AUGUST 2013 M.Sc. in Mechanical Engineering MEHMET ERKAN KÜTÜK

UNIVERSITY OF GAZİANTEP

GRADUATE SCHOOL OF

NATURAL & APPLIED SCIENCES

HYBRID MACHINE SYSTEMS: ANALYSIS AND CONTROL

M. Sc. THESIS

IN

MECHANICAL ENGINEERING

BY

MEHMET ERKAN KÜTÜK

AUGUST 2013

Hybrid Machine Systems:

Analysis and Control

M. Sc. Thesis

in

Mechanical Engineering

University of Gaziantep

Supervisor

Prof. Dr. Lale Canan DÜLGER

By

Mehmet Erkan KÜTÜK

August 2013

REPUBLIC OF TURKEY
UNIVERSITY OF GAZİANTEP
GRADUATE SCHOOL OF NATURAL & APPLIED SCIENCES
MECHANICAL ENGINEERING

Name of the thesis: Hybrid Machine Systems: Analysis and Control

Name of student : Mehmet Erkan KÜTÜK

Exam Date : 02.08.2013

Approval of the Graduate School of Natural and Applied Sciences

Assoc. Prof. Dr. Metin BEDİR
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. M. Sait SÖYLEMEZ
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Prof. Dr. L. Canan DÜLGER
Supervisor

Examining Commitee Members	Signature
Prof. Dr. Sedat BAYSEÇ	....
Prof. Dr. Sadettin KAPUCU	....
Prof. Dr. İbrahim H. GÜZELBEY	....
Prof. Dr. Mehmet TOPALBEKİROĞLU	....
Prof. Dr. L. Canan DÜLGER	....

I hereby that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Mehmet Erkan KÜTÜK

Signature

ABSTRACT

HYBRID MACHINE SYSTEMS: ANALYSIS AND CONTROL

KÜTÜK, Mehmet Erkan

M.Sc. in Mechanical Engineering

Supervisor: Prof. Dr. Lale Canan DÜLGER

August 2013, 100 pages

The main principle in hybrid driven systems is to use the conventional motors with a flywheel and the servo motors via a mechanical linkage mechanism. The hybrid systems include the advantages of the both driving systems. In the hybrid systems, a big constant velocity motor and a small servo motor are combined and non-uniform motion outputs are obtained with flexibility and decreased power requirement. A seven bar mechanism with two degree of freedom is chosen as the configuration of the system. Kinematics analysis of the hybrid driven mechanism is given as forward and inverse kinematics analysis. Loop closure equations are used to analyze them. Motion design procedures are applied with motion curve examples. Two different ram motions are designed. Fifth order polynomials are used to obtain characteristics of the motions. Curve Fitting Toolbox in Matlab® is applied as an auxiliary tool to perform motion design. A dynamic model is derived to design the control system and analyze the performance of the hybrid system. Dynamic analysis of the system is studied by two ways; Lagrangian and Kineto-static. The system equations are derived by using the energy expressions & their partial and time derivatives with respect to the defined generalized coordinates. Actuator dynamics for both axes are included with a PID control algorithm. The system simulation is performed with an explicit method; the fourth order Runge-Kutta method as an integration technique to get an approximate solution.

Key Words; hybrid driven systems, motion design, mathematical modeling, kinematic & dynamic analysis, PID control

ÖZET

HİBRİD MAKİNE SİSTEMLERİ: ANALİZİ VE DENETİMİ

KÜTÜK, Mehmet Erkan

Yüksek Lisans, Makine Müh. Bölümü

Tez Yöneticisi: Prof. Dr. Lale Canan DÜLGER

Ağustos 2013, 100 sayfa

Hibrid sürücülü sistemlerdeki temel ilke volanlı geleneksel motorların servo motorlarla birlikte bir mekanik mekanizma yardımıyla kullanılmasıdır. Hibrid sistemler her iki sürücü sisteminin avantajlarını da içermektedir. Hibrid sistemlerde büyük bir sabit hızlı motor ile küçük bir servo motor birleştirilir ve düzgün olmayan hareket çıktıları esneklik ve azaltılmış güç gereksinimi gibi özelliklerle elde edilir. İki serbestlik derecesine sahip bir yedi çubuk mekanizması sistem yapılanması olarak seçilmiştir. Sistemin kinematik analizi ters ve düz kinematik olarak sunulmuştur. Devre kapalılık denklemlerinden faydalanılmıştır. Hareket tasarım prosedürleri hareket eğrisi örnekleri ile uygulanmıştır. İki farklı koç hareketi tasarlanmıştır. Hareket özelliklerinin elde edilmesinde beşinci dereceden polinomlar kullanılmıştır. Hareket tasarımını gerçekleştirilmesinde Curve Fittting Toolbox Matlab® yardımcı bir araç olarak kullanılmıştır. Denetim sistem tasarımı ve hibrid sistem performansının analizi için bir dinamik model türetilmiştir. Sistem denklemleri enerji ifadeleri ve bu ifadelerin zamana ve genelleştirilmiş koordinatlara göre kısmi türevlerinin kullanılmasıyla elde edilmiştir. Dinamik analizde Lagrange ve Kinetostatic yöntemleri üzerinde çalışılmıştır. PID denetim algoritması her iki eksendeki sürücü dinamik denklemlerine dahil edilmiştir. Sistem benzetimi yaklaşık bir çözüm elde etmek için kullanılan bir entegrasyon tekniği olan 4. dereceden Runge – Kutta tekniğinden faydalanılanarak gerçekleştirilmiştir.

Anahtar Kelimeler: hibrid sürücülü sistemler, hareket tasarımı, matematiksel modelleme, kinematik & dinamik analiz, PID denetim.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor Prof. Dr. Lale Canan DÜLGER for the continuous support of my M.Sc. study and research, for her patience, motivation, enthusiasm, and immense knowledge. Her guidance helped me in all the time of research and writing of this thesis.

I would like to thank to Prof. Dr. Sadettin KAPUCU for his valuable comments and sharing his knowledge.

The deepest gratitude is to my parents, for their endless support and interest.

This thesis study is supported by Ministry of Science, Industry and Technology with the project number '01422.STZ.2012-1'.

TABLE OF CONTENTS

LIST OF TABLES

Page

LIST OF FIGURES

Page

CHAPTER 1

INTRODUCTION

1.1 Traditional Machines & Programmable Machines

In industrial applications, variable motion outputs desired from systems can be obtained with two different techniques; traditional machines and programmable machines.

Power is generally obtained from a constant velocity motor in traditional machines. Power transmission is carried out with gears, belts or a chain to get uniform motion outputs. The non-uniform motion outputs are achieved with cam or linkage mechanisms. The main idea in designing traditional machines is to keep the shape, size of the product and work done by the machine constant. When it is desired to make small modification for the machine function, extreme alterations are essential. They permit a limited flexibility at the output. However, energy conversion demanded from the system in traditional machines can be realized so efficiently. It is possible to reach high speeds in the systems which are well-designed and balanced.

Linkage or cam mechanisms are used to get non-uniform motions. Linkages are more favoured than the cams. They are more advantageous with the properties of high cycling speed, robustness and good dynamic performance. Number and dimensional synthesis are vital in designing a linkage. A motor and a flywheel have traditionally been used in order to obtain a fixed non-uniform motion characteristic from a linkage mechanism. In these systems, energy conservation, fast operational speed and smooth motor torques can be accepted as good specifications offered.

The programmable machines driven by servo motors offer so many variable motion outputs with the characteristics of reliability, ease to design and programmability. A designed and already used machine can be adapted to a new product. It is feasible to produce non-uniform motions directly with the availability of controllable and programmable DC and AC servo motors. This makes the mechanism a flexible motion provider. But when it is compared with traditional machines in terms of energy conversion, it is not as efficient as traditional machines. High torque requirement may force the performance limits of the servomotor. They require much more investment than traditional ones. Especially, in condition of higher capacities are needed, the cost of programmable systems is often very high. Traditional machines are cheaper and more rigid than programmable machines [1].

1.2 Hybrid Machine Systems

Hybrid machine idea is proposed to combine two techniques and to utilize from their good specifications and to remove their disadvantages. The purpose is to decrease torque/power requirement in programmable system, to make use of the energy in the system efficiently, to obtain a more flexible system having more than one degree of freedom. A hybrid machine is such a machine where its drive system integrates two types of motor; servo and constant velocity [1, 2].

The basic principle of hybrid systems is to bring together the motion of a large constant velocity motor with a small servo motor via a mechanical linkage mechanism. In these systems the constant velocity motor provides the main torque and motion requirements while the servo motor assists the modulations on the present motion. The CV motor undertakes a big amount of workload and the servo motor is like a real time regulator to change the task. Hybrid systems can be classified as hybrid actuator and hybrid mechanism. An intelligent box having two inputs and one output is considered as the *hybrid actuator*. The box is driven by a servo motor and a constant velocity motor and it provides a programmable output which may be coupled to a mechanism or a load. However, a *hybrid mechanism* is one that has at least two degrees of freedom. The mechanism has two input links which are coupled to a servo motor and a constant velocity motor [3].

The realistic benefits in using a hybrid design to obtain non-uniform motions are seen as the decrease in servo motor torque and power requirements compared to a direct servo motor drive. In the events where large loads have to be acted in direct servo drive, it is necessary to use large motor with a gearbox. Decreasing the torque and power necessity would give a remarkable cost saving in applications. Also servo motors are not particularly efficient in dissipating energy especially during deceleration periods. Hybrid machines have high flexibility at reduced cost.

One of the features of intelligent production machines is its application flexibility in a real time or offline programming. All they need is a transforming mechanism; lead screw, rack and pinion or a coupler and a slider to complement motion generation or to reach a required working position. Some motion types having extreme acceleration characteristics can bring severe torque demands for the motor. Extreme torque fluctuations can cause substantial heat in the motor windings. And in direct servo motor drive concept, some properties like range and speed of the output motions may be limited by certain motor features, such as motor rated torque peak power capability and bandwidth. In the direct generation of alternating types of non-uniform motions, these servo motors have to ensure the demanded current to get an accelerating and decelerating torque [4, 5, 6].

A hybrid machine is a typical mechatronic device that is a profitable requisite to generate smooth motion. The name '*hybrid actuation*' or '*hybrid machine*' comes from such architecture of the actuation system. Its main purpose is to form a powerful programmable drive system. So it is inevitable that Hybrid machine (HM) systems are employed in common industrial applications including pressing, cutting, printing and drawing. Different studies are seen in hybrid configurations with a five bar, slider crank, seven bar and nine bar for example. The system can be a robotic system or a machine system. Motion range can be changed and more complicated paths depending on the industrial applications may be produced.

1.3 The Scope of Research

The purpose of this thesis is to perform a study on hybrid machines including a period of two decades. All proposed configurations will be reviewed; previously used configurations will be studied. A configuration will be proposed with its mathematics available. A control system is essential to succeed design objectives. In order to design a preferable control system and analyze the performance of a hybrid machine, a dynamic model is necessary. Lagrangian approach can be applied for this purpose. A dynamic model plays a very important role in the terms of system validation, analysis, and control system design. A dynamic model will be prepared and control algorithm will be applied to show the controlled system is stable. Matlab ® will be used to get complete dynamic model of consideration. The influence of mechanism's input parameters on the output kinematics parameters will be investigated by simulation. Simulation results of the proposed configuration will be given as a design guide for its use of further applications.

A general overview is also necessary to summarize the studies done on the subject. The idea will be utilized considering alternative configurations with modeling. Preferably known facilities may be implemented for the solution depending on the problem type.

1.4 Organization of the Thesis

This study is composed of five chapters. In the first chapter, hybrid machine idea is explained and the scope of the thesis is given. In the second chapter, literature survey is given in detail. It is categorized as hybrid machine configuration, hybrid machine synthesis, modeling and control and hybrid machine applications. In addition to all, remarks on previous studies are stated. In the third chapter, kinematics analysis of the hybrid driven mechanism is given as forward and inverse kinematics analysis. The criteria in linkage synthesis are available in the third chapter. Sketch drawing and animation of the hybrid driven mechanism based on Matlab programming are given with examples. Motion design procedures are mentioned and motion curve examples are presented. In the fourth chapter, dynamic analysis of the system is studied by two ways; Lagrangian and kineto-static methods. Mathematical model for DC motor is presented. Control of the hybrid driven system is given. Simulation results of the controlled different motion types are presented. Kineto-static analysis for the hybrid driven system is given at the end of the fourth chapter. In the fifth chapter, conclusions are given. It is based on discussion on the present study and recommendations for the future works.

CHAPTER 2

LITERATURE SURVEY

2.1 Introduction

This chapter presents a review and summary on hybrid machines in general. Studies on the subject more than two decades are searched. They are given with remarks and things to be added for coming studies. It is composed of three main parts; hybrid machine configurations, hybrid machine synthesis, modelling and control and hybrid machine applications. They are given according to increasing order of years that the studies are performed.

2.2 A Review on Hybrid Machine Configurations

Formation of hybrid machine idea and development process of the proposed configurations, modifications, synthesis and design studies are presented as the following;

Tokuz and Jones (1992) have presented the first study on hybrid issue. A constant velocity motor and a servo motor were integrated by a differential gearbox which further drives a slider crank mechanism. It is shown in Figure 2.1. The study has shown that required power in the servo motor with hybrid configuration was smaller than with the configuration which has servo motor only. By inserting the servo motor into the system, flexibility was certainly obtained at the output motion (with different characteristics). Power demand was simply reduced by one third. But when the major changes were desired, the advantages in the initial case were not observed [1].

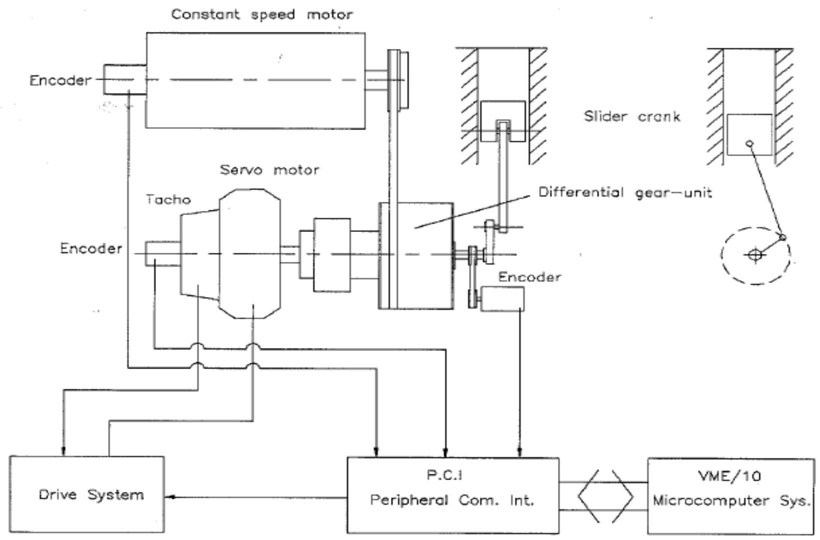


Figure 2.1. The First Hybrid Configuration [1]

Kireçci and Dülger (1995) have designed a hybrid manipulator with a 3 DOF system. This is illustrated in Figure 2.2. It is composed of DC motor, gear unit, 2 DC servo motor, servo amplifier, motion control card, a slider crank mechanism and a screw mechanism. The motions in X and Y planes are provided. Mathematical model and simulation of the designed hybrid manipulator are obtained. It is seen that power demand of the servo motor in hybrid system is less than the power requirement of servo motor in direct drive servo system [2].

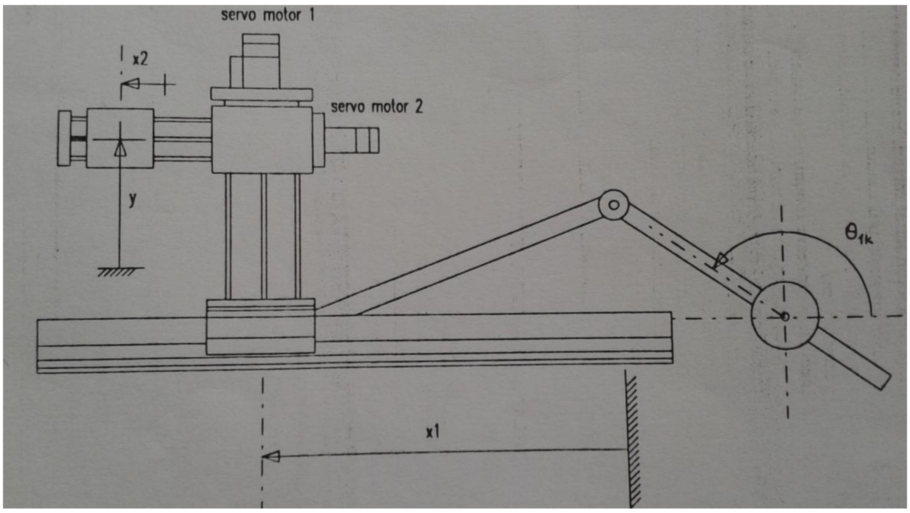


Figure 2.2. Hybrid Manipulator [2]

Kireççi and Dülger (1999) have then offered a configuration of a planar two degree of freedom, seven link mechanism. Grounded axis A is driven by a servo motor instead of a CV motor to get certain current values experimentally. The other axis remarked by E is driven by a second servo motor. The third axis G is used for the output of the mechanism that is directly connected to the load as shown in Figure 2.3. Proportional and derivative (PD) control is applied. The study has shown that the peak power requirement of servomotor was 3.5 times less than the peak power of CV motor [3].

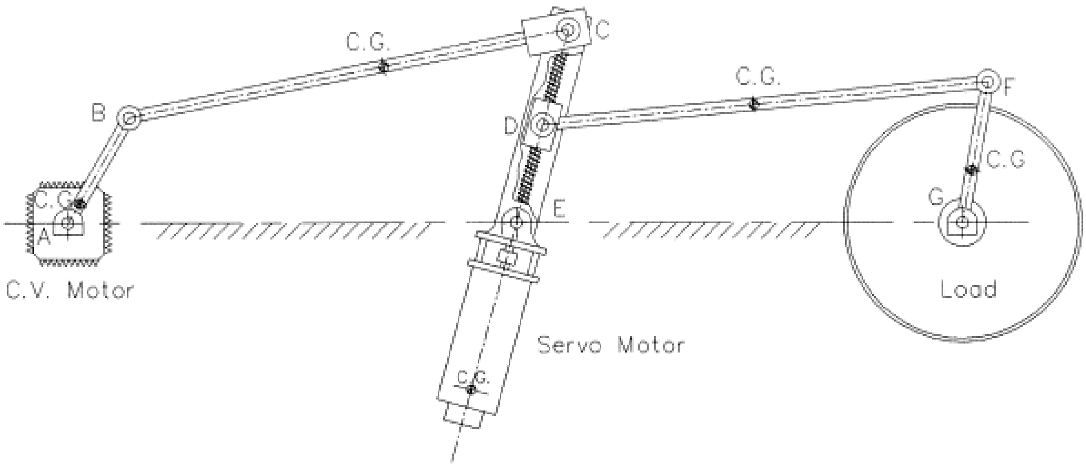
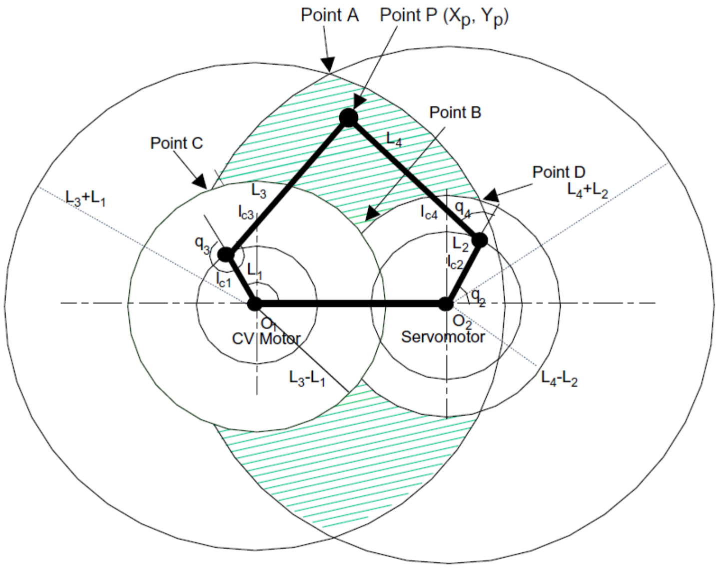


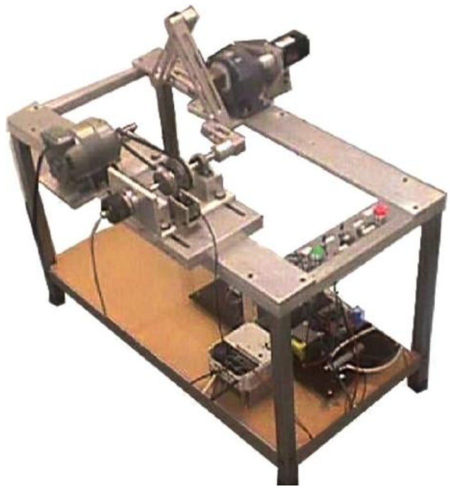
Figure 2.3. The Hybrid Actuator [3]

Seth (2004) has performed a review work about programmable hybrid mechanisms. In this paper, the studies completed were categorized into some groups; adjustable mechanisms, programmable mechanism and programmable hybrid mechanisms. Overall 25 studies were taken into consideration to form this study [4]. Development of programmable hybrid machines was studied with available conventional systems. Optimization issues were given with reduction in servo motor.

Ouyang et al. (2004) have proposed a five bar linkage consists of a five bar linkage, an AC CV motor and a frequency controller, an AC brushless servo motor and a servo amplifier with a gear transmission, a shift encoder, a flywheel and a belt. The system is shown in Figure 2.4 (a) and (b). The sliding mode control technique was applied for the hybrid machine. Modelling of mechanical system and driving system was discussed. A new control algorithm for the servomotor was developed. The speed fluctuation in the CV motor can be compensated with the aid of this new algorithm. Its stability analysis was also completed [5].



(a)



(b)

Figure 2.4. The Hybrid Five Bar Mechanism [5]

Yuan et al. (2005) have investigated two hybrid machines. Each of them is consisting of a seven link, two DOF linkage mechanisms. The configuration is shown in Figure 2.5. To ensure a range of output motion with the characteristics of minimized torque and power demands from servo motor, two different hybrid mechanisms are optimally designed. A multi objective, fuzzy based optimization method is used to achieve these purposes. The results obtained are compared with those for a direct servo motor drive system. Influences of altering inertias and masses of the links in the hybrid design were investigated (as the reduced model and full dynamics model) [6].

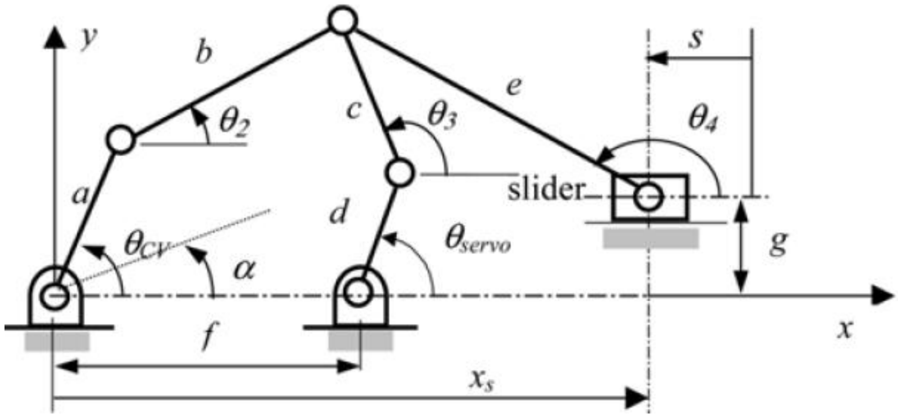


Figure 2.5. The Seven Bar Linkage [6]

Zhang (2006) has proposed a hybrid five bar mechanism. The dynamics analysis equations were derived by applying the principles of D'Alembert, virtual displacement and calculus of variations. Main and assist driving powers or reaction forces of the hybrid system were simply obtained. Dynamic design of the hybrid driven mechanism was presented in the study [7]. The configuration is illustrated in Figure 2.6.

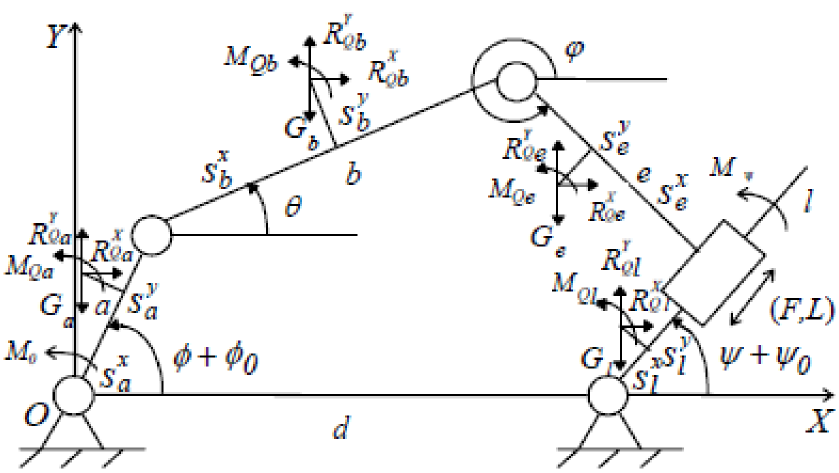


Figure 2.6. The Hybrid Five Bar Linkage [7]

Zhang (2008) has proposed a hybrid five bar mechanism. The method of kinematic and kinetic analysis of hybrid mechanism is set on power bond graph theory. According to this technique and model, dynamic equation was derived. In the mechanism, all joints except one were revolute joints. The exception was the prismatic joint for translational motion. Dimensional synthesis of the mechanism was carried out with PSO optimization method .At the end of the study, simulation results indicated that the proposed method was feasible for hybrid driven mechanism [8]. The configuration offered in Figure 2.6 was used in this study.

2.3 A Review on Hybrid Machine Synthesis, Modelling and Control

Optimal synthesis of possible configurations was carried out with the system control. In hybrid machine applications, control of the machine is carried out in two stages. The first one is the control of servo motor, the second one is the control of constant velocity motor to provide its actuation synchronously with the servomotor. In the studies completed, the proposed configurations are modelled mathematically and different PID control techniques; adaptive control, genetic algorithms (GA) and sliding mode control (SMC) techniques are applied on them. Some of the studies were documented as the following.

Greenough et al. (1995) were presented issues on the design of hybrid machines. A two DOF seven bar linkage was used. The motion design was carried out. The synthesis was completed in two parts. The first one was the kinematic synthesis. The second part was the kinetostatic synthesis. In this case, optimization was carried out with respect to inertial properties of the mechanism’s links [9].

Evolutionary algorithms were also applied on the synthesis of hybrid mechanisms. Connor et al. (1995) have presented a study on the synthesis of hybrid five bar path generating mechanisms using genetic algorithms [10].

Dulger et al. (2003) have presented a study on modeling and kinematic analysis of a hybrid actuator. A seven link mechanism with an adjustable crank has been used. A brushless DC servo motor with a gearbox was used to drive one of the inputs of the mechanism. The other one was driven by a permanent magnet DC servo motor and it was coupled to a lead screw mechanism. Equations of motion are derived by Lagrangian approach. Model is developed with PID controller. The simulation results are obtained [11]. The configuration is illustrated in Figure 2.7.

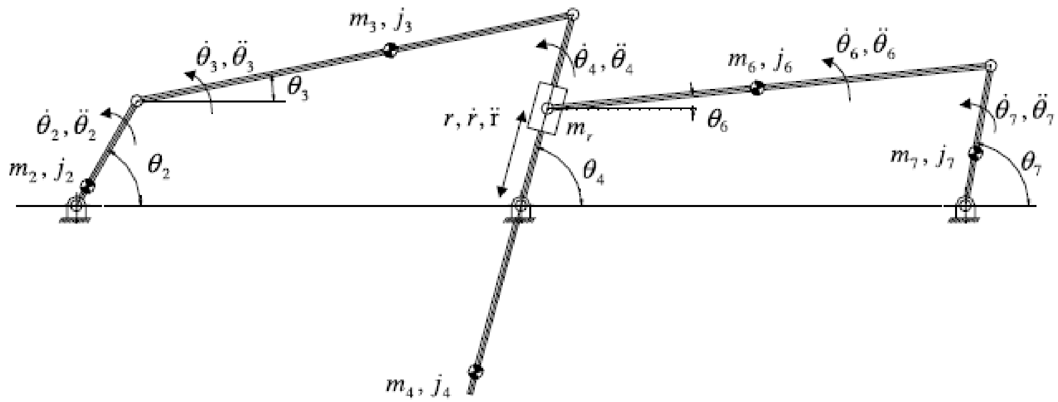


Figure 2.7. The Seven Link Hybrid Actuator [11]

Wu et al. (2005) have considered the CV motors and they have developed a control method. This new control method is solving the problem of disturbance on the trajectory of the end effector due to velocity fluctuation in the CV motor. In this study they proved that this fluctuation can be compensated by the controller for the servo motor. A five bar hybrid mechanism with 2 DOF is taken as an example in this study as shown in Figure 2.8.

At the end of the study, it was shown that proposed controller for the five bar hybrid mechanism with two DOF appeared as effective. In this case, the CV motor was not controllable while the servomotor was controllable. The inclusion of a flywheel on the shaft of the CV motor had a remarkably positive effect on the control performance of five bar hybrid machine system [12].

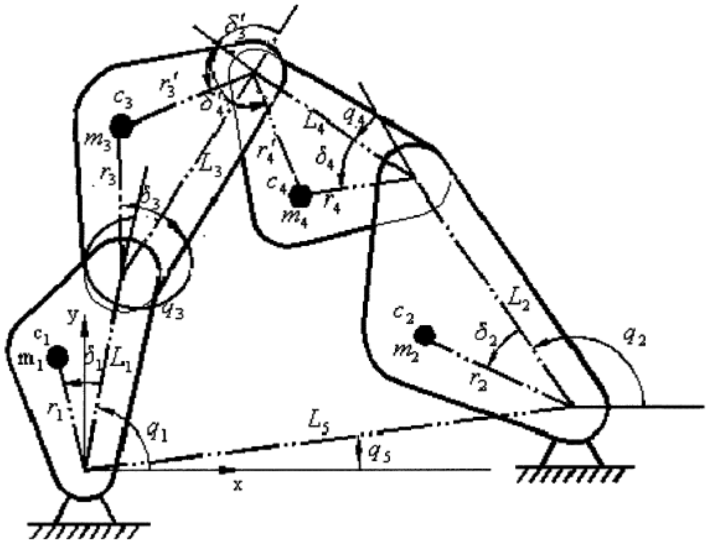


Figure 2.8. The Hybrid Five bar Mechanism [12]

Yu (2006) has offered a study composed of two parts. In the first part, a full dynamic model of an HM system with a five bar mechanism was provided. The Lagrangian formulations were based to model the system. In this model, a relation between Lagrangian formulation (including the inertial matrix, the vector of centripedal and Coriolis torque, and vector of gravitational torque) and linkages were pointed out. The second part was that two different control approaches which were computed torque control and sliding mode control are applied to control the HM system. This mechanism is shown in Figure 2.9. Simulation results have shown the effectiveness of the studied HM and control approaches [13].

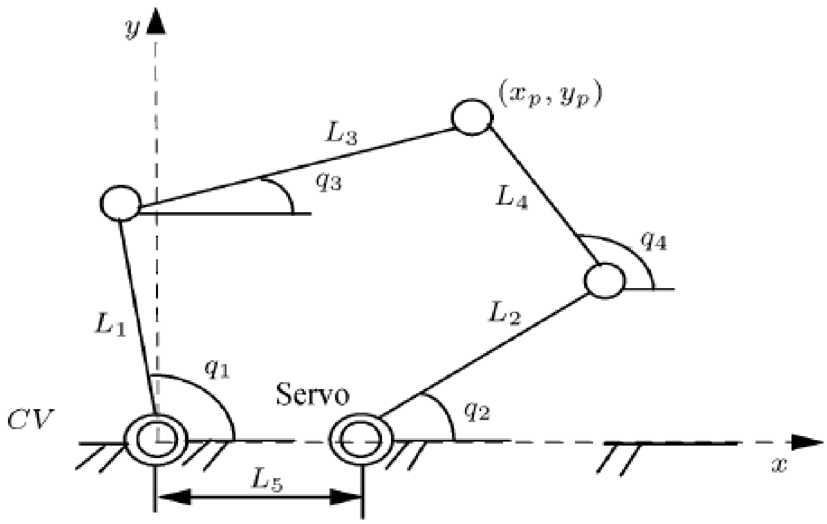


Figure 2.9. The Hybrid Mechanism [13]

Li et al. (2008) have studied on a hybrid driven mechanical press for precision drawing. The configuration of this press is a nine-bar linkage. It was a two degree of freedom system whose inputs were driven by a DC constant velocity motor and a DC servo motor. The dynamic model of the motors was developed. Lagrangian approach was used to develop the dynamic model of the hybrid driven press. Computer simulation was done by using fourth order Runge-Kutta method [14]. The mechanism is shown in Figure 2.10.

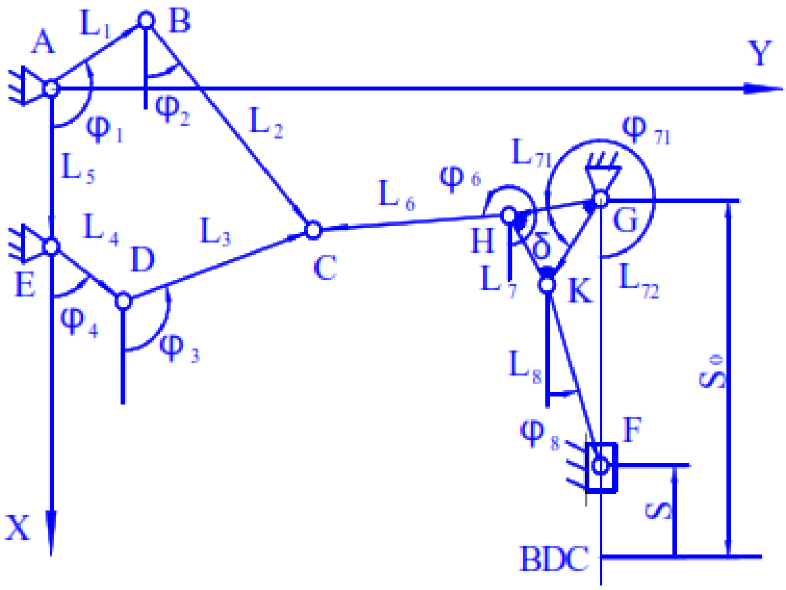


Figure 2.10. The Hybrid Nine-Bar Mechanism [14]

Chen et al. (2010) was proposed a new controller for the hybrid machine composed of one servo motor and one CV motor. Trajectory tracking at the end effector level was achieved. To show the effectiveness of the new controller, a comparison between new controller and controller developed previously was presented. In the new controller, the error which may be possible in the CV motor due to velocity fluctuation is taken into account. A five bar mechanism with 2 DOF shown in Figure 2.11 was used for the purpose. The simulation results have shown that there was an appreciable improvement in case of employing sliding mode controller and controlling at the end effector level [15].

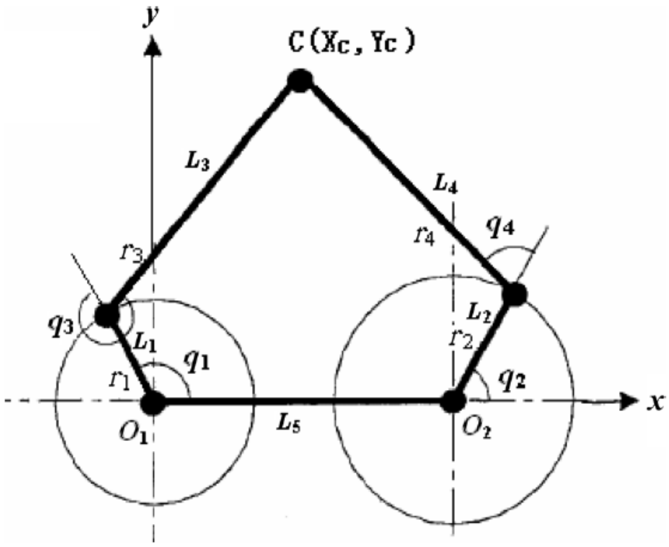


Figure 2.11. The Hybrid System Trajectory Tracking [15]

Li and Zhang (2010) have used a seven bar linkage configuration. This study was composed of two parts; the first part was kinematics analysis and optimum design of hybrid system, the second part was dynamic modelling and simulation. In the first part, forward kinematics and model were introduced. The link dimensions and servo motor trajectory were optimized. Three different motion profiles for hybrid design were shown with all graphs. In the second part, dynamic analysis and modelling of hybrid driven seven bar mechanism were presented. Dynamic model of this press was developed by Lagrangian equation. The configuration is shown in Figure 2.12 [16].

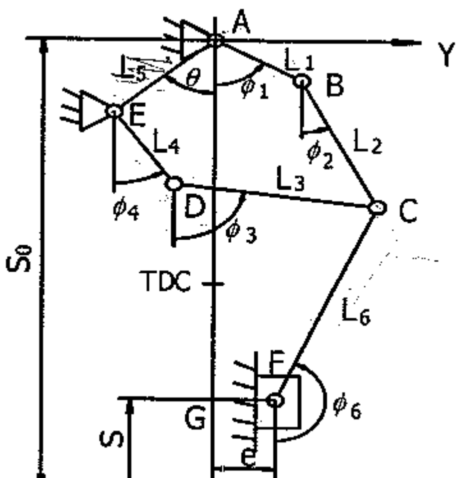


Figure 2.12. Seven Bar Linkage [16]

2.4 A Review on Hybrid Machine Applications

Hybrid machine systems are being used in cutting, stamping and pressing applications in industry. By changing the motion profiles, it is possible to obtain different trajectories and use in various applications in industry. Most of studies are taken on press application according to years of studies.

Tso and Liang (2002) were used a nine bar linkage for mechanical forming presses. A five bar linkage, a slider crank four bar linkage and a connecting link were the total parts of this press as shown in Figure 2.13. Firstly kinematic analysis has been completed for the motion of the nine bar linkage. Then a design procedure for the nine bar linkage was improved. An optimization was applied for the linkage and system synthesis [17].

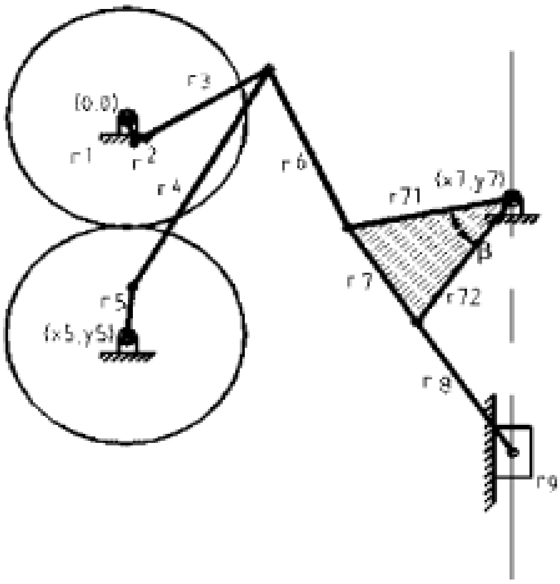


Figure 2.13. Nine Bar Linkage [17]

Du and Guo (2003) have studied on a metal forming press. Traditional configurations are investigated and a new hybrid configuration was proposed. Firstly, kinematic and kinetic properties of the design were proposed. Torque and power distribution between the motors, mechanical advantage and feasibility conditions were presented in the study. Genetic Algorithm was used in design optimization. The configuration is illustrated in Figure 2.14 [18].

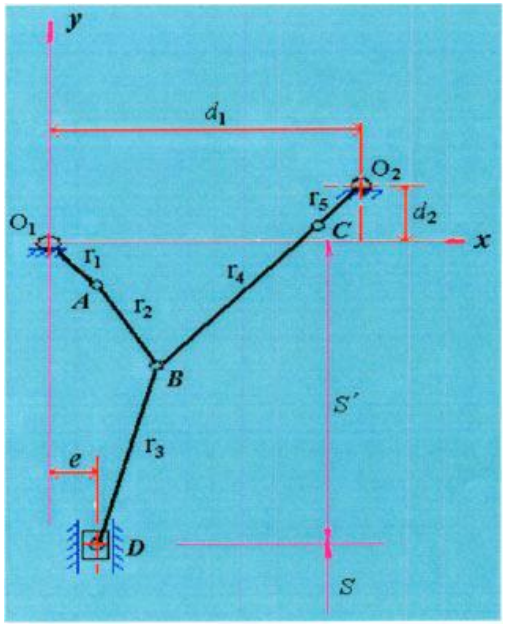


Figure 2.14. Press Mechanism [18]

Li et al. (2004) have presented hybrid driven nine bar press. In this study, firstly kinematic analysis of the linkage was completed. Secondly the model and optimization method were developed. In this configuration, while crank AB is driven by CV motor, crank ED was driven by the servo motor. The mechanism is shown in Figure 2.15. After this study, it is proved that the slider passes through a set of desired trajectory points for precision drawing application on condition that designing the input motion of the servo motor correctly [19].

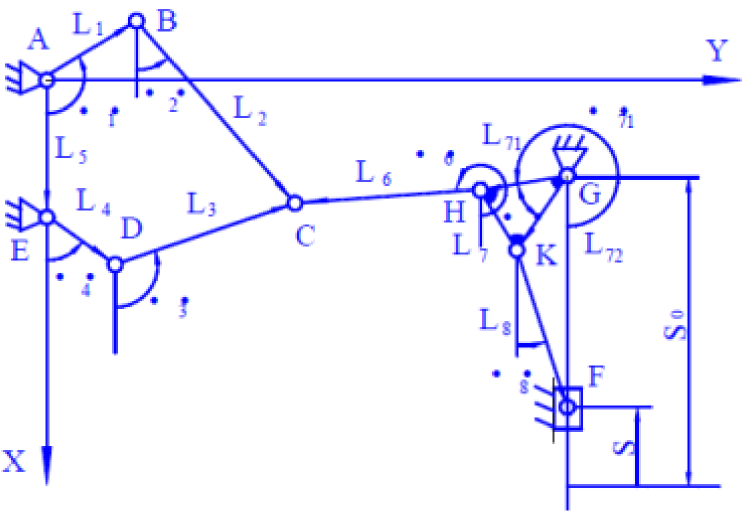


Figure 2.15. Hybrid Driven Nine Bar Press [19]

Meng et al. (2004) have offered a new press mechanism as shown in Figure 2.16. In this new mechanism with the combined input, the CV motor was driven the crank AB, the servomotor was driven the crank DE. Because of the fact that a single degree of freedom mechanism driven by a servo motor required high power need and so increased cost, a combined input mechanism has been thought as a solution for this situation. Link dimension optimization was completed and seen that there was a significant reduction in the peak value of the velocity and acceleration of the servo motor [20].

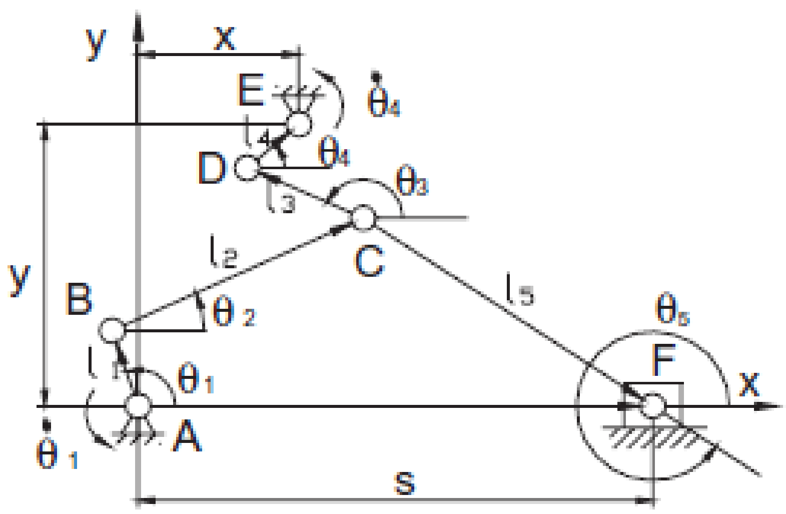


Figure 2.16. The Hybrid Mechanism [20]

Li and Tso (2008) have presented a seven bar mechanism in Figure 2.17. An iterative learning control technique was proposed for a hybrid driven press mechanism. A proportional-derivative (PD) type controller was used. The offered controller was improved and verified on a press prototype. At the end of the study, it was seen that there was a significant improvement on the punch position [21].

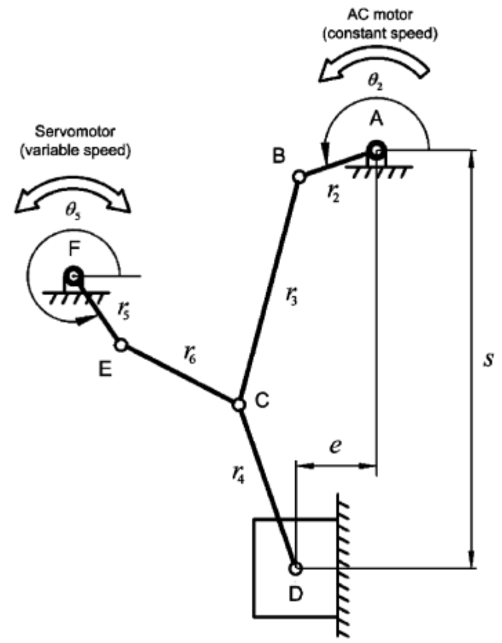


Figure 2.17. A Different Hybrid Mechanism [21]

Tso and Li (2008) have used a seven bar mechanism to investigate the stamping capacity and energy distribution between the servomotor and the flywheel. The configuration is shown in Figure 2.18. Different motion inputs were given and the capacity percentage plane was illustrated. The results have shown that the servo motor only required 12 % of the total stamping energy. The power of servo motor was remarkably saved in comparison with direct servo driven presses [22].

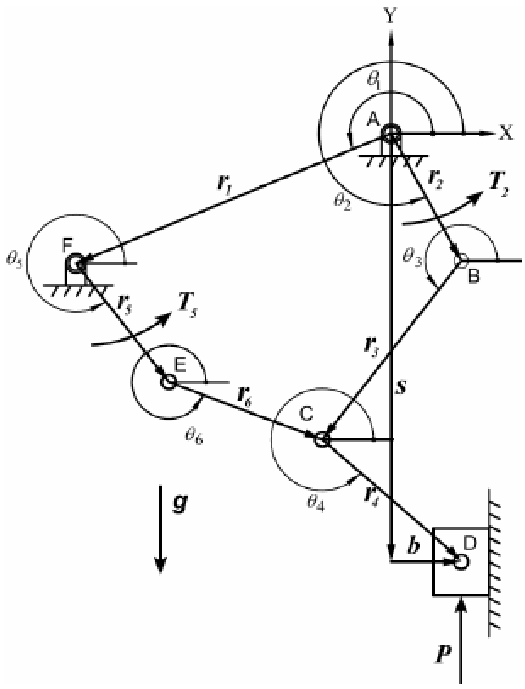


Figure 2.18. Hybrid Driven Servo Press [22]

Ouyang et al. (2008) have studied on a two degree of freedom hybrid actuation system. The configuration of the system was a five bar mechanism. A controllable servo motor and an uncontrollable constant velocity motor were used on the experimental prototype. A control system was developed to track the trajectory defined. Experimental results proved that the proposed control algorithm was effective [23]. The experimental is shown in Figure 2.19.

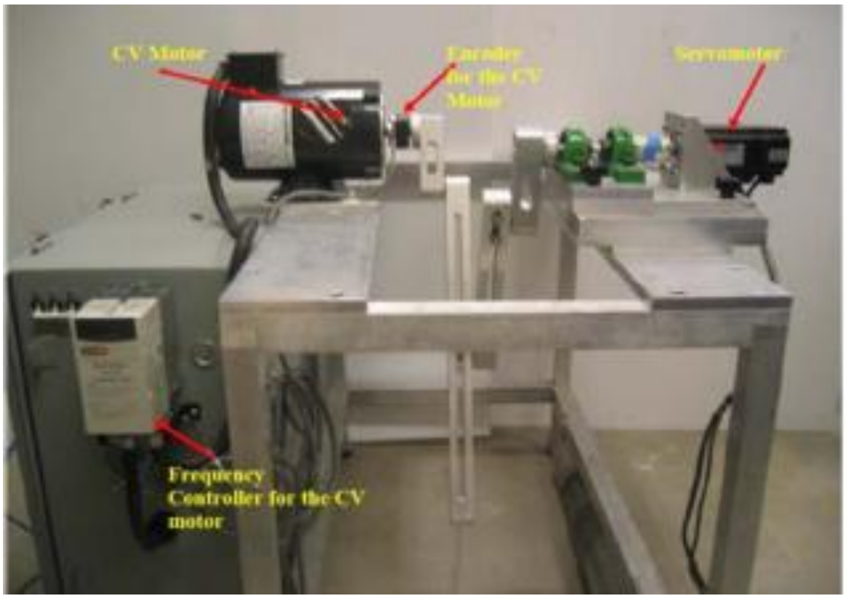


Figure 2.19. Experimental Prototype of Hybrid Five Bar Mechanism [23]

He et al. (2009) have studied on trajectory planning and optimization. In trajectory planning, cubic spline curves were used. PID control was applied. The mechanism proposed and showed in figure was applied on a 25 tons industrial prototype. In the mechanism shown, R_1 point was driven by a constant velocity motor and P_1 point was driven by a servo motor. The mechanism was a 2 DOF system shown in Figure 2.20. The first loop is a five bar linkage and composed of 4 revolute and 1 prismatic joint. There was also a connecting link and a slider in the configuration [24].

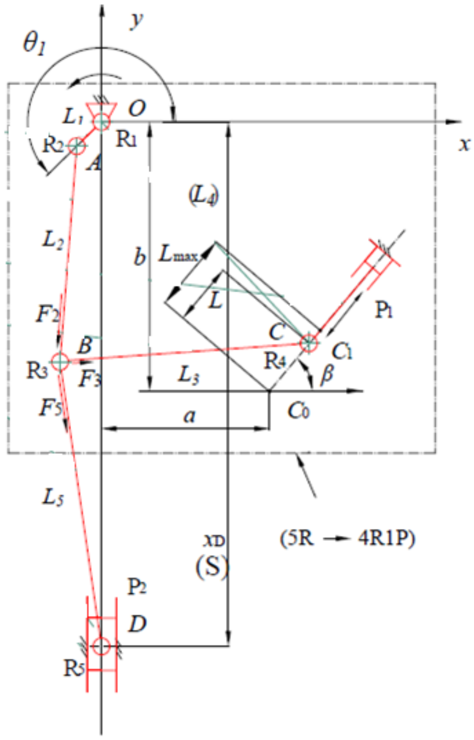


Figure 2.20. The Hybrid Press Configuration [24]

Tso (2010) has again used a seven bar mechanism like in Figure 2.21. A control system consisting of both iterative learning control and feedback control techniques was developed. All theoretical works were applied on a prototype. So influential results about stamping performance, forming capability, flexible ram speeds and energy efficiency were taken on the prototype [25].

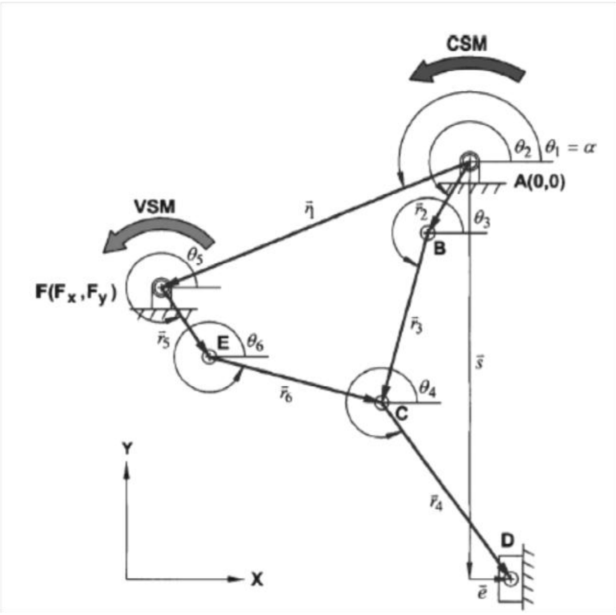


Figure 2.21. The Hybrid Driven Servo Press [25]

Li et al. (2010) have proposed a novel hybrid driven mechanical press to be used in deep drawing process. Its actuation system was composed of a CV motor and a servo motor. Adjustable motion output was obtained at the end of this study. Inverse kinematic analysis was carried out. In addition, dimension synthesis optimization by using GA was introduced in this study. The configuration is illustrated in Figure 2.22 [26].

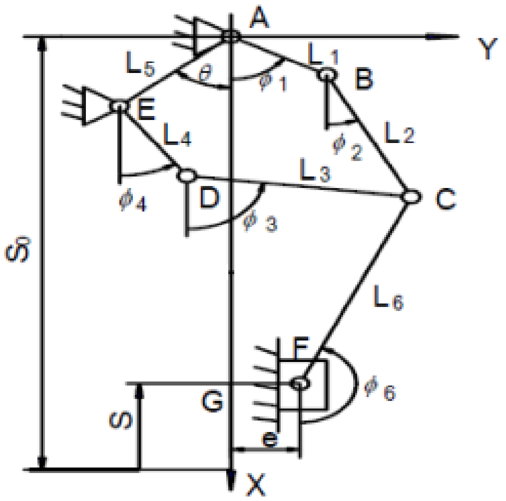


Figure 2.22. Sketch of the Hybrid Driven Press [26]

2.5. Remarks on Previous Studies

In the literature survey, the hybrid systems were examined in the last 20 years. The synthesis, analysis, controls, modelling and mathematical background were mentioned with the proposed configurations. The configurations were explained with mathematical model, simulations, controlling and dimensional synthesis. In the configurations presented, the power analysis can be carried out by utilizing the specifications of driving systems. Hybrid machine systems are currently ongoing studies. There are important points to focus on system and its details. The purpose is to decide a proper configuration for specific works and offer design specifications for the configuration given.

CHAPTER 3

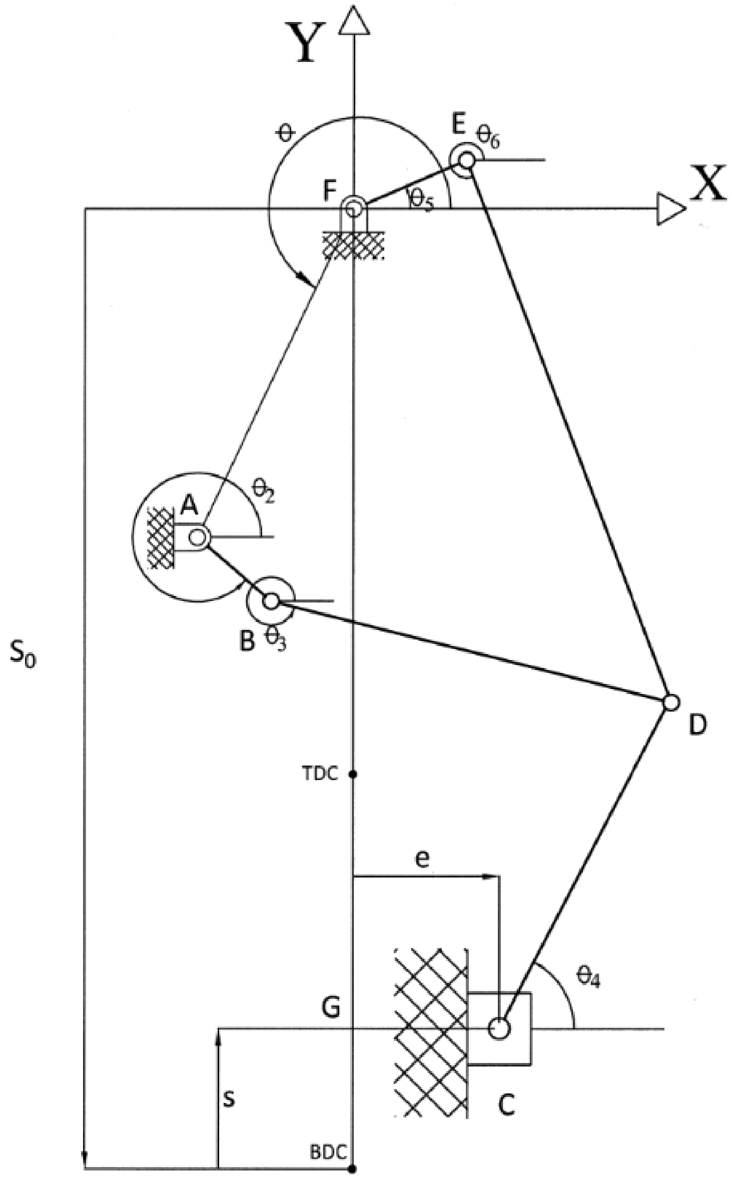
KINEMATICS ANALYSIS OF HYBRID DRIVEN SYSTEM AND MOTION DESIGN

3.1 Introduction

This chapter presents kinematics analysis and main configuration of a 2-DOF controllable hybrid driven mechanism. Many different configurations on hybrid systems are studied. Kinematic analysis and synthesis of mechanisms are performed in hybrid driven systems [16, 18, 20, 22, 26]. A seven bar configuration is taken here. The kinematics analysis includes inverse and forward analysis in detail. Linkage synthesis criterias are given. Motion design procedure is studied on hybrid systems [24]. The motion design is presented with different motion profile examples.

3.2 Description of the Hybrid Driven Mechanical System

A hybrid driven mechanical system is presented in Figure 3.1. The system is composed of a planar five bar mechanism ABDEF and a dyad of CD. All joints except one are revolute joints. The exception is a prismatic joint used in translational motion between the slider and ground. The lengths of the links are represented by r_1, r_2, r_3, r_4, r_5 and r_6 as shown in Figure 3.1. The crank r_5 is driven by a constant velocity motor. The crank r_2 is driven by a servo motor. The link r_1 is the ground link and θ is the orientation angle of it. The coordinate axes are fixed at point F. The BDC is the Bottom Dead Center point of the slider in which r_4, r_5 and r_6 are lined up in a straight line. The starting point of the slider is assumed as BDC. In addition to all, Top Dead Center of the slider is TDC point. In this mechanism, while EF link is driven by a constant velocity motor, AB link is driven by a servo motor.



$$\left|\overrightarrow{AF}\right| = r_1,\ \left|\overrightarrow{AB}\right| = r_2,\ \left|\overrightarrow{BD}\right| = r_3,\ \left|\overrightarrow{CD}\right| = r_4,\ \left|\overrightarrow{FE}\right| = r_5,\ \left|\overrightarrow{ED}\right| = r_6$$

Figure 3.1. Schematic Representation of Hybrid Driven Mechanical System

3.3 Kinematics Analysis of the Hybrid System

Here, the equations of inverse kinematics are derived and forward kinematics equations are obtained.

From the sketch of the mechanism given in Figure 3.1, two loop closure equations [27] are written as below.

Loop 1 is;

$$\vec{r_5} + \vec{r_6} = \overrightarrow{FG} + \overrightarrow{GC} + \vec{r_4} \quad (3.1a)$$

Loop 2 is;

$$\vec{r_5} + \vec{r_6} = \vec{r_1} + \vec{r_2} + \vec{r_3} \quad (3.1b)$$

These two equations can be written in Euler form [28]. Euler representation is composed of two parts. One of them is real part and the other one is imaginary part.

$$e^{i\theta} = \cos\theta + i\sin\theta \quad (3.2)$$

When the Equations of (3.1a) and (3.1b) are written in Euler form and separated into two parts, Loop 1 and Loop 2 can be written as below;

$$r_5 \sin\theta_5 + r_6 \sin\theta_6 = S_0 + s + r_4 \sin\theta_4 \quad (3.3a)$$

$$r_5 \cos\theta_5 + r_6 \cos\theta_6 = e + r_4 \cos\theta_4 \quad (3.3b)$$

$$r_5 \cos\theta_5 + r_6 \cos\theta_6 = X_A + r_2 \cos\theta_2 + r_3 \cos\theta_3 \quad (3.4a)$$

$$r_5 \sin\theta_5 + r_6 \sin\theta_6 = Y_A + r_2 \sin\theta_2 + r_3 \sin\theta_3 \quad (3.4b)$$

By differentiating Equations (3.3a, b) and (3.4a, b) with respect to time, the following equations are derived;

$$r_5 w_5 \cos\theta_5 + r_6 w_6 \cos\theta_6 = \dot{s} + r_4 w_4 \cos\theta_4 \quad (3.5a)$$

$$r_5 w_5 \sin\theta_5 + r_6 w_6 \sin\theta_6 = r_4 w_4 \sin\theta_4 \quad (3.5b)$$

$$r_5 w_5 \cos\theta_5 + r_6 w_6 \cos\theta_6 = r_2 w_2 \cos\theta_2 + r_3 w_3 \cos\theta_3 \quad (3.5c)$$

$$r_5 w_5 \sin\theta_5 + r_6 w_6 \sin\theta_6 = r_2 w_2 \sin\theta_2 + r_3 w_3 \sin\theta_3 \quad (3.5d)$$

By differentiating Equations (3.5a, b, c, d) with respect to time and rearranging them, Equations (3.6a, b, c, d) can be obtained;

$$r_5\alpha_5\cos\theta_5 - r_5w_5^2\sin\theta_5 - r_6w_6^2\sin\theta_6 + r_6\alpha_6\cos\theta_6 = \ddot{s} - r_4w_4^2\sin\theta_4 + r_4\alpha_4\cos\theta_4 \tag{3.6a}$$

$$r_5w_5^2\cos\theta_5 + r_5\alpha_5\sin\theta_5 + r_6\alpha_6\sin\theta_6 + r_6w_6^2\cos\theta_6 = r_4w_4^2\cos\theta_4 + r_4\alpha_4\sin\theta_4 \tag{3.6b}$$

$$r_5\alpha_5\cos\theta_5 - r_5w_5^2\sin\theta_5 + r_6\alpha_6\cos\theta_6 - r_6w_6^2\sin\theta_6 = r_2\alpha_2\cos\theta_2 - r_2w_2^2\sin\theta_2 + r_3\alpha_3\cos\theta_3 - r_3w_3^2\sin\theta_3 \tag{3.6c}$$

$$r_5\alpha_5\sin\theta_5 + r_5w_5^2\cos\theta_5 + r_6w_6^2\cos\theta_6 + r_6\alpha_6\sin\theta_6 = r_2\alpha_2\sin\theta_2 + r_2w_2^2\cos\theta_2 + r_3\alpha_3\sin\theta_3 + r_3w_3^2\cos\theta_3 \tag{3.6d}$$

where r_1, r_2, r_3, r_4, r_5, r_6, θ and e are the dimensional parameters of the hybrid driven mechanism as shown in Figure 3.1. X_A and Y_A are horizontal and vertical positions of point A, respectively. X_A and Y_A can be written as follows;

$$\begin{aligned} X_A &= r_1\cos\theta \\ Y_A &= r_1\sin\theta \end{aligned} \tag{3.7}$$

s is the displacement function of the slider required by different motion characteristics. S_0 is the point where r_5, r_6 and r_4 are lined up in a straight line and calculated as follows;

$$S_0 = \sqrt{(r_5 + r_6 + r_4)^2 - e^2} \tag{3.8}$$

3.3.1 Inverse Kinematics Analysis of Hybrid Driven Mechanism

In the inverse kinematics analysis, θ_5, w_5, α_5 and s, $\dot{s}$ and $\ddot{s}$ variables are the input values. The main purpose is to obtain the kinematics expressions of θ_2, w_2 and α_2 variables representing the angular position, velocity and acceleration of the servo motor.

3.3.1.1 Position Analysis

From the equations of Loop 1 and Loop 2 in Section 3.2., the angular displacement of the crank r_2 can be obtained in terms of the rotation angle θ_5 and the slider displacement s.

$$\theta_2 = f(\theta_5, s) \tag{3.9}$$

Loop 1 equations are transformed into the following ones;

$$r_4 \sin\theta_4 = r_5 \sin\theta_5 + r_6 \sin\theta_6 - S_0 - s \tag{3.10a}$$

$$r_4 \cos\theta_4 = r_5 \cos\theta_5 + r_6 \cos\theta_6 - e \tag{3.10b}$$

In order to eliminate θ_4 , the square of these two equations are added to each other and the following equation is resulted [20].

$$A\sin\theta_6 + B\cos\theta_6 = C \tag{3.11}$$

where

$$A = 2(S_0 + s)r_6 - 2r_5 r_6 \sin\theta_5 \tag{3.12a}$$

$$B = -2r_5 r_6 \cos\theta_5 + 2er_6 \tag{3.12b}$$

$$C = r_5^2 + r_6^2 + (S_0 + s)^2 + e^2 - r_4^2 - 2(S_0 + s)r_5 \sin\theta_5 - 2\cos\theta_5 r_5 e \tag{3.12c}$$

Taking $\beta = \tan(\theta_6 / 2)$, thus $\sin\theta_6 = \dfrac{2\beta}{(1+\beta^2)}$, $\cos\theta_6 = \dfrac{1-\beta^2}{1+\beta^2}$.

Then Equation (3.13) can be obtained.

$$\beta = \frac{A \pm \sqrt{(A^2 + B^2 - C^2)}}{(B + C)} \tag{3.13}$$

and $\beta = \tan(\theta_6 / 2)$, hence

$$\theta_6 = 2\tan^{-1}\left[\frac{A \pm \sqrt{(A^2 + B^2 - C^2)}}{(B+C)}\right] \tag{3.14}$$

From Equation (3.3b)

$$\theta_4 = \cos^{-1}\left[\frac{r_5 \cos\theta_5 + r_6 \cos\theta_6 - e}{r_4}\right] \tag{3.15}$$

Loop 2 equations are then transformed into the following equations;

$$r_3 \sin\theta_3 = r_5 \sin\theta_5 + r_6 \sin\theta_6 - Y_A - r_2 \sin\theta_2 \tag{3.16a}$$

$$r_3 \cos\theta_3 = r_5 \cos\theta_5 + r_6 \cos\theta_6 - X_A - r_2 \cos\theta_2 \tag{3.16b}$$

In order to eliminate θ_3 , the square of these two equations are added to each other and the following equation is resulted.

$$D\sin\theta_2 + E\cos\theta_2 = F \tag{3.17}$$

where

$$D = -2(2r_2 r_5 \sin\theta_5 - 2r_2 Y_A + 2r_2 r_6 \sin\theta_6) \tag{3.18a}$$

$$E = -2(2r_2 r_5 \cos\theta_5 + 2r_2 r_6 \cos\theta_6 - 2r_2 X_A) \tag{3.18b}$$

$$F = r_3^2 - r_5^2 - r_2^2 - r_6^2 - X_A{}^2 - Y_A{}^2 + 2Y_A r_6 \sin\theta_6 + 2Y_A r_5 \sin\theta_5 - 2r_5 r_6 \cos(\theta_5 - \theta_6) + 2X_A r_5 \cos\theta_5 + 2X_A r_6 \cos\theta_6 \tag{3.18c}$$

Similarly, θ_2 is then derived by using Equation (3.17)

$$\theta_2 = 2\tan^{-1}\left[\frac{D \pm \sqrt{(D^2 + E^2 - F^2)}}{E+F}\right] \tag{3.19}$$

The angular displacement of the third link can be found from Equation (3.4b);

$$\theta_3 = \sin^{-1}\left[\frac{(r_5 \sin\theta_5 + r_6 \sin\theta_6 - Y_A - r_2 \sin\theta_2)}{r_3}\right] \tag{3.20}$$

3.3.1.2 Velocity Analysis

The angular velocity of each link of the mechanism can be found by using Equations (3.5a, b, c, d) as follows;

$$w_6 = \frac{r_5 w_5 \sin(\theta_4 - \theta_5) - \dot{s}\sin\theta_4}{r_6 \sin(\theta_6 - \theta_4)} \tag{3.21a}$$

$$w_4 = \frac{r_5 w_5 \sin(\theta_5 - \theta_6) - \dot{s}\sin\theta_6}{r_4 \sin(\theta_4 - \theta_6)} \tag{3.21b}$$

$$w_2 = \frac{r_5 w_5 \sin(\theta_3 - \theta_5) + r_6 w_6 \sin(\theta_3 - \theta_6)}{r_2 \sin(\theta_3 - \theta_2)} \tag{3.21c}$$

$$w_3 = \frac{r_5 w_5 \sin(\theta_2 - \theta_5) + r_6 w_6 \sin(\theta_2 - \theta_6)}{r_3 \sin(\theta_2 - \theta_3)} \tag{3.21d}$$

3.3.1.3 Acceleration Analysis

The angular acceleration of each link of the hybrid driven mechanical system can be found by using Equations (3.6a, b, c, d) as follows

$$\alpha_4 = \frac{r_5 w_5^2 \cos(\theta_5 - \theta_6) - r_4 w_4^2 \cos(\theta_4 - \theta_6) + \ddot{s}\sin\theta_6 + r_6 w_6^2 + r_5 \alpha_5 \sin(\theta_5 - \theta_6)}{r_4 \sin(\theta_4 - \theta_6)} \tag{3.22a}$$

$$\alpha_6 = \frac{r_4 w_4^2 - \ddot{s}\sin\theta_4 - r_5 w_5^2 \cos(\theta_4 - \theta_5) - r_6 w_6^2 \cos(\theta_4 - \theta_6) + r_5 \alpha_5 \sin(\theta_4 - \theta_5)}{r_6 \sin(\theta_6 - \theta_4)} \tag{3.22b}$$

$$\alpha_2 = \frac{-r_5 w_5^2 \cos(\theta_5 - \theta_3) - r_6 w_6^2 \cos(\theta_3 - \theta_6) + r_6 \alpha_6 \sin(\theta_3 - \theta_6) + r_2 w_2^2 \cos(\theta_2 - \theta_3) + r_3 w_3^2 + r_5 \alpha_5 \sin(\theta_3 - \theta_5)}{r_2 \sin(\theta_3 - \theta_2)} \tag{3.22c}$$

$$\alpha_3 = \frac{-r_5 w_5^2 \cos(\theta_2 - \theta_5) - r_6 w_6^2 \cos(\theta_2 - \theta_6) + r_6 \alpha_6 \sin(\theta_2 - \theta_6) + r_2 w_2^2 + r_3 w_3^2 \cos(\theta_2 - \theta_3) + r_5 \alpha_5 \sin(\theta_2 - \theta_5)}{r_3 \sin(\theta_2 - \theta_3)}$$

(3.22d)

Therefore, according to Equations (3.21a, b, c, d) and (3.22a, b, c, d), the angular velocity w_2 and α_2 of the crank r_2 driven by servo motor can be written as the followings;

$$w_2 = f(\theta_5,\ w_5,\ s,\ \dot{s}) \quad (3.23a)$$

$$\alpha_2 = f(\theta_5,\ w_5,\ \alpha_5,\ s,\ \dot{s},\ \ddot{s}) \quad (3.23b)$$

where θ_5, w_5 and α_5 are the angular displacement, angular velocity and angular acceleration of link r_5 driven by a constant velocity motor. The displacement, velocity and acceleration of the slider are s, $\dot{s}$ and $\ddot{s}$, respectively.

3.3.2 Forward Kinematics Analysis of Hybrid Driven Mechanism

In the inverse kinematics, θ_5, w_5, α_5 and $s, \dot{s}$ and $\ddot{s}$ variables are the input variables. The main purpose is to get the kinematics expressions of θ_2, w_2 and α_2 variables. In forward kinematics, θ_5, w_5, α_5 and θ_2, w_2, α_2 variables are the input values. In this approach, the main purpose is to obtain kinematics characteristics of the slider link ($s, \dot{s}, \ddot{s}$).

3.3.2.1 Position Analysis

From the equations of Loop 1 and Loop 2, the displacement of the slider (s) can be obtained in terms of the rotation angle (θ_5) and the rotation angle (θ_2). Equations (3.4a, b) are arranged and the equations are written as;

$$r_6 \sin\theta_6 = Y_A + r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_5 \sin\theta_5 \quad (3.24a)$$

$$r_6 \cos\theta_6 = X_A + r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_5 \cos\theta_5 \quad (3.24b)$$

Taking the squares of both sides of two equations and adding them to each other, θ_6 can be eliminated and θ_3 is found. Similarly with the inverse kinematics performed above, an equation can be obtained as;

$$K \sin\theta_3 + L\cos\theta_3 = M \tag{3.25}$$

where

$$K = 2Y_A r_3 - 2r_3 r_5 \sin\theta_5 + 2r_2 r_3 \sin\theta_2 \tag{3.26a}$$

$$L = 2X_A r_3 + 2r_2 r_3 \cos\theta_2 - 2r_3 r_5 \cos\theta_5 \tag{3.26b}$$

$$\begin{aligned} M = {} & r_6^2 - X_A{}^2 - Y_A{}^2 - r_2^2 - r_3^2 - r_5^2 - 2Y_A r_2 \sin\theta_2 - 2X_A r_2 \cos\theta_2 + 2Y_A r_5 \sin\theta_5 + 2X_A r_5 \cos\theta_5 \\ & + 2r_2 r_5 \sin\theta_2 \sin\theta_5 + 2r_2 r_5 \cos\theta_2 \cos\theta_5 \end{aligned} \tag{3.26c}$$

So the angular displacement of third link can be given as;

$$\theta_3 = 2\tan^{-1}\left[\frac{K \pm \sqrt{(K^2 + L^2 - M^2)}}{L + M}\right] \tag{3.27}$$

Now θ_6, θ_4 and s can be found by using Loop1 and Loop 2 given in Equations (3.4b), (3.3b) and (3.3a), respectively;

$$\theta_6 = \sin^{-1}\left[\frac{(Y_A + r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_5 \sin\theta_5)}{r_6}\right] \tag{3.28a}$$

$$\theta_4 = \cos^{-1}\left[\frac{(r_5 \cos\theta_5 + r_6 \cos\theta_6 - e)}{r_4}\right] \tag{3.28b}$$

$$s = r_5 \sin\theta_5 + r_6 \sin\theta_6 - S_0 - r_4 \sin\theta_4 \tag{3.28c}$$

All angular and slider displacement expressions of the hybrid driven mechanism and are obtained.

3.3.2.2 Velocity Analysis

By applying some mathematical operations on Equations (3.5a, b, c, d), the equations of angular velocities of the links, w_6, w_4, w_3 and the slider velocity, $\dot{s}$ can be arranged as follows;

$$w_6 = \frac{r_2 w_2 \sin(\theta_3 - \theta_2) - r_5 w_5 \sin(\theta_3 - \theta_5)}{r_6 \sin(\theta_3 - \theta_6)} \tag{3.29a}$$

$$\dot{s} = \frac{r_5 w_5 \sin(\theta_4 - \theta_5) - r_6 w_6 \sin(\theta_6 - \theta_4)}{\sin\theta_4} \tag{3.29b}$$

$$w_4 = \frac{r_5 w_5 \sin(\theta_5 - \theta_6) + \dot{s}\sin\theta_6}{r_4 \sin(\theta_4 - \theta_6)} \tag{3.29c}$$

$$w_3 = \frac{r_5 w_5 \sin(\theta_2 - \theta_5) + r_6 w_6 \sin(\theta_2 - \theta_6)}{r_3 \sin(\theta_2 - \theta_3)} \tag{3.29d}$$

3.3.2.3 Acceleration Analysis

Similarly, by applying some mathematical operations on Equations (3.6), the equations of angular accelerations of the links and acceleration of the slider can be written as follows;

$$\alpha_6 = \frac{r_2\alpha_2 \sin(\theta_3 - \theta_2) + r_5 w_5^2 \cos(\theta_5 - \theta_3) + r_6 w_6^2 \cos(\theta_3 - \theta_6) - r_2 w_2^2 \cos(\theta_2 - \theta_3) - r_3 w_3^2 - r_5\alpha_5 \sin(\theta_3 - \theta_5)}{r_6 \sin(\theta_3 - \theta_6)} \tag{3.30a}$$

$$\ddot{s} = \frac{r_4 w_4^2 - r_6\alpha_6 \sin(\theta_6 - \theta_4) - r_5 w_5^2 \cos(\theta_4 - \theta_5) - r_6 w_6^2 \cos(\theta_4 - \theta_6) + r_5\alpha_5 \sin(\theta_4 - \theta_5)}{\sin\theta_4} \tag{3.30b}$$

$$\alpha_4 = \frac{r_5 w_5^2 \cos(\theta_5 - \theta_6) - r_4 w_4^2 \cos(\theta_4 - \theta_6) + \ddot{s}\sin\theta_6 + r_6 w_6^2 + r_5\alpha_5 \sin(\theta_5 - \theta_6)}{r_4 \sin(\theta_4 - \theta_6)} \tag{3.30c}$$

$$\alpha_3 = \frac{-r_5 w_5^2 \cos(\theta_2-\theta_5) - r_6 w_6^2 \cos(\theta_2-\theta_6) + r_6\alpha_6 \sin(\theta_2-\theta_6) + r_2 w_2^2 + r_3 w_3^2 \cos(\theta_2-\theta_3) + r_5\alpha_5 \sin(\theta_2-\theta_5)}{r_3 \sin(\theta_2-\theta_3)}$$

(3.30d)

At the end of all derivations for forward kinematics, it is seen that s, $\dot{s}$ and $\ddot{s}$ can be written as;

$$s = f(\theta_2, \theta_5) \tag{3.31a}$$

$$\dot{s} = f(\theta_5, w_5, \theta_2, w_2) \tag{3.31b}$$

$$\ddot{s} = f(\theta_5, w_5, \alpha_5, \theta_2, w_2, \alpha_2) \tag{3.31c}$$

3.4 Linkage Synthesis of Hybrid Driven Mechanism

In order to satisfy different motion characteristics of the slider, the input link r_2 and r_5 must make complete rotations. In a previous study, Li [19] has given the classifications of hybrid five-bar linkages by using the unrestraint double-crank type. The input link r_2 and r_5 are unrestraint cranks. Therefore, the inequality constraints are defined as follows:

$$r_1 + r_2 + r_5 < r_3 + r_6 \tag{3.32a}$$
$$r_2 + r_5 + r_6 < r_1 + r_3 \tag{3.32b}$$
$$r_2 + r_3 + r_5 < r_1 + r_6 \tag{3.32c}$$

By considering Figure 3.1, the following inequality constraints should be satisfied to provide slider mobility condition.

$$r_4 > X_D - e \tag{3.33}$$

where X_D is the coordinate of point *D* in x direction.

Link length condition; in order to have appropriate link lengths, we define the link lengths inequality constraints as follows:

$$r_{i\,\min} \le r_i \le r_{i\,\max} \qquad r_i = 1, 2, \ldots 6 \tag{3.34}$$

When these constraints are taken into account, the link dimensions of the hybrid driven mechanism are found and tabulated in Table 3.1.

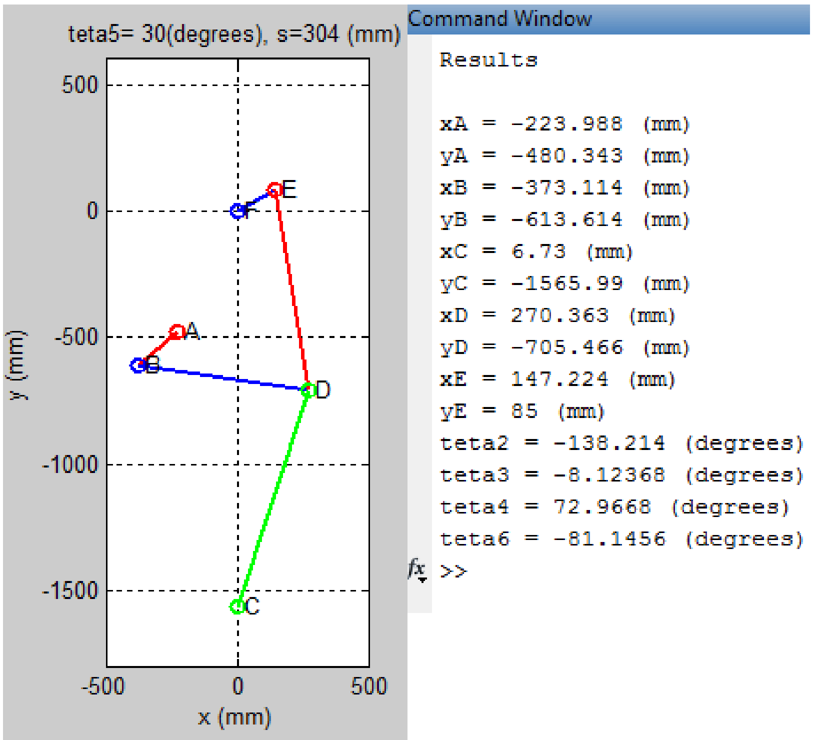
Table 3.1. Link Dimensions of the Hybrid Driven Mechanism

r_1	r_2	r_3	r_4	r_5	r_6	e	θ
530	200	650	900	170	800	6.73	245

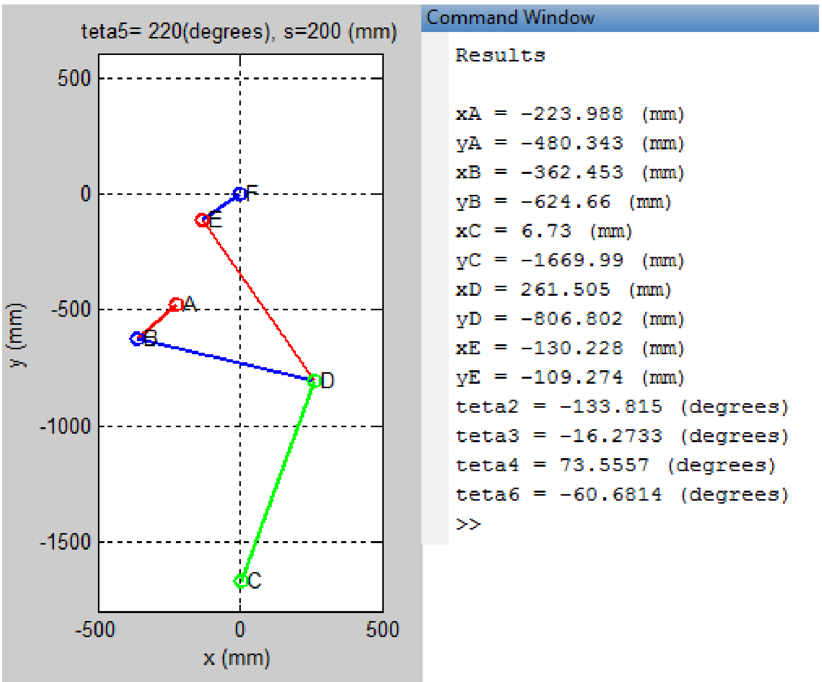
All dimensions are in mm. The orientation angle, θ, is in degrees.

3.5 Sketch Drawing and Creating Animation of the Hybrid Mechanism in Matlab®

A program based on inverse kinematics equations is written in Matlab® in order to see the simultaneous positions of the hybrid driven mechanism. So the position of the slider link and the angular position of the fifth link must be given to the program as the input values. Sketches of the hybrid system with two different values of θ_5 and s are illustrated in Figures 3.2a and 3.2b. In Figure 3.2a, $\theta_5 = 30^0$ and the slider displacement s=304 mm. In Figure 3.2b, $\theta_5 = 220^0$ and the slider displacement s=200 mm. The positions of the other points (A, B, C, D, E) in the Cartesian plane and orientations of the links are shown in the Figure 3.2.



(a)

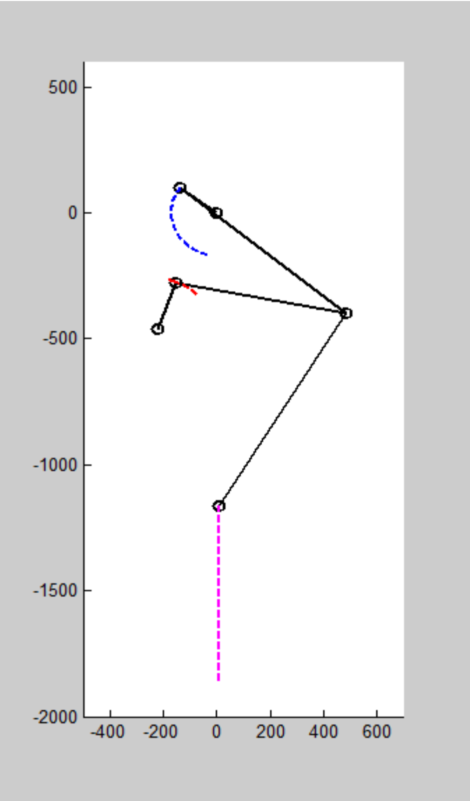


(b)

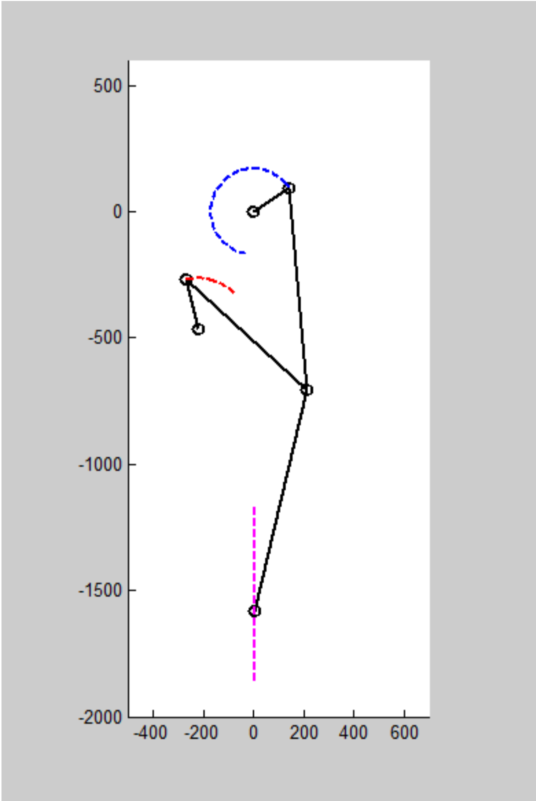
Figure 3.2 Sketching Examples of the Hybrid Driven Mechanism

When the kinematics datas of the slider link and the fifth link driven by the constant velocity motor are known, it is possible to reach the kinematics datas of servo motor link r_2 by using the inverse kinematics analysis. With the availability of the angular displacement datas of constant velocity motor and servo motor, an animation on hybrid driven mechanism motion can be created.

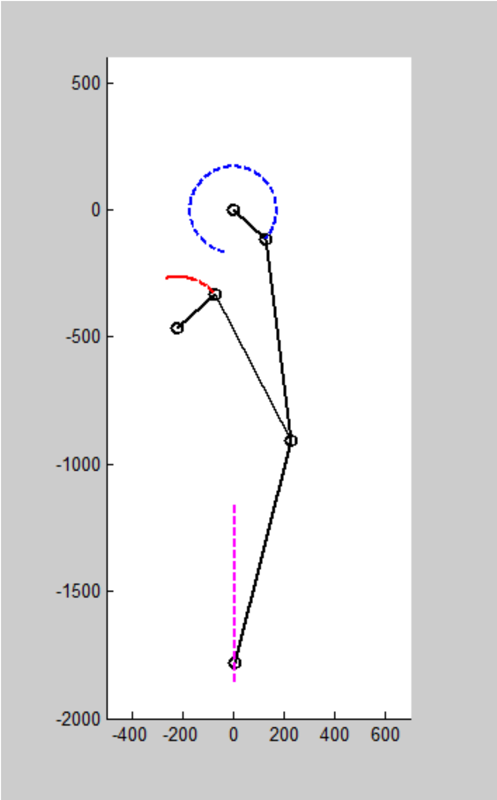
Sketches of the different moments of the motion are illustrated in Figure 3.3(a, b, c and d). In Figure 3.3a, the slider is at the Top Dead Center point. In Figure 3.3(b, c), the slider moves down. In Figure 3.3(d), the slider is at the Bottom Dead Center point. The traced circles of the tip points of the two cranks and the traced line of the slider link are shown with hidden lines.



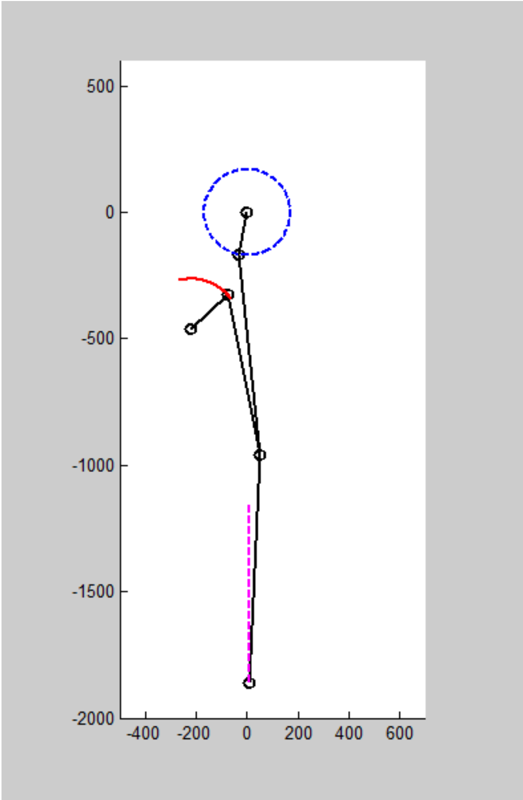
(a)



(b)



(c)



(d)

Figure 3.3. Different Sketches from the Animation Program

3.6 Motion Design and Motion Curve Examples

The motion design is explained with all details. The specifications of the designed motion types are comprehensively defined with the figures and tables [24].

3.6.1 Motion Design

In this study, two different motion types of slider link are formed. The first one is characterized by a quick rise and a slow return motion and the second one is slow return-dwell-quick rise motion. The motion design is composed of obtaining the position, velocity and acceleration of the slider link. After obtaining the datas of these parameters, the angular displacement θ_2 , the angular velocity w_2 and the angular acceleration α_2 of the second link (driven by servo motor) of the mechanism can easily be found by using inverse kinematics equations.

Fifth order polynomials are used to design the motion. The slider displacement is expressed in fifth order polynomial as below;

$$s = ft^5 + et^4 + dt^3 + ct^2 + bt + a \tag{3.35}$$

when this equation is differentiated with respect to time, the slider velocity equation is obtained as follows;

$$\dot{s} = 5ft^4 + 4et^3 + 3dt^2 + 2ct + b \tag{3.36}$$

And similarly, this equation is differentiated with respect to time; the slider acceleration equation is obtained as below;

$$\ddot{s} = 20ft^3 + 12et^2 + 6dt + 2c \tag{3.37}$$

Kinematics characteristics of the slider link can be found by using Equations (3.35, 3.36 and 3.37). It is given in Equation (3.38).

$$\underbrace{\begin{bmatrix} 1 & t_i & t_i^2 & t_i^3 & t_i^4 & t_i^5 \\ 0 & 1 & 2t_i & 3t_i^2 & 4t_i^3 & 5t_i^4 \\ 0 & 0 & 2 & 6t_i & 12t_i^2 & 20t_i^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix}}_{A} * \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} p_i \\ v_i \\ a_i \\ p_f \\ v_f \\ a_f \end{bmatrix}}_{C} \qquad (3.38)$$

A*B = C

where p_i, v_i and a_i are the initial displacement, velocity and acceleration, p_f, v_f and a_f are the final displacement, velocity and acceleration of the slider link. t_i and t_f are the initial and final values of the motion time, respectively.

a, b, c, d, e, f are the coefficients of Equation (3.38).

So by writing the initial and final values of motion time, the displacement, velocity and acceleration of the slider link, the coefficients of the fifth order polynomial are found like in Equation (3.39).

$$B = A^{-1} * C \qquad (3.39)$$

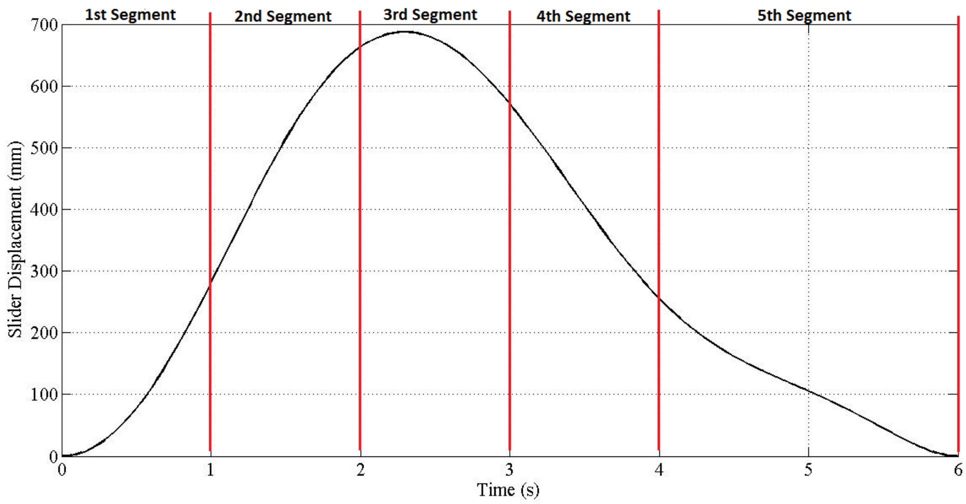
According to the complexity of the kinematics characteristics of the slider link, it may be needed to divide the kinematics specifications of the slider link into segments. It means that total motion time is divided into parts. Initial and final values of each time segment and initial and final values of position, velocity and acceleration of the slider link for each segment are specified. There is one important point that must be taken into consideration that initial values of a segment must be adjusted as the final values of the previous segment. Equation (3.38) is rewritten for each segment. From Equation (3.39), B vectors of each segment are found (Table 3.3 and Table 3.5). After that, the displacement, velocity and acceleration of the slider link are found for each segment from Equation (3.35, 3.36 and 3.37) by using related time intervals and B vectors. At the end of these operations, all datas of displacement found from Equation (3.38) from all segments are summed successively. Thus displacement curve of the slider link is obtained. The same operation is carried out for the velocity and acceleration curve of the slider link by using Equations (3.36) and (3.37), respectively.

Number of segments and the points where the segments are separated from each other are directly related with the complexity of the designed motion and choice of the designer. There is no doubt that the main objective is to form the datas accurately as desired.

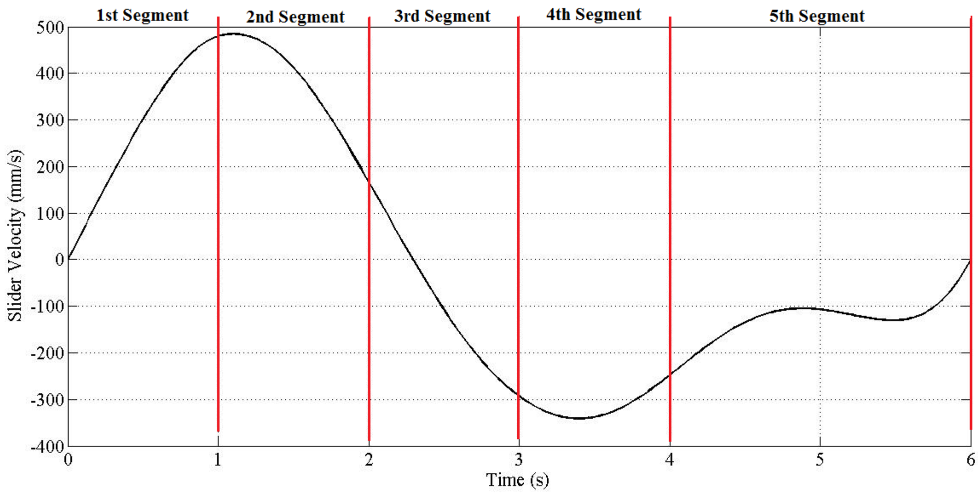
3.6.2 Motion Curve Examples

The first designed slider motion is a quick rise and slow return motion. Stroke of the slider is 687 mm and stroke per minute is 10. The displacement, velocity and acceleration curves of the slider are shown in Figure 3.4(a, b, c).

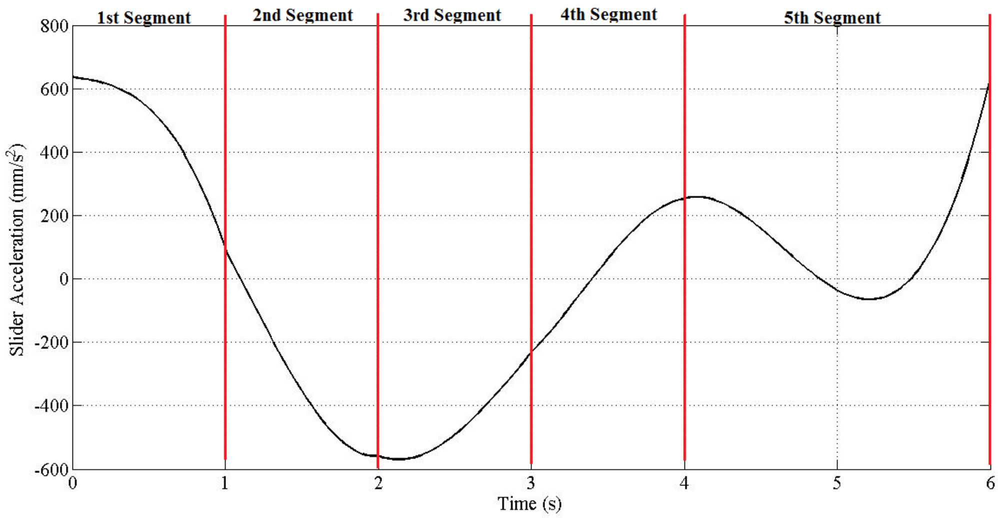
All curves are composed of five segments as shown in Figures 3.4. The kinematics specifications of each segment are illustrated in Table 3.2. These datas are used in Equation (3.38). It is formed of segment number, time interval and initial and final values of position, velocity and acceleration of each segment.



(a)



(b)



(c)

Figure 3.4. The Displacement, Velocity and Acceleration Curves of the Motion1

Table 3.2. Kinematics Specifications of the Segments (Motion Type 1)

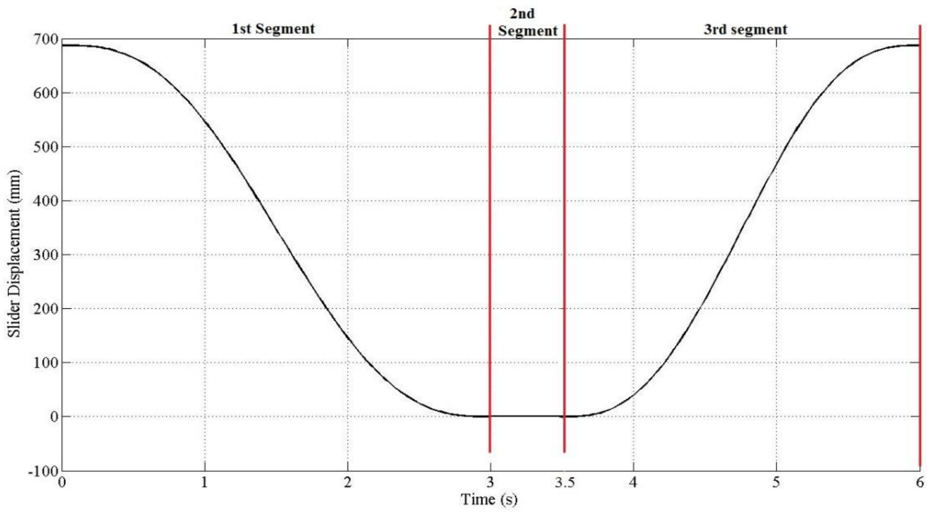
Segment Number	*Time Interval (s)*	*Position (mm)*	*Velocity (mm/s)*	*Acceleration* (mm/s^2)
1	0≤ t ≤1	0 , 281.4	0 ,479.8	635.3 , 94.11
2	1≤ t ≤2	281.4 , 662.7	479.8 , 166.6	94.11 , -558
3	2≤ t ≤3	662.7 , 570.22	166.6 , -293.3	-558 , -233
4	3≤ t ≤4	570.22 , 255.4	-293.3 , -248.5	-233 , 252.75
5	4≤ t ≤6	255.44 , 0	-248.5 , 0	252.75 , 635.3

Table 3.3. Coefficients of the B vector (Motion Type 1)

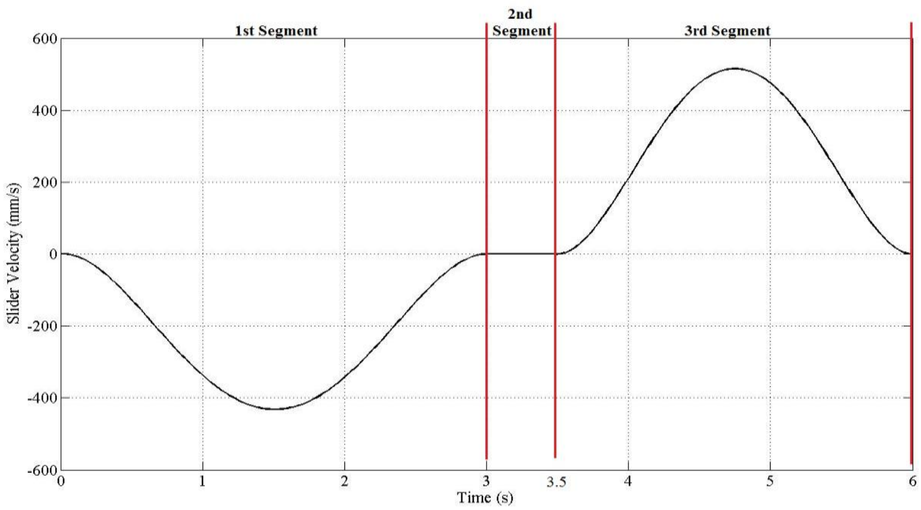
Segment Number	*Coefficients of the B vector*					
	a	*b*	*c*	*d*	*e*	*f*
1	0	0,318	318,16	-10,33	5,15	20,91
2	35,41	97,26	194,85	128,34	125,77	22,18
3	869,97	2105,23	2417,25	1033,26	187,33	12,40
4	5861,19	9262,93	6463,07	2155,88	339,44	20,39
5	40038,48	47474,7	21629,97	4809,11	525,23	22,60

The coefficients of B vector are shown in Table 3.3. Equation (3.39) is used for obtaining the coefficients of B vector.

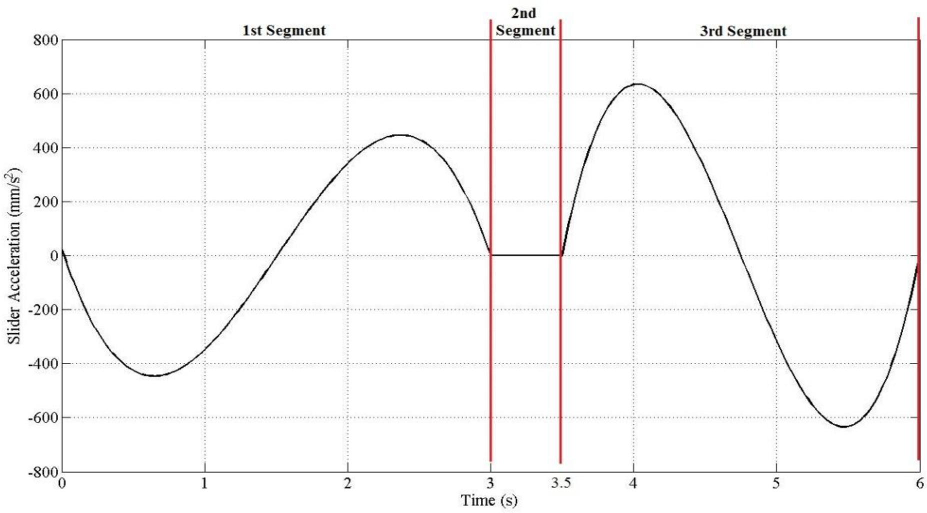
The second designed slider motion is a slow return-dwell and quick rise motion. Stroke of the slider is 687 mm and stroke per minute is accepted as 10. The displacement, velocity and acceleration curves of the slider are shown in Figure 3.5(a, b, c).



(a)



(b)



(c)

Figure 3.5. The Displacement, Velocity and Acceleration Curves of the Motion 2

All curves in Motion 2 are composed of three segments. They are shown in Figure 3.5. The kinematics specifications of each segment are illustrated in Table 3.4. These datas are used in Equation (3.38). It is formed of segment number, time interval and initial and final values of position, velocity and acceleration of each segment.

Table 3.4. Kinematics Specifications of the Segments (Motion Type 2)

Segment Number	*Motion Time (s)*	*Position (mm)*	*Velocity (mm/s)*	*Acceleration (mm/s^2)*
1	$0 \leq t \leq 3$	687 , 0	0 , 0	0 , 0
2	$3 \leq t \leq 3.5$	0 , 0	0 , 0	0 , 0
3	$3.5 \leq t \leq 6$	0 , 687	0 , 0	0 , 0

Table 3.5. Coefficients of the B vector (Motion Type 2)

Segment Number	*Coefficients of the B vector*					
	a	*b*	*c*	*d*	*e*	*f*
1	687	0,0134	13,391	267,826	131,682	17,458
2	0	0	0	0	0	0
3	80608,07	93071,46	42103,75	9303,62	1002,47	42,20

The coefficients of the B vector for Motion 2 are given in Table 3.5. Equation (3.39) is for obtaining the coefficients of the B vector. Different coefficients are available for each segment.

3.6.3 Application in Curve Fitting Toolbox

Curve Fitting Toolbox in Matlab© is used as a tool in this study to design the motion characteristics of the slider. The Curve Fitting Toolbox is not directly used in motion design operation. It is like an auxiliary method. It is possible to explain it on an example. When the via points desired to be tracked are defined, Curve Fitting

Toolbox is run. Let have a data set in x and y direction. These data set includes eighteen numerical values. The datas in x and y direction are loaded and data set is formed as shown in Figure 3.6. It is necessary to select the type of fit. The *shape preserving* technique is chosen from the *interpolant* option of *type of fit*. The completed fit operation is shown in Figure 3.7.

In Curve Fitting Toolbox, it is feasible to obtain more numerical value from a fitted curve. It means that, if only eighteen numerical values are available, the user can get more numerical value by using *Analysis* option of the Curve Fitting Toolbox. It is shown in Figure 3.8. that the curve formed only eighteen datas is transformed into a curve formed by three hundred and sixty one numerical datas. It is visible that two curves are identical curves. The curve with three hundred and sixty one datas is given in Figure 3.9. As it is seen in Figure 3.8, differentiation in first and second order or integration of the fitted curve is possible in *Analysis* option.

The first order differentiation operation of the given curve gives the results of velocity characteristics of the motion in Figure 3.10. In addition to all, it is noticed that some undesired points are available. It is inevitable that this discontinuities in velocity characteristics cause undesirable peaks in acceleration and jerk characteristics. So it is noticed that the motion is needed to be improved. The aim is to eliminate discontinuties on condition that the main characteristics of the motion is provided . The procedure given in Section 3.6.1. is applied. With the availability of the selected new via points, the motion is formed. The revised displacement and velocity characteristics of the motion are given in Figure 3.11 and Figure 3.12 [29] .

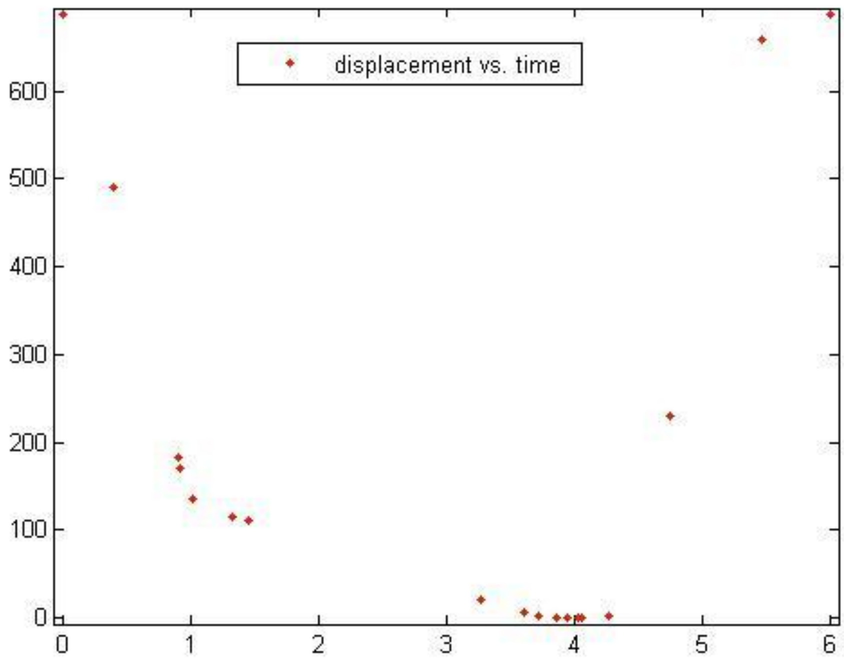


Figure 3.6. Loaded datas in CF Toolbox

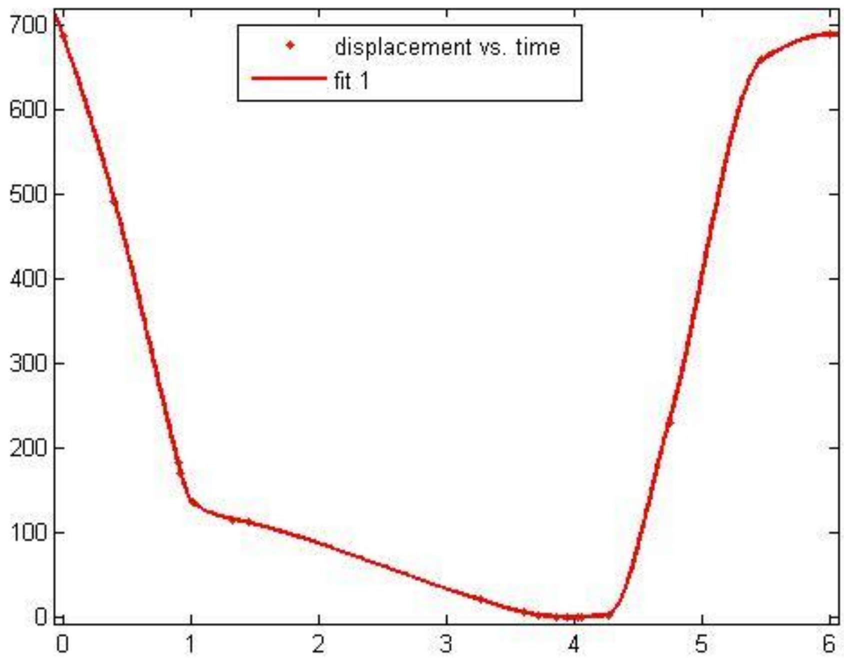


Figure 3.7. Fitted Curve in CF Toolbox

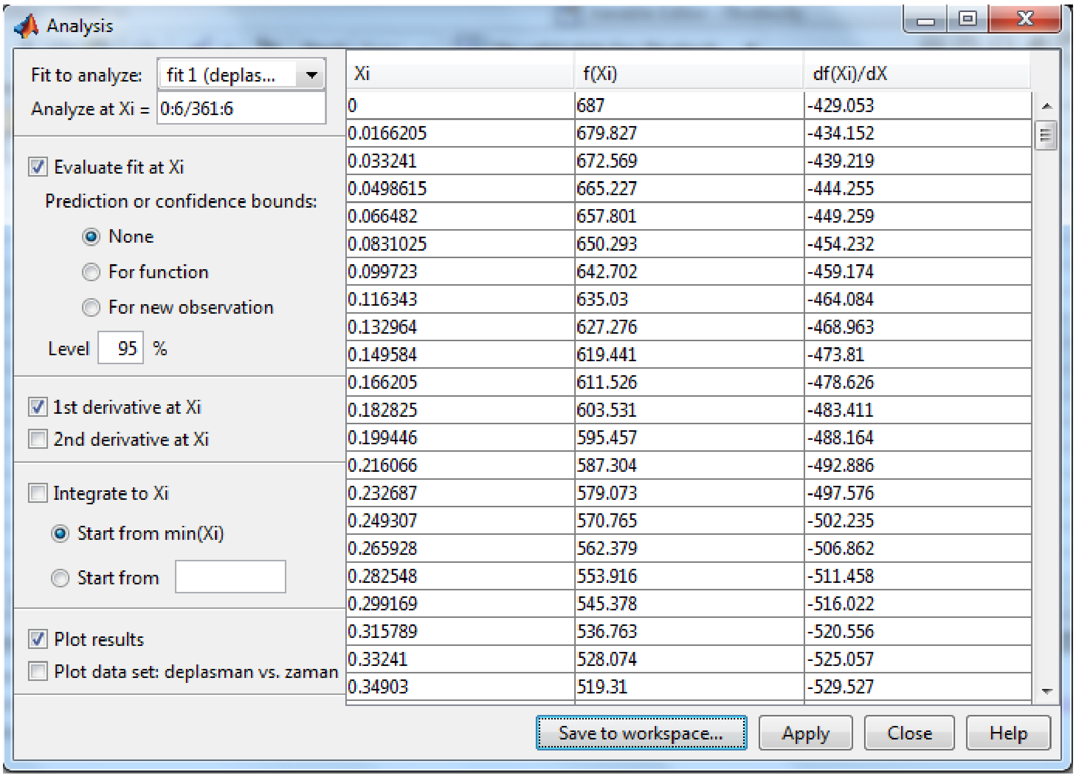


Figure 3.8. *Analysis* Option in CF Toolbox

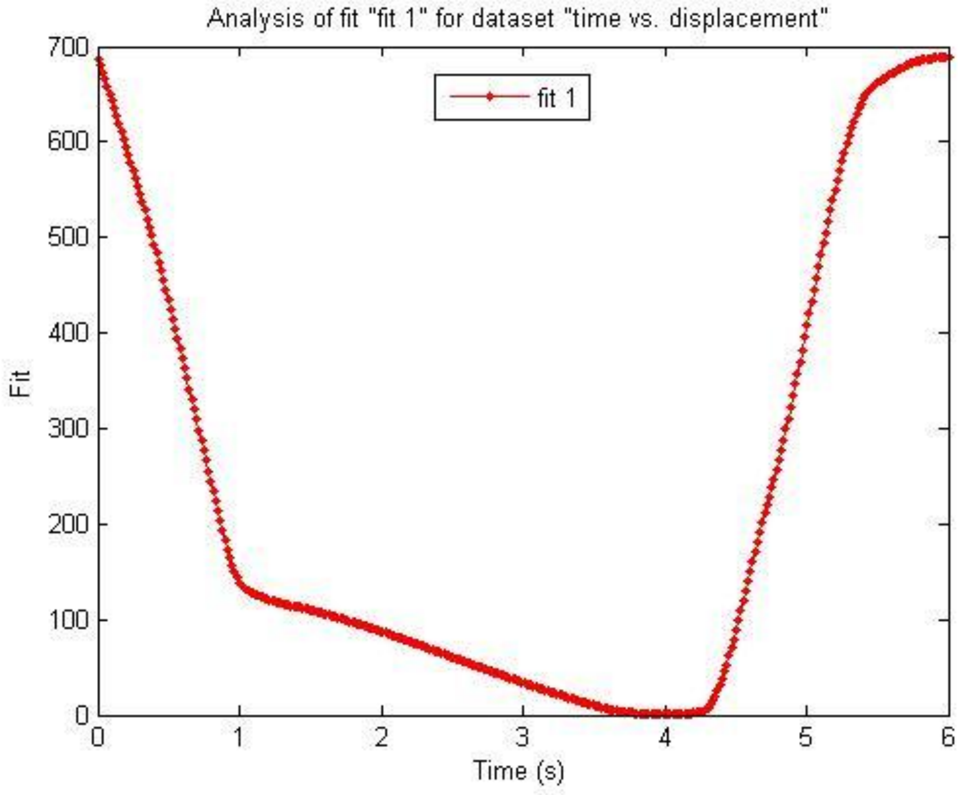


Figure 3.9. The Curve with three hundred and sixty one Datas

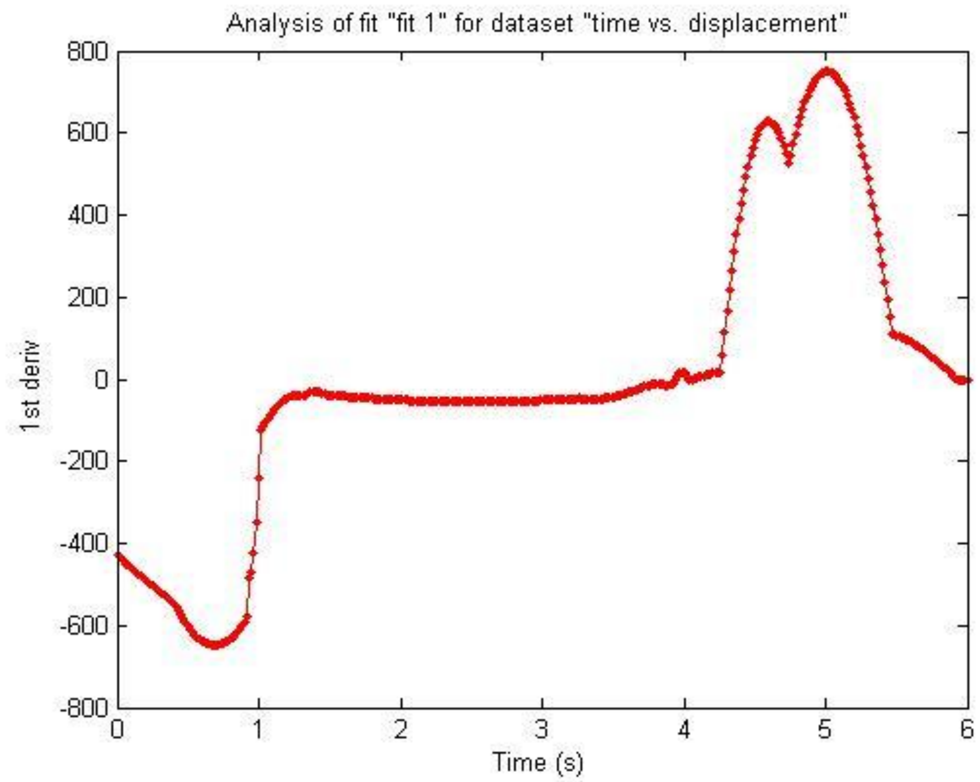


Figure 3.10. The Velocity of the Fitted Curve from CF Toolbox

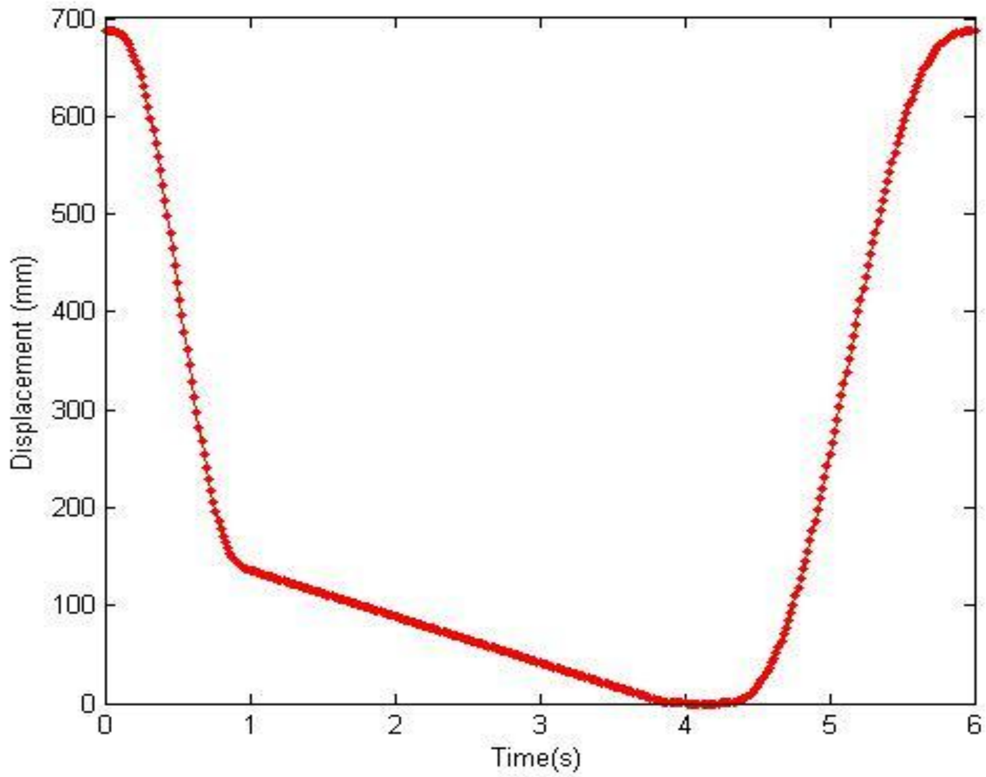


Figure 3.11. The Revised Displacement Curve

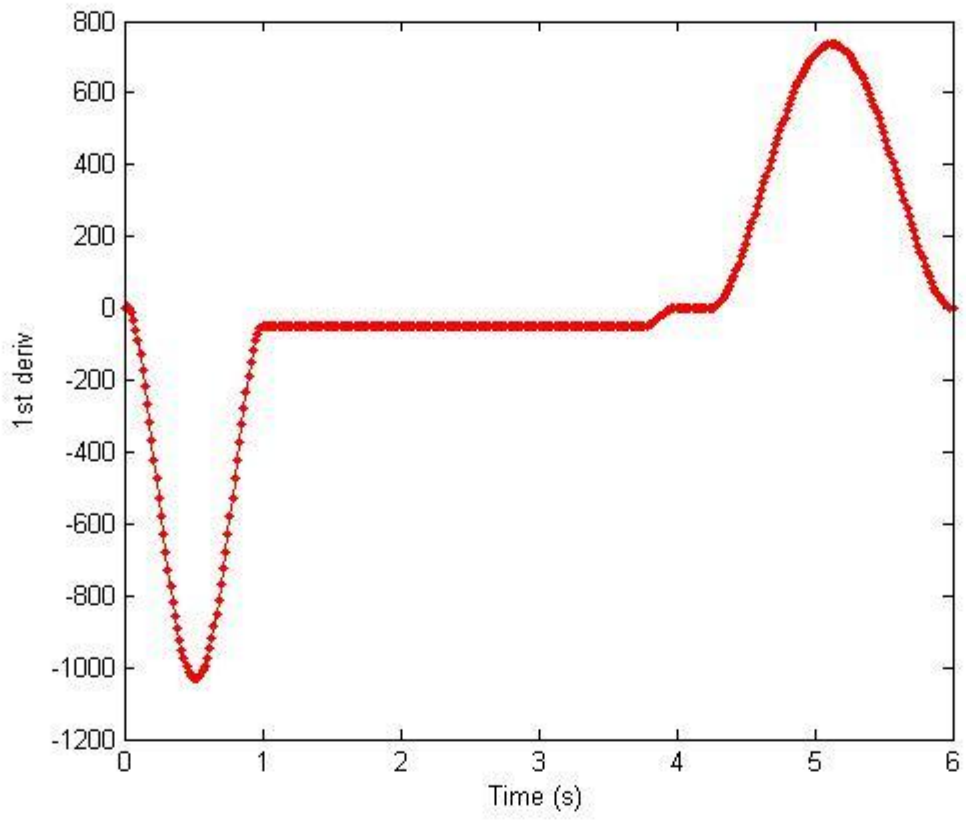


Figure 3.12. The Revised Velocity Curve

3.7 Required Motion Inputs-Outputs

The main objective of this part is to obtain the kinematics specifications of the second link driven by the servo motor shown in Figure 3.1 [26]. The kinematics characteristics of fifth link driven by the constant velocity motor shown in Figure 3.1 and the slider link are known. Inverse kinematics equations given in Section 3.3.1 are used to realize this purpose.

3.7.1 Solution for Motion 1

The first designed slider motion is a quick rise and slow return motion. Stroke of the slider is 687 mm and stroke per minute is 10. The displacement, velocity and acceleration of the slider are shown in Figure 3.4 (a, b, c).

The fifth link is driven by a constant velocity motor. It will rotate 2*pi radian in clockwise direction. Its angular velocity is a constant value, 1.0472 radian/sec in negative sense. Due to the fact that it has a constant angular velocity, its angular acceleration is zero. Kinematic specifications of the fifth link are shown in Figure 3.13.

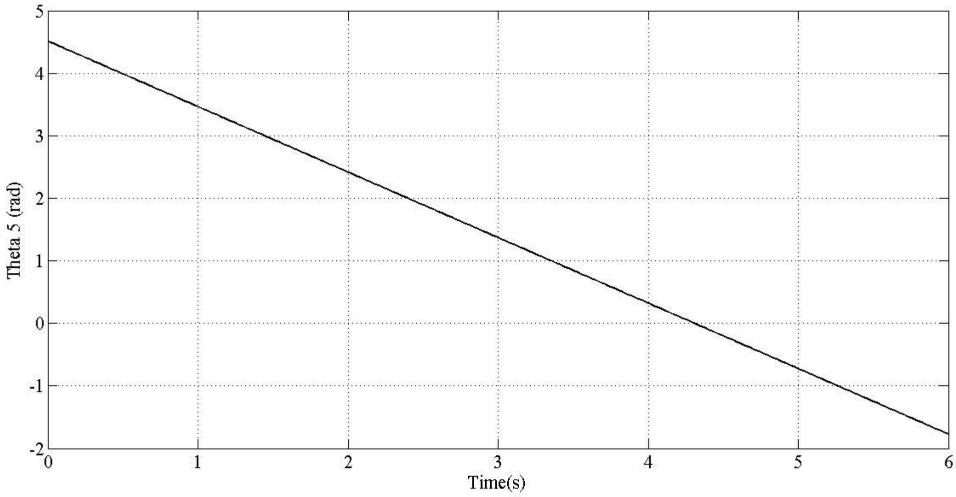
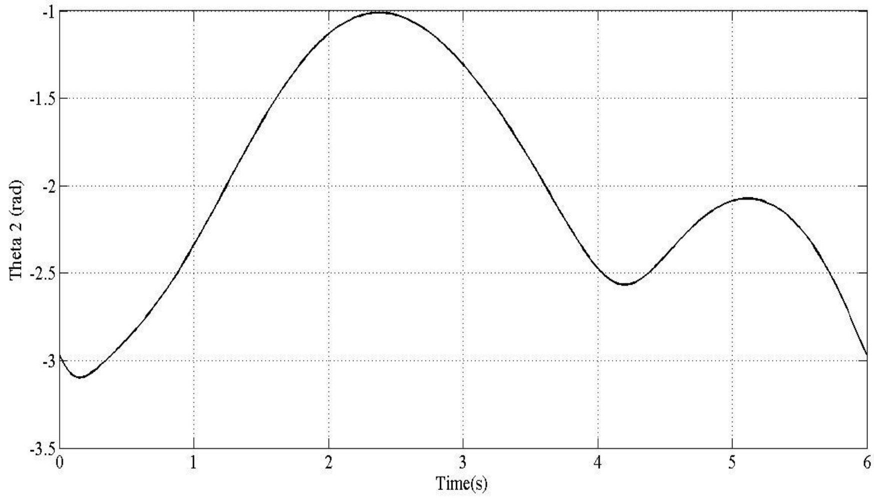
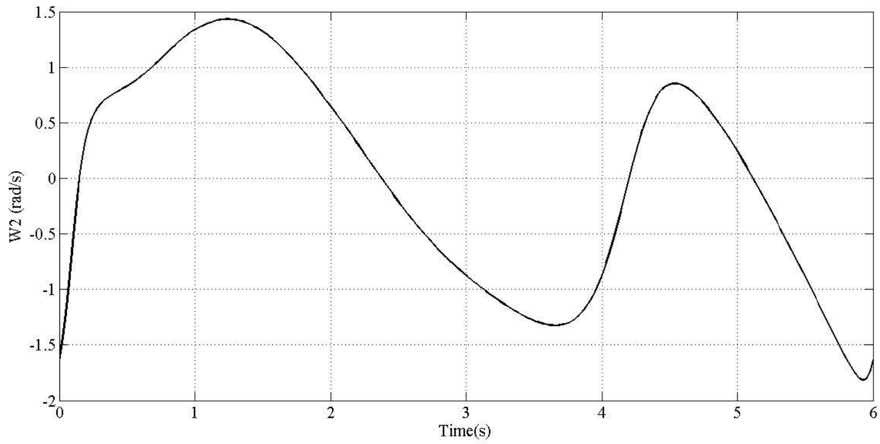


Figure 3.13. Angular Displacement of the CVM for Motion1

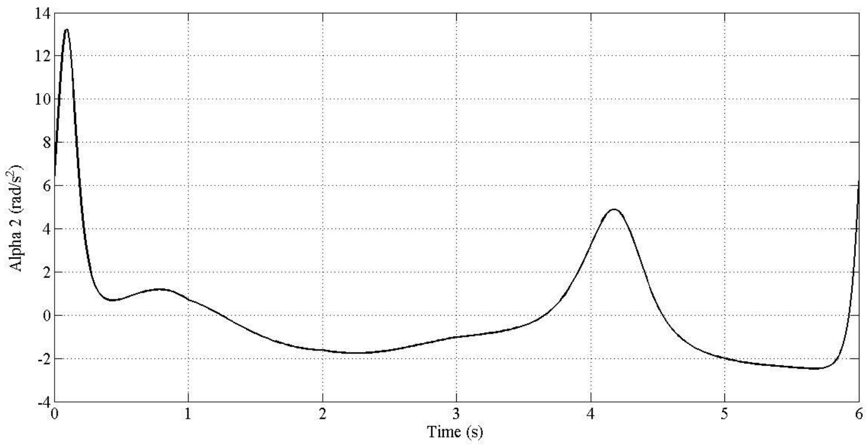
Inverse kinematics equations given in Section 3.3.1 are used to find the kinematic characteristics of the second link. They are shown in Figure 3.14.



(a)



(b)



(c)

Figure 3.14. Angular Displacement, Velocity and Acceleration of SM in Motion1

3.7.2 Solution for Motion 2

The second designed slider motion is a slow return-dwell and quick rise motion. Stroke of the slider is 687 mm and stroke per minute is 10. The displacement, velocity and acceleration of the slider link are shown in Figures 3.5 (a, b, c).

The fifth link is driven by a constant velocity motor. It will rotate 2*pi radian in counterclockwise (CCW) direction. Its angular velocity is a constant value, 1.0472 radian/sec in positive sense. Because of the fact that it has a constant angular velocity, its angular acceleration is zero. Kinematic specification of the fifth link is shown in Figure 3.15.

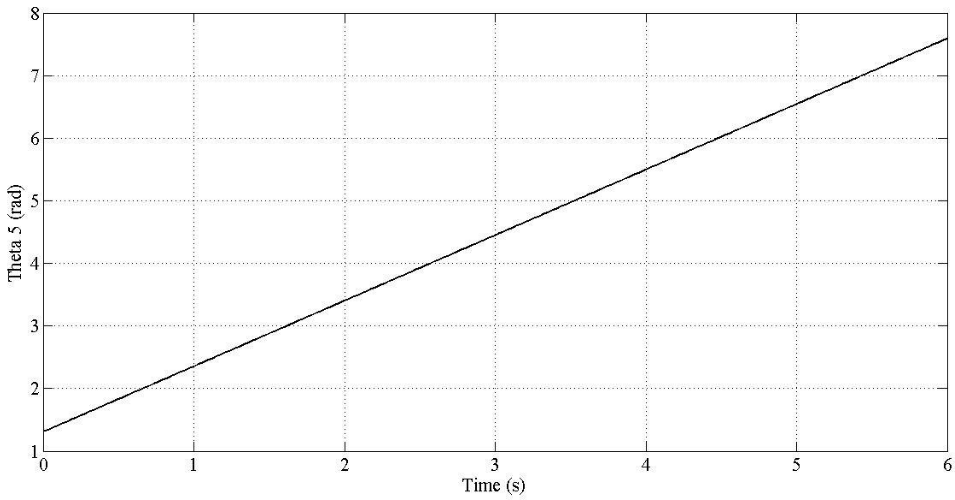
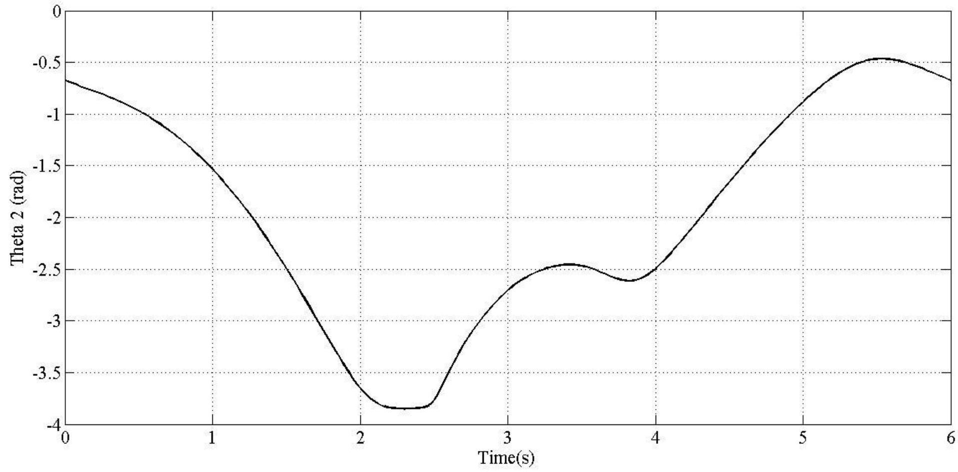
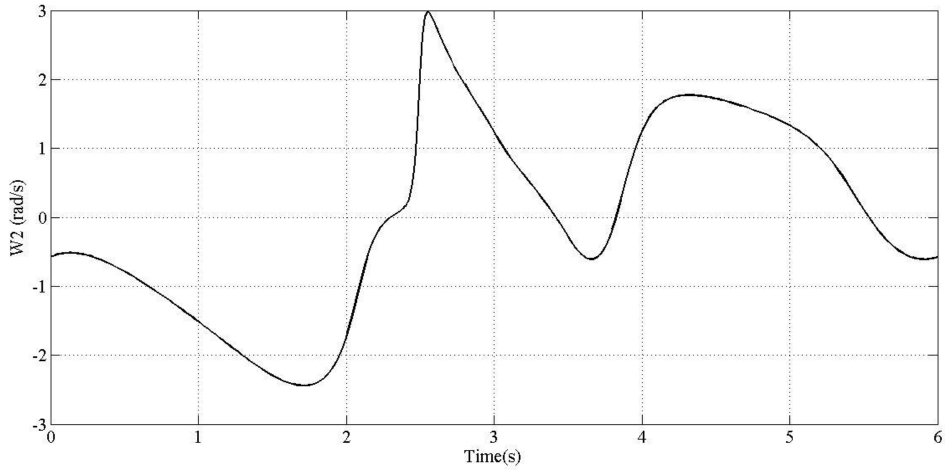


Figure 3.15. Angular Displacement of the CVM for Motion2

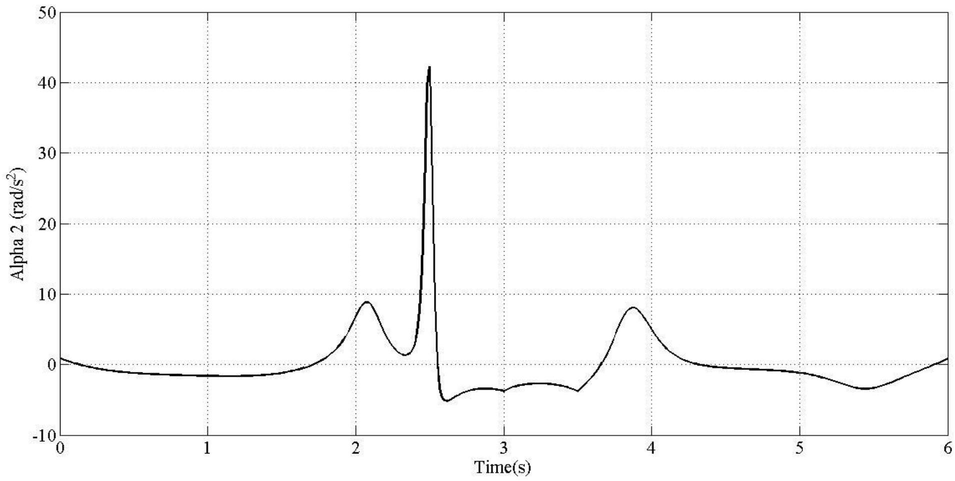
By utilizing from the inverse kinematics equation given in Section 3.3.1, angular displacement, velocity and acceleration of the second link can be obtained as shown in Figure 3.16.



(a)



(b)



(c)

Figure 3.16. Angular Displacement, Velocity and Acceleration of Servo Motor in Motion 2

CHAPTER 4

DYNAMIC MODELING OF THE HYBRID DRIVEN MECHANISM

4.1 Introduction

The dynamic modeling of the hybrid driven mechanism is presented in this chapter. A dynamic model plays a very important role in the terms of system validation, analysis, and control system. It is very essential to study the dynamic modeling of the hybrid mechanism in order to provide the desirable performance. Two main approaches are available to study dynamics of the mechanism. They are named as Newton-Euler method and Lagrange method. In Newton-Euler method, all the forces and torques applied to each link of the mechanism must be taken into account. So it is a bit complex even for simple problems but it is more intuitive with physical meanings. The Lagrange method is a systematic method obtaining the Lagrangian equation and computing it analytically [30].

In this chapter, dynamic model of the hybrid driven mechanism is derived by using Lagrange method. The simulation results are presented by considering PID control technique. Next, the kineto-static method is given for dynamic model of the hybrid driven mechanism.

4.2 The Dynamic Model from Lagrange Equation

Lagrangian mechanics is formed by differentiation of the energy expressions with respect to the variables of system and time. Lagrangian expression is obtained by subtracting the potential energy of the system from the kinetic energy of the system. It is shown in Equation (4.1) where L refers to Lagrangian, T refers to kinetic energy of the system and V refers to the potential energy of the system. For simple examples, it may take more time to use this method than Newtonian mechanics but as the details of the system increase, the Lagrangian technique becomes relatively easier to use. As a result, to obtain the equation of motion, it is necessary to derive the energy expressions of the system and differentiate the Lagrangian according to the Equation (4.2) given below.

$$L=T-V \tag{4.1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=Q_i \qquad (i=1,2,...,n\ ) \tag{4.2}$$

Q_i is the generalized torque or force and q_i is the generalized coordinate. There are basically two generalised coordinates in the Lagrangian mechanics. One of them is for linear motion and the other one is for rotational motion. Q_i and q_i in linear motions are force (F) and linear displacement (x), respectively. F is the summation of all external forces and x is the system variable. Q_i and q_i in rotational motions are torque (τ) and angular displacement (θ), respectively. τ is the summation of all torques in a rotational motion and θ is the system variable [30].

The schematic illustration of the hybrid driven mechanism which is shown in Figure 4.1 is a 2-DOF planar mechanism. So θ_2 and θ_5 are given, the kinematics of the hybrid mechanism is determined. The rotation angles of the cranks can be selected as the generalized coordinates; $q_1=\theta_2\,, q_2=\theta_5$. The Lagrange equations of the 2-DOF hybrid system are given in Equation (4.3) and (4.4).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\frac{\partial L}{\partial \theta_2}=\tau_2 \tag{4.3}$$

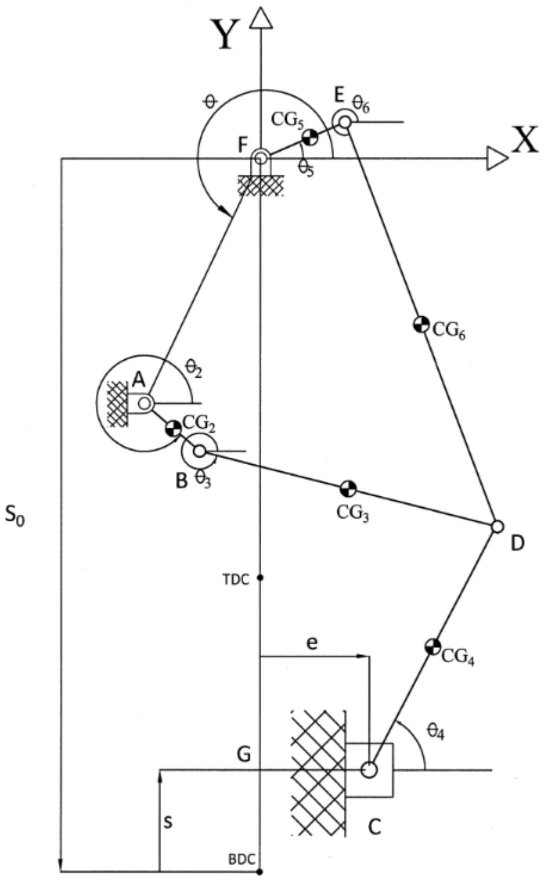
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_5}\right)-\frac{\partial L}{\partial \theta_5}=\tau_5 \tag{4.4}$$

4.2.1 The Calculation of the Kinetic Energy of the System

The velocities of the mass center of every links should be obtained to calculate the kinetic energy of the mechanism.

$$\theta_i = f(q_1, q_2) \tag{4.5}$$

where θ_i is the rotation angle of the link i. X_{cg_i} and Y_{cg_i} are respectively the horizontal and vertical mass center positions of the i^{th} link of the hybrid mechanism. r_{cg_i} is the location of the center of mass of the i^{th} link. Angular displacements of the links are given in Section 3.3.2.1. The positions of center of mass of the links in X and Y directions are given in Equations (4.6- 4.11) according to Figure 4.1. The centers of mass of the links are taken as the midpoints of them.



$$\left|\overrightarrow{AF}\right| = r_1,\ \left|\overrightarrow{AB}\right| = r_2,\ \left|\overrightarrow{BD}\right| = r_3,\ \left|\overrightarrow{CD}\right| = r_4,\ \left|\overrightarrow{FE}\right| = r_5,\ \left|\overrightarrow{ED}\right| = r_6$$

Figure 4.1. Schematic Representation of Hybrid System with Center of Masses

$$X_{cg_2} = X_A + r_{cg_2} \cos\theta_2 \tag{4.6a}$$

$$Y_{cg_2} = Y_A + r_{cg_2} \sin\theta_2 \tag{4.6b}$$

$$X_{cg_3} = X_A + r_2 \cos\theta_2 + r_{cg_3} \cos\theta_3 \tag{4.7a}$$

$$Y_{cg_3} = Y_A + r_2 \sin\theta_2 + r_{cg_3} \sin\theta_3 \tag{4.7a}$$

$$X_{cg_4} = r_5 \cos\theta_5 + r_6 \cos\theta_6 - r_{cg_4} \cos\theta_4 \tag{4.8a}$$

$$Y_{cg_4} = r_5 \sin\theta_5 + r_6 \sin\theta_6 - r_{cg_4} \sin\theta_4 \tag{4.8a}$$

$$X_{cg_5} = r_{cg_5} \cos\theta_5 \tag{4.9a}$$

$$Y_{cg_5} = r_{cg_5} \sin\theta_5 \tag{4.9a}$$

$$X_{cg_6} = r_5 \cos\theta_5 + r_{cg_6} \cos\theta_6 \tag{4.10a}$$

$$Y_{cg_6} = r_5 \sin\theta_5 + r_{cg_6} \sin\theta_6 \tag{4.10b}$$

$$X_{slider} = e \tag{4.11a}$$

$$Y_{slider} = S_0 + s \tag{4.11b}$$

The velocities of the mass centers of each link are obtained by differentiating the Equations (4.6-4.11). The angular velocities of each link are given in Section 3.3.2.2 in explicit form. So the velocity components are presented in Equations (4.12-4.17)

$$\dot{X}_{cg_2} = -r_{cg_2} w_2 \sin\theta_2 \tag{4.12a}$$

$$\dot{Y}_{cg_2} = r_{cg_2} w_2 \cos\theta_2 \tag{4.12b}$$

$$\dot{X}_{cg_3} = -r_2 w_2 \sin\theta_2 - r_{cg_3} w_3 \sin\theta_3 \tag{4.13a}$$

$$\dot{Y}_{cg_3} = r_2 w_2 \cos\theta_2 + r_{cg_3} w_3 \cos\theta_3 \tag{4.13b}$$

$$\dot{X}_{cg_4} = -r_5 w_5 \sin\theta_5 - r_6 w_6 \sin\theta_6 + r_{cg_4} w_4 \sin\theta_4 \tag{4.14a}$$

$$\dot{Y}_{cg_4} = r_5 w_5 \cos\theta_5 + r_6 w_6 \cos\theta_6 - r_{cg_4} w_4 \cos\theta_4 \tag{4.14b}$$

$$\dot{X}_{cg_5} = -r_{cg_5} w_5 \sin\theta_5 \quad (4.15a)$$

$$\dot{Y}_{cg_5} = r_{cg_5} w_5 \cos\theta_5 \quad (4.15b)$$

$$\dot{X}_{cg_6} = -r_5 w_5 \sin\theta_5 - r_{cg_6} w_6 \sin\theta_6 \quad (4.16a)$$

$$\dot{Y}_{cg_6} = r_5 w_5 \cos\theta_5 + r_{cg_6} w_6 \cos\theta_6 \quad (4.16b)$$

$$V_{slider} = \dot{s} \quad (4.17)$$

The velocity of a mass center is given in Equation (4.18).

$$V_{cg_i} = \sqrt{\dot{X}_{cg_i}{}^2 + \dot{Y}_{cg_i}{}^2} \quad (4.18)$$

The accelerations of the mass centers of each links are obtained by differentiating the Equations (4.12-4.17). The angular accelerations of each link are given in Section 3.3.2.3 in explicit form.

$$\ddot{X}_{cg_2} = -r_{cg_2}(\alpha_2 \sin\theta_2 + w_2{}^2 \cos\theta_2) \quad (4.19a)$$

$$\ddot{Y}_{cg_2} = r_{cg_2}(\alpha_2 \cos\theta_2 - w_2{}^2 \sin\theta_2) \quad (4.19b)$$

$$\ddot{X}_{cg_3} = -r_2(\alpha_2 \sin\theta_2 + w_2{}^2 \cos\theta_2) - r_{cg_3}(\alpha_3 \sin\theta_3 + w_3{}^2 \cos\theta_3) \quad (4.20a)$$

$$\ddot{Y}_{cg_3} = r_2(\alpha_2 \cos\theta_2 - w_2{}^2 \sin\theta_2) + r_{cg_3}(\alpha_3 \cos\theta_3 - w_3{}^2 \sin\theta_3) \quad (4.20b)$$

$$\ddot{X}_{cg_4} = -r_5(\alpha_5 \sin\theta_5 + w_5{}^2 \cos\theta_5) - r_6(\alpha_6 \sin\theta_6 + w_6{}^2 \cos\theta_6) + r_{cg_4}(\alpha_4 \sin\theta_4 + w_4{}^2 \cos\theta_4) \quad (4.21a)$$

$$\ddot{Y}_{cg_4} = r_5(\alpha_5 \cos\theta_5 - w_5{}^2 \sin\theta_5) + r_6(\alpha_6 \cos\theta_6 - w_6{}^2 \sin\theta_6) - r_{cg_4}(\alpha_4 \cos\theta_4 - w_4{}^2 \sin\theta_4) \quad (4.21b)$$

$$\ddot{X}_{cg_5} = -r_{cg_5}(\alpha_5 \sin\theta_5 + w_5{}^2 \cos\theta_5) \quad (4.22a)$$

$$\ddot{Y}_{cg_5} = r_{cg_5}(\alpha_5 \cos\theta_5 - w_5{}^2 \sin\theta_5) \quad (4.22b)$$

$$\ddot{X}_{cg_6} = -r_5(\alpha_5 \sin\theta_5 + w_5{}^2 \cos\theta_5) - r_{cg_6}(\alpha_6 \sin\theta_6 + w_6{}^2 \cos\theta_6) \quad (4.23a)$$

$$\ddot{Y}_{cg_6} = r_5(\alpha_5 \cos\theta_5 - w_5{}^2 \sin\theta_5) + r_{cg_6}(\alpha_6 \cos\theta_6 - w_6{}^2 \sin\theta_6) \tag{4.23b}$$

$$a_{slider} = \ddot{s} \tag{4.24}$$

The acceleration of mass center is;

$$a_{cg_i} = \sqrt{\ddot{X}_{cg_i}{}^2 + \ddot{Y}_{cg_i}{}^2} \tag{4.25}$$

The kinetic energy of the system is;

$$T = \sum_{i=2}^{6} \frac{1}{2}(m_i V_{cg_i}^2 + I_i \dot{\theta}_i^2) + \frac{1}{2} m_{slider} V_{slider}^2 \tag{4.26}$$

where I_i is mass moment of inertia of the i^{th} link of the mechanism. The links used in the mechanism are taken as solid cylinders and given in Equation (4.27) [31].

$$I_i = \frac{1}{12} m_i (r_i^2 + 3d_i{}^2) \tag{4.27}$$

where m_i, r_i and d_i are mass, length and radius of the i^{th} link, respectively.

The total kinetic energy of the hybrid driven mechanism is given in Equation (4.28). Angular displacement and angular velocity expressions in the Equation (4.28) are available in Section 3.2.2.1 and 3.2.2.2. When these expressions are substituted into Equation (4.28), total kinetic energy of the mechanical system is obtained.

$$\begin{aligned} T = {} & (m_3((r_2 w_2 \cos\theta_2 + r_{cg_3} w_3 \cos\theta_3)^2 + (r_2 w_2 \sin\theta_2 + r_{cg_3} w_3 \sin\theta_3)^2)/2 + (m_6((r_5 w_5 \cos\theta_5 + \\ & r_{cg_6} w_6 \cos\theta_6)^2 + (r_5 w_5 \sin\theta_5 + r_{cg_6} w_6 \sin\theta_6)^2))/2 + (m_4((r_5 w_5 \cos\theta_5 - r_{cg_4} w_4 \cos\theta_4 + r_6 w_6 \\ & \cos\theta_6)^2 + (r_5 w_5 \sin\theta_5 - r_{cg_4} w_4 \sin\theta_4 + r_6 w_6 \sin\theta_6)^2))/2 + (m_2(r_{cg_2}{}^2 w_2{}^2))/2 + (m_5(r_{cg_5}{}^2 w_5{}^2))/ \\ & 2 + (m_{slider}(r_5 w_5 \cos\theta_5 - r_4 w_4 \cos\theta_4 + r_6 w_6 \cos\theta_6)^2)/2 + (m_2 w_2{}^2(3d_2{}^2 + r_2{}^2))/24 + (m_3 w_3{}^2 \\ & (3d_3{}^2 + r_3{}^2))/24 + (m_4 w_4{}^2(3d_4{}^2 + r_4{}^2))/24 + (m_5 w_5{}^2(3d_5{}^2 + r_5{}^2))/24 + (m_6 w_6{}^2(3d_6{}^2 + r_6{}^2))/ \\ & 24 \end{aligned} \tag{4.28}$$

4.2.2 The Calculation of the Potential Energy of the System

By looking at Figure 4.1, point F is the reference point in the calculation of the potential energy of the system. Then the potential energy of the system is;

$$V = \sum_{i=2}^{6} m_i g Y_{cg_i} + m_{slider} g Y_{slider}$$

The total potential energy of the hybrid driven mechanism is given in Equation (4.29).

$$V = m_6 g(r_5 \sin\theta_5 + r_{cg_6} \sin\theta_6) + m_3 g(Y_A + r_2 \sin\theta_2 + r_{cg_3} \sin\theta_3) + m_3 g(Y_A + r_2 \sin\theta_2 + r_{cg_3} \sin\theta_3) + m_4 g(r_5 \sin\theta_5 - r_{cg_4} \sin\theta_4 + r_6 \sin\theta_6) + m_{slider} g(r_5 \sin\theta_5 - r_4 \sin\theta_4 + r_6 \sin\theta_6) + m_2 g(Y_A + r_{cg_2} \sin\theta_2) + m_5 g(Y_A + r_{cg_5} \sin\theta_5) \tag{4.29}$$

Angular displacement expressions in Equation (4.29) are available in Section 3.3.2.1. When these expressions are substituted into Equation (4.29), total potential energy of the system is obtained.

Lagrangian expression given in Equation (4.1) can be obtained by subtracting Equation (4.29) from Equation (4.28). After obtaining Lagrangian, equations of motions given in Equation (4.3) and Equation (4.4) can be formed.

Open forms of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_5}\right)$, $\frac{\partial L}{\partial \theta_2}$ and $\frac{\partial L}{\partial \theta_5}$ expressions are given in Appendix . These expressions contain a lot of partial derivatives in them. They are also included in Appendix.

4.3 Mathematical Model for DC Motor

The electric equivalent circuit of the armature and the free-body diagram of the rotor are shown in the following figure. The rotor and shaft are assumed to be rigid. [16, 32]

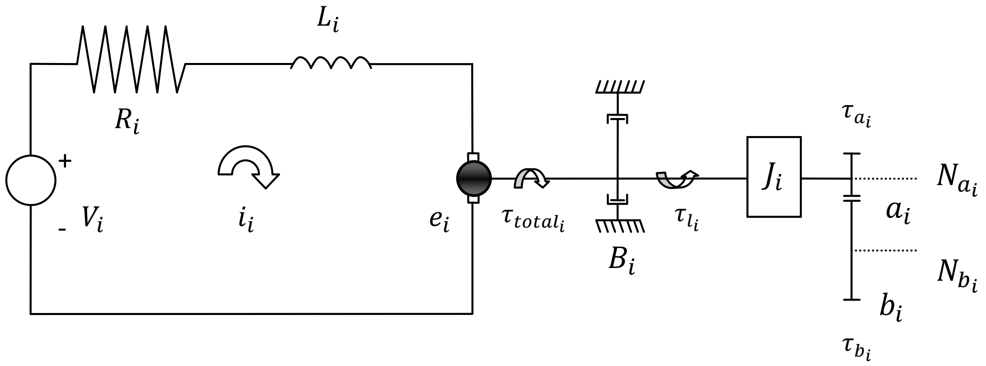


Figure 4.2. Schematic Representation of a DC Motor

Gear ratio of the speed reducer,

$$n_i = \frac{\tau_{b_i}}{\tau_{a_i}} = \frac{w_{a_i}}{w_{b_i}} = \frac{N_{b_i}}{N_{a_i}} \qquad i = (2,\ 5) \tag{4.30}$$

where i=2 and 5 for speed reducer of servo and constant velocity motor, respectively.

where w_{a_i} and w_{b_i} are the angular velocities of the input shaft a and the output shaft b, respectively. N_{a_i} and N_{b_i} are the numbers of teeth of the gears a and b. It is assumed that the input of the system is the voltage source (V_i) applied to the motor's armature, the output is torque (τ_{b_i}). i_i, R_i and L_i represent the armature current, resistance and inductance, respectively. Shaft b is connected to the crank of the mechanism in order to drive the mechanism. B_i is the viscous friction coefficient. τ_{l_i} is constant mechanical torque caused of friction of gear and brush. J_i *is* the total mass moment of inertia including flywheel, reducer and rotor of motor.

$$J_i = J_{f_i} + J_{r_i} + J_{m_i} \tag{4.31}$$

where J_{f_i} is the moment of inertia of the flywheel, J_{r_i} is the moment of inertia of the speed reducer and J_{m_i} is the moment of inertia of the motor rotor.

Kirchhoff voltage law is applied around windings of the armature [32];

$$V_i = i_i R_i + L_i \frac{di_i}{dt} + e_i \tag{4.32}$$

where e_i is back electromotive force of the motor.

Newtonian equation is used to get the torque equation below [16];

$$\tau_{b_i} = n_i(\tau_{total_i} - \tau_{l_i} - B_i w_{a_i} - J_i \frac{dw_{a_i}}{dt}) \tag{4.33}$$

where τ_{total_i} is the magnetic motor torque. τ_{total_i} and e_i are defined as;

$$\tau_{total_i} = K_{t_i} i_i \tag{4.34}$$

$$e_i = K_{e_i} w_{a_i} \tag{4.35}$$

where K_t is the motor torque constant and K_e is the motor voltage constant.

In general,

$$w_{a_i} = n_i w_{b_i} = n_i \dot{q}_i \tag{4.36}$$

$$\frac{di_i}{dt} = \frac{V_i - R_i - n_i K_{e_i} \dot{q}_i}{L_i} \tag{4.37}$$

$$\tau_{b_i} = n_i K_{t_i} i_i - n_i \tau_{l_i} - n_i^2 B_i \dot{q}_i - n_i^2 J_i \ddot{q}_i \tag{4.38}$$

τ_{b_i} for servo motor is τ_2 which is given in Equation (4.3) and τ_{b_i} for constant velocity motor is τ_5 which is given in Equation (4.4).

4.3.1 Control of the Hybrid Driven System

PID controllers are commonly used controller type in industrial applications due to their simplicity, effectiveness, reliability and cost advantage. PID controllers use a control signal including position, velocity and integral of position error functions [16]. The mathematical representation of standard PID control is described as follows;

$$V_i = k_{p_i}(q_{i_i} - q_{i_a}) + k_{d_i}(\dot{q}_{i_i} - \dot{q}_{i_a}) + k_{I_i}\int_0^t (q_{i_i} - q_{i_a})dt \tag{4.39}$$

where k_{p_i} is the proportional gain constant, k_{d_i} is the derivative gain constant and k_{I_i} is the integral gain constant. The required angular displacement and angular velocity of the crank *i* are q_{i_i} and $\dot{q}_{i_i}$. The actual angular displacement and angular velocity of the crank *i* are q_{i_a} and $\dot{q}_{i_a}$, respectively. So the motor current equation is modified as follows;

$$\frac{d_{i_i}}{dt} = \frac{k_{p_i}(q_{i_i} - q_{i_a}) + k_{d_i}(\dot{q}_{i_i} - \dot{q}_{i_a}) + k_{I_i}\int_0^t (q_{i_i} - q_{i_a})dt - R_i i_i - n_i K_{e_i} \dot{q}_{i_a}}{L_i} \tag{4.40}$$

Equation (4.40) is same for servo motor, but it is simplified in CV motor as an algebraic equation which only represents the proportional gain in reality.

4.3.2 State Space Representation

The dynamic behaviour of the hybrid driven system is studied by using a numerical method to get an approximate solution. The fourth order Runge-Kutta is used as the integration technique. It is an explicit method used in integration of nonlinear systems. It is necessary to form state-space representation. The aim in state-space representation is to introduce a suitable set of state variables. Then the equations of motion of the system are formed as a system of first order differential equations [33]. The angular displacements, the angular velocities and the currents are treated as the state variables. By letting;

$$X_1 = \theta_2 \qquad \dot{X}_1 = X_2$$

$$X_2 = w_2 \qquad \dot{X}_2 = \alpha_2$$

$$X_3 = \theta_5 \qquad \dot{X}_3 = X_4$$

$$X_4 = w_5 \qquad \dot{X}_4 = \alpha_5$$

$$X_5 = i_2 \qquad \dot{X}_5 = \frac{di_2}{dt}$$

$$X_6 = i_5 \qquad \dot{X}_6 = \frac{di_5}{dt}$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ X_5 \\ X_6 \end{bmatrix} = \begin{bmatrix} X_2 \\ \alpha_2 \\ X_4 \\ \alpha_5 \\ \left(k_{p_2}(\theta_{2_i} - X_1) + k_{d_2}(w_{2_i} - X_2) + k_{I_2}\int_0^t (\theta_{2_i} - X_1)dt - R_2 X_5 - n_2 K_{e_2} X_2 \right) / L_2 \\ \left(k_{p_5}(\theta_{5_i} - X_3) + k_{d_5}(w_{5_i} - X_4) + k_{I_5}\int_0^t (\theta_{5_i} - X_3)dt - R_5 X_6 - n_5 K_{e_5} X_4 \right) / L_5 \end{bmatrix}$$

(4.41)

α_2 and α_5 expressions are taken from Equations (4.3) and (4.4), respectively. Their explicit forms with addition of the motor dynamics are given in Appendix (A.32) and (A.33).

4.4 Simulation Results

Modeling is carried out to understand the dynamic behaviour of the complex systems. It is used to investigate the relationship between the system variables. The mathematical model of the system is described by a set of mathematical expressions. Then simulation is performed on a computer [32].

Fourth order Runge-Kutta method is an explicit method which is commonly used in integration of the systems composed of nonlinear differential equations. This technique provides good accuracy [34]. The properties of the motors and links of the mechanism are given in Table 4.1, Table 4.2 and Table 4.3. The gear ratio given in Equation (4.30) for DC motor is 60. Servo motor is directly driven to the crank r_2 . A flywheel is coupled to DC motor. Total mass moment of inertia of the flywheel and the speed reducer given in Equation (4.31) is 50 kgm^2 $(J_{r_5} + J_{f_5})$. The time required for one cycle is 6 seconds. Incremental time of simulation is taken as 0.166 seconds. Viscous friction coefficient, B_i and the constant mechanical torque, τ_{l_i} given in Equation (4.38) are assumed as zero in calculations. Simulation results of two designed motions are then presented.

Table 4.1. The Parameters of Mechanism

Parameters	*Link 2*	*Link 3*	*Link 4*	*Link 5*	*Link 6*	*Slider*
Radius (m)	0.025	0.025	0.025	0.025	0.025	-
Mass (kg)	3	9.95	13.78	2.6	12.25	20
Moment of inertia (kg m^2)	0.0105	0.3519	0.9323	0.0067	0.6552	-

Table 4.2. DC Motor Parameters

Rated Power (P_5)	18.5 KW
Rated Voltage (V_5)	440 V
Rated Current (i_5)	52 A
Rated Speed/Maximum Speed $(W_{\max})$	600 rpm
Moment of Inertia (J_{m_5})	1.72 kg m^2
Moment of Inertia $(J_{r_5} + J_{f_5})$	50 kg m^2
Gear Ratio (n_5)	60
Winding Resistance (R_5)	0.973 Ω
Winding Inductance (L_5)	$19.9*10^{-3}$ H
Motor Voltage Constant (K_{e_5})	6.2 V/rad/s
Motor Torque Constant (K_{t_5})	5.6627 Nm/A

Table 4.3. DC Servo Motor Parameters

Rated Power (P_2)	3 KW
Maximum Speed $(W_{\max})$	3000 rpm
Moment of Inertia (J_{m_2})	$6.8*10^{-4}$ kg m^2
Winding Resistance (R_2)	0.8 Ω
Winding Inductance (L_2)	$5.8*10^{-3}$ H
Motor Voltage Constant (K_{e_2})	0.8598 V/rad/s
Motor Torque Constant (K_{t_2})	0.76 Nm/A

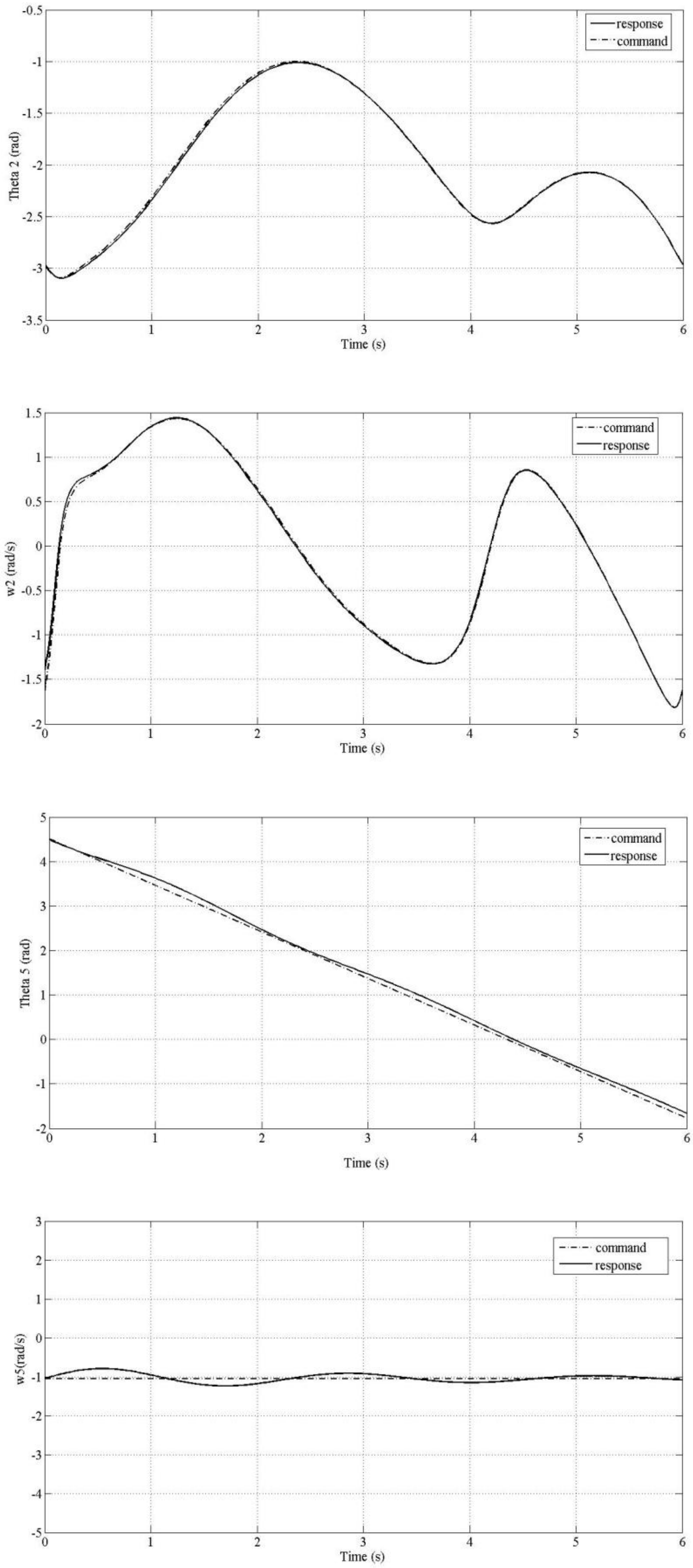
4.4.1. Results for Motion 1

The initial conditions of the angular displacement and velocity of the servo motor, the angular displacement and velocity of the constant velocity motor and the currents of the servo motor and constant velocity motor are given as X_{ic_1} = [-2.96(rad) -1.64(rad/sec) 4.51(rad) -1.0472(rad/sec) 0(A) 0(A)], respectively. The gain constants to control the system are found by using Ziegler Nichols method [33]. Then they are modified by trial and error. The proportional (k_{p_2}), derivative (k_{d_2}) and integral constant (k_{i_2}) for the servo motor are taken as 1000, 100 and 50, respectively. The proportional gain constant (k_{p_5}) for the constant velocity motor is

4000. Since it is constant velocity motor, no derivative and integral gain constants are used.

The simulation results of Motion 1 are shown in Figure 4.3. The visible lines in Figure 4.3 represent the response results of the simulation. They are the actual results. The hidden lines in Figure 4.3 are for command values. They are obtained from the kinematics analysis of the hybrid driven mechanism (Chapter 3). Figure 4.3 is composed of four parts. The angular displacement of the servo motor, θ_2 , is illustrated in Figure 4.3a. The commanded data set for θ_2 have already given in Figure 3.14a in previous chapter. The angular velocity of the servo motor, w_2 , is given in Figure 4.3b. The required data set for w_2 have already given in Figure 3.14b in previous chapter. The angular displacement of the constant velocity motor, θ_5 , is given in Figure 4.3c. The required values of θ_5 have previously shown in Figure 3.13. The angular velocity of the constant velocity motor, w_5, is a constant value, -1.0472 rad/sec. The required and actual values of the angular velocity of constant velocity motor are given in Figure 4.3d. The required and actual slider displacement curves for Motion 1 are demonstrated in Figure 4.5.

As it is noticed from the figures, the hybrid driven system can be controlled for Motion 1. The Figure 4.3a, the angular displacement of the servo motor, there is a continuous but very few deviations whose quantity is 0.03 radians between the starting point and 2.7th seconds. After the 2.7th seconds, it is visible that two lines coincide until the end of the motion. This deviation normally reflects the angular velocity of the servo motor in Figure 4.3b. There is a very little deviation at the beginning of the motion. Figure 4.3c, the angular displacement of the constant velocity motor, the beginning of the motion is very well, there is no shift until 0.3th seconds. After that, two lines start to diverge and difference gets maximum at 1.2th seconds with 0.168 radians. Then, the lines tend to converge. The lines go on parallel to each other between 2.1th and 2.5th seconds and difference between the lines is about 0.04 radians. Later, the lines tend to diverge again and the difference gets a stable characteristic whose quantity varies between 0.05 and 0.13 radians. The deviations on the angular displacements of the constant velocity motor result in the



(a) (b) (c) (d)

Figure 4.3. Simulation Results (Motion 1)

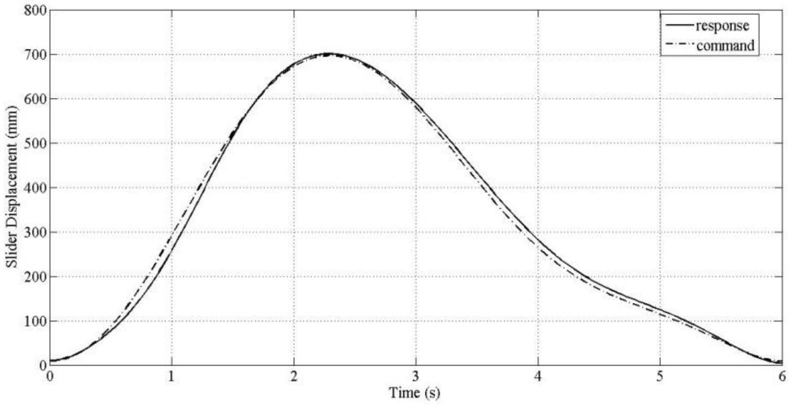


Figure 4.4. Simulated Results of Slider Output (Motion 1)

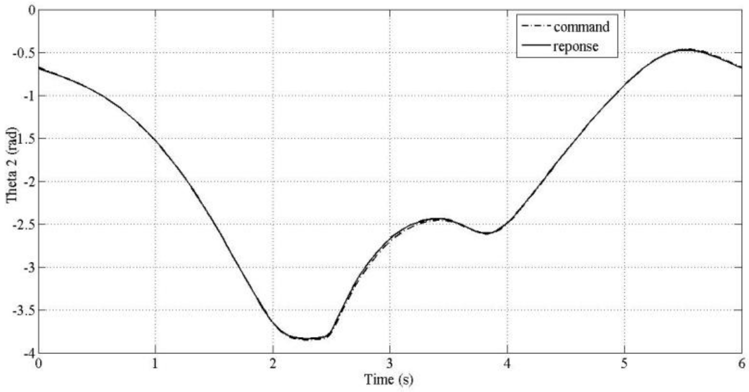
shifts on the constant angular velocity of the constant velocity motor as shown in Figure 4.3d. The slider displacement of Motion 1 is given in Figure 4.4. The controllability of the angular displacements of the cranks of the hybrid driven mechanism is directly effects the accuracy of the slider. The response line in the Figure 4.4 is obtained by using forward kinematics as mentioned in Section 3.3.2. The input parameters of it are the response values of the Figures (4.3a) and (4.3c). The response and command lines start together at the beginning of the motion. It goes on until the 0.3th seconds. After that, the lines diverge. The convergence and coincidence occur at 1.6th seconds. The maximum deviation during this period is 33.2 mm at 1st second. It traces well between the 1.6th and 2.5th seconds. Then the lines start to diverge and go on parallel to each other. The maximum divergence during this period is 17 mm.

4.4.2. Results for Motion 2

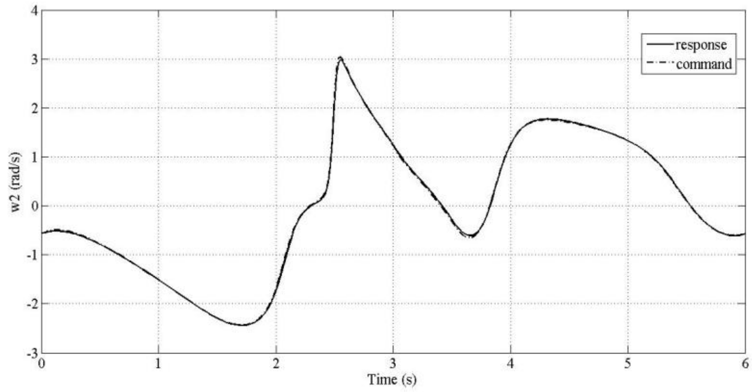
The initial conditions of the angular displacement and velocity of the servo motor, the angular displacement and velocity of the constant velocity motor and the currents of the servo motor and constant velocity motor are given as X_{ic_2} = [-0.6727 (rad) -0.5731 (rad/sec) 1.3090 (rad) 1.0472(rad/sec) 0(A) 0(A)], respectively. The gain constants to control the system are found by using Ziegler Nichols method. Then they are modified by trial and error. The proportional (k_{p_2}), derivative (k_{d_2}) and integral constant (k_{i_2}) for the servo motor are taken as 1000, 100 and 50, respectively. The proportional gain constant (k_{p_5}) for the constant velocity motor is 4000. Since it is constant velocity motor, no derivative and integral gain constants are used.

The simulation results of Motion 2 are shown in Figure 4.5 where the visible lines represent the response results of the simulation. The hidden lines are for the command values. They are obtained from the kinematics analysis of the hybrid driven mechanism in Chapter 3. Figure 4.5 is given in four parts; the angular displacement of the servo motor, θ_2 in Figure 4.5a, the angular velocity of the servo motor, w_2 is given in Figure 4.5b. The angular displacement of the constant velocity motor, θ_5 , with response and command are illustrated in Figure 4.5c. The angular velocity of the constant velocity motor, w_5, is a constant value, 1.0472 rad/sec given in Figure 4.5d. The slider displacement for Motion 2 is demonstrated in Figure 4.6.

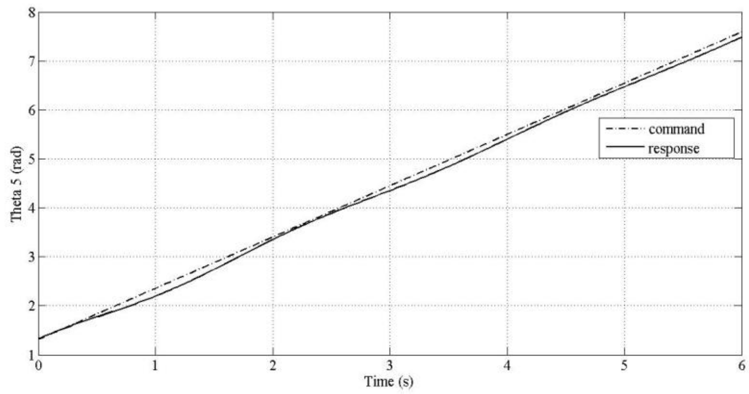
There is a little numerical difference between command and response lines in Figure 4.5a. It is between 2.5th and 3.8th seconds and its maximum quantity is 0.036 radians. The angular displacement of the servo motor is controlled well in the rest of the motion. The angular velocity of the servo motor is perfectly traced in Figure 4.5b. The Figure 4.5c, the angular displacement of the constant velocity motor, the beginning of the motion is very well, there is no shift until 0.3th seconds, two lines then start to diverge and difference gets maximum at 1.2th seconds. Its quantity is about 0.165 radians. Tracing capability is very well between 2.1th seconds and 2.6th seconds. Later, the lines tend to diverge again and the difference gets a stable condition whose quantity changes between 0.05 and 0.12 radians. The deviations on the angular displacements of the constant velocity motor cause inconsistencies on the constant angular velocity of the constant velocity motor as illustrated in Figure 4.5d. The slider displacement of Motion 2 is given in Figure 4.6. The command line in Figure 4.4e has already obtained in Figure 3.5a in the Chapter 3. The input parameters of the response line are the response values of the Figure 4.5a and Figure 4.5c. The response and command lines starts together at the beginning of the motion up to 1.1th seconds. The lines then diverge from each other and go on parallel until 2.5th seconds. The difference between the lines is maximum 18 mm and minimum 2 mm during this period. A fluctuation is visible in dwell segment of the motion. The



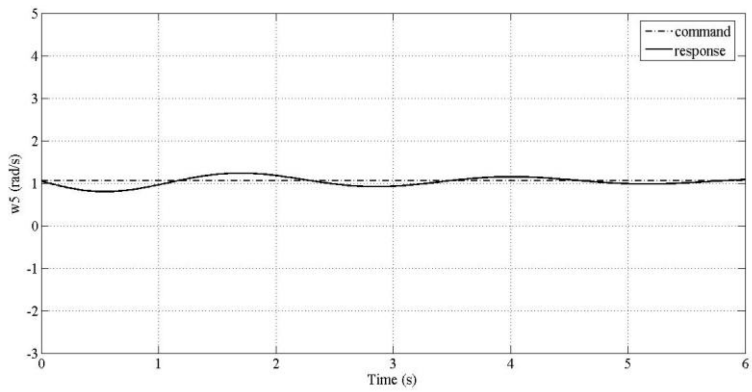
(a)



(b)



(c)



(d)

Figure 4.5. Simulation Results (Motion 2)

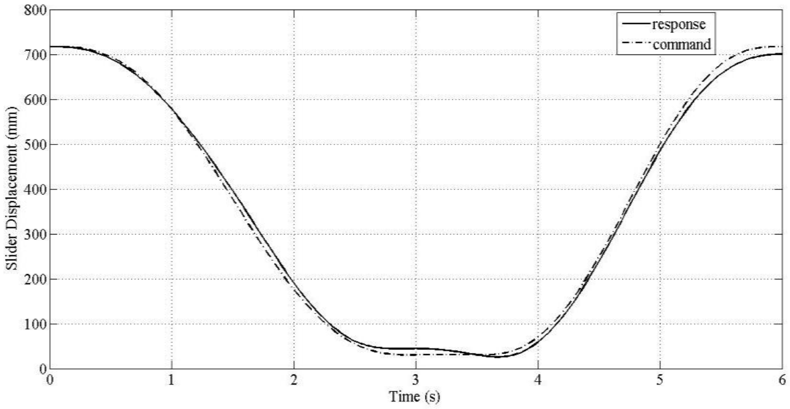


Figure 4.6. Simulated Slider Output (Motion 2)

maximum difference between command and response is 14.5 mm at 3rd seconds. After that, the lines go on parallel to each other. The difference is approximately 10 mm. A divergence is seen at the end of the motion which is about 20 mm.

4.5. The Dynamic Model from Kineto-Static Method

The Kineto-static method so called D'Alembert technique. This method rearranges the Newton's second law. A dynamic problem is transformed into static one by adding fictitious inertial forces and inertial torques onto the system. One fictitious force on each moving body which is equal to the mass of that body times the acceleration of its mass center. The direction is opposite to its acceleration. It is directly applied onto the center of gravity, apart from the already existing real forces. One fictitious torque on each moving body which is equal to the centroidal inertia of that body times its angular acceleration. The sense is opposite to that of acceleration. The already existing real torques are also applied. The model is then analyzed with the rules of statics [31].

In the dynamic model, the velocity fluctuation of the constant velocity motor, the elastic deformation and friction force of the links are assumed as zero to make the system simple. Kineto-static diagram of each link in hybrid mechanism is shown in the Figure (4.7a,b,c,d,e,f,g). The inertia forces and inertia torques are acted on every link of the hybrid driven mechanism. They are shown by dashed vectors. So the static equilibrium equations of each link can be constructed. [35]

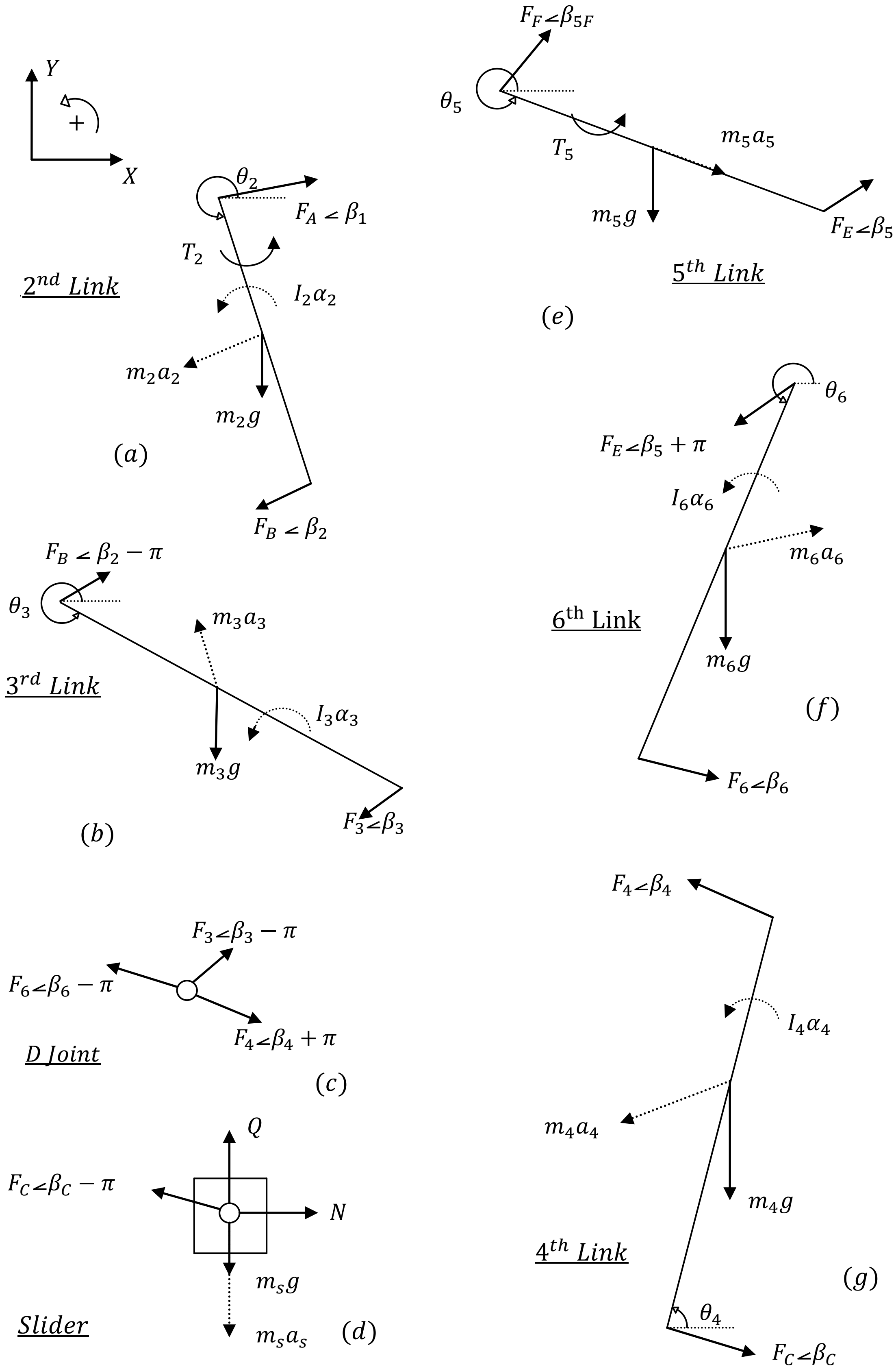


Figure 4.7. Kinetostatic Analysis of Hybrid System

Link AB

$$F_{A_x} - F_{B_x} = m_2 \ddot{X}_{cg_2} \quad (4.42)$$

$$F_{A_y} - F_{B_y} - m_2 g = m_2 \ddot{Y}_{cg_2} \quad (4.43)$$

$$r_2' = \left[r_{Acg_2} \cos(\theta_2 - \pi),\ r_{Acg_2} \sin(\theta_2 - \pi),\ 0 \right] \quad (4.44)$$

$$F_A = \left[F_{A_x},\ F_{A_y},\ 0 \right] \quad (4.45)$$

where r_{Acg_2} is the distance between the mass center of the second link and A point.

$$r_2'' = \left[r_{Bcg_2} \cos\theta_2,\ r_{Bcg_2} \sin\theta_2,\ 0 \right] \quad (4.46)$$

$$F_B = \left[-F_{B_x},\ -F_{B_y},\ 0 \right] \quad (4.47)$$

where r_{Bcg_2} is the distance between the mass center of the second link and B point.

$$\sum M_{cg_2} = (r_2' \times F_A) + (r_2'' \times F_B) + T_2 = I_2 \ddot{\theta}_{cg_2} \quad (4.48)$$

Link BD

$$F_{B_x} - F_{3_x} = m_3 \ddot{X}_{cg_3} \quad (4.49)$$

$$F_{B_y} - F_{3_y} - m_3 g = m_3 \ddot{Y}_{cg_3} \quad (4.50)$$

$$r_3' = \left[r_{Bcg_3} \cos(\theta_3 - \pi),\ r_{Bcg_3} \sin(\theta_3 - \pi),\ 0 \right] \quad (4.51)$$

$$F_B = \left[F_{B_x},\ F_{B_y},\ 0 \right] \quad (4.52)$$

where r_{Bcg_3} is the distance between the mass center of the third link and B point.

$$r_3'' = \left[r_{Dcg_3} \cos\theta_3,\ r_{Dcg_3} \sin\theta_3,\ 0 \right] \quad (4.53)$$

$$F_D = \left[-F_{3_x},\ -F_{3_y},\ 0 \right] \quad (4.54)$$

where r_{Dcg_3} is the distance between the mass center of the third link and D point.

$$\sum M_{cg_3} = (r_3' \times F_B) + (r_3'' \times FD) = I_3 \ddot{\theta}_{cg_3} \quad (4.55)$$

Link DC

$$-F_{4_x} + F_{C_x} = m_4 \ddot{X}_{cg_4} \quad (4.56)$$

$$F_{4_y} - F_{C_y} - m_4 g = m_4 \ddot{Y}_{cg_4} \quad (4.57)$$

$$r_4' = \left[r_{Dcg_4} \cos\theta_4,\ r_{Dcg_4} \sin\theta_4,\ 0 \right] \quad (4.58)$$

$$F_D = \left[-F_{4_x},\ F_{4_y},\ 0 \right] \quad (4.59)$$

where r_{Dcg_4} is the distance between the mass center of the fourth link and D point.

$$r_4'' = \left[r_{Ccg_4} \cos(\theta_4 + \pi),\ r_{Ccg_4} \sin(\theta_4 + \pi),\ 0 \right] \quad (4.60)$$

$$F_C = \left[F_{C_x}, -F_{C_y},\ 0 \right] \quad (4.61)$$

where r_{Ccg_4} is the distance between the mass center of the fourth link and C point.

$$\sum M_{cg_4} = (r_4' \times F_D) + (r_4'' \times F_C) = I_4 \ddot{\theta}_{cg_4} \quad (4.62)$$

<u>EF Link</u>

$$F_{F_x} + F_{E_x} = m_5 \ddot{X}_{cg_5} \quad (4.63)$$

$$F_{F_y} - F_{E_y} - m_5 g = m_5 \ddot{Y}_{cg_5} \quad (4.64)$$

$$r_5' = \left[r_{Fcg_5} \cos(\theta_5 - \pi),\ r_{Fcg_5} \sin(\theta_5 - \pi),\ 0 \right] \quad (4.65)$$

$$F_F = \left[F_{F_x},\ F_{F_y},\ 0 \right] \quad (4.66)$$

where r_{Fcg_5} is the distance between the mass center of the fifth link and F point.

$$r_5'' = \left[r_{Ecg_5} \cos\theta_5,\ r_{Ecg_5} \sin\theta_5,\ 0 \right] \quad (4.67)$$

$$F_E = \left[F_{E_x},\ F_{E_y},\ 0 \right] \quad (4.68)$$

where r_{Ecg_5} is the distance between the mass center of the fifth link and E point.

$$\sum M_{cg_5} = (r_5' \times F_F) + (r_5'' \times F_E) + T_5 = I_5 \ddot{\theta}_{cg_5} \quad (4.69)$$

<u>ED Link</u>

$$-F_{E_x} + F_{6_x} = m_6 \ddot{X}_{cg_6} \quad (4.70)$$

$$-F_{E_y} - F_{6_y} - m_6 g = m_6 \ddot{Y}_{cg_6} \quad (4.71)$$

$$r_6' = \left[r_{Ecg_6} \cos(\theta_6 - \pi),\ r_{Ecg_6} \sin(\theta_6 - \pi),\ 0 \right] \quad (4.72)$$

$$F_E = \left[-F_{E_x},\ -F_{E_y},\ 0 \right] \quad (4.73)$$

where r_{Ecg_6} is the distance between the mass center of the sixth link and E point.

$$r_6'' = \left[r_{Dcg_6} \cos\theta_6,\ r_{Dcg_6} \sin\theta_6,\ 0 \right] \quad (4.74)$$

$$F_D = \left[F_{6_x}, -F_{6_y},\ 0 \right] \quad (4.75)$$

where r_{Dcg_6} is the distance between the mass center of the sixth link and D point.

$$\sum M_{cg_6} = (r_6' \times F_E) + (r_6'' \times F_D) = I_6 \ddot{\theta}_{cg_6} \quad (4.76)$$

D Joint

$$F_{3_x} + F_{4_x} - F_{6_x} = 0 \quad (4.77)$$

$$F_{3_y} - F_{4_y} + F_{6_y} = 0 \quad (4.78)$$

Slider

$$-F_{C_x} + N = 0 \quad (4.79)$$

$$Q + F_{C_y} - m_{slider} g = m_{slider} a_{slider}$$

(4.80)

The static equilibrium equations are given above. Nineteen equations are obtained with nineteen unknowns. These equations are linear sets and they can be expressed in matrix form as;

$$DE=F \quad (4.81)$$

where

$$
D=\begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
D_{3,1} & D_{3,2} & D_{3,3} & D_{3,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & D_{6,3} & D_{6,4} & D_{6,5} & D_{6,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & D_{9,7} & D_{9,8} & D_{9,9} & D_{9,10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{12,11} & D_{12,12} & D_{12,13} & D_{12,14} & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{15,13} & D_{15,14} & D_{15,15} & D_{15,16} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
E=\left[F_{A_x},\ F_{A_y},\ F_{B_x},\ F_{B_y},\ F_{3_x},\ F_{3_y},\ F_{4_x},\ F_{4_y},\ F_{C_x},\ F_{C_y},\ F_{F_x},\ F_{F_y},\ F_{E_x},\ F_{E_y},\ F_{6_x},\ F_{6_y},\ T_2,\ T_5,\ N\right]^T
$$

$$
F=\begin{bmatrix}
m_2\ddot{X}_{cg_2},\ m_2\ddot{Y}_{cg_2}+m_2 g,\ I_2\ddot{\theta}_{cg_2},\ m_3\ddot{X}_{cg_3},\ m_3\ddot{Y}_{cg_3}+m_3 g,\ I_3\ddot{\theta}_{cg_3},\ m_4\ddot{X}_{cg_4},\ m_4\ddot{Y}_{cg_4}+m_4 g, I_4\ddot{\theta}_{cg_4}, \\
m_5\ddot{X}_{cg_5},\ m_5\ddot{Y}_{cg_5}+m_5 g,\ I_5\ddot{\theta}_{cg_5},\ m_6\ddot{X}_{cg_6},\ m_6\ddot{Y}_{cg_6}+m_6 g,\ I_6\ddot{\theta}_{cg_6},\ 0,\ 0,\ 0,\ m_{slider}a_{slider}+m_{slider}g-Q
\end{bmatrix}^T
$$

D is a known matrix. Some components of the D matrix are given in a symbolic form. *E* is an unknown matrix including the reaction forces in all joints and torques in the cranks of the mechanism. *F* is a known matrix formed by inertia torques, inertia forces and an external force. Equations (4.19-4.24) are used to obtain *F* matrix. E matrix can easily be found by an inverse operation. It is illustrated in Equation 4.82.

$$E=D^{-1}*F \qquad (4.82)$$

$$D_{3,1}=r_{Acg_2}\sin\theta_2\ ,\ D_{3,2}=-r_{Acg_2}\cos\theta_2\ , D_{3,3}=r_{Bcg_2}\sin\theta_2\ ,\ D_{3,4}=-r_{Bcg_2}\cos\theta_2 \qquad (4.83)$$

$$D_{6,3}=r_{Bcg_3}\sin\theta_3\ ,\qquad D_{6,4}=-r_{Bcg_3}\cos\theta_3\ ,\qquad D_{6,5}=r_{Dcg_3}\sin\theta_3\ ,\qquad D_{6,6}=-r_{Dcg_3}\cos\theta_3$$
(4.84)

$$D_{9,7}=r_{Dcg_4}\sin\theta_4\ ,\ D_{9,8}=r_{Dcg_4}\cos\theta_4\ ,\ D_{9,9}=r_{Ccg_4}\sin\theta_4\ ,\ D_{9,10}=r_{Ccg_4}\cos\theta_4 \qquad (4.85)$$

$$D_{12,11}=r_{Fcg_5}\sin\theta_5\ ,\ D_{12,12}=-r_{Fcg_5}\cos\theta_5\ ,\ D_{12,13}=-r_{Ecg_5}\sin\theta_5\ ,\ D_{12,14}=r_{Ecg_5}\cos\theta_5$$

(4.86)

$$D_{15,13} = -r_{Ecg_5}\sin\theta_6\,,\; D_{15,14} = r_{Ecg_6}\cos\theta_6\,,\; D_{15,15} = -r_{Dcg_6}\sin\theta_6\;\;, D_{15,16} = -r_{Dcg_6}\cos\theta_6$$

(4.87)

An example based on kinetostatic analysis here ;

The reaction forces in each joint and the cranks' torques can be obtained as illustrated in Figure 4.8. by using a Matlab code written for kinetostatic equations. The datas including the slider displacement, velocity and acceleration and the angular displacement, velocity and acceleration of the crank driven by the constant velocity motor and forming force of the slider are given as input parameters.

```
Command Window
                     Results of the Kinetostatic Analysis
                                    of
                           Hybrid Driven Mechanism

*** System inputs ***

s      = 9*10^-3 (m),  sdot= 0.3182*10^-3 (m/s), sdotdot= 636.33*10^-3 (m/s^2)
theta5 = 4.5124 (rad), w5  = -pi/3 (rad/s),      alpha5 = 0 (rad/s^2)
Q      = 100 (Newton)  (Forming force of the slider)

                              ***Results ***

Fax = 84.2387 (N)                            F4x = -16.6441 (N)
Fay = 11.226 (N)                             F4y = 237.23 (N)
Fa = 84.9834 (N)                             F4 = 237.813 (N)
Angle of Fa = 7.59078 (degrees)              Angle of F4 = 94.0133 (degrees)

Fbx = 83.1175 (N)                            Fcx = -4.09047 (N)
Fby = -16.5197 (N)                           Fcy = 96.8363 (N)
Fb = 84.7433 (N)                             Fc = 96.9227 (N)
Angle of Fb = -11.2411 (degrees)             Angle of Fc = 92.4188 (degrees)


F3x = 70.3345 (N)                            Ffx = -42.2557 (N)
F3y = -132.934 (N)                           Ffy = 519.72 (N)
F3 = 150.395 (N)                             Ff = 521.435 (N)
Angle of F3 = -62.1171 (degrees)             Angle of Ff = 94.6482 (degrees)

Fex = 42.3038 (N)                            F6x = 53.6904 (N)
Fey = -493.976 (N)                           F6y = 370.165 (N)
Fe = 495.784 (N)                             F6 = 374.038 (N)
Angle of Fe = -85.1052 (degrees)             Angle of F6 = 81.7471 (degrees)

T2 = 3.50812 (Nm)
T5 = -24.1615 (Nm)
N = -4.09047 (N)
>>
```

Figure 4.8. An Example on the Kinetostatic Analysis of Hybrid Driven Mechanism

CHAPTER 5

CONCLUSIONS

5.1 Discussions on the Present Study

This study has a speciality of having a solution method for kinematics and dynamic analysis of hybrid systems and motion design. The main contribution of this thesis is to give a new direction of use for hybrid systems. A basic methodology is given for system configuration, design, kinematics and dynamic issues with its control. Discussions are combined with different groups based on names of chapters as hybrid configuration, motion design, kinematics analysis & dynamic analysis & modeling & control.

(i) About Hybrid Configuration

A two degree of freedom hybrid driven system was presented in this thesis. It was composed of a two degrees of freedom planar mechanism driven by a constant velocity motor with a flywheel and a servo motor. The mechanism was a seven link one. It is a two dof planar mechanism with seven revolute joints and one prismatic joint. The crank driven by a constant velocity motor had a full rotation for each cycle. The use of servo motor in the hybrid systems has provided flexible ram outputs.

(ii) About Motion Design

Motion design procedure was explained and two different motion characteristics were designed. Segmentation technique was used. The motion was divided into segments and the fifth order polynomials were used to characterize them. The numerical value for the initial and final conditions of the segments are selected by considering requirements of the customer. The position, velocity and acceleration values for a segment must be compatible between them. Otherwise, smooth passes between the segments is not possible. Curve Fitting Toolbox in Matlab® was used as a tool to make the motion smoother. It is certainly provided to be an effective tool for motion curves.

(iii) Discussions on KinematicsAnalysis&Dynamic Analysis&Modeling &Control

The position, velocity and acceleration expressions of the mechanism were obtained by using the loop closure equations. Kinematics analysis of the hybrid driven mechanism was given as forward and inverse kinematics analysis.

Inverse kinematics was applied in order to get the kinematics specifications of the servo motor . In this technique, kinematics properties of the crank driven by constant velocity motor and slider link are used as input parameters of the system. In this operation, it not always possible to get an integer number for angular displacement of the crank driven by servo motor. Starting point of angle of the crank driven by the constant velocity motor is playing a vital role. Because, there is a trajectory to be followed by the ram. It is a stroke. During the time which it takes, the crank driven by constant velocity motor is completing a full rotation. The crank driven by the servo motor has a working envelope. If it is exceeded, the angular displacement value for servo motor will not be feasible. It means that the ram trajectory to be traced is not provided. It must be modified after all.

Forward kinematics analysis was also applied to obtain the kinematic characteristics of the slider. The kinematics specifications of the two cranks were used as input parameters of the system. Forward kinematics analysis was used in dynamic analysis and control. Because, system variables in dynamic analysis and control were chosen as the cranks of the system. So all mathematical expressions must have been expressed as a function of the crank variables.

Dynamic model of the hybrid system is derived by two techniques; Lagrangian and Kineto-static. The former is a technique based on derivations of the energy expressions. Forward kinematics is used in derivation of dynamic model of the hybrid driven mechanism. All differentiation operations and derivations were sequentially completed in order to get the general torque and angular acceleration expressions. It was really a laborious task. Because of the fact that the system has a property of highly nonlinearity, there were many embedded expressions in it. It was actually hard to follow. They were all included in Appendix. Latter is operated by adding all forces and torques to all free bodies of the mechanism, also called as D'Alembert technique. An example for kineto-static analysis for hybrid system is presented. The fourth order Runge-Kutta which is an explicit method used in integration of nonlinear systems is used as the integration technique to get an approximate solution. State space representation of the system is carried out to express the system in first order differential equations. A control system is studied to succed design objectives. PID control techniques is applied on the system. Controllability of the system is demonstrated by throughout the simulation results. Matlab ® is used to get complete analysis and design stages.

5.2 Recommendations for the Future Work

This study has presented a work on the controllable hybrid driven mechanism, including kinematics analysis, dynamic modeling, motion design and control. The benefits of both traditional and programmable machines are synthesized. However, it was obvious that there were still some limitations depending on the configuration used for hybrid system. The following subjects can be regarded as future work in order to get a better performance from the hybrid driven mechanical system.

(i) An evolutionary algorithm like genetic algorithm (GA) can be applied in linkages synthesis of the hybrid driven mechanism. Minimization of angular acceleration of the second link driven by servo motor will be chosen as the objective function of the algorithm. In hybrid machine idea, a low capacity servo motor must be used. Trajectory tracking capability of servo motor must be taken into consideration in order to achieve this purpose. If the slider moves in a complicated trajectory, a better tracing capacity servo motor is necessary to be used. The mass moment of inertia of the second link driven by the servo motor can be lowered. It is directly related with the torque provided by servo motor. Constant velocity motor is used to provide main power in the system. Servo motor has got a very important role to overcome the inertia forces or torques. If the mass moment of inertia is got lower, the inertia torque or force would be automatically reduced.

(ii) The hybrid machine systems offer motion flexibility to users. As it is already shown in Chapter 3, two different motion profiles are obtained from an already designed hybrid machine. The motion limits of hybrid machine can be determined from the beginning.

(iii) An industrial prototype will be produced as a metal forming press. As it is implied from its name, a force is required from the slider link. So force-displacement graphs will be formed for each one of the designed motion type. Torques and power characteristics belonging to cranks driven by constant velocity motor and servo motor for each motion will successively be obtained.

(iv) PID parameters can be optimized by using an evolutionary algorithm. Simulation results can be improved by using different evolutionary techniques. This is certainly considered to give a base for comparison as future study.

REFERENCES

[1] Tokuz L. C. (1992). Hybrid Machine Modelling and Control Ph.D. Thesis, Liverpool Polytechnic.

[2] Dülger L.C., Kireçci A. (1995). Hibrid Manipülatörün Modellenmesi ve Simülasyonu, 7. Ulusal Makine Teorisi Sempozyumu, YTU, Istanbul, 311-320.

[3] Dülger L.C., Kireçci A. (1999). A Study on a Hybrid Machine, *Mechanism and Machine Theory*, **35**, 1141-1149.

[4] Seth B. (2004). Programmable Hybrid Mechanisms, *Pr. of the 11th World Congress in Mech. and Machine Science*, Tianjin, China.

[5] Ouyang P.R., Li Q., Zhang W.J. (2004). Design, Modelling and Control of a Hybrid Machine System, *Mechatronics*, **14**, 1197-1217.

[6] Yuan Z., Gilmartin M. J., Douglas S.S. (2005). Design of Hybrid Machines for Nonuniform Motion Production, *Proc. IMechE*, **219** Part C, Mechanical Engineering Science, 491-499.

[7] Zhang, K. (2006). Optimization Dynamics Design of Hybrid Driving Mechanisms, Pr. of 2006 IEEE, Int. Conf. On Mechatronics & Automation China, 1914-1919.

[8] Zhang, K. (2008). Synthesis of a Hybrid Five-Bar Mechanism with Particle Swarm Optimization Algorithm, LNCS 5263, Springer –Verlag Berlin Heidelberg, 873-882.

[9] Greenough J.D., Bradshaw W.K., Gilmartin M.J., Douglas S.S., Jones J.R. (1995). Design of Hybrid Machines, *9th World Congress on the Theory of Machines & Mechanisms*, V.4, 2501-2505.

[10] Connor A.M., Douglas S.S., Gilmartin M.J. (1995). The Synthesis of Hybrid Five Bar Path Generating Mechanisms using Genetic Algorithms, *Genetic Algorithms in Engineering Systems: Innovations and Applications, IEE* No: 414, 313-316.

[11] Dülger L.C., Kireçci A., Topalbekiroglu M. (2003). Modeling and Simulation of a Hybrid Actuator, *Mechanism and Machine Theory*, **38**, 395-407.

[12] Wu F.X., Zhang W.J., Li Q., Ouyang P.R., Zhou Z.X. (2005). Control of Hybrid Machines with 2 dof for trajectory Tracking Problems, *IEEE Transactions on Control Systems Technology*, **13**, No.2, 338-342.

[13] Yu H. (2006). Modelling and Control of Hybrid Machine Systems ,A Five-bar Mechanism Case, *International Journal of Automation and Computing*, **3**, 235-243.

[14] Li H., Zhang Y., Zheng H. (2008). Dynamics Modeling and Simulation of a New Nine-Bar Press with Hybrid Driven Mechanism, *Journal of Mechanical Science and Technology*, **22**, 2436-2444.

[15] Chen Z.H., Wang Y., Ouyang P., Huang J., Zhang J. (2010). A Novel Iteration-based Controller for Hybrid Machine Systems for Trajectory Tracking at the End-effector Level, *Robotica*, 1-8.

[16] Li H., Zhang Y. (2010). Seven Bar Mechanical Press with Hybrid Driven Mechanism for Deep Drawing; Part 1: Kinematics Analysis and Optimum Design, Part 2: Dynamic Modeling and Simulation, *Journal of Mechanical Science and Technology*, 24 (11), 2153-2160, 2161-2167.

[17] Tso P.L., Liang K.C. (2002). A Nine Bar Linkage for Mechanical Forming Press, *Int. J. of Machine Tools and Manufacturing*, **42**, 139–145.

[18] Du R., Guo W.Z. (2003). The Design of a New Metal Forming Press with Controllable Mechanism, *Transactions of the ASME*, **125**, 582–591.

[19] Li H., Zhang C., Meng C. (2004). A Hybrid-Driven Nine-Bar Press for Precision Drawing, *Pr. of the 11th World Congress in Mech. And Machine Science*, Tianjin-China, 1141-1145.

[20] Meng C.F., Zhang C., Lu Y.H., Shen Z.G. (2004). Optimal Design and Control of a Novel Press with an Extra Motor, *Mechanism and Machine Theory*, **39**, 811-818.

[21] Li C. H., Tso P.L. (2008). Experimental Study on a Hybrid Driven Press using Iterative Learning Control, *Int J. of Machine Tools and Manufacture*, **48**, 209-219.

[22] Tso P. L., Li C. H. (2008). Study of Servo Press with a Flywheel, *Journal of Advanced Mechanical Design*, *Systems, and Manufacturing*, **2**, No. 1, 1-11.

[23] Ouyang P.R., Zhang W.J., Moazed R. (2008).Experimental Study on a Hybrid Actuation System, *Proceeding of the IEEE/ASME International Conference on Advanced Intelligent Mechatronics* , 465-470.

[24] He K., Luo Y., Kong C. T., Du R. (2009). Trajectory Planning, Optimization and Control of a Hybrid Mechanical Press, *WSEAS Transactions on Systems*, **8**, 614-627.

[25] Tso P. L. (2010). Optimal Design of a Hybrid-Driven Servo Press and Experimental Verification, *Journal of Mechanical Design*, **132**, 034503.

[26] Li H., Fu L., Zhang Y. (2010). Optimum Design of a Hybrid-Driven Mechanical Press Based on Inverse Kinematics, *Journal of Mechanical Engineering*, **56**, 301–306.

[27] Soylemez E. 1999. Mechanisms, M.E.T.U Mechanical Engineering Department, Publication Number: 64, Ankara.

[28] Norton N. R. 2004. Design of Machinery: An Introduction to the Synthesis and Analysis of Mechanisms and Machines. Third Edition. McGraw Hill Companies.

[29] Curve Fitting Toolbox for Use with Matlab© User's Guide. 2002. The MathWorks Inc.

[30] Niku Saeed B. 2001. Introduction to Robotics: Analysis, Systems, Applications. First Edition. Prentice Hall.

[31] Shigley J.E., Uicker J.J. 1994. Theory of Machines and Mechanisms. Second Edition. McGraw-Hill Company.

[32] Dorf R.C., Bishop R.H. 2008. Modern Control Engineering. Eleventh Edition. Pearson Prentice Hall.

[33] Ogata K. 2002. Modern Control Systems. Fourth Edition. Pearson Prentice Hall.

[34] D'Souza A. Frank, Gang Vijay K., 1984. Advanced Dynamics (Modelling and Analysis), Prentice Hall Inc.

[35] Kai H., 2008. Design and Control of a Controllable Hybrid Mechanical Metal Forming Press. PhD Thesis. The Chinese University of Hong Kong.

APPENDIX

Appendix A

Partial Derivatives of Link Angular Displacements to θ_2

$$\frac{\partial\theta_3}{\partial\theta_2} = \frac{-r_2\sin\theta_2 + r_2\cos\theta_2\tan\theta_6}{-r_3\cos\theta_3\tan\theta_6 + r_3\sin\theta_3} \tag{A.1}$$

$$\frac{\partial\theta_6}{\partial\theta_2} = \frac{-r_2\cos\theta_2\tan\theta_3 + r_2\sin\theta_2}{r_6(\sin\theta_6 - \tan\theta_3\cos\theta_6)} \tag{A.2}$$

$$\frac{\partial\theta_4}{\partial\theta_2} = \frac{r_6\sin\theta_6(\partial\theta_6/\partial\theta_2)}{r_4\sin\theta_4} \tag{A.3}$$

Partial Derivatives of Link Angular Displacements to θ_5

$$\frac{\partial\theta_6}{\partial\theta_5} = \frac{r_5(\sin\theta_5 - \cos\theta_5\tan\theta_3)}{r_6(\cos\theta_6\tan\theta_3 - \sin\theta_6)} \tag{A.4}$$

$$\frac{\partial\theta_3}{\partial\theta_5} = \frac{r_5\cos\theta_5 + r_6\cos\theta_6(\partial\theta_6/\partial\theta_5)}{r_3\cos\theta_3} \tag{A.5}$$

$$\frac{\partial\theta_4}{\partial\theta_5} = \frac{r_5\sin\theta_5 + r_6\sin\theta_6(\partial\theta_6/\partial\theta_5)}{r_4\sin\theta_4} \tag{A.6}$$

Partial Derivatives of Link Angular Velocities to $\dot{\theta}_2$

$$\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_2} = \frac{r_2 \sin(\theta_3 - \theta_2)\sin(\theta_2 - \theta_6)}{r_3 \sin(\theta_2 - \theta_3)\sin(\theta_3 - \theta_6)} \tag{A.7}$$

$$\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2} = -\frac{r_2 \sin(\theta_3 - \theta_2)\sin(\theta_6 - \theta_4)\sin\theta_6}{r_4 \sin(\theta_3 - \theta_6)\sin(\theta_4 - \theta_6)\sin\theta_4} \tag{A.8}$$

$$\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2} = \frac{r_2 \sin(\theta_3 - \theta_2)}{r_6 \sin(\theta_3 - \theta_6)} \tag{A.9}$$

Partial Derivatives of Link Angular Velocities to $\dot{\theta}_2$

$$\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_5} = \frac{r_5 \sin(\theta_2 - \theta_5) - \dfrac{r_5 \sin(\theta_2 - \theta_6)\sin(\theta_3 - \theta_5)}{\sin(\theta_3 - \theta_6)}}{r_3 \sin(\theta_2 - \theta_3)} \tag{A.10}$$

$$\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_5} = \frac{r_5 \sin(\theta_5 - \theta_6) + \dfrac{\sin\theta_6 \left(r_5 \sin(\theta_4 - \theta_5) + \dfrac{r_5 \sin(\theta_3 - \theta_5)\sin(\theta_6 - \theta_4)}{\sin(\theta_3 - \theta_6)}\right)}{\sin\theta_4}}{r_4 \sin(\theta_4 - \theta_6)} \tag{A.11}$$

$$\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5} = -\frac{r_5 \sin(\theta_3 - \theta_5)}{r_6 \sin(\theta_3 - \theta_6)} \tag{A.12}$$

Time Derivatives of Partial Derivatives of Link Angular Velocities to $\dot{\theta}_2$

$$\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_2}\right) = (r_2\cos(\theta_2-\theta_6)\sin(\theta_3-\theta_2)(\dot{\theta}_2-\dot{\theta}_6))/(r_3\sin(\theta_2-\theta_3)\sin(\theta_3-\theta_6)) - (r_2\cos(\theta_3-\theta_2)\sin(\theta_2-\theta_6)(\dot{\theta}_2-\dot{\theta}_3))/(r_3\sin(\theta_2-\theta_3)\sin(\theta_3-\theta_6)) - (r_2\cos(\theta_2-\theta_3)\sin(\theta_3-\theta_2)\sin(\theta_2-\theta_6)(\dot{\theta}_2-\dot{\theta}_3))/(r_3\sin(\theta_2-\theta_3)\sin(\theta_3-\theta_6)) - (r_2\cos(\theta_2-\theta_3)\sin(\theta_3-\theta_2)\sin(\theta_2-\theta_6)(\dot{\theta}_2-\dot{\theta}_3))/(r_3\sin^2(\theta_2-\theta_3)\sin(\theta_3-\theta_6)) - (r_2\cos(\theta_3-\theta_6)\sin(\theta_3-\theta_2)\sin(\theta_2-\theta_6)(\dot{\theta}_3-\dot{\theta}_6))/(r_3\sin(\theta_2-\theta_3)\sin^2(\theta_3-\theta_6))$$

(A.13)

$$\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2}\right) = (r_2\cos(\theta_3-\theta_2)\sin(\theta_6-\theta_4)\sin\theta_6(\dot{\theta}_2-\dot{\theta}_3))/(r_4\sin(\theta_3-\theta_6)\sin(\theta_4-\theta_6)\sin\theta_4) - (r_2\dot{\theta}_6\sin(\theta_3-\theta_2)\sin(\theta_6-\theta_4)\cos\theta_6)/(r_4\sin(\theta_3-\theta_6)\sin(\theta_4-\theta_6)\sin^2\theta_4) + (r_2\cos(\theta_3-\theta_6)\sin(\theta_3-\theta_2)\sin(\theta_6-\theta_4)\sin\theta_6(\dot{\theta}_3-\dot{\theta}_6))/(r_4\sin^2(\theta_3-\theta_6)\sin(\theta_4-\theta_6)\sin\theta_4) + (r_2\cos(\theta_4-\theta_6)\sin(\theta_3-\theta_2)\sin(\theta_6-\theta_4)\sin\theta_6(\dot{\theta}_4-\dot{\theta}_6))/(r_4\sin(\theta_3-\theta_6)\sin^2(\theta_4-\theta_6)\sin\theta_4)$$

(A.14)

$$\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right) = -(r_2\cos(\theta_3-\theta_2)(\dot{\theta}_2-\dot{\theta}_3))/(r_6\sin(\theta_3-\theta_6)) - (r_2\cos(\theta_3-\theta_6)\sin(\theta_3-\theta_2)(\dot{\theta}_3-\dot{\theta}_6))/(r_6\sin^2(\theta_3-\theta_6))$$

(A.15)

Time Derivatives of Partial Derivatives of Link Angular Velocities to $\dot{\theta}_5$

$$\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_5}\right) = (r_5\cos(\theta_2-\theta_5)(\dot{\theta}_2-\dot{\theta}_5) - (r_5\cos(\theta_2-\theta_6)\sin(\theta_3-\theta_5)(\dot{\theta}_2-\dot{\theta}_6))/\sin(\theta_3-\theta_6) - (r_5\cos(\theta_3-\theta_5)\sin(\theta_2-\theta_6)(\dot{\theta}_3-\dot{\theta}_5))/\sin(\theta_3-\theta_6) + (r_5\cos(\theta_3-\theta_6)\sin(\theta_2-\theta_6)\sin(\theta_3-\theta_5)(\dot{\theta}_3-\dot{\theta}_6))/\sin^2(\theta_3-\theta_6))/(r_3\sin(\theta_2-\theta_3)) - (\cos(\theta_2-\theta_3)(\dot{\theta}_2-\dot{\theta}_3)(r_5\sin(\theta_2-\theta_5) - (r_5\sin(\theta_2-\theta_6)\sin(\theta_3-\theta_5))/\sin(\theta_3-\theta_6)))/(r_3\sin^2(\theta_2-\theta_3))$$

(A.16)

$$\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_5}\right)=(r_5\cos(\theta_5-\theta_6)(\dot{\theta}_5-\dot{\theta}_6)+(\sin\theta_6(r_5\cos(\theta_4-\theta_5)(\dot{\theta}_4-\dot{\theta}_5)+(r_5\cos(\theta_3-\theta_5)$$
$$\sin(\theta_6-\theta_4)(\dot{\theta}_3-\dot{\theta}_5))/\sin(\theta_3-\theta_6)-(r_5\cos(\theta_6-\theta_4)\sin(\theta_3-\theta_5)(\dot{\theta}_4-\dot{\theta}_6))/\sin(\theta_3-\theta_6)$$
$$-(r_5\cos(\theta_3-\theta_6)\sin(\theta_3-\theta_5)\sin(\theta_6-\theta_4)(\dot{\theta}_3-\dot{\theta}_6))/\sin^2(\theta_3-\theta_6)))/\sin\theta_4+(\dot{\theta}_6\cos\theta_6(r_5$$
$$\sin(\theta_4-\theta_5)+(r_5\sin(\theta_3-\theta_5)\sin(\theta_6-\theta_4))/\sin(\theta_3-\theta_6)))/\sin\theta_4-(\dot{\theta}_4\cos\theta_4\sin\theta_6(r_5\sin(\theta_4-\theta_5)$$
$$+(r_5\sin(\theta_3-\theta_5)\sin(\theta_6-\theta_4))/\sin(\theta_3-\theta_6)))/\sin^2\theta_4)/(r_4\sin(\theta_4-\theta_6))-(\cos(\theta_4-\theta_6)(r_5\sin(\theta_5-$$
$$\theta_6)+\sin\theta_6(r_5\sin(\theta_4-\theta_5)+(r_5\sin(\theta_3-\theta_5)\sin(\theta_6-\theta_4))/\sin(\theta_3-\theta_6)))/\sin\theta_4))(\dot{\theta}_4-\dot{\theta}_6))/(r_4$$
$$\sin^2(\theta_4-\theta_6))$$

(A.17)

$$\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)=(r_5\cos(\theta_3-\theta_6)\sin(\theta_3-\theta_5)(\dot{\theta}_3-\dot{\theta}_6))/(r_6\sin^2(\theta_3-\theta_6))-(r_5\cos(\theta_3-\theta_5)(\dot{\theta}_3-\dot{\theta}_5))/$$
$$(r_6\sin(\theta_3-\theta_6))$$

(A.18)

Partial Derivatives of Link Angular Velocities to θ_2

$$\frac{\partial\dot{\theta}_3}{\partial\theta_2}=((\sin(\theta_2-\theta_6)(r_2\dot{\theta}_2\cos(\theta_3-\theta_2)((\partial\theta_3/\partial\theta_2)-1)-(\partial\theta_3/\partial\theta_2)r_5\dot{\theta}_5\cos(\theta_3-\theta_5)))/\sin(\theta_3-\theta_6)$$
$$+r_5\dot{\theta}_5\cos(\theta_2-\theta_5)-(\cos(\theta_2-\theta_6)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5))((\partial\theta_6/\partial\theta_2)-1))/\sin(\theta_3-$$
$$\theta_6)-(\cos(\theta_3-\theta_6)\sin(\theta_2-\theta_6)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5))((\partial\theta_3/\partial\theta_2)-(\partial\theta_6/\partial\theta_2)))/$$
$$\sin^2(\theta_3-\theta_6))/(r_3\sin(\theta_2-\theta_3))+(\cos(\theta_2-\theta_3)((\partial\theta_3/\partial\theta_2)-1)((\sin(\theta_2-\theta_6)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)$$
$$-r_5\dot{\theta}_5\sin(\theta_3-\theta_5)))/\sin(\theta_3-\theta_6)+r_5\dot{\theta}_5\sin(\theta_2-\theta_5)))/(r_3\sin^2(\theta_2-\theta_3))$$

(A.19)

$$\frac{\partial \dot{\theta}_4}{\partial \theta_2} = ((\sin\theta_6((\cos(\theta_6-\theta_4)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5))((\partial\theta_4/\partial\theta_2)-(\partial\theta_6/\partial\theta_2)))$$
$$/\sin(\theta_3-\theta_6)-(\sin(\theta_6-\theta_4)(r_2\dot{\theta}_2\cos(\theta_3-\theta_2)((\partial\theta_3/\partial\theta_2)-1)-((\partial\theta_3/\partial\theta_2)-1)-(\partial\theta_3/\partial\theta_2)$$
$$r_5\dot{\theta}_5\cos(\theta_3-\theta_5)))/\sin(\theta_3-\theta_6)+(\partial\theta_4/\partial\theta_2)r_5\dot{\theta}_5\cos(\theta_4-\theta_5)+(\cos(\theta_3-\theta_6)\sin(\theta_6-\theta_4)$$
$$(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5))((\partial\theta_3/\partial\theta_2)-(\partial\theta_6/\partial\theta_2)))/\sin^2(\theta_3-\theta_6)))/\sin\theta_4-$$
$$(\partial\theta_6/\partial\theta_2)r_5\dot{\theta}_5\cos(\theta_5-\theta_6)-((\partial\theta_6/\partial\theta_2)\cos\theta_6((\sin(\theta_6-\theta_4)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5$$
$$\sin(\theta_3-\theta_5)))/\sin(\theta_3-\theta_6)-r_5\dot{\theta}_5\sin(\theta_4-\theta_5)))/\sin\theta_4+((\partial\theta_4/\partial\theta_2)\cos\theta_4\sin\theta_6((\sin(\theta_6-$$
$$\theta_4)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5)))/\sin(\theta_3-\theta_6)-r_5\dot{\theta}_5\sin(\theta_4-\theta_5)))/\sin^2\theta_4)/(r_4\sin$$
$$(\theta_4-\theta_6))-(\cos(\theta_4-\theta_6)(r_5\dot{\theta}_5\sin(\theta_5-\theta_6)-(\sin\theta_6((\sin(\theta_6-\theta_4)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin$$
$$(\theta_3-\theta_5)))/\sin(\theta_3-\theta_6)-r_5\dot{\theta}_5\sin(\theta_4-\theta_5)))/\sin\theta_4)((\partial\theta_4/\partial\theta_2)-(\partial\theta_6/\partial\theta_2)))/(r_4\sin^2$$
$$(\theta_4-\theta_6))$$

(A.20)

$$\frac{\partial \dot{\theta}_6}{\partial \theta_2} = (r_2\dot{\theta}_2\cos(\theta_3-\theta_2)((\partial\theta_3/\partial\theta_2)-1)-(\partial\theta_3/\partial\theta_2)r_5\dot{\theta}_5\cos(\theta_3-\theta_5))/(r_6\sin(\theta_3-\theta_6))-$$
$$(\cos(\theta_3-\theta_6)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5))((\partial\theta_3/\partial\theta_2)-(\partial\theta_6/\partial\theta_2)))/(r_6\sin^2(\theta_3-\theta_6))$$

(A.21)

Partial Derivatives of Link Angular Velocities to θ_5

$$\frac{\partial \dot{\theta}_3}{\partial \theta_5} = ((\partial\theta_3/\partial\theta_5)\cos(\theta_2-\theta_3)((\sin(\theta_2-\theta_6)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5)))/\sin(\theta_3-\theta_6)$$
$$+r_5\dot{\theta}_5\sin(\theta_2-\theta_5)))/(r_3\sin^2(\theta_2-\theta_3))-((\sin(\theta_2-\theta_6)(r_5\dot{\theta}_5\cos(\theta_3-\theta_5)((\partial\theta_3/\partial\theta_5)-1)-$$
$$(\partial\theta_3/\partial\theta_5)r_2\dot{\theta}_2\cos(\theta_3-\theta_2)))/\sin(\theta_3-\theta_6)+r_5\dot{\theta}_5\cos(\theta_2-\theta_5)+((\partial\theta_6/\partial\theta_5)\cos(\theta_2-\theta_6)(r_2\dot{\theta}_2$$
$$\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5)))/\sin(\theta_3-\theta_6)+(\cos(\theta_3-\theta_6)\sin(\theta_2-\theta_6)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-$$
$$r_5\dot{\theta}_5\sin(\theta_3-\theta_5))((\partial\theta_3/\partial\theta_5)-(\partial\theta_6/\partial\theta_5))/\sin^2(\theta_3-\theta_6))/(r_3\sin(\theta_2-\theta_3))$$

(A.22)

$$
\begin{gathered}
\frac{\partial\dot{\theta}_4}{\partial\theta_5} = ((\sin\theta_6((\sin(\theta_6-\theta_4)(r_5\dot{\theta}_5\cos(\theta_3-\theta_5)((\partial\theta_3/\partial\theta_5)-1)-(\partial\theta_3/\partial\theta_5)r_2\dot{\theta}_2\cos(\theta_3-\theta_2))) \\
/\sin(\theta_3-\theta_6)+r_5\dot{\theta}_5\cos(\theta_4-\theta_5)((\partial\theta_4/\partial\theta_5)-1)+(\cos(\theta_6-\theta_4)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3 \\
-\theta_5))((\partial\theta_4/\partial\theta_5)-(\partial\theta_6/\partial\theta_5)))/\sin(\theta_3-\theta_6)+(\cos(\theta_3-\theta_6)\sin(\theta_6-\theta_4)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2) \\
-r_5\dot{\theta}_5\sin(\theta_3-\theta_5))((\partial\theta_3/\partial\theta_5)-(\partial\theta_6/\partial\theta_5)))/\sin^2(\theta_3-\theta_6)))/\sin\theta_4-r_5\dot{\theta}_5\cos(\theta_5-\theta_6) \\
((\partial\theta_6/\partial\theta_5)-1)-((\partial\theta_6/\partial\theta_5)\cos\theta_6((\sin(\theta_6-\theta_4)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5))) \\
/\sin(\theta_3-\theta_6)-r_5\dot{\theta}_5\sin(\theta_4-\theta_5)))\sin\theta_4+((\partial\theta_4/\partial\theta_5)\cos\theta_4\sin\theta_6((\sin(\theta_6-\theta_4)(r_2\dot{\theta}_2\sin \\
(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5)))/\sin(\theta_3-\theta_6)-r_5\dot{\theta}_5\sin(\theta_4-\theta_5)))/\sin^2\theta_4)/(r_4\sin(\theta_4-\theta_6))- \\
(\cos(\theta_4-\theta_6)(r_5\dot{\theta}_5\sin(\theta_5-\theta_6)-(\sin\theta_6((\sin(\theta_6-)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5)))/ \\
\sin(\theta_3-\theta_6)-r_5\dot{\theta}_5\sin(\theta_4-\theta_5)))/\sin\theta_4)((\partial\theta_4/\partial\theta_5)-(\partial\theta_6/\partial\theta_5)))/(r_4\sin^2(\theta_4-\theta_6))
\end{gathered}
$$

(A.23)

$$
\begin{gathered}
\frac{\partial\dot{\theta}_6}{\partial\theta_5} = -(r_5\dot{\theta}_5\cos(\theta_3-\theta_5)((\partial\theta_3/\partial\theta_5)-1)-(\partial\theta_3/\partial\theta_5)r_2\dot{\theta}_2\cos(\theta_3-\theta_2))/(r_6\sin(\theta_3-\theta_6))- \\
(\cos(\theta_3-\theta_6)(r_2\dot{\theta}_2\sin(\theta_3-\theta_2)-r_5\dot{\theta}_5\sin(\theta_3-\theta_5))((\partial\theta_3/\partial\theta_5)-(\partial\theta_6/\partial\theta_5)))/(r_6\sin^2(\theta_3-\theta_6))
\end{gathered}
$$

(A.24)

Lagrange Equation for the configuration

$$
\begin{gathered}
L = (m_3((r_2\dot{\theta}_2\sin\theta_2+r_{cg_3}\dot{\theta}_3\sin\theta_3)^2+(r_2\dot{\theta}_2\cos\theta_2+r_{cg_3}\dot{\theta}_3\cos\theta_3)^2))/2+(m_6((r_5\dot{\theta}_5\sin\theta_5+r_{cg_6} \\
\dot{\theta}_6\sin\theta_6)^2+(r_5\dot{\theta}_5\cos\theta_5+r_{cg_6}\dot{\theta}_6\cos\theta_6)^2/2+(m_4((r_5\dot{\theta}_5\cos\theta_5-r_{cg_4}\dot{\theta}_4\cos\theta_4+r_6\dot{\theta}_6\cos\theta_6)^2+ \\
(r_5\dot{\theta}_5\sin\theta_5-r_{cg_4}\dot{\theta}_4\sin\theta_4+r_6\dot{\theta}_6\sin\theta_6)^2))/2+(m_2(r_{cg_2}{}^2\dot{\theta}_2{}^2\cos^2\theta_2+r_{cg_2}{}^2\dot{\theta}_2{}^2\sin^2\theta_2))/2+(m_5 \\
(r_{cg_5}^2\dot{\theta}_5{}^2\cos^2\theta_5+r_{cg_5}^2\dot{\theta}_5{}^2\sin^2\theta_5))/2+(m_{slider}(r_5\dot{\theta}_5\cos\theta_5-r_4\dot{\theta}_4\cos\theta_4+r_6\dot{\theta}_6\cos\theta_6)^2)/2-m_6g \\
(r_5\sin\theta_5+r_{cg_6}\sin\theta_6)+m_{slider}g(r_4\sin\theta_4-r_5\sin\theta_5-r_6\sin\theta_6)+(m_2r_2{}^2\dot{\theta}_2{}^2)/24+(m_3r_3{}^2\dot{\theta}_3{}^2)/24 \\
+(m_4r_4{}^2\dot{\theta}_4{}^2)/24+(m_5r_5{}^2\dot{\theta}_5{}^2)/24+(m_6r_6{}^2\dot{\theta}_6{}^2)/24-m_3g(Y_A+r_2\sin\theta_2+r_{cg_3}\sin\theta_3)-m_4g(r_5 \\
\sin\theta_5-r_{cg_4}\sin\theta_4+r_6\sin\theta_6)-m_2g(Y_A+r_{cg_2}\sin\theta_2)-m_5gr_{cg_5}\sin\theta_5
\end{gathered}
$$

(A.25)

Partial Derivative of Lagrange Equation to θ_2

$$\frac{\partial L}{\partial \theta_2} = (m_6((2(\partial\dot{\theta}_6/\partial\theta_2)r_{cg_6}\cos\theta_6 - 2(\partial\theta_6/\partial\theta_2)r_{cg_6}\dot{\theta}_6\sin\theta_6)(r_5\dot{\theta}_5\cos\theta_5 + r_{cg_6}\dot{\theta}_6\cos\theta_6) +$$
$$(2(\partial\dot{\theta}_6/\partial\theta_2)r_{cg_6}\sin\theta_6 + 2(\partial\theta_6/\partial\theta_2)r_{cg_6}\dot{\theta}_6\cos\theta_6)(r_5\dot{\theta}_5\sin\theta_5 + r_{cg_6}\dot{\theta}_6\sin\theta_6)))/2 - (m_3((2$$
$$r_2\dot{\theta}_2\cos\theta_2 + 2r_{cg_3}\dot{\theta}_3\cos\theta_3)(r_2\dot{\theta}_2\sin\theta_2 - (\partial\dot{\theta}_3/\partial\theta_2)r_{cg_3}\cos\theta_3 + (\partial\theta_3/\partial\theta_2)r_{cg_3}\dot{\theta}_3\cos\theta_3) - (2$$
$$r_2\dot{\theta}_2\sin\theta_2 + 2r_{cg_3}\dot{\theta}_3\sin\theta_3)((\partial\dot{\theta}_3/\partial\theta_2)r_{cg_3}\dot{\theta}_3\sin\theta_3 + r_2\dot{\theta}_2\cos\theta_2 + (\partial\theta_3/\partial\theta_2)r_{cg_3}\dot{\theta}_3\cos\theta_3)))/2$$
$$-(m_4((2r_5\dot{\theta}_5\cos\theta_5 - 2r_{cg_4}\dot{\theta}_4\cos\theta_4 + 2r_6\dot{\theta}_6\cos\theta_6)((\partial\dot{\theta}_4/\partial\theta_2)r_{cg_4}\cos\theta_4 - (\partial\dot{\theta}_6/\partial\theta_2)r_6\cos\theta_6$$
$$-(\partial\theta_4/\partial\theta_2)r_{cg_4}\dot{\theta}_4\sin\theta_4 + (\partial\theta_6/\partial\theta_2)r_{cg_6}\dot{\theta}_6\sin\theta_6) + (2r_5\dot{\theta}_5\sin\theta_5 - 2r_{cg_4}\dot{\theta}_4\sin\theta_4 + 2r_6\dot{\theta}_6\sin\theta_6)$$
$$(\partial\dot{\theta}_4/\partial\theta_2)r_{cg_4}\sin\theta_4 - (\partial\dot{\theta}_6/\partial\theta_2)r_6\sin\theta_6 + (\partial\theta_4/\partial\theta_2)r_{cg_4}\dot{\theta}_4\cos\theta_4 - (\partial\theta_6/\partial\theta_2)r_{cg_6}\dot{\theta}_6\cos\theta_6)$$
$$))/2 - m_{slider}(r_5\dot{\theta}_5\cos\theta_5 - r_4\dot{\theta}_4\cos\theta_4 + r_6\dot{\theta}_6\cos\theta_6)((\partial\dot{\theta}_4/\partial\theta_2)r_4\cos\theta_4 - (\partial\dot{\theta}_6/\partial\theta_2)r_6\cos\theta_6$$
$$-(\partial\theta_4/\partial\theta_2)r_4\dot{\theta}_4\sin\theta_4 + (\partial\theta_6/\partial\theta_2)r_6\dot{\theta}_6\sin\theta_6) - m_3g(r_2\cos\theta_2 + (\partial\theta_3/\partial\theta_2)r_{cg_3}\cos\theta_3) + m_4g$$
$$(\partial\theta_4/\partial\theta_2)r_{cg_4}\cos\theta_4 - (\partial\theta_6/\partial\theta_2)r_6\cos\theta_6) + m_{slider}g((\partial\theta_4/\partial\theta_2)r_4\cos\theta_4 - (\partial\theta_6/\partial\theta_2)r_6$$
$$\cos\theta_6) + ((\partial\dot{\theta}_6/\partial\theta_2)m_3r_3^2\dot{\theta}_3)/12 + ((\partial\dot{\theta}_4/\partial\theta_2)m_4r_4^2\dot{\theta}_4)/12 + ((\partial\dot{\theta}_6/\partial\theta_2)m_6r_6^2\dot{\theta}_6)/12 -$$
$$m_2gr_{cg_2}\cos\theta_2 - (\partial\theta_6/\partial\theta_2)m_6gr_{cg_6}\cos\theta_6$$

(A.26)

Partial Derivative of Lagrange Equation to θ_5

$$\frac{\partial L}{\partial \theta_5} = (m_4((2r_5\dot{\theta}_5\sin\theta_5 - 2r_{cg_4}\dot{\theta}_4\sin\theta_4 + 2r_6\dot{\theta}_6\sin\theta_6)((\partial\dot{\theta}_6/\partial\theta_5)r_6\sin\theta_6 - (\partial\dot{\theta}_4/\partial\theta_5)r_{cg_4}\sin\theta_4 +$$
$$r_5\dot{\theta}_5\cos\theta_5 - (\partial\theta_4/\partial\theta_5)r_{cg_4}\dot{\theta}_4\cos\theta_4 + (\partial\theta_6/\partial\theta_5)r_6\dot{\theta}_6\cos\theta_6) - (2r_5\dot{\theta}_5\cos\theta_5 - 2r_{cg_4}\dot{\theta}_4\cos\theta_4 + 2r_6$$
$$\dot{\theta}_6\cos\theta_6)((\partial\dot{\theta}_4/\partial\theta_5)r_{cg_4}\cos\theta_4 - (\partial\dot{\theta}_6/\partial\theta_5)r_6\cos\theta_6 + r_5\dot{\theta}_5\sin\theta_5 - (\partial\theta_4/\partial\theta_5)r_{cg_4}\dot{\theta}_4\sin\theta_4 + (\partial\theta_6/$$
$$\partial\theta_5)r_6\dot{\theta}_6\sin\theta_6)))/2 + (m_3((2(\partial\dot{\theta}_3/\partial\theta_5)r_{cg_3}\cos\theta_3 - 2(\partial\theta_3/\partial\theta_5)r_{cg_3}\dot{\theta}_3\sin\theta_3)(r_2\dot{\theta}_2\cos\theta_2 + r_{cg_3}\dot{\theta}_3$$
$$\cos\theta_3) + (2(\partial\dot{\theta}_3/\partial\theta_5)r_{cg_3}\sin\theta_3 + 2(\partial\theta_3/\partial\theta_5)r_{cg_3}\dot{\theta}_3\cos\theta_3)(r_2\dot{\theta}_2\sin\theta_2 + r_{cg_3}\dot{\theta}_3\sin\theta_3)))/2 - (m_6($$
$$(2r_5\dot{\theta}_5\cos\theta_5 + 2r_{cg_6}\dot{\theta}_6\cos\theta_6)(r_5\dot{\theta}_5\sin\theta_5 - (\partial\dot{\theta}_6/\partial\theta_5)r_{cg_6}\cos\theta_6 + (\partial\theta_6/\partial\theta_5)r_{cg_6}\dot{\theta}_6\sin\theta_6) - (2r_5\dot{\theta}_5$$
$$\sin\theta_5 + 2r_{cg_6}\dot{\theta}_6\sin\theta_6)((\partial\dot{\theta}_6/\partial\theta_5)r_{cg_6}\sin\theta_6 + r_5\dot{\theta}_5\cos\theta_5 + (\partial\theta_6/\partial\theta_5)r_{cg_6}\dot{\theta}_6\cos\theta_6)))/2 - m_6g(r_5$$
$$\cos\theta_5 + (\partial\theta_6/\partial\theta_5)r_{cg_6}\cos\theta_6) - m_4g(r_5\cos\theta_5 - (\partial\theta_4/\partial\theta_5)r_{cg_4}\cos\theta_4 + (\partial\theta_6/\partial\theta_5)r_6\cos\theta_6) -$$
$$m_{slider}g(r_5\cos\theta_5 - (\partial\theta_4/\partial\theta_5)r_4\cos\theta_4 + (\partial\theta_6/\partial\theta_5)r_6\cos\theta_6) - m_{slider}(r_5\dot{\theta}_5\cos\theta_5 - r_4\dot{\theta}_4\cos\theta_4 + r_6\dot{\theta}_6$$
$$\cos\theta_6)((\partial\dot{\theta}_4/\partial\theta_5)r_4\cos\theta_4 - (\partial\dot{\theta}_6/\partial\theta_5)r_6\cos\theta_6 + r_5\dot{\theta}_5\sin\theta_5 - (\partial\theta_4/\partial\theta_5)r_4\dot{\theta}_4\cos\theta_4 + (\partial\theta_6/\partial\theta_5)$$
$$r_6\dot{\theta}_6\sin\theta_6) + ((\partial\dot{\theta}_3/\partial\theta_5)m_3r_3^2\dot{\theta}_3)/12 + (\partial\dot{\theta}_4/\partial\theta_5)m_4r_4^2\dot{\theta}_4)/12 + ((\partial\dot{\theta}_6/\partial\theta_5)m_6r_6^2\dot{\theta}_6)/12 - m_5gr_{cg_5}$$
$$\cos\theta_5 - (\partial\theta_3/\partial\theta_5)gm_3r_{cg_3}\cos\theta_3$$

(A.27)

Partial Derivative of Lagrange Equation to $\dot{\theta}_2$

$$\frac{\partial L}{\partial \dot{\theta}_2} = (m_3((2r_2 \cos\theta_2 + 2(\partial\dot{\theta}_3 / \partial\dot{\theta}_2)r_{cg_3} \cos\theta_3(r_2\dot{\theta}_2 \cos\theta_2 + r_{cg_3}\dot{\theta}_3 \cos\theta_3) + (2r_2 \sin\theta_2 + 2(\partial\dot{\theta}_3$$
$$/\partial\dot{\theta}_2)r_{cg_3} \sin\theta_3)(r_2\dot{\theta}_2 \sin\theta_2 + r_{cg_3}\dot{\theta}_3 \sin\theta_3))) / 2 - (m_4((2(\partial\dot{\theta}_4 / \partial\dot{\theta}_2)r_{cg_4} \cos\theta_4 - 2(\partial\dot{\theta}_6 / \partial\dot{\theta}_2)r_6$$
$$\cos\theta_6)(r_5\dot{\theta}_5 \cos\theta_5 - r_{cg_4}\dot{\theta}_4 \cos\theta_4 + r_6\dot{\theta}_6 \cos\theta_6) + (2(\partial\dot{\theta}_4 / \partial\dot{\theta}_2)r_{cg_4} \sin\theta_4 - 2(\partial\dot{\theta}_6 / \partial\dot{\theta}_2)r_6 \sin\theta_6)$$
$$(r_5\dot{\theta}_5 \sin\theta_5 - r_{cg_4}\dot{\theta}_4 \sin\theta_4 + r_6\dot{\theta}_6 \sin\theta_6))) / 2 + (m_6((2(\partial\dot{\theta}_6 / \partial\dot{\theta}_2)r_{cg_6} \cos\theta_6(r_5\dot{\theta}_5 \cos\theta_5 + r_{cg_6}\dot{\theta}_6$$
$$\cos\theta_6) + 2(\partial\dot{\theta}_6 / \partial\dot{\theta}_2)r_{cg_6} \sin\theta_6(r_5\dot{\theta}_5 \sin\theta_5 + r_{cg_6}\dot{\theta}_6 \sin\theta_6))) / 2 - m_{slider}((\partial\dot{\theta}_4 / \partial\dot{\theta}_2)r_4 \cos\theta_4 -$$
$$(\partial\dot{\theta}_6 / \partial\dot{\theta}_2)r_6 \cos\theta_6)(r_5\dot{\theta}_5 \cos\theta_5 - r_4\dot{\theta}_4 \cos\theta_4 + r_6\dot{\theta}_6 \cos\theta_6) + (m_2 r_2{}^2\dot{\theta}_2) / 12 + m_2 r_{cg_2}{}^2\dot{\theta}_2 +$$
$$((\partial\dot{\theta}_3 / \partial\dot{\theta}_2)m_3 r_3{}^2\dot{\theta}_3) / 12 + ((\partial\dot{\theta}_4 / \partial\dot{\theta}_2)m_4 r_4{}^2\dot{\theta}_4) / 12 + ((\partial\dot{\theta}_6 / \partial\dot{\theta}_2)m_6 r_6{}^2\dot{\theta}_6) / 12$$

(A.28)

Partial Derivative of Lagrange Equation to $\dot{\theta}_5$

$$\frac{\partial L}{\partial \dot{\theta}_5} = (m_4((2r_5 \cos\theta_5 - 2(\partial\dot{\theta}_4 / \partial\dot{\theta}_5)r_{cg_4} \cos\theta_4 + 2(\partial\dot{\theta}_6 / \partial\dot{\theta}_5)r_6 \cos\theta_6)(r_5\dot{\theta}_5 \cos\theta_5 - r_{cg_4}\dot{\theta}_4$$
$$\cos\theta_4 + r_6\dot{\theta}_6 \cos\theta_6) + (2r_5 \sin\theta_5 - 2(\partial\dot{\theta}_4 / \partial\dot{\theta}_5)r_{cg_4} \sin\theta_4 + 2(\partial\dot{\theta}_6 / \partial\dot{\theta}_5)r_6 \sin\theta_6)(r_5\dot{\theta}_5 \sin\theta_5$$
$$-r_{cg_4}\dot{\theta}_4 \sin\theta_4 + r_6\dot{\theta}_6 \sin\theta_6))) / 2 + (m_6((2r_5 \cos\theta_5 + 2(\partial\dot{\theta}_6 / \partial\dot{\theta}_5)r_{cg_6} \cos\theta_6)(r_5\dot{\theta}_5 \cos\theta_5 + r_{cg_6}$$
$$\dot{\theta}_6 \cos\theta_6) + (2r_5 \sin\theta_5 + 2(\partial\dot{\theta}_6 / \partial\dot{\theta}_5)r_{cg_6} \sin\theta_6)(r_5\dot{\theta}_5 \sin\theta_5 + r_{cg_6}\dot{\theta}_6 \sin\theta_6))) / 2 + (m_3(2$$
$$(\partial\dot{\theta}_3 / \partial\dot{\theta}_5)r_{cg_3} \cos\theta_3(r_2\dot{\theta}_2 \cos\theta_2 + r_{cg_3}\dot{\theta}_3 \cos\theta_3) + 2(\partial\dot{\theta}_3 / \partial\dot{\theta}_5)r_{cg_3} \sin\theta_3(r_2\dot{\theta}_2 \sin\theta_2 + r_{cg_3}\dot{\theta}_3$$
$$\sin\theta_3))) / 2 + (m_5 r_5{}^2\dot{\theta}_5) / 12 + m_5 r_{cg_5}{}^2\dot{\theta}_5 + m_{slider}(r_5 \cos\theta_5 - (\partial\dot{\theta}_4 / \partial\dot{\theta}_5)r_4 \cos\theta_4 + (\partial\dot{\theta}_6 / \partial\dot{\theta}_5)$$
$$r_6 \cos\theta_6)(r_5\dot{\theta}_5 \cos\theta_5 - r_4\dot{\theta}_4 \cos\theta_4 + r_6\dot{\theta}_6 \cos\theta_6) + ((\partial\dot{\theta}_3 / \partial\dot{\theta}_5)m_3 r_3{}^2\dot{\theta}_3) / 12 + ((\partial\dot{\theta}_4 / \partial\dot{\theta}_5)m_4$$
$$r_4{}^2\dot{\theta}_4) / 12 + ((\partial\dot{\theta}_6 / \partial\dot{\theta}_5)m_6 r_6{}^2\dot{\theta}_6) / 12$$

(A.29)

Time Derivative of Partial Derivative of Lagrange Equation to $\dot{\theta}_2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = (m_3((2r_2\cos\theta_2 + 2(\partial\dot{\theta}_3/\partial\dot{\theta}_2)r_{cg_3}\cos\theta_3)(-r_2\dot{\theta}_2^{\,2}\sin\theta_2 - r_{cg_3}\dot{\theta}_3^{\,2}\sin\theta_3 + \ddot{\theta}_2 r_2\cos\theta_2$$
$$+\ddot{\theta}_3 r_{cg_3}\cos\theta_3) + (2r_2\sin\theta_2 + 2(\partial\dot{\theta}_3/\partial\dot{\theta}_2)r_{cg_3}\sin\theta_3)(r_2\dot{\theta}_2^{\,2}\cos\theta_2 + r_{cg_3}\dot{\theta}_3^{\,2}\cos\theta_3 + r_2\ddot{\theta}_2\sin\theta_2 +$$
$$r_{cg_3}\ddot{\theta}_3\sin\theta_3) - (\ddot{\theta}_2 r_2\cos\theta_2 + \ddot{\theta}_3 r_{cg_3}\cos\theta_3)(2r_2\ddot{\theta}_2\sin\theta_2 - \frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_2}\right)r_{cg_3}\cos\theta_3 + 2(\partial\dot{\theta}_3/\partial\dot{\theta}_2)r_{cg_3}$$
$$\dot{\theta}_3\sin\theta_3) + (r_2\dot{\theta}_2\sin\theta_2 + r_{cg_3}\dot{\theta}_3\sin\theta_3)(2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_2}\right)r_{cg_3}\sin\theta_3 + 2r_2\ddot{\theta}_2\cos\theta_2 + 2(\partial\dot{\theta}_3/\partial\dot{\theta}_2)r_{cg_3}\dot{\theta}_3$$
$$\cos\theta_3)))/2 - (m_4((2(\partial\dot{\theta}_4/\partial\dot{\theta}_2)r_{cg_4}\sin\theta_4 - 2(\partial\dot{\theta}_6/\partial\dot{\theta}_2)r_6\sin\theta_6)(-r_{cg_4}\dot{\theta}_4^{\,2}\cos\theta_4 + r_5\dot{\theta}_5^{\,2}\cos\theta_5$$
$$+r_5\dot{\theta}_6^{\,2}\cos\theta_6 - r_{cg_4}\ddot{\theta}_4\sin\theta_4 + r_5\ddot{\theta}_5\sin\theta_5 + r_6\ddot{\theta}_6\sin\theta_6) - (2(\partial\dot{\theta}_4/\partial\dot{\theta}_2)r_{cg_4}\cos\theta_4 - 2(\partial\dot{\theta}_6/\partial\dot{\theta}_2)$$
$$r_6\cos\theta_6)(-r_{cg_4}\dot{\theta}_4^{\,2}\sin\theta_4 + r_5\dot{\theta}_5^{\,2}\sin\theta_5 + r_6\dot{\theta}_6^{\,2}\sin\theta_6 + r_{cg_4}\ddot{\theta}_4\cos\theta_4 - r_5\ddot{\theta}_5\cos\theta_5 - r_6\ddot{\theta}_6\cos\theta_6) +$$
$$(r_5\dot{\theta}_5\cos\theta_5 - r_{cg_4}\dot{\theta}_4\cos\theta_4 + r_6\dot{\theta}_6\cos\theta_6)(2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2}\right)r_{cg_4}\cos\theta_4 - 2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)r_6\cos\theta_6 -$$
$$2(\partial\dot{\theta}_4/\partial\dot{\theta}_2)r_{cg_4}\dot{\theta}_4\sin\theta_4 + 2(\partial\dot{\theta}_6/\partial\dot{\theta}_2)r_6\dot{\theta}_6\sin\theta_6) + (r_5\dot{\theta}_5\sin\theta_5 - r_{cg_4}\dot{\theta}_4\sin\theta_4 + r_6\dot{\theta}_6\sin\theta_6)$$
$$(2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2}\right)r_{cg_4}\sin\theta_4 - 2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)r_6\sin\theta_6 + 2(\partial\dot{\theta}_4/\partial\dot{\theta}_2)r_{cg_4}\dot{\theta}_4\cos\theta_4 - 2(\partial\dot{\theta}_6/\partial\dot{\theta}_2)r_6\dot{\theta}_6$$
$$\cos\theta_6)))/2 - m_{slider}(r_5\dot{\theta}_5\cos\theta_5 - r_4\dot{\theta}_4\cos\theta_4 + r_6\dot{\theta}_6\cos\theta_6)(\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2}\right)r_4\cos\theta_4 - \frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)r_6$$
$$\cos\theta_6 - (\partial\dot{\theta}_4/\partial\dot{\theta}_2)r_4\dot{\theta}_4\sin\theta_4 + (\partial\dot{\theta}_6/\partial\dot{\theta}_2)r_6\dot{\theta}_6\sin\theta_6) + m_{slider}((\partial\dot{\theta}_4/\partial\dot{\theta}_2)r_4\cos\theta_4 -$$
$$(\partial\dot{\theta}_6/\partial\dot{\theta}_2)r_6\cos\theta_6)(r_4\dot{\theta}_4^{\,2}\sin\theta_4 + r_5\dot{\theta}_5^{\,2}\sin\theta_5 + r_6\dot{\theta}_6^{\,2}\sin\theta_6 + r_4\ddot{\theta}_4\cos\theta_4 - r_5\ddot{\theta}_5\cos\theta_5 -$$
$$r_6\ddot{\theta}_6\cos\theta_6) + (\ddot{\theta}_2 m_2 r_2^{\,2})/12 + \ddot{\theta}_2 m_2 r_{cg_2}^{\,2} + (\ddot{\theta}_3(\partial\dot{\theta}_3/\partial\dot{\theta}_2)m_3 r_3^{\,2})/12 + (\ddot{\theta}_4(\partial\dot{\theta}_4/\partial\dot{\theta}_2)m_4 r_4^{\,2})$$
$$/12 + (\ddot{\theta}_6(\partial\dot{\theta}_6/\partial\dot{\theta}_2)m_6 r_6^{\,2})/12$$
$$+\ddot{\theta}_6(\partial\dot{\theta}_6/\partial\dot{\theta}_2)m_6 r_{cg_6}^{\,2} + (\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_2}\right)m_3 r_3^{\,2}\dot{\theta}_3)/12 + (\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2}\right)m_4 r_4^{\,2}\dot{\theta}_4)/12 + (\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)$$
$$m_6 r_6^{\,2}\dot{\theta}_6)/12 + \frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)m_6 r_{cg_6}^{\,2}\dot{\theta}_6 - (\partial\dot{\theta}_6/\partial\dot{\theta}_2)m_6 r_5 r_{cg_6}\dot{\theta}_5^{\,2}\sin(\theta_5-\theta_6) + \ddot{\theta}_5(\partial\dot{\theta}_6/\partial\dot{\theta}_2)m_6 r_5 r_{cg_6}$$
$$\cos(\theta_5-\theta_6) + \frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)m_6 r_5 r_{cg_6}\dot{\theta}_5\cos(\theta_5-\theta_6) + (\partial\dot{\theta}_6/\partial\dot{\theta}_2)m_6 r_5 r_{cg_6}\dot{\theta}_5\dot{\theta}_6\sin(\theta_5-\theta_6)$$

(A.30)

Time Derivative of Partial Derivative of Lagrange Equation to $\dot{\theta}_5$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_5}\right) = (m_3(2(\partial\dot{\theta}_3/\partial\dot{\theta}_5)r_{cg_3}\cos\theta_3(-r_2\dot{\theta}_2^2\sin\theta_2 - r_{cg_3}\dot{\theta}_3^2\sin\theta_3 + \ddot{\theta}_2 r_2\cos\theta_2 + \ddot{\theta}_3 r_{cg_3}\cos\theta_3)$$
$$+2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_5}\right)r_{cg_3}(r_2\dot{\theta}_2^2\cos\theta_2 + r_{cg_3}\dot{\theta}_3\cos\theta_3) + 2(\partial\dot{\theta}_3/\partial\dot{\theta}_5)r_{cg_3}\sin\theta_3(r_2\dot{\theta}_2^2\cos\theta_2 + r_{cg_3}$$
$$\dot{\theta}_3^2\cos\theta_3 + \ddot{\theta}_2 r_2\sin\theta_2 + \ddot{\theta}_3 r_{cg_3}\sin\theta_3) + 2(\partial(\partial\dot{\theta}_3/\partial\dot{\theta}_5)/\partial t)r_{cg_3}\sin\theta_3(\ddot{\theta}_2 r_2\sin\theta_2 + r_{cg_3}\dot{\theta}_3\sin\theta_3) -$$
$$2(\partial\dot{\theta}_3/\partial\dot{\theta}_5)r_{cg_3}\dot{\theta}_3\sin\theta_3(r_2\dot{\theta}_2\cos\theta_2 + r_{cg_3}\dot{\theta}_3\cos\theta_3) + 2(\partial\dot{\theta}_3/\partial\dot{\theta}_5)r_{cg_3}\dot{\theta}_3\cos\theta_3(r_2\dot{\theta}_2\cos\theta_2 + r_{cg_3}$$
$$\dot{\theta}_3\sin\theta_3)))/2 + (m_4((r_5\dot{\theta}_5\sin\theta_5 - r_{cg_4}\dot{\theta}_4\sin\theta_4 + r_6\dot{\theta}_6\sin\theta_6)(2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)r_6\sin\theta_6 - 2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_5}\right)$$
$$\dot{\theta}_4\sin\theta_4 + 2r_5\dot{\theta}_5\cos\theta_5 - 2(\partial\dot{\theta}_4/\partial\dot{\theta}_5)r_{cg_4}\dot{\theta}_4\cos\theta_4 + 2(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_6\dot{\theta}_6\cos\theta_6) - (2r_5\dot{\theta}_5\cos\theta_5 -$$
$$2(\partial\dot{\theta}_4/\partial\dot{\theta}_5)r_{cg_4}\cos\theta_4 + 2cr_6\cos\theta_6)(-r_{cg_4}\dot{\theta}_4^2\sin\theta_4 + r_5\dot{\theta}_5^2\sin\theta_5 + r_6\dot{\theta}_6^2\sin\theta_6 + r_{cg_4}\ddot{\theta}_4\cos\theta_4 -$$
$$r_5\ddot{\theta}_5\cos\theta_5 - r_6\ddot{\theta}_6\cos\theta_6) + (2r_5\sin\theta_5 - 2(\partial\dot{\theta}_4/\partial\dot{\theta}_5)r_{cg_4}\sin\theta_4 + 2(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_6\sin\theta_6)(-r_{cg_4}\dot{\theta}_4^2\cos\theta_4$$
$$+r_5\dot{\theta}_5^2\cos\theta_5 + r_6\dot{\theta}_6^2\cos\theta_6 - r_{cg_4}\ddot{\theta}_4\sin\theta_4 + r_5\ddot{\theta}_5\sin\theta_5 + r_6\ddot{\theta}_6\sin\theta_6) - (r_5\dot{\theta}_5\cos\theta_5 - r_{cg_4}\dot{\theta}_4\cos\theta_4 +$$
$$r_6\dot{\theta}_6\cos\theta_6)(2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_5}\right)r_{cg_4}\cos\theta_4 - 2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)r_6\cos\theta_6 + 2r_5\dot{\theta}_5\sin\theta_5 - 2(\partial\dot{\theta}_4/\partial\dot{\theta}_5)r_{cg_4}\dot{\theta}_4\sin\theta_4 +$$
$$2(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_6\dot{\theta}_6\sin\theta_6)))/2 + (m_6((2r_5\cos\theta_5 + 2(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_{cg_6}\cos\theta_6)(-r_5\dot{\theta}_5^2\sin\theta_5 - r_{cg_6}\dot{\theta}_6^2\sin\theta_6$$
$$+r_5\ddot{\theta}_5\cos\theta_5 + r_{cg_6}\ddot{\theta}_6\cos\theta_6) + (2r_5\sin\theta_5 + 2(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_{cg_6}\sin\theta_6)(r_5\dot{\theta}_5^2\cos\theta_5 + r_{cg_6}\dot{\theta}_6^2\cos\theta_6 +$$
$$r_5\ddot{\theta}_5\sin\theta_5 + r_{cg_6}\ddot{\theta}_6\sin\theta_6) - (r_5\dot{\theta}_5\cos\theta_5 + r_{cg_6}\dot{\theta}_6\cos\theta_6)(2r_5\dot{\theta}_5\sin\theta_5 - 2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)r_{cg_6}\cos\theta_6 +$$
$$2(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_{cg_6}\dot{\theta}_6\sin\theta_6) + r_5\dot{\theta}_5\sin\theta_5 + r_{cg_6}\dot{\theta}_6\sin\theta_6)(2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)r_{cg_6}\sin\theta_6 + 2r_5\dot{\theta}_5\cos\theta_5 +$$
$$2(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_{cg_6}\dot{\theta}_6\cos\theta_6)))/2 - m_{slider}(r_5\cos\theta_5 - (\partial\dot{\theta}_4/\partial\dot{\theta}_5)r_4\cos\theta_4 + (\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_6\cos\theta_6)$$
$$(-r_4\dot{\theta}_4^2\sin\theta_4 + r_5\dot{\theta}_5^2 + r_6\dot{\theta}_6^2\sin\theta_6 + r_4\ddot{\theta}_4\cos\theta_4 - r_5\ddot{\theta}_5\cos\theta_5 - r_6\ddot{\theta}_6\cos\theta_6) - m_{slider}(r_5\dot{\theta}_5\cos\theta_5$$
$$-r_4\dot{\theta}_4\cos\theta_4 + r_6\dot{\theta}_6\cos\theta_6)(\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_5}\right)r_4\cos\theta_4 - \frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)r_6\cos\theta_6 + r_5\dot{\theta}_5\sin\theta_5 - (\partial\dot{\theta}_4/\partial\dot{\theta}_5)$$
$$r_4\dot{\theta}_4\sin\theta_4 + (\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_6\dot{\theta}_6\sin\theta_6) + (m_5 r_5^2\ddot{\theta}_5)/12 + m_5 r_{cg_5}{}^2\ddot{\theta}_5 + (m_3 r_3^2\ddot{\theta}_3(\partial\dot{\theta}_3/\partial\dot{\theta}_5)/12 +$$
$$(m_4 r_4^2\ddot{\theta}_4(\partial\dot{\theta}_4/\partial\dot{\theta}_5)/12 + (m_6 r_6^2\ddot{\theta}_6(\partial\dot{\theta}_6/\partial\dot{\theta}_5)/12 + (m_3 r_3^2\dot{\theta}_3\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_5}\right))/12 + (m_4 r_4^2\dot{\theta}_4$$
$$\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_5}\right))/12 + (m_6 r_6^2\dot{\theta}_6\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right))/12$$

(A.31)

Explicit Form of the Angular Acceleration of the Second Link including Motor Dynamics

$$\alpha_2 = -1/J_2 n_2^2 + 1/2m_3((2r_2\cos\theta_2 + 2(\partial\dot\theta_3/\partial\dot\theta_2)r_{cg_3}\cos\theta_3)(r_2\cos\theta_2 + r_2/r_3 r_{cg_3}/\sin(\theta_2-\theta_3)\sin(\theta_3$$
$$-\theta_2)\sin(\theta_2-\theta_6)/\sin(\theta_3-\theta_6)\cos\theta_3) + (2r_2\sin\theta_2 + 2(\partial\dot\theta_3/\partial\dot\theta_2)r_{cg_3}\sin\theta_3))(r_2\sin\theta_2 + r_2/r_3 r_{cg_3}\sin(\theta_2$$
$$-\theta_3)\sin(\theta_3-\theta_2)\sin(\theta_2-\theta_6)/\sin(\theta_3-\theta_6)\sin\theta_3)) + 1/12m_2r_2^2 + m_2 r_{cg_2}{}^2 - 1/2m_4((2(\partial\dot\theta_4/\partial\dot\theta_2)r_{cg_4}\cos\theta_4 -$$
$$(2(\partial\dot\theta_6/\partial\dot\theta_2)r_6\cos\theta_6)(r_2\sin(\theta_3-\theta_2)/\sin(\theta_3-\theta_6)\cos\theta_6 + r_2/r_4 r_{cg_4}\sin(\theta_3-\theta_2)/\sin(\theta_3-\theta_6)/\sin(\theta_4 -$$
$$\theta_6)\sin(\theta_6-\theta_4)\cos\theta_4/\sin\theta_4\sin\theta_6) + (2(\partial\dot\theta_4/\partial\dot\theta_2)r_{cg_4}\sin\theta_4 - 2(\partial\dot\theta_6/\partial\dot\theta_2)r_6\sin\theta_6)(r_2\sin(\theta_3-\theta_2)/$$
$$\sin(\theta_3-\theta_6)\sin\theta_6 + r_2/r_4 r_{cg_4}\sin(\theta_3-\theta_2)/\sin(\theta_3-\theta_6)/\sin(\theta_4-\theta_6)\sin(\theta_6-\theta_4)\sin\theta_6)) - m_{slider}((\partial\dot\theta_4$$
$$/\partial\dot\theta_2)r_4\cos\theta_4 - (\partial\dot\theta_6/\partial\dot\theta_2)r_6\cos\theta_6)(r_2\sin(\theta_3-\theta_2)/\sin(\theta_3-\theta_6)\cos\theta_6 + r_2\sin(\theta_3-\theta_2)/\sin(\theta_3-\theta_6)/$$
$$\sin(\theta_4-\theta_6)\sin(\theta_6-\theta_4)\cos\theta_4/\sin\theta_4\sin\theta_6) + 1/12(\partial\dot\theta_6/\partial\dot\theta_2)m_6r_2r_6\sin(\theta_3-\theta_2)/\sin(\theta_3-\theta_6) + (\partial\dot\theta_6$$
$$/\partial\dot\theta_2)m_6r_2/r_6 r_{cg_6}{}^2\sin(\theta_3-\theta_2)\sin(\theta_2-\theta_6)/\sin(\theta_3-\theta_6) - 1/12(\partial\dot\theta_4/\partial\dot\theta_2)m_4r_2r_4\sin(\theta_3-\theta_2)/\sin(\theta_3 -$$
$$\theta_2)/\sin(\theta_3-\theta_6)/\sin(\theta_4-\theta_6)\sin(\theta_6-\theta_4)/\sin\theta_4\sin\theta_6)(1/2m_3((2r_2\dot\theta_2\cos\theta_2 + 2r_{cg_3}\dot\theta_3\cos\theta_3)(r_2\dot\theta_2$$
$$\sin\theta_2 - (\partial\dot\theta_3/\partial\theta_2)r_{cg_3}\cos\theta_3 + (\partial\theta_3/\partial\theta_2)r_{cg_3}\dot\theta_3\sin\theta_3) - (2r_2\dot\theta_2\sin\theta_2 + 2r_{cg_3}\dot\theta_3\sin\theta_3)((\partial\dot\theta_3/\partial\theta_2)r_{cg_3}\sin\theta_3$$
$$+r_2\dot\theta_2\cos\theta_2 + (\partial\theta_3/\partial\theta_2)r_{cg_3}\dot\theta_3\cos\theta_3)) - 1/2m_6((2(\partial\dot\theta_6/\partial\theta_2)r_{cg_6}\cos\theta_6 - 2(\partial\theta_6/\partial\theta_2)r_{cg_6}\dot\theta_6\sin\theta_6))(r_5$$
$$\dot\theta_5\cos\theta_5 + r_{cg_6}\dot\theta_6\cos\theta_6) + (2(\partial\dot\theta_6/\partial\theta_2)r_{cg_6}\sin\theta_6 + (\partial\theta_6/\partial\theta_2)r_{cg_6}\dot\theta_6\cos\theta_6)(r_5\dot\theta_5\sin\theta_5 + r_{cg_6}\dot\theta_6\sin\theta_6)) - 1/2$$
$$m_3((r_2\dot\theta_2\cos\theta_2 + r_{cg_3}\dot\theta_3\cos\theta_3)(2r_2\dot\theta_2\sin\theta_2 - 2\frac{d}{dt}\left(\frac{\partial\dot\theta_3}{\partial\dot\theta_2}\right)r_{cg_3}\cos\theta_3 + 2(\partial\dot\theta_3/\partial\dot\theta_2)r_{cg_3}\dot\theta_3\sin\theta_3) - (r_2\dot\theta_2\sin\theta_2 +$$
$$r_{cg_3}\dot\theta_3\sin\theta_3)(2\frac{d}{dt}\left(\frac{\partial\dot\theta_3}{\partial\dot\theta_2}\right)r_{cg_3}\sin\theta_3 + 2r_2\dot\theta_2\cos\theta_2 + 2(\partial\dot\theta_3/\partial\dot\theta_2)r_{cg_3}\dot\theta_3\cos\theta_3) + (2r_2\cos\theta_2 + 2(\partial\dot\theta_3/\partial\dot\theta_2)r_{cg_3}$$
$$\cos\theta_3)(r_2\dot\theta_2^2\sin\theta_2 + r_{cg_3}\dot\theta_3^2\sin\theta_3 - 1/r_3 r_{cg_3}/\sin(\theta_2-\theta_3)\cos\theta_3(r_2\dot\theta_2^2\sin\theta_2 + \alpha_5 r_5\sin(\theta_2-\theta_5) - \sin(\theta_2-\theta_6)$$
$$/\sin(\theta_3-\theta_6)(r_2\dot\theta_2^2\cos(\theta_2-\theta_3) + r_3\dot\theta_3^2 - r_5\dot\theta_5^2\cos(\theta_5-\theta_3) - r_6\dot\theta_6^2\cos(\theta_3-\theta_6) + \alpha_5 r_5\sin(\theta_3-\theta_5)) + r_3\dot\theta_3^2$$
$$\cos(\theta_2-\theta_3) - r_5\dot\theta_5^2\cos(\theta_2-\theta_5) - r_6\dot\theta_6^2\cos(\theta_2-\theta_6))) - (2r_2\sin\theta_2 + 2(\partial\dot\theta_3/\partial\dot\theta_2)r_{cg_3}\sin\theta_3)(r_2\dot\theta_2^2\cos\theta_2 +$$
$$r_{cg_3}\dot\theta_3^2\cos\theta_3 + 1/r_3 r_{cg_3}/\sin(\theta_2-\theta_3)\sin\theta_3(r_2\dot\theta_2^2 + \alpha_5 r_5\sin(\theta_2-\theta_5) - \sin(\theta_2-\theta_6)/\sin(\theta_3-\theta_6)(r_2\dot\theta_2^2\cos$$
$$(\theta_2-\theta_3) + r_3\dot\theta_3^2 - r_5\dot\theta_5^2\cos(\theta_5-\theta_3) - r_6\dot\theta_6^2\cos(\theta_3-\theta_6) + \alpha_5 r_5\sin(\theta_3-\theta_5)) + r_3\dot\theta_3^2\cos(\theta_3-\theta_5) - r_5\dot\theta_5^2$$
$$\cos(\theta_2-\theta_5) - r_6\dot\theta_6^2\sin(\theta_2-\theta_6)))) + 1/2m_4((2r_5\dot\theta_5\cos\theta_5 - 2r_{cg_4}\dot\theta_4\cos\theta_4 + 2r_6\dot\theta_6\cos\theta_6)((\partial\dot\theta_4/\partial\theta_2)$$
$$r_{cg_4}\cos\theta_4 - (\partial\dot\theta_6/\partial\theta_2)r_6\cos\theta_6 - (\partial\theta_4/\partial\theta_2)r_{cg_4}\dot\theta_4\sin\theta_4 + (\partial\theta_6/\partial\theta_2)r_6\dot\theta_6\sin\theta_6) + (2r_5\dot\theta_5\sin\theta_5 - 2r_{cg_4}\dot\theta_4$$
$$\sin\theta_4 + 2r_6\dot\theta_6\sin\theta_6)((\partial\dot\theta_4/\partial\theta_2)r_{cg_4}\sin\theta_4 - (\partial\dot\theta_6/\partial\theta_2)r_6\sin\theta_6 + (\partial\theta_4/\partial\theta_2)r_{cg_4}\dot\theta_4\cos\theta_4 - (\partial\theta_6/\partial\theta_2)r_6\dot\theta_6$$
$$\cos\theta_6)) - 1/2m_4((2(\partial\dot\theta_4/\partial\theta_2)r_{cg_4}\sin\theta_4 - 2(\partial\dot\theta_6/\partial\theta_2)r_6\sin\theta_6)(\alpha_5 r_5\sin\theta_5 - 1/\sin(\theta_3-\theta_6)\sin\theta_6(r_2\dot\theta_2^2$$
$$\cos(\theta_2-\theta_3) + r_3\dot\theta_3^2 - r_5\dot\theta_5^2\cos(\theta_5-\theta_3) - r_6\dot\theta_6^2\cos(\theta_3-\theta_6) + \alpha_5 r_5\sin(\theta_3-\theta_5)) - r_{cg_4}\dot\theta_4^2\cos\theta_4 + r_5\dot\theta_5^2\cos\theta_5$$
$$+r_6\dot\theta_6^2\cos\theta_6 - 1/r_4 r_{cg_4}/\sin(\theta_4-\theta_6)\sin\theta_4(r_6\dot\theta_6^2 + \alpha_5 r_5\sin(\theta_5-\theta_6) + 1/\sin\theta_4\sin\theta_6(r_4\dot\theta_4^2 + \alpha_5 r_5\sin(\theta_4-\theta_5)$$
$$+1/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4)(r_2\dot\theta_2^2\cos(\theta_2-\theta_3) + r_3\dot\theta_3^2 - r_5\dot\theta_5^2\cos(\theta_5-\theta_3) - r_6\dot\theta_6^2\cos(\theta_3-\theta_6) + \alpha_5 r_5$$
$$\sin(\theta_3-\theta_5)) - r_5\dot\theta_5^2\sin(\theta_3-\theta_5)) - r_5\dot\theta_5^2\cos(\theta_4-\theta_5) - r_6\dot\theta_6^2\cos(\theta_4-\theta_6)) - r_4\dot\theta_4^2\cos(\theta_4-\theta_6) + r_5\dot\theta_5^2$$
$$\cos(\theta_5-\theta_6))) - (2(\partial\dot\theta_4/\partial\dot\theta_2)r_{cg_4}\cos\theta_4 - 2(\partial\dot\theta_6/\partial\dot\theta_2)r_6\cos\theta_6)(1/\sin(\theta_3-\theta_6)\cos\theta_6(r_2\dot\theta_2^2\cos$$
$$(\theta_2-\theta_3) + r_3\dot\theta_3^2 - r_5\dot\theta_5^2\cos(\theta_5-\theta_3) - r_6\dot\theta_6^2\cos(\theta_3-\theta_6) + \alpha_5 r_5\sin(\theta_3-\theta_5)) - \alpha_5 r_5\cos\theta_5 - r_{cg_4}\dot\theta_4^2\sin\theta_4$$

$$+r_5\dot{\theta}_5^{\,2}\sin\theta_5+r_6\dot{\theta}_6^{\,2}\sin\theta_6+1/r_4r_{cg_4}\sin(\theta_4-\theta_6)\cos\theta_4(r_6\dot{\theta}_6^{\,2}+\alpha_5r_5\sin(\theta_5-\theta_6)+1/\sin\theta_4\sin\theta_6(r_4\dot{\theta}_4^{\,2}$$

$$+\alpha_5r_5\sin(\theta_4-\theta_5)+1/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4)(r_2\dot{\theta}_2^{\,2}\cos(\theta_2-\theta_3)+r_3\dot{\theta}_3^{\,2}-r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_3)-r_6\dot{\theta}_6^{\,2}$$

$$\cos(\theta_3-\theta_6)+\alpha_5r_5\sin(\theta_3-\theta_5))-r_5\dot{\theta}_5^{\,2}\cos(\theta_4-\theta_5)-r_6\dot{\theta}_6^{\,2}\cos(\theta_4-\theta_6))-r_4\dot{\theta}_4^{\,2}\cos(\theta_4-\theta_6)+r_5\dot{\theta}_5^{\,2}$$

$$\cos(\theta_5-\theta_6)))+(r_5\dot{\theta}_5\cos\theta_5-r_{cg_4}\dot{\theta}_4\cos\theta_4+r_6\dot{\theta}_6\cos\theta_6)(2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2}\right)r_{cg_4}\cos\theta_4-2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)r_6\cos\theta_6$$

$$-2(\partial\dot{\theta}_4/\partial\theta_2)r_{cg_4}\dot{\theta}_4\sin\theta_4+2(\partial\dot{\theta}_6/\partial\theta_2)r_6\dot{\theta}_6\sin\theta_6)+(r_5\dot{\theta}_5\sin\theta_5-r_{cg_4}\dot{\theta}_4\sin\theta_4+r_6\dot{\theta}_6\sin\theta_6)(2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2}\right)$$

$$r_{cg_4}\sin\theta_4-2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)r_6\cos\theta_6+2(\partial\dot{\theta}_4/\partial\theta_2)r_{cg_4}\dot{\theta}_4\cos\theta_4-2(\partial\dot{\theta}_6/\partial\theta_2)r_6\dot{\theta}_6\cos\theta_6))-m_{slider}(r_5\dot{\theta}_5\cos\theta_5$$

$$-r_4\dot{\theta}_4\cos\theta_4+r_6\dot{\theta}_6\cos\theta_6)(\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2}\right)r_4\cos\theta_4-\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)r_6\cos\theta_6-(\partial\dot{\theta}_4/\partial\dot{\theta}_2)r_4\dot{\theta}_4\sin\theta_4+(\partial\dot{\theta}_6/\partial\dot{\theta}_2)r_6\dot{\theta}_6$$

$$\sin\theta_6)+m_{slider}(r_5\dot{\theta}_5\cos\theta_5-r_4\dot{\theta}_4\cos\theta_4+r_6\dot{\theta}_6\cos\theta_6)((\partial\dot{\theta}_4/\partial\theta_2)r_4\cos\theta_4-(\partial\dot{\theta}_6/\partial\theta_2)r_6\cos\theta_6-(\partial\theta_4/\partial\theta_2)$$

$$r_4\dot{\theta}_4\sin\theta_4+(\partial\theta_6/\partial\theta_2)r_6\dot{\theta}_6\sin\theta_6)+m_3g(r_2\cos\theta_2+(\partial\theta_3/\partial\theta_2)r_{cg_3}\cos\theta_3)-K_{t_2}i_2n_2+m_{slider}((\partial\dot{\theta}_4/\partial\theta_2)r_4$$

$$\cos\theta_4-(\partial\dot{\theta}_6/\partial\theta_2)r_6\cos\theta_6)(1/\sin(\theta_4-\theta_5)\cos\theta_4(r_6\dot{\theta}_6^{\,2}+\alpha_5r_5\sin(\theta_5-\theta_6)+1/\sin\theta_4\sin\theta_6(r_4\dot{\theta}_4^{\,2}+\alpha_5r_5\sin$$

$$(\theta_4-\theta_5)+1/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4)(r_2\dot{\theta}_2^{\,2}\cos(\theta_2-\theta_3)+r_3\dot{\theta}_3^{\,2}-r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_3)-r_6\dot{\theta}_6^{\,2}\cos(\theta_3-\theta_6)+\alpha_5r_5$$

$$\sin(\theta_3-\theta_5))-r_5\dot{\theta}_5^{\,2}\cos(\theta_4-\theta_5)-r_6\dot{\theta}_6^{\,2}\cos(\theta_4-\theta_6))-r_4\dot{\theta}_4^{\,2}\cos(\theta_4-\theta_6)+r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_6))-\alpha_5r_5\cos\theta_5+$$

$$1/\sin(\theta_3-\theta_6)\cos\theta_6(r_2\dot{\theta}_2^{\,2}\cos(\theta_2-\theta_3)+r_3\dot{\theta}_3^{\,2}-r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_3)-r_6\dot{\theta}_6^{\,2}\cos(\theta_3-\theta_6)+\alpha_5r_5\sin(\theta_3-\theta_5))-$$

$$r_4\dot{\theta}_4^{\,2}\sin\theta_4+r_5\dot{\theta}_5^{\,2}\sin\theta_5+r_6\dot{\theta}_6^{\,2}\sin\theta_6)-m_4g((\partial\theta_4/\partial\theta_2)r_{cg_4}\cos\theta_4-(\partial\theta_6/\partial\theta_2)r_6\cos\theta_6)-m_{slider}g((\partial\theta_4/\partial\theta_2)$$

$$r_4\cos\theta_4-(\partial\theta_6/\partial\theta_2)r_6\cos\theta_6)+1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_2}\right)m_3r_3^{\,2}\dot{\theta}_3+1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_2}\right)m_4r_{34}^{\,2}\dot{\theta}_4+1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)m_6r_6^{\,2}\dot{\theta}_6+$$

$$\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)m_6r_{cg_6}^{\,2}\dot{\theta}_6-1/12(\partial\dot{\theta}_3/\partial\theta_2)m_3r_3^{\,2}\dot{\theta}_3-1/12(\partial\dot{\theta}_4/\partial\theta_2)m_4r_4^{\,2}\dot{\theta}_4-1/12(\partial\dot{\theta}_6/\partial\theta_2)m_6r_6^{\,2}\dot{\theta}_6-m_2gr_{cg_2}$$

$$\cos\theta_2+1/12(\partial\dot{\theta}_4/\partial\dot{\theta}_2)m_4r_4/\sin(\theta_4-\theta_6)(r_6\dot{\theta}_6^{\,2}+\alpha_5r_5\sin(\theta_5-\theta_6)+1/\sin\theta_4\sin\theta_6(r_4\dot{\theta}_4^{\,2}+\alpha_5r_5\sin(\theta_4-\theta_5)$$

$$+1/\sin(\theta_3-6\theta_6)\sin(\theta_6-\theta_4)(r_2\dot{\theta}_2^{\,2}\cos(\theta_2-\theta_3)+r_3\dot{\theta}_3^{\,2}-r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_3)-r_6\dot{\theta}_6^{\,2}\cos(\theta_3-\theta_6)+\alpha_5r_5\sin$$

$$(\theta_3-\theta_5))-r_5\dot{\theta}_5^{\,2}\cos(\theta_4-\theta_5)-r_6\dot{\theta}_6^{\,2}\cos(\theta_4-\theta_6))-r_4\dot{\theta}_4^{\,2}\cos(\theta_4-\theta_6)+r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_6))-1/12(\partial\dot{\theta}_6/\partial\dot{\theta}_2)$$

$$m_6r_6/\sin(\theta_3-\theta_6)(r_2\dot{\theta}_2^{\,2}\cos(\theta_2-\theta_3)+r_3\dot{\theta}_3^{\,2}-r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_3)-r_6\dot{\theta}_6^{\,2}\cos(\theta_3-\theta_6)+\alpha_5r_5\sin(\theta_3-\theta_5))+$$

$$(\partial\theta_6/\partial\theta_2)m_6gr_{cg_6}\cos\theta_6+1/12(\partial\dot{\theta}_3/\partial\dot{\theta}_2)m_3r_3/\sin(\theta_2-\theta_3)(r_2\dot{\theta}_2^{\,2}+\alpha_5r_5\sin(\theta_2-\theta_5)-\sin(\theta_2-\theta_6)/\sin$$

$$(\theta_3-\theta_6)(r_2\dot{\theta}_2^{\,2}\cos(\theta_2-\theta_3)+r_3\dot{\theta}_3^{\,2}-r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_3)-r_6\dot{\theta}_6^{\,2}\cos(\theta_3-\theta_6)+\alpha_5r_5\sin(\theta_3-\theta_5))+r_3\dot{\theta}_3^{\,2}\cos(\theta_2-\theta_3)$$

$$-r_5\dot{\theta}_5^{\,2}\cos(\theta_2-\theta_5)-r_6\dot{\theta}_6^{\,2}\cos(\theta_2-\theta_6))-(\partial\dot{\theta}_6/\partial\dot{\theta}_2)m_6r_5r_{cg_6}\dot{\theta}_5^{\,2}\sin(\theta_5-\theta_6)+\alpha_5(\partial\dot{\theta}_6/\partial\dot{\theta}_2)m_6r_5r_{cg_6}\cos(\theta_5-\theta_6)$$

$$+\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_2}\right)m_6r_5r_{cg_6}\dot{\theta}_5\cos(\theta_5-\theta_6)-(\partial\dot{\theta}_6/\partial\dot{\theta}_2)m_6/r_6r_{cg_6}^{\,2}/\sin(\theta_3-\theta_6)(r_2\dot{\theta}_2^{\,2}\cos(\theta_2-\theta_3)+r_3\dot{\theta}_3^{\,2}-r_5\dot{\theta}_5^{\,2}\cos$$

$$(\theta_5-\theta_3)-r_6\dot{\theta}_6^{\,2}\cos(\theta_3-\theta_6)+\alpha_5r_5\sin(\theta_3-\theta_5))+(\partial\dot{\theta}_6/\partial\dot{\theta}_2)m_6r_5r_{cg_6}\dot{\theta}_5\dot{\theta}_6\sin(\theta_5-\theta_6)$$

(A.32)

Explicit Form of the Angular Acceleration of the Fifth Link including Motor Dynamics

$$\alpha_5 = -1/(1/2m_3(2(\partial\dot\theta_3/\partial\dot\theta_5)/r_3 r_{cg_3}{}^2/\sin(\theta_2-\theta_3)\cos^2\theta_3(r_5\sin(\theta_2-\theta_5)-r_5\sin(\theta_2-\theta_6)\sin(\theta_3-\theta_5)/$$
$$\sin(\theta_3-\theta_6))+2(\partial\dot\theta_3/\partial\dot\theta_5)/r_3 r_{cg_3}{}^2/\sin(\theta_2-\theta_3)\sin^2\theta_3(r_5\sin(\theta_2-\theta_5)-r_5\sin(\theta_2-\theta_6)\sin(\theta_3-\theta_5)$$
$$/\sin(\theta_3-\theta_6)))-1/2m_4((2r_5\cos\theta_5-2(\partial\dot\theta_4/\partial\dot\theta_5)r_{cg_4}\cos\theta_4+2(\partial\dot\theta_6/\partial\dot\theta_5)r_6\cos\theta_6)(r_5\sin(\theta_3-\theta_5)$$
$$/\sin(\theta_3-\theta_6)\cos\theta_6-r_5\cos\theta_5+1/r_4 r_{cg_4}/\sin(\theta_4-\theta_6)\cos\theta_4(r_5\sin(\theta_5-\theta_6)+1/\sin\theta_4\sin\theta_6(r_5\sin$$
$$(\theta_4-\theta_5)+r_5\sin(\theta_3-\theta_5)/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4))))+(r_5\sin(\theta_3-\theta_5)/\sin(\theta_3-\theta_6)\sin\theta_6-r_5\sin\theta_5$$
$$+1/r_4 r_{cg_4}/\sin(\theta_4-\theta_6)\sin\theta_4(r_5\sin(\theta_5-\theta_6)+1/\sin\theta_4\sin\theta_6(r_5\sin(\theta_4-\theta_5)+r_5\sin(\theta_3-\theta_5)/\sin(\theta_3-$$
$$\theta_6)\sin(\theta_6-\theta_4))))(2r_5\sin\theta_5-2(\partial\dot\theta_4/\partial\dot\theta_5)r_{cg_4}\sin\theta_4+2(\partial\dot\theta_6/\partial\dot\theta_5)r_6\sin\theta_6))+1/2m_6((2r_5\cos\theta_5)+$$
$$2(\partial\dot\theta_6/\partial\dot\theta_5)r_{cg_6}\cos\theta_6))(r_5\cos\theta_5-r_5/r_6 r_{cg_6}\sin(\theta_3-\theta_5)/\sin(\theta_3-\theta_6)\cos\theta_6)+(2r_5\sin\theta_5+2(\partial\dot\theta_6/\partial\dot\theta_5)$$
$$r_{cg_6}\sin\theta_6)(r_5\sin\theta_5-r_5/r_6 r_{cg_6}\sin(\theta_3-\theta_5)/\sin(\theta_3-\theta_6)\sin\theta_6))+J_5 n_5{}^2+1/12m_5 r_5{}^2+m_5 r_{cg_5}{}^2-m_{slider}(r_5$$
$$\cos\theta_5-(\partial\dot\theta_4/\partial\dot\theta_5)r_4\cos\theta_4+(\partial\dot\theta_6/\partial\dot\theta_5)r_6\cos\theta_6)(1/\sin(\theta_4-\theta_6)\cos\theta_4(r_5\sin(\theta_5-\theta_6)+1/\sin\theta_4\sin\theta_4$$
$$\sin\theta_6(r_5\sin(\theta_4-\theta_5)+r_5\sin(\theta_3-\theta_5)/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4)))-r_5\cos\theta_5+r_5\sin(\theta_3-\theta_5)/\sin(\theta_3-\theta_6)$$
$$\cos\theta_6)+1/12(\partial\dot\theta_4/\partial\dot\theta_5)m_4 r_4\sin(\theta_4-\theta_6)(r_5\sin(\theta_5-\theta_6)+1/\sin\theta_4\sin\theta_6(r_5\sin(\theta_4-\theta_5)+r_5\sin(\theta_3-\theta_5)$$
$$/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4)))+1/12(\partial\dot\theta_3/\partial\dot\theta_5)m_3 r_3\sin(\theta_2-\theta_3)(r_5\sin(\theta_2-\theta_5)-r_5\sin(\theta_2-\theta_6)\sin(\theta_3-\theta_5)$$
$$/\sin(\theta_3-\theta_6))-1/12(\partial\dot\theta_6/\partial\dot\theta_5)m_6 r_5 r_6\sin(\theta_3-\theta_5)/\sin(\theta_3-\theta_6))-1/12(\partial\dot\theta_6/\partial\dot\theta_5)m_6 r_5 r_6\sin(\theta_3-\theta_5)/$$
$$\sin(\theta_3-\theta_6))(1/2m_4((r_5\dot\theta_5\sin\theta_5-r_{cg_4}\dot\theta_4\sin\theta_4+r_6\dot\theta_6\sin\theta_6)(2\frac{d}{dt}\left(\frac{\partial\dot\theta_6}{\partial\dot\theta_5}\right)r_6\sin\theta_6-2\frac{d}{dt}\left(\frac{\partial\dot\theta_4}{\partial\dot\theta_5}\right)r_{cg_4}\sin\theta_4$$
$$+2r_5\dot\theta_5\cos\theta_5-2(\partial\dot\theta_4/\partial\dot\theta_5)r_{cg_4}\dot\theta_4\cos\theta_4+2(\partial\dot\theta_6/\partial\dot\theta_5)r_6\dot\theta_6\cos\theta_6)-(2r_5\cos\theta_5-2(\partial\dot\theta_4/\partial\dot\theta_5)r_{cg_4}\cos\theta_4$$
$$+2(\partial\dot\theta_6/\partial\dot\theta_5)r_6\cos\theta_6)(r_5\dot\theta_5{}^2\sin\theta_5-r_{cg_4}\dot\theta_4{}^2\sin\theta_4-1/\sin(\theta_3-\theta_6)\cos\theta_6(-r_2\cos(\theta_2-\theta_3)\dot\theta_2{}^2-r_3\dot\theta_3{}^2+r_5$$
$$\dot\theta_5{}^2\cos(\theta_5-\theta_3)+r_6\dot\theta_6{}^2\cos(\theta_3-\theta_6)+\alpha_2 r_2\sin(\theta_3-\theta_2))+r_6\dot\theta_6{}^2\sin\theta_6+1/r_4 r_{cg_4}/\sin(\theta_4-\theta_6)\cos\theta_4(r_6\dot\theta_6{}^2$$
$$-1/\sin\theta_4\sin\theta_6(1/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4)(-r_2\sin(\theta_2-\theta_3)\dot\theta_2{}^2-r_3\dot\theta_3{}^2+r_5\dot\theta_5{}^2\cos(\theta_5-\theta_6)))+(2r_5\sin\theta_5$$
$$-2(\partial\dot\theta_4/\partial\dot\theta_5)r_{cg_4}\sin\theta_4+2(\partial\dot\theta_6/\partial\dot\theta_5)r_6\sin\theta_6)(1/\sin(\theta_3-\theta_6)\sin\theta_6(-r_2\dot\theta_2{}^2\cos(\theta_2-\theta_3)-r_3\dot\theta_3{}^2+r_5\dot\theta_5{}^2$$
$$\cos(\theta_5-\theta_3)+r_6\dot\theta_6{}^2\cos(\theta_3-\theta_6)+\alpha_2 r_2\sin(\theta_3-\theta_2)-r_{cg_4}\dot\theta_4{}^2\cos\theta_4+r_5\dot\theta_5{}^2\cos\theta_5+r_6\dot\theta_6{}^2\cos\theta_6-1/r_4 r_{cg_4}/$$
$$\sin(\theta_4-\theta_6)\sin\theta_4(r_6\dot\theta_6{}^2-1/\sin\theta_4\sin\theta_6(1/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4)(-r_2\dot\theta_2{}^2\cos(\theta_2-\theta_3)-r_3\dot\theta_3{}^2+r_5\dot\theta_5{}^2$$
$$\cos(\theta_5-\theta_3)+r_6\dot\theta_6{}^2\cos(\theta_3-\theta_6)+\alpha_2 r_2\sin(\theta_3-\theta_2))-r_4\dot\theta_4{}^2+r_5\dot\theta_5{}^2\cos(\theta_4-\theta_5)+r_6\dot\theta_6{}^2\cos(\theta_4-\theta_6))-$$
$$r_4\dot\theta_4{}^2\cos(\theta_4-\theta_6)+r_5\dot\theta_5{}^2\cos(\theta_5-\theta_6)))-(r_5\dot\theta_5\cos(\theta_5-\theta_6)-r_{cg_4}\dot\theta_4\cos\theta_4+r_6\dot\theta_6\cos\theta_6)(2\frac{d}{dt}\left(\frac{\partial\dot\theta_4}{\partial\dot\theta_5}\right)r_{cg_4}$$
$$\cos\theta_4-2\frac{d}{dt}\left(\frac{\partial\dot\theta_6}{\partial\dot\theta_5}\right)r_6\cos\theta_6+2r_5\dot\theta_5\sin\theta_5-2(\partial\dot\theta_4/\partial\dot\theta_5)r_{cg_4}\dot\theta_4\sin\theta_4+2(\partial\dot\theta_6/\partial\dot\theta_5)r_6\dot\theta_6\sin\theta_6))-1/2$$
$$m_4((2r_5\dot\theta_5\sin\theta_5-2r_{cg_4}\dot\theta_4\sin\theta_4+2r_6\dot\theta_6\sin\theta_6)((\partial\dot\theta_6/\partial\theta_5)r_6\sin\theta_6-(\partial\dot\theta_4/\partial\theta_5)r_{cg_4}\sin\theta_4+r_5\dot\theta_5\cos\theta_5-$$
$$(\partial\theta_4/\partial\theta_5)r_{cg_4}\dot\theta_4\cos\theta_4+r_5\dot\theta_5\sin\theta_5-(\partial\theta_4/\partial\theta_5)r_{cg_4}\dot\theta_4\sin\theta_4+(\partial\theta_6/\partial\theta_5)r_6\dot\theta_6\sin\theta_6))-1/2m_3((2$$
$$(\partial\dot\theta_3/\partial\theta_5)r_{cg_3}\cos\theta_3-2(\partial\theta_3/\partial\theta_5)r_{cg_3}\dot\theta_3\sin\theta_3)(r_2\dot\theta_2\cos\theta_2+r_{cg_3}\dot\theta_3\cos\theta_3)+(2(\partial\dot\theta_3/\partial\theta_5)r_{cg_3}\sin\theta_3$$

$$+2(\partial\theta_3/\partial\theta_5)r_{cg_3}\dot{\theta}_3\cos\theta_3)(r_2\dot{\theta}_2\sin\theta_2+r_{cg_3}\dot{\theta}_3\sin\theta_3))-1/2m_6((r_5\dot{\theta}_5\cos\theta_5+r_6\dot{\theta}_6\cos\theta_6)(2r_5\dot{\theta}_5\sin\theta_5-$$

$$2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)r_{cg_6}\cos\theta_6+2(\partial\dot{\theta}_6/\partial\theta_5)r_{cg_6}\dot{\theta}_6\sin\theta_6)-(r_5\dot{\theta}_5\sin\theta_5+r_{cg_6}\dot{\theta}_6\sin\theta_6)(2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)r_{cg_6}\sin\theta_6+$$

$$2r_5\dot{\theta}_5\cos\theta_5+2(\partial\dot{\theta}_6/\partial\theta_5)r_{cg_6}\dot{\theta}_6\cos\theta_6)+(2r_5\cos\theta_5+2(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_{cg_6}\cos\theta_6)(r_5\dot{\theta}_5^2\sin\theta_5+r_{cg_6}\dot{\theta}_6^2\sin\theta_6$$

$$-1/r_6r_{cg_6}/\sin(\theta_3-\theta_6)\cos\theta_6(-r_2\dot{\theta}_2^2\cos(\theta_2-\theta_3)-r_3\dot{\theta}_3^2+r_5\dot{\theta}_5^2\cos(\theta_5-\theta_3)+r_6\dot{\theta}_6^2\cos(\theta_3-\theta_6)+$$

$$\alpha_2r_2\sin(\theta_3-\theta_2)))-(2r_5\sin\theta_5+2(\partial\dot{\theta}_6/\partial\theta_5)r_{cg_6}\sin\theta_6)(r_5\dot{\theta}_5^2\cos\theta_5+r_{cg_6}\dot{\theta}_6^2\cos\theta_6+1/r_6r_{cg_6}/$$

$$\sin(\theta_3-\theta_6)\sin\theta_6(-2r_2\dot{\theta}_2^2\cos(\theta_2-\theta_3)-r_3\dot{\theta}_3^2+r_5\dot{\theta}_5^2\cos(\theta_5-\theta_3)+r_6\dot{\theta}_6^2(\theta_3-\theta_6)+\alpha_2r_2$$

$$\sin(\theta_3-\theta_6))))+1/2m_3r_{cg_3}(2(\partial\dot{\theta}_6/\partial\dot{\theta}_5)\cos\theta_3(\alpha_2r_2\cos\theta_2-r_2\dot{\theta}_2^2\sin\theta_2-r_{cg_3}\dot{\theta}_3^2\sin\theta_3+1/r_3r_{cg_3}$$

$$\sin(\theta_2-\theta_3)\cos\theta_3(r_2\dot{\theta}_2^2+\sin(\theta_2-\theta_6)/\sin(\theta_3-\theta_6)(-r_2\dot{\theta}_2^2\cos(\theta_2-\theta_3)-r_3\dot{\theta}_3^2+r_5\dot{\theta}_5^2\cos(\theta_5-\theta_3)$$

$$+r_6\dot{\theta}_6^2\cos(\theta_3-\theta_6)+\alpha_2r_2\sin(\theta_3-\theta_2))+r_3\dot{\theta}_3^2\cos(\theta_2-\theta_3)-r_5\dot{\theta}_5^2\cos(\theta_2-\theta_5)-r_6\dot{\theta}_6^2\cos(\theta_2-\theta_6)))$$

$$+2(\partial\dot{\theta}_3/\partial\dot{\theta}_5)r_{cg_3}\sin\theta_3(\alpha_2r_2\sin\theta_2+r_2\dot{\theta}_2^2\cos\theta_2+r_{cg_3}\dot{\theta}_3^2\cos\theta_3+1/r_{cg_3}r_3/\sin(\theta_2-\theta_3)\sin\theta_3(r_2\dot{\theta}_2^2+$$

$$\sin(\theta_2-\theta_6)/\sin(\theta_3-\theta_6)(-r_2\dot{\theta}_2^2\cos(\theta_2-\theta_3)-r_3\dot{\theta}_3^2+r_5\dot{\theta}_5^2\cos(\theta_5-\theta_3)+r_6\dot{\theta}_6^2\cos(\theta_3-\theta_6)+\alpha_2r_2$$

$$\sin(\theta_3-\theta_2))+r_3\dot{\theta}_3^2\cos(\theta_2-\theta_3)-r_5\dot{\theta}_5^2\cos(\theta_2-\theta_5)-r_6\dot{\theta}_6^2\cos(\theta_2-\theta_6)))+2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_5}\right)r_{cg_3}\cos\theta_3$$

$$(r_2\dot{\theta}_2^2\cos\theta_2+r_{cg_3}\dot{\theta}_3\cos\theta_3)+2\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_5}\right)r_{cg_3}\sin\theta_3(r_2\dot{\theta}_2\sin\theta_2+r_{cg_3}\dot{\theta}_3\cos\theta_3)-2(\partial\dot{\theta}_3/\partial\dot{\theta}_5)r_{cg_3}\dot{\theta}_3$$

$$\sin\theta_3(r_2\dot{\theta}_2\cos\theta_2+r_{cg_3}\dot{\theta}_3\cos\theta_3)+2(\partial\dot{\theta}_3/\partial\dot{\theta}_5)r_{cg_3}\dot{\theta}_3\cos\theta_3(r_2\dot{\theta}_2\sin\theta_2+r_{cg_3}\dot{\theta}_3\cos\theta_3))+1/2m_6((2$$

$$r_5\dot{\theta}_5\cos\theta_5+2r_{cg_6}\dot{\theta}_6\cos\theta_6)(r_5\dot{\theta}_5\sin\theta_5-(\partial\dot{\theta}_6/\partial\theta_5)r_{cg_6}\cos\theta_6+(\partial\theta_6/\partial\theta_5)r_{cg_6}\dot{\theta}_6\sin\theta_6)-(2r_5\dot{\theta}_5\sin\theta_5$$

$$+2r_{cg_6}\dot{\theta}_6\sin\theta_6)((\partial\dot{\theta}_6/\partial\theta_5)r_{cg_6}\sin\theta_6+r_5\dot{\theta}_5\cos\theta_5+(\partial\theta_6/\partial\theta_5)r_{cg_6}\dot{\theta}_6\cos\theta_6))+m_6g(r_5\cos\theta_5+$$

$$(\partial\theta_6/\partial\theta_5)r_{cg_6}\cos\theta_6)+m_4g(r_5\cos\theta_5-(\partial\theta_4/\partial\theta_5)r_{cg_4}\cos\theta_4+(\partial\theta_6/\partial\theta_5)r_6\cos\theta_6)+m_{slider}g(r_5\cos\theta_5$$

$$-(\partial\theta_4/\partial\theta_5)r_4\cos\theta_4+(\partial\theta_6/\partial\theta_5)r_6\cos\theta_6)-m_{slider}(r_5\dot{\theta}_5\cos\theta_5-r_4\dot{\theta}_4\cos\theta_4+r_6\dot{\theta}_6\cos\theta_6))(\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_5}\right)$$

$$r_4\cos\theta_4-\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)r_6\cos\theta_6+r_5\dot{\theta}_5\sin\theta_5-(\partial\dot{\theta}_4/\partial\dot{\theta}_5)r_4\dot{\theta}_4\sin\theta_4+(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_6\dot{\theta}_6\sin\theta_6)+m_{slider}(r_5\dot{\theta}_5$$

$$\cos\theta_5-r_4\dot{\theta}_4\cos\theta_4+r_6\dot{\theta}_6\cos\theta_6)((\partial\dot{\theta}_4/\partial\theta_5)r_4\cos\theta_4-(\partial\dot{\theta}_6/\partial\theta_5)r_6\cos\theta_6+r_5\dot{\theta}_5\sin\theta_5-(\partial\theta_4/\partial\theta_5)$$

$$r_4\dot{\theta}_4\sin\theta_4+(\partial\theta_6/\partial\theta_5)r_6\dot{\theta}_6\sin\theta_6-Kt_5\ i_5\ n_5-m_{slider}(r_5\cos\theta_5)-(\partial\dot{\theta}_4/\partial\dot{\theta}_5)r_4\cos\theta_4+(\partial\dot{\theta}_6/\partial\dot{\theta}_5)r_6$$

$$\cos\theta_6)(1/\sin(\theta_4-\theta_6)\cos\theta_4(r_6\dot{\theta}_6^2-1/\sin\theta_4\sin\theta_6(1/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4)(-r_2\dot{\theta}_2^2\cos(\theta_2-\theta_3)$$

$$-r_3\dot{\theta}_3^2+r_5\dot{\theta}_5^2\cos(\theta_5-\theta_3)+r_6\dot{\theta}_6^2\cos(\theta_3-\theta_6)+c)-r_4\dot{\theta}_4^2+r_5\dot{\theta}_5^2\cos(\theta_4-\theta_5)+r_6\dot{\theta}_6^2$$

$$\cos(\theta_4-\theta_6))-r_4\dot{\theta}_4^2\cos(\theta_4-\theta_6)+r_5\dot{\theta}_5^2\cos(\theta_5-\theta_6))-1/\sin(\theta_3-\theta_6)\cos\theta_6(-r_2\dot{\theta}_2^2\cos(\theta_2-\theta_3)-$$

$$r_3\dot{\theta}_3^2+r_5\dot{\theta}_5^2\cos(\theta_5-\theta_3)+r_6\dot{\theta}_6^2\cos(\theta_3-\theta_6)+\alpha_2r_2\sin(\theta_3-\theta_2))-r_4\dot{\theta}_4^2\sin\theta_4+r_5\dot{\theta}_5^2\sin\theta_5+r_6\dot{\theta}_6^2$$

$$\sin\theta_6)+1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\dot{\theta}_5}\right)m_3\dot{\theta}_3r_3^2+1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\dot{\theta}_5}\right)m_4\dot{\theta}_4r_4^2+1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)m_6\dot{\theta}_6r_6^2-1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_3}{\partial\theta_5}\right)$$

$$m_3\dot{\theta}_3r_3^2-1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_4}{\partial\theta_5}\right)m_4\dot{\theta}_4r_4^2-1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\theta_5}\right)m_6\dot{\theta}_6r_6^2+m_5gr_{cg_5}\cos\theta_5+1/12\frac{d}{dt}\left(\frac{\partial\dot{\theta}_6}{\partial\dot{\theta}_5}\right)m_6\dot{\theta}_6r_6/$$

$$\sin(\theta_3-\theta_6)(-r_2\cos(\theta_2-\theta_3)\dot{\theta}_2^2-r_3\dot{\theta}_3^2+r_5\cos(\theta_5-\theta_3)\dot{\theta}_5^2+r_6\cos(\theta_3-\theta_6)\dot{\theta}_6^2+\alpha_2r_2\sin(\theta_3-\theta_2))+$$

$$1/12(\partial\dot{\theta}_3/\partial\dot{\theta}_5)m_3 r_3/\sin(\theta_2-\theta_3)(r_2\dot{\theta}_2^{\,2}+\sin(\theta_2-\theta_6)/\sin(\theta_3-\theta_6)(-r_2\dot{\theta}_2^{\,2}\cos(\theta_2-\theta_3)-r_3\dot{\theta}_3^{\,2}+$$
$$r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_3)+r_6\dot{\theta}_6^{\,2}\cos(\theta_3-\theta_6)+\alpha_2 r_2\sin(\theta_3-\theta_2))+r_3\dot{\theta}_3^{\,2}\cos(\theta_2-\theta_3)-r_5\dot{\theta}_5^{\,2}\cos(\theta_2-\theta_5)$$
$$-r_6\dot{\theta}_6^{\,2}\cos(\theta_2-\theta_6))+(\partial\theta_3/\partial\theta_5)m_3 g\cos\theta_3+1/12(\partial\dot{\theta}_4/\partial\dot{\theta}_5)m_4 r_4/\sin(\theta_4-\theta_6)(r_6\dot{\theta}_6^{\,2}-1/\sin\theta_4$$
$$\sin\theta_6(1/\sin(\theta_3-\theta_6)\sin(\theta_6-\theta_4)(-r_2\dot{\theta}_2^{\,2}\cos(\theta_2-\theta_3)-r_3\dot{\theta}_3^{\,2}+r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_3)+r_6\dot{\theta}_6^{\,2}\cos(\theta_3-\theta_6)$$
$$+\alpha_2 r_2\sin(\theta_3-\theta_2))-r_4\dot{\theta}_4^{\,2}+r_5\dot{\theta}_5^{\,2}\cos(\theta_4-\theta_5)+r_6\dot{\theta}_6^{\,2}\cos(\theta_4-\theta_6))-r_4\dot{\theta}_4^{\,2}$$
$$\cos(\theta_4-\theta_6)+r_5\dot{\theta}_5^{\,2}\cos(\theta_5-\theta_6)))$$

(A.33)