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GRADUATE SCHOOL OF
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**APPLICATION OF ASYMPTOTIC
ITERATION METHOD TO SOME
PT-SYMMETRIC POTENTIALS**

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IN
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**Application of Asymptotic Iteration Method to Some
PT-Symmetric Potentials**

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**Supervisor
Assoc. Prof. Dr. Okan Özer**

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ABSTRACT

APPLICATION of ASYMPTOTIC ITERATION METHOD to SOME PT-SYMMETRIC POTENTIALS

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In this thesis we investigate the application of the asymptotic iteration method, which has received a lot of attention in the literature recently, to some PT-symmetric potentials. We obtain the energy eigenvalues and wavefunctions for the corresponding potentials and compare our results with those in the related literature. We have observed that the asymptotic iteration method is a good candidate for the solution of the PT-symmetric potentials if an appropriate ansatz wavefunction is suggested in the framework of the model.

Key Words: Asymptotic iteration method, PT-symmetric potentials, energy eigenvalues

ÖZET

ASİMTOTİK İTERASYON METODU'NUN BAZI PT-SİMETRİK POTANSİYELLERE UYGULANMASI

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Bu tez çalışmasında son zamanlarda oldukça dikkat çeken asimtotik iterasyon metodunun bazı PT-simetrik potansiyellere uygulanması incelendi. Göz önüne alınan potansiyellerin enerji özdeğerleri ve dalga fonksiyonları elde edilmiş ve literatürde karşı gelen değerlerle karşılaştırılması yapılmıştır. PT-simetrik potansiyellerin çözümü için asimtotik iterasyon metodunun, uygun bir dalga fonksiyonu önermesi ile iyi bir çözüm metodu olacağı görüldü.

Anahtar Kelimeler: Asimtotik iterasyon metodu, PT-simetrik potansiyeller, enerji özdeğerleri

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CHAPTER 1

INTRODUCTION

Since a Hamiltonian H specifies the energy levels and time evolution of a quantum system, one generally requires that H be Hermitian because Hermiticity guarantees that the energy spectrum is real and that time evolution is unitary (probability preserving). However, in some cases the physical situation is such that the application of non-Hermitian Hamiltonians is justified. These conditions generally occur for complex potentials used in a variety of phenomena in different fields of physics and chemistry. These Hamiltonians, for example, are used in the context of the optical potential in nuclear physics (especially, in accounting for absorption of incident particles), to study the delocalization transitions in condensed matter systems (such as a vortex flux line depinning in type –II superconductors), to study the population biology, in the description of a Bose system of hard spheres, to study the energy spectra of complex Toda lattice, quantum cosmology, quantum field theory, super symmetric quantum mechanics, etc. [1,2]. The discrete energy eigenvalues of these complex potentials become complex, in general. Until the paper of Bender and Boettcher in 1998 [1], it was supposed that the Hermiticity of an Hamiltonian was the necessary condition for having real spectrum. Their paper has introduced the concept of PT -symmetric Hamiltonians, i.e, a complex (non-Hermitian) Hamiltonian can give real and bounded eigenvalues if Hamiltonian is invariant under the simultaneous action of space (P) and time (T) reversal. For one-dimensional potentials of non-relativistic quantum mechanics this invariance requires $(V(x))^* = V(x)$. These potentials are a great deal of interest in the study of scattering theory because of their applications in scattering problems, and the complex potentials are also being called optical or average nuclear potentials [3,4].

Although the potentials exhibiting PT -invariance are usually complex, their bound-state energy eigenvalues were often found to be real. In the analysis of PT -invariant potentials various approaches have been applied such as the Fourier transformation [5], semi classical estimates [6], numerical calculations [7,8], Sturm-Liouville-like theory [9,10], variational techniques [11] or perturbation methods [12].

On the other hand, a certain part of quantum-mechanical potentials are called *quasi-exactly solvable* (QES) if one obtains a finite portion of the spectrum and associated eigenfunctions exactly and in closed form [13]. They are the intermediate potentials between exactly solvable equations, such as the harmonic oscillator equations, the Coulomb equation, etc. whose all analytic solutions can be obtained, and the analytically unsolvable ones requiring a numerical treatment. QES potentials are defined in terms of a parameter J and then one can obtain exactly the first J^{th} energy values and wavefunctions, for positive integer (and half-integer) values of J . There is in fact a Lie-algebraic formalism behind most QES systems and all one-dimensional QES systems have been particularly classified based on $sl(2)$ algebra [14, 15].

In recent years, a simple technique called the asymptotic iteration method (AIM) has been taken much attention to obtain energy eigenvalues and eigenfunctions of the class of differential equations [16, 17]. By using this technique, one can reproduce exact solutions to many differential equations which are important in applications to many problems in physics, such as the equations of Hermite, Laguerre, Legendre and Bessel [22]. In the case of most solvable potentials the AIM has reproduced the exact spectrum [18, 19] while for non exactly solvable potentials it yields reasonably good approximate values [20, 21].

The purpose of the present thesis is to apply the AIM to some PT -symmetric potentials and to obtain their energy eigenvalues and eigenfunctions. The organization of the thesis is as following: In Chapter 2 we introduce the AIM. In Chapter 3, solutions of Khare-Mandal (KM) potential and its PT -symmetric partner are presented by using AIM. The KM potential and its PT - symmetric partner

potential is studied in order to see which potential has real eigenvalues and convenient eigenfunctions, for different values of potential parameters M and ζ in the framework of AIM. We also apply the method to the complex quartic potential and determine the energy eigenvalues for certain values of the potential parameters. Finally, in Chapter 4 we present a general conclusion on the results obtained in this thesis work.

CHAPTER 2

THE ASYMPTOTIC ITERATION METHOD

AIM [16,18] proposes to solve the second-order differential equations of the form of

$$f''(x) = \lambda_0(x)f'(x) + s_0(x)f(x) \quad (2.1.1)$$

where $f(x)$ is a function of x and the prime denotes the first and second derivatives with respect to x . $\lambda_0(x)$ and $s_0(x)$ are sufficiently differentiable arbitrary functions and $\lambda_0(x) \neq 0$. To obtain a general solution to this equation, AIM suggests to differentiate Eq.(2.1.1) with respect to x , then one finds

$$f'''(x) = \lambda_1(x)f'(x) + s_1(x)f(x) \quad (2.1.2)$$

in which

$$\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2 \quad (2.1.3)$$

$$s_1(x) = s_0'(x) + s_0(x)\lambda_0(x)$$

Similarly, taking the the second derivative of the Eq. (2.1.1) we have

$$f^{(4)}(x) = \lambda_2(x)f'(x) + s_2(x)f(x) \quad (2.1.4)$$

where

$$\begin{aligned} \lambda_2(x) &= \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x) \\ \text{and} \\ s_2(x) &= s_1'(x) + s_0(x)\lambda_1(x). \end{aligned} \quad (2.1.5)$$

Eq.(2.1.1) can be easily iterated up to $(k+1)$ th and $(k+2)$ th derivatives, where $k = 1, 2, 3, \dots$. Therefore, we obtain

$$\begin{aligned} f^{(k+1)}(x) &= \lambda_{k-1}(x)f'(x) + s_{k-1}(x)f(x) \\ \text{and} \\ f^{(k+2)}(x) &= \lambda_k(x)f'(x) + s_k(x)f(x) \end{aligned} \quad (2.1.6)$$

where we can write

$$\begin{aligned} \lambda_k(x) &= \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x) \\ \text{and} \\ s_k(x) &= s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x) \end{aligned} \quad (2.1.7)$$

which are called the “recurrence relation” for Eq.(2.1.1). Taking the ratio of the $(k+2)$ th and $(k+1)$ th derivatives in Eq.(2.6.1), we obtain

$$\frac{f^{(k+2)}(x)}{f^{(k+1)}(x)} = \frac{d}{dx} \ln[f^{(k+1)}(x)] = \frac{\lambda_k \left| f'(x) + \left(\frac{s_k(x)}{\lambda_k(x)} \right) f(x) \right|}{\lambda_{k-1} \left| f'(x) + \left(\frac{s_{k-1}(x)}{\lambda_{k-1}(x)} \right) f(x) \right|} \quad (2.1.8)$$

Assuming that for sufficiently large k , if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x) \quad (2.1.9)$$

is satisfied, which is the ‘asymptotic’ aspect of the method, then, Eq.(2.1.8) is directly reduced to

$$\frac{d}{dx} \ln[f^{(k+1)}(x)] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)} \quad (2.1.10)$$

If we substitute $\lambda_k(x)$ from Eq.(2.1.7) and then using $\alpha(x)$ in the right hand side of Eq.(2.1.10), one sees

$$\begin{aligned} f^{(k+1)}(x) &= C_1 \exp \left(\int \frac{\lambda_k(x)}{\lambda_{k-1}(x)} dx \right) \\ &= C_1 \exp \left(\int \frac{\lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x)}{\lambda_{k-1}(x)} dx \right) \\ &= C_1 \exp \left(\int \frac{\lambda'_{k-1}(x)}{\lambda_{k-1}(x)} dx + \int \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} dx + \int \frac{\lambda_0(x)\lambda_{k-1}(x)}{\lambda_{k-1}(x)} dx \right) \\ &= C_1 \exp \left(\int \frac{\lambda'_{k-1}(x)}{\lambda_{k-1}(x)} dx + \int \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} dx + \int \frac{\lambda_0(x)\lambda_{k-1}(x)}{\lambda_{k-1}(x)} dx \right) \\ &= C_1 \lambda_{k-1}(x) \exp \left(\int [\alpha(x) + \lambda_0(x)] dx \right) \end{aligned} \quad (2.1.11)$$

where C_1 is the integration constant. Inserting Eq.(2.1.11) into Eq.(2.1.6), a first-order differential equation for $f(x)$ is obtained as

$$f'(x) + \alpha(x)f(x) = C_1 \exp \left\{ \int [\alpha(x) + \lambda_0(x)] dx \right\} \quad (2.1.12)$$

Eq.(2.1.12) is in the form of the first-order linear differential equations [22] as

$$y'(x) + p(x)y(x) = r(x) \quad (2.1.13)$$

whose general solution is given as

$$y(x) = e^{-\int p(x)} \left[\int e^{\int p(x)} r(x) dx + c \right] \quad (2.1.14)$$

where c is integral constant. Using the general result in Eq.(2.1.14), we get the general solution of Eq.(2.1.12) as:

$$f(x) = \exp \left\{ - \int^x \alpha(x_1) dx_1 \right\} C_1 + C_2 \int^x \exp \left(\int^{x_1} [\lambda_0(x_2) + 2\alpha(x_2)] dx_2 \right) dx_1 \quad (2.1.15)$$

Since one-dimensional time-independent Schrödinger equation ($\hbar = 2m = 1$) is

$$-\psi''(x) + [V(x) - E]\psi = 0, \quad (2.1.16)$$

one can transform Eq. (2.1.16) into Eq.(2.1.1) by the use of an appropriate coordinate transformation, if required, considering also a suitable form of the wavefunction given below

$$\psi(x) = g(x)f(x). \quad (2.1.17)$$

We note here that in the wide applications of AIM the function $g(x)$ is in the asymptotic behavior for the system under consideration. The function $f(x)$ is obtained in the polynomial form using Eq.(2.1.15) that does not disturb the asymptotic behavior. In the method, the energy eigenvalues can be determined by the quantization condition, given by the termination condition in Eq.(2.1.9). Thus one can write the quantization condition combined with Eq.(2.1.7) as

$$\delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0, \quad k = 1, 2, 3, \dots \quad (2.1.18)$$

After transforming the Schrodinger equation into the form of Eq.(2.1.1), the energy spectrum and wave function of the quantum system can be obtained analytically (or numerically). Using Eq.(2.1.1) one can determine $s_0(x)$ and $\lambda_0(x)$ and then $s_k(x)$ and $\lambda_k(x)$ parameters in an iterative procedure. The energy eigenvalues of the potential interested are obtained by the quantization condition Eq.(2.1.18) and the wave functions are determined using the following wave function generator

$$f_n(x) = C_2 \exp\left(-\int^x \frac{s_k(x')}{\lambda_k(x')} dx'\right) \quad (2.1.19)$$

Although the general solution of Eq. (2.1.1) is given by Eq. (2.1.15), the first part of Eq. (2.1.15) gives the polynomial solutions that are convergent and physical, whereas the second part gives non-physical solutions that are divergent. Therefore, the

corresponding eigenfunctions can be derived from the wave function generator given in Eq.(2.1.19) by choosing $C_2 = 0$ in Eq.(2.1.15).

On the other hand, Eq.(2.1.9) implies that the wave functions are truncated for sufficiently large values of k and the roots of the expression given in Eq.(2.1.18), which has been obtained from Eq.(2.1.7), belong indirectly to the spectrum of Eq.(2.1.16). However, for each iteration of the expression Eq.(2.1.18) depends on different variables, such as E , x and possible potential parameters. We also note that the iterations should be terminated by imposing the quantization condition $\delta_k(x) = 0$ as an approximation to Eq.(2.1.9) to obtain the eigenenergies. Therefore, the calculated eigenenergies E_n by means of this condition should be independent of the choice of the coordinate. The energy eigenvalues can easily be obtained from the roots of Eq.(2.1.18) if the problem is exactly solvable. If not, for a specific n quantum number, one has to choose an appropriate x_0 point (which satisfies $\delta_k(x) = 0$), determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential [16, 20, 23] and then the approximate energy eigenvalues are obtained from the roots of this equation for sufficiently large values of k with iteration. Thus the choice of x_0 is observed to be critical only to the speed of the convergence of the eigenenergies, as well as for the stability of the process.

CHAPTER 3

APPLICATIONS

3.1 The Khare-Mandal and its PT-Symmetric QES Partner Potential

Khare *et al.* [24] have defined a one-dimensional quasi-exactly solvable (QES) complex potential as

$$V(x) = -(\zeta \cosh 2x - iM)^2 \quad (3.1.1)$$

where ζ is real and M is positive integer value. The potential is introduced as *PT*-invariant ($i \rightarrow -i, x \rightarrow a - x$) where $a/2$ is the origin about which one is performing and the Hermiticity of the Hamiltonian for that potential is determined by the even and odd integer values of the parameter M and some critical values of ζ . On the other hand, Bagchi *et al.* [25] has introduced a *PT*-symmetric partner potential given as

$$V(x) = -(\zeta \sinh 2x - iM)^2 \quad (3.1.2)$$

and they have also shown that the potential in Eq.(3.1.2) is *PT*-invariant potential exhibiting real energy eigenvalues both for *even* and *odd* integer values of M and any value of ζ . To proceed we consider both potentials and write the corresponding Hamiltonians as

$$H^+ = -\frac{d^2}{dx^2} - (\zeta \cosh 2x - iM)^2 \quad (3.1.3)$$

and

$$H^- = -\frac{d^2}{dx^2} - (\zeta \sinh 2x - iM)^2 \quad (3.1.4)$$

If we write trigonometric functions in terms of their exponential forms as $\cosh(x) = (e^{2x} + e^{-2x})/2$ and $\sinh(x) = (e^{2x} - e^{-2x})/2$, then we can express both equations together as

$$H^\pm = -\frac{d^2}{dx^2} - \left[\frac{\zeta}{2} (e^{2x} \pm e^{-2x}) - iM \right]^2 \quad (3.1.5)$$

If we change the variable $x = \frac{1}{2} \ln z$, then we obtain Hamiltonians in the form of

$$-\frac{d^2}{dz^2} \psi^\pm(z) - \left(\frac{1}{z} \right) \frac{d}{dz} \psi^\pm(z) - \left(\frac{4Ez^2 - [2Mz + i(z^2 \pm 1)\zeta]^2}{16z^4} \right) \psi^\pm(z) = 0. \quad (3.1.6)$$

To apply the asymptotic iteration method, we set the ansatz wavefunctions $\psi^\pm(z)$ as

$$\psi^\pm(z) = z^{\frac{1-M}{2}} \exp \left[\frac{i\zeta}{4} \left(z \pm \frac{1}{z} \right) \right] f_\pm(z) \quad (3.1.7)$$

Inserting Eq.(3.1.7) into Eq.(3.1.6), one obtains a second-order linear homogeneous differential equation in the form of Eq.(2.1.1)

$$\begin{aligned}
-f_{\pm}''(z) - \frac{i(z^2 \pm 1)\zeta - 2(M-2)z}{2z^3} f_{\pm}'(z) \\
- \frac{1+E-2M-2i(M-1)\zeta z \pm \zeta^2}{4z^2} f_{\pm}(z) = 0
\end{aligned} \tag{3.1.8}$$

We can now apply the AIM for the solution of the Eq.(3.1.8). If we compare Eq.(3.1.1) and Eq.(3.1.8), we find

$$\lambda_0(z) = -\frac{i(z^2 \pm 1)\zeta - 2(M-2)z}{2z^2}$$

and

$$s_0(z) = -\frac{1+E-2M-2i(M-1)\zeta z \pm \zeta^2}{4z^2}. \tag{3.1.9}$$

By means of Eq.(2.1.11) in the previous Chapter, one can also calculate $\lambda_k(z)$ and $s_k(z)$ easily. Finally, one can use the quantization condition given by Eq.(2.1.18) to find the energy eigenvalues of these potentials. Since potentials depend on the parameter M and it is defined as $M = 2j + 1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$) in the true spirit of quasi-exact solvability [13], then we observe that the quantization condition in Eq.(2.1.18) can be applied for certain values of M . Thus, one finds only a finite portion of the energy spectrum and associated eigenfunctions, in the closed form. To find energy eigenvalues we apply the quantization condition as following:

If $j = 0$, then $M = 1$ and one finds

$$\begin{aligned}
\lambda_0(z) &= -\frac{2z + i(z^2 \pm 1)\zeta}{2z^2}, \\
s_0(z) &= -\frac{E - 1 \pm \zeta^2}{4z^2}
\end{aligned} \tag{3.1.10}$$

Applying Eq.(2.1.18) for $k = 3$, we determine the energy eigenvalues for both potentials as

$$E^\pm = 1 \pm \zeta^2 \quad (3.1.11)$$

It is seen that the spectrum corresponding to potential Eq.(3.1.1) as well as its PT -symmetric version one Eq.(3.1.2) are completely real if $M = 1$ ($j = 0$). Using Eq. (2.1.19) and Eq.(2.1.7) by determining $\lambda_k(z)$ and $s_k(z)$, and then re-changing the coordinate z in terms of x , one can easily find the accompanying wave functions for both potentials as

$$\begin{aligned} \psi^+(x) &= \exp\left(\frac{i\zeta}{2} \cosh 2x\right), \\ \psi^-(x) &= \exp\left(\frac{i\zeta}{2} \sinh 2x\right). \end{aligned} \quad (3.1.12)$$

Next, we consider the case $M = 2$ where $j = 1/2$, leading to

$$\begin{aligned} \lambda_0(z) &= -\frac{i(z^2 \pm 1)\zeta}{2z^2}, \\ s_0(z) &= -\frac{E - 3 - 2i\zeta z \pm \zeta^2}{4z^2}. \end{aligned} \quad (3.1.13)$$

Applying Eq.(2.1.18) for $k = 4$, one can now determine the energy eigenvalues and corresponding eigenfunctions as

$$\begin{aligned} E^+ &= 3 \pm 2i\zeta - \zeta^2 \\ \psi^+(x) &= \exp\left(\frac{i\zeta}{2} \cosh 2x\right) (e^{-x} \pm e^x) \end{aligned} \quad (3.1.14)$$

for KM potential and

$$E^- = 3 \pm 2i\zeta - \zeta^2$$

$$\psi^-(x) = \exp\left(\frac{i\zeta}{2} \sinh 2x\right) (e^{-x} \pm e^x) \quad (3.1.15)$$

for the PT -symmetric partner of KM potential.

It is realized that the energy eigenvalues for the KM potential are obtained in the complex conjugate pairs in Eq.(3.1.14) for $M = 2$ even integer. However, we find those are real in Eq.(3.1.15) for the PT -symmetric potential. It is obvious that one can easily determine the energy eigenvalues up to $n = 2$ quantum state, in closed form, for both potentials by using AIM just for few iterations. We have presented some energy eigenvalues and accompanying eigenfunctions in the Table 3.1.

3.2 Conclusion

In this part, we have used AIM to obtain the energy spectrum of Khare-Mandal potential and its PT -symmetric partner. We note that the method gives the eigenvalues directly if one can transform the Schrodinger equation into a form of $f''(x) = \lambda_0(x)f'(x) + s_0(x)f(x)$. The energy eigenvalues we have obtained are in excellent agreement with the existing results and we have also introduced the wave functions. We note that the energy eigenvalues for the KM potential strongly depends on one of the potential parameter, ζ . One can observe that if $\zeta > \frac{1}{2}$, then it is seen in Table 3.1 the whole spectrum (E^+) of KM potential becomes completely complex both for even and odd integer values of parameter, M . This is not true for the PT -symmetric partner one and it exhibits a real spectrum for any value of ζ and both for even and odd integer values M .

Table 3.1 Eigenvalues and eigenfunctions for the Khare-Mandal potential and its PT -symmetric partner for $M = 1, 2, 3, 4$. We note that the PT -symmetric partner potential energy values (E^- , where $q = \pm$ and $p = \pm$) are real both for even and odd integer values of M , and any values of ζ

M	E^\pm	$\psi^\pm(x)$
1	$E^+ = 1 \pm \zeta^2$ $E^- = 1 \pm \zeta^2$	$\psi^+ = \exp\left(\frac{i\zeta}{2} \cosh 2x\right)$ $\psi^- = \exp\left(\frac{i\zeta}{2} \sinh 2x\right)$
2	$E^+ = 3 \pm 2i\zeta - \zeta^2$ $E^- = 3 \pm 2i\zeta + \zeta^2$	$\psi^+ = \exp\left(\frac{i\zeta}{2} \cosh 2x\right) (e^{-x} \pm e^x)$ $\psi^- = \exp\left(\frac{i\zeta}{2} \sinh 2x\right) (e^{-x} \pm ie^x)$
3	$E^+ = 5 - \zeta^2$ $E^+ = 7 - \zeta^2 \pm 2\sqrt{1 - 4\zeta^2}$ $E^- = 5 + \zeta^2$ $E^- = 7 + \zeta^2 \pm 2\sqrt{1 + 4\zeta^2}$	$\psi^+ = \exp\left(\frac{i\zeta}{2} \cosh 2x\right) \sinh 2x$ $\psi^+ = \exp\left(\frac{i\zeta}{2} \cosh 2x\right) \left(2 \cosh 2x - \frac{i}{\zeta} (1 \pm \sqrt{1 - 4\zeta^2})\right)$ $\psi^- = \exp\left(\frac{i\zeta}{2} \sinh 2x\right) \cosh 2x$ $\psi^- = \exp\left(\frac{i\zeta}{2} \sinh 2x\right) \left(2 \sinh 2x - \frac{i}{\zeta} (1 \pm \sqrt{1 + 4\zeta^2})\right)$
4	$E_{q,p}^+ = 11 - \zeta^2 - q2i\zeta + 4p\sqrt{1 - qi\zeta - \zeta^2}$ $E_{q,p}^+ = 11 + \zeta^2 - q2\zeta + 4p\sqrt{1 - q\zeta + \zeta^2}$	$\psi_{q,p}^+ = \exp\left(\frac{i\zeta}{2} \cosh 2x\right) x (e^{-x} - qe^x) \left(\cosh 2x - \frac{i}{\zeta} (1 + p\sqrt{1 - q\zeta - \zeta^2})\right)$ $\psi_{q,p}^- = \exp\left(\frac{i\zeta}{2} \sinh 2x\right) x (e^{-x} - qie^x) \left(\sinh 2x - \frac{i}{\zeta} (1 + p\sqrt{1 - q\zeta + \zeta^2})\right)$

3.3 Quartic Complex Potential

When one chooses a real potential, that is the Hermitian Hamiltonian, then it is ensured that the corresponding Schrödinger equation has real eigen-energies. Bender and Boettcher [1] introduced complex potentials which are *invariant* under the combined symmetry of PT , and showed that even in all these cases the eigenenergies of the differential equation are real. This seems to suggest that, instead of Hermiticity, it may be enough to have the weaker condition of PT -symmetry in order to have real eigenenergies.

Bender and Boettcher generalized to a new kind of PT -symmetric quartic complex potential of the form

$$V(x) = -x^4 + 2iax^3 + (a^2 - 2b)x^2 + 2i(ab - J)x, \quad a, b \in \mathbf{R}, j = 1, 2, \dots \quad (3.3.1)$$

defined on a certain complex curve $x = x(t) \in \mathbf{C}, t \in (-\infty, \infty)$ [26]. In order to obtain the eigenenergies of the quartic complex potential Eq.(3.3.1), Bender and Boettcher solved the Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E_n\psi(x), \quad n = 0, 1, 2, \dots \quad (3.3.2)$$

where the wave function $\psi(x)$ satisfied the boundary condition Eq.(3.3.2)

$$\lim_{|x| \rightarrow \infty} \psi(x) = 0. \quad (3.3.3)$$

and the primes of $\psi(x)$ in Eq. (3.3.2) denote derivatives with respect to x .

By introducing the exponential representation for eigenfunctions as

$$\psi(x) = e^{-i\frac{x^3}{3} - a\frac{x^2}{2} - ibx} p_{j-1}(x) \quad (3.3.4)$$

and using QES for Eq.(3.3.2), they obtained polynomials $Q_J(E)$ of degree J in the eigenenergies E_n (See Ref. [1] for details).

We apply AIM to determine the eigenenergies of the quartic complex potential given in Eq.(3.3.1). To do that we write the Schrödinger equation in its general form in the one-dimensional case for the eigenvalue equation (where we only set now $\hbar = 1$ for the comparison of our work with the related literature):

$$-\frac{1}{2m}\psi''(x) + V(x)\psi(x) = E_n\psi(x) \quad (3.3.5)$$

in which $V(x)$ is the general representation of the quartic complex potential with the following form:

$$V(x) = -A_4x^4 + iA_3x^3 + A_2x^2 + iA_1x, \quad (3.3.6)$$

which is defined on a certain complex curve $x = x(t) \in \mathcal{C}, t \in (-\infty, \infty)$ and $\{A_4, A_3, A_2, A_1\} \in \mathbf{R}$.

To solve the Schrödinger Eq.(3.3.5) involving the potential in Eq.(3.3.6) we introduce the following wave function:

$$\psi(x) = e^{-(\alpha ix^3 + \beta x^2 + \gamma ix)} f(x) \quad (3.3.7)$$

where

$$\alpha = \frac{1}{3}A_4, \quad \beta = \frac{1}{4}A_3, \quad \gamma = \frac{1}{8}A_3^2 - \frac{1}{2}A_2 \quad (3.3.8)$$

Substituting the wave function $\psi(x)$ into the Schrödinger equation in Eq.(3.3.5) , that leads to second-order differential equation in the form of

$$f''(x) = \lambda_0 f'(x) + s_0 f(x) \quad (3.3.9)$$

where $\lambda_0(x)$ and $s_0(x)$ is obtained by

$$\lambda_0(x) = -iA_2 + \frac{iA_3^2}{4} + A_3x + 2iA_4x^2, \quad (3.3.10)$$

and

$$\begin{aligned}
s_0(x) = \frac{1}{64} & \left[(-4A_2 + A_3^2)^2 + 8i(4A_2A_3 - A_3^3 + 16A_4)x \right. \\
& - 16(4A_2(-2 + A_4) - A_3^2(-1 + A_4))x^2 \\
& - 64iA_3(-2 + A_4)x^3 + 64(-2 + A_4)A_4x^4 \\
& \left. + 32(A_3 - 4E_1 + 4iA_1x) \right]
\end{aligned}$$

We can now determine $\lambda_k(x)$ and $s_k(x)$ by Eq.(2.1.18) to obtain the energy eigenvalues of the potential Eq.(3.3.6). Using a MATHEMATICA program, the energy eigenvalues can be found by Eq.(2.1.18). Since the quantization condition $\delta_k(x)$ will depend on different variables, such as E , x and potential parameters, then we first eliminate the coordinate parameter at the end of the iteration procedure. Because the potential is not exactly solvable, then we have to choose a suitable x_0 point, satisfying $\delta_k(x) = 0$, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential [16,20,23], and then the approximate energy eigenvalues are obtained from the roots of this equation for sufficiently large values of k with iteration. Therefore, we set $x_0 = 0$ at the end of the iterations.

Additionally, we have to find the minimum number of iterations, k , in order to get an idea regarding the rate of convergence of the AIM. We calculate the eigenenergies for the ground state, the first excited state and the second excited state of the general form of the quartic complex potential for the non-exactly solvable case. Since the potential parameters are certain, then we run the computer program to choose the most appropriate iteration number of k . We present our results in Table 3.2. It is observed that the ground state eigenvalues is converged just after $k=30$ iterations but the first excited state is converged after for $k=40$ iterations. Further, the second excited state is found to be converged for $k=55$ iterations. It is clear that in order to get accurate numerical results for the energy eigenvalues of the non-exactly solvable case of the quartic complex potential, one must enlarge the number of iterations k for the polynomial solutions of the energy eigenvalues.

Table 3.2 The rate of convergence of the AIM for the eigenenergies of the lowest states E_0 , E_1 and E_2 with a quartic complex potential in Eq.(3.3.6) for the non-exactly solvable case, with $A_4 = 3$, $A_3 = 5$, $A_2 = -9$, $A_1 = 12$, and $m=1$, computed for a number of iterations up to $k=70$

k	E_0	E_1	E_2
5	16.2707181	24.6671759	38.53489152
10	16.86119231	24.74668096	31.61541052
15	16.86815204	24.79143816	33.10654982
20	16.86807354	24.79069593	33.19707558
25	16.86807313	24.79068708	33.19858373
30	16.86807311	24.79068906	33.19777764
35	16.86807311	24.79068886	33.19770049
40	16.86807311	24.79068884	33.19770637
45	16.86807311	24.79068884	33.19770748
50	16.86807311	24.79068884	33.19770737
55	16.86807311	24.79068884	33.19770736
60	16.86807311	24.79068884	33.19770736
65	16.86807311	24.79068884	33.19770736
70	16.86807311	24.79068884	33.19770736
Numerical Results [26]	16.86807311	24.79068884	33.19770736

We also determine the energy eigenvalues for the QES-case of the quartic complex potential for the certain values of potential parameters. We observe that the iteration number k is just equal to the quantum number n of the state we are looking for. We compare our results, with the corresponding quasi-exact solutions in the literature, in Table 3.3.

Table 3.3 Comparison between the quasi-exact eigenenergies E_n and the eigenenergies computed by means of the **AIM** for the quartic complex potential [26] in Eq.(3.3.6) for the QES case, with $A_4 = 1$, $A_3 = 14$, $A_2 = 41$, $A_1 = 40$, $2m=1$ and $k=7$, for the first eight energy states

State	E_{QES} [26]	E_{AIM}
0	16.22516783	16.22516783
1	31.77572464	31.77572464
2	47.52714458	47.52714458
3	63.47231750	63.47231750
4	79.60475184	79.60475184
5	95.91849206	95.91849206
6	112.4080506	112.4080506
7	129.0683510	129.0683510

3.4 Conclusion

The numerically solvable case of the quartic complex potential needs more iteration to determine the energy eigenvalues. It is observed that the number of iterations must be enlarged to obtain the accurate numerical results. After a certain value of the iteration number, the energy eigenvalues are converged and then it is not necessary much more iteration of the quantization condition equation. On the other hand, this is not true for the QES case of the potential in hand: For this case, iteration number of the method AIM is just equal to the quantum number n of the energy eigenvalue in question we have shown in Table 3.3. Generally, the iteration number is always greater than the quantum number n for the nontrivial potentials which are not exactly or quasi-exactly solvable ones, and the approximate energy eigenvalues are obtained from the roots of the Eq.(2.1.18) for sufficiently large values of k with iteration procedure. Since the general solution of the method is based on the condition solution of Eq.(2.1.8), then one has to satisfy the condition in Eq.(2.1.9). If the problem is exactly solvable potential, then the condition in Eq.(2.1.9) is satisfied. If not, one requires numerical treatment to obtain the energy eigenvalues by increasing the iteration number to enforce Eq.(2.1.9) to be satisfied.

CHAPTER 4

CONCLUSION

In this thesis work, we have applied a recent technique called the Asymptotic Iteration Method to some PT-symmetric potentials and obtained their energy spectrum and eigenfunctions. We have studied on two different potential types: The former one is called the QES Khare-Mandal potential (and we have also taken its partner into account) and the latter is called the quartic complex potentials. Both potentials are complex and they obey the PT-symmetry. We have first applied the method to the QES Khare-Mandal and its PT-symmetric partner potential and obtained that their energy eigenvalues and eigenfunctions. AIM is observed as a good candidate to solve the PT-symmetric QES potentials if one can suggest an appropriate ansatz wavefunction for the transformation of the Schrödinger equation into the second-order differential equation in present form of the AIM method. We have observed that the energy eigenvalues of the Khare-Mandal potential is strongly affected by one of the potential parameter that can turn the whole spectrum into complex if it is greater than a certain value. We have also applied the method to its PT-symmetric partner and obtained the corresponding energy spectra. For the partner potential, it is obviously seen that the energy eigenvalues are real for any value of the potential parameters and both for even and odd integer values of M .

As a second example, we have used the method to solve the quartic complex potential considering two different cases: QES quartic complex potential and the non-exactly solvable quartic complex potential. In both cases, AIM gave quite

accurate results for the energy eigenvalues of the both potentials in the question. Here it is noted that when the method is applied to QES potentials, the iteration number is just equal to the quantum number, as seen in Table 3.3. Therefore, it is enough if one should run the computer program for few values of the iteration number for the first few eigenvalues of the QES potential interested. On the other hand, if the numerically solvable potential case is taken into account then one must enlarge the iteration number for the convergence of energy eigenvalues. In the method, the quantization condition equation determines the energy eigenvalues of the potential in question. Therefore, the calculated eigenenergies by means of this condition should be independent of the choice of the coordinate parameter such as r , x , y , etc. To eliminate the coordinate parameter in the equation, one should choose an appropriate value of x : The choice of x is observed to be critical only to the speed of convergence of the eigenenergies, as well as for the stability of the process. In our examples we have observed that the optimal choice for x is $x = 0$.

The results we have obtained in this thesis are in excellent agreement with the corresponding published results in the literature. The method used here is simple and flexible, which can be readily employed to solve other complex potentials in physics.

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