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APPLICATION OF ASYMPTOTIC ITERATION METHOD TO CERTAIN SUPERSYMMETRIC PARTNER POTENTIALS

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Application of Asymptotic Iteration Method to Certain Supersymmetric Partner Potentials

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Assoc. Prof. Dr. Okan ÖZER

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ABSTRACT

APPLICATION OF THE ASYMPTOTIC ITERATION METHOD TO CERTAIN SUPERSYMMETRIC PARTNER POTENTIALS

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M.Sc. in Engineering Physics
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The Asymptotic Iteration Method (AIM) that has received a lot of attention in the literature recently is introduced. The method is applied to obtain the energy eigenvalues of some non-shape and shape invariant supersymmetric partner potentials, for which exact analytical solutions are not obtainable, with unbroken and broken symmetry. The numerical results obtained in this thesis are compared with the previous studies and it is also noted that the AIM condition preserves the supersymmetric energy degeneracy.

Keywords: Asymptotic iteration method, supersymmetric quantum mechanics, partner potentials.

ÖZET

ASİMTOTİK İTERASYON METODUNUN BAZI SÜPERSİMETRİK EŞPOTANSİYELLERE UYGULANMASI

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Bu tezde, son zamanlarda literatürde oldukça ilgi uyandıran Asimptotik İterasyon Metod'u (AIM) anlatılmıştır. Bu metod, bazı şekil değişmezliği olan ve olmayan "bozulmuş" ve "bozulmamış" simetri ile tam analitik sonuçlar elde edilemeyen süpersimetrik eş potansiyellerin enerji özdeğerlerini elde etmek için uygulanır. Bu tezde elde edilen nümerik sonuçlar önceki çalışmalar ile karşılaştırılmış ve AIM koşulunun süpersimetrik enerji dejenerasyonuna (bozulmasına) engel olduğu belirtilmiştir.

Anahtar Kelimeler: Asimptotik iterasyon metodu, süpersimetrik kuantum mekaniği, eş potansiyeller

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CHAPTER 1

INTRODUCTION

There is no doubt that it is an interesting problem in fundamental quantum mechanics for lecturers, advanced undergraduate and graduate students in physics and applied mathematics to obtain the exact solutions of the Schrödinger equation for any type of potential. It is well known that the Schrödinger equation proposed by Erwin Schrödinger in 1926 is a second-order differential equation that describes how the quantum state of a physical system changes in time (Schrödinger, 1926). It is as central to quantum mechanics as Newton's laws are to classical mechanics. In non-relativistic quantum mechanics, the Schrödinger equation in general is given as

$$\hat{H}\,\psi(r,t) = E\,\psi(r,t)$$

or

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(r,t) = i\hbar\frac{\partial\psi(r,t)}{\partial t} \tag{1.1}$$

Where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$V(r)$ is potential of system, $\psi(r,t)$ is wave function and $i\hbar\frac{\partial}{\partial t}$ denotes energy.

If one can solve Equation (1.1) for a given potential and obtain the analytical results for E and ψ, then the potential is called "exactly solvable potential". Unfortunately there are not many potentials which admit exact solutions. Consequently there exist several means to obtain approximate solutions of them, e.g. Variational method (Griffiths, 1995), WKB (Wentzel, Kramers, Brillouin) approximation, time-independent Perturbation theory (Schiff, 1968), the Numerical shooting method (Giordano, 1997), the Finite-element method (Ram-Mohan, 2002; Ka-oey, 2004), etc. One of the methods mentioned above, WKB approximation, is for solving the Schrödinger equation via the wave function expansion in the power series of $\hbar$. Although it is widely used in quantum mechanics and in many other branches of theoretical physics such as the theory of graded-index optical waveguides (Srivastava et al., 1987), the problem of exactness of the WKB approximation has arisen and various refinements have been developed to improve the accuracy of the method (Xiang and Yip, 1994; Popov et al., 1996; Zivanovic et al., 1997).

In the frame of the Supersymmetric quantum mechanics (Cooper et al., 2001) one can also solve Equation (1.1) by factorizing the initial Hamiltonian $\hat{H}$: The initial Hamiltonian, call $\hat{H}^-$, can be factorized by using the differential operators and can have a partner Hamiltonian, call $\hat{H}^+$. This partner Hamiltonian $\hat{H}^+$, again, can be factorized and a new partner Hamiltonian is obtained. If a hierarchy of Hamiltonians can be constructed, one can also obtain a relation among the energy eigenvalues and the eigenfunctions of these Hamiltonians. If the ground state energy of the initial potential is zero, then it can be easily written in factorizable form. Thus, the ground state energy of the partner Hamiltonian $\hat{H}^+$ will be the energy of the first excited state of the initial one $\hat{H}^-$. As a result, the differential operators in factorization acts as "raising" and "lowering" operators for the eigenfunctions of these Hamiltonians. If the ground state energy eigenvalue of the initial Hamiltonian is zero, then it has "unbroken symmetry". Applying the differential operators on the excited state wavefunction of the initial Hamiltonian, one obtains the ground state wavefunction of the partner Hamiltonian. Finally, each of the new Hamiltonians will have one fewer bound-state, so that this process can be continued until the number of bound-states is exhausted. On the other hand, if the ground state energy of the initial

potential is not zero, then it has “broken symmetry” and the operation of differential operators on the wavefunctions of Hamiltonians does not change the number of nodes of the partner wavefunctions. On the other hand, potentials in supersymmetric quantum mechanics are called “shape-invariant” or “non-shape-invariant” potentials for which exact analytic solutions are obtainable or not, respectively. For the shape-invariant potentials one can obtain the energy and the wavefunction expressions in their explicit forms. For the non-shape-invariant potentials, one of the method given above is applied in the framework of supersymmetric quantum mechanics: Comtet *et al.* formulated a supersymmetric version of the WKB method (SWKB) and demonstrated that the SWKB can give exact energy eigenvalues for several solvable potentials with unbroken symmetry (Comtet et al., 1985). The method proposed by Comtet *et al*. was applied in the cases of some exactly solvable potentials (Khare, 1985) (for which the exact bound-state spectrum is reproduced) as well as some non-exactly solvable models (Dutt et al., 1986). Later, it was observed that the SWKB method yields the exact bound-state spectra only for *some* shape-invariant potentials (Dutt et al., 1991) and it is also found that the method fails to reproduce the energy levels for *some* non-shape-invariant potentials (Adhikari et al., 1988; Khare, 1989; Varshni, 1992). On the other hand, Dutt et al. proposed an alternative quantization condition known as broken supersymmetric WKB (BSWKB) for quantum mechanical bound-state problems with broken symmetry (Dutt et al., 1993). Unfortunately, it also failed to give exact results in good agreement for the broken symmetry.

In this thesis study, none of the methods given above are reviewed or applied but a recent technique called the Asymptotic Iteration Method (AIM) is introduced (Ciftci et al., 2003) to solve some potentials which are non-shape-invariant with unbroken or broken symmetry. AIM is proposed to obtain eigenvalues of second order homogeneous differential equations. In the case of Schrödinger equation, it has been found that AIM exactly reproduces the energy spectrum for most exactly solvable potentials (Ciftci et al., 2005; Saad et al., 2006; Ozer and Roy, 2009; Ozer, 2008; Ozer and Aslan, 2008; Ozer, 2009) and for non-exactly solvable potentials it produces very good results (Soylu et al., 2007; Barakat, 2005b; Sous, 2006; Koc et al., 2007, 2008).

In the next Chapter, we present the solution of a second-order differential by the AIM method. We obtain the quantization condition and the general solution of the method. In Chapter 3, we briefly give the basic definitions of the supersymmetric quantum mechanics and show the hierarchy of Hamiltonians for the potential with unbroken or broken symmetry. In Chapter 4, we apply the AIM to some non-shape-invariant supersymmetric potentials and obtain the energy eigenvalues of these potentials. We determine the best choice of the adjustment parameter in the method to obtain the energy eigenvalues by using the minimum iteration number. We show how the adjustment parameter affects the iteration number of the method. We compare our results with the literature and it is found that our results are in excellent agreement with the numerical values. It is also noted that AIM condition preserves the supersymmetric energy degeneracy. Chapter 5 is devoted to Conclusion.

CHAPTER 2

ASYMPTOTIC ITERATION METHOD

AIM is proposed to solve the second-order differential equations of the form of (Ciftci et al., 2003; Ciftci et al., 2005)

$$f''(x) = \lambda_0(x) f'(x) + s_0(x) f(x) \tag{2.1}$$

where $\lambda_0(x) \neq 0$ and the prime denotes the derivative respect to *x*. The variables, $s_0(x)$ and $\lambda_0(x)$, are arbitrary functions and sufficiently differentiable. To obtain a general solution to this equation, we differentiate Equation (2.1) with respect to *x*, then we find

$$f'''(x) = \lambda_1(x) f'(x) + s_1(x) f(x) \tag{2.2}$$

where

$$\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2(x) \tag{2.3a}$$

$$s_1(x) = s_0'(x) + s_0(x)\lambda_0(x) \tag{2.3b}$$

Similarly, the second derivative of the Equation (2.1) gives

$$f^{(4)}(x) = \lambda_2(x) f'(x) + s_2(x) f(x) \tag{2.4}$$

where

$$\lambda_2(x) = \lambda_1'(x) + s_1(x) + \lambda_0\ (x)\lambda_1(x) \tag{2.5a}$$

$$s_2(x) = s_1'(x) + s_0(x)\lambda_1\ (x) \tag{2.5b}$$

Equation (2.1) can be easily iterated up to (k+1)th and (k+2)th derivatives, k=1,2,3,....Therefore, we have

$$f^{(k+1)}(x) = \lambda_{k-1}(x) f'(x) + s_{k-1}(x) f(x) \tag{2.6a}$$

$$f^{(k+2)}(x) = \lambda_k(x) f'(x) + s_k(x) f(x) \tag{2.6b}$$

where

$$\lambda_k(x) = \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x) \tag{2.7a}$$

$$s_k(x) = s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x) \tag{2.7b}$$

which are called the “recurrence relation". From the ratio of the (k+2)th and (k+1)th derivatives, we obtain

$$\frac{d}{dx}\ln\left[f^{(k+1)}(x)\right] == \frac{f^{(k+2)}(x)}{f^{(k+1)}(x)} = \frac{\lambda_k\left[f'(x) + \left(\frac{s_k(x)}{\lambda_k(x)}\right) f(x)\right]}{\lambda_{k-1}\left[f'(x) + \left(\frac{s_{k-1}(x)}{\lambda_{k-1}(x)}\right) f(x)\right]} \tag{2.8}$$

For sufficiently large values of k, if the following condition is satisfied

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x) \tag{2.9}$$

which is the ‘asymptotic’ aspect of the method, then Equation (2.8) is reduced to

$$\frac{d}{dx}\ln\left[f^{(k+1)}(x)\right] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)} \tag{2.10}$$

which gives directly

$$f^{(k+1)}(x) = C_1 \exp\left(\int \frac{\lambda_k(x)}{\lambda_{k-1}(x)} dx \right) = C_1 \lambda_{k-1}(x) \exp\left(\int [\alpha(x) + \lambda_0(x)] dx \right) \tag{2.11}$$

where C_1 is the integration constant. By inserting Equation (2.11) into Equation (2.6a), the first-order differential equation is obtained as

$$f'(x) + \alpha(x) f(x) = C_1 \exp\left(\int [\alpha(x) + \lambda_0(x)] dx \right) \tag{2.12}$$

This first-order differential Equation can easily be solved and the general solution of the Equation (2.1) can be obtained as:

$$f(x) = \exp\left(-\int^{x} \alpha(x_1) dx_1 \right) \left[C_1 + C_2 \int^{x} \exp\left(\int^{x_1} [\lambda_0(x_2) + 2\alpha(x_2)] dx_2 \right) dx_1 \right] \tag{2.13}$$

Since the one-dimensional time-independent Schrödinger equation ($\hbar^2 = 2m = 1$) is

$$-\psi''(x) + [V(x) - E]\psi(x) = 0 \tag{2.14}$$

one can transform it into Equation (2.1) by applying possible change of coordinate if it is necessary and then performing a change of wave function in the form

$$\psi(x) = g(x) f(x) \tag{2.15}$$

We note here that in most applications of AIM the function $g(x)$ asymptotic behavior of the system. The function $f(x)$ is obtained in the polynomial form by using Equation (2.13) that does not disturb the asymptotic behavior. In the method, the energy eigenvalues can be determined by the quantization condition, given by the termination condition in Equation (2.9). Thus one can write the quantization condition combined with Equation (2.7a and 2.7b) as

$$\delta_k(x) = \lambda_k(x) s_{k-1}(x) - \lambda_{k-1}(x) s_k(x) = 0, \qquad k = 1,2,3... \tag{2.16}$$

After transforming the Schrödinger equation into the form of Equation (2.1), the energy spectrum and wave functions of the quantum system can be obtained analytically (or numerically). Using Equation (2.1) one can determine $s_0(x)$ and $\lambda_0(x)$ and then one can determined $s_k(x)$ and $\lambda_k(x)$ parameters iteratively by the aid of Equation (2.7a and 2.7b). The energy eigenvalues of the potential in interest are obtained by the quantization condition in Equation (2.17) and the wave functions are determined by using the following wave function generator

$$\mathrm{f_n(x)} = \mathrm{C_2}\ \exp\left(-\int^{x} \frac{s_k(x')}{\lambda_k(x')}\, dx' \right). \tag{2.17}$$

Although the general solution of Equation (2.1) is given by Equation (2.13), the first part of Equation (2.13) gives the polynomial solutions that are convergent and physical, whereas the second part gives non-physical solutions that are divergent. Therefore, the corresponding eigenfunctions can be derived from the wave function generator given in Equation (2.17) by choosing $C_1 = 0$ in Equation (2.13). On the other hand, the equation given in Equation (2.9) implies that the wave functions of the system are truncated for sufficiently large values of *k* and the roots of the relation given in Equation (2.16), which has been obtained from Equation (2.7a and 2.7b), belong indirectly to the spectrum of Equation (2.14). However, for each iteration the expression Equation (2.16) depends on different variables, such as *E*, *x* and possible potential parameters. It is also noticed that the iterations should be terminated by imposing the quantization condition $\delta(x) = 0$, as an approximation to Equation (2.9) to obtain the eigenenergies. Therefore, the calculated eigenenergies E_n by means of this condition should be independent of the choice of the coordinate. Thus, the choice of x_0 is observed to be critical only to the speed of the convergence of the eigenenergies, as well as for the stability of the process. The energy eigenvalues can easily be obtained from the roots of Equation (2.16) if the problem is exactly solvable. If not, for a special *n* quantum number, we choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential (Ciftci et al., 2003; Fernandez, 2004; Barakat, 2006; Barakat, 2005a; Barakat, 2005b) and the approximate energy eigenvalues are

obtained from the roots of this equation for sufficiently large values of k with iteration.

CHAPTER 3

SUPERSYMMETRIC PARTNER POTENTIALS AND THE FACTORIZATION OF HAMILTONIAN

Since we apply the asymptotic iteration method to some certain supersymmetric partner potentials, we briefly give some information about the supersymmetric quantum mechanics (SUSYQM) in this Chapter. It is well known that the time-independent Schrödinger equation describing a particle of mass *m* moving in a one-dimensional potential *V(x)* is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x)\,. \tag{3.1}$$

Now, we ask that if this equation can be factorized? According to the basic quantum mechanics, the Scrödinger equation can be factorized if one suggests the appropriate quantum mechanical "*operators*" (Griffiths, 1995). Then, the Equation (3.1) can be factorized. Let's focus on Equation (3.1) and its ground state wave function

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0(x)}{dx^2}+V^-(x)\psi_0^-(x)=E_0^-\psi_0(x)\,. \tag{3.2}$$

We use the *'superscripts'* because our purpose is to obtain a "*partner*" of that equation, and these two systems should be distinguished from each other shortly. We start with the ground state wave function, which has a great importance in the procedure and the reason will be clearly seen in next steps. If we assume that H^- is the Hamiltonian operator for this system, then we can write

$$H^{-} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{-}(x)\,. \tag{3.3}$$

Since H^{-} is operator and E_0^{-} is its eigenvalue, then it can be convenient to rewrite the equation as following:

$$\left(H^{-} - E_0^{-}\right)\psi_0^{-}(x) = 0 \tag{3.4}$$

where E_0^{-} is the ground-state energy eigenvalue and $\psi_0^{-}(x)$ is the *normalizable* ground-state eigenfunction of the operator H^{-}. Thus, we obtain an equation in terms of Hamiltonian operator for the system. Therefore, one can now factorize the operator on the left-hand side of the wave function as:

$$H^{-} - E_0^{-} = A^{-}A^{+} \tag{3.5}$$

and it is guessed that these operators can be given in the following forms (Dutt et al., 1988)

$$A^{-} = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x), \tag{3.6a}$$

$$A^{+} = +\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x), \tag{3.6b}$$

where *W(x)* is some function, called *superpotential* in SUSYQM, whose exact form will be determined soon. As it is seen in the progress that these operations do not change anything in the physics. It is also clear that the wave functions of the system are not affected and the potential of the system is just rescaled by the constant term E_0^{-} in Equation (3.4). One can ask to examine what exactly A^{-} and A^{+} operators '*do*' on any function *f(x)*, for example. To see the effect, we just apply these operators on the function as it is done in quantum mechanics:

$$\begin{aligned}
\left(A^{-}A^{+}\right)f(x) &= \left[-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)\right]\left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)\right]f(x) \\
&= \left[-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)\right]\left[\frac{\hbar}{\sqrt{2m}}\frac{df(x)}{dx}+W(x)f(x)\right] \\
&= -\frac{\hbar^2}{2m}\frac{d^2f(x)}{dx^2}-\left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}W(x)f(x)\right]+W(x)\frac{\hbar}{\sqrt{2m}}\frac{df(x)}{dx}+W^2(x)f(x) \\
&= \left[-\frac{\hbar^2}{2\mathrm{m}}\frac{d^2}{dx^2}-\frac{\hbar}{\sqrt{2m}}W'(x)+W^2(x)\right]f(x) \qquad (3.6c)
\end{aligned}$$

For this result, it is required

$$W^2(x)-\frac{\hbar}{\sqrt{2m}}W'(x)=V^{-}(x)-E_0^{-} \qquad (3.7)$$

At that point it is clearly seen that if one can find a function *W(x)* satisfying this equation, then the Hamiltonian in Equation (3.5) will have been successfully factorized. Now, the question is how one can determine the function *W(x).* We can suppose that it were the case A^{+} has annihilated the ground state wave function ψ_0^{-}, that means, $A^{+}\psi_0^{-}(x)=0$. Then $A^{-}\left(A^{+}\psi_0^{-}(x)\right)$would automatically be zero, and Equation (3.4) would be satisfied. At his point, it is clearly seen as a reason why we have started by the ground-state wave function $\psi_0^{-}(x)$. Then we can look for a function *W(x)* which forces this condition to be satisfied. If we explicitly write the condition in terms of Equation (3.7), then we get

$$A^{+}\psi_0^{-}(x)=\frac{\hbar}{\sqrt{2m}}\frac{d\psi_0^{-}(x)}{dx}+W(x)\psi_0^{-}(x)=0 \qquad (3.8)$$

and this equation leads us to find out that

$$W(x)=-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\ln\psi_0^{-}(x). \qquad (3.9)$$

It is obvious that one can also obtain $\psi_0^-(x)$ from this equation if *W(x)* is known:

$$\psi_0^-(x) = N\exp\left[-\frac{\sqrt{2m}}{\hbar}\int^x W(y)dy\right] \tag{3.10}$$

where *N* is the normalization constant. Equation (3.9) says that we can find *W(x)* explicitly if we know the ground state wave function. To do this, we are to solve the Schrödinger equation completely. We find that we need to know the solutions already. On the other hand, the Equation (3.10) says something quite different: It says that if one somehow can find *W(x)* independently of any knowledge of ψ_0^-, then one can obtain the ground state wave function of the system by using Equation (3.10). In essence, we have transformed the Schrödinger equation into a completely different equation, but equivalent one. In more general terms, we have found a first-order, nonlinear differential equation equivalent to our original second-order linear differential equation. Such first-order nonlinear equivalents are classified under the general name of *Riccati equations*.

In order to obtain a partner potential and to factorize the Schrödinger equation, the only thing we do is to reverse the order of the factors $A^{\pm}$ and get something interesting. Reversing the order of factors leads to a new Hamiltonian such as (Dutt et al., 1988; Cooper et al., 2001)

$$A^+A^- = H^+ - E_0^- \tag{3.11}$$

where $H^- = -\frac{\hbar^2}{2m}\frac{d}{dx^2} + V^+(x)$ and it is found that

$$\begin{aligned}\left(A^+A^-\right)f(x) &= \left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right]\left[-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right]f(x)\\ &= \left[-\frac{\hbar^2}{2\mathrm{m}}\frac{d^2}{dx^2} + \frac{\hbar^2}{\sqrt{2m}}W'(x) + W^2(x)\right]f(x),\end{aligned}$$

for this result, it is required that the potential of this new Hamiltonian H^+ must satisfy

$$W^2(x)+\frac{\hbar}{\sqrt{2m}}W'(x)=V^+(x)-E_0^-. \tag{3.12}$$

From that equation, it is seen that there is a relation between partner potentials and it can be found as following: If we rewrite that equation with a little difference,

$$V^+(x)=W^2(x)+\frac{\hbar}{\sqrt{2m}}W'(x)+\frac{\hbar}{\sqrt{2m}}W'(x)-\frac{\hbar}{\sqrt{2m}}W'(x)+E_0^- \tag{3.13}$$

and using another fact,

$$V^-(x)=W^2(x)-\frac{\hbar}{\sqrt{2m}}W'(x)+E_0^-, \tag{3.14}$$

one obtains

$$V^+(x)=V^-(x)+2\frac{\hbar}{\sqrt{2m}}W'(x), \tag{3.15}$$

When the superpotential term is replaced in terms of wave function, we get finally,

$$V^+(x)=V^-(x)-\frac{\hbar^2}{m}\frac{d^2}{dx^2}ln\psi_0^-. \tag{3.16}$$

We can easily find interesting connections between these two systems described by H^- and H^+. First, consider an eigenfunction of H^- satisfying

$$H^-\psi_n^- = E_n^-\psi_n^-, \tag{3.17}$$

where E_n^- is the n^{th} energy of H^-. In terms of $A^{\pm}$, we get

$$\left(A^-A^+ + E_0^-\right)\psi_n^- = E_0^-\psi_n^- \tag{3.18}$$

Applying A^+ to the left of both equations, we obtain

$$A^+A^-A^+\psi_n^- + E_0^-A^+\psi_n^- = E_n^-A^+\psi_n^-, \tag{3.19}$$

then we can group terms

$$\left(A^{+}A^{-}+E_{0}^{-}\right)\left(A^{+}\psi_{n}^{-}\right)=E_{n}^{-}\left(A^{+}\psi_{n}^{-}\right) \tag{3.20}$$

which we notice Equation (3.11) to get

$$H^{+}\left(A^{+}\psi_{n}^{-}\right)=E_{n}^{-}\left(A^{+}\psi_{n}^{-}\right) \tag{3.21}$$

This is a quite interesting result and it says that $A^{+}\psi_{n}^{-}$ is a solution to the Schrödinger equation for H^{+}, with energy E_{n}^{-}. It is obvious that we can go the other way around. Starting with an eigenfunction ψ_{m}^{+} of H^{+} with energy E_{m}^{+} leads us to obtain

$$\left(A^{+}A^{-}+E_{0}^{-}\right)\psi_{m}^{+}=E_{m}^{+}\psi_{m}^{+} \tag{3.22}$$

Applying A^{-} to the left of both sides and grouping terms appropriately we get

$$\left(A^{-}A^{+}+E_{0}^{+}\right)\left(A^{-}\psi_{m}^{+}\right)=E_{m}^{+}\left(A^{-}\psi_{m}^{+}\right) \tag{3.23}$$

or shortly

$$H^{-}\left(A^{-}\psi_{m}^{+}\right)=E_{n}^{+}\left(A^{-}\psi_{m}^{+}\right) \tag{3.24}$$

so $A^{-}\psi_{m}^{+}$ is an eigenfunction of H^{-}, with an energy E_{m}^{+}.

In summary, A^{+} applied to any eigenfunction of H^{-} gives an eigenfunction of H^{+}, with the same energy eigenvalue. A^{-} applied to any eigenfunction of H^{+} gives an eigenfunction of H^{+}, again with the same eigenvalue. This guarantees that the two systems in fact have identical energy spectra, almost. Notice that A^{+} kills the ground state wave function ψ_{0}^{-} of H^{-}, by construction, so H^{+} has no corresponding eigenstate at the same energy. The lowest state of H^{+} must then correspond to the first 'excited' state of H^{-}, or $E_{0}^{+}=E_{1}^{-}$. In general,

$$E_{n+1}^{-}=E_{n}^{+} \tag{3.25}$$

As we have shown above if the ground-state eigenvalue of a potential can be shifted to zero and the ground-state eigenfunction can be normalized, then the system is called “unbroken symmetry”. This is why we obtain Equation (3.25). Therefore, one can conclude that partner potentials have identically the same bound spectra, except the ground state of $V^{-}(x)$ which does not appear in the spectrum of $V^{+}(x)$. This procedure is sometimes useful if one of the partner potentials is quite complex for to be solved than the other one.

On the other hand, if the partner potentials have ground-state energy eigenvalues greater than zero, then the system is called “broken symmetry”, and the energy eigenvalues satisfy the condition:

$$E_n^{-} = E_n^{+} \tag{3.26}$$

As we have shown above one can obtain the eigenvalue and the eigenfunction relations between the partner potentials if the initial potential is in the exactly solvable form such as Simple Harmonic Oscillator, Coulomb, Morse and Pösch-Teller potentials. One can easily obtain the partner potentials for them by following the procedure given above. At this point a question is asked that what the general relation among these potentials is and why they can be solved? In the paper by Gendenshtein (1983), it is shown that these potentials have a property of “shape invariance” that is of importance point in the SUSYQM to obtain eigenvalues of the partner potentials. In the procedure we have given above if the potentials $V^{-}(x)$ and $V^{+}(x)$ can be written in a more general case than Equation (3.16) as

$$V^{+}(x;a_0) = V^{-}(x;a_1) + R(a_1) \tag{3.27}$$

where a_0 is a set of parameters and a_1 is a function of a_0 and the last term $R(a_1)$ is a independent function of x, then the potentials $V^{-}(x)$ and $V^{+}(x)$ are called “shape invariant” potentials: They are similar in shape and differ only in the parameters appearing in their function forms. Following the hierarch process given above one can write the Hamiltonians in the form of the following structure

$$H^k = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^-\left(x; a_k\right) + \sum_{i=1}^{k} R(a_i) \tag{3.28}$$

where $a_k = f^k(a_0)$, the function f applied k times. If we take the next member of the hierarchy and compare its spectrum with the previous one, we obtain

$$\begin{aligned} H^{k+1} &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^-\left(x; a_{k+1}\right) + \sum_{i=1}^{k+1} R(a_i) \\ &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^+\left(x; a_k\right) + \sum_{i=1}^{k} R(a_i) \end{aligned} \tag{3.29}$$

From Equations (3.28) and (3.29), we obtain that the Hamiltonians have identical bound-state spectre except for the lovest level of the member H^k whose energy is given by

$$E_0^k = \sum_{i=1}^{k} R(a_i) \tag{3.30}$$

This can be written from Equation (3.28) and it is known that $E_0^- = 0$. The complete energy spectrum of H^- is found as

$$E_n^- = \sum_{i=1}^{n} R(a_i), \quad E_0^- = 0 \tag{3.31}$$

CHAPTER 4

APPLICATIONS

4.1 One-Dimensional Anharmonic Oscillator

As we have studied in the previous Chapter, the Hamilton hierarchy in SUSYQM is described by:

$$H_{\pm} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W^2(x) \pm \frac{\hbar}{\sqrt{2m}}W'(x)$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{\pm}(x) \qquad (4.1)$$

where *W(x)* is called the superpotential and $V_{\pm}(x)$ are called supersymmetric partner potentials and defined by

$$V_{\pm} = W^2(x) \pm \frac{\hbar}{\sqrt{2m}}W'(x) \qquad (4.2)$$

in which $W'(x)$ is the derivative with respect to *x*. If the ground-state eigenvalue of a potential can be shifted to zero and the ground-state eigenfunction can be normalized, then the system has unbroken symmetry. Thus, one can conclude that partner potentials have identically the same bound-spectra, except the ground-state of V_{-}(x) which does not appear in the spectrum of V_{+} (x). The energy eigenvalues of partner potentials satisfy the condition:

$$E_{n+1}^{-} = E_n^{+} \qquad (4.3)$$

In the case that the partner potentials have ground-state energy eigenvalues greater than zero, then the system is called the broken symmetry.Thus, the energy eigen values satisfy the condition:

$$E_n^+ = E_n^- \tag{4.4}$$

Using Equation (4.2) for the partner potentials,we study the bound-state spectra for some certain potentials with unbroken or broken symmetry by the quantization condition given in Equation (2.16), in the framework of AIM.

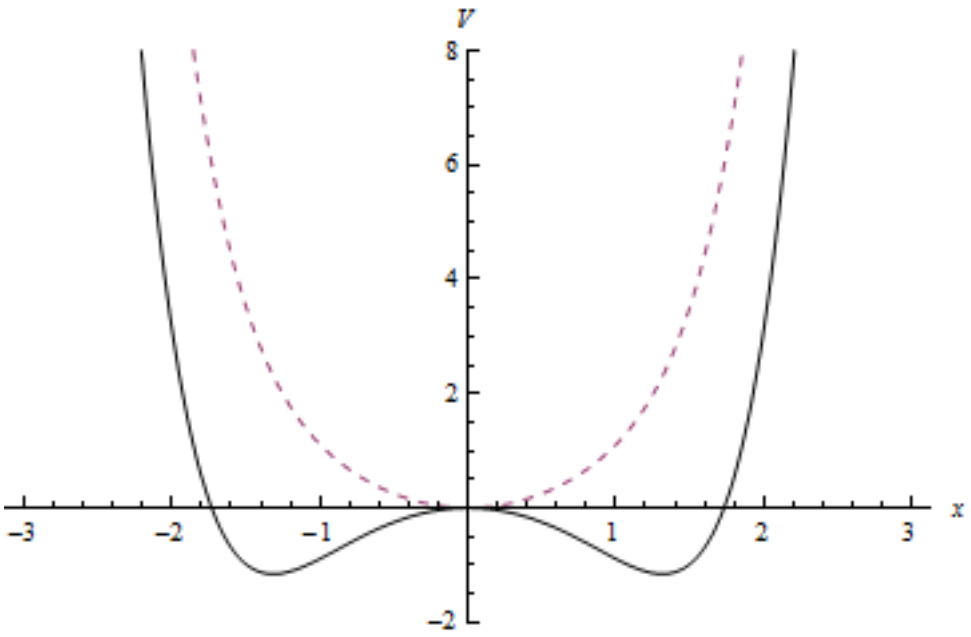

Figure 1: The potentials $V_{-}(x)$ (solid), and V_{+} (x) (dashed) for the anharmonic oscillator (Equation (4.6)) with $V_0 = 1, a = 1$

We first analyze the one-dimensional anharmonic oscillator that is a non-shape-invariant potential with unbroken symmetry (Fricke et al., 1988; Adhikari et al., 1988): Since the superpotential is defined as

$$W(x) = \frac{\sqrt{V_0}}{3}\left[\frac{x}{a}\right]^3 \tag{4.5}$$

Then the partner potentials are found to be

$$V_{\pm} = \frac{V_0}{9}\left[\frac{x}{a}\right]^6 \pm \frac{\hbar}{a}\sqrt{\frac{V_0}{2m}}\left(\frac{x}{a}\right)^2 \tag{4.6}$$

As seen in Figure 1, the potential $V_{-}(x)$ is a symmetric double well potential whereas the partner potential $V_{+}(x)$ is a single well one. If one writes the Schrödinger equation for the partner potentials $V_{\pm}(x)$

$$-\frac{\hbar^2}{2m}\psi''(x) + \left[\frac{V_0}{9}\left[\frac{x}{a}\right]^6 \pm \frac{\hbar}{a}\sqrt{\frac{V_0}{2m}}\left(\frac{x}{a}\right)^2 - E^{\pm}\right]\psi(x) = 0 \tag{4.7}$$

Then one needs to transform it into Equation (2.1) to apply AIM: For the potential $V_{-}(x)$ in Equation (4.7), we consider a generic wave function (satisfying the boundary conditions) in the form of

$$\psi(x) = g(\beta, x)\, f(x) \tag{4.8}$$

where $g(\beta, x) = e^{-x^4/12 - \beta x^2/2}$ in which β is the adjustment parameter. Substituting Equation (4.8) in the eigenvalue Equation (4.7), one obtains (we set $\hbar = 2\mathrm{m} = 1$ after that step in the following calculations.)

$$f''(x) = \left(\frac{2}{3}\left(x^2 + 3\beta\right)x\right) f'(x) + \left(\beta - E - \beta\left(\frac{2x^2}{3} + \beta\right)x^2\right) f(x) \tag{4.9}$$

Now comparing Equation (4.9) , and Equation (2.1) one finds

$$\lambda_0(x) = \frac{2}{3}\left(x^2 + 3\beta\right)x,$$

$$s_0(x) = \beta - E - \beta\left(\frac{2x^2 + \beta}{3}\right)x^2 \tag{4.10}$$

Table 1: First four energy eigenvalues of the anharmonic potential $V_{\pm}(\mathrm{x})$ [Equation (4.6)], with $\mathrm{V}_0 = 1, a = 1$,and $\hbar = 2\mathrm{m} = 1$, determined by the numerical method, the SWKB method and the AIM. $E_n^+ = E_{n+1}^-$ relation is also satisfied by AIM

n^-	Numerical (Adhikari et al., 1988)	SWKB (Adhikari et al., 1988)	AIM: $V_-(x)$	n^+	AIM: $V_+(x)$
1	1.1175	1.3077	1.1175	0	1.1175
2	3.6364	3.6989	3.6364	1	3.6364
3	6.7440	6.7953	6.7440	2	6.7440
4	10.417	10.452	10.417	3	10.417

By means of Equation (2.7a and 2.7b), one can calculate $\lambda_{\mathrm{k}}(x)$ and $s_k(x)$. Finally , one can find the energy eigenvalues of the potential in Equation (4.6) by using the quantization condition given in Equation (2.16). The calculated energy eigenvalues E_n by means of this condition should, however, be independent of the choice of *x*. If the problem is exactly solvable, then the energy eigenvalues can directly be obtained from the roots of Equation (2.16) for any value *x* (Ozer and Aslan, 2008; Ozer, 2009). If it is not, one has to choose a suitable x_0 point determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential (Soylu et al., 2007; Barakat, 2005; Sous 2006; Koc et al., 2007, 2008), and then the approximate energy eigenvalues are obtained from the roots of this equation for sufficiently large values of *k* with iteration procedure. We also note that the choice of x_0 is observed to be critical only to the speed of the convergence of the eigenvalues, as well as for the stability of the process.In this example it has been observed that the optimal choice for x_0 is x_0=0. We present energy eigenvalues determined by AIM and we also compare our result with Exact and SWKB values, in Table 1. It is seen that the AIM gives exatly the same energy eigenvalues without any percentage error. This is not true for SWKB approximation. In addition to these calculations, we also obtain the energy eigenvalues for the partner potential $V_+(\mathrm{x})$ via the AIM quantization condition. The energy hierarchy, $E_n^+ = E_{n+1}^-$,between two

partner potentials in the framework of Aim can obviously be observed between the fourth and the fifth column in Table 1, where we show the quantum states as n and l for potentials $V_{-}(x)$ and $V_{+}(x)$ respectively. It is seen that Aim can be an effective tool to determine the eigen value problem for such problems if one properly determines the adjustment parameter, β.

4.2. Polynomial Potential

The second example which we consider is the unsolvable polynomial potential with unbroken symmetry: If the superpotential is given as (Varshni, 1992)

$$W(x) = -\frac{\phi_0'(x)}{\phi_0(x)} = 2bx + 4cx^3 \tag{4.11}$$

where *b* and *c* are real constants, then the partner potentials can be found as

$$V_{\pm}(x) = \left(4b^2 \pm 12c\right)x^2 + 16bcx^4 + 16c^2x^6 \pm 2b \tag{4.12}$$

For the partner potential $V_{-}(x)$, the values of parameters *b* and *c* affect the number of minimum points of the potential: If $b^2 > 3c$, then the potential has one minimum (Varshni, 1992). Therefore, we select the parameters $b = 0.60$ and $c = 0.10$. The partner potentials $V_{\pm}(x) = \left(4b^2 \pm 12c\right)x^2 + 16bcx^4 + 16c^2x^6 \pm 2b$ are shown in Fig. 2. These class of potentials are of importance in physics and have been investigated by a variety of methods (Khare and Varshni, 1989; Sinha et al., 1996; Chakrabarti, 2008). To apply the quantization condition Equation (2.16) to the potentials in Equation (4.12) we need to transform the eigenvalue Equation (4.7) to Equation (2.1) for these partner potentials. Therefore one can write the ansatz wave function, by following the same procedure in the previous example, as

$$\psi(x) = \exp\left(-\beta x^2 - c x^4\right) f(x) \tag{4.13}$$

Inserting Equation (4.13) into (4.7) for the potentials $V_{\pm}(x)$, we obtain

$$f''(x) = 4\left(\beta + 2cx^2\right)x\ f'(x) + \left(2\beta \pm 2b - E + 4(b^2 - \beta^2)x^2 + 16c(b - \beta)x^4 + q24cx^2\right) f(x) \tag{4.14}$$

where q=0 or q=1 for $V_{-}(x)$ or $V_{+}(x)$ potentials, respectively. If we compare Equation (4.14) with Equation (2.1), then we set

$\lambda_0(x) = 4(\beta + 2cx^2)x$ and

$$s_0(x) = 2\beta \pm 2b - E + 4(b^2 - \beta^2)x^2 + 16c(b-\beta)x^4 + q24cx^2 \qquad (4.15)$$

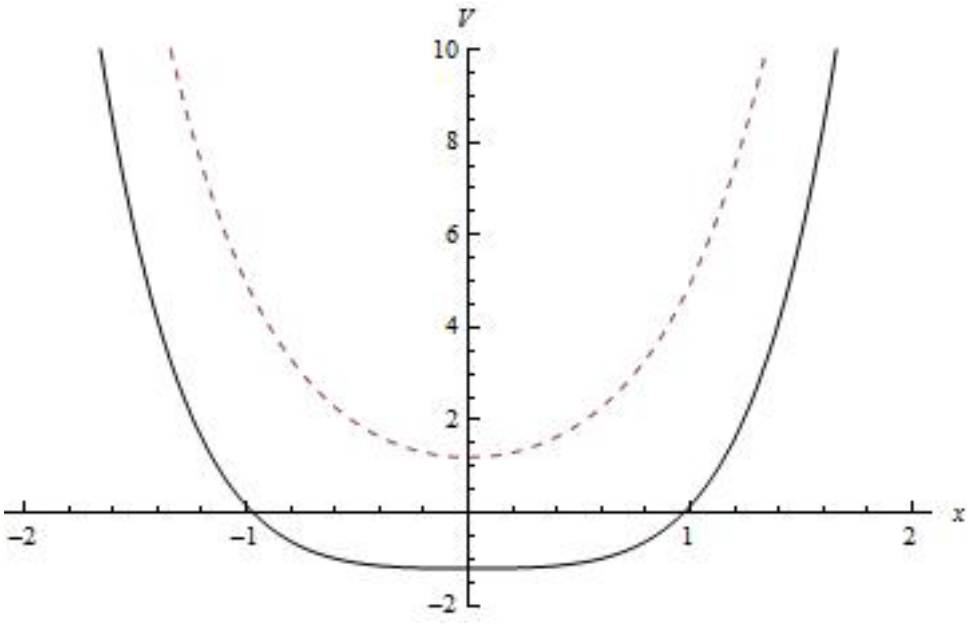

Figure 2: The potentials V_{-}(x) (solid), and V_{+} (x) (dashed) associated with the superpotential W(x) [in Equation (4.11)] $b = 0.60$ and $c = 0.10$.

Now $\lambda_k(x)$ and $s_k(x)$ can be determined by using Equation (2.7a and 2.7b) and then one can calculate the energy eigenvalues of the potential, for example, $V_{-}(x)$ in Equation (4.12). Since we have already set $b = 0.60$ and $c = 0.10$, then we have to determine the most convenient value of the adjustment parameter β in the iteration. In our calculations, if one follows the same procedure progressed in previous problem, we find that β = 2.6 for the most convenient value. At the end of the iterations, we have again set $x = 0$ which is the maximum value of the asymptotic wave function. It has been experienced that the minimum iteration number occurs when k = 72 for the potential $V_{-}(x)$. The results by AIM for the potentials $V_{+}(x)$,

together with the exact ones from solving the Schrödinger equation numerically and those of SWKB are all shown in Table 2. We obtain that our results are in excellent agreement with the exact ones for all the energy levels. We notice that the number of iterations required for the partner potential $V+(r)$ should be k = 68. It is also seen in Table 2 that AIM preserves the supersymmetric energy degeneracy.

Table 2: Energy eigenvalues of the polynomial potential for the superpotential $W(x)= 2bx + 4cx^3$ with $b = 0.60$ and $c = 0.10$, as obtained by the Numerical method, the SWKB method and the AIM

n^-	Numerical (Varshni, 1992)	SWKB (Varshni, 1992)	AIM: $V_-(x)$	n^+	AIM: $V_+(x)$
1	3.06619	3.12274	3.06619	0	3.06619
2	7.19498	7.24206	7.19498	1	7.19498
3	12.04113	12.08354	12.04113	2	12.04113
4	17.48396	17.52287	17.48396	3	17.48396
5	23.4480	23.4841	23.4480	4	23.4480
6	29.8807	29.9146	29.8807	5	29.8807
7	36.7429	36.7750	36.7429	6	36.7429
8	44.0040	44.0344	44.0040	7	44.0040

4.3. Spherically Symmetric Potential

As a last application, a non-shape-invariant potential with broken symmetry is studied. In (Dutt et al., 1993), Dutt *et al*. has investigated the energy eigenvalues of partner potentials obtained by the spherically symmetric superpotential

$$W(x) = -\frac{2}{r} - r^2 \tag{4.16}$$

Using (4.16), one can obtain the two partners as

$$V_-(r) = \frac{2}{r^2} + 6r + r^4 \quad \text{an d} \quad V_+(r) = \frac{6}{r^2} + 2r + r^4 \tag{4.17}$$

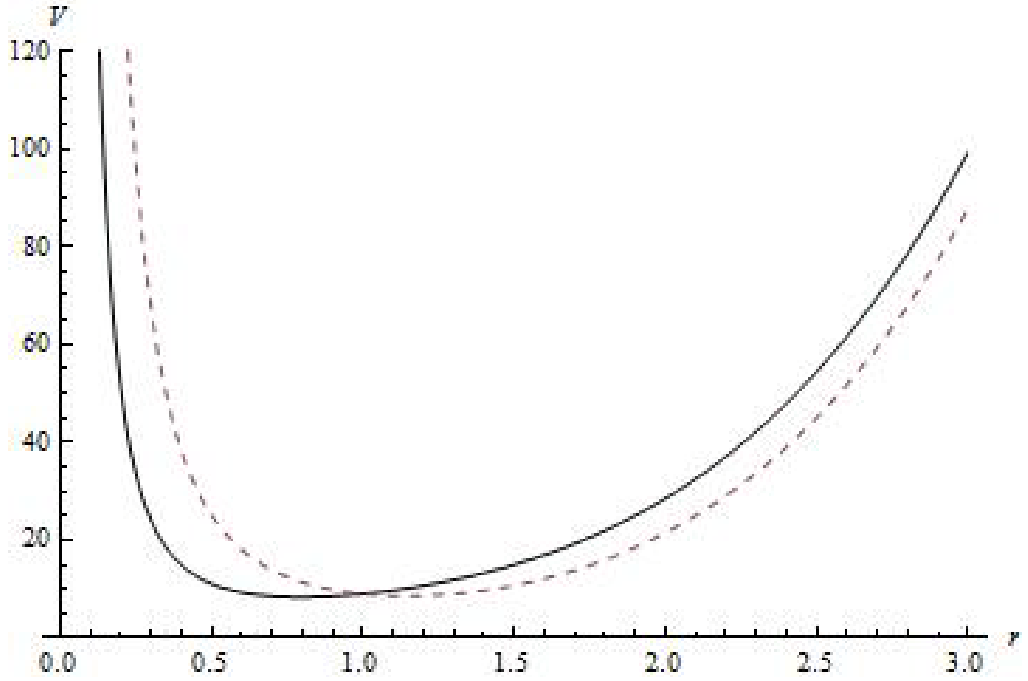

Figure 3: The potentials V_{-}(x) (solid), and V_{+} (x) (dashed) associated with the superpotential W(x) [in Equation (4.18)].

To find the eigenvalues of these potentials by using AIM, one can suggest the ansatz wave function for the solution of partner potential, $V_{-}(x)$

$$\psi(r) = r^2 \, e^{-\frac{1}{2}\alpha r^2 - \frac{1}{3}\beta r^3} \, f(r) \tag{4.18}$$

where α *and* β are adjustment parameters (to be determined by AIM). Substituting Equation (4.18) in the eigenvalue Equation (4.7) for the partner potential $V_{-}(x)$, we obtain

$$\begin{aligned} f''(r) = (2\beta r^2 + 2\alpha r - 4/r)\ f'(r) + \\ \left((6 + r^3 + \beta(6 - \beta r^3))r + \alpha(5 - 2\beta r^3) - \alpha^2 r^2 - E\right) f(r) \end{aligned} \tag{4.19}$$

Comparing Equation (4.19) and Equation (2.1), one can set

$$\lambda_0(x) = 2\beta r^2 + 2\alpha r - 4/r$$

and

$$s_0(x) = (6 + r^3 + \beta(6 - \beta r^3))r + \alpha(5 - 2\beta r^3) - \alpha^2 r^2 - E \tag{4.20}$$

Using Equation (4.20), we can compute $\lambda_k(x)$ and $s_k(x)$ and then we can determine the energy eigenvalues of the potential $V_-(x)$ in Equation (4.17) by using the quantization condition given in Equation (2.16). In our calculations we have observed that the best choice for the adjustment parameters are $\alpha = 4$ and $\beta = 1/8$, and the optimal choice for r_0 must be $r_0 = 0.6695$; the maximum value of the asymptotic wave function.

Table 3: Comparison of energy eigenvalues, for the partner potentials $V_{\pm}(x)$, obtained by the Numerical method, the SWKB method and the AIM

$n^{\pm}$	Numerical [Dutt1845]	SWKB [Dutt1845]	AIM: $V_{\pm}(x)$ Present Results
1	13.34	13.46	13.34
2	23.39	23.46	23.39
3	34.58	34.64	34.58
4	46.74	46.78	46.74
5	59.73	59.76	59.73
6	73.45	73.48	73.45
7	87.84	87.86	87.84
8	102.83	102.86	102.83

We present our results for both partner potentials, together with the numerical ones from solving the Schrödinger equation numerically and those of SWKB in Table 3. We obtain that our results are in excellent agreement with the exact ones for all the energy levels, without any percentage error. It is also noted that our results which agree with numerical ones are obtained by means of $k = 80$ iterations for the potential $V_{-}(x)$ by using the parameters presented above. On the other hand, the number of iterations required for the partner potential $V_{+}(x)$ is found to be $k = 78$.

CHAPTER 5

CONCLUSION

In this thesis we applied asymptotic iteration method to some non-shape-invariant supersymmetric partner potentials with unbroken and broken symmetry. We solved the second order differantial equations iteratively and then we transformed Schrödinger equation into those of differantial equations. After that transforming the energy spectrum and wave function of the quantum system obtained by using AIM analytically (or numerically).

We applied the AIM to the three different potentials to calculate the energy eigenvalues of the potentials. One of is One-Dimesional Anharmonic Oscillator, the other is Polynomial Potential and lastly Spherically Symmetric Potential. The obtained results are given in table 1, 2 and 3, respectively. The results clearly showed that the AIM preserves the supersymmetric energy degeneracy.

Although supersymmetric partner potentials have no analitical solutions in literature, there exist some approximate solutions to calculate them, for instance Variational method WKB (Wentzel, Kramers, Brillouin) approximation, time-independent Perturbation theory, the Numerical shooting method, the Finite-element method. The performance of the proposed method was compared with the SWKB Method in terms of closest to the exact results. The obtained results, which are given in tables 1, 2 and 3, proved that the proposed method provided much better results than the SWKB Method.

A disadvantage of the proposed method is might be its computational time. While determining the best parameter values it might be require long computaional time due to the number of iterations. Hovewer, it is worth the mention that this disadvantage can be minimized and the computational time might be reduced by using a computer which has multi-core CPUs.

Based on these results, we believe that the proposed method has great potential in calculating the non-shape-invariant supersymmetric partner potentials, for which exact analytical solutions are not obtainable, with unbroken and broken symmetry.

APPENDIX

SOLUTION OF THE ANHARMONIC OSCILLATOR POTENTIAL BY AIM USING A MATHEMATICA PROGRAM

```
(*
This is the program written in Mathematica™ for the solution of
the quantum mechanical Anharmonic Osciallator Potential by using
the Asymptotic Iteration Method
*)
(*
contact: Tugba AYDEMIR
        E-mail: tsarioglu@windowslive.com
*)

(*********************************************************************)

Clear["Global`*"]

(*Define the potential parameters*)

setparapot={V0→1,a→1,m→1/2,h→1}

(*Define the potentials*)

vplus=v0/9 (x/a)^6+h*Sqrt[v0]/Sqrt[2*m*a]*(x/a)^2  //.setparapot;

vmin=v0/9 (x/a)^6-h*Sqrt[v0]/Sqrt[2*m*a]*(x/a)^2  //.setparapot;

(*Set V here by using Vplus or Vmin*)
V=Vplus
(*Draw potentials*)
graphV=Plot[{Vmin,Vplus},{x,-3,3},PlotStyle→{Black,Dashed},
PlotRange→{-2,8},Axes→True,AxesLabel→{x,Potentials},
Epilog→{Text["V+(x)",{1.5,4},{1,1}],Text["V-(x)",{2.5,4},{1,1}]}]
```

```
Export["D:\FigureAnharmonic1.eps",graphV]
(*Find min and max points of the potential*)
Solve[D[V,x]=0,x]//N

(*Write Schrödinger, place V into it*)
```

$$Ham = -\frac{h^2}{2m} T''[x] + (V - E1)T[x];$$

```
H=Ham//.setparapot
(*Write the ansatz wavefunction*)
```

$$psi = Exp\left[-\frac{x^4}{12} - \beta x^2 / 2 \right] \text{ //.setparapot}$$

```
(*Take the derivatives of the ansatz wavefunction
and insert them into the Schrödinger equation above*)
st1=FullSimplify[{T''[x]→D[(psi*f[x]),x,x],T[x]→(psi*f[x])}]
Ht1=H//.st1/.f[x]→T[x]/.f'[x]→T'[x]/.f''[x]→T''[x];
HAM=FullSimplify[FullSimplify[Ht1]]/.x→y
HR3AIM=Collect[Simplify[Solve[HAM=0,T''[y]]],T[y]]

(*The final term T''[y] is in the form of the main equation of AIM.*)
(*Set lambda_0 and s_ 0 bu using T''[y]*)
(*Give an arbitrary positive real number for the beta value*)
setpara={β→7/2};
s[x][0]=FullSimplify[Coefficient[HR3AIM[[1,1,2]],T[y]]//.setpara]/.y→x
Lb[x][0]=FullSimplify[Coefficient[HR3AIM[[1,1,2]],T'[y]]//.setpara]/.y→x

(*Now, iteration can be done. One can solve the ∂_k equation for the E values*)
(*If the resultant equation includes "x",
then set "x=0" of set it for the min/max points of the potential interest.*)
(*If E values are not found,
try the max value of the ansatz wavefunction above for the "x" value*)

Clear[i,k];
limit=56;
Do[k=i;Print[i];
  Do[s[x][n]=Simplify[D[s[x][n-1],x]+Lb[x][n-1]s[x][0]];
   Lb[x][n]=Simplify[D[Lb[x][n-1],x]+s[x][n-1]+Lb[x][0]Lb[x][n-1]],{n,1,k}];
  enr=Numerator[Factor[s[x][k]*Lb[x][k-1]-Lb[x][k]*s[x][k-1]]]/.x□0//N;
  sonuc=Chop[Solve[enr□0,E1]]//N//TableForm;
  Print[NumberForm[sonuc,6]]
   If[k≥limit,Break[ ]],{i,46,limit,2}];

(**********************************************)
```

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PUBLICATIONS

1) T. Aydemir (Sarıoğlu) and O. Özer, *Application of Asymptotic Iteration Method to Certain Supersymmetric Partner Potentials*, ***accepted by*** Turkish Physical Society 27th International Physics Conference (TFD-27) as ***Oral presentation***.