two fuzzy numbers can be written as follows.

$$\tilde{A}(+)\tilde{B} = (a_1 + a_2, b_1 + b_2, \alpha + \gamma, \beta + \delta)_{LR} \tag{1.24}$$

$$-\tilde{A} = -(a_1, b_1, \alpha, \beta) = (-b_1, -a_1, \beta, \alpha)_{RL} \tag{1.25}$$

For defining $\tilde{A}(-)\tilde{B}$ the original fuzzy number $\tilde{B}$ should be a *R-L* fuzzy number so that $-\tilde{B}$ becomes a *L-R* fuzzy number and $\tilde{A}(-)\tilde{B}$ can be computed.

$$\begin{aligned}\tilde{A}(-)\tilde{B} &= (a_1, b_1, \alpha, \beta)_{LR}(+)(-(a_2, b_2, \gamma, \delta)_{RL}) \\ &= (a_1 - b_2, b_1 - a_2, \alpha + \delta, \beta + \gamma)_{LR}\end{aligned} \tag{1.26}$$

$\tilde{A}(\cdot)\tilde{B}$ and $\tilde{A}(\div)\tilde{B}$ are not L-R fuzzy numbers in general and will need certain L-R approximations if they are to be used as approximate L-R fuzzy numbers (Bector and Chandra, 2005). Using in this way will add more uncertainty. But, the question of tradeoff between accuracy and simplicity is no easy question to answer. The user must choose using the extension principle or using the special fuzzy numbers and approximation formulas. $\tilde{A}(\cdot)\tilde{B}$ and $\tilde{A}(\div)\tilde{B}$ can be written as follows (Chen and Hwang, 1992).

$$\tilde{A}(\cdot)\tilde{B} = (a_1a_2, b_1b_2, a_1\gamma + a_2\alpha - \alpha\gamma, b_1\delta + b_2\beta + \beta\delta) \;\; if \;\; \tilde{A} > 0, \tilde{B} > 0 \tag{1.27}$$

$$\tilde{A}(\cdot)\tilde{B} = (a_1b_2, b_1a_2, b_2\alpha - a_1\delta + \alpha\delta, -b_1\gamma + a_2\beta - \beta\gamma) \;\; if \;\; \tilde{A} < 0, \tilde{B} > 0 \tag{1.28}$$

$$\tilde{A}(\cdot)\tilde{B} = (b_1 b_2, a_1 a_2, -b_1\delta - b_2\beta - \beta\delta, -a_1\gamma - a_2\alpha + \alpha\gamma) \ \ if \ \ \tilde{A} < 0, \tilde{B} < 0 \tag{1.29}$$

$$\tilde{A}(\div)\tilde{B} = (\frac{a_1}{b_2}, \frac{b_1}{a_2}, \frac{a_1\delta + b_2\alpha}{b_2(b_2+\delta)}, \frac{b_1\gamma + a_2\beta}{a_2(a_2-\gamma)}) \ \ if \ \ \tilde{A} > 0, \tilde{B} > 0 \tag{1.30}$$

$$\tilde{A}(\div)\tilde{B} = (\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{a_2\alpha - a_1\gamma}{a_2(a_2-\gamma)}, \frac{b_2\beta - b_1\delta}{b_2(b_2+\delta)}) \ \ if \ \ \tilde{A} < 0, \tilde{B} > 0 \tag{1.31}$$

$$\tilde{A}(\div)\tilde{B} = (\frac{b_1}{a_2}, \frac{a_1}{b_2}, \frac{-b_1\gamma - a_2\beta}{a_2(a_2-\gamma)}, \frac{-a_1\delta - b_2\alpha}{b_2(b_2+\delta)}) \ \ if \ \ \tilde{A} < 0, \tilde{B} < 0 \tag{1.32}$$

1.3.2.3.2. Triangular fuzzy number

A fuzzy number $\tilde{A}$ is called a triangular fuzzy number if its membership function $\mu_{\tilde{A}}$ is defined as follows.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a_l, x > a_u \\ \dfrac{x - a_l}{a - a_l}, & a_l \leq x \leq a_u \\ \dfrac{a_u - x}{a_u - a}, & a \leq x \leq a_u \end{cases} \tag{1.33}$$

Triangular fuzzy number is a more restricted form than L-R fuzzy number, because left and right reference functions must be linear. $\tilde{A}$ is denoted by the triplet $\tilde{A} = (a_l, a, a_u)$ with a_l and a_u are the lower and upper bounds of $\tilde{A}$. The triangular fuzzy number $\tilde{A}$ has the shape of a triangle.

The α-cut of the triangular fuzzy number $\tilde{A} = (a_l, a, a_u)$ is the closed interval

$$A_\alpha = [a_\alpha^L, a_\alpha^R] = [(a - a_l)\alpha + a_l, -(a_u - a)\alpha + a_u], \quad \alpha \in [0,1]. \tag{1.34}$$

If $\tilde{A} = (a_l, a, a_u)$ and $\tilde{B} = (b_l, b, b_u)$ are two triangular fuzzy numbers, $\tilde{A} * \tilde{B}$ where * denotes any of the four basic arithmetic operations can be computed using the

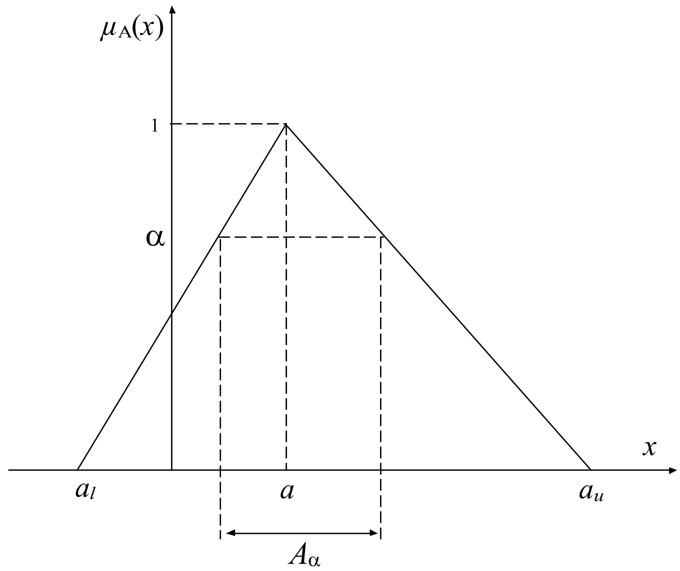


Figure1.4. Triangular fuzzy number

α-cuts, A_α and B_α for $\alpha \in$ [0, 1]. Arithmetic operations of fuzzy numbers $\widetilde{A} = (a_l, a, a_u)$ and $\widetilde{B} = (b_l, b, b_u)$ can be written as follows (Bector and Chandra, 2005).

$$\widetilde{A}(+)\widetilde{B} = (a_l + b_l, a + b, a_u + b_u) \quad (1.35)$$

$$\widetilde{A}(-)\widetilde{B} = (a_l - b_u, a - b, a_u - b_l) \quad (1.36)$$

$$k\widetilde{A} = (ka_l, ka, ka_u),\ k > 0 \quad (1.37)$$

$\widetilde{A}(\cdot)\widetilde{B}$ and $\widetilde{A}(\div)\widetilde{B}$ will not be triangular fuzzy numbers but triangular shaped fuzzy numbers (Buckley et al., 2002). $\widetilde{A}(\cdot)\widetilde{B}$ and $\widetilde{A}(\div)\widetilde{B}$ can be written as follows (Chen and Hwang, 1992).

$$\widetilde{A}(\cdot)\widetilde{B} = (a_l b_l, ab, a_u b_u) \quad if\ \widetilde{A} > 0, \widetilde{B} > 0 \quad (1.38)$$

$$\widetilde{A}(\cdot)\widetilde{B} = (a_l b_u, ab, a_u b_l) \quad if\ \widetilde{A} < 0, \widetilde{B} > 0 \quad (1.39)$$

$$\widetilde{A}(\cdot)\widetilde{B} = (a_u b_u, ab, a_l b_l) \quad if\ \widetilde{A} < 0, \widetilde{B} < 0 \quad (1.40)$$

$$\widetilde{A}(\div)\widetilde{B} = (\frac{a_l}{b_u}, \frac{a}{b}, \frac{a_u}{b_l}) \quad if\ \widetilde{A} > 0, \widetilde{B} > 0 \quad (1.41)$$

$$\widetilde{A}(\div)\widetilde{B} = (\frac{a_u}{b_u}, \frac{a}{b}, \frac{a_l}{b_l}) \quad if \ \widetilde{A} < 0, \widetilde{B} > 0 \tag{1.42}$$

$$\widetilde{A}(\div)\widetilde{B} = (\frac{a_u}{b_l}, \frac{a}{b}, \frac{a_l}{b_u}) \quad if \ \widetilde{A} < 0, \widetilde{B} < 0 \tag{1.43}$$

1.3.2.3.3. Trapezoidal fuzzy number

A fuzzy number $\widetilde{A}$ is called a trapezoidal fuzzy number if its membership function $\mu_{\widetilde{A}}$ is defined as follows (Bector and Chandra, 2005).

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & x < a_l, x > a_u \\ \dfrac{x - a_l}{\underline{a} - a_l}, & a_l \le x \le \underline{a} \\ 1, & \underline{a} \le x \le \overline{a} \\ \dfrac{a_u - x}{a_u - \overline{a}}, & \overline{a} < x \le a_u \end{cases} \tag{1.44}$$

In triangular fuzzy number there is one peak, in trapezoidal fuzzy number there are multiple peaks. The trapezoidal fuzzy number $\widetilde{A}$ is denoted by the quadruplet $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ with a_l and a_u are the lower and upper bounds, and the $[\underline{a}, \overline{a}]$ interval is the most likely values for $\widetilde{A}$. The trapezoidal fuzzy number $\widetilde{A}$ has the shape of a trapezoid.

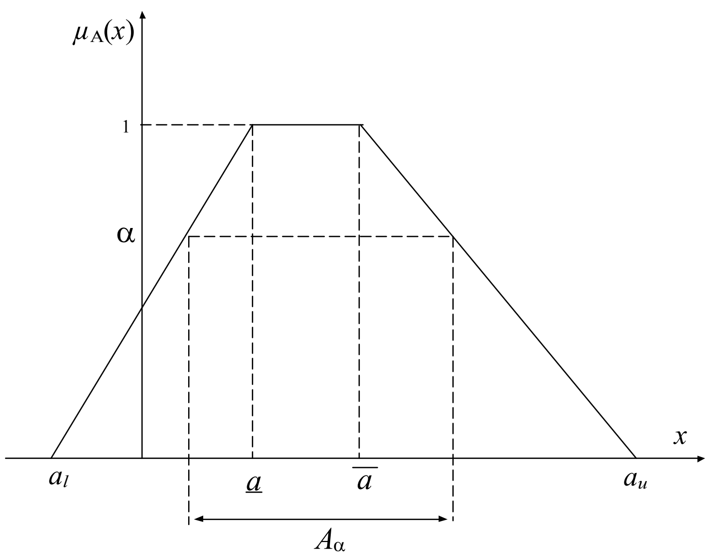


Figure 1.5. Trapezoidal fuzzy number

The α-cut of the trapezoidal fuzzy number $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ is the closed interval

$$A_\alpha = [a_\alpha^L, a_\alpha^R] = [(\underline{a} - a_l)\alpha + a_l, -(a_u - \overline{a})\alpha + a_u], \quad \alpha \in [0,1]. \tag{1.45}$$

If $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ and $\widetilde{B} = (b_l, \underline{b}, \overline{b}, b_u)$ are two trapezoidal fuzzy numbers, $\widetilde{A} * \widetilde{B}$ where * denotes any of the four basic arithmetic operations can be computed using the α-cuts, A_α and B_α for $\alpha \in [0, 1]$. Arithmetic operations of fuzzy numbers $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ and $\widetilde{B} = (b_l, \underline{b}, \overline{b}, b_u)$ can be written as follows (Bector and Chandra, 2005).

$$\widetilde{A}(+)\widetilde{B} = (a_l + b_l, \underline{a} + \underline{b}, \overline{a} + \overline{b}, a_u + b_u) \tag{1.46}$$

$$\widetilde{A}(-)\widetilde{B} = (a_l - b_u, \underline{a} - \overline{b}, \overline{a} - \underline{b}, a_u - b_l) \tag{1.47}$$

$$k\widetilde{A} = (ka_l, k\underline{a}, k\overline{a}, ka_u), \ k > 0 \tag{1.48}$$

$\widetilde{A}(\cdot)\widetilde{B}$ and $\widetilde{A}(\div)\widetilde{B}$ will not be trapezoidal fuzzy numbers but trapezoidal shaped fuzzy numbers (Buckley et al., 2002). $\widetilde{A}(\cdot)\widetilde{B}$ and $\widetilde{A}(\div)\widetilde{B}$ can be written as follows (Chen and Hwang, 1992).

$$\widetilde{A}(\cdot)\widetilde{B} = (a_l b_l, \underline{ab}, \overline{a}\overline{b}, a_u b_u) \quad if \ \widetilde{A} > 0, \widetilde{B} > 0 \tag{1.49}$$
$$\widetilde{A}(\cdot)\widetilde{B} = (a_l b_u, \underline{a}\overline{b}, \overline{a}\underline{b}, a_u b_l) \quad if \ \widetilde{A} < 0, \widetilde{B} > 0 \tag{1.50}$$

$$\widetilde{A}(\cdot)\widetilde{B} = (a_u b_u, \overline{a}\overline{b}, \underline{ab}, a_l b_l) \quad if \ \widetilde{A} < 0, \widetilde{B} < 0 \tag{1.51}$$

$$\widetilde{A}(\div)\widetilde{B} = (\frac{a_l}{b_u}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{a_u}{b_l}) \quad if \ \widetilde{A} > 0, \widetilde{B} > 0 \tag{1.52}$$

$$\widetilde{A}(\div)\widetilde{B} = (\frac{a_u}{b_u}, \frac{\overline{a}}{\overline{b}}, \frac{\underline{a}}{\underline{b}}, \frac{a_l}{b_l}) \quad if \ \widetilde{A} < 0, \widetilde{B} > 0 \tag{1.53}$$

$$\widetilde{A}(\div)\widetilde{B} = (\frac{a_u}{b_l}, \frac{\overline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{a_l}{b_u}) \quad if \ \widetilde{A} < 0, \widetilde{B} < 0 \tag{1.54}$$

1.4. Fuzzy Mathematical Programming

Fuzzy mathematical programming is suggested to solve problems which could be formulated as mathematical programming models, the parameters of which are fuzzy rather than crisp numbers (Zimmermann, 1983).

Most of the real life problems and models contain linguistic and/or imprecise variables and constraints. This can be due to different causes; usually, decision makers can state parameters on a system in terms of linguistic variables more easily and properly. Generally, collecting precise data is very hard, because the environment of the system is unstable or collecting precise data requires high information costs. In addition, decision maker might not be able to express his/her goals or constraints precisely but rather in a fuzzy sense.

The mentioned impreciseness in a system does not exist because of randomness but rather because of fuzziness. In the past, to deal with imprecision the concepts and techniques of probability theory are used, so, it is accepted as if the imprecision is equal to randomness. Fuzziness is the major source of imprecision in many decision processes. The type of imprecision, fuzziness is associated with fuzzy set in which there is no sharp boundaries from membership to nonmembership. In fuzziness, there are grades of membership intermediate between full membership and nonmembership. But, in randomness, there is uncertainty concerning membership or nonmembership of an element in a nonfuzzy set. The evidence associated with whether or not a particular element belongs to the nonfuzzy set is incomplete or hard to obtain (Bellman and Zadeh, 1970, Lodwick and Jamison, 2007).

Fuzzy set theory gives an opportunity to handle linguistic terms and vagueness in real life systems. For modeling systems which are imprecise by nature or which can not be defined precisely, fuzzy mathematical programming that is based on fuzzy set theory is generally used.

In a mathematical programming problem, the fuzziness may appear in many different ways; the aspiration values of the objective(s), the limit values of resources (the right hand value of the constraints), the coefficients of the objective(s) and the coefficients of the constraints can be stated as fuzzy numbers.

In the literature, first, only the objectives (goal(s)) and the right hand value of constraint are defined as fuzzy. Bellman and Zadeh considered that in a fuzzy environment there is no distinction between objectives and constraints, as well as no difference between single and multiple objectives (symmetric approach). Following this symmetric fuzzy decision concept Zimmermann developed the first approach for solving fuzzy linear programming problems.

Like conventional decision making, in fuzzy systems, the optimal decision is the selection of the activities which simultaneously satisfy objective function(s) and constraints. The fuzzy objective functions and the constraints are defined by their membership functions. The space of the solutions will be a fuzzy set and defined by its membership function. The logical '*and*' corresponds to the 'set-theoretic intersection' in fuzzy environments. The fuzzy decision is the intersection of fuzzy constraints and fuzzy objective function(s). The relationship between constraints and objective function(s) in a fuzzy environment is fully symmetric; there is no difference between the constraints and objective(s) (Zimmermann, 1976, 1983).

The membership function of the fuzzy decision set $\mu_D(x)$ is defined as follows;

$$\mu_D(x) = \mu_0(x) \Lambda \mu_C(x) = \min\{\mu_0(x), \mu_C(x)\} \tag{1.55}$$

where $\mu_O(x)$ are the membership functions of the fuzzy goals and $\mu_C(x)$ are the membership functions of the fuzzy constraints.

For finding the optimal fuzzy decision, the decision that is preferable to the others should be found. The decision x_{opt} is defined as follows by Zimmermann (1976);

$$\mu_D(x_{opt}) = \max_x \mu_D(x)\,. \tag{1.56}$$

The fuzzy decision process is graphically shown in Figure 1.6 (Zimmermann, 1976).

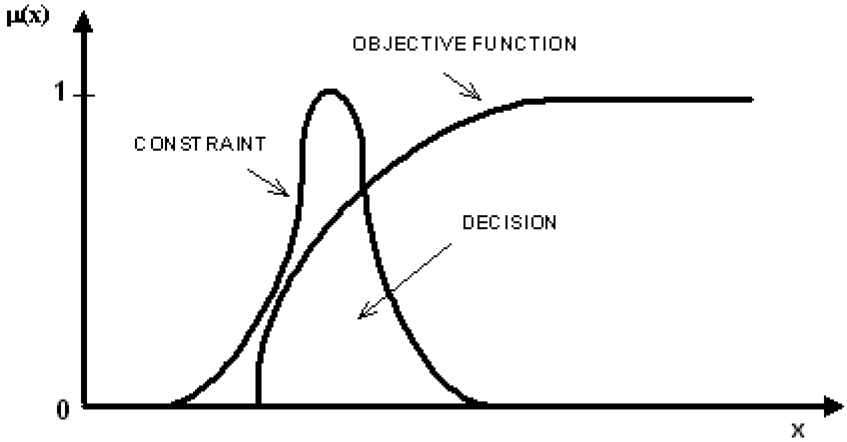


Figure 1.6. Fuzzy decision

The fuzzy decision that is defined by Zimmermann (1976) is the origin of the fuzzy optimization, which is constituted on the symmetrical approach. But, if the other parameters of the model (objective coefficients, coefficients of constraints) are also fuzzy, the symmetrical approach can not be used easily. Therefore, for fuzzy mathematical programming models with various fuzzy parameters, different optimization algorithms are proposed. Ranking of fuzzy numbers can be used when the coefficients of constraints or objective coefficients are defined as fuzzy numbers.

A fuzzy mathematical programming problem in which all of the parameters are fuzzy (objective function coefficients, coefficients of constraints and requirements or resources) can be stated as follows;

$$\begin{aligned} &\max / \min \quad f(x_j, \tilde{c}_j) \\ &s.t. \qquad g(x_j, \tilde{a}_{ij}) \{\leq, \geq, =\} \tilde{b}_i \quad i = 1,...,m, \;\; j = 1,...,n \end{aligned} \tag{1.57}$$

where x_j are the decision variables, $\tilde{c}_j$ are the fuzzy objective function coefficients, $\tilde{a}_{ij}$ are the fuzzy coefficients of the constraints and $\tilde{b}_i$ are right hand values of the constraints (limit values of resources, requirements etc.). The functions $f(x_j, \tilde{c}_j)$ and $g(x_j, \tilde{a}_{ij})$ can be linear or nonlinear functions.

For solving fuzzy mathematical programming models various approaches have been proposed. The types of fuzzy mathematical programming models and the solution approaches will be discussed in the chapter 3. The time needed for solving the fuzzy model depends on the assumptions made. When the membership functions are assumed to be linear the least effort is needed, so, generally, the fuzzy parameters in the mathematical programming models are defined as triangular or trapezoidal fuzzy numbers.

1.5. Objectives of the Research

The fuzzy mathematical programming models can reflect the real life systems more realistically. Therefore, there are many studies on fuzzy mathematical programming and various solution approaches are proposed for different fuzzy models, in the literature. In this research, fuzzy mathematical programming models are reviewed and classified according to the fuzzy components. The proposed solution approaches for fuzzy mathematical programming models are investigated. It has been reported that nearly in all proposed solution approaches, the fuzzy model is firstly transformed into its crisp equivalent and then solved with an appropriate method. One of the objectives of this study is to solve fuzzy mathematical programming problems directly.

The existing methods for the solution of fuzzy mathematical programming problems can be divided into two groups, depending on the fuzziness of decision variables. In the first group, it is assumed that the parameters of the problem are fuzzy numbers while the decision variables are crisp ones. In the second group, the decision variables are assumed as fuzzy numbers. Finding fuzzy solutions instead of crisp solutions in an uncertain environment that provide ranges of flexibility look more appropriate. In the literature, there are few examples on fuzzy mathematical programming problems with fuzzy decision variables. The other objective of this study is to solve fully fuzzy mathematical programming problems (in which all parameters and decision variables are defined as fuzzy numbers) directly.

This study mainly concentrates on the following themes;

- review and classification of fuzzy mathematical programming models,
- solution of fuzzy mathematical programming problems directly using the ranking methods for fuzzy numbers and metaheuristic algorithms,

- solution of fully fuzzy mathematical programming problems directly using the ranking methods for fuzzy numbers and metaheuristic algorithms,
- for presenting the solution process and the effectiveness of the proposed direct solution method solving different fuzzy mathematical programming problems as examples.

1.6. Methodology and Materials

In this study, firstly the literature on fuzzy mathematical programming is reviewed and a new classification is presented according to the fuzzy components. This review has revealed that in all proposed solution approaches the fuzzy mathematical programming model is transformed to its crisp equivalent and solved by conventional methods. Besides, in the literature, there are few examples on fuzzy mathematical programming problems with fuzzy decision variables. However, it is more suitable to find fuzzy solutions instead of crisp solutions in an uncertain environment. In this study, to solve fuzzy mathematical programming problems and fully fuzzy mathematical programming problems directly, use of ranking methods for fuzzy numbers and metaheuristic algorithms is proposed.

For presenting the effectiveness of the proposed direct solution method, a fuzzy peak load pricing problem, a fuzzy product mix problem, a fuzzy multi-item economic order quantity problem, a fuzzy multi-objective aggregate production planning problem and a fully fuzzy product mix problem are solved. In the proposed direct solution method, a ranking method for fuzzy numbers and a metaheuristic algorithm should be used. In this thesis, different ranking methods and the particle swarm optimization algorithm and the tabu search algorithm are used. For each example, computer programs in C language are prepared.

1.7. Organization of the Thesis

The brief contents of the following chapters in this thesis can be stated as follows:

- In Chapter 2, some existing methods for ranking fuzzy numbers in the literature are briefly explained.

- In Chapter 3, a new classification of fuzzy mathematical programming models according to its components is presented and a review is given.
- In Chapter 4, the proposed direct solution method is explained. The used metaheuristic algorithms are described briefly. Then, two small examples are solved using the proposed direct solution method with different fuzzy components.
- In Chapter 5, a fuzzy multi-item economic order quantity problem and a fuzzy multi-objective aggregate production planning problem is solved using the proposed direct solution method.
- In Chapter 6, solution of fully fuzzy mathematical programming problems using the proposed direct solution method is presented. A fully fuzzy product mix problem is solved.
- In Chapter 7, a brief review of this study and the conclusions drawn from the study are presented.

CHAPTER 2

RANKING OF FUZZY NUMBERS

2.1. Introduction

In this chapter, ranking of fuzzy numbers is reviewed. Some of the ranking methods exist in the literature and the selected ranking methods are briefly explained.

2.2. Ranking Fuzzy Numbers

Ranking of fuzzy numbers is one of very important area in fuzzy set theory. In decision making if there are fuzzy parameters, operations on fuzzy numbers and ranking are needed to be used. So, in the literature, ranking of fuzzy numbers have been studied widely.

Ranking fuzzy numbers is not a simple process. Unlike real numbers, fuzzy numbers have no natural order (Wang and Kerre, 2001). Fuzzy numbers are not in linear order; they are usually in partial order. In ranking fuzzy numbers, it is not always possible to obtain a totally ordered set. Frequently, overlap of membership functions or small differences in the support of fuzzy numbers make comparison of fuzzy numbers a very difficult task (Chang and Lee, 1994).

Since the study of fuzzy ranking began, various ranking methods that yield a totally ordered set have been proposed. However, there is no best method agreed. All the proposed ranking methods have advantages as well as disadvantages. The proposed ranking methods are based on extracting various features from fuzzy sets (numbers). The ranking method orders the fuzzy sets based on a specific feature. Because different ranking methods order fuzzy sets according to different features, normally the obtained ranking order for the same sample of fuzzy sets (numbers) can be different (Prodanovic, 2001).

There are various ranking methods for fuzzy numbers in the literature, and these methods are classified in different manners by authors. Delgado et al. (1988) have classified ranking methods as belonging to two different approaches.

- Ranking fuzzy numbers using crisp relations. In this case, the ranking methods use a ranking function and a crisp total order relation between fuzzy numbers is obtained. Fuzzy numbers are then ordered by ranking crisp numbers.
- Giving a comparison index for each pair of fuzzy numbers. In this case, a fuzzy relation is obtained. The ordered results will be in the form like that 'fuzzy number $\widetilde{A}$ is slightly better than fuzzy number $\widetilde{B}$'.

Chen and Hwang (1992) classified ranking methods into four categories.

- Preference relation is utilized for ranking. Degree of optimality, hamming distance, α-cut and comparison function are the techniques involved in this category.
- Fuzzy mean and spread. In this category probability distributions are applied.
- Fuzzy scoring is used for ranking. Proportion to optimal, left/right scores, centroid index and area measurement are the techniques involved in this category.
- Linguistic expression. This category includes intuition and linguistic approximation.

Chang and Lee (1994) classified ranking methods as follows.

- Methods using an α-cut. Fuzzy numbers are ranked by comparing their α-cuts.
- Methods using the possibility concept. The possibility or necessity concepts are used to compare fuzzy numbers.
- Methods with integration. In this case, fuzzy ranking method measures a fuzzy number by its mean value.
- Methods using multiple indices. The results of multiple ranking functions are used as references to compare fuzzy numbers.

- Linguistic approach.

In the literature, as specified before, there are various ranking methods for fuzzy numbers. In the selection of the ranking method, the shape of the fuzzy numbers, the obtained relation from the ranking method (crisp or fuzzy) and the ease of computation of the ranking method can be the major factors to be considered. As stated before, there is no best method agreed.

2.3. A Review of Some Existing Methods for Ranking Fuzzy Numbers

Some of the ranking methods exists in the literature will be briefly explained in this section.

2.3.1. The preference order of fuzzy numbers

Chen and Lu (2002) have defined a signal/noise ratio to evaluate quality of a fuzzy number. The defined ratio considers the middle point and spread of each α-cut of fuzzy numbers as the signal and noise. A fuzzy number with the stronger signal and the weaker noise is considered better.

In the study of Chen and Lu, a few α-cuts of a fuzzy number is used to measure quantity, because a α-cut can signify the fuzzy number's location on the x-axis at the specified α level.

In the ranking method, a specified number of α-cut is used; for example k α-cuts. For the k^{th} α-cut of fuzzy number $\widetilde{A}_i$, the quantity of $\widetilde{A}_i$ on the x-axis can be specified as

$$\Delta_{i,k}(\beta) = \beta r_{i,k} + (1-\beta) l_{i,k} \tag{2.1}$$

where $l_{i,k}$ is the minimum value of the k^{th} α-cut of fuzzy number $\widetilde{A}_i$, $r_{i,k}$ is the maximum value of the k^{th} α-cut of fuzzy number $\widetilde{A}_i$ and $\beta(\in [0,1])$ is the index of optimism that reflects a decision maker's degree of optimism. The large index of optimism infers that the decision maker is more optimistic. When β is taken as 1, only the maximum value of the cut is considered. The small index of optimism infers that the decision maker is more pessimistic.

The middle point of the k^{th} α-cut of the fuzzy number $\widetilde{A}_i$ (${}^{\alpha_k}\widetilde{A}_i$) is defined as

$$m_{i,k} = \frac{(r_{i,k} + l_{i,k})}{2}. \tag{2.2}$$

The spread of ${}^{\alpha_k}\widetilde{A}_i$ is defined as

$$\delta_{i,k} = r_{i,k} - l_{i,k}. \tag{2.3}$$

The α-cuts of a fuzzy number can be thought as crisp intervals and the values in the crisp interval have been statistically considered to be a uniform distribution by Chen and Lu. Following this, the middle point and spread of each α-cut of fuzzy numbers are treated as the mean and approximate standard deviation of the uniform distribution.

The signal/noise (S/N) ratio for the ${}^{\alpha_k}\widetilde{A}_i$ is defined as

$$\eta_{i,k} = m_{i,k} / \delta_{i,k} \tag{2.4}$$

where $\eta_{i,k} \in [0, \infty)$.

The value of S/N ratio is infinite when the spread is equal to zero. So, S/N ratio is adjusted as

$$\eta_{i,k}^{a} = 1 - \frac{1}{1 + \eta_{i,k}}. \tag{2.5}$$

The comparison index for two fuzzy numbers $\widetilde{A}_i$ and $\widetilde{A}_j$ is defined as

$$R_{i,j}(\beta) = \frac{\sum_{k=1}^{n} \alpha_k \cdot [\Delta_{i,k}(\beta) - \Delta_{j,k}(\beta)] \cdot \eta_{i,k}^{a} / \eta_{j,k}^{a}}{\sum_{k=1}^{n} \alpha_k} \tag{2.6}$$

where $\alpha_k = k/n$, $0 \le k \le n$, $k, n \in N$ and n is the number of α-cuts. The ratio $\eta_{i,k}^{a} / \eta_{j,k}^{a}$ at each α level is included to measure the relative quality of α-cuts between two fuzzy numbers. The α levels are used as weights for strengthening the influence of α-cut with higher α levels.

For two fuzzy numbers $\widetilde{A}_i$ and $\widetilde{A}_j$, the preference order is determined as follows.

If $R_{i,j}(\beta) > 0$ then $\widetilde{A}_i > \widetilde{A}_j$,

if $R_{i,j}(\beta) = 0$ then $\widetilde{A}_i = \widetilde{A}_j$,

if $R_{i,j}(\beta) < 0$ then $\widetilde{A}_i < \widetilde{A}_j$.

In this ranking method, the number of α-cuts is not significant to final order. The degree of optimism β has close correlation with the order. For different β values different orderings are obtained. The fuzzy numbers with the larger mean and the smaller spread are considered better.

2.3.2. Ranking fuzzy numbers according to centroids

Cheng (1998) has proposed using a centroid based distance for ranking fuzzy numbers in his study. Chu and Tsao (2002) have proposed using the area between the centroid point and the origin to rank fuzzy numbers, because the method of Cheng cannot give true orders if the fuzzy numbers are negative. Wang et al. (2006) have found that the centroid formula of Cheng is incorrect and have given the correct formula.

For a fuzzy number $\widetilde{A}$, suppose that the membership function is defined as follows.

$$\mu_{\widetilde{A}}(x) = \begin{cases} f_{\widetilde{A}}^{L}(x), & a \le x \le b \\ w, & b \le x \le c \\ f_{\widetilde{A}}^{R}, & c \le x \le d \\ 0, & otherwise \end{cases} \tag{2.7}$$

The inverse functions of $f_{\widetilde{A}}^{L}$ and $f_{\widetilde{A}}^{R}$ are $g_{\widetilde{A}}^{L}$ and $g_{\widetilde{A}}^{R}$.

The centroid point of a fuzzy number corresponds to an $\bar{x}$ value on the horizontal axis and a $\bar{y}$ value on the vertical axis (Chu and Tsao, 2002). The formula of centroid of $\widetilde{A}$ is defined as

$$\bar{x}_0(\widetilde{A}) = \frac{\int_{-\infty}^{\infty} x f_{\widetilde{A}}(x)dx}{\int_{-\infty}^{\infty} f_{\widetilde{A}}(x)dx} = \frac{\int_a^b x f_{\widetilde{A}}^{L}(x)dx + \int_b^c (xw)dx + \int_c^d x f_{\widetilde{A}}^{R}(x)dx}{\int_a^b f_{\widetilde{A}}^{L}(x)dx + \int_b^c w dx + \int_c^d f_{\widetilde{A}}^{R}(x)dx} \tag{2.8}$$

$$\bar{y}_0(\widetilde{A}) = \frac{\int_0^w y(g_{\widetilde{A}}^{R}(y) - g_{\widetilde{A}}^{L}(y))dy}{\int_0^w (g_{\widetilde{A}}^{R}(y) - g_{\widetilde{A}}^{L}(y))dy} \tag{2.9}$$

where the denominator $\int_0^w (g_{\widetilde{A}}^{R}(y) - g_{\widetilde{A}}^{L}(y))dy$ represents the area under the inverse function of trapezoidal fuzzy number, while the numerator $\int_0^w y(g_{\widetilde{A}}^{R}(y) - g_{\widetilde{A}}^{L}(y))dy$ is the weighted average of the area (Wang et al., 2006).

For a general trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d; w)$ the centroids can be written as follows (Wang et al., 2006).

$$\bar{x}_0(\widetilde{A}) = \frac{1}{3}\left[a + b + c + d - \frac{dc - ab}{(d + c) - (a + b)}\right] \tag{2.10}$$

$$\bar{y}_0(\widetilde{A}) = w\frac{1}{3}\left[1 + \frac{c - b}{(d + c) - (a + b)}\right] \tag{2.11}$$

The triangular fuzzy numbers are special forms of trapezoidal fuzzy numbers with $b = c$. for the triangular fuzzy number $\widetilde{A} = (a, b, d; w)$ with a piecewise linear membership function the centroids are written as follows (Wang et al., 2006).

$$\bar{x}_0(\widetilde{A}) = \frac{1}{3}(a + b + d) \tag{2.12}$$

$$\bar{y}_0(\widetilde{A}) = \frac{1}{3}w. \tag{2.13}$$

For normal triangular fuzzy numbers it will be $\bar{y}_0(\widetilde{A}) = 1/3$.

The area between the centroid point $(\bar{x}_0, \bar{y}_0)$ and the original point (0, 0) of the fuzzy number $\widetilde{A}$ is defined as (Chu and Tsao, 2002),

$$S(\widetilde{A}) = \bar{x}_0(\widetilde{A}) \cdot \bar{y}_0(\widetilde{A}) \tag{2.14}$$

The area *S* is used to rank fuzzy numbers. The bigger fuzzy number will be the fuzzy number with the bigger value of *S*. For any two fuzzy numbers $\widetilde{A}_i$ and $\widetilde{A}_j$,

if $S(\widetilde{A}_i) > S(\widetilde{A}_j)$ then $\widetilde{A}_i > \widetilde{A}_j$,

if $S(\widetilde{A}_i) = S(\widetilde{A}_j)$ then $\widetilde{A}_i = \widetilde{A}_j$,

if $S(\widetilde{A}_i) < S(\widetilde{A}_j)$ then $\widetilde{A}_i < \widetilde{A}_j$.

According to definition of the S value, normal triangular fuzzy numbers can be compared or ranked directly in terms of their centroid coordinates on horizontal axis, since the centroid coordinates on vertical axis are same for all normal triangular fuzzy numbers (Wang et al., 2006).

2.3.3. Ranking fuzzy numbers according to mean value and variance

Hashemi et al. (2006) have proposed a solution method for fully fuzzified linear programming problems in their study. For using in the solution approach, they have proposed a ranking method for fuzzy numbers based on comparison of mean and standard deviation of fuzzy numbers.

In the study, it is accepted that a fuzzy number $\widetilde{A}$ is a convex normalized fuzzy subset of the real line R with bounded support. The set of all fuzzy numbers is denoted by $F(R)$.

For using in the ranking method the following definition is given in the study of Hashemi et al. (2006).

A nonzero vector X is called lexicographically positive and denoted by $X >_{LG} 0$ if the first nonzero component of X is positive. Also, a lexicographically nonnegative vector is either a zero vector or else a lexicographically positive vector, denoted by $X \geq_{LG} 0$.

The possibilistic mean value and variance of the fuzzy number $\widetilde{A}$ denoted by $M(\widetilde{A})$ and $Var(\widetilde{A})$ are defined as follows.

$$M(\widetilde{A}) = \int_0^1 \alpha(\inf \widetilde{A}_\alpha + \sup \widetilde{A}_\alpha) d\alpha \tag{2.15}$$

$$Var(\widetilde{A}) = 1/2 \int_0^1 \alpha(\sup \widetilde{A}_\alpha - \inf \widetilde{A}_\alpha)^2 d\alpha \tag{2.16}$$

The standard deviation of $\widetilde{A}$ is defined by

$$SD(\widetilde{A}) = \sqrt{Var(\widetilde{A})}\,. \tag{2.17}$$

A symmetric fuzzy number $\widetilde{A}$ can be denoted by $\widetilde{A} = (a, \gamma)_L$ and defined as

$$\widetilde{A}(x) = L((x-a)/\gamma), \qquad \gamma \geq 0 \tag{2.18}$$

where a and γ are the center and spread of $\widetilde{A}$, and L is the reference function.

If the fuzzy number $\widetilde{A}$ is a symmetric fuzzy number, then the possibilistic mean value and variance of $\widetilde{A}$ can be defined as;

$$M(\widetilde{A}) = a \tag{2.19}$$

$$Var(\widetilde{A}) = \frac{\gamma^2}{6} \tag{2.20}$$

$$SD(\widetilde{A}) = \frac{\gamma}{\sqrt{6}}\,. \tag{2.21}$$

Hashemi et al. (2006) have proposed a method for ranking fuzzy numbers using the concept of possibilistic mean value and standard deviation of fuzzy numbers.

Consider the following function,

$$R : F(R) \to R \times R$$
$$\widetilde{A} \mapsto \begin{bmatrix} M(\widetilde{A}) \\ -SD(\widetilde{A}) \end{bmatrix}$$

The ordering of two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ is defined as follows.

$$\widetilde{A} \geq \widetilde{B} \Leftrightarrow R(\widetilde{A}) - R(\widetilde{B}) = \begin{bmatrix} M(\widetilde{A}) - M(\widetilde{B}) \\ -SD(\widetilde{A}) + SD(\widetilde{B}) \end{bmatrix} \geq_{LG} 0$$

$$\widetilde{A} > \widetilde{B} \Leftrightarrow R(\widetilde{A}) - R(\widetilde{B}) = \begin{bmatrix} M(\widetilde{A}) - M(\widetilde{B}) \\ -SD(\widetilde{A}) + SD(\widetilde{B}) \end{bmatrix} >_{LG} 0$$

$$\widetilde{A} = \widetilde{B} \Leftrightarrow R(\widetilde{A}) - R(\widetilde{B}) = \begin{bmatrix} M(\widetilde{A}) - M(\widetilde{B}) \\ -SD(\widetilde{A}) + SD(\widetilde{B}) \end{bmatrix} = 0$$

Also, $\widetilde{A} \leq \widetilde{B}$ can be written if and only if $\widetilde{B} \geq \widetilde{A}$. The function R is called lexicographic ranking function.

2.3.4. Ranking method of Dorohonceanu and Marin

In the study of Dorohonceanu and Marin (2002), a ranking method for fuzzy numbers and its extension which are based on the idea of determining the degree of membership that fuzzy relation "$\widetilde{A}$ is greater than $\widetilde{B}$" belongs to the set of "greater" have been proposed.

The ranking method of Dorohonceanu and Marin based on the confidence interval comparison, since fuzzy numbers can be represented as ordered set of confidence intervals. A confidence interval is an interval of real numbers that provides a representation for an imprecise numerical value by means of its sharpest enclosing range (Dorohonceanu and Marin, 2002).

From the linearity of the greater than relationship, using interpolation, for intervals $[a_1, a_2]$ and $[b_1, b_2]$ it is obtained that

$$\gamma_{A>B} = \frac{a_2 - b_1}{b_2 - b_1 + a_2 - a_1}. \tag{2.22}$$

For any two confidence intervals A and B, the following equations are valid (Dorohonceanu and Marin, 2002).

$$\gamma_{A>B} + \gamma_{B>A} = 1$$

$$A \geq B \Leftrightarrow \gamma_{A>B} \geq \gamma_{B>A}$$

$$(A \geq B \, and \, B \geq C) \Rightarrow A \geq C, \quad with \; \gamma_{A>C} \geq \max(\gamma_{A>B}, \gamma_{B>C})$$

$$A \leq B \, and \, B \leq C) \Rightarrow A \leq C, \quad with \; \gamma_{A<C} \geq \max(\gamma_{A<B}, \gamma_{B<C})$$

The fuzzy number comparison method B2 is based on the fuzzy number representation in fuzzy arithmetic. It relies on the representation of fuzzy numbers as ordered set of confidence intervals, each of them providing the related numerical value at a given presumption level $\alpha \in [0,1]$. It is stated by Dorohonceanu and Marin that the shapes of the convex fuzzy numbers do not need special computations during comparisons.

For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, to compute the value of $\tilde{A}$ greater than $\tilde{B}$ the following algorithm is applied.

For each level of presumption, α, the corresponding confidence intervals, A_α and B_α, are compared and the degree that A_α is greater than B_α is computed as $\gamma_{A\alpha > B\alpha}$.

The degree of membership that fuzzy number $\tilde{A}$ greater than fuzzy number $\tilde{B}$ is computed by

$$\gamma_{A>B} = \frac{\sum_{\alpha=0}^{NOPL-1} \gamma_{A_\alpha > B_\alpha} \cdot length(A_\alpha) \cdot length(B_\alpha)}{\sum_{\alpha=0}^{NOPL-1} length(A_\alpha) \cdot length(B_\alpha)} \tag{2.23}$$

where NOPL is the number of presumption levels and length denotes the length of the respective confidence interval.

In their study, Dorohonceanu and Marin have proposed a variant of the B2 method to compare fuzzy numbers. In the variant of the B2 method (B2x), each part of constant monotony of the fuzzy number $\tilde{A}$ is separately compared with the fuzzy number $\tilde{B}$.

In the variant method, the following algorithm is applied. Fuzzy number $\tilde{A}$ is decomposed into two fuzzy intervals, A_1 and A_2. Then, the first interval A_1 is compared with fuzzy number $\tilde{B}$ and the degree that A_1 is greater than $\tilde{B}$ is computed. After that, the second interval A_2 is similarly compared with fuzzy number $\tilde{B}$ and the degree that A_2 is greater than $\tilde{B}$ is computed. The mean of these two resulted values will be considered as the degree that $\tilde{A}$ is greater than $\tilde{B}$.

$$\gamma_{\tilde{A}>\tilde{B}} = \frac{\gamma_{A_1>\tilde{B}} + \gamma_{A_2>\tilde{B}}}{2} \tag{2.24}$$

The generalization of the above procedure is defined as follows.

$$\gamma_{\tilde{A}>\tilde{B}} = \frac{1}{n}\sum_{i=1}^{n} \gamma_{A_i>\tilde{B}} \tag{2.25}$$

where n represents the number of distinctive intervals of absolute monotony of fuzzy numbers $\tilde{A}$ (Dorohonceanu and Marin, 2002).

2.3.5. Ranking fuzzy numbers by distance minimization

In their study, Asady and Zendehnam (2007) have obtained the nearest point with respect to a fuzzy number and by considering the nearest point have proposed a ranking method.

It is defined that, a fuzzy number $\widetilde{A}$ in parametric form is a pair $(\underline{a},\overline{a})$ of function $\underline{a}(r)$ and $\overline{a}(r)$, 0 ≤ r ≤ 1. $\underline{a}(r)$ is a bounded increasing left continuous function and $\overline{a}(r)$ is a bounded decreasing left continuous function, $\underline{a}(r) \le \overline{a}(r),\ \ 0 \le r \le 1$.

For arbitrary fuzzy numbers $\widetilde{A}=(\underline{a},\overline{a})$ and $\widetilde{B}=(\underline{b},\overline{b})$, the distance is defined as follows.

$$D(\widetilde{A},\widetilde{B}) = \left[\int_0^1 (\underline{a}(r)-\underline{b}(r))^2\,dr + \int_0^1 (\overline{a}(r)-\overline{b}(r))^2\,dr \right]^{1/2} \tag{2.26}$$

Interval of a fuzzy number $\widetilde{A}$ is defined as follows.

$$EI(\widetilde{A}) = \left[\int_0^1 \underline{a}(r), \int_0^1 \overline{a}(r) \right] \tag{2.27}$$

The middle point of interval $EI(\widetilde{A})$ can be defined as follows.

$$M(\widetilde{A}) = 1/2\int_0^1 (\underline{a}(r)+\overline{a}(r))dr \tag{2.28}$$

The interval $EI(\widetilde{A})$ is the nearest interval to the fuzzy number $\widetilde{A}$. It is tried to find a crisp point $C(\widetilde{A})$, which is the nearest to $\widetilde{A}$ with respect to metric *D*. the distance between the fuzzy number $\widetilde{A}$ and a crisp point $C(\widetilde{A})$ is defined as follows (Asady and Zendehnam, 2007).

$$D(\widetilde{A},C(\widetilde{A})) = \left[\int_0^1 (\underline{a}(r)-C(\widetilde{A}))^2\,dr + \int_0^1 (\overline{a}(r)-C(\widetilde{A}))^2\,dr \right]^{1/2} \tag{2.29}$$

It is stated by Asady and Zendehnam that, if $\widetilde{A}$ is a fuzzy number and $C(\widetilde{A})$ is a crisp point then the function $D(\widetilde{A}, C(\widetilde{A}))$ with respect to $C(\widetilde{A})$ is minimum value if $C(\widetilde{A}) = M(\widetilde{A})$ and $M(\widetilde{A})$ is unique.

According to above theorem, if $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ is a trapezoidal fuzzy number, then the nearest point to $\widetilde{A}$ can be defined as follows.

$$M(\widetilde{A}) = 1/2\,(\underline{a} + \overline{a}) + \frac{(a_u - \overline{a}) - (\underline{a} - a_l)}{4} \tag{2.30}$$

If $\widetilde{A} = (a_l, a, a_u)$ is a triangular fuzzy number, then the nearest point will be

$$M(\widetilde{A}) = a + \frac{(a_u - a) - (a - a_l)}{4}\,. \tag{2.31}$$

If $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ is a symmetric trapezoidal fuzzy number, then

$$M(\widetilde{A}) = 1/2\,(\underline{a} + \overline{a})\,. \tag{2.32}$$

If $\widetilde{A} = (a_l, a, a_u)$ is a symmetric triangular fuzzy number, then

$$M(\widetilde{A}) = a\,. \tag{2.33}$$

The ranking of fuzzy numbers associated with the distance minimization have been defined as follows. For two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$,

$M(\widetilde{A}) > M(\widetilde{B})$ if and only if $\widetilde{A} > \widetilde{B}$,

$M(\widetilde{A}) < M(\widetilde{B})$ if and only if $\widetilde{A} < \widetilde{B}$,

$M(\widetilde{A}) = M(\widetilde{B})$ if and only if $\widetilde{A} = \widetilde{B}$.

The order "≥" and "≤" can be formulated as

$\widetilde{A} \geq \widetilde{B}$ if and only if $\widetilde{A} > \widetilde{B}$ *or* $\widetilde{A} = \widetilde{B}$,

$\widetilde{A} \leq \widetilde{B}$ if and only if $\widetilde{A} < \widetilde{B}$ *or* $\widetilde{A} = \widetilde{B}$.

2.3.6. Ranking fuzzy numbers with an area method using radius of gyration

Deng at al. (2006) have proposed a modified area method to rank fuzzy numbers. It is said in the study that the modified method can effectively rank various fuzzy numbers and their images. In the proposed method fuzzy numbers are ranked with the radius of gyration (ROG) points. Radius of gyration is a concept in mechanics.

An area A located in the xy plane and the element of area dA of coordinates x and y are considered. The moment of inertia of the area A with respect to the x axis, and the moment of inertia of the area A with respect to the y axis are defined as follows.

$$I_x = \int_A y^2 dA \tag{2.34}$$

$$I_y = \int_A x^2 dA \tag{2.35}$$

The radius of gyration of an area A with respect to the x axis is defined as the quantity r_x, the radius of gyration of an area A with respect to the y axis is defined as the quantity r_y, that satisfies the relation,

$$I_x = r_x^2 A \tag{2.36}$$

$$I_y = r_y^2 A . \tag{2.37}$$

For an area made up of a number of simple shapes, the moment of inertia of the entire area is the sum of the moments of inertia of each of the individual area about the axis desired.

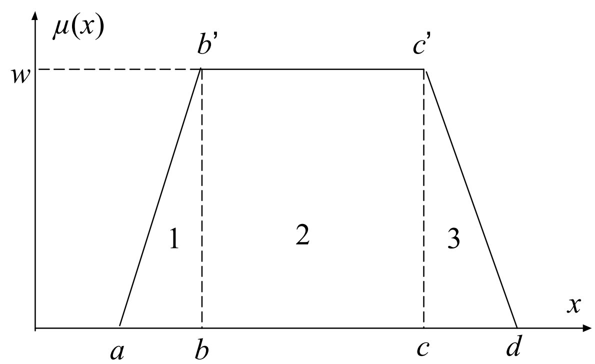


Figure 2.1. A generalized trapezoidal fuzzy number ($a,b,c,d;w$)

The moment of inertia of the generalized trapezoidal fuzzy number in figure 2.1 can be defined as follows.

$$(I_x) = (I_x)_1 + (I_x)_2 + (I_x)_3 \tag{2.38}$$

$$(I_y) = (I_y)_1 + (I_y)_2 + (I_y)_3 \tag{2.39}$$

$$(I_x)_1 = \int_{abb'} y^2 dA = \int_0^w y^2 \frac{(b-a)(w-y)}{w} dy = \frac{(b-a)w^3}{12} \tag{2.40}$$

$$(I_y)_1 = \int_{abb'} x^2 dA = \frac{(b-a)^3 w}{4} + \frac{(b-a)a^2 w}{2} + \frac{2(b-a)^2 aw}{3} \tag{2.41}$$

$$(I_x)_2 = \frac{(c-b)w^3}{3} \tag{2.42}$$

$$(I_y)_2 = \frac{(c-b)^3 w}{3} + (c-b)b^2 w + (c-b)^2 bw \tag{2.43}$$

$$(I_x)_3 = \frac{(d-c)w^3}{12} \tag{2.44}$$

$$(I_y)_3 = \frac{(d-c)^3 w}{12} + \frac{(d-c)c^2 w}{2} + \frac{(d-c)^2 cw}{3} \tag{2.45}$$

The radius of gyration point of a generalized trapezoidal fuzzy number $(a,b,c,d;w)$ can be calculated as follows.

$$r_x = \sqrt{\frac{(I_x)_1 + (I_x)_2 + (I_x)_3}{(((c-b)+(d-a)\cdot w)/2)}} \tag{2.46}$$

$$r_y = \sqrt{\frac{(I_y)_1 + (I_y)_2 + (I_y)_3}{(((c-b)+(d-a)\cdot w)/2)}} \tag{2.47}$$

The area between the radius of gyration point and original points of a fuzzy number will be;

$$S(\widetilde{A}) = r_x^{\widetilde{A}} \cdot r_y^{\widetilde{A}} \tag{2.48}$$

The area *S* is used to rank fuzzy numbers. The bigger fuzzy number will be the fuzzy number with the bigger value of *S*. For any two fuzzy numbers $\widetilde{A}_i$ and $\widetilde{A}_j$ (Deng et al., 2006),

if $S(\widetilde{A}_i) > S(\widetilde{A}_j)$ then $\widetilde{A}_i > \widetilde{A}_j$,

if $S(\widetilde{A}_i) = S(\widetilde{A}_j)$ then $\widetilde{A}_i = \widetilde{A}_j$,

if $S(\widetilde{A}_i) < S(\widetilde{A}_j)$ then $\widetilde{A}_i < \widetilde{A}_j$.

2.3.7. Ranking of fuzzy numbers using possibility programming

Ranking of fuzzy numbers by using the possibility programming approach has been presented by Negi and Lee (1993) and Iskander (2002) stated some comments. Comparing fuzzy numbers using possibility programming approach depends on the type of the fuzzy numbers.

The membership function of a triangular fuzzy number can be written as follows (Iskander, 2002).

$$\mu_{\widetilde{A}}(x;w) = \begin{cases} 0 & , \quad x < a_l \;\; or \;\; x > a_u \\ \dfrac{(x-a_l)w}{a-a_l}, & a_l \le x \le a \\ \dfrac{(a_u-x)w}{a_u-a}, & a \le x \le a_u \end{cases} \tag{2.49}$$

where, *w* is the maximum value of the membership function ($w \in [0,1]$). If the fuzzy number is a normal fuzzy number *w* will be 1.

Ranking of fuzzy numbers is given using the exceedance possibility and the strict exceedance possibility.

If $\widetilde{A}$ and $\widetilde{B}$ are two triangular fuzzy numbers, $\widetilde{A} = (a_l, a, a_u)$ and $\widetilde{B} = (b_l, b, b_u)$, then the possibility that $\widetilde{A} \ge \widetilde{B}$, according to the exceedance possibility, is given by

$$Poss(\widetilde{A} \geq \widetilde{B}) = \begin{cases} w, & a \geq b \\ \dfrac{(a_u - b)w}{(\overline{a} - a) + (b - \underline{b})}, & b \geq a,\ a_u \geq b_l \\ 0, & b_l \geq a_u \end{cases} \tag{2.50}$$

and this possibility, in the case of the strict exceedance possibility can be presented as (Iskander, 2002);

$$Poss(\widetilde{A} > \widetilde{B}) = \begin{cases} w, & a \geq b_u \\ \dfrac{(a_u - b)w}{(a_u - a) + (b_u - b)}, & b_u \geq a,\ a_u \geq b\,. \\ 0, & b \geq a_u \end{cases} \tag{2.51}$$

The membership function of a trapezoidal fuzzy number can be written as follows (Iskander, 2002).

$$\mu_{\widetilde{A}}(x; w) = \begin{cases} 0 \quad , & x < a_l \ \ or \ \ x > a_u \\ \dfrac{(x - a_l)w}{\underline{a} - a_l}, & a_l \leq x \leq \underline{a} \\ w, & \underline{a \leq x \leq \overline{a}} \\ \dfrac{(a_u - x)w}{a_u - \overline{a}}, & \overline{a} \leq x \leq a_u \end{cases} \tag{2.52}$$

If $\widetilde{A}$ and $\widetilde{B}$ are two trapezoidal fuzzy numbers, $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ and $\widetilde{B} = (b_l, \underline{b}, \overline{b}, b_u)$, then the possibility that $\widetilde{A} \geq \widetilde{B}$, according to the exceedance possibility, is given by

$$Poss(\widetilde{A} \geq \widetilde{B}) = \begin{cases} w, & \overline{a} \geq \underline{b} \\ \dfrac{(a_u - b_l)w}{(a_u - \overline{a}) + (\underline{b} - b_l)}, & \underline{b} \geq \overline{a},\ a_u \geq b_l \\ 0, & b_l \geq a_u \end{cases} \tag{2.53}$$

and this possibility, in the case of the strict exceedance possibility can be presented as (Iskander, 2002);

$$Poss(\widetilde{A} > \widetilde{B}) = \begin{cases} w, & \overline{a} \geq b_u \\ \dfrac{(a_u - \overline{b})w}{(a_u - \overline{a}) + (b_u - \overline{b})}, & b_u \geq \overline{a},\ a_u \geq \overline{b} \\ 0, & \overline{b} \geq a_u \end{cases}. \quad (2.54)$$

If two different fuzzy numbers (triangular and trapezoidal) are compared the following situations will be occurred.

If $\widetilde{A}$ is a trapezoidal fuzzy number and $\widetilde{B}$ is a triangular fuzzy number, $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ and $\widetilde{B} = (b_l, b, b_u)$, then the possibility that $\widetilde{A} \geq \widetilde{B}$, according to the exceedance possibility, is given by

$$Poss(\widetilde{A} \geq \widetilde{B}) = \begin{cases} w, & \overline{a} \geq b \\ \dfrac{(a_u - b_l)w}{(a_u - \overline{a}) + (b - b_l)}, & b \geq \overline{a},\ a_u \geq b_l \\ 0, & b_l \geq a_u \end{cases} \quad (2.55)$$

and this possibility, in the case of the strict exceedance possibility can be presented as (Iskander, 2002);

$$Poss(\widetilde{A} > \widetilde{B}) = \begin{cases} w, & \overline{a} \geq b_u \\ \dfrac{(a_u - b)w}{(a_u - \overline{a}) + (b_u - b)}, & b_u \geq \overline{a},\ a_u \geq b \\ 0, & b \geq a_u \end{cases}. \quad (2.56)$$

If $\widetilde{A}$ is a triangular fuzzy number and $\widetilde{B}$ is a trapezoidal fuzzy number, $\widetilde{A} = (a_l, a, a_u)$ and $\widetilde{B} = (b_l, \underline{b}, \overline{b}, b_u)$, then the possibility that $\widetilde{A} \geq \widetilde{B}$, according to the exceedance possibility, is given by

$$Poss(\widetilde{A} \geq \widetilde{B}) = \begin{cases} w, & a \geq \underline{b} \\ \dfrac{(a_u - b_l)w}{(a_u - a) + (\underline{b} - b_l)}, & \underline{b} \geq a,\ a_u \geq b_l \\ 0, & b_l \geq a_u \end{cases} \quad (2.57)$$

and this possibility, in the case of the strict exceedance possibility can be presented as (Iskander, 2002);

$$Poss(\widetilde{A} > \widetilde{B}) = \begin{cases} w, & a \geq b_u \\ \dfrac{(a_u - \overline{b})w}{(a_u - a) + (b_u - \overline{b})}, & b_u \geq a,\ a_u \geq \overline{b}\ . \\ 0, & \overline{b} \geq a_u \end{cases} \tag{2.58}$$

The α-cut approach can be utilized in possibility programming, $\alpha \in (0,w]$, by making any of the probabilities greater than or equal to α.

2.4. Selected Ranking Methods

Five ranking methods are selected to use in the solution of fuzzy mathematical programming models in our study. The ranking methods are selected according to the ease of computation and their accomplishment and consistency in ranking of fuzzy numbers. The selected five ranking methods will be briefly explained in the following.

2.4.1. The signed distance method

Yao and Wu (2000) have used signed distance to define ranking of fuzzy numbers. The signed distance used for fuzzy numbers has some similar properties to the properties induced by the signed distance in real numbers. The signed distance method for ranking fuzzy numbers can be explained briefly as follows.

Let F be the family of the fuzzy numbers on R. The sign distance is defined as $d^*(a,0) = a$ on R. Then for a, $b \in R$, $d^*(a, b) = a\text{-}b$. For $\widetilde{A}$, $\widetilde{B} \in F$, with α-cut $(0 \leq \alpha \leq 1)$, there is a closed interval $A(\alpha) =[A_L(\alpha), A_R(\alpha)]$. Then, the signed distance of $\widetilde{A}$, $\widetilde{B}$ is defined as (Yao and Wu, 2000),

$$d(\widetilde{A},\widetilde{B}) = \frac{1}{2}\int_0^1 [A_L(\alpha) + A_R(\alpha) - B_L(\alpha) - B_R(\alpha)]d\alpha \tag{2.59}$$

It can be proved that d is an extension of d^*. And,

$d(\widetilde{A},\widetilde{B}) > 0 \quad iff \quad d(\widetilde{A},0) > d(\widetilde{B},0) \quad iff \quad \widetilde{B} < \widetilde{A}$

$d(\widetilde{A},\widetilde{B}) < 0 \quad iff \quad d(\widetilde{A},0) < d(\widetilde{B},0) \quad iff \quad \widetilde{A} < \widetilde{B}$

$d(\widetilde{A},\widetilde{B}) = 0 \quad iff \quad d(\widetilde{A},0) = d(\widetilde{B},0) \quad iff \quad \widetilde{A} = \widetilde{B}$

According to these definitions, the signed distance of a triangular fuzzy number $\widetilde{A} = (a_l, a, a_u)$ is defined as,

$$d(\widetilde{A},0) = \frac{1}{2}\int_0^1 [a_l + (a - a_l)\alpha + a_u - (a_u - a)\alpha]d\alpha = \frac{1}{4}(2a + a_l + a_u)\,. \tag{2.60}$$

The signed distance of a trapezoidal fuzzy number $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ is defined as

$$d(\widetilde{A},0) = \frac{1}{2}\int_0^1 [a_l + (\underline{a} - a_l)\alpha + a_u - (a_u - \overline{a})\alpha]d\alpha = \frac{1}{4}(a_l + \underline{a} + \overline{a} + a_u)\,. \tag{2.61}$$

Let $\widetilde{A}$ and $\widetilde{B}$ are two triangular or trapezoidal fuzzy numbers, their ranking relation is defined as $\widetilde{A} \le \widetilde{B} \Leftrightarrow d(\widetilde{A},0) \le d(\widetilde{B},0)$ (Yao and Wu, 2000).

2.4.2. Ranking of fuzzy numbers with integral value

Liou and Wang (1992) proposed the method of ranking fuzzy numbers with integral value. Some ranking methods assume that the membership functions are normal. In the method of ranking fuzzy numbers with integral value the assumption of normality of membership functions is not required. Ranking fuzzy numbers with integral value is relatively simple in computation, especially in ranking of triangular and trapezoidal fuzzy numbers, and can be used to rank more than two fuzzy numbers simultaneously (Liou and Chen, 2006).

Let the left and right side membership functions of a triangular fuzzy number $\widetilde{A} = (a_l, a, a_u)$ are defined as follows (Liou and Chen, 2006).

$$\mu^L_{\widetilde{A}}(x) = \begin{cases} (x - a_l)/(a - a_l), & a_l \le x \le a,\ a_l \ne a \\ 1, & a_l = a \end{cases} \tag{2.62}$$

and

$$\mu^R_{\tilde{A}}(x) = \begin{cases} (x - a_u)/(a - a_u), & a \le x \le a_u,\ a \ne a_u \\ 1, & a = a_u \end{cases} \quad (2.63)$$

Then $u^L_{\tilde{A}} : [a_l, a] \to [0,1]$ and $u^R_{\tilde{A}} : [a, a_u] \to [0,1]$. Since $\mu^L_{\tilde{A}}$ and $\mu^R_{\tilde{A}}$ are continuous and strictly increasing, the inverse function of $\mu^L_{\tilde{A}}$ and $\mu^R_{\tilde{A}}$ exist, denoted by $g^L_{\tilde{A}}$ and $g^R_{\tilde{A}}$, and $g^L_{\tilde{A}} : [0,1] \to [a_l, a]$ and $g^R_{\tilde{A}} : [0,1] \to [a, a_u]$, respectively. Both $g^L_{\tilde{A}}$ and $g^R_{\tilde{A}}$ are as follows (Liou and Chen, 2006).

$$g^L_{\tilde{A}}(u) = \begin{cases} a_l + (a - a_l)u, & a_l \ne a, u \in [0,1] \\ a_l, & a_l = a \end{cases} \quad (2.64)$$

and

$$g^R_{\tilde{A}}(u) = \begin{cases} a_u + (a - a_u)u, & a \ne a_u, u \in [0,1] \\ a_u, & a = a_u \end{cases} \quad (2.65)$$

The definition of integral values for the triangular fuzzy number $\tilde{A}$ is written as follows (Liou and Chen, 2006).

$$I(\tilde{A}) = (1-\alpha)\int_0^1 g^L_{\tilde{A}}(u)du + \alpha\int_0^1 g^R_{\tilde{A}}(u)du = \frac{1-\alpha}{2}a_l + \frac{1}{2}a + \frac{\alpha}{2}a_u \quad (2.66)$$

where $0 \le \alpha \le 1$.

The index of optimism α is representing the degree of optimism for a person. A larger α indicates a higher degree of optimism (Liou and Chen, 2006). The fuzzy numbers are ranked according to their integral values; the fuzzy number with the larger integral value is the bigger fuzzy number.

2.4.3. Chen and Chen's ranking method

Chen and Chen (2003) has presented a method to evaluate the ranking order between generalized fuzzy numbers based on center of gravity points and standard deviations of generalized fuzzy numbers. The proposed ranking method can overcome the drawbacks of the existing centroid-index ranking methods.

If $\widetilde{A}_i = (a_1, a_2, a_3, a_4)$, i = 1,…,n are normal trapezoidal fuzzy numbers, the proposed ranking method of fuzzy numbers can be presented as follows.

Step 1: Use the following equations to calculate the center of gravity point ($x^*_{\widetilde{A}_i}, y^*_{\widetilde{A}_i}$) of each fuzzy number $\widetilde{A}_i$, where $1 \le i \le n$.

$$y^*_{\widetilde{A}} = \begin{cases} \dfrac{1 \times (\dfrac{a_3 - a_2}{a_4 - a_1} + 2)}{6} & if\ a_1 \ne a_4 \\ 1/2 & if\ a_1 = a_4 \end{cases} \tag{2.67}$$

$$x^*_{\widetilde{A}} = \frac{y^*_{\widetilde{A}}(a_3 + a_2) + (a_4 + a_1)(1 - y^*_{\widetilde{A}})}{2} \tag{2.68}$$

Step 2: Use the following equation to calculate the standard deviation $\hat{s}_{\widetilde{A}_i}$ of each fuzzy number $\widetilde{A}_i$, where $1 \le i \le n$.

$$\hat{s}_{\widetilde{A}} = \sqrt{\frac{\sum_{i=1}^{4}(a_i - \overline{a})^2}{4-1}} = \sqrt{\frac{\sum_{i=1}^{4}(a_i - \overline{a})^2}{3}} \tag{2.69}$$

$$\overline{a} = \frac{a_1 + a_2 + a_3 + a_4}{4} \tag{2.70}$$

Step 3: Use equation () to calculate the ranking value $Rank(\widetilde{A}_i)$ of each fuzzy number $\widetilde{A}_i$, where $1 \le i \le n$.

$$Rank(\widetilde{A}_i) = x^*_{\widetilde{A}} + (1 - y^*_{\widetilde{A}})^{\hat{s}_{\widetilde{A}}} \tag{2.71}$$

Step 4: Compute the ranking order of fuzzy numbers. For fuzzy numbers $\widetilde{A}_i$ and $\widetilde{A}_j$ the ranking order is evaluated as follows (Chen and Chen, 2003).

If $Rank(\widetilde{A}_i) < Rank(\widetilde{A}_j)$, then $\widetilde{A}_i < \widetilde{A}_j$,

if $Rank(\widetilde{A}_i) = Rank(\widetilde{A}_j)$, then $\widetilde{A}_i = \widetilde{A}_j$,

if $Rank(\widetilde{A}_i) > Rank(\widetilde{A}_j)$, then $\widetilde{A}_i > \widetilde{A}_j$.

2.4.4. Ranking of fuzzy numbers through the comparison of their expected intervals

Jimenez (1996) has proposed a ranking method of fuzzy numbers based on the comparison of their expected intervals.

The membership function of a fuzzy number $\widetilde{A} = (a_l, \underline{a}, \overline{a}, a_u)$ can be written as;

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & \forall x \in (-\infty, a_l] \\ f_A(x), & \forall x \in [a_l, \underline{a}] \\ 1, & \forall x \in [\underline{a}, \overline{a}] \\ g_A(x), & \forall x \in [\overline{a}, a_u] \\ 0, & \forall x \in (a_u, \infty] \end{cases} \tag{2.72}$$

In order to warrant the existence and integrability of the inverse functions $f_A^{-1}(x)$ and $g_A^{-1}(x)$, it is assumed that $f_A(x)$ is continuous and increasing, and $g_A(x)$ is continuous and decreasing.

The expected interval of a fuzzy number is defined as follows (Jimenez, 1996);

$$EI(\widetilde{A}) = [E_1^{\widetilde{A}}, E_2^{\widetilde{A}}] = [\int_{a_l}^{\underline{a}} x df_A(x), -\int_{\overline{a}}^{a_u} x dg_A(x)] \tag{2.73}$$

Integrating by parts and changing the variable $\alpha = f_A(x)$, $\alpha = g_A(x)$:

$$EI(\widetilde{A}) = [E_1^{\widetilde{A}}, E_2^{\widetilde{A}}] = [\int_0^1 f_A^{-1}(\alpha) d\alpha, -\int_0^1 g_A^{-1}(\alpha) d\alpha] \tag{2.74}$$

If f_A and g_A are linear, that is if the fuzzy number $\widetilde{A}$ is triangular or trapezoidal, its expected interval will be (Jimenez, 1996):

$$EI(\widetilde{A}) = [\frac{1}{2}(a_l + \underline{a}), \frac{1}{2}(\overline{a} + a_u)] \tag{2.75}$$

If there are two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, the expected interval of $\widetilde{A} - \widetilde{B}$ is:

$$EI(\widetilde{A}-\widetilde{B}) = [E_1^{\widetilde{A}} - E_2^{\widetilde{B}}, E_2^{\widetilde{A}} - E_1^{\widetilde{B}}] = EI(\widetilde{A}) - EI(\widetilde{B}) \tag{2.76}$$

According to the ranking method of Jimenez, for any pair of fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, the degree in which $\widetilde{A}$ is bigger than $\widetilde{B}$ is defined as (Jimenez, 1996; Jimenez et al., 2007);

$$\mu_M(\widetilde{A},\widetilde{B}) = \begin{cases} 0, & \text{if } E_2^{\widetilde{A}} - E_1^{\widetilde{B}} < 0 \\ \dfrac{E_2^{\widetilde{A}} - E_1^{\widetilde{B}}}{E_2^{\widetilde{A}} - E_1^{\widetilde{B}} - (E_1^{\widetilde{A}} - E_2^{\widetilde{B}})}, & \text{if } 0 \in [E_1^{\widetilde{A}} - E_2^{\widetilde{B}}, E_2^{\widetilde{A}} - E_1^{\widetilde{B}}] \\ 1, & \text{if } E_1^{\widetilde{A}} - E_2^{\widetilde{B}} > 0 \end{cases} \tag{2.77}$$

where, $\mu_M(\widetilde{A},\widetilde{B})$ is the degree of preference of $\widetilde{A}$ over $\widetilde{B}$. When $\mu_M(\widetilde{A},\widetilde{B}) = 0.5$ it will be said that $\widetilde{A}$ and $\widetilde{B}$, are equal.

2.4.5. Ranking fuzzy numbers based on left and right dominance

Chen and Lu (2001) have proposed an approximate ranking approach based on the left and right dominance. This ranking approach follows the concept of are measurement. The ranking approach is useful when the membership functions of the fuzzy numbers cannot be acquired.

The ranking method uses a few left and right spreads at some α-levels of fuzzy numbers to determine the dominance of one fuzzy number over the other. In the ranking method, the left (right) dominance is determined by summing the difference of the left (right) spreads at each α-level to calculate the degree to which one fuzzy number dominates the other at the left (right) hand side (Chen and Lu, 2001).

For a fuzzy number $\widetilde{A}$, the α-cuts are convex subsets of R. In the ranking method, first, the number of α-cuts that are used are determined. The lower and upper limits of the k^{th} α-cut for the fuzzy number $\widetilde{A}_i$ are defined as

$$l_{i,k} = \inf_{x \in R}\{x \mid \mu_{\widetilde{A}}(x) \geq \alpha_k\} \tag{2.78}$$

$$r_{i,k} = \sup_{x \in R}\{x \mid \mu_{\widetilde{A}}(x) \geq \alpha_k\} \tag{2.79}$$

where $l_{i,k}$ and $r_{i,k}$ are left and right spreads.

For two fuzzy numbers $\widetilde{A}_i$ and $\widetilde{A}_j$, the left (right) dominance $D_{i,j}^L$ ($D_{i,j}^R$) of $\widetilde{A}_i$ over $\widetilde{A}_j$ is the average difference of the left (right) spreads at some α-levels. The left and right dominance can be written as following formulas.

$$D_{i,j}^L = \frac{1}{n+1}\sum_{k=0}^{n}(l_{i,k} - l_{j,k}) \tag{2.80}$$

$$D_{i,j}^R = \frac{1}{n+1}\sum_{k=0}^{n}(r_{i,k} - r_{j,k}) \tag{2.81}$$

(n+1) is the number of α-cuts used to calculate the dominance. α_k denotes the k^{th} α-level and $\alpha_k = k/n$, $k \in \{0,1,\ldots,n\}$. the distance between each two adjacent α-levels is equal; $\alpha_k - \alpha_{k-1} = 1/n$, $k \geq 1$. as $n \to \infty$, $D_{i,j}^L$ ($D_{i,j}^R$) approximates the area difference of $\widetilde{A}_i$ over $\widetilde{A}_j$ according to the membership axis to the left (right) membership function.

The total dominance is determined by combining the left and right dominance based on a decision maker's degree of optimism. The total dominance of $\widetilde{A}_i$ over $\widetilde{A}_j$ with the index of optimism $\beta \in [0,1]$ can be defined as the convex combination of $D_{i,j}^L$ and $D_{i,j}^R$ by the following formula.

$$\begin{aligned} D_{i,j}(\beta) &= \beta D_{i,j}^R + (1-\beta)D_{i,j}^L \\ &= \beta\left[\frac{1}{n+1}\sum_{k=0}^{n}(r_{i,k} - r_{j,k})\right] + (1-\beta)\left[\frac{1}{n+1}\sum_{k=0}^{n}(l_{i,k} - l_{j,k})\right] \\ &= \frac{1}{n+1}\left\{\left[\beta\sum_{k=0}^{n} r_{i,k} + (1-\beta)\sum_{k=0}^{n} l_{i,k}\right] - \left[\beta\sum_{k=0}^{n} r_{j,k} + (1-\beta)\sum_{k=0}^{n} l_{j,k}\right]\right\} \end{aligned} \tag{2.82}$$

Two fuzzy numbers $\widetilde{A}_i$ and $\widetilde{A}_j$ can be ranked using $D_{i,j}(\beta)$ as follows.

If $D_{i,j}(\beta) > 0$ then $\widetilde{A}_i > \widetilde{A}_j$,

if $D_{i,j}(\beta) = 0$ then $\widetilde{A}_i = \widetilde{A}_j$, if $D_{i,j}(\beta) < 0$ then $\widetilde{A}_i < \widetilde{A}_j$.

In the study, it is shown that the ranking orders are consistent regardless of the number of α-cuts. Chen and Lu (2001) have stated that, if the membership functions of fuzzy numbers are simple, only a small number of α-cuts is necessary. The use of a greater number of α-cuts can obviously produce more accurate ordering.

CHAPTER 3

REVIEW AND CLASSIFICATION OF FUZZY MATHEMATICAL PROGRAMING MODELS

3.1. Introduction

In this chapter, a new classification of fuzzy mathematical programming models according to the fuzzy components is presented. In section 3.2, existing classifications for fuzzy mathematical programming models are given. In section 3.3, the new classification of fuzzy mathematical programming models according to the fuzzy components is explained and in the consequent subsections, some of the studies from the literature on each type of fuzzy models are reviewed briefly.

3.2. Existing Classifications for Fuzzy Mathematical Programming Models

In the literature, various authors classified fuzzy mathematical programs with respect to different criteria. Zimmermann (1987) classified fuzzy mathematical programs into two groups according to the solution method; symmetric and non-symmetric models (Kuruüzüm, 1998).

Table 3.1. Zimmermann's classification of fuzzy mathematical programming problems (Kuruüzüm, 1998)

		OBJECTIVE(S) or GOALS	
		CRISP	*FUZZY*
CONSTRAINTS	*CRISP*	Conventional Decision	Symmetric Model
	FUZZY	Non-symmetric Model	Symmetric Model

Leung (1988) classified fuzzy mathematical programming models into four categories:

i. A precise objective and fuzzy constraints,
ii. A fuzzy objective and precise constraints,
iii. A fuzzy objective and fuzzy constraints, and
iv. Robust programming (Lai and Hwang, 1994).

Luhandjula (1989) categorized fuzzy mathematical programming models into three classes;

i. Flexible programming,
ii. Mathematical programming with fuzzy parameters, and
iii. Fuzzy stochastic programming.

Luhandjula (1989) also classified flexible programming methods into symmetrical and asymmetrical approaches, as did Zimmermann. He grouped mathematical programming problems with fuzzy parameters into two major classes; problems with a deterministic objective function and problems with a fuzzy objective function. Fuzzy stochastic programming concerns parameters involving both fuzzy and stochastic natures (Lai and Hwang, 1994).

According to Fedrizzi et al. (1991), the fuzzification of the single or multiple objective linear programming model usually includes four forms of imprecision;

i. Models with fuzzy constraints,
ii. Models with fuzzy objectives (goals imposed on the objective functions),
iii. Models with fuzzy coefficients on the variables, and
iv. Combinations of the above (Lai and Hwang, 1994, Pires et al., 1996).

Negoita (1981) distinguishes two types of imprecision in fuzzy optimization problems:

i. Flexible programming for problems with fuzzy equations and objectives (vague nature) and
ii. Robust programming for fuzzy coefficients or parameters (of an ambiguous nature) (Pires et al., 1996).

Rommelfanger and Slowinski (1999) presents the general fuzzy linear programming problem as where all parameters (coefficients of objective, coefficients of constraints and right hand values of constraints) and the goal(s) are defined as fuzzy. They state that the general system includes the following special cases:

i. The objective function is crisp,
ii. Some or all constraints are crisp,
iii. Some or all constraints are soft.

Rommelfanger and Slowinski (1999) point out that these special cases may be combined. When the objective function(s) and the left-hand sides of constraints are defined as crisp, the fuzzy sets express preferences concerning attainment of goal(s) and satisfaction of flexible constraints. This case corresponds to flexible programming. When the coefficients of objective function(s) and constraints are defined as fuzzy numbers, then fuzzy sets represent vague state of data under the form of possibility distributions (Rommelfanger and Slowinski, 1999).

Inuiguchi and Ramik (2000) classify fuzzy mathematical programming into three categories based on uncertainties treated in the method;

i. Fuzzy mathematical programming with vagueness,
ii. Fuzzy mathematical programming with ambiguity,
iii. Fuzzy mathematical programming with vagueness and ambiguity.

Inuiguchi and Ramik (2000) point out that, two major types of uncertainties exist in real life, ambiguity and vagueness. Ambiguity is associated with one-to-relations, that is, situations in which the choice between two or more alternatives is left unspecified. Vagueness is associated with the difficulty of defining sharp boundaries. For example, the uncertain description 'about 2 minutes' shows the ambiguity of the true value, one value around 2 is true but not known exactly. The uncertain description, 'substantially smaller than 900 minutes' shows the vagueness of the aspiration level, does not define a sharp boundary of a set of satisfactory values but shows that values around 900 and smaller than 900 are to some extent and completely satisfactory.

Inuiguchi and Ramik (2000) also described the categories of their classification as follows. The fuzzy mathematical programming problems in the first category are decision-making problems with fuzzy goals and constraints. The fuzzy goals and constraints represent the flexibility of the target values of the objective functions and the elasticity of the constraints, and this type of fuzzy mathematical programming is called as ‘flexible programming’. The second includes ambiguous coefficients of objective functions and constraints but does not include fuzzy goals and constraints. The third type handles ambiguous coefficients as well as vague decision maker’s preference. This fuzzy mathematical programming is called as ‘robust programming’.

As it can be seen from the literature the authors who dealt with fuzzy mathematical programming have made various classifications according to different criteria. In the following section, a new classification of fuzzy mathematical programming models is given according to the fuzzy components (the objective values, the objective coefficients, the right hand values, etc.).

3.3. Classification of Fuzzy Mathematical Programming Models

In this thesis, fuzzy mathematical programming models are classified according to the fuzzy components. In a fuzzy mathematical programming model, the aspiration level(s) of the objective(s) **(z)**, the limit values of the constraints **(b)**, the coefficients of the objectives **(c)** and the coefficients of the constraints **(A)** can be uncertain. In Table 3.2, the classes of the fuzzy mathematical programming models according to the fuzzy components. The classes are named as ‘types’ in Table 2 and fifteen types of the fuzzy mathematical programming models are defined in the classification. Different types of fuzzy mathematical programs are formed considering all possible situations. The possible situations are determined according to the counting technique. The number of combinations of *r* objects selected from *n* objects is denoted ${}_nC_r$ and is given by the formula ${}_nC_r = n!/(n-r)!r!$. For example if one of the four parameters is fuzzy, there will be four types; $4!/(4-1)!1! = 4$. If two of the four parameters is fuzzy, there will be six types; $4!/(4-2)!2! = 6$. If three of the four parameters is fuzzy, there will be four types; $4!/(4-3)!3! = 4$. Finally, if all parameters are fuzzy, there will be one type. Hence, it is determined that there are fifteen types of fuzzy mathematical programming models according to the fuzzy components. Type one fuzzy models (fuzzy models with fuzzy objectives (z)), include the fuzzy goal programming models. The aspiration levels of the objectives are imprecise in fuzzy goal programming and these models are similar to type one fuzzy models. In the study of Baykasoğlu and Göçken (2008) the solution approaches for the classes of fuzzy mathematical programming models are reviewed and presented in the table of classes exhaustively.

Table 3.2. Classification of fuzzy mathematical programming models

	Objective(s) **(z)**	Constraints **(b)**	Coefficients of the objectives **(c)**	Coefficients of the constraints **(A)**
Type 1	✔			
Type 2		✔		
Type 3			✔	
Type 4				✔
Type 5	✔	✔		
Type 6		✔	✔	
Type 7			✔	✔
Type 8	✔		✔	
Type 9		✔		✔
Type 10	✔			✔
Type 11	✔	✔	✔	
Type 12		✔	✔	✔
Type 13	✔	✔		✔
Type 14	✔		✔	✔
Type 15	✔	✔	✔	✔

In the following subsections, some of the studies from the literature on each type of fuzzy models are reviewed briefly.

3.3.1. Type 1 fuzzy models

Type 1 models include problems with fuzzy objectives and fuzzy goal programming models. Problems with fuzzy objective(s) and fuzzy limit values of constraints can be solved with Zimmermann's max-min method. From the beginning of the studies on fuzzy decision-making, the authors solved this type of fuzzy models with various techniques but most of these techniques are based on Zimmermann's max-min method. Zimmermann used the max-min operator of Bellman and Zadeh (Lai and

Hwang, 1994) for the first time to solve fuzzy multi-objective linear programming problems. Since Zimmermann's max-min method is the basic of most of the studies on fuzzy mathematical programming, the max-min method is going to be described briefly. Type-1 fuzzy multi-objective mathematical program can be stated in the following form;

$$\begin{aligned} & f_k(x_j, c_j)\{\geq, \leq, =\}\tilde{f}_k^0 \quad k = 1,\ldots,p \\ & s.t. \quad g(x_j, a_{ij})\{\leq, \geq, =\}\ b_i \quad i = 1,\ldots,m,\ j = 1,\ldots,n \end{aligned} \tag{3.1}$$

where, $\tilde{f}_k^0$, $\forall k$, are fuzzy goals, x_j are the decision variables, c_j are coefficients of the objective function, a_{ij} are the coefficients of the constraints and b_i are right hand values of the constraints. The functions $f_k(x_j, c_j)$ and $g(x_j, a_{ij})$ can be linear or nonlinear functions.

The decision set is defined as the intersection of the fuzzy objective(s). The decision set is characterized by its membership $\mu_D(x)$ as,

$$\mu_D(x) = \min\ (\mu_1(x_j,c_j), \mu_2(x_j,c_j), \ldots, \mu_k(x_j,c_j)) \tag{3.2}$$

The optimal decision is the maximum value of the feasible decision set;

$$\mu_D(x_{opt}) = \max \mu_D(x)\ . \tag{3.3}$$

The fuzzy multi-objective mathematical programming problems become 'maximize $\mu_D(x)$, subjective $x \in$ X',

$$\begin{aligned} & \max[\min_k\ \mu_k(x_j, c_j)] \quad k = 1,\ldots,p \\ & s.t. \quad g(x_j, a_{ij})\{\leq, \geq, =\}b_i \quad i = 1,\ldots,m,\ j = 1,\ldots,n \end{aligned} \tag{3.4}$$

This max-min problem is generally transformed to a conventional mathematical programming problem. It is taken that α is the overall satisfactory level of compromise and $\alpha = \min_k \mu_k(x_j, c_j)$ (α is the minimum of the membership values of the objectives and it is tried to maximize α, the overall satisfactory level). The max-min problem becomes a conventional mathematical programming problem such that;

$$
\begin{aligned}
\max \quad & \alpha \\
s.t. \quad & \alpha \le \min\{\mu_k(x_j, c_j)\} \qquad k = 1,..,p \\
& \alpha \in [0,1] \\
& g(x_j, a_{ij})\{\le, \ge, =\}b_i \quad i = 1,...,m,\ j = 1,...,n
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
\max \quad & \alpha \\
s.t. \quad & \alpha \le \mu_k(x_j, c_j) \qquad k = 1,..., p \\
& \alpha \in [0,1] \\
& g(x_j, a_{ij})\{\le, \ge, =\}b_i \quad i = 1,...,m,\ j = 1,...,n
\end{aligned} \tag{3.6}
$$

Another approach for solving multi-objective fuzzy linear programming problems is parametric approach that is proposed by Chanas (1983). For example a multi-objective fuzzy linear programming problem can be written as follows.

$$
\begin{aligned}
\max \quad & \tilde{z}_l = c_j^l x_j \qquad l = 1,...,k \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1,...,m,\ j = 1,...,n \\
& x_j \ge 0
\end{aligned} \tag{3.7}
$$

In Chanas's parametric approach, the problem is first assumed as a fuzzy linear program with a single objective and the other objectives are transferred into constraints. The fuzzy multi-objective linear-programming problem becomes a fuzzy linear programming problem with single objective as follows;

$$
\begin{aligned}
\widetilde{\max} \quad & z_1 = c_1^1 x_1 + c_2^1 x_2 + ... + c_n^1 x_n \\
s.t. \quad & \sum_{j=1}^{n} c_i^l x_j \underset{\sim}{>} z_l^* \qquad l = 2,3,...,k \\
& \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1,2,...,m \\
& x_j \ge 0 \qquad j = 1,2,...,n
\end{aligned} \tag{3.8}
$$

and the aspiration levels are z_i^* and tolerances are $p_l = z_i^* - z_i'$.

Then, this fuzzy linear programming problem is transferred into a crisp parametric-programming problem as follows;

$$\begin{aligned}
\max \quad & z_1 = c_1^1 x_1 + c_2^1 x_2 + \ldots + c_n^1 x_n \\
s.t. \quad & \sum_{j=1}^{n} c_i^l x_j \geq z_l^* - \theta(z_i^* - z_i^{'}) \qquad l = 2,3,\ldots,k \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1,2,\ldots,m \\
& x_j \geq 0 \qquad j = 1,2,\ldots,n
\end{aligned} \tag{3.9}$$

The decision which objective is taken as only objective in the crisp problem is decided by the decision maker according to which objective is the most important.

As indicated before, many of the previous studies in the literature are based on Zimmermann's max-min approach. For example, Gen et al. (1997) solved a fuzzy nonlinear goal programming problem via firstly converting the problem into crisp form by using max-min method and then solved the converted crisp nonlinear problem by using a genetic algorithm. Abboud et al. (1998) solved a manpower allocation problem. In the problem the target values are specified imprecisely. The fuzzy multi-objective manpower allocation problem converted into a crisp problem by using the max-min method and the resultant problem was solved by using a genetic algorithm based approach. Sinha (2003) used the max-min method for solving multi-level programming problems. Chakraborty and Gupta (2002) solved fuzzy multi objective linear fractional programming problems by using the max-min method in order to convert the problem into a crisp form and then the crisp problem is solved with classical linear programming methods.

Baykasoğlu and Göçken (2006) proposed a multiple-objective tabu-search-based solution method to solve fuzzy goal programs. A fuzzy aggregate production-planning problem is solved directly (not transforming into a crisp equivalent) in the study. The goals of the problem are defined as fuzzy numbers. The membership functions of the goals are defined as linear functions. Three different methods, namely the preemptive method, the max-min method and the additive method, are used to handle fuzzy goals within the proposed tabu search algorithm. The tabu search algorithm of Baykasoğlu et al. (1999) that is developed to solve goal programs is used as a base for the proposed tabu search algorithm. Three versions of the tabu search algorithm that implements the above methods are developed. In the developed three versions, the selection and updating stages are redefined; the other stages are identical with the tabu search algorithm of Baykasoğlu. In their study, it is shown that fuzzy goal programs can be solved directly by using meta-heuristic algorithms.

3.3.2. Type 2 fuzzy models

A fuzzy mathematical program in which the right hand side parameters (b) of the constraints are defined to be uncertain can be stated in the following form;

$$\begin{aligned} &\max/\min \quad f(x_j, c_j) \\ &s.t. \quad g(x_j, a_{ij})\{\leq, \geq, =\}\tilde{b}_i \quad i = 1,\ldots,m_1,\ j = 1,\ldots,n \\ &\qquad\quad g(x_j, a_{ij})\{\leq, \geq, =\}b_i \quad i = m_1 + 1,\ldots,m,\ j = 1,\ldots,n \end{aligned} \tag{3.10}$$

For the solution of this type of fuzzy problems different approaches are proposed and used in the literature. Most of these methods are based on the symmetric approach of Belman and Zadeh and the max-min method.

Verdegay (1984) used the duality theory of linear programming for solving fuzzy linear programming problems with fuzzy constraints. This solution is based on the idea that the dual of a fuzzy linear programming problem with fuzzy objective must be a fuzzy linear programming problem with a fuzzy constraint set, and reciprocally. This idea is based on the general fact that in fuzzy mathematical programming, the objectives and constraints play the same roles.

Rommelfanger (1996) used a solution method for type 2 fuzzy problems in which each soft constraint adds an additional objective to decision problem that is called fuzzy objective. A fuzzy linear programming problem written as follows

$$\begin{aligned} &\max \quad f(x_j, c_j) \\ &s.t. \quad g(x_j, a_{ij}) \leq \tilde{b}_i \quad i = 1,\ldots,m_1,\ j = 1,\ldots,n \\ &\qquad\ g(x_j, a_{ij}) \leq b_i \quad i = m_1 + 1,\ldots,m,\ j = 1,\ldots,n \end{aligned} \tag{3.11}$$

turns into a multi objective optimization problem as follows,

$$\begin{aligned} &\max_{x \in X_U} \quad (f(x_j, c_j), \mu_i(x_j, a_{ij})) \qquad i = 1,...,m_1, \; j = 1,...,n \\ &X_U = \; \{x_j \in R_0^n \big| g(x_j, a_{ij}) \le b_i + d_i, \; \forall i = 1,...,m_1, \; and \; g(x_j, a_{ij}) \le b_i, \; \forall i = m_1 + 1,...,m\} \end{aligned} \tag{3.12}$$

For comparing $f(x_j,c_j)$ with the fuzzy objective functions $\mu_i(x_j,a_{ij})$, $f(x_j,c_j)$ is substituted with a function $\mu_f(x_j,c_j)$. $\bar{f} = \max_{x \in X_U} f(x_j,c_j)$ and $\underline{f} = \max_{x \in X_L} f(x_j,c_j)$ $X_L = \{x_j \in R_0^n \big| g(x_j,a_{ij}) \le b_i, \; \forall i = 1,2,...,m\}$ are computed. Then, the membership functions $\mu_f(x_j,c_j)$ are defined as Zimmermann did (1976). To determine a compromise solution for the multi-objective optimization problem $\max_{x \in X_U} (f(x_j,c_j), \mu_1(x_j,a_{ij}),..., \mu_{m_1}(x_j,a_{ij}))$, it is usually assumed in the literature that the total satisfaction of a decision maker may be described by

$$\mu_D(x) = \alpha = \min(\mu_f(x_j,c_j), \mu_1(x_j,a_{ij}),...\mu_{m_1}(x_j,a_{ij})) \tag{3.13}$$

The objective is treated in the same manner as the soft constraints (symmetric). Consequently, a linear programming problem with soft constraints can be solved by a classical linear programming problem as follows;

$$\begin{aligned} &\max \quad \alpha \\ &s.t. \quad \alpha \le \mu_f(x_j,c_j) \\ &\qquad \alpha \le \mu_i(x_j,a_{ij}) \qquad i = 1,...,m_1 \\ &\qquad x_j \in X_U \;\; and \;\; \alpha \in [0,1] \end{aligned} \tag{3.14}$$

Tang and Wang (1997) are proposed a non-symmetric model for the solution of fuzzy nonlinear programming problems. They used r-power type nonlinear membership function to describe the fuzzy available resources and fuzzy constraints, and penalty coefficients to describe the additional expenses brought by the tolerances of resources. In the proposed solution method first a satisfying solution method is found by using α- level cuts and then by using the max-min method a crisp optimal solution is found. With the max-min method an unconstrained optimization problem is obtained. But its objective function is not continuous and derivable, so it cannot be solved by traditional optimization methods. To solve this unconstrained optimization problem a genetic algorithm with mutation along the weighted gradient direction is suggested.

Shih et al. (2003) developed three alternative α-level-cuts approaches: single-cut, double-cuts, and multiple-cuts, for solving nonlinear programming design problems of structuring engineering with fuzzy resources in their paper. After transforming fuzzy model into crisp model with α-level cuts, if each α value for the objective and constraint functions have the same value than the method is single α-cut approach. If a final unique α_f value exists in the objective function and a final common α_g value exists in all constraints functions, than the method is double α-cuts approach. If each objective function and constraint has its own final optimum α value [i.e. α_f and α_i (i =1,2,...,m)], the method is called multiple α-cuts approach.

In their article, Tanaka et al. (1984) solved fuzzy linear programming problems with fuzzy parameter b and fuzzy decision variables. They concerned to obtain a fuzzy solution reflecting ambiguity of fuzzy parameters.

Tanaka et al. used triangular membership functions for the fuzzy parameters. They defined the membership function of a fuzzy parameter as,

$$\mu_{a_j}(a_j) = \begin{cases} 1 - \left|2a_j - ({}^0a_j + {}_0a_j)\right| / ({}^0a_j - {}_0a_j), & {}_0a_j \le a_j \le {}^0a_j \\ 0, & \textit{otherwise} \end{cases} \tag{3.15}$$

where ${}^0a_j$, ${}_0a_j$ denote the upper limit and lower limit of the 0-level set of a_j, respectively.

The membership function of a fuzzy linear function

$$y = a_1x_1 + a_2x_2 + \ldots + a_nx_n \stackrel{\Delta}{=} ax \tag{3.16}$$

is defined as follows.

$$\mu_y(y) = \begin{cases} 1 - \left|2y - ({}^0a + {}_0a)x\right| / ({}^0a - {}_0a)|x|, & x \neq 0 \\ 1, & x = 0, y = 0 \\ 0, & x = 0, y \neq 0 \end{cases} \tag{3.17}$$

where ${}^0a = [{}^0a_1, \ldots, {}^0a_n]$, ${}_0a = [{}_0a_1, \ldots, {}_0a_n]$, $|x| = [|x_1|, \ldots, |x_n|]^t$.

The α-level set of the fuzzy set y is as follows,

$$R_\alpha(y) = [1/2\{(1-\alpha)({}^0a - {}_0a)|x| + ({}^0a + {}_0a)x\},\ 1/2\{(\alpha-1)({}^0a - {}_0a)|x| + ({}^0a + {}_0a)x\}]. \qquad (3.18)$$

Assuming that $x \geq 0$, it is obtained that

$$R_\alpha(y) = [\{(1-\alpha/2)\,{}^0a + (\alpha/2)\,{}_0a\}x,\ \{(\alpha/2)\,{}^0a + (1-\alpha/2)\,{}_0a\}x\}]. \qquad (3.19)$$

Tanaka et al. (1984) written that "a is greater than b" can be defined as $a \geq b \Leftrightarrow \max[a,b] = a$. And $a \,\widetilde{\geqq}^{\alpha}\, b$ if $({}^0a)_k > ({}^0b)_k$ and $({}_0a)_k > ({}_0b)_k$ hold for all $k \in [\alpha,1]$.

The maximization of a fuzzy set y is defined as follows (Tanaka et al., 1984),

$$\max\ y \Leftrightarrow \max\ (w_1\,{}^0y + w_2\,{}_0y) \qquad (3.20)$$

where $w_1 + w_2 = 1$ and $w_1, w_2 \in [0,1]$.

Using α-level sets of fuzzy parameters and linear equations and the comparison of fuzzy numbers, the fuzzy linear programming problem with fuzzy b and x is transformed into a conventional linear programming problem with twice the number of constraints of the fuzzy linear programming problem by Tanaka et al.

The fuzzy linear programming problem with fuzzy b and x can be written as follows.

$$\begin{aligned} &\max \quad y = c\tilde{x} \\ &s.t. \quad a_i\tilde{x} \,\widetilde{\leqq}^{\alpha}\, b_i \qquad i = 1,\ldots,m \end{aligned} \qquad (3.21)$$

where $a_i = [a_{i1},\ldots,a_{in}]$, $c = [c_1,\ldots,c_n]$ and the level α is given a priori. $x = ({}_0x, {}^0x)$ and $b = ({}_0b_i, {}^0b_i)$ are triangular fuzzy numbers.

The above fuzzy linear programming problem can be reduced to the following conventional linear programming problem (Tanaka et al., 1984).

$$
\begin{aligned}
\max \quad & c(w_1{}^0x + w_2{}_0x) \\
s.t. \quad & a_i\{(1-\alpha/2)^0x + (\alpha/2)_0x\} \le (1-\alpha/2)^0b_i + (\alpha/2)_0b_i \\
& a_i\{(\alpha/2)^0x + (1-\alpha/2)_0x\} \le (\alpha/2)^0b_i + (1-\alpha/2)_0b_i \qquad i = 1,...,m
\end{aligned}
\tag{3.22}
$$

As it can be seen from the above examples, type 2 models can be solved by using the max-min method and α-cuts as type 1 models. Generally speaking by using these methods the fuzzy constrained problems are first transformed into crisp problems afterwards they are solved with an appropriate method. If membership functions are defined as nonlinear or if the problem becomes nonlinear after transformation, meta-heuristics are used to solve it.

3.3.3. Type 3 fuzzy models

A fuzzy mathematical program with fuzzy objective coefficients can be stated as follows;

$$
\begin{aligned}
\max/\min \quad & f(x_j, \tilde{c}_j) \\
s.t. \quad & g(x_j, a_{ij})\{\le, \ge, =\}b_i \quad i = 1,...,m,\ j = 1,...,n
\end{aligned}
\tag{3.23}
$$

As in the other fuzzy models, α-cuts are used frequently for solving type 3 fuzzy models. Furthermore, several other different methods are also proposed by various authors.

Verdegay (1984) transformed type 3 fuzzy models into a crisp parametric programming problem by using α-cuts. In a fuzzy linear programming with fuzzy objective coefficients there will be a membership function related to each cost taking part in the objective, which is an n-vector function of membership functions.

$$\mu_i : \; R \to [0,1], \qquad i = 1,2,...,n \qquad (\mu_1, \mu_2, ..., \mu_n) \tag{3.24}$$

In the linear case;

$$\forall c \in R^n : \; \mu(c) = \underset{i}{Inf}\; \mu_i(c_i), \quad c = (c_1, ..., c_n) \tag{3.25}$$

Verdegay (1982) has showed that the fuzzy solution of a fuzzy linear-programming problem with fuzzy objective coefficients could be obtained according to α-cuts after solving the following parametric problem;

$$\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \mu(c_j) \geq 1-\alpha \qquad j = 1,\ldots,n \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1,\ldots,m \\
& x_j \geq 0 \qquad j = 1,\ldots,n \\
& \alpha \in [0,1], \quad c_j \in R^n
\end{aligned} \tag{3.26}$$

If $\mu(c) \geq 1-\alpha$ then

$$\underset{j}{Inf}\ \mu_j(c_j) \geq 1-\alpha \Rightarrow \mu_j(c_j) \geq 1-\alpha \qquad j = 1,\ldots,n \tag{3.27}$$

As μ_j is continuous and strictly monotone, μ_j^{-1} exists and

$$\mu_j(c_j) \geq 1-\alpha \Rightarrow c_j \geq \mu_j^{-1}(1-\alpha)\,. \tag{3.28}$$

So, problem (3.26) turns into the form,

$$\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & c_j \geq \mu_j^{-1}(1-\alpha) \qquad j = 1,\ldots,n \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1,\ldots,m \\
& x_j \geq 0 \qquad j = 1,\ldots,n \\
& \alpha \in [0,1], \quad c_j \in R^n
\end{aligned} \tag{3.29}$$

This problem is equivalent to the following problem;

$$\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & c_j = \mu_j^{-1}(1-\alpha) \quad j = 1,\ldots,n \\
& \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1,\ldots,m \\
& x_j \ge 0 \quad j = 1,\ldots,n \\
& \alpha \in [0,1]
\end{aligned} \tag{3.30}$$

Each optimal solution of problem (3.30) is also optimal for problem (3.29). Then, the following problem is obtained finally;

$$\begin{aligned}
\max \quad & z(\alpha) = \sum_{j=1}^{n} \mu_j^{-1}(1-\alpha) x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1,\ldots,m \\
& x_j \ge 0 \quad j = 1,\ldots,n \\
& \alpha \in [0,1]
\end{aligned} \tag{3.31}$$

For solving type-3 fuzzy problems Delgado et al. (1990) interrelated two approaches, a general approach that is proposed by Delgado et al. (1987) and a particular method proposed by Tanaka et al. (1984), and proposed a new method. This new method is based on the α-cuts and the upper and lower values of the fuzzy objective coefficients.

In the procedure proposed by Wang and Wang (1997), the fuzzy multi objective linear programming problem is regarded to be an interval-valued mathematical programming problem. From the membership functions of the fuzzy costs, by using α-level cut, the costs intervals are obtained and the fuzzy multi-objective linear-programming problem is transformed into a crisp interval-valued problem.

Zhang et al. (2003) proposed a method that converts a type-3 fuzzy model into a multi-objective optimization model with four objectives. If the fuzzy coefficients are defined as trapezoidal fuzzy numbers, the objectives of the multi-objective optimization model are obtained by using boundary values of the fuzzy numbers. For example, for the fuzzy linear programming problem (Zhang et al., 2003);

$$\begin{aligned}
\max \quad & f(x,y) = \tilde{c}_1 x + \tilde{c}_2 y \\
s.t. \quad & x + 4y \le 14 \\
& 4x + 10y \le 38 \\
& 28x - 5y \le 56 \\
& x \ge 0,\ y \ge 0
\end{aligned} \tag{3.32}$$

where, the membership functions of $\tilde{c}_1$ and $\tilde{c}_2$ are

$$\mu_{\tilde{c}_1}(x) = \begin{cases} 0 & x < 5 \\ x-5 & 5 \le x < 6 \\ 1 & 6 \le x \le 7 \\ \dfrac{20-x}{13} & 7 < x \le 20 \\ 0 & 20 < x \end{cases} \quad (3.33) \quad \text{and} \quad \mu_{\tilde{c}_2}(x) = \begin{cases} 0 & x < 16 \\ x-16 & 16 \le x < 17 \\ 1 & 17 \le x \le 18 \\ \dfrac{40-x}{22} & 18 < x \le 40 \\ 0 & 40 < x \end{cases} \quad (3.34)$$

the following multi-objective optimization is obtained

$$\begin{aligned}
\max \quad & \{5x+16y, 6x+17y, 7x+18y, 20x+40y\} \\
s.t. \quad & \text{(constraints in problem 3.32)}
\end{aligned} \tag{3.35}$$

According to Zhang et al. the optimal solution to the multi-objective linear programming problem is an optimal solution to the fuzzy linear programming problem.

Mavrotas et al. (2003) solved a type-3 linear fuzzy problem by converting the problem into a multi-objective linear programming problem as Zhang et al. did. If a fuzzy coefficient is defined as $\tilde{c}_i = (c_{i1}, c_{i2}, c_{i3})$, then for all c_{ij} values, an objective (for example first objective is $z_1 = \sum c_{i1} x_i$) is formed. After solving this multi-objective linear problem the optimal solution will be the optimal solution of the fuzzy linear programming problem.

3.3.4. Type 4 fuzzy models

A fuzzy mathematical program with fuzzy coefficients of constraints can be written in a general form as follows;

$$\begin{array}{lll} \max/\min & f(x_j, c_j) & \\ s.t. & g(x_j, \widetilde{a}_{ij})\{\le, \ge, =\}b_i & i=1,...,m,\ j=1,...,n \end{array} \tag{3.36}$$

Buckley et al. (1999) argued that it is possible to train a layered, feed-forward neural net to approximate solutions to fuzzy linear programming problems. Various parameters of the problem can be defined as triangular fuzzy numbers. Authors made use of α-cuts and interval programming in order to solve the fuzzy linear programming problem.

3.3.5. Type 5 fuzzy models

A fuzzy mathematical program with fuzzy objectives and fuzzy right hand values can be stated as follows;

$$\begin{array}{lll} \max/\min & \widetilde{f}(x_j, c_j) & \\ s.t. & g(x_j, a_{ij})\{\le, \ge, =\}\widetilde{b}_i & i=1,...,m,\ j=1,...,n \end{array} \tag{3.37}$$

In the literature, for transforming type-5 fuzzy models into crisp models generally Zimmermann's max-min method, α-cuts and Tiwari's additive model is used. Some examples from the literature are given next.

Zimmermann's max-min approach (Kuruüzüm, 1998) is used for solving type-5 fuzzy models. The relationship between the constraints and objective function(s) in a fuzzy environment is fully symmetric; there is no difference between the constraints and objective(s). The fuzzy decision is the selection of the activities which simultaneously satisfy objective function(s) *and* constraints. The fuzzy objective function(s) and the fuzzy constraints are characterized by the membership functions. The logical '*and*' corresponds to the 'intersection' in the fuzzy environments, so the fuzzy decision is the intersection of fuzzy constraints' and fuzzy objective(s)' membership functions. The optimal point is the maximum point of this intersection area (Kuruüzüm, 1998).

Chanas (1983) proposed a parametric approach for solving type-5 fuzzy models. Chanas (1983) indicated that, in the approach of the determination of a maximizing decision, only the maximizing alternative is obtained and the information on a complete fuzzy decision is lost. A fuzzy decision provides some information on the other alternatives close to the maximizing solution. Parametric programming can analytically describe the set of solutions incorporating the whole range of possible values of the fuzzy decision.

A fuzzy linear programming with fuzzy objective and fuzzy constraints can be written as follows;

$$\begin{aligned} &\widetilde{\max} \quad \sum_{j=1}^{n} c_j x_j \\ &s.t. \quad \sum_{j=1}^{n} a_{ij} x_j \le \tilde{b}_i \qquad i = 1,2,\dots,m \\ &\qquad\quad x \ge 0 \end{aligned} \tag{3.38}$$

The membership functions of the objective and the constraints are defined as follows;

$$\mu_0(x) = \begin{cases} 1, & \sum_{j=1}^{n} c_j x_j \ge b_0 \\ [\sum_{j=1}^{n} c_j x_j - (b_0 - p_0)]/p_0, & b_0 - p_0 < \sum_{j=1}^{n} c_j x_j < b_0 \\ 0, & \sum_{j=1}^{n} c_j x_j \le b_0 - p_0 \end{cases} \tag{3.39}$$

$$\mu_i(x) = \begin{cases} 0, & \sum_{j=1}^{n} a_{ij} x_j \le b_i - p_i \\ [\sum_{j=1}^{n} a_{ij} x_j - (b_i - p_i)]/p_i, & b_i - p_i < \sum_{j=1}^{n} a_{ij} x_j < b_i \\ 1, & \sum_{j=1}^{n} a_{ij} x_j \ge b_i \end{cases} \tag{3.40}$$

where b_0 is the possible best solution and the (b_0-p_0) is the possible worst solution of the objective function. The possibly best solution can be obtained by solving the linear program with the constraints that have the margin values and the possibly worst values can be obtained by solving the linear program with the constraints that have the values (b_i - p_i). The b_i values are decided by the decision maker and the conditions, and the p_i values are the values of the violation admissible.

According to Chanas (1983), the fuzzy problem (3.38) should be transformed into a parametric programming problem as follows;

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i + \theta p_i \qquad i = 1,2,...,m \\ & x \geq 0 \end{aligned} \tag{3.41}$$

where, parameter θ ($0 \leq \theta \leq 1$) is the degree of constraints violation.

For every admissible solution θ_x with a fixed parameter θ, the condition $\mu_i(\theta_x) \geq 1-\theta$, $i=1,2,...,m$ is valid. For every non-zero basic solution (if $p_i>0$, $i=1,2,...,m$) there exists i such that $\mu_i(\theta_x)=1-\theta$ and therefore the common degree of satisfaction of constraints is $\mu_c(\theta_x) = \underset{i}{\Lambda}\, \mu_i(\theta_x) = 1-\theta$.

Chanas and Kuchta (1998) have solved the fuzzy integer transportation problem. The membership functions of the constraints are defined in different forms; linear, exponential, power shape and rational. The fuzzy integer transportation problem is solved with an iterative algorithm based on α-cuts. The problem is transformed into a crisp interval transportation problem and solved for different α values according to proposed iterative algorithm.

Roy and Maiti (1998) solved multi-objective inventory models with fuzzy nonlinear programming. The objectives and the right hand values are defined as triangular fuzzy numbers. The membership functions are defined as linear functions. The fuzzy inventory model is solved with the fuzzy nonlinear programming method based on Zimmermann (1978) and Lee and Li (1993) and fuzzy additive goal programming method (Tiwari et al., 1987). By using these two methods fuzzy inventory model is transformed into crisp model and the resultant model is solved with a computer program based on the gradient method algorithm.

Chen and Lin (2001) have used fuzzy multi-objective optimization to design the optimum dimensions of a conical convective spine. Membership functions of the objectives and the constraint are defined as linear functions. For solving the fuzzy problem the max-min method is used. The maximization of the overall membership function is performed with a genetic algorithm.

In their paper, Sasaki and Gen (2003) proposed a method for solving fuzzy multiple objective optimal system design problems with generalized upper bounding structure by hybridized genetic algorithms. The membership functions of the goals and constraints are defined as linear. The fuzzy multiple objective problem is transformed into a crisp single-objective nonlinear integer programming problem by using weighted additive method. The obtained crisp single-objective nonlinear integer programming problem was solved by using a hybrid genetic algorithm.

Baykasoğlu et al. (2004) solved a multi-item fuzzy economic order quantity problem in their study. The goal and the limit values of the constraints are defined as fuzzy numbers. The membership functions of the fuzzy umbers are defined as linear functions. The problem is solved with tabu search and simulated annealing algorithms without converting the problem into a crisp problem. In the proposed tabu search and simulated annealing algorithms the max-min method of Bellman and Zadeh is used as the selection criteria.

Amid et al. (2006) modeled a supplier selection problem as a fuzzy multi-objective problem. The objectives and demand constraint were defined as triangular fuzzy numbers. Membership functions were defined as linear according to Zimmermann (1978). The problem was solved according to the weighted additive method (Tiwari et al., 1987).

Verma et al. (2005) proposed a dc load flow-based fuzzy optimization model in their study. They have used triangular and trapezoidal fuzzy numbers for the constraints' right hand side values. The objective's membership function is defined by finding maximum and minimum values of the objective can take. After defining membership functions the fuzzy model is solved by using the max-min method.

3.3.6. Type 6 fuzzy models

Fuzzy mathematical programs with fuzzy objective coefficients and fuzzy right hand values can be stated as follows;

$$\begin{array}{ll} \max/\min & f(x_j, \tilde{c}_j) \\ s.t. & g(x_j, a_{ij})\{\leq, \geq, =\}\tilde{b}_i \quad i=1,\ldots,m,\ j=1,\ldots,n \end{array} \tag{3.42}$$

Tang et al. (2000) has modeled a multi-product aggregate production planning problem as a fuzzy quadratic programming model with fuzzy objectives and fuzzy constraints. The fuzzy equations are defuzzified with a preferred acceptable level and the fuzzy model is transformed into a crisp model. The obtained crisp model is a quadratic programming model and can be solved with any conventional optimization technique.

Liu and Kao (2004) have solved fuzzy transportation problems in their study. In their model, the cost coefficients, the supply and demand quantities were fuzzy triangular and trapezoidal numbers. The fuzzy model was solved with a method based on extension principle. A pair of crisp parametric problems is obtained for solving the fuzzy model. With this pair of problems, the lower and upper bounds of the objective function at different α levels are found. From different values of α, the membership function of the objective function is constructed.

3.3.7. Type 7 fuzzy models

A fuzzy mathematical programming model with fuzzy objective coefficients and fuzzy left hand values can be stated as follows;

$$\begin{array}{ll} \max/\min & f(x_j, \tilde{c}_j) \\ s.t. & g(x_j, \tilde{a}_{ij})\{\leq, \geq, =\}b_i \quad i=1,\ldots,m,\ j=1,\ldots,n \end{array} \tag{3.43}$$

Sakawa and Yano (1989) have presented an interactive decision making method for multi-objective nonlinear programming problems with fuzzy parameters. In the presented method, firstly, the fuzzy problem is transformed into a non-fuzzy α-multi-objective nonlinear programming problem by using α-cuts. According to the presented interactive algorithm the individual minimum and maximum of each objective function under the given constraints for $\alpha = 0$ and $\alpha = 1$ is calculated. Then, the decision maker selects an initial value of α and reference levels for the objectives. For the specified α and reference values, the obtained non-fuzzy α-multi-objective nonlinear programming problem is transformed into a minimax problem and solved. The minimax problem is used for generating an α-Pareto optimal solution. If the decision maker does not satisfy with the current values of the objectives and α, he/she updates the reference levels and/or the degree of α and the problem is solved again.

Iskander (2005) suggested an approach for solving a stochastic fuzzy linear programming problem. The parameters, objective coefficients and constraint coefficients are defined as triangular or trapezoidal fuzzy numbers. In the suggested approach, two possibility and two necessity dominance indices that have been introduced by Dubois and Prade (1983) is used. The chance-constrained approach and the α-cut are used to transform the stochastic fuzzy problem into its deterministic-crisp equivalent, according to each of the four dominance indices. The α-cut technique is utilized for the membership functions to derive closed crisp intervals. For different values of α, and by comparing the closed crisp intervals, the solutions are generated according to each of the four dominance indices and the most proper solution is chosen.

In their study, Abo-Sinna et al. (2006) have studied the stability of multi-objective dynamic programming problems with fuzzy parameters in the objective functions and in the constraints. Using α-cuts the fuzzy multi-objective dynamic programming problem is converted into a non-fuzzy parametric multi-objective dynamic programming problem. They have proposed an interactive fuzzy decision making algorithm for the determination of any subset of the parametric space which has the same corresponding α-Pareto optimal solution.

3.3.8. Type 9 fuzzy models

Type 9 fuzzy models are fuzzy mathematical programming models in which coefficients of the constraints and right hand values are defined as fuzzy numbers. Type 9 fuzzy models can be stated as follows;

$$\begin{array}{ll} \max/\min & f(x_j, c_j) \\ s.t. & g(x_j, \tilde{a}_{ij})\{\leq, \geq, =\}\tilde{b}_i \quad i=1,\ldots,m,\ j=1,\ldots,n \end{array} \tag{3.44}$$

The ranking of fuzzy numbers are generally used for the solution of type 9 fuzzy models. For a decision vector, the left hand side of the soft constraints will be a fuzzy number and the right hand side will be another fuzzy number. For this decision vector, to decide the feasibility, the left hand side and the right hand side of the constraints should be compared. Therefore, ranking methods for fuzzy numbers are generally used.

Nakahara (1998) proposed two kinds of ranking criteria for fuzzy numbers and showed how these ranking criteria can be used for solving fuzzy models in which the parameters of the constraints are fuzzy numbers. The proposed ranking criteria are specified by three parameters: type, α-level and the lowest permitted degree of fuzzy inequalities. The decision maker decides α-level and the lowest permitted degree, and the fuzzy constraint is transformed into a crisp equivalent according to the one of the ranking criteria.

Fang et al. (1999) have proposed a solution method based on ranking of fuzzy numbers for the solution of type 9 fuzzy models. The fuzzy linear programming problem with fuzzy A and b is reduced to a linear semi-infinite programming problem by using a specific ranking method. The obtained linear semi-infinite programming problem is solved with a cutting plane algorithm.

Liu (2001) has proposed a different method for solving type 9 fuzzy models based on the satisfaction degree of the constraints. The satisfaction degree of the constraints is measured according to a new ranking method. The definition of the ranking method depends on the type of the fuzzy numbers (triangular or trapezoidal). The fuzzy linear programming problem with fuzzy parameters A and b are transformed into a crisp parametric programming problem according to the ranking method. The parameters of the obtained crisp parametric programming problem are the constraint satisfaction degrees. The objective distribution function can be obtained by solving the crisp parametric problem for different constraint satisfaction degrees.

3.3.9. Type 11 fuzzy models

A fuzzy mathematical programming model with fuzzy objectives, fuzzy objective coefficients and fuzzy right hand values can be stated as follows;

$$\begin{array}{ll} \max/\min & \tilde{f}(x_j, \tilde{c}_j) \\ s.t. & g(x_j, a_{ij})\{\leq, \geq, =\}\tilde{b}_i \quad i = 1,\ldots,m, \; j = 1,\ldots,n \end{array} \tag{3.45}$$

Mondal and Maiti (2002) have solved multi-item fuzzy economic order quantity models. The membership functions of the fuzzy parameters are assumed to be non-increasing continuous linear membership functions. The fuzzy nonlinear model is transformed into a crisp nonlinear model by using the max-min method. The obtained crisp nonlinear model is solved according to a classical optimization technique (using Lagrange multipliers according to Kuhn and Tucker conditions) (Roy and Maiti, 1997) and genetic algorithm.

Yadavalli et al. (2005) solved fuzzy multi-item economic order quantity models with no constraint and with one constraint. The only constraint is the limitation of the average number of stocked units in the model with one constraint. The set-up cost, the holding cost, the total cost and the average number of stocked units are defined as fuzzy numbers. The membership functions of the set-up cost and the holding cost are defined as linear. The membership functions of the total cost and the average number of stocked units are defined as linear, parabolic and cubic in three cases. The fuzzy parameters are transformed into crisp using α-cuts and the fuzzy model is transformed into a crisp maximization (the objective is max α) model. In the study, the multi-item inventory model is applied in manpower planning problem.

3.3.10. Type 12 fuzzy models

Type 12 fuzzy models are the models with fuzzy parameters *c*, *b* and *A*. Type 12 fuzzy models are studied frequently in the literature as. Various authors proposed different solution procedures.

Type 12 fuzzy mathematical programming models can be stated as follows;

$$\begin{array}{ll} \max/\min & f(x_j, \tilde{c}_j) \\ s.t. & g(x_j, \tilde{a}_{ij})\{\leq, \geq, =\}\tilde{b}_i \quad i = 1,\ldots,m,\ j = 1,\ldots,n \end{array} \tag{3.46}$$

A parametric programming model has been proposed by Carlsson and Korhonen (1986) for type 12 problems. The fuzzy parameters are specified with intervals. The parameters are defined as R-type fuzzy numbers. The best value of the objective function, at a fixed level of precision, is tried to be found by using parameter values of the same level of precision. This process is the same as in α-cuts. The fuzzy problem is converted into a parametric programming problem by using α-cuts. The obtained crisp problem can be solved by using any conventional method.

Gen et al. (1998) proposed a neural network technique for solving fuzzy multi-objective linear programming problems in which parameters *c*, *b*, *A* are defined as fuzzy numbers. In the proposed procedure first the fuzzy multi-objective linear programming problem is transformed into crisp multi-objective linear programming problem by using α-cuts. Then the obtained crisp multi-objective linear programming problem is solved with each single objective to obtain positive and negative ideal solutions. After that, using positive and negative ideal values the crisp multi-objective linear programming problem is converted into a crisp linear programming problem. After this step the crisp linear programming problem is solved with proposed neural network technique based on a penalty method.

Wang and Fang (2001) have solved a fuzzy aggregate production planning problem with multiple objectives. Some parameters in the objectives and the constraints are defined as trapezoidal fuzzy numbers. In their study, an interactive solution procedure is developed. In the proposed interactive solution procedure first the fuzzy data are modeled (the fuzzy parameters are modeled as trapezoidal fuzzy numbers) and membership functions of fuzzy parameters and fuzzy constraints are defined. Then the membership function to obtain a satisfactory solution is determined. After defining membership functions, the fuzzy problem is converted into a crisp problem by using the max-min method. The attained crisp problem can be solved with any solution technique. After the solution is derived, the interactive decision process is implemented and the decision maker accepts the solution or modifies the problem.

Sakawa and Nishizaki (2002) introduced an interactive fuzzy programming method through genetic algorithm for solving two-level non-convex programming problems with fuzzy parameters. The fuzzy parameters in the problem formulation process are assumed to be characterized as fuzzy numbers. Using the α-level sets of fuzzy numbers, the corresponding crisp two-level non-convex programming problem is attained. Fuzzy goals are assigned for the non-convex objective functions at both levels. According to these fuzzy goals, a membership function is quantified for each objective. After the membership functions of the objectives are constituted, the minimum of the membership values are tried to be maximized. The overall satisfactory solution is balanced between both levels. The satisfactory solution well balanced between both levels is obtained through a genetic algorithm.

Iskander (2002) proposed an approach to solve fully fuzzified linear programming problems and fully fuzzified multi objective linear programming problems via comparison of fuzzy numbers using possibility programming. In the study it was assumed that the fuzzy parameters in the fuzzy model are defined as triangular or trapezoidal fuzzy numbers. First, using comparison of fuzzy numbers the possibility that x belongs to a feasible constraint i and the possibility that the objective function Z equal to any value z is presented. After that according to α-cuts and the membership functions presented for the possibilities of x belongs to a feasible constraint i and the objective function Z equal to any value z, the equivalent crisp problem is obtained. The obtained crisp problem is solved with any appropriate linear programming method.

Chen and Weng (2003) solved a quality function deployment problem in fuzzy nature. The relationships between customer requirements and engineering design requirements as well as among the design requirements are defined as fuzzy. A fuzzy model is formulated to determine the fulfillment level of each design requirement for maximizing the customer satisfaction. The membership functions of the fuzzy normalized relationships are defined via α-cuts and the extension principle. New expressions are proposed for the fuzzy numbers to obtain more shortened α-cuts, such that fuzzy numbers can be determined in terms of α-cuts with less uncertainty. After that, the formulated fuzzy model is solved using α-cuts and the defined membership functions.

Jimenez et al. (2007) proposed a ranking method for fuzzy numbers (Jimenez, 1996) and used this ranking method for solving type 12 fuzzy models. The ranking method is used so as to define the feasibility degree of the decision vector and the acceptable optimal solution concept. In the proposed ranking method the expected interval and the expected value of a fuzzy number is used. The degree of one fuzzy number is bigger than another is defined with a membership function. The constraints of a fuzzy mathematical model will be transformed into crisp by using the ranking method to obtain α-feasible solutions. The fuzzy cost coefficients (c) are replaced with expected values to obtain α-acceptable optimal solution. After those transformations, the fuzzy problem is converted into a crisp α-parametric problem. For solving the obtained crisp α-parametric problem Jimenez et al. (2007) proposed an interactive method. In the crisp α-parametric problem there are two conflicting objectives: to improve the objective function value and to improve the degree of satisfaction of constraints. A better value to the optimal objective function implies a lesser degree of feasibility of the constraints. In the interactive method a reasonable solution is tried to be obtained (Jimenez et al., 2007). In the first step of the interactive method, the crisp α-parametric problem is solved for different α values (the α values are determined by the decision maker). For each α value the solution of the problem and the possibility distribution of the objective are obtained. After seeing the possibility distribution of the objective for each α value, a goal and a tolerance threshold can be determined for the objective. Then, in accordance with these values a membership function is defined for the objective. In the second step, the degree of satisfaction of the fuzzy goal by each α-acceptable solution is calculated by using the index proposed by Yager (1979). In the third step of the method, it is tried to find a balanced solution between the feasibility degree and the degree of satisfaction using the max-min method.

3.3.11. Type 13 fuzzy models

Type 13 fuzzy models are the models with fuzzy parameters *z*, *b* and *A*. Type 13 fuzzy mathematical programming models can be stated as follows;

$$\begin{array}{ll} \max/\min & \tilde{f}(x_j, c_j) \\ s.t. & g(x_j, \tilde{a}_{ij})\{\leq, \geq, =\}\tilde{b}_i \quad i = 1,\ldots,m,\ j = 1,\ldots,n \end{array} \tag{3.47}$$

Cadenas and Jimenez (1994) proposed a solution method for type 13 fuzzy models according to Bellman and Zadeh's maximizing decision. An interactive method is introduced for the solution of type 13 multi-objective fuzzy programming problem. A decided linear ranking function will be used for transforming fuzzy constraint set into an ordinary constraint set. After transforming fuzzy constraint set into crisp constraint set, only the objective remains as fuzzy. This fuzzy model with fuzzy objectives can be solved according to Bellman and Zadeh's maximizing decision. The membership function for each objective function is formed by obtaining the individual minimum and maximum values of each objective function in the given constraint domain.

3.3.12. Type 15 fuzzy models

In type 15 fuzzy models all parameters and the objective(s) of a mathematical programming problem are defined as fuzzy. Type 15 fuzzy models can be stated as follows;

$$\begin{array}{ll} \max/\min & \tilde{f}(x_j, \tilde{c}_j) \\ s.t. & g(x_j, \tilde{a}_{ij})\{\leq, \geq, =\}\tilde{b}_i \quad i = 1,\ldots,m,\ j = 1,\ldots,n \end{array} \tag{3.48}$$

Roy and Maiti (1998) solved fuzzy multi-item inventory models of deteriorating items with stock-dependent demand. Two fuzzy models are constructed for the inventory problem. In one fuzzy model the objectives and the right hand side value of one of the constraints are accepted as imprecise. In the other fuzzy model, some of c, b, A parameters and the objectives are fuzzy. The fuzzy parameters are defined as triangular fuzzy numbers and the membership functions of the goals of the objectives are defined as linear. The fuzzy models are solved with two methods and the solutions are compared. One of the methods is the fuzzy nonlinear programming method based on Zimmermann's max-min method and the method of Lee and Li (1993). The other method is fuzzy additive goal programming method of Tiwari et al. (1987). The obtained crisp problems from the two methods are nonlinear. For the solution of the obtained crisp nonlinear programming problems, a computer program based on the gradient method algorithm is used.

3.4. Conclusions

In this section of the thesis, fuzzy mathematical programming models are reviewed and classified into fifteen types according to the fuzzy components they include. A literature review is carried out in order to see which types of problems are more frequently studied and what type of solution approaches are implemented. After the literature review, it is observed that the mostly frequently studied fuzzy mathematical programming models are;

-the fuzzy mathematical programming models with fuzzy objective(s) (type 1 fuzzy models),

-the fuzzy mathematical programming models with fuzzy right hand values of constraints (type 2 fuzzy models),

-the fuzzy mathematical programming models with fuzzy cost coefficients (type 3 fuzzy models),

-the fuzzy mathematical programming models with fuzzy objective(s) and fuzzy right hand values of constraints (type 5 fuzzy models),

-the fuzzy mathematical programming models with fuzzy cost coefficients, fuzzy right hand values of constraints and fuzzy coefficients of constraints (type 12 fuzzy models).

In the literature, different solution approaches are proposed for the solution of these fifteen types of fuzzy models. Zimmermann's max-min method and α-cuts are used very frequently as the solution approaches. These two methods can be applied to almost all types of fuzzy models. For type 9 and type 12 fuzzy models, solution approaches which are based on ranking methods of fuzzy numbers are proposed. Besides these, some authors proposed different specific techniques. The most common idea of the proposed approaches is to transform the fuzzy model into a crisp model. After the transformation process, the obtained model is solved by using a conventional method according to the form of the resultant model (linear or nonlinear). The main challenge in solving fuzzy mathematical programming problems is to solve them directly. In transformation process some information can be lost, the number of constraints can be increased etc., so transformation is not always helpful. For a direct solution, employing fuzzy number ranking functions and procedures can be a very effective approach.

CHAPTER 4

DIRECT SOLUTION OF FUZZY MATHEMATICAL PROGRAMMING MODELS

4.1. Introduction

In this chapter, the proposed direct solution of fuzzy mathematical programming models is explained and shown on two small examples. The explanation of the proposed solution method is given in section 4.2. The metaheuristic algorithms used in the solution process are explained briefly in section 4.3. Two small examples are solved using the proposed method in section 4.4.

4.2. Direct Solution Method for Fuzzy Mathematical Programming Problems

In the literature, there are various studies on solving fuzzy mathematical programming models. In a fuzzy mathematical programming model all or some of the parameters can be defined as fuzzy numbers. For fuzzy mathematical programming models with various fuzzy parameters, different optimization algorithms are proposed. However, most of the solution approaches are based on the fuzzy decision concept proposed by Zimmermann (1976). Other common approach is to use fuzzy ranking procedures as a part of the solution mechanism for solving fuzzy mathematical programs. In the literature, there are various studies in which different fuzzy ranking procedures used for the solution of fuzzy mathematical models (Tanaka et al. 1984, Campos and Verdegay 1989, Nakahara 1998, Fang et al. 1999, Cadenas and Verdegay 2000, Iskander 2002, Jimenez et al. 2000, 2007, Baykasoğlu and Göçken 2007). In all these solution approaches, fuzzy mathematical programming models were first transformed into a crisp equivalent then solved by a classical solution approach.

In this thesis, a direct solution method is proposed for solving fuzzy mathematical programming problems. In the proposed direct solution method ranking methods for fuzzy numbers and metaheuristic algorithms are used. Ranking methods for fuzzy numbers are used to rank the objective function values and to determine the feasibility of the constraints. In a fuzzy mathematical programming problem, any of the parameters can be defined as fuzzy numbers. In this part of the thesis, the decision variables are accepted as crisp. When a crisp number is multiplied with a fuzzy number, the result will be the same kind of fuzzy number. If the cost coefficients of the objective function are defined as fuzzy numbers, the objective function values of the generated solution vectors will be fuzzy numbers. Therefore, in the selection of the best solution vector, ranking of fuzzy numbers is used. If the parameters of the constraints are defined as fuzzy numbers, the right hand values and left hand values of the constraints will be fuzzy numbers. So, the feasibility of the constraints for the generated solution vectors will be determined via ranking of two fuzzy numbers (i.e. comparing right and left hand side fuzzy numbers for the constraint functions). If only some of the parameters of the constraints are defined as fuzzy numbers, still, ranking methods for fuzzy numbers can be used (i,e. left hand side coefficients are defined as crisp). Because, ranking methods for fuzzy numbers can rank a fuzzy number with a crisp number.

In the proposed direct solution method, ranking methods for fuzzy numbers are used to rank the objective function values and to determine the feasibility of the constraints. A metaheuristic algorithm is used to carry out the ranking process. The advantage of the direct solution method is that the fuzzy mathematical programming problems can be solved without any necessity to transform them into their crisp equivalents. In the transformation process, some information can be missed. Essentially, it can be very hard to transform many problems into their crisp equivalents and sometimes the obtained crisp equivalents can be nonlinear. When the obtained crisp equivalents are nonlinear, meta-heuristics algorithms should be used again for the solution. Therefore transformation might not be always advantageous. Besides, in literature, the fuzzy parameters' membership functions are accepted as linear and in this way the transformation process can be applied. For example, the ranking functions are used for transformation into crisp equivalent and usually the fuzzy parameters are defined as triangular or trapezoidal fuzzy number. The ranking methods for triangular or trapezoidal fuzzy numbers are simple. But, when the fuzzy numbers are defined in different shapes, mathematically, the ranking methods not easy to use in transformation process.

4.3. Metaheuristic Algorithms Used in the Direct Solution Method

In the thesis, for the solution of the problems, two metaheuristic algorithms are used; the particle swarm optimization algorithm and the tabu search algorithm. These two metaheuristic are selected because the implementation of them is not very hard and they are adequate for problems with continuous variables. Some of the problems are solved via particle swarm optimization algorithm, some of them are solved via tabu search algorithm. These two metaheuristic algorithms are briefly explained in this section.

4.3.1. The particle swarm optimization algorithm

Particle swarm optimization (PSO) is an extremely simple algorithm that seems to be effective for optimizing a wide range of functions (Eberhart and Kennedy, 1995).

A PSO algorithm maintains a swarm of particles, where each particle represents a potential solution. A swarm is similar to a population, while a particle is similar to an individual. The particles are flown through a multidimensional search space, where the position of each particle is adjusted according to its own experience and that of its neighbors (Engelbrecht, 2005).

Two PSO algorithms have been developed which differ in the size of their neighborhoods; global best and local best PSO. For the global best PSO, the neighborhood for each particle is the entire swarm (Engelbrecht, 2005). In local best PSO, particles have information only of their own and their nearest array neighbours' bests, rather than that of the entire group (Eberhart and Kennedy, 1995). In this thesis, global best PSO algorithm is used.

Let $x_i(t)$ denote the position of particle i in the search space at the time step t. The position of the particle is changed by adding a velocity, $v_i(t)$, to the current position,

$$x_i(t+1) = x_i(t) + v_i(t+1) \tag{4.1}$$

It is the velocity that drives the optimization process, and reflects both the experiential knowledge of the particle and socially exchanged information from the particle's neighborhood (Engelbrecht, 2005). The PSO concept consists of, at each time step, changing the velocity of each particle toward its particle best and global best. Acceleration is weighted by a random term, with separate random numbers being generated for acceleration toward particle best and global best (Eberhart and Kennedy, 1995).

For global best PSO, the velocity of particle i is calculated as (Engelbrecht, 2005);

$$v_{ij}(t+1) = wv_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] + c_2 r_{2j}(t)[\hat{y}_j(t) - x_{ij}(t)] \tag{4.2}$$

where $v_{ij}(t)$ is the velocity of particle i in dimension j=1,…,n_x at time step t, $x_{ij}(t)$ is the position of particle i in dimension j at time step t, w is the inertia weight, c_1 and c_2 positive acceleration constants, $r_{1j}(t)$ and $r_{2j}(t) \sim$ U(0,1) are random values in the range [0,1], $y_{ij}(t)$ is the personal best position of particle i, $\hat{y}_j(t)$ is the global best position at time step t.

Inertia weight w controls the impact of previous historical values of particle velocity on its current one. A larger inertia weight pressures toward global exploration while a smaller inertia weight pressures toward fine-tuning the current search area. The acceleration constants c_1 and c_2 represent the weighting of the stochastic acceleration terms that pull each particle towards *pbest* and *gbest* positions. Thus, adjustment of these constants changes the amount of tension in the system (Dong et al., 2005)

The global best PSO algorithm can be summarized as follows (Engelbrecht, 2005).

Create and initialize an n_x-dimensional swarm, S;
repeat
  for each particle i=1,…,$S.n_s$ **do**
    // set the personal best position
    if $f(S.x_i) < f(S.y_i)$ **then**
      $S.y_i = S.x_i$;
    end
    // set the global best position
    if $f(S.y_i) < f(S.\hat{y})$ **then**
      $S.\hat{y} = S.y_i$;
    end
  end
  for each particle i = 1,…,$S.n_s$ **do**
    update the velocity using equation (2);
    update the position using equation (1);
  end
until stopping condition is true.

4.3.2. Tabu search algorithm

Tabu search has its origins in combinatorial procedures applied to nonlinear covering problems in late 1970s. tabu search is a higher level heuristic procedure for solving optimization problems, designed to guide other methods to escape the trap of local optimality. Tabu search is a stochastic neighborhood search algorithm that is first suggested and applied by Glover (1990, 1993). The basic tabu search algorithm operates starts from a randomly selected or a known feasible solution. From this initial solution, a set of neighborhood solutions are generated using a number of previously determined movement strategies. The objective function is evaluated for each solution in the set of neighborhood solutions and the best neighbor replaces the current solution, even though it may be worse than the initial solution: in this way it is possible to escape from the local minima (or maxima) of the objective function. The algorithm iterates, repeating the procedure with the new solution, until some given stopping condition(s) is reached. However, the algorithm as described above may recycle. To avoid this situation, certain attributes of the last k replaced solutions are stored in a list, which is called the tabu list. The neighbors of the current solution that satisfy conditions given by the tabu list are systematically eliminated unless they meet an aspiration criterion, so at each iteration the algorithm is forced to select a point not recently selected. The main stages of a tabu search algorithm are; initial solution, generation of neighbors, selection, aspiration, and updating (Baykasoğlu and Göçken, 2006). The flowchart of the simple tabu search algorithm is given in figure 4.1.

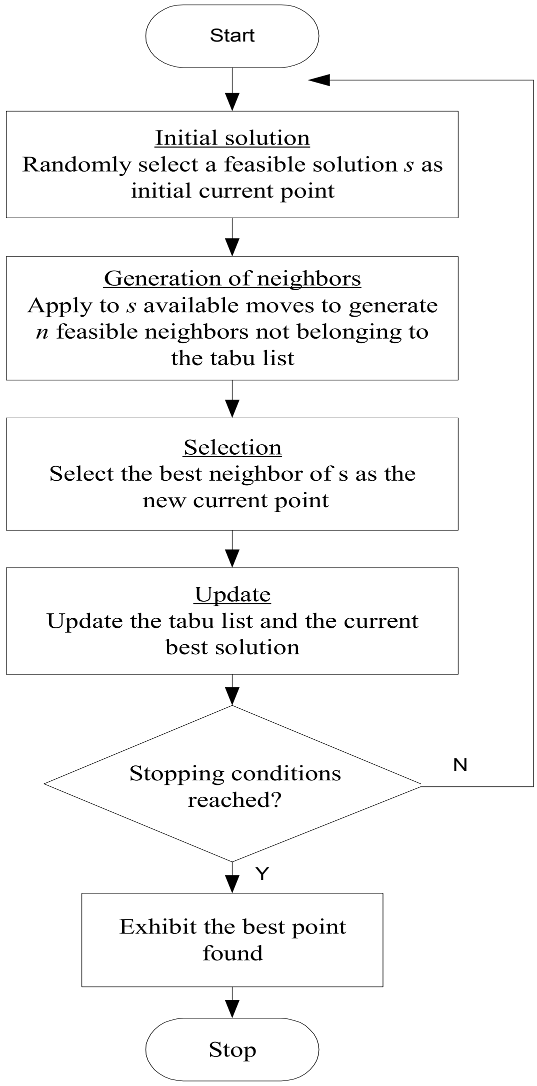


Figure 4.1. Flowchart of the simple tabu search algorithm (Baykasoğlu et al., 1999a)

Baykasoğlu et al. (1999a) adapted tabu search algorithm for solving multiple objective optimization problems. The solution structure of the tabu search algorithm enables to work with more than one solution (*neighborhood solutions*) at a time. This situation gives the opportunity to evaluate multiple goals simultaneously. To enable the original tabu search algorithm to work with more than one goal the selection and updating stages are redefined. Other stages are identical to the original tabu search algorithm (Baykasoğlu et al., 1999a). In section 5.2 for solving fuzzy aggregate production planning problem the tabu search algorithm for multiple goals is used.

The main stages of the tabu search algorithm can be defined as follows.

***Initial solution*:** An initial random feasible solution vector (*or a previously known feasible solution vector*) that satisfies all hard constraints.

***Generation of neighborhood solutions*:** A neighbor solution is obtained by changing the value of a randomly selected decision variable from the solution vector. Tabu search works with population of solutions therefore; this action is performed repetitively to obtain previously determined numbers of neighbor solutions (S*). To generate a neighbor for any type of variable, new values are formulated as follows (Baykasoglu *et al.* 1999a, 1999b).

$$\begin{array}{ll} \text{Integer variable} & x_i^* = x_i + \text{integer}[(2*random() - 1)*stepi_i] \\ \text{Zero-One variable} & x_i^* = \begin{cases} 1 & if \ x_i = 0 \\ 0 & if \ x_i = 1 \end{cases} \\ \text{Discrete variable} & x_i^* = d_{(l+\text{integer}[(2*random(\,)-1)*stepd_i])} \quad if \quad x_i = d_l \\ \text{Continuous variable} & x_i^* = x_i + (2 * random(\,) - 1) * stepc_i \end{array} \tag{4.3}$$

where:

x_i :	Value of the i^{th} variable prior to the neighborhood move.
x_i^* :	Value of the i^{th} variable after the neighborhood move.
random( ):	Random number generator, where *random*( )$\in(0,1)$.
$stepi_i$, $stepd_i$, $stepc_i$:	Step size for integer, discrete and real variables.
d_l :	The l th element of the discrete variable subset X^d.
integer[ ]:	Function to convert a real value to an integer value.

According to the types of variables used in the model, the appropriate movement strategies are used to generate a previously determined number of feasible, non-tabu, neighborhood solutions from the current seed solution (Baykasoğlu and Göçken, 2006).

***Selection of the current best solution vector*:** The selection of the current best solution vector from neighborhood solutions is performed simply as follows. For each neighborhood solution vector, the objective function values are calculated and then if the objective is maximization the one with the maximum value, if the objective is minimization the one with the minimum value is selected as the current best solution.

If the problem is a multiple objective optimization problem, selection of the current best solution is performed using the pareto optimality logic (Baykasoğlu, 2001). If the problem is a multiple objective optimization problem and priorities are given to the objectives, the current best solution is decided according to these priorities. The current best solution will be the solution that maximizes or minimizes the objective of first priority. If the values of the first prior objective for two feasible solutions are equal, the solution that maximizes or minimizes the second prior objective will be selected and so on.

***Updating the best-known solution vector*:** The initial feasible solution vector is also recorded as the best-known initial solution vector. In the subsequent iterations, the best solution vector for each method is updated as follows. The objective function value for the current best solution vector and the best-known solution vector are compared. The best-known solution vector is updated in the case of improvement. If the problem is a multiple objective optimization problem and priorities are given to the objectives, the first priority objective value for the current best solution vector and the best solution vector is checked. If the current best solution has the maximum/minimum objective function value, the best solution vector is updated and the current best solution will be the new best-known solution. If the objective function values for the first priority objective of the current best solution vector and the best-known solution vector are equal, the objective function values for second priority goal are checked. If the current best solution has the maximum/minimum objective function value, the best solution vector will be updated and the current best solution will be the new best-known solution, and so on.

***Tabu list*:** Accepted solutions for an arbitrarily defined number of previous moves are considered as tabu, because to allow one of them may trap the algorithm into cycling. The tabu list is circular; when it is full, a new item replaces the head of the list.

Aspiration criteria**:** Any move that improves the best-known solution is accepted, even if the move is tabu.

Termination**:** If a previously determined number of iterations is reached, or if there is no improvement in the best-known solution in the last *n* iterations the algorithm terminates.

4.4. Solution of Two Examples Using the Proposed Direct Solution Method

Two example problems are solved using the proposed direct solution method, a peak load pricing problem and a product mix problem. In each instance, different parameters of the problems are defined as triangular fuzzy numbers and solved using the proposed direct solution method.

4.4.1. Peak load pricing problem

A peak load pricing example is handled and solved using the proposed direct solution method to prove that fuzzy mathematical programming problems can be solved effectively by using ranking methods of fuzzy numbers without any necessity of transformation into crisp equivalent. As stated before, some information can be missed in transformation process and the obtained crisp equivalent can be nonlinear. When the obtained crisp equivalent is nonlinear, a meta-heuristics algorithm should be used again for the solution. Therefore, transformation might not be always advantageous. The handled peak load problem is a nonlinear problem, so after the transformation the obtained crisp problem will be a nonlinear problem.

The crisp mathematical programming model of the handled peak load pricing problem can be written as follows (Ribeiro and Varela, 2003).

$$\begin{aligned}
\max \quad & z = 60P - 0.5P^2 + 0.2FP + 40F - F^2 - 10C \\
\textit{subject to} & \\
& C + 0.5P - 0.1F \geq 60 \\
& C - 0.1P + F \geq 40 \\
& F, P, C \geq 0
\end{aligned} \tag{4.4}$$

The solution of the crisp mathematical programming model of the handled peak load pricing problem is F = 26.53, P = 70.31, C = 27.5 and objective function value z = 2202.3.

In the following subsections, different parameters of the peak load pricing problem are defined as triangular fuzzy numbers. In each instance, the problem is solved using the proposed direct solution method. The peak load pricing problem is solved by employing four different fuzzy ranking methods (the signed distance method, the integral value method, Chen and Chen's method and the ranking of fuzzy numbers through the comparison of their expected intervals) and the PSO algorithm. In order to solve the problem directly, the global best PSO algorithm is used. In the solution of the problem, the parameters of the algorithm are taken as follows; inertia weight w = 0.4, individual and sociality weights $c_1 = c_2$ = 1.4962, n = 20 and the number of iterations is 1000. For the solution of the problem computer programs are written in C language. For each ranking method, different computer programs are written for the solution of the problem. But, for the same ranking method for each instance only the data is changed. The ranking methods for fuzzy numbers can rank a fuzzy number with a crisp number and two crisp numbers with each other truly.

4.4.1.1. The peak load pricing problem with fuzzy objective function coefficients

In this stage, only the coefficients of the objective function are determined as triangular fuzzy numbers and solved with the proposed method. The peak load pricing problem with fuzzy objective function coefficients can be written in mathematical form as follows.

$$\max \quad z = (54; 60; 66)P - (0.45; 0.5; 0.55)P^2 + (0.18; 0.2; 0.22)FP$$
$$+ (36; 40; 44)F - (0.9; 1.0; 1.1)F^2 - (9; 10; 11)C$$

subject to

$$C + 0.5P - 0.1F \geq 60$$
$$C - 0.1P + F \geq 40 \qquad (4.5)$$
$$F, P, C \geq 0$$

The solution of the problem using the signed distance method is; F = 26.53, P = 70.31, C = 27.5 and the triangular possibility distribution of the objective function is z = (1982.066; 2002.296; 2422.525). Triangular possibility distribution of the objective function is determined after finding the optimal solution vector. The cost coefficients of the problem are triangular fuzzy numbers, so by calculating the objective function value by using minimum, middle and maximum points of the cost coefficients separately, the possibility distribution of the objective function can be obtained. The solution is same as the crisp solution.

The solution of the problem using the Chen and Chen's method is; is same as the solution obtained from the signed distance method and the crisp solution.

The problem is solved for different alpha values using the integral value method. The problem is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The obtained solutions and the possibility distributions of the objective for each α value are given in table 4.1.

Table 4.1. α-acceptable solutions of the peak load pricing problem with fuzzy objective function coefficients for the integral value method

Feasibility degree, α	Decision vector	Possibility distribution of the objective value, $\tilde{z}^0(\alpha)$
0.5	F = 26.54, P = 70.347, C = 27.48	(1982.066; 2202.295; 2422.524)
0.6	F = 26.54, P = 70.327, C = 27.49	(1982.066; 2202.296; 2422.525)
0.7	F = 26.54, P = 70.347, C = 27.48	(1982.066; 2202.295; 2422.524)
0.8	F = 26.54, P = 70.347, C = 27.48	(1982.066; 2202.295; 2422.524)
0.9	F = 26.533, P = 70.325, C = 27.49	(1982.066; 2202.296; 2422.525)
1.0	F = 26.531, P = 70.369, C = 27.468	(1982.065; 2202.294; 2422.523)

For different α values, the solutions differ slightly as seen in the table. The solutions are nearly the same with the solutions obtained from the signed distance and the Chen and Chen's method (Chen and Chen, 2003).

For the expected interval method the problem is solved for different alpha values. The problem is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The obtained solutions and the possibility distributions of the objective for each α value are given in table 4.2.

Table 4.2. α-acceptable solutions of the peak load pricing problem with fuzzy objective function coefficients for the expected interval method

Feasibility degree, α	Decision vector	Possibility distribution of the objective value, $\tilde{z}^0(\alpha)$
0.5	F = 26.522, P = 70.327, C = 27.488	(1982.066; 2202.296; 2422.525)
0.6	F = 28.357, P = 72.684, C = 27.229	(1970.679; 2189.643; 2408.607)
0.7	F = 25.393, P = 69.149, C = 28.463	(1976.048; 2195.608; 2415.169)
0.8	F = 31.398, P = 81.483, C = 25.817	(1883.553; 2092.837; 2302.121)
0.9	F = 28.983, P = 70.061, C = 30.298	(1954.643; 2171.826; 2389.009)
1.0	F = 32.531, P = 63.285, C = 37.101	(1870.478; 2078.309; 2286.140)

The solutions obtained by using the expected interval method is different from the solutions obtained by using the other methods.

4.4.1.2. The peak load pricing problem with fuzzy right hand values

In this stage, only the right hand values of the constraints are defined as triangular fuzzy numbers and solved with the proposed method. The peak load pricing problem with fuzzy right hand values of the constraints can be written in mathematical form as follows.

$$\begin{aligned}
&\max \quad z = 60P - 0.5P^2 + 0.2FP + 40F - F^2 - 10C \\
&\textit{subject to} \\
&\qquad C + 0.5P - 0.1F \geq (54;60;66) \\
&\qquad C - 0.1P + F \geq (36;40,44) \\
&\qquad F, P, C \geq 0
\end{aligned} \tag{4.6}$$

The solution of the problem obtained using the signed distance method is; $F = 26.516$, $P = 70.151$, $C = 27.576$ and the objective function value is $z = 2002.284$.

The solution of the problem obtained using the Chen and Chen's method is the same as the solution obtained using the signed distance method.

The solutions of the problem obtained using the integral value method for each α value are given in table 4.3.

Table 4.3. α-acceptable solutions of the peak load pricing problem with fuzzy right hand values for the integral value method

Feasibility degree, α	Decision vector	The objective function value
0.5	$F = 26.525$, $P = 70.258$, $C = 27.523$	2202.295
0.6	$F = 26.531$, $P = 70.272$, $C = 28.117$	2196.295
0.7	$F = 26.518$, $P = 70.284$, $C = 28.709$	2190.296
0.8	$F = 26.533$, $P = 70.171$, $C = 29.367$	2184.287
0.9	$F = 26.523$, $P = 70.266$, $C = 29.919$	2178.295
1.0	$F = 26.533$, $P = 70.298$, $C = 30.504$	2172.296

For the expected interval method the problem is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The obtained solutions and the objective fuction value for each α value are given in table 4.4.

Table 4.4. α-acceptable solutions of the peak load pricing problem with fuzzy right hand values for the expected interval method

Feasibility degree, α	Decision vector	The objective function value
0.5	$F = 26.526$, $P = 70.258$, $C = 27.523$	2202.295
0.6	$F = 26.531$, $P = 70.272$, $C = 28.117$	2196.295
0.7	$F = 26.518$, $P = 70.284$, $C = 28.709$	2190.296
0.8	$F = 26.533$, $P = 70.171$, $C = 29.367$	2184.287
0.9	$F = 26.523$, $P = 70.266$, $C = 29.919$	2178.295
1.0	$F = 26.533$, $P = 70.298$, $C = 30.504$	2172.296

The solutions obtained by using the expected interval method is same as the solutions obtained by using the integral value method.

4.4.1.3. The peak load pricing problem with fuzzy left hand values

In this stage, only the left hand values of the constraints are defined as triangular fuzzy numbers and solved with the proposed method. The peak load pricing problem with fuzzy coefficients of the constraints can be written in mathematical form as follows.

$$\begin{aligned}
&\max \quad z = 60P - 0.5P^2 + 0.2FP + 40F - F^2 - 10C \\
&\textit{subject to} \\
&\qquad (0.9;1.0;1.1)C + (0.45;0.5;0.55)P - (0.09,0.1;0.11)F \geq 60 \\
&\qquad (0.9;1.0;1.1)C - (0.09;0.1;0.11)P + (0.9;1.0;1.1)F \geq 40 \\
&\qquad\qquad F, P, C \geq 0
\end{aligned} \tag{4.7}$$

The solution of the peak load pricing problem with fuzzy left hand values obtained using the signed distance method is; F = 26.537, P = 70.316, C = 27.495 and the objective function value is z = 2002.296.

The solution of the peak load pricing problem with fuzzy left hand values obtained using the Chen and Chen's method (Chen and Chen, 2003) is; F = 26.539, P = 70.364, C = 27.471 and the objective function value is z = 2002.294. It is slightly different from the solution obtained using the signed distance method.

For the integral value method the peak load pricing problem with fuzzy left hand values is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The solutions of the problem obtained using the integral value method for each α value are given in table 4.5.

Table 4.5. α-acceptable solutions of the peak load pricing problem with fuzzy left hand values for the integral value method

Feasibility degree, α	Decision vector	The objective function value
0.5	F = 26.534, P = 70.341, C = 27.483	2202.295
0.6	F = 26.534, P = 70.297, C = 26.910	2208.237
0.7	F = 26.529, P = 70.378, C = 26.287	2214.058
0.8	F = 26.529, P = 70.268, C = 25.771	2219.771
0.9	F = 26.525, P = 70.297, C = 25.196	2225.373
1.0	F = 26.528, P = 70.328, C = 24.631	2230867

For the expected interval method the peak load pricing problem with fuzzy left hand values is solved for α = {0.5; 0.6; 0.7; 0.8; 0.9; 1.0}. The obtained solutions and the objective fuction value for each α value are given in table 4.6.

Table 4.6. α-acceptable solutions of the peak load pricing problem with fuzzy left hand values for the expected interval method

Feasibility degree, α	Decision vector	The objective function value
0.5	F = 26.528, P = 70.332, C = 27.486	2202.296
0.6	F = 26.529, P = 70.293, C = 28.112	2196.235
0.7	F = 26.522, P = 70.232, C = 28.761	2190.048
0.8	F = 26.536, P = 70.283, C = 29.367	2183.739
0.9	F = 26.528, P = 70.293, C = 30.006	2177.296
1.0	F = 26.527, P = 70.306, C = 30.658	2170.717

The solutions obtained by using the four ranking methods are different from each other.

4.4.1.4. The peak load pricing problem with fuzzy left and right hand values

In this stage, all the parameters of the constraints are defined as triangular fuzzy numbers and solved with the proposed method. The peak load pricing problem with fuzzy left and right hand values can be written in mathematical form as follows.

$$\max \quad z = 60P - 0.5P^2 + 0.2FP + 40F - F^2 - 10C$$

subject to

$$\begin{aligned}
&(0.9;1.0;1.1)C + (0.45;0.5;0.55)P - (0.09,0.1;0.11)F \geq (54;\ 60;66) \qquad (4.8)\\
&(0.9;1.0;1.1)C - (0.09;0.1;0.11)P + (0.9;1.0;1.1)F \geq (36;\ 40;44)\\
&F, P, C \geq 0
\end{aligned}$$

The solution of the peak load pricing problem with fuzzy left and right hand values obtained using the signed distance method is; F = 26.533, P = 70.253, C = 27.527 and the objective function value is z = 2002.294.

The solution of the peak load pricing problem with fuzzy left and right hand values obtained using the Chen and Chen's method is; F = 26.538, P = 70.299, C = 27.504 and the objective function value is z = 2002.296. It is slightly different from the solution obtained using the signed distance method.

For the integral value method the peak load pricing problem with fuzzy left and right hand values is solved for α = {0.5; 0.6; 0.7; 0.8; 0.9; 1.0}. The solutions of the problem obtained using the integral value method for each α value are given in table 4.7.

Table 4.7. α-acceptable solutions of the peak load pricing problem with fuzzy left and right hand values for the integral value method

Feasibility degree, α	Decision vector	The objective function value
0.5	F = 26.532, P = 70.274, C = 27.516	2202.295
0.6	F = 26.532, P = 70.274, C = 27.516	2202.295
0.7	F = 26.532, P = 70.274, C = 27.516	2202.295
0.8	F = 26.532, P = 70.274, C = 27.516	2202.295
0.9	F = 26.532, P = 70.274, C = 27.516	2202.295
1.0	F = 26.532, P = 70.278, C = 27.513	2202.296

For the integral value method except α value 1.0, for all α values the same solutions are obtained. Besides, the solution for α value 1.0 is very slightly different from others.

For the expected interval method the peak load pricing problem with fuzzy left and right hand values is solved for α = {0.5; 0.6; 0.7; 0.8; 0.9; 1.0}. The obtained solutions and the objective fuction value for each α value are given in table 4.8.

Table 4.8. α-acceptable solutions of the peak load pricing problem with fuzzy left and right hand values for the expected interval method

Feasibility degree, α	Decision vector	The objective function value
0.5	F = 26.532, P = 70.278, C = 27.513	2202.296
0.6	F = 26.533, P = 70.303, C = 28.713	2190.175
0.7	F = 26.528, P = 70.267, C = 29.968	2177.805
0.8	F = 26.539, P = 70.260, C = 31.235	2165.181
0.9	F = 26.514, P = 70.232, C = 32.535	2152.293
1.0	F = 26.531, P = 70.314, C = 33.811	2139.138

The solutions obtained by using the four ranking methods are different from each other.

4.4.1.5. The peak load pricing problem with fuzzy parameters

In this stage, all of the parameters of the peak load pricing problem are defined as triangular fuzzy numbers and solved with the proposed method. The peak load pricing problem with fuzzy parameters can be written in mathematical form as follows.

$$\begin{aligned}
\max \quad & z = (54;\ 60;66)P - (0.45;0.5;0.55)P^2 + (0.18;\ 0.2;0.22)FP \\
& \quad + (36;40;44)F - (0.9;1.0;1.1)F^2 - (9;\ 10;11)C \\
\textit{subject to} \quad & \\
& (0.9;1.0;1.1)C + (0.45;0.5;0.55)P - (0.09,0.1;0.11)F \geq (54;\ 60;66) \\
& (0.9;1.0;1.1)C - (0.09;0.1;0.11)P + (0.9;1.0;1.1)F \geq (36;\ 40;44) \\
& F, P, C \geq 0
\end{aligned} \tag{4.9}$$

The solution of the peak load pricing problem with fuzzy parameters obtained using the signed distance method is; F = 26.533, P = 70.253, C = 27.527 and the triangular possibility distribution of the objective function is z = (1982.065; 2002.294;

2422.524). This solution is same as the solution of the peak load pricing problem with fuzzy left and right hand values obtained using the signed distance method.

The solution of the peak load pricing problem with fuzzy parameters obtained using the Chen and Chen's method is; F = 26.532, P = 70.247, C = 27.529 and the triangular possibility distribution of the objective function is z = (1982.065; 2002.294; 2422.523). It is slightly different from the solution obtained using the signed distance method.

For the integral value method the peak load pricing problem with fuzzy parameters is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The solutions of the problem obtained using the integral value method for each α value are given in table 4.9.

Table 4.9. α-acceptable solutions of the peak load pricing problem with fuzzy parameters for the integral value method

Feasibility degree, α	Decision vector	Possibility distribution of the objective value, $\tilde{z}^0(\alpha)$
0.5	F = 26.532, P = 70.274, C = 27.516	(1982.066; 2202.295; 2422.525)
0.6	F = 26.532, P = 70.274, C = 27.516	(1982.066; 2202.295; 2422.525)
0.7	F = 26.529, P = 70.267, C = 27.519	(1982.066; 2202.295; 2422.524)
0.8	F = 26.529, P = 70.267, C = 27.519	(1982.066; 2202.295; 2422.524)
0.9	F = 26.529, P = 70.267, C = 27.519	(1982.066; 2202.295; 2422.524)
1.0	F = 26.529, P = 70.293, C = 27.506	(1982.066; 2202.296; 2422.525)

For the integral value method for α values 0.5 and 0.6, for α values 0.7, 0.8 and 0.9 the same solutions are obtained. Besides, the solution for α value 1.0 is very slightly different from others. For α values 0.5 and 0.6, the solutions are same as the solutions of the peak load pricing problem with fuzzy left and right hand values.

For the expected interval method the peak load pricing problem with fuzzy parameters is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The obtained solutions and the objective function value for each α value are given in table 4.10.

Table 4.10. α-acceptable solutions of the peak load pricing problem with fuzzy parameters for the expected interval method

Feasibility degree, α	Decision vector	Possibility distribution of the objective value, $\tilde{z}^0(\alpha)$
0.5	F = 26.534, P = 70.247, C = 27.529	(1982.065; 2202.294; 2422.523)
0.6	F = 27.568, P = 67.569, C = 33.479	(19362.645; 2151.827; 2367.010)
0.7	F = 33.920, P = 67.714, C = 34.324	(1883.348; 2092.608; 2301.869)
0.8	F = 36.275, P = 73.037, C = 35.239	(1824.872; 2027.635; 2230.399)
0.9	F = 38.070, P = 80.259, C = 35.058	(1735.893; 1928.770; 2121.647)
1.0	F = 32.531, P = 63.285, C = 43.101	(1816.478; 2018.309; 2220.140)

The solutions obtained by using the expected interval method are very different from the solutions obtained by using the other methods.

4.4.2. The product mix problem

A product mix problem is solved using the proposed direct solution method in this subsection. In the problem, three products are produced and these three products are processed through three departments. The constraints are time capacity constraints for the departments and it is wanted to determine the number of units to produce for each product to maximize the revenue. The crisp mathematical programming model of the product mix problem can be written as follows (Buckley et al., 2002).

$$\begin{aligned} \max \quad & z = 6x_1 + 8x_2 + 6x_3 \\ \textit{subject to} \quad & \\ & 6x_1 + 8x_2 + 3x_3 \leq 288 \\ & 12x_1 + 8x_2 + 6x_3 \leq 312 \\ & 2x_1 + 4x_2 + x_3 \leq 124 \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{4.10}$$

The solution of the crisp mathematical programming model of the handled product mix problem is $x_1 = 0$, $x_2 = 27$, $x_3 = 16$ and objective function value $z = 312$.

In the following subsections, different parameters of the product mix problem are defined as triangular fuzzy numbers. In each instance, the problem is solved using the proposed direct solution method. The product mix problem is solved by employing four different fuzzy ranking methods and the PSO algorithm. In order to solve the problem directly, the global best PSO algorithm is used. In the solution of the problem, the parameters of the algorithm are taken as follows; inertia weight w = 0.4, individual and sociality weights $c_1 = c_2$ = 1.4962, the number of particles n = 20 and the number of iterations is 1000. For the solution of the problem computer programs are written in C language. For each ranking method, different computer programs are written for the solution of the problem.

4.4.2.1. The product mix problem with fuzzy objective function coefficients

In this stage, only the coefficients of the objective function of the product mix problem are determined as triangular fuzzy numbers and solved with the proposed method. The product mix problem with fuzzy objective function coefficients can be written in mathematical form as follows.

$$\begin{aligned}
&\max \quad z = (5.8;6;6.2)x_1 + (7.5;8;8.5)x_2 + (5.6;6;6.4)x_3 \\
&\textit{subject to} \\
&\qquad 6x_1 + 8x_2 + 3x_3 \le 288 \\
&\qquad 12x_1 + 8x_2 + 6x_3 \le 312 \\
&\qquad 2x_1 + 4x_2 + x_3 \le 124 \\
&\qquad x_1, x_2, x_3 \ge 0
\end{aligned} \tag{4.11}$$

The solution of the product mix problem with fuzzy objective function coefficients using the signed distance method is; X_1 = 0, X_2 = 24.141, X_3 = 19.812 and the triangular possibility distribution of the objective function is z = (292.005; 312.000; 331.995). The solution of the problem using the Chen and Chen's method is same as the solution obtained from the signed distance method.

The product mix problem with fuzzy objective function coefficients is solved for different alpha values using the integral value method. The problem is solved for α = {0.5; 0.6; 0.7; 0.8; 0.9; 1.0}. The obtained solutions and the possibility distributions of the objective for each α value are given in table 4.11.

Table 4.11. α-acceptable solutions of the product mix problem with fuzzy objective function coefficients for the integral value method

Feasibility degree, α	Decision vector	Possibility distribution of the objective value, $\tilde{z}^0(\alpha)$
0.5	$X_1 = 0, X_2 = 12.5891, X_3 = 35.2145$	(291.620; 312.000; 332.380)
0.6	$X_1 = 0, X_2 = 12.5896, X_3 = 35.2139$	(291.620; 312.000; 332.380)
0.7	$X_1 = 0, X_2 = 12.5891, X_3 = 35.2146$	(291.620; 312.000; 332.380)
0.8	$X_1 = 0, X_2 = 12.5765, X_3 = 35.2314$	(291.619; 312.000; 332.381)
0.9	$X_1 = 0, X_2 = 8.7694, X_3 = 40.3074$	(291.492; 312.000; 332.508)
1.0	$X_1 = 0.0009, X_2 = 3.1449, X_3 = 47.8051$	(291.300; 311.995; 332.689)

For different α values, the solutions differ slightly as seen in the table. The solutions are nearly the same with each other for different α values.

For the expected interval method the product mix problem with fuzzy objective function coefficients is solved for different alpha values. The problem is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The obtained solutions and the possibility distributions of the objective for each α value are given in table 4.12.

Table 4.12. α-acceptable solutions of the product mix problem with fuzzy objective function coefficients for the expected interval method

Feasibility degree, α	Decision vector	Possibility distribution of the objective value, $\tilde{z}^0(\alpha)$
0.5	$X_1 = 0, X_2 = 24.1412, X_3 = 19.8118$	(292.005; 312.000; 331.995)
0.6	$X_1 = 0.0591, X_2 = 24.5409, X_3 = 19.0027$	(290.814; 310.698; 330.581)
0.7	$X_1 = 0.8580, X_2 = 24.2326, X_3 = 17.6937$	(285.805; 305.171; 324.536)
0.8	$X_1 = 0.8580, X_2 = 24.2326, X_3 = 17.6937$	(285.805; 305.171; 324.536)
0.9	$X_1 = 2.2467, X_2 = 23.9888, X_3 = 15.2532$	(278.365; 296.910; 315.455)
1.0	$X_1 = 2.2467, X_2 = 23.9888, X_3 =1\ 5.2532$	(278.365; 296.910; 315.455)

The solutions obtained by using the integral value method are different from the solutions obtained by using the other methods. The solutions obtained by using the signed distance method, Chen and Chen's method and the expected interval method for $\alpha = 0.5$ are same.

4.4.2.2. The product mix problem with fuzzy right hand values

In this stage, only the right hand values of the constraints of the product mix problem are defined as triangular fuzzy numbers and solved with the proposed method. The product mix problem with fuzzy right hand values of the constraints can be written in mathematical form as follows.

$$\begin{aligned}
\max \quad & z = 6x_1 + 8x_2 + 6x_3 \\
\textit{subject to} \quad & \\
& 6x_1 + 8x_2 + 3x_3 \leq (283;288;293) \\
& 12x_1 + 8x_2 + 6x_3 \leq (306;312;318) \\
& 2x_1 + 4x_2 + x_3 \leq (121;124;127) \\
& x_1, x_2, x_3 \geq 0
\end{aligned} \tag{4.12}$$

The solution of the product mix problem with fuzzy right hand values obtained using the signed distance method is; X_1 = 0, X_2 = 19.6643, X_3 = 25.7810 and the objective function value is z = 312.000. The solution of the problem obtained using the Chen and Chen's method is same as the solution obtained using the signed distance method.

The product mix problem with fuzzy right hand values is solved for different α = {0.5; 0.6; 0.7; 0.8; 0.9; 1.0} using the integral value method. The obtained solutions and the objective function values for each α value are given in table 4.13.

Table 4.13. α-acceptable solutions of the product mix problem with fuzzy right hand values for the integral value method

Feasibility degree, α	Decision vector	The objective function value
0.5	X_1 = 0, X_2 = 22.2248, X_3 = 22.3670	312.000
0.6	X_1 = 0, X_2 = 26.2476, X_3 = 17.1032	312.600
0.7	X_1 = 0, X_2 = 21.7944, X_3 = 23.1408	313.200
0.8	X_1 = 0, X_2 = 21.8811, X_3 = 23.1252	313.800
0.9	X_1 = 0, X_2 = 22.1201, X_3 = 22.9066	314.400
1.0	X_1 = 0, X_2 = 20.7411, X_3 =24.8451	315.000

For different α values, different solutions are obtained using the integral value method.

For the expected interval method the product mix problem with fuzzy right hand values is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The obtained solutions and the objective function value for each α value are given in table 4.14.

Table 4.14. α-acceptable solutions of the product mix problem with fuzzy right hand values for the expected interval method

Feasibility degree, α	Decision vector	The objective function value
0.5	$X_1 = 0, X_2 = 19.6641, X_3 = 25.7812$	312.000
0.6	$X_1 = 0, X_2 = 21.3210, X_3 = 23.4720$	311.400
0.7	$X_1 = 0, X_2 = 17.7334, X_3 = 28.1554$	310.800
0.8	$X_1 = 0, X_2 = 20.7038, X_3 = 24.0949$	310.200
0.9	$X_1 = 0, X_2 = 19.3631, X_3 = 25.7826$	309.600
1.0	$X_1 = 0, X_2 = 19.6799, X_3 =25.2601$	309.000

The solutions obtained by using the signed distance method, Chen and Chen's method and the expected interval method for $\alpha = 0.5$ are nearly the same.

4.4.2.3. The product mix problem with fuzzy left hand values

In this stage, only the left hand values of the constraints of the product mix problem are defined as triangular fuzzy numbers and solved with the proposed method. The product mix problem with fuzzy coefficients of the constraints can be written in mathematical form as follows.

$$\begin{aligned}
&\max \quad z = 6x_1 + 8x_2 + 6x_3 \\
&\textit{subject to} \\
&\qquad (5.6;6;6.4)x_1 + (7.5;8;8.5)x_2 + (2.8;3;3.2)x_3 \le 288 \\
&\qquad (11.4;12;12.6)x_1 + (7.6;8;8.4)x_2 + (5.7;6;6.3)x_3 \le 312 \\
&\qquad (1.8;\ 2;2.2)x_1 + (3.8;4;4.2)x_2 + (0.9;1.0;1.1)x_3 \le 124 \\
&\qquad x_1, x_2, x_3 \ge 0
\end{aligned} \tag{4.13}$$

The solution of the product mix problem with fuzzy left hand values obtained using the signed distance method is; $X_1 = 0$, $X_2 = 13.4892$, $X_3 = 34.0144$ and the objective function value is $z = 312.000$. The solution of the problem obtained using the Chen and Chen's method is same as the solution obtained using the signed distance method.

The product mix problem with fuzzy left hand values is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$ using the integral value method. The obtained solutions and the objective function values for each α value are given in table 4.15.

Table 4.15. α-acceptable solutions of the product mix problem with fuzzy left hand values for the integral value method

Feasibility degree, α	Decision vector	The objective function value
0.5	$X_1 = 0, X_2 = 18.7373, X_3 = 27.0169$	312.000
0.6	$X_1 = 0, X_2 = 14.0802, X_3 = 32.9677$	310.448
0.7	$X_1 = 0, X_2 = 12.9824, X_3 = 34.1753$	308.911
0.8	$X_1 = 0, X_2 = 19.5804, X_3 = 25.1244$	307.389
0.9	$X_1 = 0, X_2 = 13.5666, X_3 = 32.8916$	305.882
1.0	$X_1 = 0, X_2 = 12.1909, X_3 = 34.4772$	304.390

For different α values, different solutions are obtained using the integral value method.

For the expected interval method the product mix problem with fuzzy left hand values is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The obtained solutions and the objective function value for each α value are given in table 4.16.

Table 4.16. α-acceptable solutions of the product mix problem with fuzzy left hand values for the expected interval method

Feasibility degree, α	Decision vector	The objective function value
0.5	$X_1 = 0, X_2 = 13.4892, X_3 = 34.0144$	312.000
0.6	$X_1 = 0, X_2 = 14.1450, X_3 = 32.8812$	310.448
0.7	$X_1 = 0, X_2 = 26.0091, X_3 = 16.8064$	308.911
0.8	$X_1 = 0, X_2 = 12.1460, X_3 = 35.0369$	307.389
0.9	$X_1 = 0, X_2 = 12.2193, X_3 = 34.6880$	305.882
1.0	$X_1 = 0, X_2 = 0.5983, X_3 = 49.9340$	304.390

For different α values, different solutions are obtained using the expected interval method. The solutions obtained by using the signed distance method, Chen and Chen's method and the expected interval method for $\alpha = 0.5$ are same.

4.4.2.4. The product mix problem with fuzzy left and right hand values

In this stage, all the parameters of the constraints of the product mix problem are defined as triangular fuzzy numbers and solved with the proposed method. The product mix problem with fuzzy left and right hand values can be written in mathematical form as follows.

$$
\begin{aligned}
&\max \quad z = 6x_1 + 8x_2 + 6x_3 \\
&\textit{subject to} \\
&\qquad (5.6;6;6.4)x_1 + (7.5;8;8.5)x_2 + (2.8;3;3.2)x_3 \le (283;288;293) \\
&\qquad (11.4;12;12.6)x_1 + (7.6;8;8.4)x_2 + (5.7;6;6.3)x_3 \le (306;312;318) \\
&\qquad (1.8;2;2.2)x_1 + (3.8;4;4.2)x_2 + (0.9;1.0;1.1)x_3 \le (121;124;127) \\
&\qquad x_1, x_2, x_3 \ge 0
\end{aligned} \tag{4.14}
$$

The solution of the product mix problem with fuzzy left and right hand values obtained using the signed distance method is; $X_1 = 0$, $X_2 = 19.1466$, $X_3 = 26.4712$ and the objective function value is $z = 312.000$. The solution of the problem obtained using the Chen and Chen's method is same as the solution obtained using the signed distance method.

The product mix problem with fuzzy left and right hand values is solved for α = {0.5; 0.6; 0.7; 0.8; 0.9; 1.0} using the integral value method. The obtained solutions and the objective function values for each α value are given in table 4.17.

Table 4.17. α-acceptable solutions of the product mix problem with fuzzy left and right hand values for the integral value method

Feasibility degree, α	Decision vector	The objective function value
0.5	$X_1 = 0$, $X_2 = 25.9560$, $X_3 = 17.3921$	312.000
0.6	$X_1 = 0$, $X_2 = 26.3263$, $X_3 = 16.7391$	311.045
0.7	$X_1 = 0$, $X_2 = 26.8014$, $X_3 = 15.9479$	310.099
0.8	$X_1 = 0$, $X_2 = 26.7328$, $X_3 = 15.8834$	309.163
0.9	$X_1 = 0$, $X_2 = 25.4564$, $X_3 = 17.4306$	308.235
1.0	$X_1 = 0$, $X_2 = 26.4571$, $X_3 = 15.9434$	307.317

For different α values, different solutions are obtained using the integral value method. As seen from the table, the objective function value decreases as the α value increases.

For the expected interval method the product mix problem with fuzzy left and right hand values is solved for $\alpha = \{0.5;\ 0.6;\ 0.7;\ 0.8;\ 0.9;\ 1.0\}$. The obtained solutions and the objective function value for each α value are given in table 4.18.

Table 4.18. α-acceptable solutions of the product mix problem with fuzzy left and right hand values for the expected interval method

Feasibility degree, α	Decision vector	The objective function value
0.5	$X_1 = 0, X_2 = 19.1466, X_3 = 26.4712$	312.000
0.6	$X_1 = 0, X_2 = 18.1533, X_3 = 27.4373$	309.851
0.7	$X_1 = 0, X_2 = 22.1857, X_3 = 21.7062$	307.723
0.8	$X_1 = 0, X_2 = 25.7349, X_3 = 16.6228$	305.616
0.9	$X_1 = 0, X_2 = 25.9221, X_3 = 16.0254$	303.529
1.0	$X_1 = 0, X_2 = 25.8301, X_3 = 15.8037$	301.463

For different α values, different solutions are obtained using the expected interval method. As seen from the table, the objective function value decreases as the α value increases. The solutions obtained by using the signed distance method, Chen and Chen's method and the expected interval method for $\alpha = 0.5$ are same.

4.4.1.5. The product mix problem with fuzzy parameters

In this stage, all of the parameters of the product mix problem are defined as triangular fuzzy numbers and solved with the proposed method. The product mix problem with fuzzy parameters can be written in mathematical form as follows.

$$
\begin{aligned}
&\max \quad z = (5.8;6;6.2)x_1 + (7.5;8;8.5)x_2 + (5.6;6;6.4)x_3 \\
&\textit{subject to} \\
&\quad (5.6;6;6.4)x_1 + (7.5;8;8.5)x_2 + (2.8;3;3.2)x_3 \le (283;288;293) \\
&\quad (11.4;12;12.6)x_1 + (7.6;8;8.4)x_2 + (5.7;6;6.3)x_3 \le (306;312;318) \\
&\quad (1.8;2;2.2)x_1 + (3.8;4;4.2)x_2 + (0.9;1.0;1.1)x_3 \le (121;124;127) \\
&\quad x_1, x_2, x_3 \ge 0
\end{aligned}
\tag{4.15}
$$

The solution of the product mix problem with fuzzy parameters obtained using the signed distance method is; X_1 = 0, X_2 = 19.1466, X_3 = 26.4712 and the triangular possibility distribution of the objective function is z = (291.838; 312.000; 332.162). The solution of the problem obtained using the Chen and Chen's method is same as the solution obtained using the signed distance method. This solution is same as the solution of the product mix problem with fuzzy left and right hand values obtained using the signed distance method and the Chen and Chen's method.

The product mix problem with fuzzy parameters is solved for α = {0.5; 0.6; 0.7; 0.8; 0.9; 1.0} using the integral value method. The obtained solutions and the triangular possibility distribution of the objective function for each α value are given in table 4.19.

Table 4.19. α-acceptable solutions of the product mix problem with fuzzy parameters for the integral value method

Feasibility degree, α	Decision vector	Possibility distribution of the objective value, $\tilde{z}^0(\alpha)$
0.5	X_1 = 0, X_2 = 25.9560, X_3 = 17.3921	(292.065; 312.000; 331.935)
0.6	X_1 = 0, X_2 = 26.3262, X_3 = 16.7392	(291.186; 311.045; 330.904)
0.7	X_1 = 0, X_2 = 26.8014, X_3 = 15.9480	(290.319; 310.099; 329.879)
0.8	X_1 = 0.0018, X_2 = 3.8664, X_3 = 46.3681	(288.670; 309.151; 329.632)
0.9	X_1 = 0, X_2 = 25.4564, X_3 = 17.4307	(288.535; 308.235; 327.936)
1.0	X_1 = 0, X_2 = 26.4571, X_3 = 15.9434	(287.711; 307.317; 326.923)

The solutions obtained for each α value except α = 0.8 are same as the solutions of the product mix problem with fuzzy left and right hand values obtained using the integral value method.

The product mix problem with fuzzy parameters is solved using the expected interval method for α = {0.5; 0.6; 0.7; 0.8; 0.9; 1.0}. The obtained solutions and the triangular possibility distribution of the objective function value for each α value are given in table 4.20.

Table 4.20. α-acceptable solutions of the product mix problem with fuzzy parameters for the expected interval method

Feasibility degree, α	Decision vector	Possibility distribution of the objective value, $\tilde{z}^0(\alpha)$
0.5	$X_1 = 0, X_2 = 19.1466, X_3 = 26.4713$	(291.838; 312.000; 332.162)
0.6	$X_1 = 0.0323, X_2 = 11.4793, X_3 = 36.1552$	(288.751; 308.959; 329.167)
0.7	$X_1 = 0.4776, X_2 = 8.0200, X_3 = 39.5244$	(284.257; 304.172; 324.087)
0.8	$X_1 = 0.1147, X_2 = 10.8593, X_3 = 35.9602$	(283.488; 303.324; 323.161)
0.9	$X_1 = 1.6433, X_2 = 23.1035, X_3 = 16.1113$	(273.031; 291.356; 309.681)
1.0	$X_1 = 4.1314, X_2 = 8.6822, X_3 = 30.2628$	(258.551; 275.823; 293.096)

For different α values, different solutions are obtained using the expected interval method. As seen from the table, the objective function value decreases as the α value increases. The solutions obtained by using the signed distance method, Chen and Chen's method and the expected interval method for $\alpha = 0.5$ are same.

4.5. Conclusion

In this chapter, the proposed direct solution method is presented. In the literature, fuzzy mathematical programming problems are generally solved after a transformation process into crisp equivalent. This transformation process can be hard and after transformation some information can be missed. In literature, the fuzzy parameters' membership functions are accepted as linear and in this way the transformation process can be applied. For example, the ranking functions are used for transformation into crisp equivalent and usually the fuzzy parameters are defined as triangular or trapezoidal fuzzy number. The ranking methods for triangular or trapezoidal fuzzy numbers are simple. But, when the fuzzy numbers are defined in different shapes, mathematically, the ranking methods not easy to use in transformation process. Using the proposed direct solution method, a fuzzy mathematical programming problem can be solved without any necessity of transformation into its crisp equivalent. So, the fuzzy parameters can be defined in different shapes. In the proposed direct solution method, to rank the objective function values and to determine the feasibility of the constraints ranking methods for fuzzy numbers are used. A metaheuristic algorithm is used to carry out the ranking process. In this chapter, two mathematical programming problems are solved using the proposed direct solution method. The problems are solved by the proposed direct solution method using four different ranking methods of fuzzy numbers. The problems are solved by defining different parameters of the problems as fuzzy triangular fuzzy numbers. From the solutions, it has been observed that fuzzy decision making problems can be solved effectively by using ranking methods of fuzzy numbers without any necessity of transformation into crisp equivalent.

CHAPTER 5

APPLICATION OF THE PROPOSED DIRECT SOLUTION METHOD TO SOME INDUSTRIAL ENGINEERING PROBLEMS

5.1. Introduction

In this chapter, a fuzzy multi-item economic order quantity problem and a fuzzy multi-objective aggregate production planning problem is solved using the proposed direct solution method. The solution of the fuzzy multi-item economic order quantity problem is given in section 5.2 and the solution of the fuzzy multi-objective aggregate production planning problem is given in section 5.3.

5.2. Solution of Fuzzy Multi-item Economic Order Quantity Problem

5.2.1. The fuzzy multi-item economic order quantity problem

Inventory management is very important for many service and manufacturing industries. A proper control of inventory can significantly enhance a company's profit. The purpose of the economic order quantity (EOQ) model is to find the optimal order quantity of inventory items at each time such that the combination of the order cost and the stock cost is minimal (Wang et al., 2007). EOQ models are used for determining the quantity of item(s) to purchase from suppliers or to process through a production facility (Stockton and Quinn, 1993). Inventory management is used to decide when and how much to replenish the companies' inventory under a minimum of total cost. An EOQ model can be defined as only the minimization of the cost function or minimization of cost function under limitations like that budget, warehouse space, number of orders, etc.

There are a variety of EOQ models available and all originate from the classical EOQ model (Stockton and Quinn, 1993). In reality, it is very hard to define parameters of the EOQ model precisely. Moreover, it is very hard to estimate the probability distribution of these parameters due to a lack of historical data. Instead, these parameters are often estimated based on experience and subjective managerial judgment (Wang et al., 2007). However, these nonstochastic and illformed inventory models can be realistically represented in the fuzzy environment (Roy and Maiti, 1997).

In the literature, various authors solved different fuzzy versions of EOQ problem. Lee and Yao (1999) have solved the EOQ problem without backorder in which the order quantity is defined as a fuzzy variable. The fuzzy order quantity is defined as a normal triangular fuzzy number. Because the order quantity is defined as fuzzy variable, the cost function will be fuzzy and the economic order quantity is generated by defuzzification with the centroid method.

Yao and Su (2000) have studied fuzzy inventory with backorder in which demand is defined as fuzzy. In their article, the fuzzy demand is defined in three different forms. First the fuzzy demand is defined as triangular fuzzy number, in second form the fuzzy demand is defined with interval-valued fuzzy set based on two triangular fuzzy numbers, and in third form the fuzzy demand is defined with interval-valued fuzzy set based on two trapezoidal fuzzy numbers. Then, the economic order quantity is generated by defuzzification with the centroid method.

Wang et al. (2007) have solved the fuzzy EOQ problem without backordering. Wang et al. (2007) have characterized the order cost and the stock cost as fuzzy variables and constructed a fuzzy expected value model and a fuzzy dependent chance programming model. They have designed fuzzy simulations and a PSO algorithm based on the fuzzy simulation.

Syed and Aziz (2007) have studied an inventory model without shortages. The ordering cost and the holding cost are represented by fuzzy triangular numbers and fuzzy total cost is obtained by applying the sign distance method.

Roy and Maiti (1997) solved fuzzy EOQ model with limited storage capacity where unit price varies inversely with the demand and the setup cost varies with the quantity produced. In the model, the objective function and right hand of the constraint are introduced as fuzzy. The membership functions of the fuzzy parameters were assumed to be non-increasing continuous linear membership functions. The fuzzy nonlinear model was transformed into a crisp nonlinear model by using the max-min method.

Roy and Maiti (1998) solved fuzzy multi-item inventory models of deteriorating items with stock-dependent demand. Two fuzzy models were constructed for the inventory problem. In the first fuzzy model, the objectives and the right hand side value of one of the constraints were accepted as fuzzy numbers. In the second fuzzy model, some of the parameters and the objective function were considered as fuzzy. The fuzzy parameters were defined as triangular fuzzy numbers and the membership functions of the goals of the objectives were defined as linear. The fuzzy models were solved with two different methods, fuzzy nonlinear programming method based on Zimmermann's max-min method and the method of Lee and Li (1993) and fuzzy additive goal programming method of Tiwari et al. (1987), and the solutions are compared.

Mondal and Maiti (2002) have solved multi-item fuzzy economic order quantity model with fuzzy objective function and fuzzy parameters. The membership functions of the fuzzy parameters were assumed to be non-increasing continuous linear membership functions. The fuzzy nonlinear model was transformed into a crisp nonlinear model by using the max-min method.

Yadavalli et al. (2005) solved fuzzy multi-item EOQ models with no constraint and with one constraint. The only constraint was the limitation of the average number of stocked units in the model with one constraint. The objective function and some parameters were defined as fuzzy numbers. The membership functions of the objective function coefficients were defined as linear. The membership functions of the objective and the limit value of the constraint were defined as linear, parabolic and cubic in three cases. The fuzzy parameters were transformed into crisp using α-cuts.

Baykasoğlu and Göçken (2007) have studied the fuzzy multi-item EOQ problem with two constraints; available warehouse space and number of orders placed during a time period. All parameters of the EOQ model are accepted as triangular fuzzy numbers. The fuzzy model is solved via transformation into crisp problem using different ranking methods.

In this thesis, a multi-item EOQ model with two constraints (available warehouse space and number of orders placed during a time period) is handled. The problem is to decide the order levels Q_i, i = 1, 2,…, m which minimize the average total cost. The studied fuzzy multi-item EOQ model is defined as follows;

$$\min \quad C(Q) = \sum_{i=1}^{m} \left(\frac{\tilde{c}_{1i} Q_i}{2} + \frac{\tilde{c}_{2i} D_i}{Q_i}\right)$$

subject to

$$\sum_{i=1}^{m} \tilde{a}_i Q_i \leq \tilde{W} \qquad (5.1)$$

$$\sum_{i=1}^{m} \frac{\tilde{M}_i}{Q_i} \leq \tilde{n}$$

$$Q_i \geq 0$$

where, Q_i is the economic order quantity for i^{th} item, $\tilde{c}_{1i}$ is the holding cost per unit quantity per unit time for i^{th} item, $\tilde{c}_{2i}$ is the set up cost per period for i^{th} item, D_i is the demand per unit time for i^{th} item, $\tilde{a}_i$ is the space required by each unit of product i (in sq.m), $\tilde{M}_i$ is the total demand of product i during some given time interval, $\tilde{W}$ is the maximum available warehouse space (in sq.m.), $\tilde{n}$ is the maximum number of orders placed during the given time period and m is the number of items.

The parameters of the problem are defined as triangular fuzzy numbers. It is accepted that there are two items in the problem. The input data of the problem is given in table 5.1 and table 5.2. The input data is similar to the data that was used by Mondal and Maiti's (2002) multi-item EOQ problem, except $\tilde{a}_i$ and $\tilde{M}_i$ which are not considered fuzzy in their work. In the study of Mondal and Maiti (2002), the objective function, the cost coefficients and the right hand values of the constraints were defined as fuzzy numbers. Moreover, they did not employ triangular fuzzy numbers and set aspiration value for the objective function. Therefore the present model is different than Mondal and Maiti's model.

Table 5.1. Input data1

Item	$\tilde{c}_{1i}=(\overline{c}_{1i},c_{1i},\underline{c}_{1i})$	$\tilde{c}_{2i}=(\overline{c}_{2i},c_{2i},\underline{c}_{2i})$	D_i	$\tilde{a}_i=(\overline{a}_i,a_i,\underline{a}_i)$	$\tilde{M}_i=(\overline{M}_i,M_i,\underline{M}_i)$
1	(200; 250; 300)	$(90.10^3;10^5;110.10^3)$	200	(0.8; 1; 1.1)	(7500; 8000; 8500)
2	(150; 200; 250)	$(225.10^3; 245.10^3; 265.10^{3)}$	800	(0.8; 1; 1.1)	(3500; 4000; 4500)

Table 5.2. Input data2

$\tilde{W}=(\overline{W},W,\underline{W})$	$\tilde{n}=(\overline{n},n,\underline{n})$
(1450; 1500; 1550)	(18; 20; 22)

The fuzzy multi-item EOQ problem of the present study can be explicitly stated as follows;

$$\begin{aligned}
\min C(Q) = {} & (200;250;300)\frac{Q_1}{2}+(90.10^3;10^5;110.10^3)\frac{200}{Q_1}+(150;200;250)\frac{Q_2}{2}+ \\
& (225.10^3;245.10^3;265.10^3)\frac{800}{Q_2} \\
\textit{subject to} & \\
& (0.8;1;1.1)Q_1+(0.8;1;1.1)Q_2 \le (1450;1500;1550) \\
& (7500;8000;8500)\frac{1}{Q_1}+(3500;4000;4500)\frac{1}{Q_2} \le (18;20;22) \\
& Q_1,Q_2 \ge 0
\end{aligned} \tag{5.2}$$

5.2.2. Solution of the fuzzy multi-item economic order quantity problem

The fuzzy multi-item EOQ problem is solved directly with the proposed solution method by employing four fuzzy ranking methods, the signed distance method, Chen and Chen's method, the integral value method and the expected interval method, and the PSO algorithm. Ranking methods for fuzzy numbers are used to rank the objective function values and to determine the feasibility of the constraints.

As the cost coefficients of the objective functions are fuzzy the objective function values of the generated solution vectors will be fuzzy numbers. Therefore, in the selection of the best solution vector, ranking of fuzzy numbers is used. Similarly, the feasibility of the constraints for the generated solution vectors will be determined via ranking of two fuzzy numbers (i.e. comparing right and left hand side fuzzy numbers for the constraint functions).

In order to solve the fuzzy multi-item EOQ problem directly, the global best PSO algorithm is used. In the solution of the problem, the parameters of the algorithm are taken as follows; number of particles n = 20, inertia weight w = 0.4, individual and sociality weights $c_1 = c_2 = 1.4962$, and the number of iterations is 1000.

The solution of the crisp EOQ model is Q_1 = 500, Q_2 = 1000 and the objective function value $C(Q)$ = 398500. In the crisp model, the data are the mid values of the fuzzy data.

The solutions obtained using the signed distance method and Chen and Chen's ranking method are given in table 5. 3.

Table 5.3. The solutions of the fuzzy multi-item EOQ problem obtained using the signed distance method and Chen and Chen's ranking method

Ranking method	Decision vector, Q^0	Possibility distribution of the objective value, $\tilde{z}^0$
Signed distance method	$Q_1 = 494.827$ $Q_2 = 1043.634$	(336605.92; 394440.32; 452274.72)
Chen and Chen's method	$Q_1 = 493.181$ $Q_2 = 1058.546$	(335251.38; 393214.93; 451178.48)

The fuzzy multi-item EOQ problem is solved for different alpha values using the integral value method. The problem is solved for α = {0.5; 0.6; 0.7; 0.8; 0.9; 1.0}. The obtained solutions and the possibility distributions of the objective for each α value are given in table 5.4.

Table5.4. α-acceptable optimal solutions of the fuzzy multi-item EOQ problem obtained using the integral value method

Feasibility degree, α	Decision vector, $Q^0(\alpha)$	Possibility distribution of the objective value, $\tilde{z}^0(\alpha)$
0.5	$Q_1 = 494.8279$ $Q_2 = 1043.634$	(336605.92; 394440.32; 452274.72)
0.6	$Q_1 = 495.4511$ $Q_2 = 1024.751$	(338384.40; 396039.73; 453695.05)
0.7	$Q_1 = 496.1461$ $Q_2 = 1006.341$	(340235.60; 397728.03; 455220.47)
0.8	$Q_1 = 496.9146$ $Q_2 = 988.3795$	(342159.74; 399505.04; 456850.34)
0.9	$Q_1 = 497.7585$ $Q_2 = 970.8405$	(344157.36; 401370.92; 458584.47)
1.0	$Q_1 = 498.6800$ $Q_2 = 953.7010$	(346229.27; 403326.13; 460422.99)

The fuzzy multi-item EOQ problem is solved for different alpha values using the expected interval method. The problem is solved for $\alpha = \{0.5; 0.6; 0.7; 0.8; 0.9; 1.0\}$. The obtained solutions and the possibility distributions of the objective for each α value are given in table 5.5.

Table 5.5. α-acceptable optimal solutions of the fuzzy multi-item EOQ problem obtained using the expected intervals method

Feasibility degree, α	Decision vector, $Q^0(\alpha)$	Possibility distribution of objective value, $\tilde{z}^0(\alpha)$
0.5	$Q_1 = 494.828$ $Q_2 = 1043.634$	(336605.92; 394440.32; 452274.72)
0.6	$Q_1 = 514.735$ $Q_2 = 994.318$	(342045.41; 399748.68; 457451.94)
0.7	$Q_1 = 584.182$ $Q_2 = 872.178$	(361023.77; 419201.23; 477378.69)
0.8	$Q_1 = 644.855$ $Q_2 = 724.459$	(395194.43; 454614.19; 514033.95)
0.9	$Q_1 = 644.855$ $Q_2 = 724.459$	(395194.43; 454614.19; 514033.95)
1.0	$Q_1 = 644.855$ $Q_2 = 724.459$	(395194.43; 454614.19; 514033.95)

Only the solutions obtained for α = 0.5 using the integral value method and the expected interval method, and the solution obtained using the signed distance method are same. The other solutions are different from each other.

5.3. Solution of Fuzzy Multi-objective Aggregate Production Planning Problem

5.3.1. The fuzzy multi-objective aggregate production planning problem

Aggregate planning is the determination of production rate and the best strategy to meet the demand by considering sales forecasts, production capacity, inventory levels and work force for a medium period, often from 3 to 18 months in advance (Sipper and Bulfin 1997). The aims of aggregate production planning (APP) are; to set overall production levels for each product category to meet fluctuating or uncertain demand in the near future and to set decisions concerning hiring, layoffs, overtime, backorders, subcontracting, inventory level and determining appropriate resources to be used (Lai and Hwang, 1992, Wang and Liang, 2004). APP is an important upper level planning activity in a production management system. Other forms of family disaggregation plans, such as master production schedule, capacity plan, material requirements plan all depend on APP in a hierarchical way (Tang et al., 2000, Fung et al., 2003).

All traditional models of APP problems can be classified into six categories according to Saad (1982); linear programming, linear decision rule, transportation method, management coefficient approach, search decision rule and simulation. In practical production planning systems, aggregate planning generally have conflicting objectives with respective to the use of the resources (Wang and Liang, 2004).

In real world APP problems, the input data or parameters, such as demand, resources, cost etc. are imprecise because some information is incomplete or unobtainable (Wang and Liang, 2004). These imprecise parameters can be defined as random numbers with probability distribution, fuzzy numbers or interval numbers (Tang et al., 2000). A great deal of knowledge about the statistical distribution of the uncertain parameters is required to define the parameters as random numbers with probability distribution. So, using fuzzy numbers for imprecise parameters is more efficient in real world.

In the literature, there are various works on the solution of fuzzy multi-objective APP problem. But, in these studies generally only the goals are defined as fuzzy values and the fuzzy model is solved by transforming the fuzzy model into classical crisp mathematical programming problems. Wang and Fang (1997) handled a fuzzy APP problem which tries to maximize profit and find the production quantity of products. In their study, the APP problem is defined as a fuzzy linear programming problem; the resources are defined as fuzzy numbers and for the objective function a fuzzy goal is defined. The fuzzy APP problem is transformed into a crisp linear programming problem using Zimmermann's max-min approach. After the transforming process, instead of finding one exact solution, Wang and Fang proposed an inexact approach which imitates the human decision making process by generating a family of inexact solutions within an acceptable level as candidates for decision maker to consider. In order to generate a family of inexact solutions, a genetics based algorithm is proposed in their work.

In the study of Tang et al. (2000), a multi-product APP problem with fuzzy demands and fuzzy capacities is modeled. The demand and capacities as fuzzy numbers and for the solution the fuzzy model is converted into its crisp equivalent using defuzzification of soft equations according to satisfaction of membership functions at a defined degree of truth.

Wang and Fang (2001) presented a novel fuzzy linear programming method for solving the fuzzy APP problem with multiple objectives. In the problem, some of the parameters in the objective functions , in the left hand and right hand side of the constraints are defined as trapezoidal fuzzy numbers. For the fuzzy objectives, L type and R type fuzzy numbers are defined as fuzzy goals and the upper and lower bounds of fuzzy goals are determined according to Rommelfanger (1991). The fuzzy APP problem is converted into its crisp equivalent according to Zimmermann's approach.

Fung et al. (2003) solved a fuzzy multi-product APP problem in their study. They accepted demands and capacities as triangular fuzzy numbers. The fuzzy multi-product APP problem has only one objective (minimizing the total cost) and only the demands and the capacities are defined as fuzzy. Using the membership functions of the fuzzy parameters, the problem is converted into a crisp parametric programming problem. The obtained crisp problem is solved using parametric programming and the proposed interactive method.

Wang and Liang (2004) have developed a fuzzy multiple objective linear programming model for solving the multi-product APP decision problem in a fuzzy environment. The handled problem has three objectives, minimizing total production costs, minimizing carrying and backordering costs and minimizing rate of change in labor levels, and for the objectives fuzzy aspiration levels are defined. Piecewise linear membership functions are specified for the fuzzy goals. The other parameters of the problem are accepted as certain. The fuzzy multiple objective linear programming problem is converted into its crisp equivalent using Zimmermann's approach. The decision maker can interactively modify the membership functions of the objectives until a satisfactory solution is obtained.

Wang and Liang (2005) have handled a fuzzy multi-product APP problem in their work. The multi-product APP problem is modeled as a multiple objective linear programming problem and all of the objectives are defined as fuzzy with imprecise aspiration levels. The other parameters of the problem are accepted as crisp. The fuzzy multiple objective linear programming problem is transformed into its crisp equivalent according to Zimmermann's max-min approach. The obtained crisp equivalent linear programming problem is solved with classical approaches. After solution process, the decision maker can interactively modify the fuzzy data and related model parameters until a satisfactory solution is obtained.

In their study, Moghaddam et al. (2007) presented a fuzzy APP model for make-to-stock environments. In the model, the aspiration level of the objective, one of the right hand parameters of the constraints (the demand) and one of the left hand parameters of the constraints (average usage rate) are accepted as fuzzy numbers. The fuzzy problem is converted into its crisp equivalent using Zimmermann's approach and a defuzzification method for the left hand parameter. Then the crisp equivalent is solved using any classical approach.

In this thesis, a fuzzy multi-objective APP problem is solved using the proposed direct solution method. In Baykasoglu (2001a), the crisp multiple product, multiple period APP model of Masud and Hwang (1980) was stated as a pre-emptive goal programming model. Baykasoglu (2001a) solved his model by using the multiple objective tabu search algorithm. In this thesis, the parameters of APP problem are stated as triangular fuzzy numbers. The model is a general model and the number of goals, constraints and variables are relatively high. Therefore, the size of the problem is relatively big. The mathematical model of fuzzy multi-objective APP model is written as follows.

$$
\begin{aligned}
&\text{maximize} \quad \sum_t \sum_i \tilde{r}_i S_{it} - \sum_t \sum_i \tilde{c}_{1i} (P_{it} + O_{it}) - \sum_t \tilde{c}_{2t} W_t - \sum_t \sum_i c_{3t} (\tilde{a}_i O_{it}) \\
&\text{minimize} \quad 1/T \left\{ \sum_i \sum_t \tilde{r}_i I_{it}^- \right\} \\
&\text{minimize} \quad \sum_t (H_t + L_t) \qquad (5.3) \\
&\text{minimize} \quad 1/T \left\{ \sum_i \sum_t \tilde{c}_{4i} I_{it}^+ \right\} \\
&s.t. \\
&W_t = W_{t-1} + H_t - L_t \qquad \forall t \\
&W_t \le W_{t\max} \qquad \forall t \\
&\sum_i \tilde{a}_i P_{it} \le \tilde{\delta} W_t \qquad \forall t \\
&\sum_i \tilde{a}_i O_{it} \le \tilde{\delta} \beta_t W_t \qquad \forall t \\
&H_t L_t = 0 \qquad \forall t \\
&I_{it}^+ - I_{it}^- = I_{it-1}^+ - I_{it-1}^- + P_{it} + O_{it} - S_{it} \qquad \forall i,t \\
&S_{it\min} \le S_{it} \le S_{it\max} \qquad \forall i,t \\
&I_{it-1}^+ - I_{it-1}^- + P_{it} + O_{it} \ge S_{it\min} \qquad \forall i,t \\
&I_{it}^+ I_{it}^- = 0 \qquad \forall i,t \\
&\sum_i \tilde{b}_i P_{it} \le \tilde{M}_t \qquad \forall t \\
&\sum_i \tilde{b}_i O_{it} \le \alpha_t \tilde{M}_t \qquad \forall t \\
&\sum_i \tilde{b}_i P_{it} \ge \tilde{M}_{t\min} \qquad \forall t \\
&I_{it}^+, I_{it}^-, P_{it}, O_{it}, S_{it}, W_t, H_t, L_t \ge 0 \qquad \forall i,t
\end{aligned}
$$

The symbols used in the mathematical model of the fuzzy multi-objective aggregate production planning model are as follows.

Decision variables

P_{it} : Regular time production of product *i* in period *t* (units),

O_{it} : Overtime production of product *i* in period *t* (units),

S_{it} : Product *i* sold in period *t* (units),

H_t : Worker hired in period *t* (man-day),

L_t : Worker laid-off in period *t* (man-day).

State variables

I_{it}^+ : Inventory of product *i* at the end of period *t* (units),

I_{it}^{-} : Backorder of product i at the end of period t (units),

W_t : Work force level in period t (units).

Parameters and constants

$\widetilde{a}_i$: Labor time for product i (man-hour/unit),

$\widetilde{b}_i$: Machine time for product i (machine-hour/unit),

$\widetilde{c}_{1i}$: Production cost (other than labor cost) for product i ($/unit),

$\widetilde{c}_{2t}$: Labor cost in period t ($/man-day),

c_{3t} : Overtime labor cost in period t ($/man-hour),

$\widetilde{c}_{4i}$: Standard cost per unit of product i ($/unit),

$\widetilde{M}_t$: Regular time machining capacity in period t (machine-hour),

$\widetilde{M}_{t\min}$: Lower bound on the utilization of machine capacity in period t (machine-hour),

$\widetilde{r}_i$: Per unit sales revenue of product i ($/unit),

$S_{it\min}$: Minimum sales (which cannot be backordered) of product i in period t (units),

$S_{it\max}$: Maximum forecasted sales of product i in period t (units),

$W_{t\max}$: Maximum work force available in period t (man-day),

∂_t : Fraction of regular machine capacity available for use in overtime in period t,

β_t : Fraction of regular work force available for use in overtime in period t,

$\widetilde{\delta}$: Regular time per worker (man-hour/man-day),

T : Number of periods,

Initial values

I_{i0}^{+} : Inventory of product i at the start of planning horizon (units),

I_{i0}^{-} : Outstanding backorder of product i at the start of planning horizon (units),

W_0 : Work force at the start of planning horizon (man-day).

In this study, the fuzzy multi-objective APP problem is solved for two products and eight periods using the proposed direct solution method. The input data of the problem is given in Table 5.6-5.8.

Table 5.6. Operating and cost data

Product	Labor production time ($\tilde{a}_i$, h/unit)	Machining time ($\tilde{b}_i$, h/unit)	Production cost (other than labour) ($\tilde{c}_{1i}$, $/unit)	Value added ($\tilde{c}_{4i}$, $/unit)	Sales revenue ($\tilde{r}_i$, $/unit)
1	(1.8;2;2.2)	(1.35;1.5;1.8)	(13.5;15;16.5)	(36;40;44)	(63;70;77)
2	(2.7;3;3.3)	(1.8;2;2.2)	(18;20;22)	(54;60;66)	(90;100;110)

Table 5.7. Sales, work force and machine capacity data

Period	$S_{1t\min}$ **(unit)**	$S_{1t\max}$ **(unit)**	$S_{2t\min}$ **(unit)**	$S_{2t\max}$ **(unit)**	$W_{t\max}$ **(man-day)**	$\tilde{M}_t$ **(machine-hour)**
1	3000	12000	2000	6000	5000	(28800; 32000; 35200)
2	4500	19000	3000	18000	4000	(25560; 28400; 31240)
3	3000	15000	3000	15000	4500	(26640; 29600; 32560)
4	3000	16000	1500	5000	3000	(18000; 20000; 22000)
5	4500	18000	4500	12000	5000	(28800; 32000; 35200)
6	3500	16000	1000	4000	5500	(30240; 33600; 36960)
7	2000	10000	2000	10000	4500	(26640; 29600; 32560)
8	4500	14000	4000	12000	4000	(23760; 26400; 29040)

Table 5.8. Miscellaneous data

Period	1	2	3	4	5	6
α_t	0.5	0.6	0.5	0.6	0.4	0.4
β_t	0.3	0.3	0.3	0.3	0.3	0.3
Other data	$\tilde{c}_{2t} = (57.6;64;70.4)$, $c_{3t} = 15$, $I_{10}^{-} = 500$, $I_{20}^{+} = 500$, $w_0 = 3500$, $\tilde{\delta} = (14.4;16;17.6)$					

5.3.2. Solution of the fuzzy multi-objective aggregate production planning problem

The fuzzy multi-objective APP problem is solved directly with the proposed solution method by employing four fuzzy ranking methods and the tabu search algorithm. Ranking methods for fuzzy numbers are used to rank the objective functions' values and to determine the feasibility of the constraints.

In the solution of fuzzy APP problem, for the comparison of the objective functions' values and evaluation of feasibility of the constraints, the selected four ranking methods of fuzzy numbers are used. Feasibility of the constraints for the obtained solution vectors is determined with ranking of fuzzy numbers. For the right hand and left hand of the constraints, by the multiplication of the solution vectors with fuzzy parameters, again a fuzzy number will be obtained and the feasibility of the constraints will be determined by ranking of fuzzy numbers. For the objective functions, priorities are assigned. First priority is assigned to first objective, second priority is assigned to second objective and so on. The best solution will be the solution that maximizes the objective of first priority. If the values of the first prior objective for two feasible solutions are equal, the solution that minimizes the second prior objective will be selected and so on. The values of the objective functions for the solution vectors will be fuzzy values owing to the fuzzy parameters in the objective functions. So, for the selection of the best solution, ranking of fuzzy numbers is used.

Tabu search algorithm is used for the solution of fuzzy multi-objective APP problem. The problem is solved for three different parameter sets. In table 5.9, the parameter sets which are used for the solution of the problem are given.

Table 5.9. Tabu search parameter sets

Tabu search parameter sets	Step sizes for $(S_t, P_t, O_t, H_t, L_t)$	Tabu list size, p	Neighborhood size, S^*	Max number of iterations
TSP1	50, 50, 60, 60, 60	30	7	10000
TSP2	90, 50, 50, 90, 50	15	6	10000
TSP3	90, 50, 50, 90, 50	25	30	10000

The crisp solution of the multi-objective APP problem is given in the table 5.10. In the crisp model, the data are the mid values of the fuzzy data.

Table 5.10. The crisp solution of fuzzy multi-objective APP problem

	TSP 1	**TSP2**	**TSP3**
Objective function 1	5407274	5414522	5632425
Objective function 2	352940	364761	389926
Objective function 3	15262	15329	15052
Objective function 4	195733	190160	180428

For the three of the selected ranking methods – the signed distance method, Chen and Chen's method and the integral value method – the same solutions are obtained. For the integral value method the problem is solved for alpha values between 0.5 and 1.0. Only for the tabu search parameter set 1 and alpha value 1.0 different solution is obtained. Except this, for all alpha values and tabu search parameters same solutions are obtained. The obtained solutions using the signed distance method, Chen and Chen's method and the integral value method are given in table 5.11.

Table 5.11. The solutions of fuzzy multi-objective APP problem obtained using the signed distance method, Chen and Chen's method and the integral value method

	TSP 1	**TSP2**	**TSP3**
Objective func. 1	(5309177; 5899085; 6488994)	(5345984; 5939982; 6533980)	(5769327; 6410363; 7051400)
Objective func. 2	(230456; 256063; 281669)	(234540; 260600; 286660)	(272723; 303025; 333328)
Objective func. 3	13512	13548	13791
Objective func. 4	(28093; 312145; 343360)	(283455; 314950; 346445)	(240950; 267723; 294495)

Only for the integral value method and for the tabu search parameter set 1 and alpha value 1.0 a different solution is obtained; objective function 1 = (5301481; 5890534; 6479588), objective function 2 = (229860; 255400; 280940), objective function 3 = 13456, objective function 4 = (279709; 310788; 341866).

The obtained solutions using ranking of fuzzy numbers through the comparison of their expected intervals are differs according to the used tabu search parameter sets and the alpha values.

Table 5.12. The solutions of fuzzy multi-objective APP problem obtained using expected interval method

Alpha value	TSP1	TSP2	TSP3
0.5	(5309177; 5899085; 6488994) (230456; 256063; 281669) 13512 (280931; 312145; 343360)	(5345984; 5939982; 6533980) (234540; 260600; 286660) 13548 (283455; 314950; 346445)	(5769327; 6410363; 7051400) (272723; 303025; 333328) 13791 (241855; 268728; 295600)
0.6	(5025885; 5584316; 6142748) (249634; 277371; 305108) 7179 (155693; 172993; 190292)	(5034364; 5593737; 6153111) (261090; 290100; 319110) 7576 (148248; 164720; 181192)	(5331890; 5924322; 6516754) (286422; 318246; 350071) 7797 (134746; 149718; 164689)
0.7	(5131968; 5702186; 6272405) (303471; 337190; 370909) 11932 (110795; 123105; 135416)	(5194597; 5771774; 6348952) (321544; 357271; 392998) 12043 (109190; 121323; 133455)	(5661544; 6290604; 6919665) (386578; 429531; 472484) 12397 (95589; 106210; 116831)
0.8	(5015356; 5572618; 6129880) (244058; 271175; 298293) 10030 (177278; 196975; 216673)	(5008212; 5564680; 6121148) (253676; 281863; 310049) 10012 (170093; 188993; 207892)	(5190402; 5767113; 6343824) (261135; 290150; 319165) 9881 (154413; 171570; 188727)
0.9	(4970841; 5523157; 6075473) (243360; 270400; 297440) 9959 (170757; 189730; 208703)	(4980462; 5533846; 6087231) (251359; 279288; 307216) 9994 (173214; 192460; 211706)	(5200238; 5778042; 6355846) (263408; 292675; 321943) 9920 (152420; 169355; 186291)
1.0	(4977333; 5530370; 6083407) (244755; 271950; 299145) 10014 (169614; 188460; 207306)	(5005813; 5562014; 6118216) (255566; 283963; 312359) 10000 (172575; 191750; 210925)	(5185492; 5761658; 6337824) (257276; 285863; 314449) 9906 (150354; 167060; 183766)

5.4. Conclusion

In this section, two real world problems – economic order quantity problem and aggregate production planning problem- in fuzzy environments are solved using the proposed direct solution method. In the solution process, four different ranking methods and for the fuzzy EOQ problem the PSO algorithm and for the fuzzy APP problem the tabu search algorithm are used. It has been observed that fuzzy decision making problems can be solved effectively by using ranking methods of fuzzy numbers without any necessity of transformation into crisp equivalent. When crisp solutions are compared with fuzzy solutions, it is seen that with fuzzy data better solutions can be obtained for both the EOQ problem and the APP problem.

CHAPTER 6

SOLUTION OF FUZZY MATHEMATICAL PROGRAMMING PROBLEMS WITH FUZZY DECISION VARIABLES

6.1. Introduction

In this chapter, solution of fuzzy mathematical programming problems with fuzzy decision variables is presented. The proposed direct solution method is used for solving fuzzy mathematical programming problems with fuzzy decision variables. The fuzzy product mix problem with fuzzy variables is solved with proposed direct solution method.

6.2. Fuzzy Mathematical Programming Problems with Fuzzy Decision Variables

In real world applications, the nature of the parameters of the decision making problems are imprecise. Fuzzy set theory gives an opportunity to handle linguistic terms and vagueness in real life systems. After Bellman and Zadeh (1970) proposed the concept of decision making in fuzzy environments, various researchers studied on fuzzy mathematical programming problems frequently. A review of these studies is given in chapter 3. In another point of view, existing methods can be divided into two groups, depending on the fuzziness of decision variables. In the first group, it is assumed that the parameters of the problem are fuzzy numbers while the decision variables are crisp ones. This means that in an uncertain environment, a crisp decision is made to meet some decision criteria (Hashemi et al., 2006, Allahviranloo, 2008). In the first group, the fuzzy characteristic of the decision can be partly lost and the decision making process is constrained to crisp solutions. Finding fuzzy solutions instead of crisp solutions in an uncertain environment that provide ranges of flexibility to decision maker looks more attractive (Hashemi et al., 2006). In the second group, the decision variables are assumed as fuzzy numbers. Tanaka and Asai (1984a) are the pioneers of the second group. In the literature, there are few examples on fuzzy mathematical programming problems with fuzzy decision variables.

Tanaka and Asai (1984a) presented how fuzzy solution can be obtained for fuzzy linear programming problems. They handled fuzzy linear programming problems with fuzzy satisfaction criteria and fuzzy right hand values of constraints. The fuzzy linear programming problem is converted into crisp problem and solved. Tanaka and Asai (1984b) solved fully fuzzy linear programming problems in their paper. They handled fuzzy linear programming problems with fuzzy satisfaction criteria and fuzzy parameters. The goals and constraints are accepted as identical concepts. The fuzzy linear programming problem is converted into crisp problem as in Tanaka and Asai (1984a).

Tanaka et al. (2000) studied obtaining fuzzy decision to fuzzy decision making problems using possibility distributions of fuzzy decision variables. In their study, Tanaka et al. solved fuzzy linear programming problems with fuzzy right hand values and fuzzy decision variables. For solving the fuzzy problem, they used possibility distributions of fuzzy decision variables and transformed fuzzy problems into crisp problem.

Buckley and Feuring (2000) proposed a solution method for fully fuzzy linear programming problems. All of the parameters and decision variables are defined as triangular fuzzy numbers. The fuzzy linear programming problem is changed into a multi-objective fuzzy linear programming problem. For example, for a maximizing problem it is tried to maximize the mid point of the fuzzy objective, minimize the area between the mid point and the minimum point of the membership function of fuzzy objective and maximize the area between the mid point and the maximum point of the membership function of fuzzy objective. The fuzzy constraints are handled using fuzzy ranking methods. Fuzzy flexible programming problem is used to find the whole undominated set of the multi-objective fuzzy linear programming problem. Buckley and Feuring (2000) designed an evolutionary algorithm to solve the fuzzy flexible program.

Buckley et al. (2001) solved multi-objective fully fuzzified linear programming problems in their study. All the parameters and variables are defined as triangular fuzzy numbers. The same solution procedure proposed in Buckley and Feuring (2000) is used for the multi-objective fully fuzzified linear programming problems. The multi-objective fully fuzzified linear programming problem is changed into a single objective fuzzy linear programming problem and then solved using the proposed solution procedure. An evolutionary algorithm is used to generate undominated solutions.

Tsakiris and Spiliotis (2004) presented a methodology for solving the problem of water allocation to various users under uncertainty. In one instance of the study, they defined decision variables as fuzzy. In the study of Tsakiris and Spiliotis, the water allocation problem with fuzzy decision variables is solved using the method proposed by Tanaka et al. (2000).

Hashemi et al. (2006) proposed a solution method for fully fuzzified linear programming problems in which all parameters and decision variables are defined as symmetric fuzzy numbers. The solution procedure is constructed on a ranking method which is based on the comparison of mean and standard deviation of fuzzy numbers. In fuzzy arithmetic based on Zadeh's extension principle, the shape of L-R fuzzy number is not preserved. So, Hashemi et al. used the new fuzzy arithmetic operations on symmetric fuzzy numbers introduced by Nasrabadi and Nasrabadi (2004). Hashemi et al. (2006) proposed a two phase approach for the solution of the fully fuzzified linear programming problems. In the first phase, the possibilistic mean value of fuzzy objective function is tried to be maximized and a set of feasible solutions are obtained. In the second phase, the standard deviation of the original fuzzy objective function is tried to be minimized by considering all basic feasible solutions obtained at the end of the first phase.

Allahviranloo et al. (2008) proposed a method to solve fuzzy mathematical programming problems with fuzzy variables. They concentrated on fully fuzzy linear programming problems in their work; they defined all of the parameters of the problem and the decision variables as fuzzy numbers. The parameters and the decision variables of the fuzzy mathematical programming problems are defined as triangular fuzzy numbers. The fully fuzzy linear programming problem is defuzzified using a linear ranking function and the crisp equivalent of the fully fuzzy linear programming problem is obtained. After the defuzzification process, the number of constraints and the number of decision variables are increasing.

6.3. Solution of Fuzzy Mathematical Programming Problems with Fuzzy Decision Variables

In this thesis, fuzzy mathematical programming problems in which all of the parameters and the decision variables are defined as fuzzy numbers are handled. When all parameters of the problem are fuzzy, it can be more appropriate that the decision variables are fuzzy too. In a fuzzy mathematical programming problem, when the parameters and the decision variables are fuzzy numbers, the arithmetic of two fuzzy numbers will be occurred. When two fuzzy numbers added or subtracted, the type of the obtained fuzzy number will not change. But, when two fuzzy numbers multiplied or divided by each other, the type of the obtained fuzzy number will not be same with the formers. For example, when two triangular fuzzy numbers are multiplied, the resultant will be a triangular shaped fuzzy number, but not a triangular fuzzy number. The left and right reference functions will not be linear functions. So, for these fuzzy numbers, general forms of fuzzy ranking methods can be used, not the special forms of fuzzy ranking methods which are generated for special types of fuzzy numbers.

The proposed direct solution method can be used for solving fuzzy mathematical programming problems with fuzzy decision variables. A ranking method for fuzzy numbers is used to rank the objective function values and to determine the feasibility of the constraints. In the previous sections, the fuzzy parameters are defined as triangular fuzzy numbers and the decision variables are accepted as crisp. So, after the arithmetic of triangular fuzzy parameters with crisp decision variables (multiplication, division etc.), the obtained objective function values and the obtained right and left hand values of the constraints are triangular fuzzy numbers too. Hence, for the solution of fuzzy problems in previous sections, special forms of ranking methods for triangular fuzzy numbers are used in the proposed direct solution method. When decision variables are defined as fuzzy, general forms of the ranking methods should be used in the proposed direct solution method.

In this thesis, the product mix problem is solved as an example for the solution of fuzzy mathematical programming problems with fuzzy decision variables. The fully fuzzy product mix problem can be written in mathematical form as follows.

$$
\begin{aligned}
\max \quad & z = \tilde{6}\tilde{x}_1 + \tilde{8}\tilde{x}_2 + \tilde{6}\tilde{x}_3 \\
\textit{subject to} & \\
& \tilde{6}\tilde{x}_1 + \tilde{8}\tilde{x}_2 + \tilde{3}\tilde{x}_3 \le \widetilde{288} \\
& \widetilde{12}\tilde{x}_1 + \tilde{8}\tilde{x}_2 + \tilde{6}\tilde{x}_3 \le \widetilde{312} \\
& \tilde{2}\tilde{x}_1 + \tilde{4}\tilde{x}_2 + \tilde{x}_3 \le \widetilde{124} \\
& \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \ge 0
\end{aligned}
\tag{6.1}
$$

All parameters of the product mix problem are defined as triangular fuzzy numbers. The decision variables are accepted as symmetric triangular fuzzy numbers and the left and right difference are defined as determined percentage of the mid point of the fuzzy decision variable. The fuzzy decision variables can be presented as in figure 6.1.

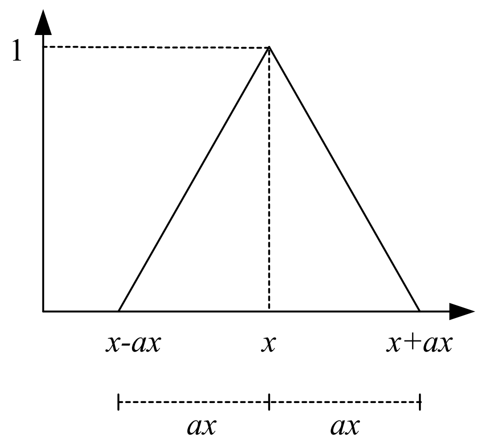


Figure 6.1. The shape of the fuzzy decision variables

The fully fuzzy product mix problem can be written in an open form as follows.

$$
\begin{aligned}
&\max \quad z=(5.8;6;6.2)(x_1-ax_1;x_1;x_1+ax_1)+(7.5;8;8.5)(x_2-ax_2;x_2;x_2+ax_2)+(5.6;6;6.4)(x_3-ax_3;x_3;x_3+ax_3)\\
&\mathit{subject\,to}\\
&(5.6;6;6.4)(x_1-ax_1;x_1;x_1+ax_1)+(7.5;8;8.5)(x_2-ax_2;x_2;x_2+ax_2)+(2.8;3;3.2)(x_3-ax_3;x_3;x_3+ax_3)\le(283;288;293)\\
&(11.4;12;12.6)(x_1-ax_1;x_1;x_1+ax_1)+(7.6;8;8.4)(x_2-ax_2;x_2;x_2+ax_2)+(5.7;6;6.3)(x_3-ax_3;x_3;x_3+ax_3)\le(306;312;318)\\
&(1.8;2;2.2)(x_1-ax_1;x_1;x_1+ax_1)+(3.8;4;4.2)(x_2-ax_2;x_2;x_2+ax_2)+(0.9;1.0;1.1)(x_3-ax_3;x_3;x_3+ax_3)\le(121;124;127)\\
&x_1,x_2,x_3\ge 0
\end{aligned}
\tag{6.2}
$$

where a is the percentage value.

The fully fuzzy product mix problem is solved using the proposed direct solution method. The ranking method based on left and right dominance (Chen and Lu, 2001) is used to rank the objective function values and to determine the feasibility of the constraints. The ranking method based on left and right dominance is explained in chapter 2. The ranking method based on left and right dominance is a general ranking method which can be used for all types of fuzzy numbers. This ranking approach follows the concept of area measurement and useful when the membership functions of the fuzzy numbers cannot be acquired. The ranking method based on left and right dominance is selected according to the ease of computation among the others. In the solution of the fully fuzzy product mix problem, the PSO algorithm is used to carry out the ranking process. The parameters of the algorithm are taken as follows; inertia weight w = 0.4, individual and sociality weights $c_1 = c_2 = 1.4962$, the number of particles n = 20 and the number of iterations is 1000. For the solution of the problem a computer program is written in C language.

The left and right difference of the fuzzy decision variables are defined as determined percentage of the mid point of the fuzzy decision variables. The problem is solved for different percentage values. In the ranking method based on left and right dominance, the index of optimism β is determined by the decision maker. The problem is solved for three values of β (0, 0.5, 1). The ranking method uses a few left and right spreads at some α-levels of fuzzy numbers to determine the dominance of one fuzzy number over the other. In the ranking method, (n+1) is the number of α-cuts used to calculate the dominance. In the solution of the fully fuzzy product mix problem n is accepted as 5 ($\alpha_0 = 0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\alpha_3 = 0.6$, $\alpha_5 = 0.8$, $\alpha_6 = 1.0$) and 10 ($\alpha_0 = 0$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\alpha_3 = 0.3$, $\alpha_5 = 0.4$, $\alpha_6 = 0.5$, $\alpha_7 = 0.6$, $\alpha_8 = 0.7$,

$\alpha_9 = 0.8$, $\alpha_{10} = 0.9$, $\alpha_{11} = 1.0$). The solutions obtained for different percentage values, β values and number of α-cuts are shown in the tables 6.1-6.6.

Table 6.1. The solution of the fully fuzzy product mix problem for percentage value = 10% and $n = 5$

β	X_1	X_2	X_3	The possibility distribution of the objective function
0.0	(0; 0; 0)	(24.769; 27.521; 30.273)	(16.984; 18.871; 20.758)	(280.876; 333.393; 390.172)
0.5	(0; 0; 0)	(24.179; 26.866; 29.553)	(14.475; 16.084; 17.692)	(262.406; 311.429; 364.425)
1.0	(0.00009; 0.0001; 0.00011)	(2.045; 2.272; 2.499)	(41.152; 45.724; 50.296)	(245.788; 292.524; 343.145)

Table 6.2. The solution of the fully fuzzy product mix problem for percentage value = 20% and $n = 5$

β	X_1	X_2	X_3	The possibility distribution of the objective function
0.0	(0; 0; 0)	(16.828; 21.036; 25.243)	(24.451; 30.564; 36.677)	(263.141; 351.669; 449.295)
0.5	(0; 0; 0)	(20.922; 26.152; 31.383)	(13.552; 16.940; 20.328)	(232.806; 310.860; 396.855)
1.0	(0; 0; 0)	(9.894; 12.3676; 14.841)	(24.020; 30.025; 36.030)	(208.717; 279.090; 356.741)

Table 6.3. The solution of the fully fuzzy product mix problem for percentage value = 30% and $n = 5$

β	X_1	X_2	X_3	The possibility distribution of the objective function
0.0	(0; 0; 0)	(22.555; 32.222; 41.889)	(13.333; 19.048; 24.762)	(243.834; 372.065; 514.534)
0.5	(0; 0; 0)	(14.189; 20.269; 26.351)	(17.282; 24.689; 32.096)	(203.198; 310.293; 429.395)
1.0	(0; 0; 0)	(3.209; 4.584; 5.959)	(26.852; 38.360; 49.868)	(174.440; 266.836; 369.814)

Table 6.4. The solution of the fully fuzzy product mix problem for percentage value = 10% and $n = 10$

β	X_1	X_2	X_3	The possibility distribution of the objective function
0.0	(0; 0; 0)	(26.284; 29.204; 32.125)	(15.601; 17.3339; 19.067)	(284.491; 337.637; 395.090)
0.5	(0; 0; 0)	(24.179; 26.866; 29.552)	(14.474; 16.083; 17.691)	(262.398; 311.420; 364.414)
1.0	(0; 0; 0)	(5.730; 6.367; 7.003)	(36.242; 40.269; 44.296)	(245.930; 292.547; 343.021)

Table 6.5. The solution of the fully fuzzy product mix problem for percentage value = 20% and n = 10

β	X_1	X_2	X_3	The possibility distribution of the objective function
0.0	(0; 0; 0)	(23.121; 28.901; 34.681)	(17.180; 21.475; 25.770)	(269.613; 360.057; 459.717)
0.5	(0; 0; 0)	(20.854; 26.068; 31.282)	(13.639; 17.049; 20.459)	(232.790; 310.841; 396.834)
1.0	(0.00008; 0.0001; 0.00012)	(7.118; 8.898; 10.677)	(27.726; 34.658; 41.589)	(208.655; 279.131; 356.932)

Table 6.6. The solution of the fully fuzzy product mix problem for percentage value = 30% and n = 10

β	X_1	X_2	X_3	The possibility distribution of the objective function
0.0	(0; 0; 0)	(22.750; 32.500; 42.250)	(14.661; 20.944; 27.228)	(252.727; 385.667; 533.383)
0.5	(0; 0; 0)	(14.150; 20.214; 26.279)	(17.331; 24.758; 32.186)	(203.178; 310.265; 429.359)
1.0	(0; 0; 0)	(3.383; 4.833; 6.283)	(26.626; 38.038; 49.449)	(174.483; 266.893; 369.882)

When the obtained results are analyzed, it is seen that the fuzzy mathematical programming problems with fuzzy decision variables can be solved effectively by using the proposed direct solution method and significant results are obtained.

It is seen from the tables, as expected the objective values for $\beta = 0$ is bigger than the others and the objective values are decreasing while β values are increasing. When the percentage value increases, for $\beta = 0$ the objective value increases, but for $\beta = 0.5$ and 1.0 the objective value decreases. When the number of α-cuts used in the ranking method increases, for $\beta = 0$ and 1.0 the objective value increases, $\beta = 0.5$ the objective value decreases. The differences for $\beta = 0.5$ and 1.0 are very small. Chen and Lu (2001) showed that the ranking orders are consistent regardless of the number of α-cuts. It is stated that, if the membership functions of fuzzy numbers are simple, only a small number of α-cuts is necessary. The use of a greater number of α-cuts can obviously produce more accurate ordering.

6.4. Conclusion

In this chapter, the solution of fuzzy mathematical programming problems with fuzzy decision variables is shown. A fully fuzzy mathematical programming problem is solved using the proposed direct solution method. In the literature, there are a few studies on the solution of fully fuzzy mathematical programming problems. Because, after the multiplication or division of two fuzzy numbers, the type of the obtained fuzzy number will be change and the membership function will not be a linear function. So, the solution of fully fuzzy mathematical programming problems is not easy. But in this chapter, it is shown that fully fuzzy mathematical programming problems can be solved easily using the proposed direct solution method. In this chapter, a fully fuzzy product mix problem is solved as an example. In the solution, the ranking method based on left and right dominance and the PSO algorithm are used and significant solutions are obtained. It has been observed from this study that fully fuzzy decision making problems can be solved effectively by using the proposed direct solution method. In this study, the fuzzy decision variables are accepted as symmetric triangular fuzzy numbers and the left and right difference are defined as determined percentage of the mid point of the fuzzy decision variable. The fuzzy decision variables can be defined in different types of fuzzy numbers and solved using the proposed direct solution method.

CHAPTER 7

DISCUSSION and CONCLUSION

7.1. Introduction

In this chapter, section 7.2 gives a discussion of the study. Section 7.3 summarizes the conclusions obtained from the study. In section 7.4 some recommendations for future research are given.

7.2. Discussion

Most of the real life problems and models contain linguistic and/or imprecise variables and constraints. This can be due to different causes; usually, decision makers can state parameters on a system in terms of linguistic variables more easily and properly. Generally, collecting precise data is very hard, because the environment of the system is unstable or collecting precise data requires high information costs. In addition, decision maker might not be able to express his/her goals or constraints precisely but rather in a fuzzy sense.

The mentioned impreciseness in a system does not exist because of randomness but rather because of fuzziness. Fuzziness is the major source of imprecision in many decision processes. Fuzzy set theory gives an opportunity to handle linguistic terms and vagueness in real life systems. For modeling systems which are imprecise by nature or which can not be defined precisely, fuzzy mathematical programming that is based on fuzzy set theory is generally used.

Fuzzy mathematical programming is suggested to solve problems which could be formulated as mathematical programming models, the parameter of which are fuzzy rather than crisp numbers (Zimmermann, 1983).

Fuzzy mathematical programming is used frequently for modeling the systems after Zimmermann (1976) first used the max-min operator of Bellman and Zadeh to solve fuzzy linear programming problems. Several researchers proposed various solution methods for solving fuzzy mathematical programming models. The cause of this interest is that; fuzzy mathematical programming provides modeling of a system in accordance to current state of information and fuzzy models can reflect the real life systems more properly. Therefore, the notion of fuzzy set theory is widely spread to various fields, the history of fuzzy decision making and fuzzy mathematical programming are very rich.

7.2.1. The need for the present work

In the literature, various authors classified fuzzy mathematical programs with respect to different criteria. But, there is not a detailed classification according to the fuzzy components included in a fuzzy mathematical programming model. In this study, a new classification of fuzzy mathematical programming models is given according to the fuzzy components. The proposed solution approaches for fuzzy mathematical programming models are reviewed.

In the literature, there are various studies on solving fuzzy mathematical programming models. In a fuzzy mathematical programming model all or some of the parameters can be defined as fuzzy numbers. For fuzzy mathematical programming models with various fuzzy parameters, different optimization algorithms are proposed. But, most of the solution approaches are based on the fuzzy decision concept proposed by Zimmermann (1976). Other common approach is using fuzzy ranking procedures as a part of the solution mechanism for solving fuzzy mathematical programs. In all these solution approaches, fuzzy mathematical programming models were first transformed into a crisp equivalent then solved by a classical solution approach. In the literature, there is not any study on solving fuzzy mathematical programming problems directly. In this thesis, a direct solution method is proposed for solving fuzzy mathematical programming problems. In the proposed direct solution method ranking methods for fuzzy numbers and metaheuristic algorithms are used. Essentially, it can be very hard to transform many problems into crisp equivalent and sometimes the obtained crisp equivalent can be nonlinear. When the obtained crisp equivalent is nonlinear, a meta-heuristics algorithm should be used again for the solution. Therefore transformation might not be always advantageous. The use of transformation into crisp equivalent can limit the application of fuzzy mathematical programming. To make the transformation process easier, generally the membership functions of fuzzy parameters are defined as linear. So, generally, the fuzzy parameters are defined as triangular or trapezoidal fuzzy numbers. Besides, in the transformation process some information can be missed. Consequently, using a direct solution method can be very advantageous.

Existing methods for fuzzy mathematical programming can be divided into two groups, depending on the fuzziness of decision variables. In the first group, it is assumed that the parameters of the problem are fuzzy numbers while the decision variables are crisp ones. This means that in an uncertain environment, a crisp decision is made to meet some decision criteria (Hashemi et al., 2006, Allahviranloo, 2008). In the first group, the fuzzy characteristic of the decision can be partly lost and the decision making process is constrained to crisp solutions. In the second group, the decision variables are assumed as fuzzy numbers. In the literature, there are few examples on fuzzy mathematical programming problems with fuzzy decision variables. The solution of fully fuzzy mathematical programming problems is not easy. In the literature, as in other fuzzy mathematical programming problems, the fully fuzzy mathematical programming problems are converted into their crisp equivalents and then solved. In this thesis, it is shown that fully fuzzy mathematical programming problems can be solved easily using the proposed direct solution method.

7.2.2. The structure of the proposed direct solution method

In this thesis, a direct solution method is proposed for solving fuzzy mathematical programming problems. In the proposed direct solution method ranking methods for fuzzy numbers and metaheuristic algorithms are used. Ranking methods for fuzzy numbers are used to rank the objective function values and to determine the feasibility of the constraints. In a fuzzy mathematical programming problem, any of the parameters can be defined as fuzzy numbers. If the cost coefficients of the objective function are defined as fuzzy numbers, the objective function values of the generated solution vectors will be fuzzy numbers. Therefore, in the selection of the best solution vector, ranking of fuzzy numbers is used. If the parameters of the constraints are defined as fuzzy numbers, the right hand values and left hand values of the constraints will be fuzzy numbers. So, the feasibility of the constraints for the generated solution vectors will be determined via ranking of two fuzzy numbers. If only some of the parameters of the constraints are defined as fuzzy numbers, still, ranking methods for fuzzy numbers can be used. Because, ranking methods for fuzzy numbers can rank a fuzzy number with a crisp number. To carry out the ranking process a metaheuristic algorithm is used. For presenting the effectiveness of the proposed direct solution method, a fuzzy peak load pricing problem, a fuzzy product mix problem, a fuzzy multi-item economic order quantity problem and a fuzzy multi-objective aggregate production planning problem are solved. For the solution of the problems, two metaheuristic algorithms are used; the particle swarm optimization algorithm and the tabu search algorithm. The problems are solved by employing four different fuzzy ranking methods; the signed distance method, the integral value method, Chen and Chen's method and the ranking of fuzzy numbers through the comparison of their expected intervals. In the thesis, different instances of the problems are solved. For the solution of the problems, computer programs are written in C language.

In the thesis, the proposed direct solution method is used for solving fully fuzzy mathematical programming problems. In a fuzzy mathematical programming problem, when the parameters and the decision variables are fuzzy numbers, the arithmetic of two fuzzy numbers will be occurred. When two fuzzy numbers multiplied or divided by each other, the type of the obtained fuzzy number will not be same with the formers. For example, when two triangular fuzzy numbers are multiplied, the resultant will be a triangular shaped fuzzy number, but not a triangular fuzzy number. The left and right reference functions will not be linear functions. So, for these fuzzy numbers, general forms of fuzzy ranking methods can be used, not the special forms of fuzzy ranking methods which are generated for special types of fuzzy numbers. The product mix problem is solved as an example for the solution of fuzzy mathematical programming problems with fuzzy decision variables. The ranking method based on left and right dominance is used to rank the objective function values and to determine the feasibility of the constraints. The PSO algorithm is used to carry out the ranking process. For the solution of the problem a computer program is written in C language.

7.3. Conclusion

In this study, fuzzy mathematical programming models are reviewed and classified into fifteen types according to the fuzzy components they include. A literature review is carried out in order to see which types of problems are more frequently studied and what type of solution approaches are implemented. After the literature review, it is observed that the mostly frequently studied fuzzy mathematical programming models are;

-the fuzzy mathematical programming models with fuzzy objective(s),

-the fuzzy mathematical programming models with fuzzy right hand values of constraints,

-the fuzzy mathematical programming models with fuzzy cost coefficients,

-the fuzzy mathematical programming models with fuzzy objective(s) and fuzzy right hand values of constraints,

-the fuzzy mathematical programming models with fuzzy cost coefficients, fuzzy right hand values of constraints and fuzzy coefficients of constraints.

From the literature review it is observed that, the most common idea of the existing solution approaches is to transform the fuzzy model into a crisp model. After the transformation process, the obtained model is solved by using a conventional method according to the form of the resultant model (linear or nonlinear).

In this study, a direct solution method is proposed to solve fuzzy mathematical programming problems and fully fuzzy mathematical programming problems. In the proposed direct solution method, to rank the objective function values and to determine the feasibility of the constraints a ranking method for fuzzy numbers is used. A metaheuristic algorithm is used to carry out the ranking process. For demonstration, different fuzzy mathematical programming problems are solved using different ranking methods and metaheuristic algorithms. From the solutions, it has been observed that fuzzy mathematical programming problems with different fuzzy parameters and fully fuzzy mathematical programming problems can be solved effectively by using ranking methods of fuzzy numbers without any necessity of transformation into crisp equivalent.

7.4. Recommendations for Future Research

In this study, a direct solution method is proposed to solve fuzzy mathematical programming problems and fully fuzzy mathematical programming problems. Different problems are solved with the proposed direct solution method and it has been observed that fuzzy mathematical programming problems with different fuzzy parameters and fully fuzzy mathematical programming problems can be solved effectively. In future works, different problems can be solved using the proposed direct solution method. The implementation of other ranking methods for fuzzy numbers and other metaheuristic algorithms can be considered. Different types of fuzzy numbers can be used as parameters in fuzzy mathematical programming problems and solved using appropriate ranking method in the proposed direct solution method.

REFERENCES

Abboud, N., Inuiguchi, M., Sakawa, M. and Uemura, Y. (1998). Manpower allocation using genetic annealing. *European Journal of Operational Research*, **111**, 405-420.

Abo-Sinna, M.A., Amer, A.H., El Sayed, H.H. (2006). An interactive algorithm for decomposing the parametric space in fuzzy multiobjective dynamic programming problem. *Applied Mathematics and Computation*, **174**, 684-699.

Allahviranloo, T., Hosseinzadeh Lotfi, F., Kiasary, M.Kh., Kiani, N.A., Alizadeh, L. (2008). Solving fully fuzzy linear programming problem by the ranking function. *Applied Mathematical Sciences*, **2**, 19-32.

Amid, A., Ghodsypour, S.H., O'Brien, C. (2006). Fuzzy multiobjective linear model for supplier selection in a supply chain. *International Journal of Production Economics*, **104**, 394-407.

Asady, B. and Zendehnam, A. (2007). Ranking fuzzy numbers by distance minimization. *Applied Mathematical Modelling*, **31**, 2589-2598.

Baykasoglu, A. (2001a). MOAPPS 1.0: aggregate production planning using the multiple-objective tabu search. *International Journal of Production Research*, **29**, 3685-3702.

Baykasoglu, A. (2001b). Goal programming using multiple objective tabu search. *Journal of the Operational Research Society*, **52**, 1359-1369.

Baykasoğlu, A., Dereli, T., Daş, G.S., Göçken, T. (2004). Solution of multi-item fuzzy economic order quantity problem using tabu search and simulated annealing algorithms, YA/EM'2004 Yöneylem Araştırması / Endüstri Mühendisliği XXIV. Ulusal Kongresi, Adana, 15-18 June 2004, 157-159.

Baykasoğlu, A. and Göçken, T. (2006). A tabu search approach to fuzzy goal programs and an application to aggregate production planning. *Engineering Optimization*, **38**, 155-177.

Baykasoğlu, A. and Göçken, T. (2007). Solution of a Fully Fuzzy Multi-Item Economic Order Quantity Problem by Using Fuzzy Ranking Functions. *Engineering Optimization*, **39**, 919-939.

Baykasoğlu, A. and Göçken, T. (2008). A review and classification of fuzzy mathematical programs. *Journal of Intelligent and Fuzzy Systems*, **19**, 205-229.

Baykasoglu, A., Owen, S., Gindy, N. (1999a). Solution of goal programming models using a basic taboo search algorithm. *Journal of the Operational Research Society*, **50**, 960-973.

Baykasoğlu, A., Owen, S., Gindy, N. (1999b). A taboo search based approach to find the Pareto optimal set in multiple objective optimization. *Engineering Optimization*, **31**, 731-748.

Bector, C.R. and Chandra, S. (2005). *Fuzzy mathematical programming and fuzzy matrix games,* Studies in fuzziness and soft computing, Volume 169. Germany: Springer-Verlag Berlin Heidelberg.

Bellman, R.E. and Zadeh, L.A. (1970). Decision-making in a fuzzy environment. *Management Science*, **17**, B141-B164.

Buckley, J.J., Eslami, E., Feuring, T. (2002). *Fuzzy mathematics in economics and engineering*. Germany: Physica-Verlag Heidelberg.

Buckley, J.J., and Feuring, T. (2000). Evolutionary algorithm solution to fuzzy problems: fuzzy linear programming. *Fuzzy Sets and Systems*, **109**, 35-53.

Buckley, J.J., Feuring, T., Hayashi, Y. (1999). Neural net solutions to fuzzy linear programming. *Fuzzy Sets and Systems*, **106**, 99-111.

Buckley, J.J., Feuring, T., Hayashi, Y. (2001). Multi-objective fully fuzzified linear programming. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **9**, 605-621.

Cadenas, J.M. and Jimenez, F. (1994). Interactive decision-making in multiobjective fuzzy programming. *Mathware & Soft Computing*, **3**, 219-230.

Cadenas, J.M. and Verdegay, J.L. (2000). Using Ranking Functions in Multiobjective Fuzzy Linear Programming. *Fuzzy Sets and Systems*, **111**, 47-53.

Campos, L. and Verdegay, J.L. (1989). Linear Programming Problems and Ranking of Fuzzy Numbers. *Fuzzy Sets and Systems*, **32**, 1-11.

Carlsson, C. and Korhonen, P. (1986). A parametric approach to fuzzy linear programming. *Fuzzy Sets and Systems*, **20**, 17-30.

Chakraborty, M. and Gupta, S. (2002). Fuzzy mathematical programming for multi objective linear fractional programming problem. *Fuzzy Sets and Systems*, **125**, 335-342.

Chanas, S. (1983). The use of parametric programming in fuzzy linear programming. *Fuzzy Sets and Systems*, **11**, 243-251.

Chanas, S. and Kuchta, D. (1998). Fuzzy integer transportation problem. *Fuzzy Sets and Systems*, **98**, 291-298.

Chang, P.-T. and Lee, E.S. (1994). Ranking of fuzzy sets based on the concept of existence. *Computers & Mathematics with Applications*, **27**, 1-21.

Chen, S.-J. and Chen S.-M. (2003). A new method for handling multicriteria fuzzy decision making problems using FN_IOWA operators. *Cybernetics and Systems: An International Journal*, **34**, 109-137.

Chen, S.-J. and Hwang, C.L. (1992). *Fuzzy multiple attribute decision making methods and applications*. Germany: Springer-Verlag Berlin Heidelberg.

Chen, C.-K. and Lin, J.-M. (2001). A multi objective fuzzy optimization for optimum dimensions design of a convective spine. *International Communications in Heat and Mass Transfer*, **28**, 67-76.

Chen, L.H. and Lu, H.W. (2001). An approximate approach for ranking fuzzy numbers based on left and right dominance. *Computers and Mathematics with Applications*, **41**, 1580-1602.

Chen, L.H. and Lu, H.W. (2002). The preference order of fuzzy numbers. *Computers and Mathematics with Applications*, **44**, 1455-1465.

Chen, L.-H. and Weng, M.-C. (2003). A fuzzy model for exploiting quality function deployment. *Mathematical and Computer Modeling*, **38**, 559-570.

Cheng, C.H. (1998). A new approach for ranking fuzzy numbers by distance method. *Fuzzy Sets and Systems*, **95**, 307-317.

Chu, T.C. and Tsao, C.T. (2002). Ranking fuzzy numbers with an area between the centroid point and original point. *Computers and Mathematics with Applications*, **43**, 111-117.

Delgado, M., Verdegay, J.L., Vila, M.A. (1987). Imprecise costs in mathematical programming problems. *Control and Cybernet*, **16**, 113-121.

Delgado, M., Verdegay, J.L., Vila, M.A. (1988). A procedure for ranking fuzzy numbers using fuzzy relations. *Fuzzy Sets and Systems*, **26**, 49-62.

Delgado, M., Verdegay, J.L., Vila, M.A. (1990). Relating different approaches to solve linear programming problems with imprecise costs. *Fuzzy Sets and Systems*, **37**, 33-42.

Deng, Y., Zhenfu, Z., Qi, L. (2006). Ranking fuzzy numbers with an area method using radius of gyration. *Computers and Mathematics with Applications*, **51**, 1127-1136.

Dong, Y., Tang, J., Xu, B., Wang, D. (2005). An Application of Swarm Optimization to Nonlinear Programming. *Computers and Mathematics with Applications*, **49**, 1655-1668.

Dorohonceanu, B. and Marin, B. (2002). A simple method for comparing fuzzy numbers. http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.17.9044

Dubois, D. and Prade, H. (1983). Ranking fuzzy numbers in the setting of possibility theory. *Information Sciences*, **30**, 183–224.

Eberhart, R. and Kennedy, J. (1995). A New Optimizer Using Particle Swarm Theory. *Proceedings of the Sixth International Symposium on Micro Machine and Human Science*, 39-43.

Engelbrecht, A.P. (2005). *Fundamentals of computational swarm intelligence.* England: John Wiley & Sons Ltd.

Fedrizzi, M., Kacprzyk, J., Verdegay, J.L. (1991). *A survey of fuzzy optimization and mathematical programming.* In: Interactive Fuzzy Optimization, M. Fedrizzi, J. Kacprzyk, M. Roubens, ed., Heidelberg, pp. 15-28.

Fang, S.-C., Hu, C.-F., Wang, H.-F., Wu, S.-Y. (1999). Linear programming with fuzzy coefficients in constraints. *Computers and Mathematics with Applications*, **37**, 63-76.

Fung, R.Y.K., Tang, J. , Wang, D. (2003). Multiproduct aggregate production planning with fuzzy demands and fuzzy capacities. *IEEE Transactions on Systems, Man, and Cybernetics – Part A: Systems and Humans*, **33**, 302-313.

Gen, M., Ida, K., Lee, J., Kim, J. (1997). Fuzzy nonlinear goal programming using genetic algorithm. *Computers & Industrial Engineering*, **33**, 39-42.

Gen, M., Ida, K., Kobuchi, R. (1998). Neural network technique for fuzzy multiobjective linear programming. *Computers & Industrial Engineering*, **35**, 543-546.

Glover, F. (1990). Tabu search: a tutorial. *Interfaces,* **20**, 74-94.

Glover, F. (1993). A User's Guide to Tabu Search. *Annals of Operations Research*, **41**, 3-28.

Hashemi, S.M., Modarres, M., Nasrabadi, E., Nasrabadi, M.M. (2006). Fully fuzzified linear programming, solution and duality. *Journal of Intelligent & Fuzzy Systems*, **17**, 253-261.

Inuiguchi, M. and Ramik, J. (2000). Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. *Fuzzy Sets and Systems*, **111**, 3-28.

Iskander, M.G. (2002). Comparison of fuzzy numbers using possibility programming: comments and new concepts. *Computers and Mathematics with Applications*, **43**, 833-840.

Iskander, M.G. (2005). A suggested approach for possibility and necessity dominance indices in stochastic fuzzy linear programming. *Applied Mathematics Letters*, **18**, 395-399.

Jimenez, M. (1996). Ranking fuzzy numbers through the comparison of its expected intervals. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **4**, 379-388.

Jimenez, M., Arenas, M., Bilbao, A., Rodriguez, M.V. (2007). Linear programming with fuzzy parameters: an interactive method resolution. *European Journal of Operational Research*, **177**, 1599-1609.

Jimenez, L.M., Rodriguez, M.V., Arenas, M., Bilbao, A. (2000). Solving a Possibilistic Linear Program through Compromise Programming. *Mathware and Soft Computing*, **7**, 175-184.

Klir, G.J. and Yuan, B. (1995). *Fuzzy sets and fuzzy logic: theory and applications*. USA: Prentice Hall PTR.

Kuruüzüm, A. (1998). *Karar Destek Sistemlerinde Çok Amaçlı Yöntemler*. Antalya: Akdeniz Üniversitesi Basımevi.

Lai, Y.J. and Hwang, C.L. (1994). *Fuzzy Multiple Objective Decision Making, Methods and Applications*. Berlin Heidelberg: Lecture Notes in Economics and Mathematical Systems, Springer-Verlag.

Lee, E.S. and Li, R.J. (1993). Fuzzy multiple objective programming and compromise programming with Pareto optima. *Fuzzy Sets and Systems*, **53**, 275-288.

Lee, H.-M., and Yao, J.-S. (1999). Economic Order Quantity in Fuzzy Sense for Inventory without Backorder Model. *Fuzzy Sets and Systems*, **105**, 13-31.

Leung, Y. (1988). *Spatial Analysis and Planning under Imprecision*. Amsterdam: North-Holland.

Liou, T.-S. and Chen, C.-W. (2006). Fuzzy decision analysis for alternative selection using a fuzzy annual worth criterion. *The Engineering Economist*, **51**, 19-34.

Liou, T.-S. and Wang, M.J. (1992). Ranking fuzzy numbers with integral value. *Fuzzy Sets and Systems*, **50**, 247-255.

Liu, S.T. and Kao, C. (2004). Solving fuzzy transportation problems based on extension principle. *European Journal of Operational Research*, **153**, 661-674.

Liu, X. (2001). Measuring the satisfaction of constraints in fuzzy linear programming. *Fuzzy Sets and Systems*, **122**, 263-275.

Lodwick, W.A. and Jamison, K.D. (2007). Theoretical and semantic distinctions of fuzzy, possibilistic, and mixed fuzzy/possibilistic optimization. *Fuzzy Sets and Systems*, **158**, 1861-1872.

Luhandjula, M.K. (1989). Fuzzy optimization an appraisal. *Fuzzy Sets and Systems*, **30**, 257-282.

Mavrotas, G., Demertzis, H., Meintani, A., Diakoulaki, D. (2003). Energy planning in buildings under uncertainty in fuel costs: The case of a hotel unit in Greece. *Energy Conversion and Management*, **44**, 1303-1321.

Mondal, S. and Maiti, M. (2002). Multi-item fuzzy EOQ models using genetic algorithm. *Computers & Industrial Engineering*, **44**, 105-117.

Nakahara, Y. (1998). User oriented ranking criteria and its application to fuzzy mathematical programming problems. *Fuzzy Sets and Systems*, **94**, 275-286.

Nasrabadi, M.M. and Nasrabadi, E. (2004). A mathematical-programming approach to fuzzy linear regression analysis. *Applied Mathematics and Computation*, **155**, 873-881.

Negi, D.S. and Lee, E.S. (1993). Possibility programming by the comparison of fuzzy numbers. *Computers and Mathematics with Applications*, **25**, 43-50.

Negoita, C.V. (1981). The current interest in fuzzy optimization. *Fuzzy Sets and Systems*, **6**, 261-269.

Pires, F.M., Pires, J.M., Ribeiro, R.A. (1996). *Solving fuzzy optimization problems: flexible approaches using simulated annealing*. In: Proceedings of the World Automation Congress, WAC'96, TSI Press.

Prodanovic, P. (2001). *Fuzzy set ranking methods and multiple expert decision making*. The University of Western Ontario, Faculty of Engineering Science

Ribeiro, R.A. and Varela, L.R. (2003). Fuzzy optimization using simulated annealing: an example set. In: Fuzzy Sets Based Heuristics for Optimization, J.L. Verdegay, ed., Studies in Fuzziness and Soft Computing Series, Springer, Vol. 126, pp. 159-180.

Rommelfanger, H. (1991). FULP-APC supported procedure for solving multicriteria linear programming problems with fuzzy data. In: Interactive Fuzzy Optimization and Mathematical Programming, Fedrizzi, M., Kacprzyk, J., Rubens, M., eds., Springer, Berlin, pp. 154-167.

Rommelfanger, H. (1996). Fuzzy linear programming and applications. *European Journal of Operational Research*, **92**, 512-527.

Rommelfanger, H. and Slowinski, R. (1999). Fuzzy Linear Programming with single or multiple Objective Functions. In: Fuzzy Sets in Decision Analysis, Operations Research and Statistics, Slowinski, R., ed., Kluwer Academic Publishers, Norwell, Massachusetts, USA 1998, pp. 179-213.

Ross, T.J. (2004). *Fuzzy logic with engineering applications*. (2nd ed.). England: John Wiley & Sons Ltd.

Roy, T.K. and Maiti, M. (1997). A fuzzy EOQ model with demand-dependent unit cost under limited storage capacity. *European Journal of Operational Research*, **99**, 425-432.
Roy, T.K. and Maiti, M. (1998). Multi-objective inventory models of deteriorating items with some constraints in a fuzzy environment. *Computers & Operations Research*, **25**, 1085-1095.

Saad, G. (1982). An overview of production planning model: structure classification and empirical assessment. *International Journal of production Research*, **20**, 105-114.

Sakawa, M. and Nishizaki, I. (2002). Interactive fuzzy programming for two-level nonconvex programming problems with fuzzy parameters through genetic algorithms. *Fuzzy Sets and Systems*, **127**, 185-197.

Sakawa, M. and Yano, H. (1989). Interactive decision making for multi objective nonlinear programming problems with fuzzy parameters. *Fuzzy Sets and Systems*, **29**, 315-326.

Sasaki, M. and Gen, M. (2003). Fuzzy multiple objective optimal system design by hybrid genetic algorithm. *Applied Soft Computing*, **2/3F**, 189-196.

Shih, C.J., Chi, C.C., Hsiao, J.H. (2003). Alternative α-level-cuts methods for optimum structural design with fuzzy resources. *Computers & Structures*, **81**, 2579-2587.

Sinha, S. (2003). Fuzzy mathematical programming applied to multi-level programming problems. *Computers & Operations Research*, **30**, 1259-1268.

Sipper, D., Bulfin, R. L. Jr. (1997). *Production Planning, Control, and Integration*. New York: McGraw-Hill.

Stockton, D.J., and Quinn, L. (1993). Identifying Economic Order Quantities Using Genetic Algorithms. *International Journal of Operations & Production Management*, **13**, 92-103.

Syed, J.K., and Aziz, L.A. (2007). Fuzzy Inventory Model without Shortages Using Signed Distance Method. *Applied Mathematics & Information Sciences – An International Journal*, **1**, 203-209.

Tanaka, H., and Asai, K. (1984a). Fuzzy solution in fuzzy linear programming problems. *IEEE Transactions on Systems, Man, and Cybernetics*, **SMC-14**, 325-328.

Tanaka, H., and Asai, K. (1984b). fuzzy linear programming problems with fuzzy numbers. *Fuzzy Sets and Systems*, **13**, 1-10.

Tanaka, H., Guo, P., Zimmermann, H.J. (2000). Possibility distributions of fuzzy decision variables obtained from possibilistic linear programming problems. *Fuzzy Sets and Systems*, **113**, 323-332.

Tanaka, H., Hichihashi, H., Asai, K. (1984). A formulation of fuzzy linear programming problems based on comparison of fuzzy numbers. *Control and Cybernet*, **13**, 185-194.

Tang, J. and Wang, D. (1997). A nonsymmetric model for fuzzy nonlinear programming problems with penalty coefficients. *Computers & Operations Research*, **24**, 717-725.

Tang, J., Wang, D., Fung, R.Y.K. (2000). Fuzzy formulation for multi-product aggregate production planning. *Production Planning and Control*, **11**, 670-676.

Tavakkali-Moghaddam, R., Rabbani, M., Gharehgozli, A.H., Zaerpour, N. (2007). A fuzzy aggregate production planning model for make-to-stock environments. Proceedings of the 2007 IEEE International Conference on Industrial Engineering and Engineering Management, Singapore, 2-4 December 2007, pp. 1609-1613.

Tiwari, R.N., Dharmar, S., Rao, J.R. (1987). Fuzzy goal programming: an additive model. *Fuzzy Sets and Systems*, **24**, 27-34.

Tsakiris, G. and Spiliotis, M. (2004). Fuzzy linear programming for problems of water allocation under uncertainty. *European Water*, **7/8**, 25-37.

Verdegay, J.L. (1982). Fuzzy mathematical programming. In: Fuzzy Information and Decision Process, M.M. Gupta and E. Sanchez, ed., North-Holland, Amsterdam, pp. 231-237.

Verdegay, J.L. (1984). A dual approach to solve the fuzzy linear programming problem. *Fuzzy Sets and Systems*, **14**, 131-141.

Verma, A.K., Srividya, A., Deka, B.C. (2005). Composite System Reliability Assessment Using Fuzzy Linear Programming. *Electric Power Systems Research*, **73**, 143-149.

Yadavalli, V.-S.S., Jeeva, M., Rajagopalan, R. (2005). Multi-item deterministic fuzzy inventory model. *Asia-Pasific Journal of Operational Research*, **22**, 287-295.

Wang, D. and Fang, S.-C. (1997). A genetics-based approach for aggregated production planning in a fuzzy environment. *IEEE Transactions on Systems, Man, and Cybernetics – Part A: Systems and Humans*, **27**, 636-645.

Wang, R.-C. and Fang, H.-H. (2001). Aggregate production planning with multiple objectives in a fuzzy environment. *European Journal of Operational Research*, **133**, 521-536.

Wang, X. and Kerre, E.E. (2001). Reasonable properties for the ordering of fuzzy quantities (I). *Fuzzy Sets and Systems*, **118**, 375-385.

Wang, R.-C. and Liang, T.-F. (2004). Application of fuzzy multi-objective linear programming to aggregate production planning. *Computers & Industrial Engineering*, **46**, 17-41.

Wang, R.-C. and Liang, T.-F. (2005). Aggregate production planning with multiple fuzzy goals. *The Intelligent Journal of Advanced Manufacturing Technology*, **25**, 589-597.

Wang, X., Tang, W., Zhao, R. (2007). Fuzzy Economic Order Quantity Inventory Models without Backordering. *Tsinghua Science and Technology*, **12**, 91-96.

Wang, H.F. and Wang, M.L. (1997). A fuzzy multi objective linear programming. *Fuzzy Sets and Systems*, **86**, 61-72.

Wang, Y.-M., Yang, J.-B., Xu, D.-L., Chin, K.-S. (2006). On the centroids of fuzzy numbers. *Fuzzy Sets and Systems*, **157**, 919-926.

Yager, R.R. (1979). Ranking fuzzy subsets over the unit interval. Proceedings of 17th IEEE International Conference on Decision and Control, San Diego, CA, pp. 1435-1437.

Yao, J.-S., and Su, J.-S. (2000). Fuzzy Inventory with Backorder for Fuzzy Total Demand based on Interval-Valued Fuzzy Set. *European Journal of Operational Research*, **124**, 390-408.

Yao, J.-S. and Wu, K. (2000). Ranking fuzzy numbers based on decomposition principle and signed distance. *Fuzzy Sets and Systems*, **116**, 275-288.

Zadeh, L.A. (1965). Fuzzy sets. *Information and Control*, **8**, 338-353.

Zhang, G., Wu, Y.-H., Remios, M., Lu, J. (2003). Formulation of fuzzy linear programming problems as four-objective constrained optimization problems. *Applied Mathematics and Computation*, **139**, 383-399.

Zimmermann, H.J. (1983). Fuzzy mathematical programming. *Computers & Operations Research*, **10**, 291-298.

Zimmermann, H.J. (1976). Description and optimization of fuzzy systems. *International Journal of General Systems*, **2**, 209-215.

Zimmermann, H.J. (1978). Fuzzy linear programming with several objective functions. *Fuzzy Sets and Systems*, **1**, 46-55.

Zimmermann, H.J. (1987). *Fuzzy Sets, Decision Making and Expert Systems*. Boston: Kluwer Academic Publishers.

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FOREIGN LANGUAGES

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PUBLICATIONS RELATED TO THESIS

International Journal Papers

1. Baykasoğlu, A., Göçken T., A review and classification of fuzzy mathematical programs, Journal of Intelligent and Fuzzy Systems, 19(3), 205-229, 2008.

2. Baykasoğlu, A., Göçken T., Solution of a fully fuzzy multi-item economic order quantity problem using fuzzy ranking functions, Engineering Optimization, 39(8), 919-939, 2007.

3. Baykasoğlu, A., Göçken T., A tabu search approach to fuzzy goal programs and an application to aggregate production planning, Engineering Optimization, 38(2), 155-177, 2006.

Book Chapters

1. Baykasoğlu, A., Göçken T., Employing "particle swarm optimization" and "fuzzy ranking functions" for direct solution of EOQ problem, Modelling, Computation and Optimization in Information Systems and Management Sciences, Second International Conference MCO 2008, Metz, France-Luxemburg, September 8-10, 2008, Hoai An, Le Thi; Bouvry, Pascal; Pham Dinh, Tao (eds.), ISBN: 978-3-540-87476-8, Springer, Communications in Computer and Information Science, Vol.14, 32-42.

2. Baykasoğlu, A., Göçken T., Solving multi-item economic order quantity problems with fuzzy parameters via particle swarm optimization, Geiger, M.J., Habenicht, W., Sevaux, M., Sörensen, K. (eds.), Metaheuristics in the Service Industry, in Lecture Notes in Economics and Mathematical Systems, Springer Verlag, ISSN 0075-8442, accepted.

International Conference Papers

1. Baykasoğlu, A., Göçken, T., Solving Fuzzy Multi-Item Economic Order Quantity Problems via Fuzzy Ranking Functions and Particle Swarm Optimization, Proceedings of EU/ME 2007 (eds., Geiger, M.J., Habenicht, W.): "Metaheuristics in the Service Industry", 8th Workshop of the EURO Working Group "EU/ME, the European Chapter on Metaheuristics", October 4-5, 2007, Stuttgart, Germany, pp. 90-95.

2. Baykasoğlu, A., Göçken, T., Solving fully fuzzy multi-item EOQ models, IMS-2006, 5th Int. Symposium on Intelligent Manufacturing Systems, pp. 784-791, 29-31 May 2006, Sakarya, Turkey, 2006.

3. Baykasoglu, A., Dereli, T., Göçken, T., A note on the classification of fuzzy mathematical programming models, IMS-2004, 4th Int. Symposium on Intelligent Manufacturing Systems, 6-8 September 2004, Sakarya, Turkey, pp. 1057-1070, 2004.

4. Baykasoglu, A., Sevim, T., A review and classification of fuzzy approaches to multi objective optimization, TAINN ' 2003, International Twelfth Turkish Symposium on Artificial Intelligence and Neural Networks, 02-04, July, 2003, Çanakkale, Turkey, Vol. E6, pp. 578-580, 2003.

National Conference Papers

1. Baykasoğlu, A., Göçken, T., Bulanık çok objektifli üretim planlama modelinin bulanık sayı sıralama yöntemleri ile çözülmesi, YA/EM 2007: Yöneylem Araştırması ve Endüstri Mühendisliği 27. Ulusal Kongresi, 02-04 Temmuz 2007, İzmir, 302-307.

2. Baykasoğlu, A., Göçken, T., Taxonomy of fuzzy mathematical programming and a review of fully fuzzified models, GAP V. Mühendislik Kongresi, pp. 155-162, 26-28 Nisan 2006, Şanlıurfa, 2006.

3. Baykasoglu, A., Dereli, T., Emre, S., Göçken, T., Çok ürünlü bulanık ekonomik parti büyüklüğü probleminin tabu arama ve tavlama benzetimi algoritmaları ile çözülmesi, YA/EM'2004: Yöneylem Araştırması / Endüstri Mühendisliği Kongresi XXIV. Ulusal Kongresi, 15-18 Haziran 2004, Gaziantep, Adana, pp. 157-159, 2004.

4. Baykasoglu, A., Dereli, T., Göçken, T., Emre, S., Çok objektifli üretim planlaması problemlerinin bulanık matematiksel programlama ile çözülmesi, YA/EM'2004: Yöneylem Araştırması / Endüstri Mühendisliği Kongresi XXIV. Ulusal Kongresi, 15-18 Haziran 2004, Gaziantep, Adana, pp. 500-502, 2004.