Construction of Klein-Gordon Equation (KG) with Position Dependent Mass and Application to the Physical Systems

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ABSTRACT

CONSTRUCTION OF KLEIN-GORDON EQUATION (KG) WITH POSITION DEPENDENT MASS AND APPLICATION TO THE PHYSICAL SYSTEMS

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In this study, Klein-Gordon equation with position dependent mass is formulated via Einstein's theory of relativity. The relation between scalar and vector potentials is defined as $S(x) = (\beta - 1)V(x)$ in order to be solved in the bound state solutions. Asymptotic iteration method, one of the most common methods to solve Klein-Gordon type equations, is chosen and explained.

The solutions of the equations are analyzed for both position-dependent and constant mass situations and their eigenvalues and eigenfunctions are derived. Before the solution of Klein-Gordon equations with position-dependent mass, the validity of asymptotic iteration method for Klein-Gordon equation with constant mass is examined. The spectrums are obtained by applying the method to Morse potential, Harmonic oscillator potential and Kratzer potential for the Klein-Gordon equation with constant mass, and applied to Kratzer potential and exponential potentials for Klein-Gordon equation with position-dependent mass.

Key Words: Position dependent mass Klein-Gordon equation, eigenvalue, eigenfunction, asymptotic iteration method, scalar potential, vector potential.

ÖZET

POZİSYONA BAĞLI KÜTLE İÇEREN KLEIN-GORDON DENKLEMİNİN ÇIKARIMI VE FİZİKSEL SİSTEMLERE UYGULAMALARI

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Bu çalışmada pozisyona bağlı kütle içeren Klein-Gordon (KG) denkleminin Einstein'ın temel izafiyet denkleminden yola çıkılarak çıkarımı yapıldı. Denklemdeki skalar ve vektör potansiyeli arasındaki ilişki bağlı durumdaki çözüm verebilmesi için $S(x) = (\beta - 1)V(x)$ olarak tanımlandı. Klein-Gordon tipi denklemleri çözmek için en yaygın metotların arasından asimptotik iterasyon metodu seçilerek anlatıldı.

Denklem çözümleri, pozisyona bağlı ve bağlı olmayan durumlar için ayrı ayrı ele alınarak enerji özdeğerleri ve öz fonksiyonları bulundu. Pozisyona bağlı kütle içeren Klein-Gordon denkleminin çözümünden önce sabit kütleli Klein-Gordon denklemi için asimptotik iterasyon metodunun geçerliliği incelendi. Metot, Klein Gordon denkleminde sabit kütle için Morse potansiyeline, Harmonik osilator potansiyeline ve Kratzer potansiyeline, pozisyona bağlı kütle için de Kratzer potansiyeli ve üssel tip potansiyellerine uygulanarak spektrumları elde edildi.

Anahtar Kelimeler: Pozisyona bağlı kütle içeren Klein-Gordon Denklemi, Enerji özdeğerleri, özfonksiyon, Asimptotik İterasyon metodu, vektör potansiyel, skalar potansiyel.

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CONTENTS

LIST OF SYMBOLS

T	Kinetic energy operator
$\hbar$	The Plank constant
$m(x)$	Mass function
c	Speed of light
∇	Gradient operator
$\vec{p}$	Momentum operator
H	Hamiltonian
$\psi(r,t)$	Wavefunction
$S(x)$	Scalar potential
$V(x)$	Vector potential
f	Frequency
E	Energy eigenvalues
$f_n(r)$	Eigenfunction generator
$\psi(r,t)$	Wavefunction
$L_{n_l}(r)$	Laguerre polynomials
k	Iteration number
$P_n(x)$	Legendre polynomial
$P_n^m(x)$	Associated Legendre polynomial

$J_n(x)$	Bessel function
$H_{2n}(x)$	Hermite polynomial
$L_n(x)$	Laguerre polynomial
$L_n^k(x)$	Associated Laguerre polynomial
$T_n(x)$	Chebyshev polynomial of the first kind
$U_n(x)$	Chebyshev polynomial of the second kind
$C_n^\lambda(x)$	Gegenbauer polynomial
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomial
V_{eff}	Effective potential

CHAPTER 1

INTRODUCTION

Schrödinger Equation [1] is a function which is called wavefunction gives us all of the information about a quantum system like as probability, energy etc. At 1900 Max Planck's quantum assumptions [2], at 1924 de Broglie's hypothesis [3] and at 1927 Heisenberg's uncertainty principle [4] cause a stir in science world. After that years Max Planck's quantum assumptions and Schrödinger's wave mechanism merging and quantum mechanism theory was born. Schrödinger equation like as below in implicit form

$$H\psi = E\psi \tag{1.1}$$

where H is Hamiltonian operator and it gives total energy of the system

$$H = \frac{p^2}{2m} + V \tag{1.2}$$

First term is the kinetic energy operator and second one is the potential energy and p is the momentum operator $(-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{dx}})$ in Eq. (1.2).

Actually Schrödinger equation is about non-relativistic subatomic particles and Schrödinger studied about relativistic subatomic particles but he could not formalism. Relativistic subatomic particles cannot be solved with Schrödinger equation and we need a new viewpoint such as relativistic equations called Dirac equation [5] and Klein-Gordon [KG] equation. The Dirac equation is a relativistic quantum mechanical wave_equation formulated by British physicist Paul Dirac in 1928. It is provided a description of elementary spin-½ particles, such as electrons, consistent with both the principles of quantum mechanics and the theory of special relativity, and was the first theory to fully account for relativity in the context of quantum mechanics. Similarly, the Klein–Gordon equation (Klein–Fock–Gordon equation or sometimes Klein–Gordon–Fock equation) is a relativistic version of the Schrödinger equation.

The solution of relativistic equations has received a great consideration in recent years with constant mass [6-27]. Most of these studies are related with mixed scalar $\left(S(x)\right)$ and vector $\left(V(x)\right)$ potentials. The energy spectrum of correspondence potentials has been calculated by taking the scalar and vector potentials equal to each other in KG equation [8, 10, 28]. In this thesis, we define a new transformation which conforms the condition of having bound state solution, $S(x) \geq V(x)$ [29]. In other words, it is not necessary to restrict ourselves for choosing the vector and scalar potentials.

Position dependent mass (PDM) Hamiltonian has an important role in physics. If we have a quantum mechanical system, we can attain PDM. For example, in semiconductor physics, the fact that the carriers move is identified with the quantum mechanical system. In nuclear physics, designing theoretical models with effective interactions are other vital applications [30-37]. Therefore, the importance of PDM is undeniable [71]. The exact or approximate solutions of position dependent mass relativistic wave equation have received a great attention along the last few years. [8, 11, 13, 19, 26, 38-45]

For all types of problem considered above for relativistic or non relativistic wave equations, the energy spectra of physical potentials has been obtained by using different techniques. For example Lie algebraic methods [46], the supersymmetric quantum mechanics approach [47, 48], transformations method [49], series expansion method [50], Nikiforov-Uvarov method [51], Asymptotic Taylor Expansion method [52], and function analysis method [45]. In this thesis we use the Asymptotic Iteration Method (AIM) [53] that is proposed to solve second order homogeneous differential equations in the form of $y'' = \lambda_0(x)y' + s_0(x)y$ where $\lambda_0(x)$ and $s_0(x)$ are arbitrary functions. When the differential equation is reduced to this form, it means that, it is amenable to apply AIM. This method has some advantages when it is comparable with other methods. For example, we have to use more complex mathematics in other methods, but AIM is less complex for energy eigenstates and wave eigenfunctions.

The organization of thesis is as follows: We will explain Klein-Gordon equation formalism in the chapter 2 for constant mass and position dependent mass KG. The correction of the Stark effect and the Zeeman effect are also taken into consideration during the formalism.

In Chapter 3, the chosen method for solving the eigenvalues of KG equation is presented for second order differential equations and first order differential equations. Actually AIM for first order differential equation can be used solution of Dirac equations [19].

The applications of AIM for constant mass KG equation for harmonic oscillator potential, Morse potential and Kratzer potential are considered in the subsequent chapter. Additionally the eigenfunctions of PDMKG for Kratzer potential and exponential type potentials are obtained by the AIM of the hypergeometric functions using the wave generators of AIM.

The final chapter deals with the discussion of the results obtained for different type potentials KG.

CHAPTER 2

KLEIN-GORDON EQUATION

One of the most important relativistic equations is the Klein-Gordon (KG) equation. Some knowledge about the wave mechanism and Klein-Gordon equation is outlined in this section. Also, we derive the final form of KG which used in this study by considering the relation between vector potentials and scalar potentials.

2.1 Brief About Wave Mechanism and Klein-Gordon Equation

Classical physics is successful but it cannot explain some physical events like as blackbody radiation, photo-electric effect. Because classical physics think that universe is constant. But at 1900 Max Planck [55] presumed that energy cannot be constant, after that at 1905 Einstein [54] presumed that light be formed by packets namely it is not constant. Until that time scientist believe that electrons but well known atom model was Thomson's – plum pudding - atom model [54] and at that years Rutherford [57] shown that atoms have a small nucleus and nucleus cannot contain electrons. In this case if electrons moving around of nucleus after a time it must fall into nucleus because from classical electromagnetic theory when electrons accelerated around the nucleus it loses its energy after a time and fall into the nucleus. This is a very important phenomenon at that years and Bohr [55] found a solution. From Bohr's atom model electrons cannot radiate until have some energy values. So radiation energy is quantized but Bohr's atom model is satisfied for only one electron atoms for more electrons system Bohr atom model could not satisfied.

At 1900 when Max Planck tried to solve blackbody radiation (ultraviolet catastrophe), he used that $E = hf$ equation and this equation started that photon concept because Planck presumed that when an electron oscillated with f frequency, it emitted light but this light can have energy only hf and its integer numbers. Between the years 1925-1926 Werner Heisenberg, Wolfgang Pauli and Pascal Jordan studied about quantum mechanism but they did not interest in wave mechanism because their dialectic was positivist. At 1926 Schrödinger [56] regenerated wave mechanism with some equations. Finally he derived the Heisenberg's matrix mechanism and showed that two formalisms equal each other mathematical and he found a new attitude which called Schrödinger equation, about non-relativistic particles. He published an article about relativistic particles wave mechanism but relativistic wave equation was taken its final form by Oskar Klein and Walter Gordon and this equation called Klein-Gordon equation.

2.2 Formalism of Klein-Gordon Equation

Klein-Gordon relation [56-60] is an equation about relativistic subatomic particles which has no spin like as bosons, for example π mesons [55]. There are more than one ways for derive the Klein-Gordon expression. Actually the starting point of all derivation's is the fundamental energy relation used in special relations.

$$E^2 = p^2c^2 + m^2c^4 \tag{2.1}$$

where p is the momentum, c is the speed of light m is the mass of the particle and E is the energy. At this point Quantum Mechanics can help us. In Quantum Theory, observables have to turn into mathematical operators for solution. Hence following substitutions are made for energy. Energy operator in quantum mechanics is defined as

$$E = i\hbar\frac{\partial}{\partial t} \tag{2.2}$$

Similarly, the momentum operator expression is in the form of

$$p = -i\hbar\frac{\partial}{\partial x} \tag{2.3}$$

Taking the squares of both sides of equations Eq.(2) and Eq.(3), we get

$$E^2 = -\hbar^2\frac{\partial^2}{\partial t^2} \tag{2.4}$$

$$p^2 = -\hbar^2\frac{\partial^2}{\partial x^2}. \tag{2.5}$$

After these substitutions, Einstein relation between energy, mass and momentum can be written like as below with operators.

$$-\hbar^2\frac{\partial^2}{\partial t^2} = -\hbar^2 c^2\frac{\partial^2}{\partial x^2} + m^2c^4 \qquad (2.6)$$

It cannot make any sense unless wave function of space and time $\psi = \psi(x,t)$ is applied.

$$-\hbar^2\frac{\partial^2\psi(x,t)}{\partial t^2} = -\hbar^2 c^2\frac{\partial^2\psi(x,t)}{\partial x^2} + m^2c^4\psi(x,t) \qquad (2.7)$$

When $\hbar = c = 1$ (natural units) [45] is taken Klein-Gordon equation becomes in cartesian coordinate system as

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} + m^2\psi = 0. \qquad (2.8)$$

On the other hand there is a second way for derive the Klein-Gordon equation. The starting point of second way's is the Einstein's relation Eq. (2.1)

$$E^2 = p^2c^2 + m^2c^4.$$

This relation can be complex factorization like as below.

$$E^2 = (pc + imc^2)(pc - imc^2) \qquad (2.9)$$

After this step when each term multiplied, the equation becomes

$$E^2 = p^2c^2 - pcimc^2 + imc^2pc + m^2c^4. \qquad (2.10)$$

When the momentum and energy operators with squares are substituted into Eq. (2.10), we get

$$-\hbar^2\frac{\partial^2}{\partial t^2} = -\hbar^2\frac{\partial^2}{\partial x^2}c^2 + i\hbar\frac{\partial}{\partial x}cimc^2 - imc^2 i\hbar\frac{\partial}{\partial x} + m^2c^4. \qquad (2.11)$$

It is known that operators operate least significant term and if m value is not position dependent –constant- the relation takes form

$$\hbar^2 \frac{\partial^2}{\partial t^2} - \hbar^2 c^2 \frac{\partial^2}{\partial x^2} - \hbar m c^3 \frac{\partial}{\partial x} + m c^3 \hbar \frac{\partial}{\partial x} + m^2 c^4 = 0. \qquad (2.12)$$

Applying the wave function space and time $\varphi = \varphi(x,t)$ to this equation yields to

$$\hbar^2 \frac{\partial^2 \varphi(x,t)}{\partial t^2} - \hbar^2 c^2 \frac{\partial^2 \varphi(x,t)}{\partial x^2} - \hbar m c^3 \frac{\partial \varphi(x,t)}{\partial x} + m c^3 \hbar \frac{\partial \varphi(x,t)}{\partial x} + m^2 c^4 \varphi = 0. \qquad (2.13)$$

Clearly, the third term $(-\hbar m c^3 \frac{\partial \varphi(x,t)}{\partial x})$ and the fourth term $(m c^3 \hbar \frac{\partial \varphi(x,t)}{\partial x})$ are canceled each other. Taking the $(\hbar = c = 1)$, the one dimensional KG equation is reduced to

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + m^2 \varphi = 0. \qquad (2.14)$$

The (1+1) dimensional KG can be extended to (1+3) dimensional coordinated by defining nabla (∇) operator like as below

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \qquad (2.15)$$

and KG equation is take form for (1+3) dimensional coordinates

$$\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + m(r)^2 \psi = 0 \qquad (2.16)$$

KG equation has some potential corrections due to the Stark effect and the Zeeman effect. Shortly, the Stark Effect is the shifting and splitting of spectral lines of atoms and molecules due to the presence of an external static electric field. The Zeeman Effect is the splitting of a spectral line into several components in the presence of a static magnetic field. As a result from these effects, the momentum operator and the energy operator are written in the form of

$$P = P + A\varphi$$
$$i \frac{\partial}{\partial t} = i \frac{\partial}{\partial t} - V(x) \qquad (2.17)$$

where A is the vector potentials. The other correction is an additional coupling to the space-time scalar potential $S(t,r)$ which is introduced by the substitution

$$m \to m + S(t,r). \tag{2.18}$$

The term "four-vector" and "scalar" refers to the corresponding unitary irreducible representation of the Poincare space-time symmetry group Gauge invariance of the vector coupling allows for the freedom to fix the gauge (eliminate the nonphysical gauge modes) without altering the physical content of the problem. There are many choices of gauge fixing that one could impose [8]. The Lorentz gauge, $\partial \cdot A = 0$ and the Coulomb gauge, $\nabla \cdot A = 0$ are two of the most commonly used conditions. If we adapt this later choice and write the time component of the four-vector potential as $gA_0 = V(t,\vec{r})$, where g is a quantity related to the magnetic moment of an electron, nucleus or other particle, then we end up with two independent potential functions in the KG equations. These are vector potential $V(t,\vec{r})$ and the scalar potential $S(t,\vec{r})$.

In the relativistic units, the free KG equation are written as

$$(\partial^\mu \partial_\mu + m^2)\psi_{KG}(t,\vec{r}) = 0. \tag{2.19}$$

The vector and scalar couplings mentioned above introduce potential interactions by mapping the free KG equations above into following

$$\left\{-\left[i\frac{\partial}{\partial t} - V(x)\right]^2 - \frac{\partial^2}{\partial x^2} + [S(x)+m]^2\right\}\psi_{KG}(x) = 0. \tag{2.20}$$

when we simplify the equation, it turns most common using form in this thesis

$$\frac{\partial^2\psi}{\partial x^2} + (E - V(x))^2\psi - (m(x) + S(x))^2\Psi = 0. \tag{2.21}$$

When the Eq.(2.21) is rearranged

$$\frac{\partial^2\psi(x)}{\partial x^2} + V_{eff}(x)\psi(x) = 0 \tag{2.22}$$

where

$$V_{eff} = E^2 - m^2 + [V^2(x) - S^2(x)] - [mS(x) + EV(x)] \qquad (2.23)$$

The last equation is reducing form of KG equation with energy dependent effective potential.

CHAPTER 3

THE ASYMPTOTIC ITERATION METHOD

In general, the second-order differential equations play an important role in many branches of physics, especially in quantum physics and mathematical physics. As a result of this importance, lots of techniques are developed in the literature that can be used to solve second order homogeneous linear differential equations with boundary conditions. One of these methods is AIM. It has algorithm suitable for computer programming to get results directly and rapidly.

In this chapter we focus on the Asymptotic Iteration Method for obtaining the corresponding eigenvalue values and eigenfunctions for PDMKG. In addition to the formalism of AIM for second order differential equations, we also derive the method to the first order differential equations.

3.1 Asymptotic Iteration Method

The (AIM) is proposed [53] to solve the second-order homogeneous linear differential equations of the form

$$y'' = \lambda_0(x)y' + s_0(x)y \tag{3.1}$$

where $\lambda_0(x)$ and $s_0(x)$ are functions defined in $C_\infty(a,b)$ and they have sufficiently many continuous derivatives.

In order to find a general solution to this equation, we rely on the symmetric structure of the right hand side of Eq. (3.1). Actually, if we differentiate Eq. (3.1) with respect to x, we find that

$$y''' = \lambda_1(x)y' + s_1(x)y \tag{3.2}$$

where the function are in the form of

$$\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2(x), \text{ and } s_1(x) = s_0'(x) + s_0(x)\lambda_0(x).$$

By taking the second derivative of Eq. (3.2), we get

$$y'''' = \lambda_2(x)y' + s_2(x)y \tag{3.3}$$

with

$$\lambda_2(x) = \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x),$$

and

$$s_2(x) = s_1'(x) + s_0(x)\lambda_1(x).$$

With application of similar procedure, for $(n+1)^{th}$ and $(n+2)^{th}$ derivative $(n = 1,2,3, \dots)$, we reach to the forms

$$y^{(n+1)} = \lambda_{n-1}(x)y' + s_{n-1}(x)y \tag{3.4}$$

and

$$y^{(n+2)} = \lambda_n(x)y' + s_n(x)y \tag{3.5}$$

where the coefficients $\lambda_n(x)$ and $s_n(x)$ are

$$\lambda_n(x) = \lambda_{n-1}'(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x) \tag{3.6a}$$

$$s_n(x) = s_{n-1}'(x) + s_0(x)\lambda_{n-1}(x). \tag{3.6b}$$

From the ratio of the $(n+2)^{th}$ and $(n+1)^{th}$ derivatives, we get

$$\frac{d}{dx} ln\left(y^{(n+1)}\right) = \frac{y^{(n+2)}}{y^{(n+1)}} = \frac{\lambda_n(x)(y' + \frac{s_n(x)}{\lambda_n(x)}y)}{\lambda_{n-1}(x)(y' + \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}y)} \tag{3.7}$$

We now introduce the “asymptotic” aspect of the method. If we have, for sufficiently large n,

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = \alpha(x), \tag{3.8}$$

then Eq. (3.8) simplifies to

$$\frac{d}{dx}\ln(y^{n+1}) = \frac{\lambda_n(x)}{\lambda_{n-1}(x)} \tag{3.9}$$

which yields

$$y^{(n+1)}(x) = C_1 \exp\left(\int \frac{\lambda_n(t)}{\lambda_{n-1}} dt\right) = C_1\lambda_{n-1}\exp\left(\int(\alpha + \lambda_0)dt\right) \tag{3.10}$$

where C_1 is integration constant, and the right hand equation follows from Eq. (3.7) and the definition of α. If we substitute Eq. (3.10) into Eq. (3.4), we get a first order differential equation

$$y' + \alpha y = C_1\exp\left(\int \alpha + \lambda_0)dt\right) \tag{3.11}$$

which gives the general solution to Eq. (3.1) likes as below

$$y(x) = \exp(-\int \alpha\, dt)\left[C_2 + C_1 \int exp\left(\int\left(\lambda_0(\tau) + 2\alpha(\tau)\right)d\tau\right)dt\right]. \tag{3.12}$$

At this stage, the corresponding second order homogeneous differential equation should be transformed in the form of Eq. (3.1) for the reason of obtaining the energy eigenvalues and the eigenfunctions. Subsequently, the parameters $\lambda_0(x)$ and $s_0(x)$ are determined by comparing the equations and $\lambda_k(x)$ and $s_k(x)$ are calculated by the recurrence relations defined in the Eq.(3.7). At the beginning of the procedure, in calculating the parameters for $n = 0$, the initial conditions are taken as $\lambda_{-1}(x) = 1$, $s_{-1}(x) = 0$ [61] and the iterations should be terminated by imposing the quantization condition in Eq. (3.9) $\Delta_k(x) = 0$ for

$$\Delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) \tag{3.13}$$

where *k* is the iteration number.

At the final step of AIM, the energy eigenvalues of corresponding equation are obtained from the roots of the quantization condition, given by the termination condition of the method if the system is exactly solvable. If the problem is not exactly solvable, the authors [31, 53, 61, 62, 63] proposed that for a special *n* quantum number, a suitable x_0 point is chosen, then generally as the maximum value of the asymptotic wave function or the minimum value of the potential is calculated.

And finally, for sufficiently large values of iteration, the approximate energy eigenvalues are obtained from the roots of this equation.

The wavefunction in AIM is calculated by the wave function generator which can be obtained by taking the parameter $C_1 = 0$ in equation (3.12)

$$y(x) = C_2 \exp(-\int \alpha \, dt). \tag{3.14}$$

2.2 Derivation of AIM to the First Order Linear Differential Equations

To start to derive the AIM equations for the first order linear differential equation [64], we consider two different first order differential equations in the form

$$\phi'_1 = \lambda_0(x)\phi_1 + s_0(x)\phi_2 \tag{3.15}$$

$$\phi'_2 = \omega_0(x)\phi_{2+}p_0(x)\phi_2 \tag{3.16}$$

where $\omega_0(x), p_0(x)$ are differentiable functions depend on x.

If we take derivative Eq. (3.15) and Eq. (3.16), we get

$$\phi''_1 = \lambda_1(x)\phi_1 + s_1(x)\phi_2 \tag{3.17}$$

$$\phi''_2 = \omega_1(x)\phi_1 + p_1(x)\phi_2 \tag{3.18}$$

where

$$\lambda_1(x) = \lambda'_0(x) + \lambda_0^2(x) + s_0(x)\omega_0(x) \tag{3.19}$$

$$s_1(x) = s'_0(x) + \lambda_0(x)s_0(x) + s_0(x)p_0(x) \tag{3.20}$$

$$\omega_1(x) = \omega'_0(x) + \lambda_0(x)\omega_0(x) + p_0(x)\omega_0(x) \tag{3.21}$$

$$p_1(x) = p'_0(x) + p_0^2(x) + s_0(x)\omega_0(x). \tag{3.22}$$

With same way, if we take $(n+2)^{th}$ derivatives, we obtain

$$\phi_1^{(n+2)} = \lambda_{n+1}(x)\phi_1 + s_{n+1}(x)\phi_2 \tag{3.23}$$

$$\phi_2^{(n+2)} = \omega_{n+1}(x)\phi_1 + p_{n+1}(x)\phi_2 \tag{3.24}$$

where

$$\lambda_{n+1}(x) = \lambda'_n(x) + \lambda_n(x)\lambda_0(x) + s_n(x)\omega_0(x) \qquad (3.25)$$

$$s_{n+1}(x) = s'_n(x) + \lambda_n(x)s_0(x) + s_n(x)p_0(x) \qquad (3.26)$$

$$\omega_{n+1}(x) = \omega'_n(x) + \lambda_0(x)\omega_n(x) + p_n(x)\omega_0(x) \qquad (3.27)$$

$$p_{n+1}(x) = p'_n(x) + \omega_n(x)s_0(x) + p_n(x)p_0(x) \qquad (3.28)$$

From the ratio of $(n+2)^{th}$ and $(n+1)^{th}$ derivatives, we get

$$\frac{d}{dx}\ln\left(\phi_1^{(n+1)}\right) = \frac{\phi_1^{(n+2)}}{\phi_1^{(n+1)}} = \frac{\lambda_{n+1}(x)[\phi_1 + ({s_{n+1}(x)}/{\lambda_{n+1}(x)})\phi_2]}{\lambda_n[\phi_1 + ({s_n(x)}/{\lambda_n}(x))\phi_2]} \qquad (3.29)$$

Applying the same procedure for ϕ_2 yields to similar expression. When n goes infinity we can obtain

$$\frac{s_{n+1}(x)}{\lambda_{n+1}(x)} = \frac{s_n(x)}{\lambda_n(x)} = \alpha,\ n = 1,2,3, ... \qquad (3.30)$$

Using this relation, Eq. (3.30) takes form $(d/dx)\ln\left(\phi_1^{(n+1)}\right) = \lambda_{n+1}/\lambda_n$ and the function $\phi_1^{(n+1)}(x)$ is obtained as

$$\phi_1^{(n+1)}(x) = C_1 \exp\left(\int^x \frac{\lambda_{n+1}(t)}{\lambda_n(t)} dt\right) = C_1\lambda_n \exp\left(\int^x (\alpha\omega_0 + \lambda_0)dt\right). \qquad (3.31)$$

Substituting Eq. (3.31) into $\phi_1^{(n+1)} = \lambda_n(x)\phi_1 + s_n(x)\phi_2$ relation, we obtain

$$\phi_1 + \alpha(x)\phi_2 = C_1 \exp\left(\int^x (\alpha\,\omega_0 + \lambda_0)dt\right). \qquad (3.32)$$

Using Eq. (3.18) and Eq. (3.32), the general solution of $\phi_2(x)$ is obtained in the following form [19]

$$\phi_2(x) = \exp\left(\int^x (p_0 - \omega_0\alpha)dt\right)\left\{C_2 + C_1\int^x \left[\omega_0 \exp\left(\int^t (\lambda_0 - p_0 + 2\omega_0\alpha)d\tau\right)dt\right]\right\}.$$

(3.33)

This expression is the wavefunction generator for the first order differential equations.

CHAPTER 4

APPLICATION OF AIM FOR KLEIN-GORDON EQUATION

In this chapter, the solution of KG equation with constant mass for Harmonic oscillator potential, Morse potential and Kratzer potential and PDMKG with different masses for linear potential and exponential potential is outlined.

4.1 Application of AIM for Constant Mass Klein-Gordon Equation

4.1.1 Harmonic Oscillator Potential

Generally, Klein-Gordon equation with scalar and vector potential can be written as like as Eq.(4.1) in [65]

$$\{\frac{d^2}{dx^2} + [E - V(x)]^2 - [m + S(x)]^2\} f(x) = 0 \tag{4.1}$$

When $S(x) \geq V(x)$, this means that KG equations has a real bound state solutions, when $S(x) = V(x)$, KG equations reduces to the Schrödinger type equations. In this part, we consider that the case of $S(x) > V(x)$ for the KG equations. When we rearrange the Eq. (4.1), we can get

$$\{\frac{d^2}{dx^2} - \left[V_{eff}(x) - (E^2 - m^2)\right]\} f(x) = 0 \tag{4.2}$$

where

$$V_{eff} = [S^2(x) - V^2(x)] + 2[mS(x) + EV(x)].$$

We choose the potential like as below [21]

$$V(x) = V_0 + \beta S(x),$$

where V_0 and β are arbitrary constants. At this point, in order to have the conditions $S(x) > V(x)$, we have to take the arbitrary constant β is less than 1, if $V_0 = 0$. So we define the relationship between potentials as

$$S(x) = V(x)(\beta - 1), \;\; \beta \geq 0. \tag{4.3}$$

When the last two expressions are substituted in KG equation

$$\{\frac{d^2}{dx^2} - \left[\left[\left(V(x)(\beta - 1)\right)^2 - V^2(x)\right] + 2\left[m\left(V(x)(\beta - 1)\right) + EV(x)\right] - (E^2 - m^2)]\right\} f(x) = 0. \tag{4.4}$$

Expand the parameters in equation yields to

$$\{\frac{d^2}{dx^2} - ([V^2(x)\beta^2 - 2V^2(x)\beta + V^2(x) - V^2(x)] + 2[mV(x)\beta - 2mV(x) + EV(x)] - [E^2 - m^2])\}f(x) = 0. \tag{4.5}$$

Rearranging the Eq. (4.5) gives

$$\{\frac{d^2}{dx^2} - [V(x)(V(x)A^2 + B) - \varepsilon]\}f(x) \tag{4.6}$$

where $\varepsilon = E^2 - m^2$, $A^2 = \beta^2 - 2\beta$ and $B = 2(E - m - m\beta)$. When we do some algebraic simplifications Eq.(4.6) is transformed to

$$\left\{\frac{d^2}{dx^2} - \left[\left[AV(x) + \frac{B}{2A}\right]^2 - \xi\right]\right\} f(x) = 0 \tag{4.7}$$

where $\xi = \varepsilon + (B/2A)^2$.

Let us consider the linear potential form of vector potential like as

$$V(x) = x.$$

Substituting the potential in Eq. (4.7), KG becomes

$$\left\{\frac{d^2}{dx^2} - \left[\left[Ax + \frac{B}{2A}\right]^2 - \xi\right]\right\} f(x) = 0. \tag{4.8}$$

After changing of variables $y = Ax + \frac{B}{2A}$, the second order differential Eq. (4.8) yields

$$\left\{\frac{d^2}{dy^2} - [y^2 - \xi]\right\} f(y) = 0. \tag{4.9}$$

This equation is the KG harmonic oscillator potential and actually Eq.(4.9) is a Schrödinger equation with an energy dependent potential.

For applying AIM, Eq. (4.9) should be transformed to the form of differential equation of the form Eq. (3.1). Therefore it should has a solution in the form of normalized wavefunction which is found by means of the iteration procedure

$$f(y) = \exp\left(-\frac{y^2}{2}\right) \chi(y). \tag{4.10}$$

Substituting Eq. (4.10) into Eq. (4.9) yields

$$\chi''(y) = 2y\chi'(y) + (1 - \xi)\chi(y) \tag{4.11}$$

which is now amenable to apply AIM for solution. When we compare the Eq.(4.11) with Eq.(3.1), we can write $\lambda_0(y)$ and $s_0(y)$ values and by means Eq.(3.6a) and Eq.(3.6b), we can calculate $\lambda_n(y)$ and $s_n(y)$. First third functions are

$$\lambda_0 = 2y$$

$$s_0 = (1 - \xi)$$

$$\lambda_1 = 3 - \xi + 4y^2$$

$$s_1 = 2(1 - \xi)$$

$$\lambda_2 = 8y + 2(1 - \xi)y + 2y(3 - \xi + 4y^2)$$

$$s_2 = 2(1 - \xi) + (1 - \xi)(3 - \xi + 4y^2)$$

…………..

Combining these results with the quantization condition, we get the energy eigenvalues

$$\frac{\lambda_0}{s_0} = \frac{\lambda_1}{s_1} \Rightarrow \xi_0 = 1$$

$$\frac{\lambda_1}{s_1} = \frac{\lambda_2}{s_2} \Rightarrow \xi_1 = 3$$

$$\frac{\lambda_2}{s_2} = \frac{\lambda_3}{s_3} \Rightarrow \xi_2 = 5$$

$$\frac{\lambda_3}{s_3} = \frac{\lambda_4}{s_4} \Rightarrow \xi_3 = 7$$

……

When we generalize the expression, we can get

$$\xi_n = 2n + 1, \;\; n = 0,1,2,3 \ldots$$

If we use parameters which defined ξ, ε, A and, we obtain

$$E^2 - m^2 + \frac{[2(E - m(1-\beta))]^2}{4(2\beta - \beta^2)} \tag{4.12}$$

which is exactly the same as the eigenvalue equation obtained in [65]. From Eq.(4.12), one can find that

$$E_n = \mp \frac{(-1+\beta)m \mp \sqrt{\gamma_n}}{-1+(-2+\beta)\beta} \tag{4.13}$$

where

$$\gamma_n = (-2+\beta)\beta\big(-1 - 2n + (-2+\beta)\beta(1 + 2m^2 + 2n)\big).$$

As seen from Eq. (4.13), we can see that, the energy eigenvalues have two types of energy. Non relativistic Schrödinger type equations do not have negative energy values but a relativistic type equation has negative energy states. This means that positive part is particle energy and negative part is anti-particle's energy.

4.1.2 Morse Potential

In this subsection, we deal with the exponential form of scalar potential. We take $V(x) = -\exp(-\alpha x)$ and because of the relationship between scalar and vector potentials $S(x) = -\exp(-\alpha x)(1-\beta)$. Substituting these equations into Eq. (4.6), we obtain

$$\Big\{\frac{d^2}{dx^2} - [(A^2 \exp(-2\alpha x) - B\exp(-\alpha x)\Big) - \xi]\Big\} f(x) \tag{4.14}$$

where $\xi = -\varepsilon$.

This equation is the KG Morse potential. We note that, this equation is also called Schrödinger equation with an energy dependent potential. We have to change of variables with

$$y = \exp(-\alpha x)$$

and we obtain

$$\left\{\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + \left[\frac{B}{\alpha^2 y} - \frac{A}{\alpha^2} + \frac{\xi}{\alpha^2 y^2}\right]\right\} f(y) = 0. \qquad (4.15)$$

Actually, we reach a position that the differential equation is suitable for applying AIM. Therefore Eq. (4.15) should have a solution in the form of the normalized wavefunctions,

$$f(y) = y^{\sqrt{\xi}/\alpha} \exp\left(-\frac{A}{\alpha} y\right) \chi(y). \qquad (4.16)$$

Substituting Eq. (4.16) into Eq. (4.15), one gets

$$\chi''(y) = \left[\frac{-2\sqrt{\xi}-\alpha+2Ay}{\alpha y}\right]\chi'(y) + \left[\frac{-B+2A\sqrt{\xi}+A\alpha}{\alpha^2 y^2}\right]\chi(y) \qquad (4.17)$$

which is now amenable to apply AIM for solution. Comparing the Eq. (4.17) and Eq. (3.1), the functions $\lambda_0(y)$, $s_0(y)$, $\lambda_n(y)$ and $s_n(y)$ are listed as follows by means of Eq.(3.6a) and Eq.(3.6b),

$$\lambda_0 = \frac{-2\sqrt{\xi} - \alpha + 2Ay}{\alpha y}$$

$$s_0 = \frac{-B + 2A\sqrt{\xi} + A\alpha}{\alpha^2 y^2}$$

$$\lambda_1 = \frac{1}{\alpha^2 y^2}\left[4\xi - By + 4A^2y^2 - 3Ay\alpha + 2\alpha^2 + \sqrt{\xi}(-6Ay - 6\alpha)\right]$$

$$s_1 = -\frac{1}{\alpha^3 y^2}\left(\sqrt{\xi} - Ay + \alpha\right)\left(B - A(2\sqrt{\xi} + \alpha)\right)$$

…..

Applying the quantization condition yields

$$\frac{\lambda_0}{s_0} = \frac{\lambda_1}{s_1} \Rightarrow \xi_0 = (\frac{1}{2}\alpha - \frac{B}{2A})^2$$

$$\frac{\lambda_1}{s_1} = \frac{\lambda_2}{s_2} \Rightarrow \xi_1 = (\frac{3}{2}\alpha - \frac{B}{2A})^2$$

$$\frac{\lambda_2}{s_2} = \frac{\lambda_3}{s_3} \Rightarrow \xi_2 = (\frac{5}{2}\alpha - \frac{B}{2A})^2$$

$$\frac{\lambda_3}{s_3} = \frac{\lambda_4}{s_4} \Rightarrow \xi_3 = \left(\frac{7}{2}\alpha - \frac{B}{2A}\right)^2$$

From these expressions we can write general formula for ε

$$\xi_n = \left(\left(n + \frac{1}{2}\right)\alpha - \frac{B}{2A}\right)^2, \quad n < \left(\frac{B}{\alpha^2} - \frac{1}{2}\right)$$

Substituting the parameters ξ and ε, it gives

$$E_n^2 - m^2 = -\left(\left(n + \frac{1}{2}\right)\alpha - \frac{B}{2A}\right)^2. \tag{4.18}$$

and

$$E_n = \pm\sqrt{-\left(\left(n + \frac{1}{2}\right)\alpha - \frac{B}{2A}\right)^2 + m^2} \tag{4.19}$$

which is exactly the same as the eigenvalue equation in [66] through a proper choice of parameters.

4.1.3 Kratzer Potential

In this subsection, we deal Kratzer potential that has a very important role in quantum mechanics. In order to obtain the required potential form in the KG equation, we choose

$$V(x) = \frac{1}{x}$$

And then relationship between scalar and vector potentials, scalar potential takes form

$$S(x) = \frac{1}{x}(\beta - 1).$$

Substituting them into Eq. (4.6), we obtain the differential equation

$$\left\{\frac{d^2}{dx^2} - \left[\left(\frac{A^2}{x^2} - \frac{B}{x}\right) - \xi\right]\right\} f(x) = 0 \tag{4.20}$$

where $\xi = -\varepsilon$, and Eq.(20) is the form of the KG Kratzer potential. At this point, to obtain the form of differential Eq.(3.1), and apply AIM, we propose the wavefunction like as below

$$f(x) = x^A \exp\left(-\sqrt{\xi} x\right)\chi(x). \tag{4.21}$$

Substituting this proposed wavefunction into Eq.(4.20), we get

$$\chi''(x) = 2\left[\frac{\sqrt{\xi}x - A}{x}\right]\chi'(x) + \left[\frac{A + \left(B + 2A\sqrt{\xi}x\right)}{x^2}\right]\chi(x) \tag{4.22}$$

which is now amenable to apply AIM for solution. Comparing the second order differential equation with Eq. (3.1) and applying procedure that of other applications, we get the function in the form of

$$\lambda_0 = 2\left[\frac{\sqrt{\xi}x - A}{x}\right]$$

$$s_0 = \left[\frac{A + \left(B + 2A\sqrt{\xi}x\right)}{x^2}\right]$$

$$\lambda_1 = \frac{1}{y^2}\left[4A^2 + A\left(3 - 6\sqrt{\xi}x\right) + x(B + 4\xi x)\right]$$

$$s_1 = \frac{1}{x^3}\left[Bx\left(-1 + 2\sqrt{\xi}x\right) - 2A^2\left(1 + 2\sqrt{\xi}x\right) - 2A(1 + Bx - 2\xi x^2)\right]$$

…..

And combining these results with the quantization condition yields

$$\frac{\lambda_0}{s_0} = \frac{\lambda_1}{s_1} \Rightarrow \xi_0 = (\frac{B}{2A})^2$$

$$\frac{\lambda_1}{s_1}=\frac{\lambda_2}{s_2}\Rightarrow \xi_1=(\frac{B}{2(A+1)})^2$$

$$\frac{\lambda_2}{s_2}=\frac{\lambda_3}{s_3}\Rightarrow \xi_2=(\frac{B}{2(A+2)})^2$$

$$\frac{\lambda_3}{s_3}=\frac{\lambda_4}{s_4}\Rightarrow \xi_3=\left(\frac{B}{2(A+3)}\right)^2.$$

From these expressions we can write general formula of ε

$$\xi_n=\left(\frac{B}{2(A+n)}\right)^2,\quad n=0,1,2,..$$

where the relation of n is obtained by comparing our parameters with that of in [69]. Substituting the parameters ξ and ε, one can obtain

$$E_n^2-m^2=\left(\frac{B}{2(A+n)}\right)^2 \tag{4.23}$$

and

$$E_n=\pm\sqrt{\left(\frac{B}{2(A+n)}\right)^2+m^2} \tag{4.24}$$

4.1.3.1 Obtaining Wavefunctions

In this subsection we find that wavefunctions with AIM [53], but first time we illustrate Hypergeometric functions. Starting with definition of Hypergeometric functions [70] and let us start define $(\alpha)_r$,

$$(\alpha)_r=\alpha(\alpha+1)+(\alpha+2)\ldots(\alpha+r-1)$$

$$=\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)};$$

where r is a positive integer and

$$(\alpha)_0=1.$$

The general hypergeometric function is defined as

$${}_mF_n(\alpha_1,\alpha_2,\alpha_3,..\alpha_m;\beta_1,\beta_2,\beta_3,\ldots\beta_n;x).$$

In series form, it is defined as

$$ {}_mF_n(\alpha_1, \alpha_2, \alpha_3, .. \alpha_m; \beta_1, \beta_2, \beta_3, ... \beta_n; x) = \sum_{r=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r ... (\alpha_m)_r x^r}{(\beta_1)_r (\beta_2)_r (\beta_3)_r ... (\beta_n)_r r!}. $$

We shall show that many of the special functions encountered up to now may be expressed in terms of hypergeometric functions.

We have to consider convergence of the series in last equation.

(i)The confluent hypergeometric series is convergent for all values of x.

(ii)If $|x| < 1$, hypergeometric series is convergent and if $|x| > 1$, it is divergent. For $x = 1$ the series converges if $\beta > \alpha_1 + \alpha_2$, while for $x = -1$, it converges if $\beta > \alpha_1 + \alpha_2 - 1$.

The following equations show the intimate relationship which exist between the hypergeometric functions and the special functions already considered.

$$ (a) \qquad P_n(x) = {}_2F_1(-n, n+1; 1; \frac{1-x}{2}) $$

$$ (b) \qquad P_n^m(x) = \frac{(n+m)!}{(n-m)!} \frac{(1-x^2)^{m/2}}{2^m m!} \, {}_2F_1\left(m-n, m+n+1; m+1; \frac{1-x}{2}\right) $$

$$ (c) \qquad J_n(x) = \frac{e^{-ix}}{n!} \left(\frac{x}{2}\right)^n \, {}_1F_1(n + \frac{1}{2}; 2n+1; 2ix) $$

$$ (d) H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} \, {}_1F_1\left(-n; \frac{1}{2}; x^2\right). $$

$$ (e) H_{2n+1}(x) = (-1)^n \frac{2(2n+1)!}{n!} x \, {}_1F_1\left(-n; \frac{3}{2}; x^2\right) $$

$$ (f) L_n(x) = {}_1F_1(-n; 1; x) $$

$$ (g) L_n^k(x) = \frac{\Gamma(n+k+1)}{n! \, \Gamma(k+1)} \, {}_1F_1(-n; k+1; x) $$

$$ (h) T_n(x) = {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) $$

$$(i) U_n(x) = \sqrt{(1-x^2)n}\ {}_2F_1\left(-n+1, n+1; \frac{3}{2}; \frac{1-x}{2}\right)$$

$$(j) C_n^{\lambda}(x) = \frac{\Gamma(n+2\lambda)}{n!\,\Gamma(2\lambda)}\ {}_2F_1\left(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-x}{2}\right)$$

$$(k) P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}\ {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2})$$

After reminder some properties of hypergeometric functions. Let sees how we can use in obtaining the eigenfunctions of the corresponding potentials form using eigenfunction generator [53]

$$f_n(r) = \exp\,(-\int^r \frac{s_k}{\lambda_k}\,dr)$$

where n represents the radial quantum number. By this procedure, the first few $f(r)$ functions are

$$f(0) = 1$$

$$f(1) = 4 + 2(A^2 - B^2)(3 + A^2 - B^2) - By$$

$$f(2) = 2(1 + A^2 - B^2)(3 + A^2 - B^2)^2(3 + 2A^2 - 2B^2) - 2B(3 + A^2 - B^2)(3 + 2A^2 - B^2)y + B^2y^2$$

$$f(3) = 4 + (1 + A^2 - B^2)(4 + A^2 - B^2)^3(3 + 2A^2 - 2B^2) - 6B(2 + A^2 - B^2)(4 + A^2 - B^2)^2(3 + 2A^2 - 2B^2)y + 6B^2(2 + A^2 - B^2)(4 + A^2 - B^2)y^2 - B^3y^3$$

$$f(4) = 4(1 + A^2 - B^2)(2 + A^2 - B^2)(5 + 2A^2 - 2B^2)^4(3 + 2A^2 - 2B^2)(5 + 2A^2 - 2B^2) - 8B(2 + A^2 - B^2)(5 + A^2 - B^2)^3(3 + 2A^2 - 2B^2)(5 + 2A^2 - 2B^2)y + 12B^2(2 + A^2 - B^2)(5 + A^2 - B^2)^2(5 + 2A^2 - 2B^2)y^2 - 4B^3(5 + A^2 - B^2)(5 + 2A^2 - 2B^2)y^3 + B^4y^4$$

After analyzing these results, we can see that the $f_n(r)$ functions can be written in series expansion by hypergeometric functions with constant $((A^2 - B^2) + n + 1)^n$ and $\prod_{k=0}^{(n-1)}(2(A^2 - B^2) + 2 + k)$. Generalizing these expansions, we get

$$f_n(r) = ((A^2 - B^2) + n + 1)^n \left[\prod_{k=0}^{n-1} (2(A^2 - B^2) + 2 + k)\right] \times {}_1F_1(-n, 2(A^2 - B^2) + 2; 2\varepsilon_{nA^2-B^2} r)$$

4.2 Applications of AIM with Position Dependent Mass Klein-Gordon Equation

4.2.1 AIM for Energies of Inversely Linear Potential with Spatially Dependent Mass

There have been many studies considered the KG equation with mixed linearly inversely potentials within the framework of different methods. [20, 41, 43]. Similarly, the solution of KG with any mixed vector and scalar potentials has received considerable attention in the literature. Actually in these studies scalar $(S(x))$ and vector $(V(x))$ potentials equal the each other. But here we consider the transformation described in [29] to conform the condition of having bound state solution, $S(x) \geq V(x)$.

For inversely linear potential, we consider the mass distribution function in the form of $m(x) = m_0(1 + \frac{\lambda/m_0}{|x|})$ as in [41].

The time-independent Klein-Gordon equation for a spatially dependent bosonic mass in $(1 + 1)$ dimensions for is considered [45]

$$\frac{d^2\psi}{dx^2} + \left(E - V(x)\right)^2 - (m(x) + S(x))^2\psi = 0 \qquad (4.25)$$

where $m(x)$ is the position dependent mass of the particle. When we rearrange the Eq. (4.25), we obtain

$$\frac{d^2\psi}{dx^2} + [E^2 - 2EV(x) + V^2(x) - m^2 - 2mS(x) - S^2(x)]\psi = 0. \quad (4.26)$$

If we collect the terms, we obtain

$$\frac{d^2\psi}{dx^2} + [E^2 - m^2 + [V^2(x) - S^2(x) - 2[mS(x) + EV(x)]]\psi = 0. \quad (4.27)$$

Let define the terms without differential as effective potential, V_{eff} as

$$[E^2 - m^2 + [V^2(x) - S^2(x) - 2[mS(x) + EV(x)]] = V_{eff}. \qquad (4.28)$$

Then we get

$$\frac{d^2 f(x)}{dx^2} + V_{eff}(x) f(x) = 0 \qquad (4.29)$$

Recently in [65], the relationship between the corresponding potentials was proposed to

$$S(x) = V(x)(\beta - 1), \quad \beta \geq 0, \qquad (4.30)$$

where β is an arbitrary constant. In this general description of scalar potential, by choosing β parameter 0, 1 and 2 the scalar potential leads to case $S(x) = -V(x)$, $S(x) = 0$ (purely vector potential), and $S(x) = V(x)$ respectively. The other choices of β leads to the required condition for bound state solutions that correspond to the case of $S(x) > V(x)$. Substituting the expression into Eq. (4.26), we obtain

$$\frac{d^2\psi(x)}{dx^2} + \Big[E^2 - 2EV(x) + V^2(x) - m^2 - 2m(\beta - 1)V(x) - \big((\beta - 1)V(x)\big)^2\Big]\psi(x) = 0.$$

After a simple algebra we can take

$$\frac{d^2\psi(x)}{dx^2} + [E^2 - 2EV(x) + V^2(x) - m^2 - 2m\beta V(x) + 2mV(x) - \beta^2 V^2 - 2\beta V^2(x)V^2(x)]\psi(x) = 0.$$

Rearranging this equation, the effective potential transforms to

$$V_{eff} = [E^2 - m^2] - V(x)[2E + 2(\beta - 1)m + \beta V(x)(\beta - 2)]. \qquad (4.31)$$

After this formalism of KG equation using the relation in Eq. (4.30), our task is only to choose the potential form and to apply the AIM.

Let choose the linear vector potential proportional to the absolute value of the coordinate as [41]

$$V(x) = -\frac{\hbar\, c\, q}{|x|} \qquad (4.32)$$

where q is a dimensionless real parameter. With this choose of potential, $\big(S(x)\big)$ is reduced to

$$S(x) = (\beta - 1)V(x) = -\frac{(\beta-1)\hbar\, c\, q}{|x|}. \qquad (4.33)$$

We found that final type of V_{eff} in Eq.[4.32] and substitute the vector potential into Eq. (4.29), V_{eff} becomes

$$V_{eff} = [E^2 - m^2] + \frac{q}{|x|}\left[2E + 2(\beta - 1)m + \beta(\beta - 2)\frac{q}{|x|}\right] \qquad (4.34)$$

and after rearranging, it is reduced to

$$V_{eff} = [E^2 - m^2] + \frac{2qE}{|x|} + \frac{2\beta qm}{|x|} - \frac{2qm}{|x|} - \frac{\beta^2 q^2}{x^2} - \frac{2\beta q^2}{x^2}. \qquad (4.35)$$

At this point, we have to define our position dependent mass and consider the bound-state solution PDMKG. For this application, we choose the form of the effective mass distribution as [41]

$$m(x) = m_0(1 + \frac{\lambda/m_0}{|x|}) \qquad (4.36)$$

where λ is a dimensionless real parameter. With this spatially dependent mass, effective potential takes form

$$V_{eff} = \left[E^2 - (m_0(1 + \frac{\lambda/m_0}{|x|}))^2\right] + \frac{2qE}{|x|} + \frac{2\beta q m_0(1+\frac{\lambda/m_0}{|x|})}{|x|} - \frac{2q m_0(1+\frac{\lambda/m_0}{|x|})}{|x|} - \frac{\beta^2 q^2}{x^2} - \frac{2\beta q^2}{x^2}. \qquad (4.37)$$

When we simplify the Eq. (4.35), we take

$$V_{eff} = E^2 - m_0^2 - 2m_0\frac{\lambda}{|x|} - \frac{\lambda^2}{x^2} + \frac{2qE}{|x|} + \frac{2\beta q m_0}{|x|} + \frac{2\beta q\lambda}{x^2} - \frac{2q m_0}{|x|} - \frac{2q\lambda}{x^2} - \frac{\beta^2 q^2}{x^2} - \frac{2\beta q^2}{x^2}.$$

With this V_{eff} and changing variable $u = |x|$, the corresponding KG equation takes form like as below;

$$\left\{\frac{d^2}{du^2} - \frac{(\lambda^2 - 2\lambda(\beta-1)q + \beta(\beta+2)q^2)}{u^2} + \frac{-2(-\lambda m_0 + (E + (\beta-1)m_0)q)}{u} + E^2 - m_0^2\right\} f(u) = 0. \qquad (4.38)$$

One can see easily that Eq.(4.38) seems a special potential family and rearranging the equation, finally we get

$$\left\{\frac{d^2}{du^2} - \frac{B(B+1)}{u^2} + \frac{A}{u} - \xi^2\right\} f(u) = 0 \tag{4.39}$$

where

$$B(B+1) = (\lambda^2 - 2\lambda(\beta-1)q + \beta(\beta+2)q^2)$$

$$A = -2(-\lambda m_0 + (E + (\beta - 1)m_0)q)$$

and we can find B and it has two values

$$B_1 = \frac{1}{2}\left(-1 - \sqrt{1 + 8\beta q^2 - 4\beta^2 q^2 - 8q\lambda + 8\beta q\lambda - 4\lambda^2}\right)$$

$$B_2 = \frac{1}{2}\left(-1 + \sqrt{1 + 8\beta q^2 - 4\beta^2 q^2 - 8q\lambda + 8\beta q\lambda - 4\lambda^2}\right)$$

Eq. (4.39) is the similar equation form of the Klein-Gordon Kratzer potential [67]. Interestingly, we note that Eq. (4.38) is, in fact a Schrödinger equation with an energy dependent potential. At this position, we have to propose a wavefunction like as below to apply the AIM

$$f(u) = u^{B+1} \exp(-\xi u)\, \chi(u) \tag{4.40}$$

with substituting this wavefunction into Eq.(4.38), we get

$$\chi''(u) = 2\left[\frac{\xi u - B - 1}{u}\right]\chi'(u) + \left[\frac{2(B+1)\xi - A}{u}\right]\chi(u) = 0. \tag{4.41}$$

Eq.(4.41) is the second order differential equation and is suitable for apply AIM like as Eq.(3.1). We can write the $\lambda_0(u)$ and $s_0(u)$ values from Eq.(3.1) and we may calculate $\lambda_n(u)$ and $s_n(u)$ from Eq.(3.6a) and Eq.(3.6b). The first fourth terms are

$$\lambda_0 = 2\left[\frac{\xi u - B - 1}{u}\right],$$

$$s_{0=}\left[\frac{2(B+1)\xi - A}{u}\right],$$

$$\lambda_1 = \frac{u(-A + 2(1+B)\xi) + 4(1 + B - u\xi)^2}{u^2}$$

$$s_1 = \frac{2(A - 2(1 + B)\xi)(1 + B - u\xi)}{u^2}$$

$$\lambda_2 = \frac{-4(1 + B - u\xi)(2 + 2B^2 - Au - 2u\xi + 2u^2\xi^2 + B(4 - 2u\xi))}{u^3}$$

$$s_2 = \frac{(A - 2(1 + B)\xi)\big(-4 - 4B^2 + Au + 6u\xi - 4u^2\xi^2 + B(-8 + 6u\xi)\big)}{u^3}$$

$$\lambda_3 = \frac{1}{u^4}\ (8(1 + B - u\xi)^2\big(2 + 2B^2 - Au - 2u\xi + 2u^2\xi^2 + B(4 - 2u\xi)\big) + u(-A2 + (1 + B)\xi)(u(-A + 2(1 + B)\xi) + 4(1 + B - u\xi^2))$$

$$s_3 = \frac{1}{4}(4(A - 2(1 + B)\xi)(1 + B - u\xi)(2 + 2B^2 - Au - 2u\xi + 2u^2\xi^2)B(4 - 2u\xi))$$

$$\lambda_4 = -\frac{1}{u^5}(2(1 + B - u\xi)\Big(16B^4 + 3A^2u^2 - 32B^3(-2 + u\xi) - 4Au(4 - 5u\xi + 4u^2\xi^2) + B^2(96 - 16Au - 96u\xi + 44u^2\xi^2) + 4(4 - 8u\xi + 11u^2\xi^2 - 8u^3\xi^3 + 4u^4\xi^4) + 4B\big(16 - 24u\xi + 22u^2\xi^2 - 8u^3\xi^3 + Au(-8 + 5u\xi)\big)\Big)$$

$$s_4 = \frac{1}{5}((-A + 2(1 + B)\xi)\big(8(1 + B - u\xi)\big)^2\big(2 + 2B^2 - Au - 2u\xi + 2u^2\xi^2 + B(4 - 2u\xi)\big) + u(-A + 2(1 + B)\xi) + 4(1 + B - u\xi)^2)))$$

………….

Similarly, from the quantization condition, we get

$$\frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} \Rightarrow \xi_0 = \frac{A}{2(B + 1)},$$

$$\frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2} \Rightarrow \xi_1 = \frac{A}{2(B + 2)},$$

$$\frac{s_2}{\lambda_2} = \frac{s_3}{\lambda_3} \Rightarrow \xi_2 = \frac{A}{2(B + 3)},$$

$$\frac{s_3}{\lambda_3} = \frac{s_4}{\lambda_4} \Rightarrow \xi_3 = \frac{A}{2(B+4)},$$

$$\frac{s_4}{\lambda_4} = \frac{s_5}{\lambda_5} \Rightarrow \xi_4 = \frac{A}{2(B+5)},$$

$$\frac{s_5}{\lambda_5} = \frac{s_6}{\lambda_6} \Rightarrow \xi_5 = \frac{A}{2(B+6)},$$

The general formula of ξ for n values can be written as

$$\xi_n = \frac{A}{2(n+B+1)}, \qquad n = 0,1,2,\ldots \tag{4.42}$$

The eigenvalues in Eq.(4.41) is transformed into the form of E_n by the definition of the parameter ξ, the energy definition can be obtained as [68]

$$E_n^2 - m_0^2 = -(\frac{A}{2(n+A+1)})^2. \tag{4.43}$$

We can find that E_n, after a simple algebra

$$E_n = \pm\sqrt{m_0^2 - (\frac{A}{2(n+A+1)})^2} \tag{4.44}$$

4.2.1.1 Obtaining Wavefunctions

Properties of hypergeometric functions are defined in the last subsection. In order to find corresponding the energy eigenfunctions, we may use the following energy eigenfunction generator

$$f_n(r) = \exp\left(-\int^r \frac{s_k}{\lambda_k}\, dr\right)$$

By applying the function generator, the first few $f(r)$ functions can be seen

$$f(0) = 1$$

$$f(1) = 4 + 2B(3+B) - Ay$$

$$f(2) = 2(1+B)(3+B)^2(3+2B) - 2A(3+B)(3+2B)y + A^2y^2$$

$$f(3) = 4 + (1+B)(4+B)^3(3+2B) - 6A(2+B)(4+B)^2(3+2B)y + 6A^2(2+B)(4+B)y^2 - A^3y^3$$

$$f(4) = 4(1+B)(2+B)(5+2B)^4(3+2B)(5+2B) \\ -8A(2+B)(5+B)^3(3+2B)(5+2B)y \\ +12A^2(2+B)(5+B)^2(5+2B)y^2 - 4A^3(5+B)(5+2B)y^3 \\ +A^4y^4$$

$$f(5) = 8(1+B)(2+B)(3+B)(6+B)^5(3+2B)(5+2B) \\ -20A(2+B)(3+B)(6+B)^4(3+2B)(5+2B)y \\ +40A^2(2+B)(3+B)(6+B)^3(5+2B)y^2 \\ -20A^3(3+B)(6+B)^2(5+2B)y^3 + 10A^4(3+B)(6+B)y^4 \\ -A^5y^5.$$

One can see that the $f_n(r)$ functions can be written in series expansion by hypergeometric functions with constant $(B+n+1)^n$ and $\prod_{k=0}^{(n-1)}(B+2+k)$. Generalizing these expansions, we get

$$f_n(r) = (B+n+1)^n\left[\prod_{k=0}^{n-1}(2B+2+k)\right] \times {}_1F_1(-n, 2B+2; 2\varepsilon_{nB}r)$$

4.2.2 AIM for Energies of Exponential Mass and Potential with Spatially Dependent Mass

Now, let consider the bound state solutions of PDMKG with mass distribution of $m = m_0e^{-\alpha x}$. The time independent KG equation for spatially dependent mass can be written as like Eq. (4.25). After rearranging this equation we obtain Eq. (4.26), Eq. (4.29) and Eq. (4.28). The relationship between the corresponding potentials is proposed to in Eq. (4.33) and after same steps like as in part 4.1 we obtain Eq. (4.34).

Let choose the vector potential function in exponential type as [16]

$$V(x) = V_0e^{-\alpha x} \quad (4.44)$$

Using the relationship between scalar and vector potentials, we get

$$S(x) = (\beta-1)V(x) = \big((\beta-1)\big)V_0e^{-\alpha x}. \quad (4.45)$$

By substituting these potentials into effective potential equation, we obtain V_{eff} as

$$V_{eff} = E^2 - m(x)^2 - (V_0^2 - \beta^2 V_0^2 + 2\beta V_0^2 - V_0^2)e^{-2\alpha x} - (2EV_0 - 2m(x)V_0)e^{-\alpha x} . \quad (4.46)$$

After defining the scalar and vector potential, let consider the effective mass distribution is also in exponential form as

$$m(x) = m_0 e^{-\alpha x}. \quad (4.47)$$

With spatially dependent mass, effective potential takes form

$$V_{eff} = E^2 - m_0^2 e^{-2\alpha x} - (V_0^2 - \beta^2 V_0^2 + 2\beta V_0^2 - V_0^2)e^{-2\alpha x} - (2EV_0 - 2m_0 e^{-\alpha x} V_0)e^{-\alpha x} .$$

By rearranging the effective potential equation, it takes form of

$$V_{eff} = E^2 - (m_0^2 + V_0^2 - \beta^2 V_0^2 + 2\beta V_0^2 - V_0^2 + 2m_0 V_0)e^{-2\alpha x} - (2EV_0)e^{-\alpha x} .$$

When we define

$$-(m_0^2 + V_0^2 - \beta^2 V_0^2 + 2\beta V_0^2 - V_0^2 + 2m_0 V_0) = C^2$$

and the effective potential takes form

$$V_{eff} = E^2 + C^2 e^{-2\alpha x} - 2EV_0 e^{-\alpha x} .$$

Then the PDMKG equation is reduced to

$$[E^2 + C^2 e^{-2\alpha x} - 2EV_0 e^{-\alpha x}]\psi(x) + \psi''(x) = 0 \quad (4.48)$$

This is not a known potential type and we change functions and change variables at the same time like as below

$$\psi''(x) = y\alpha^2 F'(y) + y^2\alpha^2 F''(y)$$

$$\psi(x) = F(y)$$

$$x = -\log[y]/\alpha.$$

Dividing both sides with terms $y^2\alpha^2$ and we obtain

$$\frac{(E^2 - 2EV_0 y + C^2 y^2)F(y)}{y^2\alpha^2} + \frac{F'(y) + yF''(y)}{y} = 0. \quad (4.49)$$

Take the wavefunction as

$$F(y) = \exp\left[\int -\frac{1}{2y}\, dy\right] M(y) \tag{4.50}$$

and substitute into Eq. (4.49), one gets

$$-\frac{2EV_0 M(y)}{y\alpha^2} + \frac{4E^2 M(y) + \alpha^2 M(y)}{4y^2\alpha^2} + \frac{4C^2 M(y) + 4\alpha^2 M''(y)}{4\alpha^2} = 0. \tag{4.51}$$

By collecting the terms, KG equation takes form

$$\left(\frac{\frac{1}{4}+\frac{E^2}{\alpha^2}}{y^2} + \frac{C^2}{\alpha^2} - \frac{2EV_0}{y\alpha^2}\right) M(y) + M''(y) = 0. \tag{4.52}$$

Let define the constant

$$\frac{1}{4} + \frac{E^2}{\alpha^2} = A^2$$

$$\frac{C^2}{\alpha^2} = -En$$

$$\frac{2EV_0}{\alpha^2} = B$$

$$M(y) = \varphi(x)$$

to final form of KG equation as

$$-En\, \varphi(x) - \frac{A^2}{x^2}\, \varphi(x) - \frac{B}{x}\, \varphi(x) + \varphi''(x) = 0 \tag{4.53}$$

Eq. (4.52) is the similar equation form of the KG Kratzer potential [67]. Let propose a wave function as

$$\varphi(x) = x^A \exp\left(-\sqrt{En}\, x\right) \chi(x) \tag{4.54}$$

with substituting this wavefunction into Eq.(4.52), we obtain

$$\chi''(x) = \left(\frac{A}{x^2} + \frac{B + 2A\sqrt{En}}{x}\right) \chi(x) + 2\left(\sqrt{En} - \frac{A}{x}\right) \chi'(x) \tag{4.55}$$

The last term is the expected form of the second order differential equation. By applying the steps of AIM, the functions can be obtained as

$$\lambda_0 = \frac{A}{x^2} + \frac{B + 2A\sqrt{En}}{x}$$

$$S_0 = 2\sqrt{En} - \frac{2A}{x}$$

$$\lambda_1 = 2\sqrt{En} - \frac{2A}{x^3} + \frac{B + 2A\sqrt{En}}{x^2} - \frac{2A}{x} + \frac{(A + Bx + 2A\sqrt{En}\,x)^2}{x^4}$$

$$S_1 = \frac{2(Ax + (-A + \sqrt{En}x)(A + Bx + 2A\sqrt{En}\,x))}{x^3}$$

$$\begin{aligned}\lambda_2 = \frac{1}{x^6}(&\left(A + 2A\sqrt{En}\,x\right)^3 + B\,x^3\left(2 - 3B + B^2 + 4\sqrt{En}\,x^2\right) \\ &- A^2\,x\left(1 + 2\sqrt{En}\,x\right)\left(6 + 6\sqrt{En}\,x + 4x^2 - 3B\left(1 + 2\sqrt{En}\,x\right)\right) \\ &+ Ax^2(6 + 4x^2 + 8En\,x^3 + 4\sqrt{En}\,x(1 + x) + B^2\left(3 + 6\sqrt{En}\,x\right) \\ &- B(9 + 12\sqrt{En}\,x + 4x^2)))\end{aligned}$$

$$\begin{aligned}S_2 = -\frac{1}{5}(&2(A^3\left(1 + 2\sqrt{En}\,x\right)^2 - \sqrt{En}\,x^3\left(-2B + B^2 + 2\sqrt{En}\,x^2\right) \\ &- A^2x\Big(5 + 4\,En\,x + 2x^2 + 4En^{3/2}\,x^2 + \sqrt{En}\,(1 + 6x) \\ &- 2B\left(1 + 2\sqrt{En}\,x\right)\Big) + A(2 + B^2 + 4En\,x - B\left(3 + 2\sqrt{En} + 4En\,x\right) \\ &+ 4\sqrt{En}\,(1 + x^2))))\end{aligned}$$

Using the quantization conditions, the energy eigenvalues are obtained as

$$\frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} \Rightarrow En_0 = \frac{B^2}{4A^2},$$

$$\frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2} \Rightarrow En_1 = \frac{B^2}{4(1 + A)^2},$$

$$\frac{s_2}{\lambda_2} = \frac{s_3}{\lambda_3} \Rightarrow \xi_2 = En_2 = \frac{B^2}{4(2 + A)^2},$$

$$\frac{s_3}{\lambda_3} = \frac{s_4}{\lambda_4} \Rightarrow En_3 = \frac{B^2}{4(3 + A)^2},$$

$$\frac{s_4}{\lambda_4} = \frac{s_5}{\lambda_5} \Rightarrow En_4 = \frac{B^2}{4(4+A)^2},$$

$$\frac{s_5}{\lambda_5} = \frac{s_6}{\lambda_6} \Rightarrow En_5 = \frac{B^2}{4(5+A)^2},$$

Generalizing for energy eigenvalues

$$En = \frac{B^2}{4(n+1+A)^2}, \qquad n = 0,1,2,\ldots \tag{4.56}$$

After substituting the parameters B and A, we get

$$-\frac{C^2}{\alpha^2} = \frac{\frac{4E^2V^2}{\alpha^2}}{4\left(n+1+\sqrt{\frac{1}{4}+\frac{E^2}{\alpha^2}}\right)^2}$$

$$-C^2 = \frac{E^2V^2}{n^2+2n+1+2(n+1)\sqrt{\frac{1}{4}+\frac{E^2}{\alpha^2}}+\frac{1}{4}+\frac{E^2}{\alpha^2}}$$

so

$$E^2 = -\frac{C^2}{V^2}\left(n^2+2n+1+2(n+1)\sqrt{\frac{1}{4}+\frac{E^2}{\alpha^2}}+\frac{1}{4}+\frac{E^2}{\alpha^2}\right). \tag{4.57}$$

The roots of this equation gives the energy values.

4.2.2.1 Obtaining Wavefunctions

The exponential potential equation is reduced to Kratzer type potential after changing variables and functions. Therefore, using the similar procedure for eigenfunction generator, the functions of $f_n(r)$ are obtained as

$$f(0) = 1$$

$$f(1) = 4 + 2B(3+B) - Ay$$

$$f(2) = 2(1+B)(3+B)^2(3+2B) - 2A(3+B)(3+2B)y + A^2y^2$$

$$f(3) = 4 + (1 + B)(4 + B)^3(3 + 2B) - 6A(2 + B)(4 + B)^2(3 + 2B)y + 6A^2(2 + B)(4 + B)y^2 - A^3y^3$$

$$\begin{aligned} f(4) = 4(1 + B)(2 + B)(5 + 2B)^4(3 + 2B)(5 + 2B) \\ - 8A(2 + B)(5 + B)^3(3 + 2B)(5 + 2B)y \\ + 12A^2(2 + B)(5 + B)^2(5 + 2B)y^2 - 4A^3(5 + B)(5 + 2B)y^3 \\ + A^4y^4 \end{aligned}$$

$$\begin{aligned} f(5) = 8(1 + B)(2 + B)(3 + B)(6 + B)^5(3 + 2B)(5 + 2B) \\ - 20A(2 + B)(3 + B)(6 + B)^4(3 + 2B)(5 + 2B)y \\ + 40A^2(2 + B)(3 + B)(6 + B)^3(5 + 2B)y^2 \\ - 20A^3(3 + B)(6 + B)^2(5 + 2B)y^3 + 10A^4(3 + B)(6 + B)y^4 \\ - A^5y^5 \end{aligned}$$

When results are analyzed, one can see that the $f_n(r)$ functions can be written in series expansion by hypergeometric functions with constant $((A^2 - B^2) + n + 1)^n$ and $\prod_{k=0}^{(n-1)}(2(A^2 - B^2) + 2 + k)$. Generalizing these expansions, we obtain.

$$f_n(r) = ((A^2 - B^2) + n + 1)^n \left[\prod_{k=0}^{n-1}(2(A^2 - B^2) + 2 + k)\right] \times {}_1F_1(-n, 2(A^2 - B^2) + 2; 2\varepsilon_{nA^2-B^2}r)$$

where $\sqrt{\frac{1}{4} + \frac{E^2}{\alpha^2}} = A$ and $\frac{2EV_0}{\alpha^2} = B$.

It seen easily $f_n(r)$ a function for exponential type potential is same for Kratzer potential KG with constant mass with different coefficients A and B.

CHAPTER 5

CONCLUSION

In this thesis, in order to solve the Klein-Gordon equation for both constant mass and variable mass, we applied asymptotic iteration method. The corresponding differential equations for some physical potential are solved by transformed into Schrödinger-like equations. And this transformation results in solving the energy spectrum of corresponding potential using AIM.

The method firstly applied the constant mass. The energy spectrum and wavefunction of Morse oscillator, Harmonic oscillator and Kratzer potentials is obtained for constant mass. The results clearly show that AIM produces the exact analytic more simpler than the other methods.

Secondly, AIM is applied to the position dependent mass Klein-Gordon equation for inversely linear potential and exponential type potentials. After making some transformation the corresponding potentials are reduced the Kratzer type. Then using the function generator, the wavefunction are obtained in terms of hypergeometric functions.

Generally, in literature most studies on AIM is only deal with obtaining the energy eigenvalues because of difficulties in obtaining the wavefunctions.

Besides given the exact analytic results, AIM also gives the numerical results for non-solvable potentials.

In this thesis, we only applied this method for exactly solvable potentials are calculated analytically.

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