UNIVERSITY OF GAZIANTEP
GRADUATE SCHOOL OF
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THE DETERMINATION OF EIGENVALUES OF KLEIN-GORDON EQUATION WITH POSITION DEPENDENT MASS (PDMKG) FOR EXPONENTIAL TYPE POTENTIALS

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ABSTRACT

The determination of eigenvalues of Klein-Gordon equation with Position Dependent Mass (PDMKG) for Exponential Type Potentials

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The bound-state solution of Klein-Gordon equation calculated using Asymtotic Iteration Method for some exponential-type scalar and vector potential functions such as Morse potential and Wood-Saxon potential. The eigenvalues and eigenfunctions of corresponding potentials are obtained for constant mass and for the exponential type of position dependent mass functions. Also, the eigenvalues are analyzed for pure scalar potential and the vector potential in addition to the equal scalar and vector potential with respect to β values. The bound state eigenfunctions calculated after the operations are obtained in terms of the confluent hypergeometric function using the wave function generator.

Keywords: Position dependent mass Klein-Gordon equation, eigenvalue, eigenfunction, asymptotic iteration method, scalar potential, vector potential, exponential potentials, Wood-Saxon potential.

ÖZ

Pozisyona bağlı kütle içeren Klein-Gordon Denkleminin üstel tipi potaniyeller için özdeğerinin hesaplanması

Yüksek Lisans Tezi, Fizik Mühendisliği, Gaziantep Üniversitesi

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Klein-Gordon denkleminin bağlı durum çözümleri Asimtotik İterasyon Metodu kullanılarak Morse potansiyeli ve Wood-Saxon potansiyeli gibi bazı üstel fonksiyon tipindeki skaler ve vektörel potansiyelleri için hesaplandı. İlgili potansiyellerin enerji özdeğerleri ve özfonksiyonları sabit kütle ve konuma bağlı kütle fonksiyonları için elde edildi. Ayrıca, özdeğerler β değerlerine bağlı olarak eşit skaler ve vektörel potansiyellerine ilaveten salt vektörel ve skaler potansiyel için analiz edildi. İşlemlerin sonunda elde edilen bağlı durum özfonksiyonları, dalga fonksiyonu generatörü kullanılarak konfluent hipergeometrik fonksiyonları cinsinden elde edildi.

Anahtar Kelimeler: Pozisyona bağlı kütle içeren Klein-Gordon Denklemi, Enerji özdeğerleri, özfonksiyon, Asimptotik İterasyon metodu, vektör potansiyel, skaler potansiyel, üstel potansiyeller, Wood-Saxon potansiyeli.

To my parents. And to my sisters, brothers and friend.

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LIST OF SYMBOLS

T Kinetic energy operator

$\hbar$ The Plank constant

$m(x)$ Mass function

c Speed of light

∇ Gradient operator

$\vec{p}$ Momentum operator

$W(r)$ Wood-Saxon potential

$\psi(r,t)$ Wavefunction

$S(x)$ Scalar potential

$V(x)$ Vector potential

f Frequency

E Energy eigenvalues

$f_n(r)$ Eigenfunction generator

$\psi(r,t)$ Wavefunction

V_{eff} Effective potential

k Iteration number

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CHAPTER ONE

1 INTRODUCTION

One of the important relativistic equations is the Klein-Gordon (KG) equation or sometimes it is called Klein-Fock equation. It is used for spineless particles and it is the first result of effort spectral relativity to quantum mechanics due to the consisting of negative energy solutions. KG equation is also called the relativistic version of the Schrödinger equation. The negative energy solution represents the antimatter particles solutions. Nonetheless, the KG equation does have serious restriction. It is basically a single equation [1].

In literature there have been many studies tackle with a solution of KG equation exponential type potentials. In most of these studies, the [2-4] vector and scalar potential were considered as equal to each other or pure scalar and pure vector form [5-7]. In this study, in order to obey the condition for bound-state solution we consider the transformation $S\,(x) = (\beta - 1)V\,(x)$ [8]. With these choices, there is no reason to restrict ourselves for the choices of vector and scalar potential. A huge part of this study considers for constant mass [9, 10]. The bound-state solutions for exponential potential in one dimension have been reported [11-15].

In addition to the constant mass solution, position mass applications have a wide range in physics. For example impurities in crystal [16-18, 23, 40, 71], the relation of nuclear forces to the relative velocity of two nucleons [19, 20] the electrostatic properties of quantum wells and quantum dots [21], He clusters [22], quantum liquids [23] and semiconductor heterostructures [24].

In this we especially focus on the solution of Wood-Saxon potential [25]. This potential was introduced study the elastic scattering of 20 Mev proton heavy nuclei. The Wood-Saxon potential is a reasonable potential for nuclear shell model and a short-range potential. Due to these properties it has a great popularity in nuclear physics [26-35] and it is widely used in nuclear, particle, atomic, condensed matter and chemical physics [36, 37]. Recently, the relativistic and non relativistic solutions are solved for Wood-Saxon potentials with different methods [38-45].

Various methods have been used to obtain the energy eigenvalues and eigenfunction for relativistic KG and Dirac equation. These are the super symmetric quantum mechanics method [46, 47], Lie algebraic method [48], transformation method [44], Nikiforov-Uvarov method [49], series expansion method [46], and function analysis method [50].

In order to solve the spectrum of same exponential potential forms, in this study we use the Asymptotic Iteration method [51] that is introduced to solve second order homogeneous differential equation. In the last decade the [AIM] has an increasing attention to solving the spectrum of physical systems in both relativistic [52-56] and non-relativistic [57] quantum mechanics.

The organization of the thesis is arranged as follows: The derivation of the KG equation in the presence of both vector and scalar potentials is outlined. Additionally, there is short definition of Wood-Saxon potential with graphs.

Chapter 3 deals with the formulation of AIM method to calculate eigenvalues and eigenfunction with arbitrary function $s_0(x)$ and $\lambda_0(x)$.

The main part of this thesis is the Chapter 4. It consists of all applications of exponential potential in KG with constant and effective mass cases. The eigenvalues and eigenfunctions are discussed in accord to the adjusting parameter β.

Finally, the last chapter is devoted to the main results of this thesis.

CHAPTER 2

2 KLEIN-GORDON EQUATION

The Klein-Gordon equation (Klein-Fock-Gordon equation or sometimes Klein-Gordon-Fock equation) describes a particle with spin 0, which limits its usefulness [58]. It is the relativistic version of the Schrödinger equation, which is used to describe spin less particles. However, this equation was named after the physicists Oskar Klein and Walter Gordon, who in 1927 proposed that it describes relativistic electrons [59]. These relativistic equations contain two objects, the vector *V(x)* and scalar potential *S(x)* [60].

2.1 Derivation of Klein-Gordon Equation

The Klein-Gordon equation is derived from two types which are from the special relativity from and from the quantum mechanical form. The relativistic form deals with the relativistic relation between energy, mass, and momentum derived by Einstein. And the other form deals with the promotion of measurable quantities (observable) to mathematical operators in quantum mechanics.)

Consider first of all, non-relativistic the Schrödinger equation in the case of one spatial dimension as.

$$E(\psi) = -\frac{\hbar^2 \partial^2 \psi}{2m\, \partial\, x^2} + V\, \psi \tag{2.1}$$

energy variable E is defined in terms of linear momentum as

$$E = \sqrt{p^2\, c^2 + m^2\, c^4} \tag{2.2}$$

Where E is the energy, p is the momentum c is the speed of light and m is the mass of the particle. $\psi\,(x)$ represents the wave function in one dimension and V represents the potential. The

From relation of E it is possible to reach the K.G equation. Now, let focus on Eq.(2.1) by considering observable mathematical operators for $E$and p. The Schrödinger equation can be thought of as a statement of the non-relativistic definition of energy. Then instead of constant energy value, the operator form of E takes form as

$$E\,\psi = -\frac{\hbar^2 \partial^2 \psi}{2m\,\partial\, x^2} + V\,\psi \tag{2.3}$$

where E is the energy, ψ is the wave function, V is the potential, and m is the mass $\frac{\partial^2 \psi}{\partial\, x^2}$ is the second-order derivative with respect to the spatial coordinate . The Schrödinger equation can be thought of as a statement of the non relativistic definition of energy

$$E \to i\hbar \frac{\partial}{\partial t}\,. \tag{2.4}$$

That is in ordinary quantum mechanics, momentum p is given by a derivative form with respect to x as,

$$p \to -\,i\hbar \frac{\partial}{\partial x} \tag{2.5}$$

The three dimensional of the p is

$$p \to\; -i\hbar\,\nabla \tag{2.6}$$

After giving the corresponding operators, substituting these operators into Einstein relation for energy, momentum, and mass Eq. (2.2) and apply it to a wave function φ we get in operator form

$$E^2 \to -\hbar^2 \frac{\partial^2}{\partial t^2},$$

$$p^2 \to \; -i\hbar^2 \, \nabla^2$$

and

$$-\hbar^2 \frac{\partial^2}{\partial t^2} = -\hbar^2 c^2 \, \nabla^2 + m^2 c^4 \tag{2.7}$$

The operator form equation is not useful. If we consider the wave function depends time $\psi = \psi(\vec{x}, t)$, the equation in operator for K.G equation is transformed to

$$\hbar^2 \frac{\partial^2 \psi}{\partial t^2} - \hbar^2 c^2 \, \nabla^2 \psi + m^2 c^4 \psi = 0 \tag{2.8}$$

When $\hbar = c = 1$, in particle physics it will be typed in units,

$$\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + m^2 \psi = 0 \tag{2.9}$$

Simplify the appearance of the equation a little further by using different notation. In fact write it in two different ways. The first is to recall the D'Alembertian operator in Minkowski space as

$$\square = \frac{\partial^2}{\partial t^2} - \nabla^2$$

Then Eq. (2.9) is written as following with substituting $\square$

$$(\square + m^2)\varphi = 0$$

where $\square$ is a relativistic invariant, that is, it is the same in all inertial reference frames because it transforms as a scalar.

Since the mass m is the scalar so the operator given by

$$\square + m^2$$

is also a scalar. The Klein-Gordon equation will be covariant provided that the φ which interprets later as a field also transforms as a scalar. It is possible to write the coordinates,

$$x'^{\mu} = \Lambda^{\mu}{}_{v} x^{v} \qquad (2.10)$$

where Λ is the Lorentz transformation, allows to transform between different inertial reference frames. An inertial reference frame x'^{μ} moves along the x axis with respect to another inertial reference frame x^{μ} with speed $v < c$.

Under a Lorentz transformation, if a field $\varphi\,(x)$ is a scalar field, then it transforms us

$$\varphi\,(x) = \varphi(\Lambda^{-1}x) \qquad (2.11)$$

It is applied to scalar particle (actual scalar field) and these particles are spin-0 particle, which is the first important characteristic of the Klein-Gordon equation .

Eq. (2.7) can be written in a nice, compact style using the notation developed using $\partial\mu\,\partial^{\mu} = \frac{\partial^2}{\partial t^2} - \nabla^2$, it becomes

$$(\partial\mu\,\partial^{\mu} + m^2) = 0 \qquad (2.12)$$

Eq. (2.11) describes a free particle solution is given by

$$\varphi\,(\vec{x}, t) = e^{-ip.x}$$

By applying special relativity, here so p and x are 4-vectors given by $p = (E, \vec{p})$ and $x = (\,t, \vec{x})$, respectively . The scalar product in the exponent is

$$p.x = p\mu\, x^{\mu} = Et - \vec{p}\,.\vec{x} \qquad (2.13)$$

The free particle solution implies the relativistic relation between energy, mass, and momentum. This is very easy to show, so let's do it. For simplicity, only one spatial dimension is considered. Since

$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial t} e^{-i(Et-px)} = -iEe^{-i(Et-px)} = -iE\psi$$

$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial t} e^{-i(Et-px)} = -ipe^{-i(Et-px)} = -ip\psi$$

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} = -iE\psi + p^2\psi$$

Therefore, by applying the full Klein-Gordon equation, Eq. (2.8) can be written as

$$(E^2 - p^2)\psi = m^2\psi$$

countermand the wave function and rearranging terms gives $E^2 = p^2 + m^2$, the favorable result. Solving for the energy, the square root is taken, existence careful to locate both positive and negative square roots.

$$E = \mp\sqrt{p^2 + m^2} \tag{2.14}$$

This is dramatic result which is one reason Schrödinger discarded the Klein-Gordon equation. The solution for the energy of the particle tells us that it is possible to have both positive and negative energy states, a nonphysical result [61].

The vector and scalar couplings mentioned above introduce potential interactions by mapping the free KG equations above into following

$$\left\{-\left[i\frac{\partial}{\partial t} - V(x)\right]^2 - \frac{\partial^2}{\partial x^2} + [S(x) + m]^2\right\} f_{KG}(x) = 0 \tag{2.15}$$

When we simplify the equation, it turns most common using the form in this thesis

$$\frac{\partial^2 f}{\partial x^2} + (E - V(x))^2 f - (m(x) + S(x))^2 f = 0. \tag{2.16}$$

The radial wavefunction for Eq. (2.15) is expressed as $\psi(x) = \frac{f(x)}{x}$. Rearranging the last equation with effective potential $V_{eff}(x)$ yields the equation form

$$\frac{\partial^2 f(x)}{\partial x^2} + V_{eff}(x) f(x) = 0 \tag{2.17}$$

where

$$V_{eff} = E^2 - m^2 + [V^2(x) - S^2(x)] - [mS(x) + EV(x)]$$

When we focus on effective potential, it has seen that the effective potential is energy dependent potential and the K.G is reduced to the Schrödinger-like equation form.

2.2 Woods−Saxon Potential

A basic problem in the nuclear physics is the motion of the free electrons which have a conclusive influx on the abundance of metal inflorescence. These electrons are moving in well defined orbits, around the central nucleus and in a mean field potential which is produced by the positively charged ions and the rest of the electrons. In the mean field potential, the details of the potential are described by three parameters such as depth, width and the slope of the potential, which have to be fitted to experimental observation. Therefore, a mean field potential is always empirical and its an example can be given as the Woods − Saxon potential [25].

$$V(r) = \frac{-V_0}{1 + e^{\left[\frac{r - R_0}{a}\right]}}$$

where V is the potential depth, R_0 is the width of the potential and its diffuseness and a is the surface thickness which is usually adjusted to the experimental values of ionization energies.

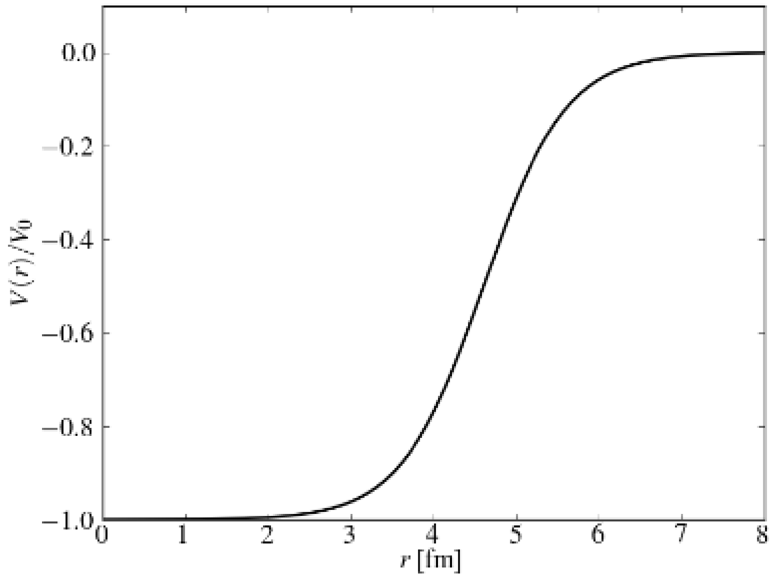

Figure 3.1 Woods–Saxon potential for A = 50, relative to V_0 with a = 0.5 fm

In Figure 3.1, V_0 (having dimension of energy) represents the potential well depth, a is a long representing the "surface thickness" of the nucleus, and $R = \frac{r_0 \, \mathrm{A}_1}{3}$ is the nuclear radius where $r_0 =$ 1.25 FM and A is the mass number.

CHAPTER 3

3 THE ASYMTOTIC ITERATION METHOD

The asymptotic iteration method was presented [63] to get exact and approximate solutions of eigenvalue equations [64]. The solve Schrödinger equations first step in applying this method is to transform these equations, with the aid of appropriate asymptotic forms, to second-order homogeneous linear differential equations of the general form.

In this chapter, the asymptotic iteration method is used for obtaining the corresponding spectrum for PDMKG. In addition to the formalism of AIM for second order differential equations, also the method is derived to the first order differential equations.

3.1 The Asymptotic Iterative Method

In this part, give a brief delineation of the AIM; particularity of the method can be obtained [63]. Let assume, we wish to solve the homogeneous linear second-order differential equation of form

$$y'' = \lambda_0(x)y' + s_0(x)y \tag{3.1}$$

For which $\lambda_0(x)$ and $s_0(x)$ are functions in $C_\infty(a,b)$. Here primes denote the derivatives with respect to x. A key feature of AIM is to note the invariant structure of the right-hand side of Eq. (3.1) under further differentiation. Indeed, if Eq. (3.1) differentiate with respect to x, it can be found that

$$y''' = \lambda_1(x)y' + s_1(x)y \tag{3.2}$$

where

$$\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2(x)$$

$$s_1(x) = s_0'(x) + s_0(x)\lambda_0(x)$$

The derivative of the equation introduced in (3.2) yields

$$y'''' = \lambda_2(x)y' + s_2(x)y \tag{3.3}$$

for which

$$\lambda_2(x) = \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x)$$

and

$$s_2(x) = s_1'(x) + s_0(x)\lambda_1(x)$$

Thus, generalizing for the $(n+1)^{th}$ and $(n+2)^{th}$ derivatives of (3.1), $n = 1, 2, 3, \ldots$, we get a relation of form

$$y^{(n+1)} = \lambda_{n-1}(x)y' + s_{n-1}(x)y \tag{3.4a}$$

$$y^{(n+2)} = \lambda_n(x)y' + s_n(x)y \tag{3.4b}$$

with the important relation

$$\lambda_n(x) = \lambda_{n-1}'(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x) \tag{3.5a}$$

$$s_n(x) = s_{n-1}'(x) + s_0(x)\lambda_{n-1}(x). \tag{3.5b}$$

From the ratio of the $(n+2)^{th}$ and $(n+1)^{th}$ derivative, the Eq. (3.4) will be transformed to

$$\frac{d}{dx}ln\left(y^{(n+1)}\right) = \frac{y^{(n+2)}}{y^{(n+1)}} = \frac{\lambda_n(x)(y' + \frac{s_n(x)}{\lambda_n(x)}y)}{\lambda_{n-1}(x)(y' + \frac{s_{n-1(x)}}{\lambda_{n-1}(x)}y)} \tag{3.4}$$

For a sufficiently large n, there is the following asymptotic expression

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = \alpha(x) \tag{3.5}$$

with the termination condition given as

$$\Delta_k(x) = \begin{vmatrix} s_n(x) & \lambda_n(x) \\ s_{n-1}(x) & \lambda_{n-1}(x) \end{vmatrix} = \lambda_{n-1}(x)s_n(x) - \lambda_n(x)s_{n-1}(x) \tag{3.6}$$

$k = 1,2,3, \dots$

Also note that the energy eigenvalues are obtained from the roots of the equation (3.3) if the problem is exactly solvable. However, for a specific n principal quantum number, a suitable x_0 point will be chosen, determined generally as the maximum value of the asymptotes wave function or the minimum value of the potential [64]. Also the approximate energy eigenvalues are obtained from the roots of this equation for sufficiently great values of k with iteration. Using Eq. (3.6), Eq. (3.4) reduces to

$$\frac{d}{dx} ln(y^{n+1}) = \frac{\lambda_n(x)}{\lambda_{n-1}(x)} \tag{3.7}$$

which yields

$$y^{(n+1)}(x) = C_1 \operatorname{Exp}\left(\int \frac{\lambda_n(t)}{\lambda_{n-1}} dt\right) = C_1 \lambda_{n-1} \operatorname{Exp}\left(\int (\alpha + \lambda_0) dt\right) \tag{3.8}$$

Note that, the relations (3.5a), (3.5b) and (3.5) have been used in obtaining the right hand side of Eq. (3.8) and C1 is the integration constant. By substituting Eq. (3.8) into Eq. (3.4a), we have the first order differential equation

$$y' + \alpha y = C_1 \operatorname{Exp}\left(\int \alpha + \lambda_0) dt\right) \tag{3.9}$$

By solving Eq. (3.9), it is possible to obtain the general solution to Eq. (3.1) as follows

$$y(x) = \mathrm{Exp}\left(-\int \alpha\, dt\right)\left[C_2 + C_1 \int \mathrm{Exp}\left(\int \big(\lambda_0(\tau) + 2\alpha(\tau)\big)d\tau\right) dt\right] \qquad (3.10)$$

For a given potential, the radial Klein-Gordon equation is converted to the form of Eq. (3.1). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by (3.5a), and (3.5b). The termination condition of the method in (3.5) can be arranged as

$$\Delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) \qquad (3.11)$$

where k shows the iteration number. For the exactly solvable potentials, the energy eigenvalues are obtained from the roots of Eq. (3.11) and the radial quantum number n is equal to the iteration number k in this case. For nontrivial potentials that have no exact solutions, for a specific n principal quantum number, a suitable x_0 point should be used, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential [65-69] and the approximate energy eigenvalues are obtained from the roots of Eq. (3.11) for sufficiently great values of k with iteration for which k is always greater than n in these numerical solutions.

The general solution of Eq. (3.1) is given by (10). The first part of Eq. (10) gives us the polynomial solutions that are converging and physical, whereas the second part of Eq. (10) gives us non-physical solutions that are divergent. However Eq. (10) is the general solution of Eq. (3.1), the coefficient of the second part $c_1 = 0$ zero is taken, in order to find the square integrable solutions. Hence, the corresponding eigenfunctions can be derived from the following wave function generator for exactly solvable potentials

$$y(x) = C_2 \mathrm{e}^{-\int \alpha dt}$$

3.2 Calculation of eigenfunction

This part, consists of obtaining the eigenfunction by using AIM. The second-order differential equation is considered as [70]

$$y^{''} = 2\left(t\, x\, p(x) - \frac{(m+1)}{x}\right) y^{'} - w\, p(x) y, \tag{3.12}$$

where t, m and w are arbitrary constants. This equation has on exact solution under some conditions related with polynomial function $p(x)$. To get exact result, the eq(3.12) has to be second order form as introduced by the outhors in [70].

$$\mathrm{y}^{''} = 2\left(\frac{t\, x^N}{1 - bx^{N+2}} - \frac{(m+1)}{x}\right) y^{'} - \frac{wx^N}{1 - bx^{N+2}} y. \tag{3.13}$$

By using AIM, this second order differential equation form produceds

$$\lambda_0(x) = 2\left(\frac{ax^{N+1}}{1-bx^{N+2}} - \frac{(m+1)}{x}\right) \text{ and } s_0(x) = -\frac{wx^N}{1-bx^{N+2}} \tag{3.14}$$

Using the wave function generator, we reach to the general formula for the exact solution $y_0(x)$ as

$$y_n(x) = (-1)^n c_2 (N+3)^n (\rho)_n \, {}_2F_1(-n, \rho + n;\ \rho; b\, x^{N+2}) \tag{3.15}$$

where the following parameters have been used

$$(\rho)_n = \frac{\Gamma(\rho + n)}{\Gamma(\alpha)}$$

$$\sigma = \frac{2m + N + 3}{N + 2}$$

$$\rho = \frac{(2m+1)b + 2a}{(N+2)b}$$

In this equation, the Gauss hypergeometric function ${}_2F_1$ is defined

$${}_2 f_1(-\mathrm{n}, \mathrm{b}; \mathrm{c}; \mathrm{y}) = \sum_{\mathrm{k}=0}^{\mathrm{n}} \frac{(-\mathrm{n})_{\mathrm{k}} (\mathrm{b})_{\mathrm{k}}}{(\mathrm{c})_{\mathrm{k}}} \mathrm{y}^{\mathrm{n}} \tag{3.16}$$

for give n degree polynomial function in y.

CHAPTER 4

4 APPLICATION OF AIM FOR KLEIN-GORDON EQUATION

In this chapter, the bound-state solution of KG equation is outlined when the mass is constant and it is spatially dependent mass. To get bound-state solution, the relationship between scalar potential and vector potential is examined by the form introduced in [8]. The energy and wavefunctions of corresponding potentials are calculated subsequently.

4.1 Bound-State Solutions.

Bound-state solution of KG equation is obtained when the relationship between vector and scalar potentials is given as $\mathrm{S(x)} \geq \mathrm{V(x)}$ with $\mathrm{S(x)} = (\beta - 1)\mathrm{V(x)}$.The KG equation without taking $\hbar = c = m = 1$, is introduce [25] as

$$\left(\frac{d^2\psi}{dx^2} + \frac{(E_n - V)^2 - (c^2 m + S)^2}{c^2\hbar^2}\right)\psi(x) = 0 \tag{4.1}$$

by substituting the scalar potentials as $(\beta - 1)\mathrm{V(x)}$, in to eq (4.1)

$$\frac{\partial^2\psi(x)}{\partial x^2} + V_{eff}(x)\psi(x) = 0 \tag{4.2}$$

where

$$V_{eff} = \frac{1}{c^2\hbar^2}\left(E^2 - m^2c^2 + (2m - 2E - 2\beta m)V + V^2(x)\left(2\beta - \beta^2\right)\right).$$

The last effective potential states that the effective potential is energy dependent potentials. In this case, the KG has a real bound-state solution under this condition

If $S(x) = V(x)$ (when $\beta = 2$) the KG equation is reduced to the Schrödinger equation forms. By using this formula for scalar potential, we can and adjust the constant $\beta > 2$ for real bound-state. In the following section, we consider the mass for exponential function, as a constant mass and position dependent mass.

4.2 Constant- Mass Applications

4.2.1 $\boldsymbol{V = V_0 e^{-\alpha x}}$ case

First of all, we take the vector potential $\mathrm{V} = \mathrm{V_0 e^{-\alpha x}}$ and after substituting in to Eq.(4.1), we get

$$\psi''(x) + \left(e^{-x\alpha} \quad \frac{(-2E_n V_0 + 2c^2 m_0 V_0 - 2\beta c^2 m_0 V_0)}{c^2 \hbar^2} + e^{-2x\alpha} \quad \frac{\left(2bV_0{}^2 - \beta^2 V_0{}^2\right)}{c^2 \hbar^2} + \frac{E_n{}^2 - c^4 m_0{}^2}{c^2 \hbar^2} \right) \psi(x) = 0 \tag{4.3}$$

The resulted potential form is similar with the Morse potential for different constants.

By changing the variables $y = e^{-x\alpha}$, the Eq. (4.3) reduces to

$$y^2 \alpha^2 \psi''(y) + y\alpha^2 \psi'(y) + \left(\frac{E_n{}^2 - c^4 \mathrm{m_0}{}^2}{c^2 \hbar^2} + \frac{(-2E_n \mathrm{V_0} + 2c^2 \mathrm{m_0 V_0} - 2\beta c^2 \mathrm{m_0 V_0})y}{c^2 \hbar^2} + \frac{\left(2\beta \mathrm{V_0}{}^2 - \beta^2 \mathrm{V_0}{}^2\right) y^2}{c^2 \hbar^2} \right) \psi(y) = 0. \tag{4.4}$$

After same algebraic arrangement, we get a second order differential equation in the form of

$$\psi''(y) + \frac{\psi'(y)}{\mathrm{y}} + \left(\frac{-\xi^2 - \mathrm{b_2}{}^2 \mathrm{y} - \mathrm{b_1}{}^2 y^2}{y^2 \alpha^2} \right) \psi(y) = 0 \tag{4.5}$$

where

$$-\xi^2 = \frac{{E_n}^2 - c^4{\mathrm{m}_0}^2}{c^2\hbar^2}$$

$$-{\mathrm{b}_2}^2 = \frac{(-2E_n\mathrm{V}_0 + 2c^2\mathrm{m}_0\mathrm{V}_0 - 2\beta c^2\mathrm{m}_0\mathrm{V}_0)}{c^2\hbar^2}$$

$$-\ {\mathrm{b}_1}^2 = \frac{\left(2\beta{\mathrm{V}_0}^2 - \beta^2{\mathrm{V}_0}^2\right)}{c^2\hbar^2}$$

At this point, we have to propose the wave function since the Eq. (4.5) should be transformed to the form introduction in second order differential form. The wave function should be satisfied the boundary conditions i.e., when y goes to zero, $\psi(0)\sim s$ proportional to $e^{b_1y/\alpha}$. Hence, the physical wave functions for KG equation is proposed as

$$\psi(\mathrm{y}) = y^{\xi/\alpha}(\mathrm{e}^{\mathrm{b}_1y/\alpha})f(y) \tag{4.6}$$

Substituting Eq. (4.6) into Eq.(4.5) yields to the equation from of

$$f^{''}(y) = -\frac{1}{y\alpha}(2\xi + 2b_1y + \alpha)f^{'}(y) + \frac{1}{y\alpha^2}\left({b_2}^2 - b_1(2\xi - \alpha)\right)f^{'}(y) \tag{4.7}$$

which is transformed to the form of differential equation to be solved by AIM. By comparing the Eq. (4.7) and Eq. (3.1), we easily obtained the function $\lambda_0(\mathrm{y})$ and $s_0(y)$ as

$$\lambda_0 = -\frac{2\xi + 2\mathrm{b}_1y + \alpha}{y\alpha}$$

$$\mathrm{s}_0 = \frac{{\mathrm{b}_2}^2 - \mathrm{b}_1(2\xi + \alpha)}{y\alpha^2}$$

With the aid of $\lambda_0(y)$ and $s_0(y)$, we may calculate $\lambda_n(y)$ and $s_n(y)$ by means of Eq. (3.5b) and Eq. (3.5a)

$$\lambda_1 = \frac{4\xi^2 + {b_2}^2y + 4{b_1}^2y^2 + 3b_1y\alpha + 2\alpha^2 + 6\xi(b_1y + \alpha)}{y^2\alpha^2}$$

$$s_1 = \frac{2(\xi + b_1 y + \alpha)(-{b_2}^2 + b_1(2\xi + \alpha))}{y^2\alpha^3}$$

$$\lambda_2 = -\frac{1}{y^3\alpha^3} 2(4\xi^3 + 4{b_1}^3 y^3 + 2{b_2}^2 y\alpha + 4{b_1}^2 y^2\alpha + 3\alpha^3 + 4\xi^2(2b_1 y + 3\alpha)$$
$$+ 2b_1 y({b_2}^2 y + 2\alpha^2) + \xi(2{b_2}^2 y + 8{b_1}^2 y^2 + 12b_1 y\alpha$$
$$+ 11\alpha^2))$$

$$s_2 = \frac{1}{y^3\alpha^4}(4\xi^2 + {b_2}^2 y + 6b_1\xi y + 4{b_1}^2 y^2 + 10\xi\alpha + 5b_1 y\alpha + 6\alpha^2)({b_2}^2 - b_1(2\xi$$
$$+ \alpha))$$

$$\lambda_3 = \frac{1}{y^4\alpha^4}(16\xi^4 + {b_2}^4 y^2 + 16{b_1}^4 y^4 + 20{b_1}^3 y^3\alpha + 27{b_1}^2 y^2\alpha^2 + 30b_1 y\alpha^3$$
$$+ 24\alpha^4 + 40\xi^3(b_1 y + 2\alpha) + 2{b_2}^2 y(6{b_1}^2 y^2 + 10b_1 y\alpha + 9\alpha^2)$$
$$+ 4\xi^2(3{b_2}^2 y + 13{b_1}^2 y^2 + 30b_1 y\alpha + 35\alpha^2) + 10\xi(4{b_1}^3 y^3$$
$$+ 3{b_2}^2 y\alpha + 8{b_1}^2 y^2\alpha + 10\alpha^3 + b_1 y(2{b_2}^2 y + 11\alpha^2)))$$

$$s_3 = -\frac{1}{y^4\alpha^5} 2({b_2}^2 - b_1(2\xi + \alpha))(4\xi^3 + 4{b_1}^3 y^3 + 6{b_1}^2 y^2\alpha + 2\xi^2(4b_1 y + 9\alpha)$$
$$+ 3\alpha({b_2}^2 y + 4\alpha^2) + b_1 y(2{b_2}^2 y + 9\alpha^2) + 2\xi({b_2}^2 y + 4{b_1}^2 y^2$$
$$+ 9b_1 y\alpha + 13\alpha^2))$$

….ect.

Combining with quantization condition, we get the values of ξ for each n

$$\frac{\lambda_0}{s_0} = \frac{\lambda_1}{s_1} \Rightarrow \xi_0 = \frac{{b_2}^2 - 1b_1\alpha}{2b_1}$$

$$\frac{\lambda_1}{s_1} = \frac{\lambda_2}{s_2} \Rightarrow \xi_1 = \frac{{b_2}^2 - 3b_1\alpha}{2b_1}$$

$$\frac{\lambda_2}{s_2} = \frac{\lambda_3}{s_3} \Rightarrow \xi_2 = \frac{{b_2}^2 - 5b_1\alpha}{2b_1}$$

$$\frac{\lambda_3}{s_3} = \frac{\lambda_4}{s_4} \Rightarrow \xi_3 = \frac{{b_2}^2 - 7b_1\alpha}{2b_1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Generalizing these ξ expressions, we can get the general formula for ξ

$$\xi_n = \frac{{b_2}^2 - (2n+1)b_1\alpha}{2b_1}$$

And using the parameters introduced for ξ , b_1 and b_2 , one can obtain

$$E_n^2 - {m_0}^2 c^4 = -c^2\hbar^2\left(\frac{{b_2}^2 - (2n+1)b_1\alpha}{2b_1}\right) \tag{4.8}$$

thus

$$E_n = \pm\sqrt{{m_0}^2 c^4 - \frac{c^2\hbar^2}{2b_1}({b_2}^2 - (2n+1)b_1\alpha)} \tag{4.9}$$

4.2.1.1 Obtaining Wave functions

The corresponding eigenfunction for the Morse potential is obtained by using the wave function generator given by the equation, it yields

$$f_n(r) = \text{Exp}\,(-\int^r \frac{s_k}{\lambda_k}\,dr)$$

where n represents the radial quantum number. By this procedure, the first few $f(r)$ functions are

$$f_0\,(y) = 1,$$

$$f_1\,(y) = \left(2b_2 - {b_1}^2\right)\left(1 - \frac{2b_2 y}{\left(\frac{{b_1}^2 - 3b_2}{b_2} - 1\right)}\right),$$

$$f_2\,(y) = \left(b_1{}^2 - 4b_2\right)\left(b_1{}^2 - 3b_2\right)(1 - \frac{4b_2 y}{\left(\frac{b_1{}^2 - 5b_2}{b_2} - 1\right)}$$

$$+ \frac{4{b_2}^2 y^2}{\left(\frac{b_1{}^2 - 5b_2}{b_2} - 1\right)\left(\frac{b_1{}^2 - 5b_2}{b_2} - 2\right)}$$

…..etc.

Using the properties of hypergeometric function, we can write the general formula for wave functions form the given function for λ_0 and s_0 as

$$f_n\,(y) = (-1)(\prod_{k=n}^{2n-1}\left(b_1{}^2 - (k+1)b_2\right), {}_2F_1\left(-n, 2\xi_n + 1; 2\,b_2 y\right).$$

For this formula, substituting y to the radial wave function, we get

$$R_n\,(y) = (-1)^n(\prod_{k=n}^{2n-1}\left(b_1{}^2 - (k+1)b_2\right)e^{-x\alpha\,\xi_n}\, e^{-b_2 e^{-x\alpha}}, {}_2F_1(-n, 2\xi_n$$

$$+ 1; 2\,b_2 e^{-x\alpha}).$$

4.2.2 $V = V_0 e^{x\alpha}$ case

When we take the potential in the form of $V = V_0 e^{\alpha x}$, and substituting $S = (\beta - 1)V$ in to KG equation, the equation (4.1) transforms to

$$\psi''(x) + \left(-\frac{-E_n{}^2 + c^4 m_0{}^2}{c^2\hbar^2} - \frac{e^{x\alpha}(2EnV_0 - 2c^2 m_0 V_0 + 2\beta c^2 m_0 V_0)}{c^2\hbar^2}\right. \tag{4.10}$$

$$\left. - \frac{e^{2x\alpha}\left(-2\beta V_0{}^2 + \beta^2 V_0{}^2\right)}{c^2\hbar^2}\right)\psi(x) = 0$$

similar to the previous example, by changing the variable $y = e^{\alpha x}$ and substitute into Eq. (4.10), we reach to

$$\psi''(y)+\frac{\psi'(y)}{y}-\left(\frac{\xi^2+b_2{}^2y+b_1{}^2y^2}{y^2\alpha^2}\right)\psi(y)=0 \tag{4.11}$$

with parameters similar to the previous application except for minus signs as

$$\xi^2=\frac{E_n{}^2-c^4m_0{}^2}{c^2\hbar^2}$$

$$b_2{}^2=\frac{1}{c^2\hbar^2}(2E_nV_0-2c^2m_0V_0+2bc^2m_0V_0) \tag{4.12}$$

$$b_1{}^2=\frac{\left(2bV_0{}^2-b^2V_0{}^2\right)}{c^2\hbar^2}$$

Generalizing these ξ expressions, we can get the general formula for ξ

$$\xi_n=\frac{b_2{}^2-(2n+1)b_1\alpha}{2b_1}$$

After this part, all steps for applying the AIM is same with those of $y=e^{-\alpha x}$. Doing same algebraic produce, the energy related to $y=e^{\alpha x}$ potential is obtained as

$$E_n=\pm\sqrt{m_0{}^2c^4-\frac{c^2\hbar^2}{2b_1}(b_2{}^2-(2n+1)b_1\alpha)} \tag{4.13}$$

In similar manner, the function f_n (y)

$$f_n(y)=(-1)(\prod_{k=n}^{2n-1}\left(b_1{}^2-(k+1)b_2\right),{}_2F_1\left(-n,2\xi_n+1;2\,b_2y\right).$$

and the total radial wavefunction becomes after substituting y,

$$R_n(x)=(-1)^n(\prod_{k=n}^{2n-1}\left(b_1{}^2-(k+1)b_2\right)ye^{\alpha x\xi_n}\,e^{-b_2e^{\alpha x}},{}_2F_1(-n,2\xi_n+1;2\,b_2e^{\alpha x}).$$

with parameters

$$\xi^2 = \frac{{E_n}^2 - c^4{m_0}^2}{c^2\hbar^2}$$

$${b_2}^2 = \frac{1}{c^2\hbar^2}(2E_n V_0 - 2c^2 m_0 V_0 + 2bc^2 m_0 V_0)$$

$${b_1}^2 = \frac{\left(2b{V_0}^2 - b^2{V_0}^2\right)}{c^2\hbar^2}$$

4.3 The Woods-Saxon potential for constant mass

The standard Woods–Saxon potential, as mean-field nuclear potential is given as,

$$V(r) = \frac{-V_0}{1 + e^{\left[\frac{x - R_0}{a}\right]}} \tag{4.14}$$

We take the vector scalar potential $S = (\beta - 1)V$ and after substituting (4.14) in to Eq. (4.1), we get

$$\psi''(x) + \frac{{E_n}^2\psi(x)}{c^2\hbar^2} - \frac{c^2{m_0}^2\psi(x)}{\hbar^2} + \frac{2E_n V_0\psi(x)}{c^2(1 + e^{\frac{-R+x}{\alpha}})\hbar^2} - \frac{2m_0 V_0\psi(x)}{(1 + e^{\frac{-R+x}{\alpha}})\hbar^2}$$
$$+ \frac{2m_0 V_0\beta\psi(x)}{(1 + e^{\frac{-R+x}{\alpha}})\hbar^2} + \frac{2{V_0}^2\beta\psi(x)}{c^2(1 + e^{\frac{-R+x}{\alpha}})^2\hbar^2} - \frac{{V_0}^2\beta^2\psi(x)}{c^2(1 + e^{\frac{-R+x}{\alpha}})^2\hbar^2} \tag{4.15}$$
$$= 0$$

By changing the variables $y = \frac{1}{1+e^{(x-R)/a}}$, the Eq. (4.15)

$$\left(\frac{{E_n}^2\alpha^2 - c^4 \mathrm{m0}^2\alpha^2}{c^2\hbar^2\alpha^2} + \frac{y\left(2E_n V_0\alpha^2 + 2c^2 m_0 V_0\alpha^2(-1+\beta)\right)}{c^2\hbar^2\alpha^2}\right.$$
$$\left. - \frac{{V_0}^2 y^2(-2+\beta)\beta}{c^2\hbar^2}\right)\psi(y) + \left(\frac{y}{\alpha^2} - \frac{3y^2}{\alpha^2} + \frac{2y^3}{\alpha^2}\right)\psi'(y) \tag{4.16}$$
$$+ \left(\frac{y^2}{\alpha^2} - \frac{2y^3}{\alpha^2} + \frac{y^4}{\alpha^2}\right)\psi''(y) = 0$$

After same algebraic arrangement, we get a second order differential equation in the form of

$$\psi''(x)+\frac{(1-2y)\psi'(y)}{(1-y)y}+\frac{\left(-\xi^2-\mathrm{b}_1y-\mathrm{b}_2y^2\right)\psi(y)}{(1-y)^2y^2}=0 \tag{4.17}$$

where

$$\xi^2=-\frac{E_n{}^2-c^4\mathrm{m}_0{}^2}{c^2\hbar^2}$$

$$\mathrm{b}_2=-\frac{(2-\beta)\beta\mathrm{V}_0{}^2\alpha^2}{c^2\hbar^2}$$

$$\mathrm{b}_1=\frac{(2E_n\mathrm{V}_0\alpha^2+2(-1+\beta)c^2\mathrm{m}_0\mathrm{V}_0\alpha^2)}{c^2\hbar^2}$$

At this point, we have to propose the wave function since the equation (4.17) should be transformed to the form introduction in second order differential form. The wave function should be satisfied the boundary conditions. Hence, the physical wave functions for KG equation is proposed as

$$\psi(y)=y^{\xi}(1-y)^{\sqrt{\xi^2+b_1+b_2}}f(y) \tag{4.18}$$

Substituting Eq. (4.18) into Eq. (4.17) yields to the equation form of

$$\begin{aligned}f''(y)=\frac{1}{(-1+y)y}&(-\mathrm{b}_1f(y)-\xi f(y)-2\xi^2f(y)-vf(y)-2\xi vf(y)\\&+f'(y)2\xi\chi'(y)-2yf'(y)-2\xi yf'(y)-2vyf'(y))\end{aligned} \tag{4.19}$$

which is transformed to the form of differential equation to be solved by AIM. By comparing the Eq. (4.19) and Eq. (3.1), we easily obtained the function $\lambda_0(\mathrm{y})$ and $s_0(y)$ as

$$\lambda_0=\frac{1-2\xi(-1+y)-2(1+v)y}{(-1+y)y}$$

$$\mathrm{s}_0=\frac{\mathrm{b}_1+\xi+2\xi^2+v+2\xi v}{y-y^2}$$

With aid of $\lambda_0(y)$ and $s_0(y)$, we may calculate $\lambda_n(y)$ and $s_n(y)$ by means of Eq. (3.5b) and Eq. (3.5a)

$$\lambda_1 = \frac{1}{(-1+y)^2y^2}(2+(-6+b_1-3v)y+(6-b_1+9v+4v^2)y^2+3\xi(-1+y)(-2+(3+2v)y)+2\xi^2(2-3y+y^2))$$

$$s_1 = \frac{2(b_1+(1+2\xi)(\xi+v))(-1+\xi(-1+y)+(2+v)y)}{(-1+y)^2y^2}$$

$$\lambda_2 = \frac{1}{(-1+y)^3y^3}2(3+4\xi^3(-1+y)^2+2(b_1-2(3+v))y+(18+15v+4v^2-2b_1(3+v))y^2+2(2+v)(-3+b_1-4v-2v^2)y^3-4\xi^2(-1+y)(3-2(3+v)y+(2+v)y^2)+\xi(-1+y)(-11+(29-2b_1+12v)y+2(-11+b_1-12v-4v^2)y^2))$$

$$s_2 = \frac{1}{(-1+y)^3y^3}({b_1}^2(-1+y)y+2b_1(-3+2\xi^2(-1+y)+(9+2v)y-(9+7v+2v^2)y^2-\xi(-1+y)(-5+(7+2v)y))-(1+2\xi)(\xi+v)(6-(18+5v)y+(18+15v+4v^2)y^2+\xi(-1+y)(-10+3(5+2v)y)+2\xi^2(2-3y+y^2)))$$

….etc.

Combining with quantization condition, we get the values of ξ for each n

$$\frac{\lambda_0}{s_0} = \frac{\lambda_1}{s_1} \Rightarrow \xi_0 + v = -\frac{1}{2} - \frac{1}{2}\sqrt{1+4b_1}$$

$$\frac{\lambda_1}{s_1} = \frac{\lambda_2}{s_2} \Rightarrow \xi_1 + v = -\frac{3}{2} - \frac{1}{2}\sqrt{1+4b_1}$$

$$\frac{\lambda_2}{s_2} = \frac{\lambda_3}{s_3} \Rightarrow \xi_2 + v = -\frac{5}{2} - \frac{1}{2}\sqrt{1+4b_1}$$

…………………………

From this recurrence relation, when we generalize ξ_n for nth term,we get

$$\xi_n + v = -\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4b_1} \tag{4.20}$$

using the parameters introduced for ξ , b_1 and v , we obtain a more explicit

$$E_n^2 - m_0{}^2 c^4 = -c^2\hbar^2\left(-\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4\frac{(2E_n V_0\alpha^2 + 2(-1+\beta)c^2 m_0 V_0\alpha^2)}{c^2\hbar^2}} - v\right) \tag{4.21}$$

By substituting for ξ_n and v we obtain a more explicit expression for the eigenvalues energy as

$$E_n = \pm(m_0{}^2 c^4 - c^2\hbar^2(-\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4\frac{(2E_n V_0\alpha^2 + 2(-1+\beta)c^2 m_0 V_0\alpha^2)}{c^2\hbar^2}} - v))^{\frac{1}{2}} \tag{4.22}$$

4.3.1.1 Obtaining Wave Function

Let us now turn to the calculation of the normalized wave functions.

$$t = \frac{2v+1}{2}; b = 1; N = -1, m = \frac{2\xi_n - 1}{2}; \sigma = 2\xi_n + 1\ ;\ \rho = 2\xi_n - 2v + 1.$$

Having determined these parameters, we can easily obtain the wave functions as

$$f_n\ (y) = (-1)^n C_2 \frac{\Gamma(2\xi_n + n + 1)}{\Gamma(2\xi_n + 1)}\ {}_2F_1(-n, 2(\xi_n - v) + 1 + n; 2\xi_n + 1; y)$$

where Γ and $2\ F_1$ are the Gamma function and hypergeometric function respectively. The total radial wave function can be written as follows:

$$R_n\ (x) = (-1)^n N_n \frac{(1+e^{\frac{-R+x}{\alpha}})^v}{(1+e^{\frac{-R+x}{\alpha}})^\xi}\ {}_2F_1(-n, 2(\xi - v) + 1 + n; 2\xi$$

$$+ 1; (1+e^{\frac{-R+x}{\alpha}})^{-1})$$

where N_n is the normalization constant and the parameters are

$$\xi^2 = -\frac{{E_n}^2 - c^4{\mathrm{m}_0}^2}{c^2\hbar^2}$$

$$\mathrm{b}_2 = -\frac{(2-\beta)\beta{\mathrm{V}_0}^2\alpha^2}{c^2\hbar^2}$$

$$\mathrm{b}_1 = \frac{(2E_n\mathrm{V}_0\alpha^2 + 2(-1+\beta)c^2\mathrm{m}_0\mathrm{V}_0\alpha^2)}{c^2\hbar^2}$$

4.3.2 Discussion

i) If β*=1*, it yields zero scalar potentials with parameters

$$\xi = \frac{{E_n}^2\alpha^2 - c^4{\mathrm{m}_0}^2\alpha^2}{c^2\hbar^2}$$

$$\mathrm{b}_1 = \frac{2\mathrm{EnV}_0\alpha^2}{c^2\hbar^2}, \mathrm{b}_2 = -\frac{{\mathrm{V}_0}^2\alpha^2}{c^2\hbar^2}$$

$$\xi_\mathrm{n} + v = -\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4\mathrm{b}_1} \tag{4.23}$$

$$E_n = \pm\sqrt{{m_0}^2c^4 - \frac{c^2\hbar^2}{\alpha^2}(-\frac{2n+1}{2} - \frac{1}{2}\sqrt{1 - 4\frac{2\mathrm{EnV}_0\alpha^2}{c^2\hbar^2}} - v)} \tag{4.24}$$

The eigenfunction of this potential is again

$$f_n\ (y) = (-1)^n C_2 \frac{\Gamma(2\xi_n + n + 1)}{\Gamma(2\xi_n + 1)}\ {}_2F_1(-n, 2(\xi_n - v) + 1 + n; 2\xi_n + 1; y)$$

For this formula, the radial wave function is

$$R_n\ (x) = (-1)^n N_n \frac{(1+e^{\frac{-R+x}{\alpha}})^{v}}{(1+e^{\frac{-R+x}{\alpha}})^{\xi}}\, {}_2F_1(-n, 2(\xi - v) + 1 + n; 2\xi$$

$$+ 1; (1+e^{\frac{-R+x}{\alpha}})^{-1})$$

ii) For $\beta = 2$, we get equal scalar and vector potentials with relations following

$$\xi = \frac{{E_n}^2\alpha^2 - c^4{\mathrm{m}_0}^2\alpha^2}{c^2\hbar^2}$$

$$\mathrm{b}_1 = \frac{2(E_n + c^2 m_0)\mathrm{V}_0\alpha^2}{c^2\hbar^2}, \mathrm{b}_2 = 0$$

$$\xi_{\mathrm{n}} + v = -\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4\mathrm{b}_1} \tag{4.25}$$

$$E_n = \pm\sqrt{{m_0}^2c^4 - \frac{c^2\hbar^2}{\alpha^2}(-\frac{2n+1}{2} - \frac{1}{2}\sqrt{1 - 4\frac{2\mathrm{EnV}_0\alpha^2}{c^2\hbar^2}} - v)} \tag{4.26}$$

Also, the eigenfunction is again obtained in a similar algebraic procedure with previous example. It result radial wave function in same form

$$R_n\ (x) = (-1)^n N_n \frac{(1+e^{\frac{-R+x}{\alpha}})^{v}}{(1+e^{\frac{-R+x}{\alpha}})^{\xi}}\, {}_2F_1(-n, 2(\xi - v) + 1 + n; 2\xi$$

$$+ 1; (1+e^{\frac{-R+x}{\alpha}})^{-1})$$

4.4 Position dependent Mass Application:

The second application part consists of the solution PDMKG equation for different mass and for different exponential vector and scalar potential.

4.4.1 $V = V_0 e^{\alpha x}$, $m = m_0 e^{\alpha x}$ case

The first choices about spatially mass is $m = m_0 e^{\alpha x}$ and the vector potential is $V = V_0 e^{\alpha x}$. Using the functions forms, the scalar potential becomes $S = (\beta - 1)\mathrm{V}$ and the relativistic equation from as

$$\psi''(x) + (\frac{En^2}{c^2\hbar^2} - (2e^{x\alpha} En\mathrm{V}_0)/(c^2\hbar^2) + \frac{1}{c^2\hbar^2} e^{2x\alpha}(-c^4\mathrm{m}_0{}^2 + 2c^2\mathrm{m}_0\mathrm{V}_0 - 2\beta c^2\mathrm{m}_0\mathrm{V}_0 + 2\beta\mathrm{V}_0{}^2 - \beta^2\mathrm{V}_0{}^2))\psi(x) = 0 \tag{4.27}$$

By changing the variable $y = e^{x\alpha}$, one gets

$$y^2\alpha^2\psi''(y) + y\alpha^2\psi'(y) + \left(\mathrm{b}_2\mathrm{y} - \xi^2 - \mathrm{b}_1{}^2 y^2\right)\psi(y) = 0 \tag{4.28}$$

Where the parameters are take form of

$$\xi^2 = -\frac{E_n{}^2}{c^2\hbar^2}$$

$$\mathrm{b}_1{}^2 = \frac{(-c^4\mathrm{m}_0{}^2 + 2c^2\mathrm{m}_0\mathrm{V}_0 - 2bc^2\mathrm{m}_0\mathrm{V}_0 + 2b\mathrm{V}_0{}^2 - b^2\mathrm{V}_0{}^2)}{c^2\hbar^2} \tag{4.29a}$$

$$\mathrm{b}_2 = -\left(\frac{2E_n\mathrm{V}_0}{c^2\hbar^2}\right)$$

Dividing the Eq. (4.28) with $\mathrm{b}_1{}^2 y^2$, we get

$$\psi''(\mathrm{y}) + \frac{\psi'(y)}{y} + \frac{\left(\mathrm{b}_2\mathrm{y} - \xi^2 - \mathrm{b}_1{}^2 y^2\right)\psi(y)}{y^2\alpha^2} = 0 \tag{4.30}$$

Let propose the physical function for case (a) as

$$\psi(y) = y^{\xi/\alpha}\, e^{-\frac{b_1 y}{\alpha}}\, f\,(y) \tag{4.31}$$

Substituting the proposed physical functions into Eq. (4.29a) one can gets the equation

$$f''(y) = \left(\frac{-b_2 + b_1(2\xi + \alpha)}{y\alpha^2}\right) f(y) + (\frac{2b_1}{\alpha} - \frac{(2\xi\alpha + \alpha^2)}{y\alpha^2}) f'(y) \tag{4.32}$$

By applying the quantization conditions, we obtain the general solution for ξ as

$$\xi_n = \frac{b_2 - (2n+1)b_1\alpha}{2b_1}$$

and using the definitions of ξ b_1 and b_2, one obtain the energy values E_n

$$E_n^2 = -\hbar^2 c^2 \left(\frac{b_2 - (2n+1)b_1\alpha}{2b_1}\right)^2 \tag{4.33}$$

$$E_n = \pm\sqrt{-\hbar^2 c^2 \left(\frac{b_2 - (2n+1)b_1\alpha}{2b_1}\right)^2} \tag{4.34}$$

with parameters introduced in Eq. (4.29a)

The corresponding eigenfunction of this potential is

$$f_n\,(y) = (-1)(\prod_{k=n}^{2n-1}\left({b_1}^2 - (k+1)b_2\right), {}_2F_1(-n, 2\xi_n + 1; 2\,b_2 y).$$

with the radial wave function, is obtained as

$$R_n\,(x) = (-1)^n(\prod_{k=n}^{2n-1}\left({b_1}^2 - (k+1)b_2\right) e^{\alpha x\,\xi_n}\, e^{-b_2 e^{\alpha x}}, {}_2F_1(-n, 2\xi_n + 1; 2\,b_2 e^{\alpha x}).$$

with parameters

$$\xi = \frac{{E_n}^2\alpha^2 - c^4{m_0}^2\alpha^2}{c^2\hbar^2}$$

$$b_1 = \frac{2(E_n + c^2 m_0)V_0\alpha^2}{c^2\hbar^2}, \text{and } b_2 = 0$$

4.4.2 Discussion

i) If $\beta = 1$, it yields $S = 0$, then we get $S = (\beta - 1)V$ the parameters have forms

$$\xi^2 = -\frac{{E_n}^2}{c^2\hbar^2}$$

$$-{b_1}^2 = \frac{(-c^4{m_0}^2 + {V_0}^2)}{c^2\hbar^2}$$

$$b_2 = -\left(\frac{2E_n V_0}{c^2\hbar^2}\right)$$

After applying the quantization conditions, we obtain the general solution for ξ as

$$\xi_n = \frac{b_2 - (2n+1)b_1\alpha}{2b_1}$$

the energy values E_n reduced to

$$E_n^2 = -\hbar^2 c^2 \left(\frac{b_2 - (2n+1)b_1\alpha}{2b_1}\right)^2 \tag{4.35}$$

$$E_n = \pm\sqrt{-\hbar^2 c^2 \left(\frac{b_2 - (2n+1)b_1\alpha}{2b_1}\right)^2} \tag{4.36}$$

The corresponding eigenfunction gets the similar form for radial wavefunction as

$$f_n\,(y) = (-1)(\prod_{k=n}^{2n-1}\left({b_1}^2 - (k+1)b_2\right), {}_2F_1\left(-n, 2\xi_n + 1; 2\,b_2 y\right).$$

and

$$R_n\,(y) = (-1)^n(\prod_{k=n}^{2n-1}\left({b_1}^2 - (k+1)b_2\right)y^{\xi_n}\,e^{-b_2 y}, {}_2F_1(-n, 2\xi_n + 1; 2\,b_2 y).$$

with parameters

$$\xi^2 = -\frac{{E_n}^2}{c^2\hbar^2}$$

$$-{b_1}^2 = \frac{(-c^4{m_0}^2 + {V_0}^2)}{c^2\hbar^2}, b_2 = -\left(\frac{2E_n V_0}{c^2\hbar^2}\right)$$

ii) When $\beta = 2$ yields, we get equal scalar and vector potential with parameters as following.

$$\xi^2 = -\frac{E_n{}^2}{c^2\hbar^2}$$

$$\mathrm{b_1}^2 = \frac{(-c^4\mathrm{m_0}^2 + 2c^2\mathrm{m_0}\mathrm{V_0}^2)}{c^2\hbar^2},$$

$$\mathrm{b_2} = -\left(\frac{2E_n\mathrm{V_0}}{c^2\hbar^2}\right)$$

And the parameters are becomes

$$E_n^2 = -\hbar^2\mathrm{c}^2\left(\frac{\mathrm{b_2} - (2\mathrm{n}+1)\mathrm{b_1}\alpha}{2\mathrm{b_1}}\right)^2 \tag{4.37}$$

$$E_n = \pm\sqrt{-\hbar^2\mathrm{c}^2\left(\frac{\mathrm{b_2} - (2\mathrm{n}+1)\mathrm{b_1}\alpha}{2\mathrm{b_1}}\right)^2} \tag{4.38}$$

After the same algebraic procedure, the eigenfunction becomes

$$f_n\ (\mathrm{y}) = (-1)(\prod_{\mathrm{k=n}}^{\mathrm{2n-1}}\left(\mathrm{b_1}^2 - (\mathrm{k}+1)\mathrm{b_2}\right), {}_2F_1\left(-n, 2\xi_\mathrm{n} + 1; 2\ \mathrm{b_2}\mathrm{y}\right).$$

For this formula, if we go past to the radial wave function, is obtained as

$$R_n\ (\mathrm{y}) = (-1)^\mathrm{n}(\prod_{\mathrm{k=n}}^{\mathrm{2n-1}}\left(\mathrm{b_1}^2 - (\mathrm{k}+1)\mathrm{b_2}\right)\mathrm{y}^{\xi_\mathrm{n}}\ \mathrm{e}^{-\mathrm{b_2}\mathrm{y}}, {}_2F_1(-n, 2\xi_\mathrm{n} + 1; 2\ \mathrm{b_2}\mathrm{y}).$$

with parameters

$$\xi^2 = -\frac{E_n{}^2}{c^2\hbar^2}$$

$$b_1{}^2 = \frac{(-c^4m_0{}^2 + 2c^2m_0V_0{}^2)}{c^2\hbar^2}, b_2 = -\left(\frac{2E_nV_0}{c^2\hbar^2}\right)$$

4.4.3 $\boldsymbol{m = m_0(1 + qe^{-\alpha x})}$, $\boldsymbol{V = V_0 e^{-\alpha x}}$ case

In this case, the vector potential function and mass functions are considered as $\mathrm{V} = \mathrm{V}_0 \mathrm{e}^{-\alpha \mathrm{x}}$ and $m = \mathrm{m}_0(1 + \mathrm{e}^{-\alpha \mathrm{x}})$ respectivity [71]. At this line, the scalar potentials become $\mathrm{V} = \mathrm{V}_0(\beta - 1)\, \mathrm{e}^{-\alpha \mathrm{x}}$ and the KG equation reads

$$\begin{aligned}\psi''(x) &+ \frac{E_n{}^2 \psi(x)}{c^2 \hbar^2} - \frac{c^2 m_0{}^2 \psi(x)}{\hbar^2} - \frac{2c^2 e^{-x\alpha} m_0{}^2 q \psi(x)}{\hbar^2} - \frac{c^2 e^{-2x\alpha} m_0{}^2 q^2 \psi(x)}{\hbar^2} \\ &- \frac{2 e^{-x\alpha} E n V_0 \psi(x)}{c^2 \hbar^2} + \frac{2 e^{-x\alpha} m_0 V_0 \psi(x)}{\hbar^2} - \frac{2\beta e^{-x\alpha} m_0 V_0 \psi(x)}{\hbar^2} \\ &+ \frac{2 e^{-2x\alpha} m_0 q V_0 \psi(x)}{\hbar^2} - \frac{2\beta e^{-2x\alpha} m_0 q V_0 \psi(x)}{\hbar^2} \\ &+ \frac{2\beta e^{-2x\alpha} V_0{}^2 \psi(x)}{c^2 \hbar^2} - \frac{\beta^2 e^{-2x\alpha} V_0{}^2 \psi(x)}{c^2 \hbar^2} = 0\end{aligned} \tag{4.39}$$

By changing the variables $\mathrm{y} = e^{-x\alpha}$, we get the same equations except for parameters ξ , b_1 and b_2 as following;

$$\psi''(y) + \frac{\psi'(y)}{y} + (-b_2{}^2 - \frac{\xi^2}{y^2} + \frac{b_1{}^2}{y})\psi(y) = 0 \tag{4.40}$$

where

$$\xi^2 = \frac{-E_n{}^2 + c^4 \mathrm{m}_0{}^2}{c^2 \hbar^2 \alpha^2}$$

$$b_2 = \frac{(-2c^4 \mathrm{m}_0{}^2 q - 2E_n \mathrm{V}_0 - 2(-1 + \beta) c^2 \mathrm{m}_0 \mathrm{V}_0)}{c^2 \hbar^2 \alpha^2}$$

$$b_1{}^2 = \frac{(c^4 \mathrm{m}_0{}^2 q^2 - 2(-1 + \beta) c^2 \mathrm{m}_0 q \mathrm{V}_0 + (2 - \beta)\beta \mathrm{V}_0{}^2)}{c^2 \hbar^2 \alpha^2}$$

Let propose the physical function for case (a) is

$$\psi(y) = y^{\xi}\, e^{-b_2 y}\, f(y) \tag{4.41}$$

Substituting Eq. (4.41) into Eq. (4.40), one gets

$$\psi''(y) = \frac{\left(-{b_1}^2 + b_2 + 2b_2\,\xi\right)\chi(y)}{y} + \frac{(-1 - 2\xi + 2b_2\,y)\chi'(y)}{y} \tag{4.42}$$

which is transformed to the form of differential equation to be solved by AIM. By comparing the Eq. (4.42) with Eq. (3.1), we easily found that the function as

$$\lambda_0 = \frac{-1 - 2\xi - 2\mathrm{b}_2 y}{y}$$

$$s_0 = \frac{-{b_1}^2 + b_2 + 2b_2\xi}{y}$$

In this case, the energy relation becomes

$$\xi_n = \frac{b_2 - (n+1)b_1\alpha}{2b_1}$$

and

$$E_n^2 = -\hbar^2 c^2 \alpha^2 \left(\frac{b_2 - (n+1)b_1\alpha}{2b_1}\right)^2 + {m_0}^2 c^4 \tag{4.43}$$

$$E_n = \pm\sqrt{-\hbar^2 \mathrm{c}^2 \alpha^2 \left(\frac{\mathrm{b}_2 - (n+1)\mathrm{b}_1\alpha}{2\mathrm{b}_1}\right)^2 + {\mathrm{m}_0}^2 \mathrm{c}^4} \tag{4.44}$$

Simplifying the wavefunction equations like previous one, radial wave function, is obtained as

$$R_n\,(y) = (-1)^n \left(\prod_{k=n}^{2n-1} \left({b_1}^2 - (k+1)b_2\right) y^{\xi_n}\, e^{-b_2 y},\, {}_2F_1(-n, 2\xi_n + 1; 2\,b_2 y).\right.$$

with parameters

$$\xi^2 = \frac{-{E_n}^2 + c^4 {\mathrm{m}_0}^2}{c^2 \hbar^2 \alpha^2}$$

$$b_2 = \frac{(-2c^4 {\mathrm{m}_0}^2 q - 2E_n \mathrm{V}_0 - 2(-1+\beta)c^2 \mathrm{m}_0 \mathrm{V}_0)}{c^2 \hbar^2 \alpha^2}$$

$$b_1{}^2 = \frac{(c^4 \mathrm{m_0}^2 q^2 - 2(-1+\beta)c^2 \mathrm{m_0} q \mathrm{V_0} + (2-\beta)\beta \mathrm{V_0}^2)}{c^2 \hbar^2 \alpha^2}$$

4.4.4 Discussion

i) When $\beta = 1$, again we have zero scalar potential and we have only vector potential function. The energy relation is reduced to

$$\xi^2 = -\frac{E_n{}^2 - c^4 \mathrm{m_0}^2}{c^2 \hbar^2}$$

$$b_2 = \frac{(-2c^4 \mathrm{m_0}^2 q - 2E_n \mathrm{V_0})}{c^2 \hbar^2}$$

$$b_1{}^2 = \frac{(-c^4 \mathrm{m_0}^2 q^2 + \mathrm{V_0}^2)}{c^2 \hbar^2}$$

These values of each ξ yields a general solution in the form of

$$\xi_n = \frac{\mathrm{b_2} - (n+1)\mathrm{b_1}\alpha}{2\mathrm{b_1}}.$$

And the energy values E_n has form

$$\mathrm{E_n^2} = -\hbar^2 \mathrm{c}^2 \left(\frac{\mathrm{b_2} - (n+1)\mathrm{b_1}\alpha}{2\mathrm{b_1}}\right)^2 + \mathrm{m_0}^2 \mathrm{c}^4 \tag{4.45}$$

$$E_n = \pm\sqrt{-\hbar^2 \mathrm{c}^2 \left(\frac{\mathrm{b_2} - (n+1)\mathrm{b_1}\alpha}{2\mathrm{b_1}}\right)^2 + \mathrm{m_0}^2 \mathrm{c}^4} \tag{4.46}$$

The corresponding eigenfunction this potential is

$$f_n\ (\mathrm{y}) = (-1)(\prod_{\mathrm{k=n}}^{\mathrm{2n-1}} (\mathrm{b_1}^2 - (\mathrm{k}+1)\mathrm{b_2}),\ {}_2F_1(-n, 2\xi_n + 1; 2\ \mathrm{b_2 y}).$$

For this formula, if we go past to the radial wave function, it is obtained as

$$R_n\ (\mathrm{y}) = (-1)^{\mathrm{n}}(\prod_{\mathrm{k=n}}^{\mathrm{2n-1}} (\mathrm{b_1}^2 - (\mathrm{k}+1)\mathrm{b_2})\mathrm{y}^{\xi_n}\ \mathrm{e}^{-\mathrm{b_2 y}},\ {}_2F_1(-n, 2\xi_n + 1; 2\ \mathrm{b_2 y}).$$

with parameters

$$\xi^2 = -\frac{{E_n}^2 - c^4 {m_0}^2}{c^2 \hbar^2}$$

$$b_2 = \frac{(-2c^4 {m_0}^2 q - 2E_n V_0)}{c^2\hbar^2}, \qquad {b_1}^2 = \frac{(-c^4 {m_0}^2 q^2 + {V_0}^2)}{c^2\hbar^2}$$

ii) When $\beta = 2$, it yields equal vector and scalar potential $S = V$.

$$\xi^2 = -\frac{{E_n}^2 - c^4 {m_0}^2}{c^2 \hbar^2}$$

$$b_2 = \frac{(-2c^4 {m_0}^2 q - 2E_n V_0 + 2c^2 m_0 V_0)}{c^2\hbar^2}$$

$${b_1}^2 = \frac{(c^4 {m_0}^2 q^2 + 2c^2 m_0 q V_0)}{c^2\hbar^2}$$

These values of each ξ yield a general solution in the form of

$$\xi_n = \frac{b_2 - (n+1) b_1 \alpha}{2b_1}$$

the energy values E_n

$$E_n^2 = -\hbar^2 c^2 \left(\frac{b_2 - (n+1) b_1 \alpha}{2b_1}\right)^2 + {m_0}^2 c^4 \tag{4.47}$$

and

$$E_n = \pm\sqrt{-\hbar^2 c^2 \left(\frac{b_2 - (n+1) b_1 \alpha}{2b_1}\right)^2 + {m_0}^2 c^4} \tag{4.48}$$

After some algebraic calculation, the corresponding eigenfunction is

$$f_n\,(y) = (-1)(\prod_{k=n}^{2n-1}\left({b_1}^2 - (k+1)b_2\right), {}_2F_1\left(-n, 2\xi_n + 1; 2\, b_2 y\right).$$

and the total radial wave function becomes

$$R_n\,(y) = (-1)^n(\prod_{k=n}^{2n-1}\left({b_1}^2 - (k+1)b_2\right) y^{\xi_n}\, e^{-b_2 y}, {}_2F_1(-n, 2\xi_n + 1; 2\, b_2 y).$$

with parameters

$$\xi^2 = -\frac{{E_n}^2 - c^4{\mathrm{m}_0}^2}{c^2\hbar^2}$$

$$b_2 = \frac{(-2c^4{\mathrm{m}_0}^2 q - 2E_n\mathrm{V}_0 + 2c^2\mathrm{m}_0\mathrm{V}_0)}{c^2\hbar^2}$$

$$b_1{}^2 = \frac{(c^4{\mathrm{m}_0}^2 q^2 + 2c^2\mathrm{m}_0 q\mathrm{V}_0)}{c^2\hbar^2}$$

4.4.5 $\boldsymbol{m = m_0(1 + qe^{\alpha x})}$, $\boldsymbol{V = V_0 e^{\alpha x}}$, case

In this case, the vector potential function and mass functions are considered as $V = V_0 e^{\alpha x}$ and $m = m_0(1 + e^{\alpha x})$ respectivity. At this line, the scalar potentials become $\mathrm{V} = \mathrm{V}_0(\beta - 1)\,\mathrm{e}^{-\alpha x}$ and the KG equation reads

$$\begin{aligned}
\psi''(x) &+ \frac{En^2\psi(x)}{c^2\hbar^2} - \frac{c^2{m_0}^2\psi(x)}{\hbar^2} - \frac{2c^2 e^{x\alpha}{m_0}^2 q\psi(x)}{\hbar^2} \\
&- \frac{c^2 e^{2x\alpha}{m_0}^2 q^2\psi(x)}{\hbar^2} - \frac{2e^{x\alpha}EnV_0\psi(x)}{c^2\hbar^2} + \frac{2e^{x\alpha}m_0V_0\psi(x)}{\hbar^2} \\
&- \frac{2\beta e^{x\alpha}m_0V_0\psi(x)}{\hbar^2} + \frac{2e^{2x\alpha}m_0qV_0\psi(x)}{\hbar^2} \\
&- \frac{2\beta e^{2x\alpha}m_0qV_0\psi(x)}{\hbar^2} + \frac{2\beta e^{2x\alpha}{V_0}^2\psi(x)}{c^2\hbar^2} - \frac{\beta^2 e^{2x\alpha}{V_0}^2\psi(x)}{c^2\hbar^2} \\
&= 0
\end{aligned} \tag{4.49}$$

Changing the variables $\mathrm{y} = e^{-x\alpha}$ to get the familiar equations except for parameters ξ , b_1 and b_2 as following

$$\psi''(y) + \frac{\psi'(y)}{y}\left(-b_2{}^2 - \frac{\xi^2}{y^2} + \frac{b_1{}^2}{y}\right)\psi(y) = 0 \tag{4.50}$$

and

$$\xi^2 = -\frac{{E_n}^2 - c^4{m_0}^2}{c^2\hbar^2\alpha^2}$$

$$b_2 = \frac{-c^4{m_0}^2q^2 - 2(-1+b)c^2m_0qV_0 + (2-b)b{V_0}^2}{c^2\hbar^2\alpha^2}$$

$${b_1}^2 = \frac{-2c^4{m_0}^2q - 2EnV_0 - 2(-1+b)c^2m_0V_0}{c^2\hbar^2\alpha^2}$$

We come a point that to propose the physical function for this case

$$\psi(y) = y^{\xi}\, e^{-b_2y}\; f\;(y) \tag{4.51}$$

Substituting Eq. (4.51) in to Eq. (4.50), one gets

$$\psi''(y) = \frac{-{b_2}^2\chi(y) + b_2\chi(y) + 2b_2\xi\chi(y)}{y} + (2b_2 + \frac{-1-2\xi}{y})\psi'(y) \tag{4.52}$$

With a similar algebraic calculation, we get

$$\xi_n = \frac{b_2 - (n+1)b_1\alpha}{2b_1}$$

and

$$E_n^2 = -\hbar^2c^2\left(\frac{b_2 - (n+1)b_1\alpha}{2b_1}\right)^2 + {m_0}^2c^4 \tag{4.53}$$

with

$$E_n = \pm\sqrt{-\hbar^2c^2\left(\frac{b_2 - (n+1)b_1\alpha}{2b_1}\right)^2 + {m_0}^2c^4} \tag{4.54}$$

4.4.5.1 Obtaining Wave functions

For this effective mass and exponential potentials form *V(x)* and *S(x)*, the solution of differential equation results in eigenfunction

$$f_n\;(y) = (-1)(\prod_{k=n}^{2n-1}\left({b_1}^2 - (k+1)b_2\right), {}_2F_1(-n, 2\xi_n + 1; 2\; b_2 e^{-x\alpha}).$$

This function yields a total radial wave function

$$R_n\ (\mathrm{y}) = (-1)^{\mathrm{n}}(\prod_{\mathrm{k=n}}^{2\mathrm{n}-1}\left(\mathrm{b_1}^2 - (\mathrm{k}+1)\mathrm{b_2}\right)e^{-x\alpha\,\xi_{\mathrm{n}}}\ \mathrm{e}^{-\mathrm{b_2}e^{-x\alpha}}\ ,{}_2F_1(-n, 2\xi_{\mathrm{n}}$$

$$+ 1; 2\ \mathrm{b_2}e^{-x\alpha}).$$

with parameters

$$\xi^2 = -\frac{{E_n}^2 - c^4\mathrm{m_0}^2}{c^2\hbar^2\alpha^2}$$

$$b_2 = \frac{-c^4\mathrm{m_0}^2q^2 - 2(-1+b)c^2\mathrm{m_0}q\mathrm{V_0} + (2-b)b\mathrm{V_0}^2}{c^2\hbar^2\alpha^2}$$

$${b_1}^2 = \frac{-2c^4\mathrm{m_0}^2q - 2\mathrm{EnV_0} - 2(-1+b)c^2\mathrm{m_0V_0}}{c^2\hbar^2\alpha^2}$$

4.4.6 Discussion

i)If $\beta = 1$, only vector potentials is included with parameters

$$\xi^2 = \frac{{E_n}^2 - c^4\mathrm{m_0}^2}{c^2\hbar^2}$$

$$b_1 = \frac{(-c^4{m_0}^2q^2 + {V_0}^2)}{c^2\hbar^2}, \qquad b_2 = \frac{(-2c^4{m_0}^2q - 2E_nV_0)}{c^2\hbar^2}$$

Then the general form of energy is

$$\xi_{\mathrm{n}} = \frac{\mathrm{b_2} - (n+1)\mathrm{b_1}\alpha}{2\mathrm{b_1}}$$

the energy values E_n

$$E_n^2 = \hbar^2c^2\left(\frac{b_2 - (n+1)b_1\alpha}{2b_1}\right)^2 + {m_0}^2c^4 \tag{4.55}$$

$$E_n = \pm\sqrt{\hbar^2 c^2 \left(\frac{b_2 - (n+1) b_1 \alpha}{2b_1}\right)^2 + m_0{}^2 c^4} \tag{4.56}$$

and the radial wave function, it becomes

$$R_n\,(\mathrm{y}) = (-1)^{\mathrm{n}} (\prod_{\mathrm{k=n}}^{\mathrm{2n-1}} \left(\mathrm{b_1}^2 - (\mathrm{k}+1)\mathrm{b_2}\right) \mathrm{y}^{\xi_{\mathrm{n}}}\ \mathrm{e}^{-\mathrm{b_2 y}}, {}_2F_1(-n, 2\xi_{\mathrm{n}} + 1; 2\ \mathrm{b_2 y}).$$

with parameters

$$\xi^2 = \frac{E_n{}^2 - c^4 \mathrm{m_0}^2}{c^2 \hbar^2}$$

$$\mathrm{b_2} = \frac{(-2c^4 \mathrm{m_0}^2 q - 2E_n \mathrm{v_0})}{c^2\hbar^2}, \mathrm{b_1} = \frac{(-c^4 \mathrm{m_0}^2 q^2 + \mathrm{v_0}^2)}{c^2\hbar^2}$$

ii) For $\beta = 2$, we get equal scalar and vector potentials with relations as following

$$-\xi^2 = \frac{E_n{}^2 - c^4 \mathrm{m_0}^2}{c^2\hbar^2}$$

$$\mathrm{b_2} = \frac{(-2c^4 \mathrm{v}^2 q - 2\mathrm{EnV_0} - 2c^2 \mathrm{m_0 V_0})}{c^2\hbar^2}$$

$$-\mathrm{b_1}^2 = \frac{(-c^4 \mathrm{m_0}^2 q^2 - 2c^2 \mathrm{m_0} q \mathrm{V_0})}{c^2\hbar^2}$$

Rearranging these relations, we get

$$\xi_{\mathrm{n}} = \frac{\mathrm{b_2} - (n+1)\mathrm{b_1}\alpha}{2\mathrm{b_1}}$$

and the energy values E_n

$$\mathrm{E_n^2} = -\hbar^2 \mathrm{c}^2 \left(\frac{\mathrm{b_2} - (n+1)\mathrm{b_1}\alpha}{2\mathrm{b_1}}\right)^2 + \mathrm{m_0}^2 \mathrm{c}^4 \tag{4.57}$$

$$E_n = \pm\sqrt{-\hbar^2 c^2\left(\frac{\mathrm{b}_2 - (n+1)\mathrm{b}_1\alpha}{2\mathrm{b}_1}\right)^2 + \mathrm{m_0}^2 \mathrm{c}^4} \tag{4.58}$$

For different parameters and variables y, we get the some form of wavefunction as

$$f_n\ (\mathrm{y}) = (-1)(\prod_{\mathrm{k=n}}^{\mathrm{2n-1}}\left(\mathrm{b_1}^2 - (\mathrm{k}+1)\mathrm{b}_2\right),\ {}_2F_1\left(-n, 2\xi_{\mathrm{n}} + 1; 2\ \mathrm{b}_2\mathrm{y}\right).$$

and the radial wave function becomes

$$R_n\ (\mathrm{y}) = (-1)^{\mathrm{n}}(\prod_{\mathrm{k=n}}^{\mathrm{2n-1}}\left(\mathrm{b_1}^2 - (\mathrm{k}+1)\right)\ {}_2F_1\mathrm{y}^{\xi_{\mathrm{n}}}\ \mathrm{e}^{-\mathrm{b_2 y}}, f_1(-n, 2\xi_{\mathrm{n}} + 1; 2\ \mathrm{b}_2\mathrm{y}).$$

with parameters

$$-\xi^2 = \frac{{E_n}^2 - c^4 \mathrm{m_0}^2}{c^2\hbar^2}$$

$$\mathrm{b}_2 = \frac{(-2c^4\mathrm{v}^2 q - 2\mathrm{EnV_0} - 2c^2\mathrm{m_0 V_0})}{c^2\hbar^2}$$

$$-\mathrm{b_1}^2 = \frac{(-c^4 \mathrm{m_0}^2 q^2 - 2c^2 \mathrm{m_0} q \mathrm{V_0})}{c^2\hbar^2}$$

4.4.7 The Woods-Saxon Potential For Position Dependent Mass Application

In this part consists of the solution PDMKG equation for different mass and for Wood-Saxon potential.

$$\frac{{E_n}^2\,\psi(x)}{c^2\hbar^2} - \frac{c^2 \mathrm{m_0}^2\,\psi(x)}{(1+e^{\frac{-R+x}{\alpha}})^2\hbar^2} + \frac{2E_n \mathrm{V_0}\,\psi(x)}{c^2(1+e^{\frac{-R+x}{\alpha}})\hbar^2} - \frac{2\mathrm{m_0 V_0}\,\psi(x)}{(1+e^{\frac{-R+x}{\alpha}})^2\hbar^2}$$
$$+ \frac{2\mathrm{m_0 V_0}\beta\,\psi(x)}{(1+e^{\frac{-R+x}{\alpha}})^2\hbar^2} + \frac{2\mathrm{V_0}^2\beta\,\psi(x)}{c^2(1+e^{\frac{-R+x}{\alpha}})^2\hbar^2} \tag{4.59}$$
$$- \frac{\mathrm{V_0}^2\beta^2\phi\,\psi(x)}{c^2(1+e^{\frac{-R+x}{\alpha}})^2\hbar^2} + \psi''(x) = 0$$

By changing the variable $y = \frac{1}{e^{(x-R)/\alpha}}$ as,

$$\left(\frac{E_n{}^2}{c^2\hbar^2}+\frac{2E_n\mathrm{V}_0 y}{c^2\hbar^2} - \frac{y^2\left(c^4\mathrm{m}_0{}^2-2c^2\mathrm{m}_0\mathrm{V}_0(-1+\beta)+\mathrm{V}_0{}^2(-2+\beta)\beta\right)}{c^2\hbar^2}\right)\psi(x) + \left(\frac{y}{\alpha^2}-\frac{3y^2}{\alpha^2}+\frac{2y^3}{\alpha^2}\right)\psi^{'}(x)+\left(\frac{y^2}{\alpha^2}-\frac{2y^3}{\alpha^2}+\frac{y^4}{\alpha^2}\right)\psi^{''}(x)=0 \tag{4.60}$$

defining following new parameters

$$\xi^2=-\frac{E_n{}^2\alpha^2}{c^2\hbar^2}$$

$$\mathrm{b}_2=\frac{(c^4\mathrm{m}_0{}^2-2c^2\mathrm{m}_0\mathrm{V}_0(-1+\beta)+\mathrm{V}_0{}^2(-2+\beta)\beta)\alpha^2}{c^2\hbar^2}$$

$$\mathrm{b}_1=\frac{2E_n\mathrm{V}_0\alpha^2}{c^2\hbar^2}$$

Dividing the (4.) with $(\frac{y^2}{\alpha^2}-\frac{2y^3}{\alpha^2}+\frac{y^4}{\alpha^2})$, we get

$$\psi^{''}(x)+\frac{(1-2y)\psi^{'}(x)}{(1-y)y}+\frac{(-\xi^2-\mathrm{b}_1y-\mathrm{b}_2y^2)\,\psi(x)}{(1-y)^2y^2}=0$$

The physical wave functions for KG equation is proposed as

$$\psi(x)=y^{\xi}(1-y)^{\sqrt{\xi^2+\mathrm{b}_1+\mathrm{b}_2}}f(x)$$

substituting Eq.(4.) into Eq.(4.) yields to the equation form of

$$f^{''}(y)=\frac{1}{(-1+y)y}(-\mathrm{b}_1f(y)-\xi f(y)-2\xi^2f(y)-vf(y)-2\xi vf(y)+f^{'}(y) + 2\xi f^{'}(y)-2yf^{'}(y)-2\xi yf^{'}(y)-2vyf^{'}(y))$$

Similar to the previous applications, by applying the quantization conditions, we obtain the general solution for ξ as

$$\xi_{\mathrm{n}}+v=-\frac{2n+1}{2}-\frac{1}{2}\sqrt{1+4\mathrm{b}_1}$$

using the parameters introduced for ξ , b_1 and v , we obtain a more explicit

$$E_n^2 - \mathrm{m_0}^2 c^4 = -c^2\hbar^2\left(-\frac{2n+1}{2} - \frac{1}{2}\sqrt{1 + 4\frac{2E_n \mathrm{V_0}\alpha^2}{c^2\hbar^2}} - v\right) \tag{4.61}$$

By substituting for ξ_n and v we obtain a more explicit expression for the eigenvalues energy as

$$E_n = \pm(m_0{}^2 c^4 - c^2\hbar^2\left(-\frac{2n+1}{2} - \frac{1}{2}\sqrt{1 + 4\frac{2E_n \mathrm{V_0}\alpha^2}{c^2\hbar^2}} - v\right))^{\frac{1}{2}} \tag{4.62}$$

With the same way, the function f_n (y)

$$f_n\ (x) = (-1)^n C_2\left(\frac{\Gamma(2\xi_n + n + 1)}{\Gamma(2\xi_n + 1)}\right) {}_2F_1\ (-n, 2(\xi_n - v) + 1 + n; 2\xi_n + 1; \frac{1}{e^{(x-R)/\alpha}})$$

where Γ and ${}_2F_1$ are the Gamma function and hypergeometric function respectively.

The total radial wave function can be written as follows:

$$R_n\ (\mathrm{x}) = (-1)^{\mathrm{n}}\mathrm{N_n}\frac{(1 + e^{\frac{-R+x}{\alpha}})^v}{(1 + e^{\frac{-R+x}{\alpha}})^{\xi}}\,{}_2F_1(-\mathrm{n}, 2(\xi - v) + 1 + \mathrm{n}; 2\xi + 1; (1 + e^{\frac{-R+x}{\alpha}})^{-1})$$

where $\mathrm{N_n}$ is the normalization constant.

With parameters

$$\xi^2 = -\frac{E_n{}^2\alpha^2}{c^2\hbar^2}$$

$$\mathrm{b_2} = \frac{(c^4\mathrm{m_0}^2 - 2c^2\mathrm{m_0}\mathrm{V_0}(-1+\beta) + \mathrm{V_0}^2(-2+\beta)\beta)\alpha^2}{c^2\hbar^2}$$

$$\mathrm{b_1} = \frac{2E_n\mathrm{V_0}\alpha^2}{c^2\hbar^2}$$

4.4.8 Discussion

i) When $\beta = 1$, this is the zero scalar potential case. The energy relation is reduced to

$$\xi^2 = -\frac{{E_n}^2\alpha^2}{c^2\hbar^2}$$

$$b_2 = \frac{\left(-c^4{m_0}^2 + {V_0}^2\right)}{c^2\hbar^2}, b_1 = \frac{2E_n V_0\alpha^2}{c^2\hbar^2}$$

and the general form of energy is

$$\xi_n + v = -\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4b_1}$$

Using the defined parameters, the energy values E_n

$$E_n^2 = -\frac{c^2\hbar^2}{\alpha^2}\left(-\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4\frac{2E_n V_0\alpha^2}{c^2\hbar^2}} - v\right) \tag{4.63}$$

$$E_n = \pm\sqrt{-\frac{c^2\hbar^2}{\alpha^2}\left(-\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4\frac{2E_n V_0\alpha^2}{c^2\hbar^2}} - v\right)} \tag{4.64}$$

and the wavefunction f_n (y)

$$f_n\,(y) = (-1)^n C_2 \frac{\Gamma(2\xi_n + n + 1)}{\Gamma(2\xi_n + 1)}\ {}_2F_1(-n, 2(\xi_n - v) + 1 + n; 2\xi_n + 1; y)$$

At this point, the total radial wave function reduces to

$$R_n\,(y) = (-1)^n N_n \frac{(1 + e^{\frac{-R+x}{\alpha}})^v}{(1 + e^{\frac{-R+x}{\alpha}})^\xi}\ {}_2F_1(-n, 2(\xi - v) + 1 + n; 2\xi + 1;\ (1 + e^{\frac{-R+x}{\alpha}})^{-1})$$

where N_n is the normalization constant and the related parameters are

$$\xi^2 = -\frac{E_n{}^2\alpha^2}{c^2\hbar^2}$$

$$b_1 = \frac{2E_nV_0\alpha^2}{c^2\hbar^2}, \qquad b_2 = \frac{\left(-c^4m_0{}^2 + V_0{}^2\right)}{c^2\hbar^2}$$

ii) When $\beta = 2$, we get equal vector and scalar potential $S(x) = V(x)$ and the parameters becomes

$$\xi^2 = -\frac{E_n{}^2\alpha^2}{c^2\hbar^2}$$

$$b_1 = \frac{2E_nV_0\alpha^2}{c^2\hbar^2}, \qquad b_2 = \frac{(-c^4m_0{}^2\alpha^2 + 2c^2m_0V_0\alpha^2)}{c^2\hbar^2}.$$

Rearranging these relations, we get

$$\xi_n + v = -\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4b_1}$$

And the energy values E_n and the wavefunction f_n (y)

$$E_n^2 = -\frac{c^2\hbar^2}{\alpha^2}\left(-\frac{2n+1}{2} - \frac{1}{2}\sqrt{1+4\frac{2E_nV_0\alpha^2}{c^2\hbar^2}} - v\right) \tag{4.65}$$

$$f_n\,(\mathrm{y}) = (-1)^n C_2 \frac{\Gamma(2\xi_n + n + 1)}{\Gamma(2\xi_n + 1)}\, {}_2F_1(-n, 2(\xi_n - v) + 1 + n; 2\xi_n + 1; y)$$

Substituting the variable *y* interims of *x*, the total radial wave function can be written as follows:

$$R_n\,(\mathrm{y}) = (-1)^n N_n \frac{(1 + q^{-1}e^{\frac{-R+x}{\alpha}})^v}{(1 + qe^{\frac{-R+x}{\alpha}})^\xi}\, {}_2F_1(-n, 2(\xi - v) + 1 + n; 2\xi + 1; (1 + qe^{\frac{-R+x}{\alpha}})^{-1})$$

where

$$\xi^2 = -\frac{E_n{}^2\alpha^2}{c^2\hbar^2}, \qquad b_1 = \frac{2E_nV_0\alpha^2}{c^2\hbar^2}, \qquad b_2 = \frac{(-c^4m_0{}^2\alpha^2 + 2c^2m_0V_0\alpha^2)}{c^2\hbar^2}$$

CHAPTER 5

5 CONCLUSION

In this study, by considering an alternative method, the asymptotic iteration method, the bound state solution KG equation is solved for exponential type of potential. During the calculation of energy eigenfunction and energy eigenvalues, the mass function is considered red in two types: i) constant masses and ii) position dependent masses. When we consider the vector and scalar potential in the form of $e^{-x\alpha}$, the transformed effective potential become Morse potential. The eigenfunction for all types of potential forms are calculated in terms of hyper geometric functions.

In addition to the constant mass case of Wood-Saxon potential, it is also analyzed for the condition of spatially dependent masses.

After applying the AIM to the both of mass function, the energy values are analyzed and discussed with respect to the values of β parameter which adjust the relation between vector and scalar potentials. Since, this relations is important to get bound-state solutions or scattering-state solutions.

The obtained spectrums for corresponding potentials show that AIM reproduces the exact results more simpler and under stable than other algebraic method which is important advantages of AIM.

As a conclusion, it is possible to extend the type of position dependent mass functions to analyze for different system using AIM.

REFFERENCES

[1] Baym, Gordo. 1990. “Lectures on Quantum Mechanics”. *Third Edition. New York*. Westview Press.

[2] Alhaidari, A.D. ,Bahlouli,H. , Al-Hasan,A. (2006). Dirac and Klein-Gordon equations with equal scalar and vector potentials, *Physics Letters A*, 349, 87-97.

[3] Xıao Yong-Jun, Long Zheng-Wen (2010). Solution of Klein–Gordon Equation for Pseudo-Coulomb Potential Plus a New Ring-Shaped Potential, *Communications Theoretical Physics (Beijing,China)*, 53, 54-56.

[4] Olğar Eser, Koç Ramazan, Tütüncüler Hayriye (2005). Bound states of the S wave equation with equal scalar and vector standart Eckart potential, *Communications Theoretical Physics (Beijing,China*), 23**,** 539-541.

[5] Alhaidari, A.D. ,Bahlouli,H. , Al-Hasan,A. (2006). Dirac and Klein-Gordon equations with equal scalar and vector potentials, *Physics Letters A*, 349, 87-97.

[6] Xıao Yong-Jun, Long Zheng-Wen (2010). Solution of Klein–Gordon Equation for Pseudo-Coulomb Potential Plus a New Ring-Shaped Potential, *Communications Theoretical Physics (Beijing,China)*, 53, 54-56.

[7] Olğar Eser, Koç Ramazan, Tütüncüler Hayriye (2008). The exact solution of the s-wave Klein-Gordon equation for the generalized Hulthem potential by the asymptotic Iteration method, *physica scrpta*,78, 0150111499 .

[8] Olgar, E. (2008). Exact Solution of Klein-Gordon Equation by Asymptotic Iteration Method. *Chinese Physics Letters*, *25*(6), 1939.

[9] Nagalakshmi, A.Rao, Kagali, B.A. (2008). Energy profile of the one-dimensional Klein-Gordon oscillator, *Physica scripta*, **77**, 15003-15006**.**

[10] Wen-Chao Qiang (2002). Bound states of two dimensional relativistic harmonic oscillators, *Communications Theoretical Physics (Beijing,China*), 13, 283-286.

[11] Diao, Y. F., Yi, L. Z., & Jia, C. S. (2004). Bound states of the Klein–Gordon equation with vector and scalar five-parameter exponential-type potentials. *Physics Letters A*, *332*(3), 157-167.

[12] Chen, G., Chen, Z. D., & Xuan, P. C. (2006). Exactly solvable potentials of the Klein–Gordon equation with the supersymmetry method. *Physics Letters A*, *352*(4), 317-320.

[13] Saad, Nasser (2007). The Klein-Gordon equation with a generalized Hulthen potential in D-dimensions, *Physica scripta*, 76, 623-627.

[14] Zou Xia, Zhong Yi Liang and Sheng Jia Chun (2005). Bound States of the Dirac equation with vector and scalar Eckart potentials, *Physics Letters A*, 346, 54-64.

[15] Diao Yong-Feng, Zhong Yi Liang and Sheng Jia Chun (2004). Bound states of the Klein-Gordon equation with vector and scalar five-parameter exponential-type potentials, *Physics Letters A*, 332**,**157-167; Zhong Yi Liang, Diao Yong-Feng,Yi Liu Jian and Sheng Jia Chun (2004). Bound states of the Klein-Gordon equation with vector and scalar Rosen-Morse-type potentials, *Physics Letters A*, 333, 212-217.

[16] Luttinger, J. M., & Kohn, W. (1955). Motion of electrons and holes in perturbed periodic fields. *Physical Review*, *97*(4), 869.

[17] Ikhdair, S. M. (2012). Effective Schrödinger equation with general ordering ambiguity position-dependent mass Morse potential. *Molecular Physics*, *110*(13), 1415-1428.

[18] Slater, J. C. (1949). Electrons in perturbed periodic lattices. *Physical Review*, *76*(11), 1592.

[19] Rojo, O., & Levinger, J. S. (1961). Integrated Cross Section for a Velocity-Dependent Potential. *Physical Review*, *123*(6), 2177.

[20] Razavy, M., Field, G., & Levinger, J. S. (1962). Analytical Solutions for Velocity-Dependent Nuclear Potentials. *Physical Review*, *125*(1), 269.

[21] Serra, L., & Lipparini, E. (2007). Spin response of unpolarized quantum dots. *EPL (Europhysics Letters)*, *40*(6), 667.

[22] Barranco, M., Pi, M., Gatica, S. M., Hernandez, E. S., & Navarro, J. (1997). Structure and energetics of mixed ^{4}He-^{3}He drops. *Physical Review B*, *56*(14), 8997.

[23] Dai, T. Q., & Cheng, Y. F. (2008). Bound state solutions of the Klein–Gordon equation with position-dependent mass for the inversely linear potential. *Physica Scripta*, *79*(1), 015007.

[24] Von Roos, O. (1983). Position-dependent effective masses in semiconductor theory. *Physical Review B, 27*(12), 7547.

[25] R. . Woods, D. S. Saxon. (1954). Di use surface optical model for nucleon-nuclei scattering, Phys. *Rev.* 95 ,577-578.

[26] R. D. Woods, D. S. (1954), Saxon, Diffuse surface optical model for nucleon-nuclei scattering, *Phys. Rev.* 95, 577-578.

[27] M. Dasgupta & al. (2004). Effect of breakup on the fusion of 6Li, 7Li and 7Be with heavy nuclei, Phys. *Rev. C 70*, 024606-024625.

[28] J. Sadeghi, M. R. Pahlavani. (2004). The hierarchy of Hamiltonian for spherical Woods- Saxon potential*, Afr. Phys. J. Math.* 1(2),195-199.

[29] V. Z. Goldberg & al. (2004). Low-lying levels in 15 F and the shell model potential for drip-line nuclei, Phys. *Rev. C 69(3)*, 031302-031306.

[30] H. Erkol, E. Demiralp. (2007). The Woods-Saxon potential with point interactions, Phys. *Lett. A* 365, 55-63.

[31] K. Khounfais, T. Boudjedaa and L. Chetouani. (2004). Scattering matrix for Feshbach- Villars equation for spin 0 and 1/2: Woods-Saxon potential, Phys. *Czech. J. 54*, 697-792.

[32] J. O. Newton & all. (2004). Systematic failure of the Woods-Saxon nuclear potential to describe both fusion and elastic scattering: Possible need for a new dynamical approach to fusion, Phys. *Rev. C 70(2)*, 024605-024619.

[33] J. Y. Guo and Q.Sheng. (2005). Solution of the Dirac equation for the Woods-Saxon potential with spin and pseudospin symmetry, Phys. *Lett. A 338(2)*, 90-96.

[34] Diaz-Torres, A., & Scheid, W. (2005). Two center shell model with Woods–Saxon potentials: Adiabatic and diabatic states in fusion. *Nuclear Physics A, 757*(3), 373-389.

[35] R. D. Woods, D. S. (1954), Saxon, Diffuse surface optical model for nucleon-nuclei scat-tering, *Phys. Rev.* 95 577-578.

[36] Guo, J. Y., & Sheng, Z. Q. (2005). Solution of the Dirac equation for the Woods–Saxon potential with spin and pseudospin symmetry. *Physics Letters A, 338*(2), 90-96.

[37] Arda, A., Aydoğdu, O., & Sever, R. (2010). Scattering of the Woods–Saxon potential in the Schrödinger equation. *Journal of Physics A: Mathematical and Theoretical*, *43*(42), 425204.

[38] Cüneyt Berkdemir, Ayse Berkdemir, Ramazan Sever. (2005). Polynomial solutions of the Schrödinger equation for The generalized Woods-Saxon potential, Phys. *Rev. C 72, Issue 2,* 027001.

[39] Ikhdair, S. M., & Sever, R. (2007). Exact solution of the Klein-Gordon equation for the PT symmetric generalized Woods-Saxon potential by the Nikiforov-Uvarov method. *Annalen der Physik*, *16*(3), 218-232.

[40] ALTUĞ, A., & Sever, R. (2009). approximate ℓ-state solutions to the klein–gordon equation for modified woods–saxon potential with position dependent mass. *International Journal of Modern Physics A*, *24*(20n21), 3985-3994.

[41] Ikhdair, S. M., & Sever, R. (2008). Solutions of Dirac Equation for Symmetric Generalized Woods-Saxon Potential by the Hypergeometric Method. *arXiv preprint arXiv:0808.1002*.

[42] Ikhdair, S. M., & Sever, R. (2010). Approximate analytical solutions of the generalized Woods-Saxon potentials including the spin-orbit coupling term and spin symmetry. *Central European Journal of Physics*, *8*(4), 652-666.

[43] Schrödinger, E. (1940, January). Further studies on solving eigenvalue problems by factorization. In *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences* (Vol. 46, pp. 183-206). Royal Irish Academy.

[44] Steeb W.H., Continuoud Symmetries, Lie Algebras.(2007) Differential equations and Computer Algebra(2nd Edition),*World Scientific Publising* Co. Pte.Ltd.

[45] Chang-Ying, C., Zhong-Zhou, R., & Guo-Xing, J. (2008). Exact solutions to three-dimensional Schrödinger equation with an exponentially position-dependent mass. *Communications in Theoretical Physics*, *43*(6), 1019.

[46] Valance, A., Morgan, T. J., & Bergeron, H. (1990). Eigensolution of the Coulomb Hamiltonian via supersymmetry. *American Journal of Physics*, *58*, 487-491.

[47] E. Schrödinger, (1940). A Method of Determining Quantum-Mechanical Eigenvalues abd Eigenfunctions, *Proc. R. Irish Acad. A*, 46, 9. E. Schrödinger, (1941). Further Studies on Solving Eigenvalue Problems by Factorization, *Proc. R. Irish Acad. A*, 46, 183.

[48] Steeb W.H, Continuoud Symmetries, Lie Algebras.(2007) Differential equations and Computer Algebra(2nd Edition),*World Scientific Publishing* Co. Pte.Ltd.

[49] Nikiforov, A. D. F., & Uvarov, V. B. (1988). *Special functions of mathematical physics*. BaselBoston: Birkhäuser.

[50] Valance, A., Morgan, T. J., & Bergeron, H. (1990). Eigensolution of the Coulomb Hamiltonian via supersymmetry. *American Journal of Physics*, *58*, 487-491.

[51] Ciftci, H., Hall, R. L., & Saad, N. (2003). Asymptotic iteration method for eigenvalue problems. *Journal of Physics A: Mathematical and General*, *36*(47), 11807.

[52] F., Durmus, A., & Boztosun, I. (2006). Exact analytical solution to the relativistic -Gordon equation with non-central equal scalar and vector potentials. *arXiv int quant-ph/0606224*.

[53] Saad, N., Hall, R. L., & Ciftci, H. (2008). The Klein-Gordon equation with the Kratzer potential in d dimensions. *Central European Journal of Physics*, *6*(3), 717-729.

[54] Taşkın, F., Boztosun, I., & Bayrak, O. (2008). Exact solutions of Klein-Gordon equation with exponential scalar and vector potentials. *International Journal of Theoretical Physics*, *47*(6), 1612-1617.

[55] Falaye, B. J., Hamzavi, M., & Ikhdair, S. M. (2012). Approximate bound state solutions of the deformed Woods-Saxon potential using asymptotic iteration method. *arXiv preprint arXiv:1207.1218*.

[56] Bayrak, O., Soylu, A., & Boztosun, I. (2010). The relativistic treatment of spin-0 particles under the rotating Morse oscillator. *Journal of Mathematical Physics*,*51*, 112301.

[57] R. Koc, O. Ozer, H. Tutunculer(2008). Asymptotic iteration method applied to bound-state problems with unbroken and broken super symmetry, Chin. J. Phys., 46,1-13.

[58] W. Strauss(1989). Nonlinear Wave Equations, CBMS Vol. 73, *American Math. Soc*.

[59] Diao, Y. F., Yi, L. Z., & Jia, C. S. (2004). Bound states of the Klein–Gordon equation with vector and scalar five-parameter exponential-type potentials.*Physics Letters A*, *332*(3), 157-167.

[60] McMahon, D. (2008). *Quantum Field Theory Demystified* (Vol. 96). McGraw-Hill.

[61] Ciftci, H., Hall, R. L., & Saad, N. (2003). Asymptotic iteration method for eigenvalue problems. *Journal of Physics A: Mathematical and General*, *36*(47), 11807.

[62] Olğar Eser (2009). An alternative method for calculating bound-state of energy eigenvalues of Klein-Gordon for quasi-exactly solvable potentials, *Communications Theoretical Physics (Beijing,China*), 26, 20302-20305.

[63] Ciftci, H., Hall, R. L., & Saad, N. (2005). Construction of exact solutions to eigenvalue problems by the asymptotic iteration method. *Journal of Physics A: Mathematical and General*, *38*(5), 1147.

[64] Francisco M Fernandez (2004). On an Iteration method for eigenvalue problems, *Journal of Physics A(Mathematical and General),* 37, 6173-6180.

[65] T. Barakat (2006). The asymptotic iteration method for eigenenergies of the Schrödinger equation with the potential $\mathrm{V(r) = -Z/r + g\,r + \lambda r^2}$,*Journal of Physics A(Mathematical and General),* 39, 823-831.

[66] Hakan Ciftci, Richard L. Hall and Nasser Saad (2003). Asymptotic iteration method for eigenvalue problem, *J. Phys. A*, 36, 11807-11816.

[67] Francisco M Fernandez (2004). On an Iteration method for eigenvalue problems, *Journal of Physics A(Mathematical and General)*, 37, 6173-6180.

[68] Barakat, T., Abodayeh, K. and Mukheimer, A. (2005a). The asymptotic iteration method for the angular spheroidal eigenvalues. *Journal of Physics*, 38, 1299-1304.

[69] Barakat, T. (2005b). The asymptotic iteration method for the eigenenergies of the anharmonic oscillator potential $V(x) = Ax^{2\alpha} + Bx^2$. *Physics Letters*, 344, 411-417.

[70] Ciftci, H., Hall, R. L., & Saad, N. (2005). Construction of exact solutions to eigenvalue problems by the asymptotic iteration method. *Journal of Physics A: Mathematical and General*, *38*(5), 1147.

[71] Dai, T. Q. (2011). Bound state solutions of the s-wave Klein-Gordon equation with position dependent mass for exponential potential. *J. At. Mol. Sci. doi, 10*.