UNIVERSITY OF GAZIANTEP
GRADUATE SCHOOL OF
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A SLIDING MODE CONTROL FOR INVERTED PENDULUM SYSTEM

M.Sc. THESIS
IN
ELECTRICAL AND ELECTRONICS ENGINEERING

BY
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A Sliding Mode Control for Inverted Pendulum System

M.Sc. Thesis
In
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Supervisor
Prof. Dr. Ahmet UÇAR

by

Bakhtyar Abdullah SHARIF

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Bakhtyar Abdullah SHARIF

ABSTRACT

A SLIDING MODE CONTROL FOR INVERTED PENDULUM SYSTEM

SHARIF, Bakhtyar Abdullah
M.Sc. in Electrical and Electronics Engineering
Supervisor: Prof. Dr. Ahmet UÇAR
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The main aim of this study is to design sliding mode control for both linear and nonlinear systems; an inverted pendulum system is selected as an example to illustrate the advantage of sliding mode control for linear and nonlinear systems with uncertainties. The inverted pendulum is linearized for upright and unstable equilibrium point and then state feedback and optimal quadratic regulator controllers are designed for the system without uncertainties the results are discussed in the seminar of thesis.

In this thesis sliding mode control is designed for the linear systems with uncertainties based on Ackermann's formula and linear matrix inequality (LMI). The special problems of the inverted pendulum system such as swing-up and stabilizing problems are solved. Furthermore stabilizing cart position with swinging up and stabilizing problems of nonlinear inverted pendulum are discussed and a new approach is developed.

Key words: Swing up and stabilizing, Inverted pendulum system, State-feedback, Sliding mode control.

ÖZET

TERS SARKAÇ SİSTEMİ İÇİN KAYMA MOD KONTROLÜ

SHARIF, Bakhtyar Abdullah
Elektrik ve Elektronik Mühendisliği Yüksek Lisans
Tez Yöneticisi: Prof. Dr. Ahmet UÇAR
Temmuz 2013
65 sayfa

Bu tezin amacı doğrusal ve doğrusal olmayan sistemler için kayma mod kontrolunun tasarlanmasıdır; kayma mod kontrolünün belirsizlik içeren doğrusal ve doğrusal olmayan sistemler için kullanılmasındaki avantajları görmek için bir ters sarkaç sistemi seçilmiştir. Ters sarkaç sistemi düşey doğrultudaki karasız denge noktası için doğrusallaştırılmış ve durum geri besleme ve LQR kontrolör tasarlanarak sonuçlar bu tezin seminerinde verilmiştir.

Bu tezde Ackerman formülü ve lineer matris eşitliği (LMI) kullanılarak belirsizlik içeren doğrusal sistemler için kayma mod kontrol tasarlanmıştır. Yukarı doğru salınım ve kararlılık ters sarkaç sısteminin özel problemleri kayma mod ile çözüldü.

Ayrıca ters sarkacın yukarı denge noktasına doğru salınımı ve kararlılığı sarkacın bulunduğu çekçek sisteminin kararlılığı ile birlikte değerlendirilmiş ve buna yönelik yeni bir metot önerilmiştir.

Anahtar kelimeler: sarkaç sistemi salınım ve kararlılığı, Durum geri besleme, Kayma mod kontrolü.

DEDICATION

First of all, praise is to ALLAH who has enlightened me and paved the way to accomplish this thesis. Then to My family, they should receive my greatest appreciation for their enormous love. They always respect what I want to do also give me their full support encouragement over the years.

In the end but not the end, to my wife AVAN, I could never have done this work without your faith, support, and constant encouragement.

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TABLE OF CONTENTS

CONTENTS **Page**

LIST OF FIGURES

Figures **Page**

LIST OF TABLES

Tables **Page**

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CHAPTER 1

INTRODUCTION

1.1 General

The inverted pendulum system studied in [1] is subject to this thesis. It is well known that the inverted pendulum system is a nonlinear system. It has been used to illustrate many of the ideas emerging in the field of nonlinear control and leads to develop and test control methods.

The inverted pendulum system that depicted in Figure 1.1 shows the cart-inverted pendulum. It consists of a rod mounted on the cart. Thus it can rotate freely around its fixed point P. The angle between the vertical axis and the rod is θ. The control force applied to the cart is u. Here a two dimensional problem in which the pendulum moves in the plane of the page is considered is considered only. In Figure 1.1, l represents the half length of rod where its mass m assumed to be at its geometric centre. The mass of the cart is represented by M.

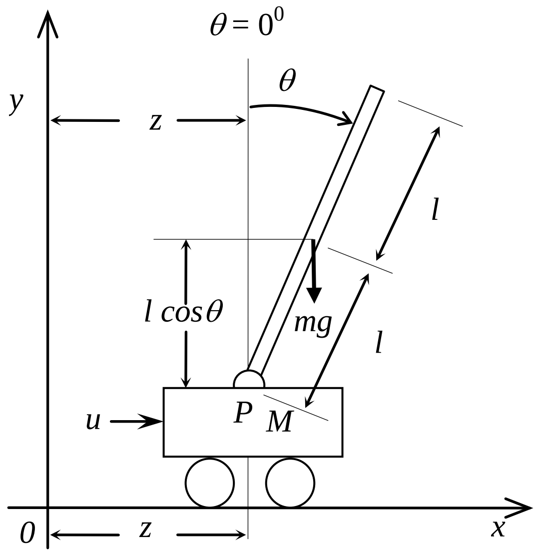

Figure 1.1: Inverted Pendulum System

The control problem is discussed in [1] is stabilizing the pendulum nearby upright position.

Here the control problem is stated as follows;

Swing-up the pendulum from pendant position and bring it to upright position at first stage and then stabilizing the pendulum at unstable equilibrium point (upright position θ=0), with the control signal u by moving cart on the direction of displacement z.

Above control problem task is considered for the linearized and nonlinear models of inverted pendulum system given in Figure 1.1. In [1] complete state feedback control and optimal state feedback namely Linear Quadratic Regulator (LQR) are designed for linear case without swing-up. This problem will be considered here again in the first phase of this thesis. In the second phase, variable structure system with sliding mode control SMC is studied and designed for both swinging up and stabilizing the pendulum subsystem with stabilizing the cart subsystem position.

1.2 Aim of this thesis

One of the old and challenging problems in the nonlinear controller field is swing-up and stabilizing of inverted pendulum system. There are many control strategies has been introduced and used for both problems [2-12].

Spong and Block [3] have used partial feedback linearization and zero dynamic for the swinging up the pendulum from rest position to upright position, however no stability analysis was discussed there and the cart position not considered.

K. Astrom and Furuta [6] have achieved swing-up by energy based control under the condition that the pendulum’s input is the cart acceleration. Note that the controller doesn’t include the cart displacement.

Matsuda and *et al.* [9] have developed an energy controller for swing-up the pendulum based on normalized energy model and normalized oscillation model but the limitation of the cart rail doesn’t considered.

Lozano and *et al.* [7] have presented a control strategy based on total energy of the system by using its passivity property for swing-up the pendulum. In this method the cart position was restricted.

Chatterjee *et al.* [13] have proposed swing-up with restricting cart displacement by using a generalized energy control and stabilizing the pendulum at upright position based on LQR technique.

Mihara and *et al.* [12] have proposed the two step control strategy based on Saeki's backstepping-like controller. The energy based control was used to swing-up the pendulum and potential function-based controller is employed to stabilize the inverted pendulum at upright position.

In this thesis, the controller is designed such that it swing-up the pendulum from its downward position to its unstable equilibrium point and controlling the cart position. Thus the aim of this thesis are;

a) Designing the swing up controller by combining the energy-function controller with sliding mode controller within the limited region of the cart traveling and.

b) Design the stabilizing controller for pendulum to stay at upright equilibrium point at the desired cart position.

1.3 Structure of thesis

This thesis consists of five chapters:

Chapter 1: Contains introduction, thesis objective, and thesis organization.

Chapter 2: Shows the system mathematical model and control problem formulation. Also include the controller design for linearized inverted pendulum system based on state feedback and LQR controller, the results are also discussed.

Chapter 3: In this chapter the basic principle about sliding mode control is discussed and inverted pendulum system is considered as an example to explain the designing based on Sliding mode control.

Chapter 4: The swing up pendulum and stabilizing it at upright position is considered in this chapter and simulation result is discussed.

Chapter5: presents conclusion of the thesis and recommendation for future studies that can be conducted in this field.

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CHAPTER 2

INVERTED PENDULUM SYSTEM AND THE CONTROL PROBLEM FORMULATION

2.1 Model of Inverted Pendulum System

To obtain a mathematical model for the inverted pendulum system as shown in the Figure 2.1 where the pendulum is suspended on P point, we assume the centre of gravity of the pendulum rod is at the centre of the rod, and its friction is zero. The angle between the vertical axis and the rod is defined as θ. The (x, y) coordinates of the centre of gravity of the rod are defined as (x_G ,y_G);

$$x_G = z + lsin\theta$$

$$y_G = lcos\theta$$

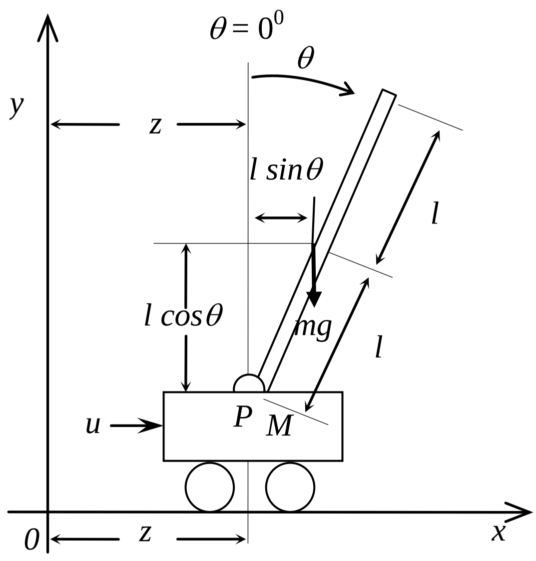

Figure 2.1: Inverted Pendulum system free body diagram

Inverted pendulum system's motion given in Figure 2.1 consists of two subsystems; one is the motion of the pendulum about its suspension point with the cart and the other is the horizontal motion related to the cart. The motions equations for the system are considered to obtain the mathematical model of the system defined in Figure 2.1. The rotational motion of the pendulum rod about its centre of gravity can be described by:

$$I\ddot{\theta} = Vl\, sin\theta - Hl\, cos\theta \tag{2.1}$$

where: I is the moment of inertia of the rod about its center of gravity and I=(ml^2/3), H is horizontal motion of center of gravity and V is vertical motion of center of gravity.

The linear motion is obtained from Newton's second law; $F = ma = m\frac{d^2z}{dt^2}$ where a is acceleration.

The horizontal motion of center of gravity, H, of pendulum rod is given by

$$H = m\frac{d^2}{dt^2}(z + l\, sin\theta)$$

$$H = m\ddot{z} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 \tag{2.2}$$

The vertical motion of center of gravity, V of pendulum rod is

$$V - mg = m\frac{d^2}{dt^2}(l\cos\theta)$$

$$V - mg = -ml\sin\theta\,\ddot{\theta} - ml\cos\theta\,\dot{\theta}^2$$

$$V = mg - ml\sin\theta\,\ddot{\theta} - ml\cos\theta\,\dot{\theta}^2 \tag{2.3}$$

The horizontal motion of cart is described by

$$M\frac{d^2z}{dt^2} = u - H \tag{2.4}$$

by substituting Equation (2.2) and Equation(2.3) in Equation(2.1), leads.

$$I\ddot{\theta} = mlg\sin\theta - ml\cos\theta\,\ddot{z} - ml^2\ddot{\theta}$$

$$(I + ml^2)\ddot{\theta} - mlg\sin\theta + ml\cos\theta\,\ddot{z} = 0$$

$$J\ddot{\theta} - mlg\sin\theta + ml\cos\theta\,\ddot{z} = 0 \tag{2.5a}$$

where J= (I + ml^2)

$$\ddot{\theta} = \frac{mlg\ \sin\theta}{J} - \frac{ml\ \cos\theta}{J}\ddot{z} \tag{2.5b}$$

also by substituting Equation (2.2) in Equation (2.4) yields

$$M\ddot{z} = u - \left(m\ddot{z} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2\right)$$

$$(M + m)\ddot{z} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 = \mathrm{u} \tag{2.6a}$$

Equation (2.5a) represents the pendulum dynamic equation, and Equation (2.6a) is represents the cart dynamic equation.

By substituting Equation (2.6a) in Equation (2.5b) then

$$\ddot{\theta} = \frac{(M+m)mlg\ \sin\theta - m^2l^2 \sin\theta \cos\theta\, \dot{\theta}^2}{J(M+m) - m^2l^2\cos^2\theta} - \frac{ml\cos\theta}{J(M+m) - m^2l^2\cos^2\theta}u \qquad (2.5c)$$

By substituting Equation (2.5c) in Equation (2.6a) leads:

$$\ddot{z} = \frac{Jml\ \sin\theta\, \dot{\theta}^2 - m^2l^2 g\ \sin\theta\cos\theta}{J(M+m) - m^2l^2\cos^2\theta} + \frac{J}{J(M+m) - m^2l^2\cos^2\theta}\ u \qquad (2.6b)$$

Finally the mathematical models of the system governed by Equations (2.5c) and Equation (2.6b) have nonlinearity. Hence it needs to be approximated by linearization technique to study the behavior of the system nearby an equilibrium point or an operation points.

Substituting the system parameters are given in [14]; the mass of cart M = 2kg, the mass of rod m= 0.1kg, l = 0.25 *meters* g=9.8ms^{-2} then the nonlinear numerical system is:

$$\ddot{\theta} = \frac{20.58\sin\theta - 0.025\sin\theta\cos\theta\dot{\theta}^2}{0.7 - 0.025\cos^2\theta} - \frac{\cos\theta}{0.7 - 0.025\cos^2\theta}u$$

$$\ddot{x} = \frac{0.1\ \sin\theta\, \dot{\theta}^2 - 2.94\ \sin\theta\cos\theta}{8.4 - 0.3\ \cos^2\theta} + \frac{4}{8.4 - 0.3\ \cos^2\theta}u$$

To write the state space model of the inverted pendulum system, let define a new state variables $[x_1, x_2, x_3, x_4] = [\theta, \dot{\theta}, z, \dot{z}]$.

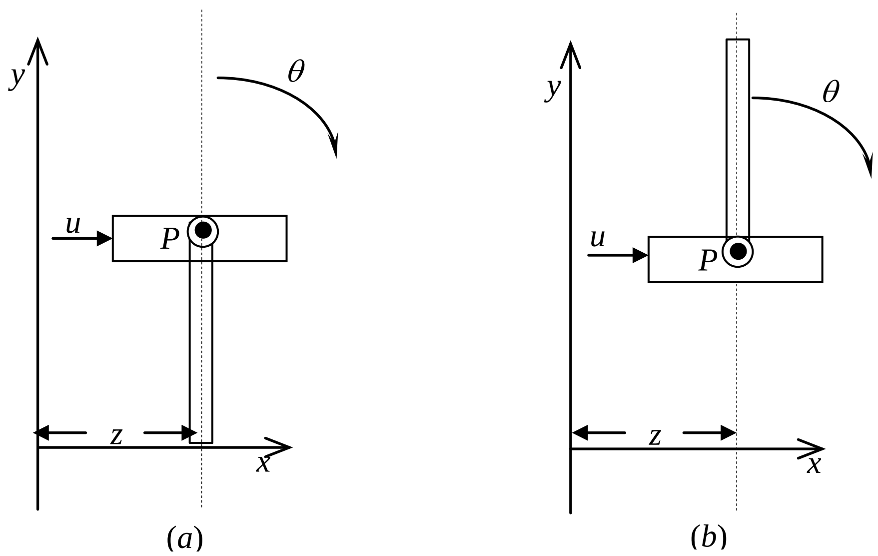

Figure 2.2: Equilibrium points of inverted pendulum rod subsystem.

The motion of the free inverted pendulum represented in Figure 2.1 physically can be divided and studied by two subsystems as depicted in Figure 2.2 (a) and (b) where Figure 2.2 (a) represents vertical up position for $(x_1,x_2)=(0^0,0)$ and Figure 2.2 (b) represents vertical down position, $(x_1,x_2)=(\pm\pi,0)$. It is clear that inverted pendulum rod subsystem, shown in Figure 2.2 is unstable at equilibrium point, $(x_1,x_2)=(0^0,0)$ and stable at equilibrium point, $(x_1,x_2)=(\pm\pi,0)$.

Since stabilizing of unstable equilibrium point, $(x_1,x_2)=(0^0,0)$ is a particular interest, it is necessary to linearize system for this point. Ones $x_1=x_2=0$ then the other subsystem, the horizontal motion of cart, is assumed has $x_3=x_4=0$ and the system has equilibrium point$(x_1, x_2, x_3, x_4) = (0,0,0,0)$. To design the regulator system the origin is selected to be the desired point.

For linearization we assume that the inverted pendulum rod is nearby to the vertical position equilibrium point, $(x_1,x_2)=(0^0,0)$ (angle between rod and vertical axis, $\theta= x_1$ is small quantities). With this assumption the nonlinearities in equations (2.5b) and (2.6b) are approximated such that;

1. $sin\ \theta \cong \theta$.
2. $cos\ \theta \cong 1$.

where the θ and 1 are the first parameter of the following Taylor series of $sin\ \theta$ and $cos\ \theta$ respectively.

$$sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \doteqdot \theta$$

$$cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \doteqdot 1$$

Since θ is a small quantity $\dot{\theta}$ is also be small quantities thus any nonlinearity that contains $\dot{\theta}$ can be neglected. Hence $\theta\dot{\theta}^2 = 0$ in Equation(2.5c) and (2.6b).

Then, the linearized form for Equations (2.5c) and Equation (2.6b) are:

$$\ddot{\theta} = \frac{(M+m)mlg}{J(M+m)-m^2l^2}\theta - \frac{ml}{J(M+m)-m^2l^2}u \tag{2.5d}$$

$$\ddot{z} = -\frac{m^2l^2g}{J(M+m)-m^2l^2}\theta + \frac{J}{J(M+m)-m^2l^2}u \tag{2.6c}$$

Equations (2.5d) and Equation (2.6c) are described the linearized motion of the inverted-pendulum-on-the-cart system.

The state space model of the inverted pendulum system, with the defined state variables$[x_1, x_2, x_3, x_4] = [\theta, \dot{\theta}, z, \dot{z}]$, where θ and z are outputs, and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Notice that x_1 is the angle indicate the rotation of the pendulum rod about point *P*, and x_3 is the position of the cart.

Then Equations (2.5d) and (2.6c) in the form of the defined state variables have the following forms.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{(M+m)mlg}{J(M+m) - m^2l^2}x_1 - \frac{ml}{J(M+m) - m^2l^2}u$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{m^2l^2g}{J(M+m) - m^2l^2}x_1 + \frac{J}{J(M+m) - m^2l^2}u$$

By using the matrix notation, the state space representation;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)mlg}{J(M+m)-m^2l^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m^2l^2g}{J(M+m)-m^2l^2} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{ml}{J(M+m)-m^2l^2} \\ 0 \\ \frac{J}{J(M+m)-m^2l^2} \end{bmatrix}u \qquad (2.7)$$

Substituting the system parameters; M = 2kg, m= 0.1kg, l = 0.25 *meters* g=9.8ms^{-2} [14] then the numerical state space model of inverted pendulum is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 30.48 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.362 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1.48 \\ 0 \\ 0.493 \end{bmatrix}u$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad (2.8)$$

The system matrix, *A*, is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 30.48 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.362 & 0 & 0 & 0 \end{bmatrix}$$

and solving the characteristic matrix $|\lambda \mathrm{I} - \mathrm{A}|=0$ where I is identity matrix.

$$|\lambda \mathrm{I} - A| = \det\begin{bmatrix} \lambda & -1 & 0 & 0 \\ -30.48 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0.362 & 0 & 0 & \lambda \end{bmatrix} = \lambda^4 - 30.48\lambda^2 = 0$$

and has the following eigen values;

$\lambda_1 = 0, \lambda_2 = 0,\ \lambda_3 = 5.52,\ \lambda_4 = -5.52$. Thus the characteristic equation of the system has one pole in positive real part and the system is unstable.

2.2. Control problem formulation

The problems task to be solved here is to swing up the pendulum and bring it from rest position to a region near upright position with limit cart displacement where the pendulum is swinging and controlling the cart position, and then balancing the pendulum at the upright position, $\theta = 0$, with the control signal u, by moving cart on the direction of displacement z.

2.3 Complete state feedback for inverted pendulum

The linearized inverted pendulum system is considered and complete state feedback is designed. Note that the inverted pendulum's open loop system is discussed in section 2.1 is unstable. To stabilizing the system and letting the closed-loop poles at desired location in $\lambda = -\sigma \pm j\omega$ complex plain, based on state feedback design a controller $u = -Kx$ to drive all system states from any initial conditions to the origin. Thus stabilizing and keeping the inverted pendulum rod at upright position $\theta = 0$ is the control problem.

The system block diagram is shown in Figure 2.3 contains the complete sate feedback control law is:

$$u = -Kx = -[k_1 \quad k_2 \quad k_2 \quad k_4]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{2.9}$$

The state space form of linearized model of inverted pendulum given in

Equation(2.8) for (0,0).

for regulator case [14].

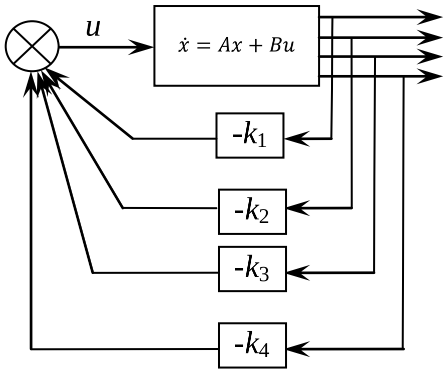

Figure 2.3: Full state feedback control of inverted pendulum control system

To determine state feedback gain K the following steps are required. First the system has to satisfy the following assumptions;

*Assumption A*1: The system has to be full state controllable. Thus controllability matrix ($[B \quad AB \quad \dots\dots \quad A^{n-1}B]$) is full rank.

*Assumption A*2: All the states of the system need to be measurable. i.e observability matrix ($[C \quad CA \quad \dots\dots \quad CA^{n-1}]^T$) is full rank.

To satisfy the first *Assumption A*1 and the second *Assumption A*2, the rank of controllability matrix must examine $M=[B\ AB\ A^2B\ A^3B]$ and observability matrix $N=[C\ CA\ CA^2\ CA^3\ ]^T$,the rank of matrix M and matrix N is four. Therefore the system described by Equation (2.8) is completely state controllable, and all state variables are observable. So pole placement technique is possible.

Next step to designing controller based on pole-placement for desired closed loop poles. To achieve a specific reasonable speed and damping in the response of the designed system the following desired poles are selected;

$\mu_{1,2} = -1 \mp j\sqrt{3}$ and $\mu_{3,4} = -6$.

where the settling time is 4 *sec* and the maximum over shoot is *18*% in the step response of the cart.

Note the other two poles are located to the left and far away from the first and second desired closed-loop poles. Therefore their effect over the total response can be neglected.

Since the system given in Equation (2.8) has single input then Ackermann's formula namely known *acker* in MATLAB Control toolbox can be used. The *M-File* 3.1 of MATLAB program is given in Table 2.1.

Table 2.1 MATLAB program to determine K matrix based on Ackermann's formula.

M-File 21:

```
% To determine state feedback gain K matrix for Inverted Pendulum by
% Ackermann's formula
J=[-1+i*sqrt(3) -1-i*sqrt(3) -6  -6] % Desired closed loop poles.
A=[0 1 0 0;30.48 0 0 0;0 0 0 1;-0.362 0 0 0]
B=[0;-1.48;0;0.493]
K=acker(A,B,J)
```

Thus the controller gains are;

$K = [k_1\ k_2\ k_3\ k_4] = [-67.1\quad -12.2\quad -9.9\quad -8.2]$

and the controller dynamic is:

$$u = -Kx = [67.1 \quad 12.2 \quad 9.9 \quad 8.2]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The closed loop performance of the system is depicted in Figures 2.4 and 2.5, where the initial conditions are $[x_1(0), x_2(0), x_3(0), x_4(0)] = [-0.3,0,0,0]$. Figure 2.4 shows the system output x_1 started from $x_1(0) = -0.3$ *rad* $\approx$-17 degrees, and it approaches to zero when $t \to \infty$ with the settling time approximately 4.8 *sec.* and x_2 where the maximum value is 1 *rad/sec* then it converges to zero.

Figure 2.5 shows the position and velocity of the cart where the cart position goes far from origin 0.4 *m* and then both of them approach to zero.

Note that all states approach to zero when $t \to \infty$. Hence the closed loop system is asymptotically stable in the sense of Lyapunov.

Figure 2.6 shows the control signal which starts from -17 *Newton* and approaches to zero when time approaches to infinity.

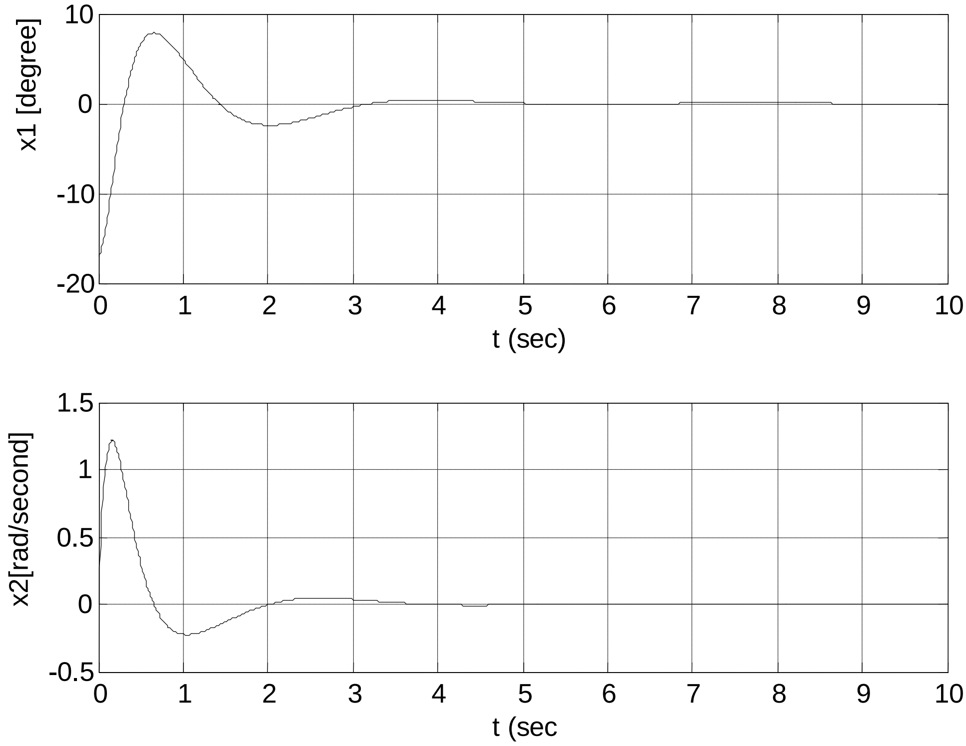

Figure 2.4: Closed loop rod angle and road angular velocity time response.

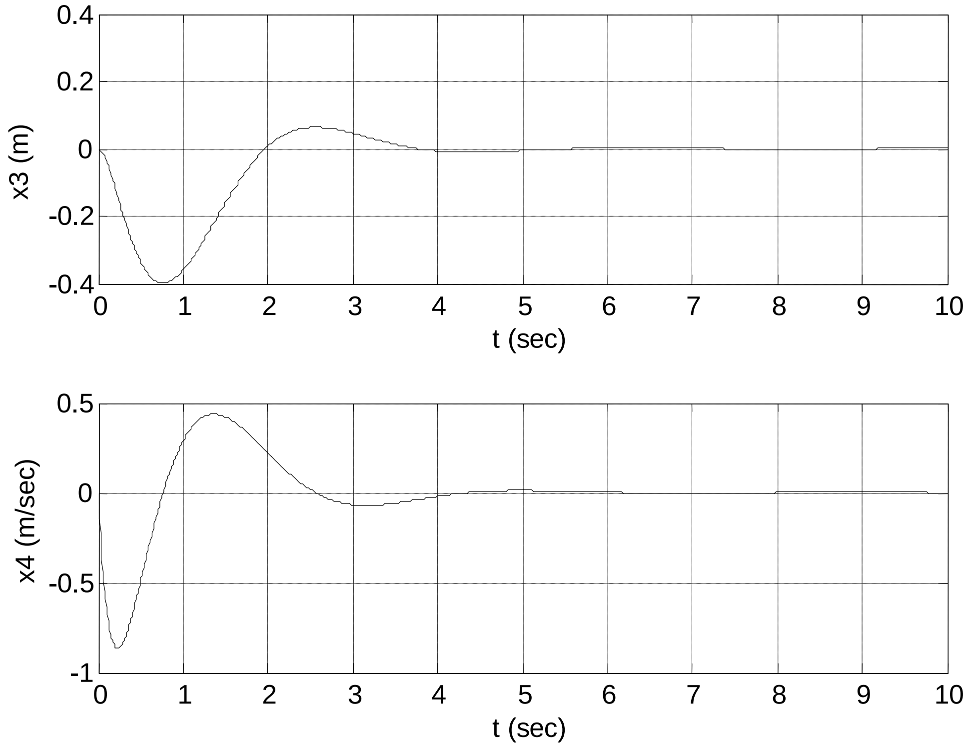

Figure 2.5: Closed loop cart position and cart velocity time response.

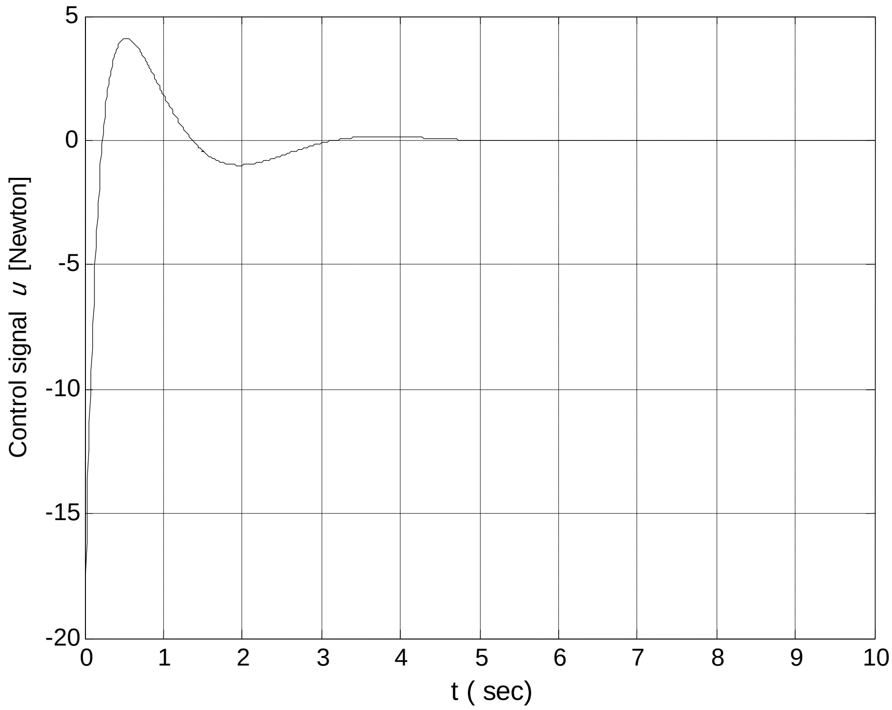

Figure 2.6: Control signal for closed-loop system.

The results show that the controller is achieves the desired performance but it may be produced a large control signal leads saturation and leads system enter in nonlinear region. This is the natural result of pole placement, it is achieved the desired performance without limiting the control energy.

Optimal control strategies need to account the control energy with constrain on the closed loop performance. That is the controller gain is obtained based on the weighting function of the control signal and steady state error. Then the closed loop performance is resulted. In the following subsection, an optimal state feedback control namely Linear Quadratic Control (LQR) based on the Lyapunov method is considered.

2.4 Design LQR control for linearized pendulum model.

The linear model of inverted pendulum system in Equation(2.8) is considered and the control gain in Equation(2.9) is designed based on linear quadratic regulator (LQR) approach [14] such that the following performance index J is minimized:

$$J = \int_0^{\infty} (e^T Q e + u^T R u) dt \qquad (2.10)$$

where $e = x - x_d$ and if $x_d=0$ then $e=x$.

Hence the performance index J is:

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt$$

where x^T and u^T represents the transpose of the state vector x and input vector u respectively, Q and R are a positive-definite (or positive-semi definite) real symmetric matrices. Q and R matrices respectively represent the cost functions placed upon reducing errors e and the cost placed upon saving energy (limiting control energy).

Q and R are usually chosen to be diagonal. Selection of the weighting matrices Q and R in the cost function J is based on some criteria determined and given in [15]. The Q matrix is usually responsible for system performance and R matrix is responsible for control effort. The physics of the problem may suggest further terms in the cost function. Another procedure to select weighting matrices is by trial and error [15-16]. The designer first specifies which outputs are important to drive to zero with this method.

To see the effect of Q and R matrices on the system performance and control signal, here three different set of matrices Q and R are chosen and their effects are discussed.

case 1: Let $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $R = 0.01$,

The *lqr* command in MATLAB Control tool Box is used and controller gain K is determined by *M-File* is given in table 2.2.

Running *M-File* 2.2 program leads to:

$K = [k_1\ k_2\ k_3\ k_4]$ =[-101.6380 -21.0469 -10.0000 -16.2955].

Table 2.2 MATLAB program to determine K matrix based on LQR.

M-File 2.2:

```
% To determine state feedback gain K matrix for Inverted Pendulum by
%calculating k matrix by LQR case 1
A=[0 1 0 0;30.48 0 0 0;0 0 0 1;-0.362 0 0 0]
B=[0;-1.48;0;0.493]
Q=[1 0 0 0 ;0 1 0 0 ;0 0 1 0 ;0 0 0 1]
R=[0.01]
K=lqr(A,B,Q,R)
```

Again the system is released from $x_1(0)$ = -0.3 *rad* ≈-17 *degree* and the other states initial conditions are zero. The closed loop performance for stabilized system based on LQR technique of case 1 changing with time is depicted in Figures 2.7 and 2.8. Figure 2.7 shows the rod angle it starts from its initial condition then it approaches to zero when $t \rightarrow \infty$ with the settling time is 2 *sec*, and the rod angular velocity reaches maximum value approximately at 1.4 *rad/sec*.

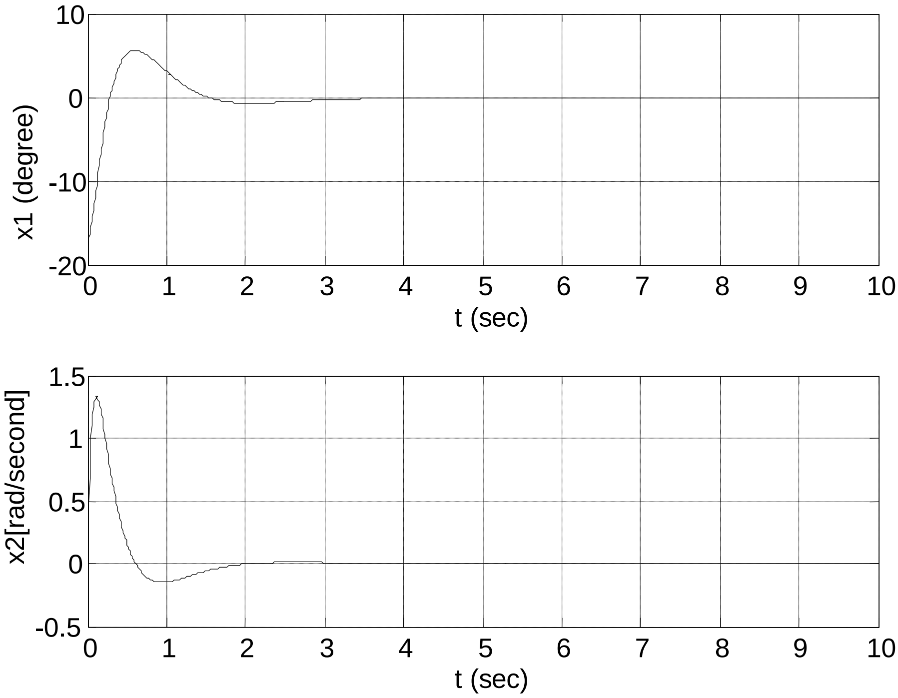

Figure 2.7: Closed loop rod angle and rod angular velocity time response based on LQR for Case 1.

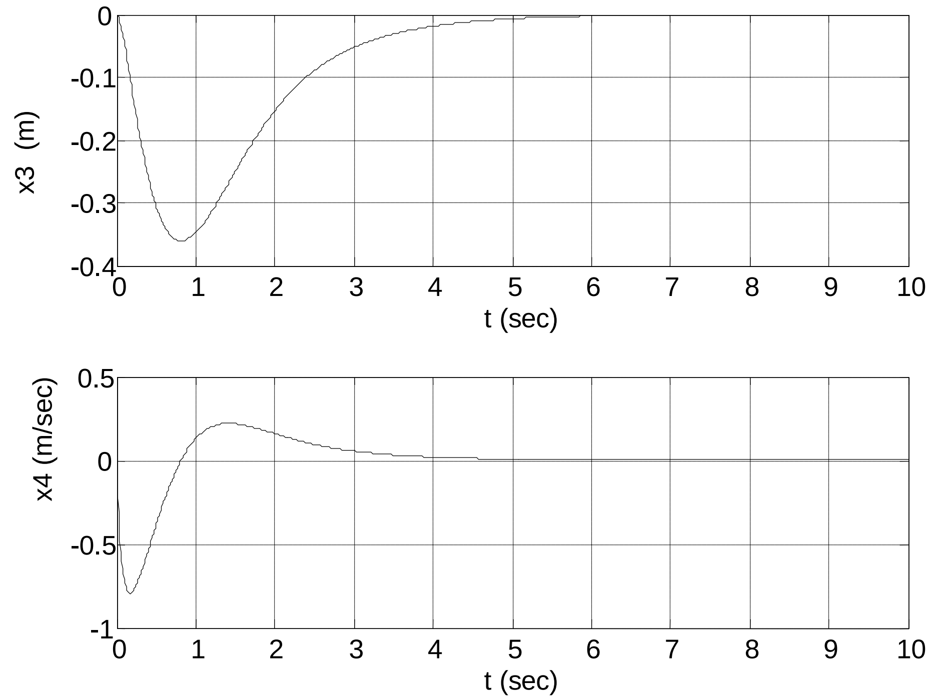

Figure 2.8: Closed loop cart position and cart velocity time response based on LQR for Case 1.

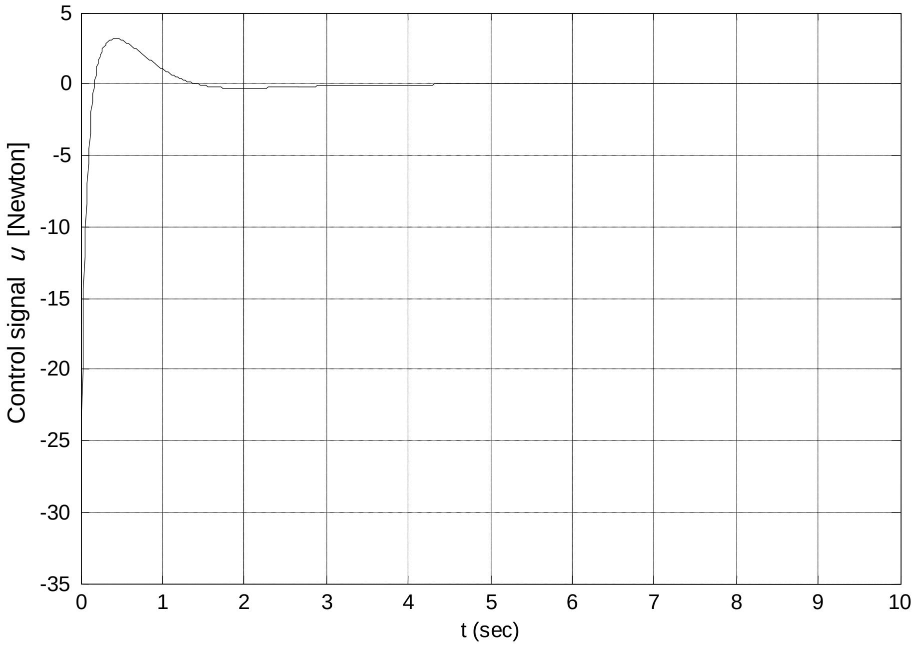

Figure 2.9: Closed-loop system input signal based on LQR for Case 1.

Case 2:

Let keep Q matrix as previous case and R = 1 *M file* 2.2 gives:

$K = [k_1\ k_2\ k_3\ k_4]$ =[-50.0313 -9.1447 -1 -2.6482]

Figure 2.10 shows the rod angle and rod angular velocity time response for case 2, Figure 2.10 (*a*) shows the rod angle start from -17^o and the controller converges it to zero with a settling time of 5 *sec* and maximum over shoot approximately % 22. Figure 2.10 (*b*) shows the rod angular velocity it reach to maximum value 0.8 *rad/sec*, then controller also converges it to zero.

The results are presented in Figures 2.11 and 2.12 are almost the same as Figure 2.5 and Figure 2.6 because of selection the Q and R matrices.

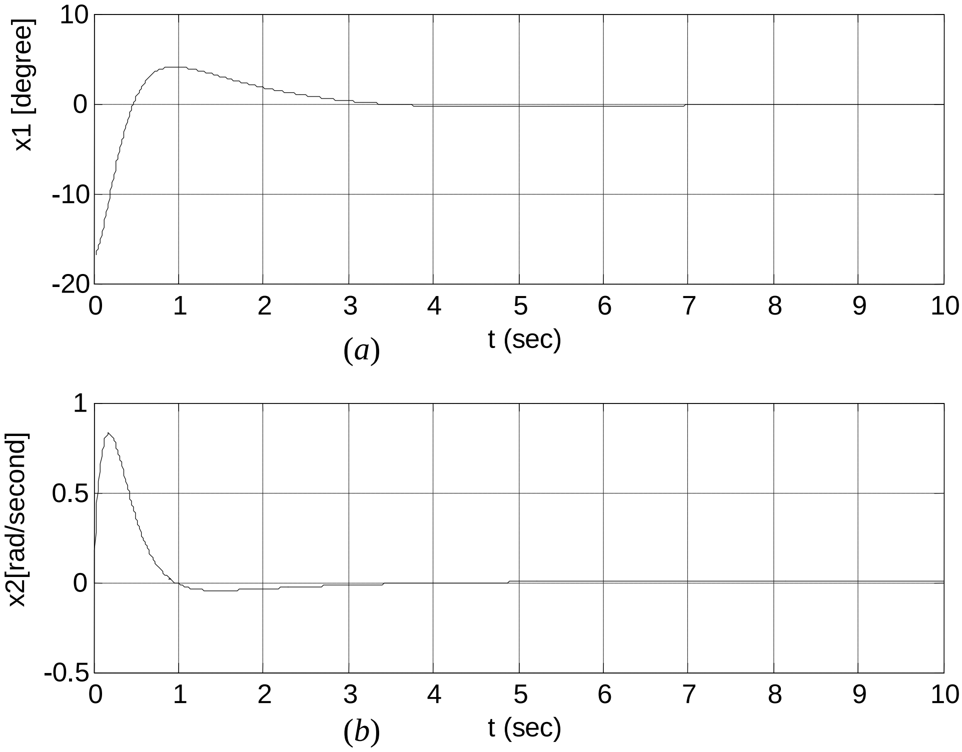

Figure 2.10: Closed loop rod angle and rod angular velocity time response for Case 2

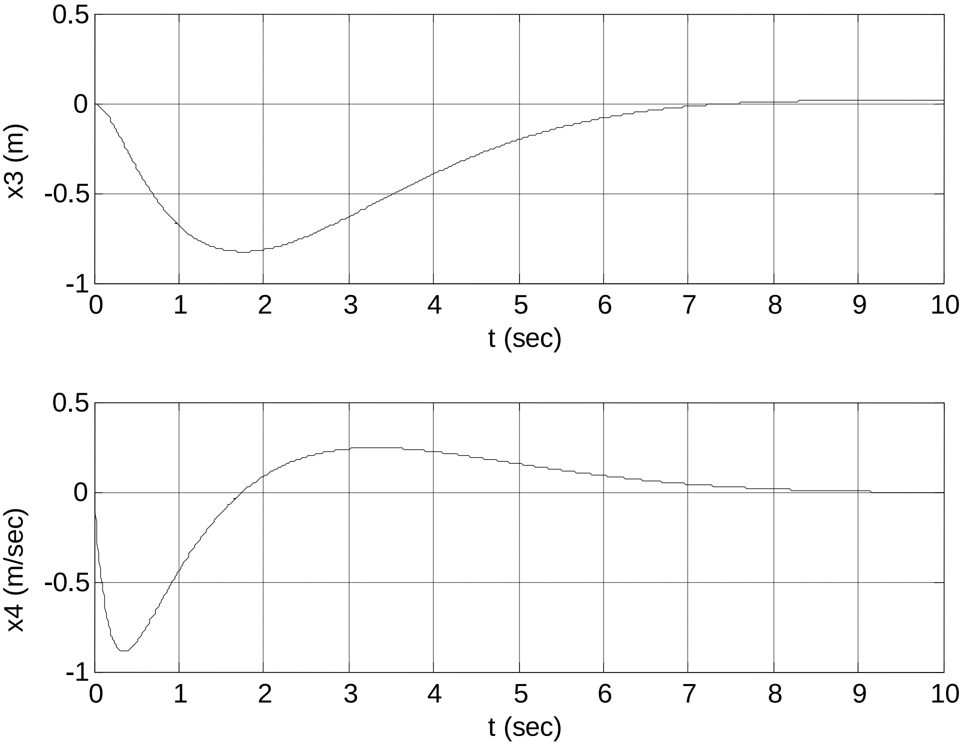

Figure 2.11: Closed loop cart position and cart velocity time response for Case 2.

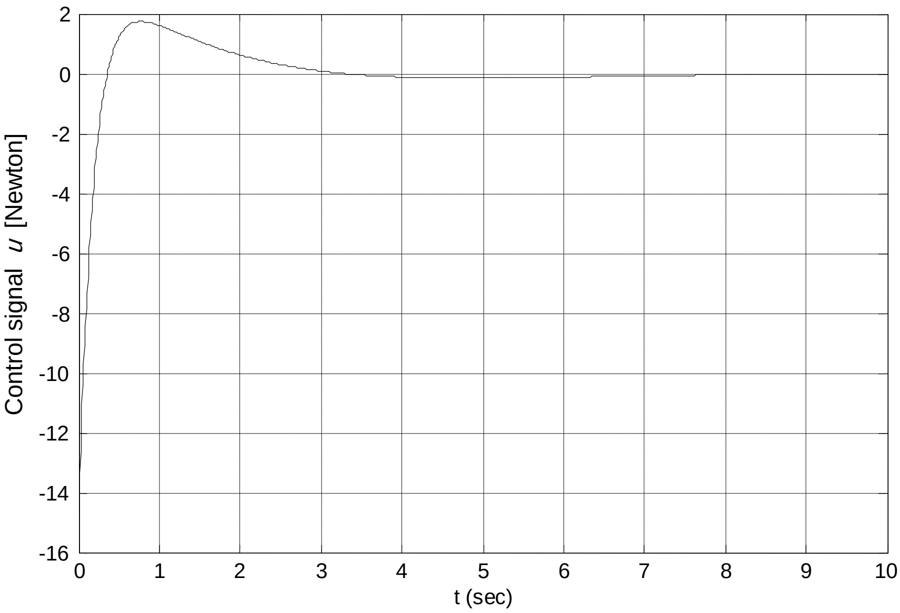

Figure 2.12: Closed-loop system input signal for case 2.

2.5 Conclusions and remarks

Mathematical model of inverted pendulum is obtained, nonlinear model is linearized and then two linear controllers are applied. In case controller designed based on the pole placement, it is noticed that the controller is achieved the desired performance but it produced a large control signal that may be saturated and lead system to behave as a nonlinear system. This is the natural result from pole placement that the desired closed loop is selected without considering the control energy. In case of design a controller based on LQR technique the effect of weighting matrixes are as follow:

- When the value of R constant is increased by keeping the same value for Q matrix, that the control signal value is decreased.
- By fixing the R constant value and increasing Q matrix element $q33$ speeded up the time response [16].

Designing controller based on pole placement and LQR technique do not perform stable behavior with presence of disturbance and cannot reject the perturbation. Robust control strategies need to be designed to reject the disturbance and perform properly where the system dynamic model contains uncertainty.

CHAPTER 3

SLIDING MODE CONTROL

3.1 Introduction

Sliding Mode Control (SMC) is adapted from Variable structure systems [17]. Sliding Mode Control is a type of discontinuous in nonlinear control, it consists of two or more continuous subsystems together by using a high-speed switching control that switches between two different values based on the defined conditions.

The advantage of using SMC is it's ability to very robust closed loop control system that gives the system fast response and insensitive to system parameter variations or uncertainty, external disturbances or noises[18]. It can be designed for both linear and non-linear systems.

3.2 Sliding Mode Controller design for linear systems

Designing of any SMC consist of two stages, Switching function $s(x)$ and discontinuous control law. A switching function is mostly assumed to be linear and its dimension equal to m (m input matrix dimension).

Designin SMC for linear system is discussed and studied in [1]. Consider a linear system described by the state Equation:

$$\dot{x} = Ax + Bu \tag{3.1}$$

where vectors $x(\mathrm{t}) \in \mathrm{R}^n$, $u(\mathrm{t}) \in \mathrm{R}^m$, and $\mathrm{rank}(B) = m$

The steps for designing any SMC are as follow:

3.2.1 Design switching function

A suitable switching surface is generally define as $s(x)=Cx$, where dim-$s=m$, C is n-dimensional constant row vector. Each switching function $s(x)$ describes a linear surface $s(x) = 0$, which is defined to be *switching surface* or *switching manifold*. If every point in $s(x)$ is an end point, that is, for every point in $s(x)$ there are trajectories reaching it from both sides of $s(x)$, then switching surface $s(x)$ called a *sliding surface.*

There are many methods has been introduced to design the sliding surface.

$$s(x)=Cx \tag{3.2}$$

Here, the method is recently introduced in [19] based on Ackermann's Formula is used at first, and then linear matrix inequality is used to design the switching surface.

3.2.2 Reaching Condition

The condition whereby the state will move toward and reach to the sliding surface is called *Reaching Condition* [17]. Reaching phase or reaching mode is the system trajectory under reaching condition. To determination the reaching condition two approaches are proposed here.

1) *The Direct Switching Function Approach*: The older of reaching condition proposed was:

$\dot{s}(x) > 0, \quad \text{when} \quad s(x) < 0$

$\dot{s}(x) < 0, \quad \text{when} \quad s(x) > 0$

where $\dot{s}(x) = C\dot{x}$ (3.3)

A similar sufficient condition was proposed by Utkin [20]:

$\lim_{s\to -0} \dot{s} > 0 \qquad \text{and} \qquad \lim_{s\to +0} \dot{s} < 0$

2) *The Lyapunov Function Approach*: By choosing the candidate the positive definite Lyapunov function $V(s)$ such that $V(s) = \frac{1}{2}(s(x))^2$, the condition for stability is the time derivative of $V(s)$ along to the switching surface and $\dot{V} = s(x)\dot{s}(x)$ needs to be negative definite.

$\dot{V}(s) < 0 \qquad \text{when} \quad s(x) \neq 0$

3.2.3 Reaching Law

The reaching law is differential equation which specifies the dynamics of a switching function $s(x)$ [21]. Sliding mode based on reaching law contains two stages, *reaching stage*, it drives system toward the sliding surface, and *sliding stage,* it can constrain the system on the sliding manifold and lead it to the origin, or equilibrium point [22].

A general form of reaching law is:

$$\dot{s}(x) = -\eta sign(s(x)) - kf(s) \tag{3.4}$$

where η is positive diagonal matrix and k is diagonal matrix with positive elements, $sgn(s) = [sgn(s_1) \ldots. \, sgn(s_m)]^T$ and $f(s) = [f_1(s_1) \ldots\ldots \, f_m(s_m)]^T$.

Selecting value of η and k identify different rates for $s(x)$ and produce different structures in the reaching law.

Three practical different reaching laws are [22-23]:

1) *Constant rate reaching law*;

$$\dot{s}(x) = -\eta sign(s(x)) \tag{3.4a}$$

where η represents a constant rate and $\eta > 0$. This reaching law forces the switching variable $s(x)$ to reach switching surface $s(x) = Cx = 0$ at a constant rate $|\dot{s}| = -\eta$. The advantage of this law is its simplicity. But if η is too small the reaching time will be too long. On the other hand by selecting η too large will cause serious chattering.

2) *Exponential reaching law*

$$\dot{s}(x) = -\eta sign(s(x)) - ks(x) \tag{3.4b}$$

where $\eta > 0$, $k > 0$ and $\dot{s}(x) = -ks(x)$ is exponential term, can be solved as:

$$\frac{ds(x)}{dt} = -ks(x) \quad \Rightarrow \quad \frac{ds(x)}{s(x)} = -kdt \quad \Rightarrow \quad \int \frac{ds(x)}{s(x)} = -k\int dt \Rightarrow \ln s(x) = -kt$$

leads to $s(x) = s(0)e^{-kt}$.

Clearly, by adding the proportional rate term $-ks(x)$ to this reaching law forces the state to approach the switching surface faster when $s(x)$ is enough large. It can be shown that the reaching time for x state is move from an initial state $x(0)$ to the switching surface $s(x) = Cx = 0$ is finite, and is given by

$$T = \frac{1}{k}\ln\frac{k|s|+\eta}{\eta}$$

3) *Power rate reaching law*

$$\dot{s}(x) = -k|s(x)|^{\alpha} sign(s(x)) \tag{3.4c}$$

where $k > 0$ and $0 < \alpha < 1$. This reaching law decreases the slow motion's time when the state is far away from the switching surface line, but when the state near to the switching surface (switching manifold) it reduced the rate, so the result from this law is reduced the chattering reaching mode and fast reaching. Integrating power rate reaching law from $s = s_0$ (where s_0 is initial value of $s(x)$) to $s(x) = Cx = 0$, yields

$$T = \frac{1}{(1-\alpha)k} s_0(1-\alpha)$$

shows the reaching time T is finite. Thus power rate reaching law gives a fine reaching time. In addition, power rate reaching law eliminate the chattering because in the right hand of this law does not have the $-\eta\, sign(s)$ term.

3.2.4 Control law u

The discontinuous control u, that enforces the sliding mode in the $s(x) = Cx = 0$ plane to provide the condition $s(x)\dot{s}(x) < 0$ and lim $s(x)\dot{s}(x) < 0$ with $s(x) \to 0$, can be obtained by two methods

a) Using the general sliding control law

$$u = \begin{cases} u^{+}(x,t) & where \quad s(x) > 0 \\ u^{-}(x,t) & where \quad s(x) < 0 \end{cases} \tag{3.5}$$

Then the control law can be obtained as:

$$u = -\eta sign(s(x)) \tag{3.6}$$

To check the reachability condition. Let select the candidate positive definite Lyapunov function $V(s)$ such that:

$$V(s) = \frac{1}{2}(s(x))^2 \tag{3.7}$$

The time derivative of $V(s)$ needs to be negative definite such that.

$$\dot{V}(s) = s(x)\dot{s}(x) \leq 0 \tag{3.8}$$

substituting Equation (3.3) in (3.8) leads to:

$$\dot{V}(s) = s(x)C\dot{x}(x)$$

substituting Equation (3.1) in above Equation has:

$$\dot{V}(s) = s(x)(CAx + CBu)$$

substituting Equation(3.6) in above Equation leads to:

$$\dot{V}(s) = s(x)(CAx + CB(-\eta sign(s(x))))$$

$\dot{V}(s) = s(x)CAx - CB\eta sign\big(s(x)\big)s(x)$ by using *sign*(*s*)***s*=|*s*|, *sign* function property

$\dot{V}(s) = s(x)CAx - CB\eta|s(x)|$

To satisfy the condition given in Equation (3.8) we have

$0 > |s(x)|CA|x| - CB\eta|s(x)|$

$$\eta > (CB)^{-1}CA|x| \quad (3.9)$$

b) Using reaching law

Using one of the reaching law and by taken time derivative for switching surface *s*(*x*) as shown by Equation (3.3) and substituting Equation (3.1) in this equation as:

$\dot{s}(x) = C\dot{x}$

$$\dot{s}(x) = C(Ax + Bu) \quad (3.10)$$

Selecting any reaching law and then equating it with Equation (3.10) for example selecting exponential reaching law as:

$\dot{s}(x) = -ks(x) - \eta sign(s(x))$

substituting Equation(3.10) in above equation Leads to

$-ks(x) - \eta sign(s(x)) = CAx + CBu$

then the control law can be obtained as:

$$u(x) = (CB)^{-1}(-ks(x) - \eta sign(s(x)) - CAx) \quad (3.11)$$

3.3 Sliding mode controller where the switching surface is designed based on Ackermann's formula.

Ackermann's Formula gives the designer to determine a linear state-feedback control law in specific form resulting in a feedback system with desired eigenvalues [24]. A similar work presents itself when designer want to design sliding mode control for linear systems with discontinuity surface. Because the corresponding sliding mode equation is linear and depends on the coefficients of the surface equation for *n* order system the sliding mode equation is of the (*n* – 1) order and does not depend on the disturbance [25].

Switching surface design;

The C-vector where $C \in R^{1\times n}$ may be obtained in an explicit form without the sliding motion equation based on Ackermann's Formula, and the procedure is illustrated in the literature [14, 19] .

The design steps to determine C-vector matrix for system (3.1) are as follows [25]:

The first step is selecting the sliding motion desired spectrum $\lambda_1, \lambda_2, \ldots. \lambda_{n-1}$ and then the equation of discontinuity plane $s(x)= Cx = 0$ is found based on Ackermann's Formula (see appendix A).

$$C = e^T P_1(A)$$

$$C = e^T(A - \lambda_1 I)(A - \lambda_2 I)\ldots(A - \lambda_{n-1} I) \qquad (3.12)$$

where $e^T = [0\ 0\ 0 \ldots .1][B\ \ AB\ \ \ A^2B\ \ A^{n-1}B\ ]^{-1}$ and I identity matrix $\in R^{n\times n}$.

Consider the linear model of inverted pendulum system with bounded disturbance ($f(t)$) for equilibrium point $(x_1, x_2) = (0,0)$.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 30.48 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.362 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1.48 \\ 0 \\ 0.493 \end{bmatrix} (u + f(t)) \qquad (3.13)$$

where $f(t)=f_o sin\omega t$ (f_o constant).

Let design a sliding mode control to keep the pendulum rod at upright position $x_1=\theta= 0$ as follow.

Sliding Surface equation for this system is:

$$s(x) = Cx$$

where $C=[c_1\ c_2\ c_3\ c_4]$

$$s(x) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \qquad (3.14)$$

Since the order of the system is n=4, determining the C matrix based on Ackerman's formula (see appendix A) that described by equation (3.12) is:

$$C = e^T(A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) \qquad (3.15)$$

where $e^T = [0\ 0\ 0\ 1][B\ \ AB\ \ \ A^2B\ \ A^3B\ ]^{-1}$ and $(\lambda_1, \lambda_2, \lambda_3)$ are the desired eigenvalues of sliding motion.

In this particular design, to achieve a specific reasonable speed and damping in the response of the closed loop system, let $\mu_1 = \lambda_1 = -1 + j\sqrt{3}$, $\mu_2 = \lambda_2 = -1 - j\sqrt{3}$ and $\mu_3 = -6$. Substituting all known parameters in Equation (3.15), leads C constants:

The *M-File* (MATLAB Program) written to obtain C matrix for this problem is given in Table (3.1)

Table 3.1 MATLAB Program to determine C matrix.

M-File 3.1:

```
%%%Calculation C* matrix %%%%

A=[0 1 0 0;30.488 0 0 0;0 0 0 1;-0.362 0 0 0]

B=[0;-1.48;0;0.495];

μ1=-1-i*sqrt(3); μ2=-1+i*sqrt(3); μ3=-6;

M= inv([B A*B (A^2)*B (A^3)*B])

E=[0 0 0 1]*M;

I=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];

P=(A-(μ1*I))*(A-(μ2*I))*(A-(μ3*I));

C=E*P
```

The result of *M-File* 3.1 is;

C=[-5.9569 -1.0433 -1.6488 -1.0992].

The switching surface for whole system is:

$$s(x) = -5.95x_1 - 1.04x_2 - 1.64x_3 - 1.09x_4$$

Second step is to design a control law based on Equation (3.5), here three different methods are considered to design the controller. First sliding mode controller is designed based on Lyapunov method. Then the controller is designed based on two different reaching laws. The performance of the closed loop system is studied for three cases and the results are discussed.

Case 1. In the first case, the most general sliding mode controller is used to stabilizing the pendulum at upright position, and the reachability condition is obtained, by selecting the sliding mode controller as:

$$u = -\eta sign(s(x)) \tag{3.16}$$

where $\eta > 0$ and the switching surface $s(x)$ is

$$s(x) = -5.95x_1 - 1.04x_2 - 1.64x_3 - 1.09x_4$$

Let f_o=1 and $\omega = 3$, leads to $f(t)=sin3t$.

By selecting the candidate positive definite Lyapunov function $V(s)$ such that: $V(s(x)) = \frac{1}{2}(s(x))^2$

The time derivative of $V(s)$ needs to be negative definite such that.

$$\dot{V}(s) = s(x)\dot{s}(x) \quad < 0$$

the time derivative of switching surface is:

$$\dot{s}(x) = -5.95\dot{x}_1 - 1.04\dot{x}_2 - 1.64\dot{x}_3 - 1.09\dot{x}_4$$

Substituting the system Equation (3.13) in to above equation leads to.

$$\dot{s}(x) = -31.36x_1 - 5.95x_2 - 1.64x_4 + u + f(t)$$

where u is the control law defined in Equation(3.16) and $f(t)$ is bonded disturbance ($f(t)=sin3t$).

substituting the controller given in Equation (3.16) in above equation leads to:

$$\dot{s}(x) = -31.36x_1 - 5.95x_2 - 1.64x_4 - \eta sign(s) + f(t) \tag{3.17}$$

To see the negatives of derivative of Lyapunov Equation (3.4) let substituting Equation (3.17) in to Equation (3.4) then.

$$\dot{V}(s) = s(-31.36x_1 - 5.95x_2 - 1.64x_4 - \eta sign(s) + f(t))$$

$$\dot{V}(s) = -31.36x_1 s - 5.95x_2 s - 1.64x_4 s - \eta sign(s)s + f(t)s \quad \text{since } sign(s)s=|s|$$

$$\dot{V}(s) = -31.36x_1 s - 5.95x_2 s - 1.64x_4 s - \eta|s| + |f(t)||s| \quad \text{by using } |xy| \le |x|.|y|$$

$$\dot{V}(s) \le -31.36|x_1||s| - 5.95|x_2||s| - 1.64|x_4||s| - \eta|s| + |f(t)||s|$$

$$\dot{V} \le -|s|(31.36|x_1| + 5.95|x_2| + 1.64|x_4| + \eta - |f(t)|)$$

$$0 < (31.36|x_1| + 5.95|x_2| + 1.64|x_4| + \eta - |f(t)|$$

$$\eta > -31.36|x_1| - 5.95|x_2| - 1.64|x_4| + |f(t)| \tag{3.18}$$

Let select the state variables initial conditions are:

[$x_1(0)$, $x_2(0)$, $x_3(0)$, $x_4(0)$]=[-0.35, 0, 0, 0] and η=18 then the control signal is

$$u = -18\ sign\left(s(x)\right) = -18\ sign(-5.95x_1 - 1.04x_2 - 1.64x_3 - 1.09x_4)$$

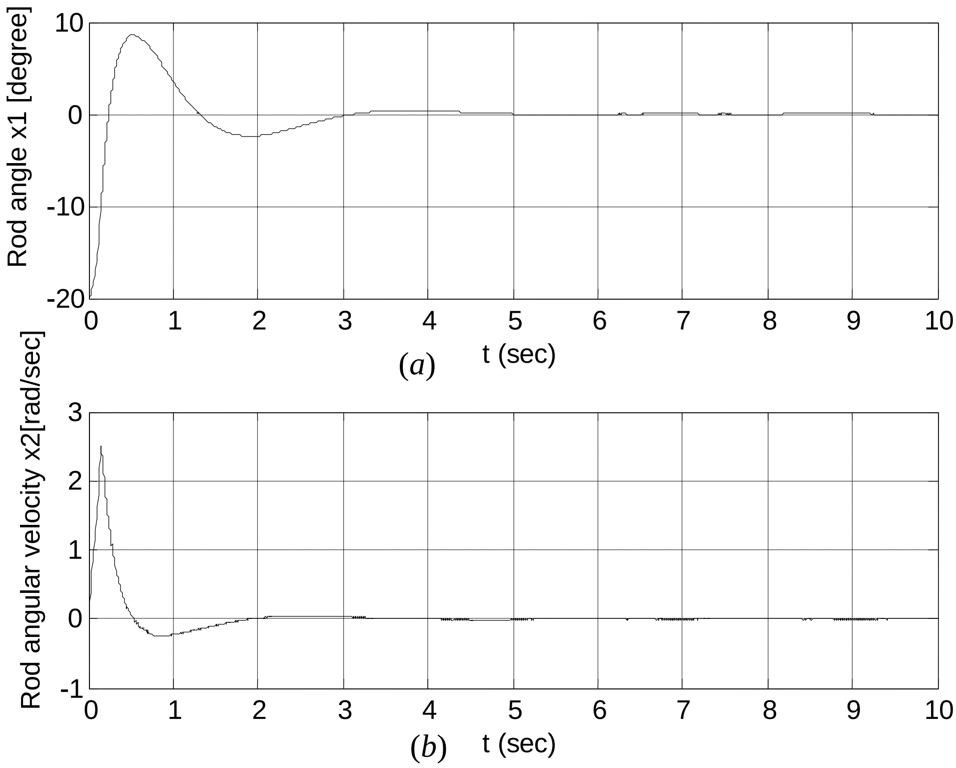

Figure 3.1: Rod angle and rod angular velocity time response for Case 1

The closed loop performance of the stabilized system based on sliding mode of case1 are depicted in Figures 3.1 and 3.2, where the states initial conditions are [$x_1(0)$, $x_2(0)$, $x_3(0)$, $x_4(0)$] = [-0.35,0,0,0]. Figure 3.1(a) shows the system output state x_1 starts from -20^o, and it approaches to zero when $t \to \infty$ with the settling time approximately 4 *sec*. Note that closed loop system exhibits the desired behavior and the disturbance is rejected. Figure 3.1(b) shows the rod angular velocity the when the rod angle converges to zero also it approaches to zero when the time approaches to infinity $t \to \infty$.

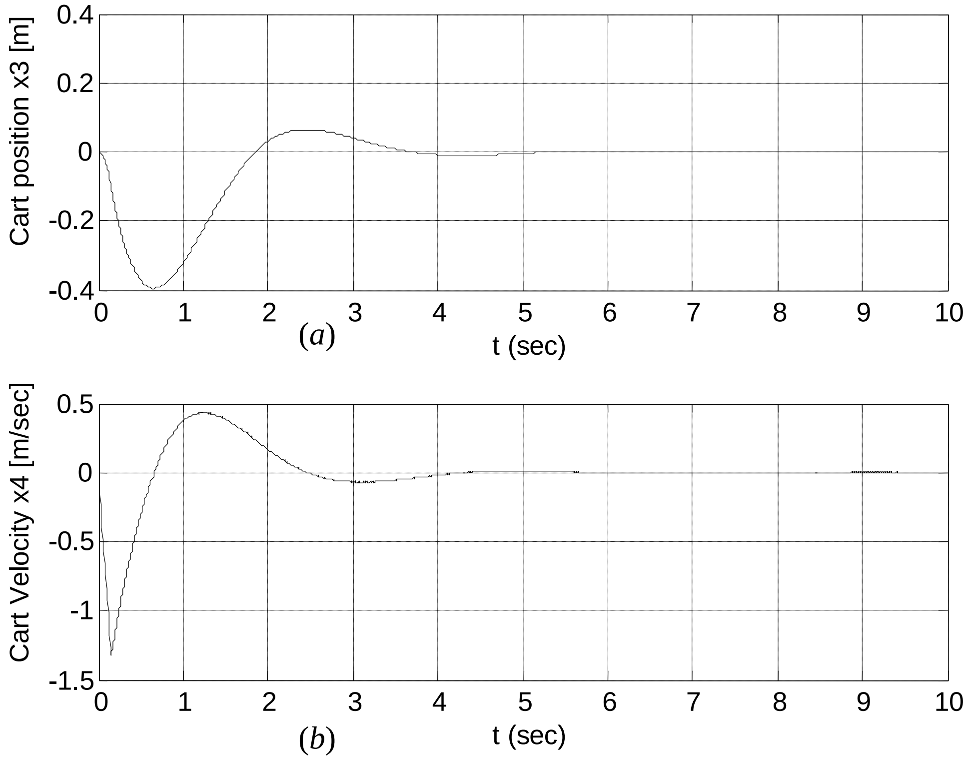

Figure 3.2: Cart position and cart velocity time response for Case 1

Figure 3.2 *a* shows the cart position displacement where it starts from origin and it go far from its origin approximately 0.4 *meters* then the controller can bring it again to zero. Figure 3.2 *b* shows the cart velocity time response, it reaches maximum value (1.3 *meters/sec*) after 0.3 *seconds* from beginning and then it's value is go to zero when time approach to infinity $t \to \infty$.

Switching surface curve and control input signal versus time, are depicted in Figure3.3 illustrate that the switching function value is decreased to zero and the control signal contains the chattering effect.

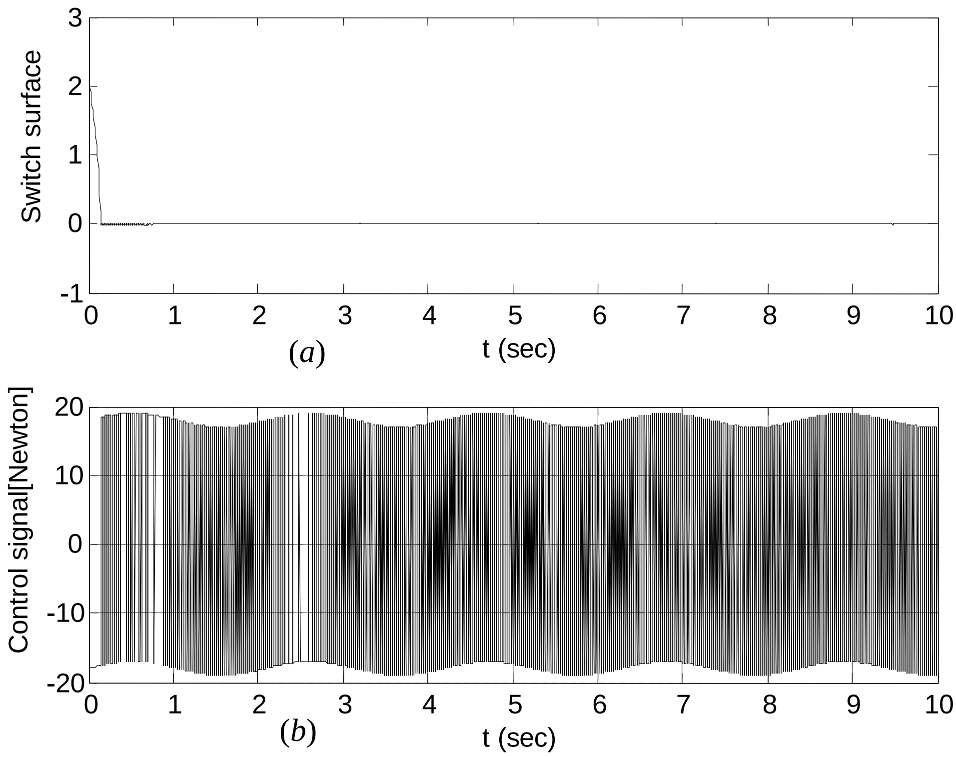

Figure 3.3: Switching surface and control signal for Case 1

Case 2. In the second case, the controller is designed based on the following exponential reaching law given in Equation (3.4.b).

$\dot{s}(x) = -\eta sgn(s) - ks$ where $\eta > 0$ and $k > 0$

Let η=8, k = 1, and $f(t)=sin3t$, then the control law can be found which is based on equation (3.11) is:

$u(x) = (CB)^{-1}(-ks(x) - \eta sign(s(x)) - CAx)$

The terms (CB), $(CB)^{-1}$ and CA are:

CB = [-5.9569 -1.0433 -1.6488 -1.099]*[0 -1.48 0 0.495]^T

$CB \cong 1$, since $(CB)^{-1} \cong 1$ and CA=[-31.4108 -5.9569 0 -1.6488]

Substituting the control law yields:

$u = - ks(x) - \eta\ sign(s) + 31.4\ x_1 + 5.9569\ x_2 + 1.6488\ x_4$

$\quad = -s(x) - 8\ sign(s) + 31.4\ x_1 + 5.9569\ x_2 + 1.6488\ x_4$

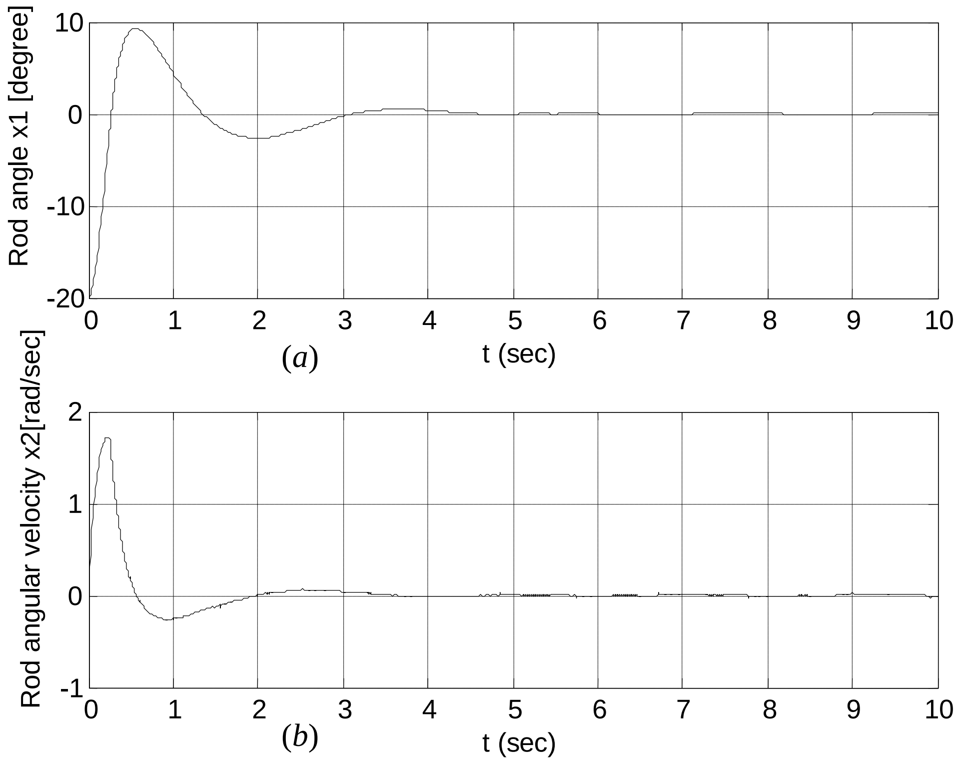

Figure 3.4: Rod angle and rod angular velocity time response for Case 2

The closed loop performance of the stabilized system based on sliding mode case2 is depicted in Figures 3.4 and 3.5 for the same initial condition of [$x_1(0)$, $x_2(0)$, $x_3(0)$, $x_4(0)$] = [-0.35,0,0,0]. Figure 3.4(*a*) shows the rod angle (system output state x_1) that the controller brings it from its initial value (-20^o) to zero. Figure 3.4(*b*) shows the rod angular velocity it's reaching to maximum value 1.8 *rad/sec*, and then it is approached to zero when the rod angle converges to zero. Note that the maximum value of angular velocity in this case is smaller than at Case 1.

Figure 3.5 (*a*) shows the cart position displacement where it starts from origin and it goes far from its origin approximately 0.42 *meter* then the controller can bring it again to zero. Figure 3.5(*b*) shows the cart velocity time response, it reach maximum value (1.2 *meter/sec*) after 0.3 *sec* from beginning. Then its value is decreased to zero when time approach to infinity $t \to \infty$. Control input signal versus time, are depicted in Figure 3.6(*b*) and shows that it started with a value is approximately 20 Newton and it has the chattering effect. Note that the chattering amplitude in this case is smaller than at the previous case, and the rachability condition does not depend on the states initial conditions.

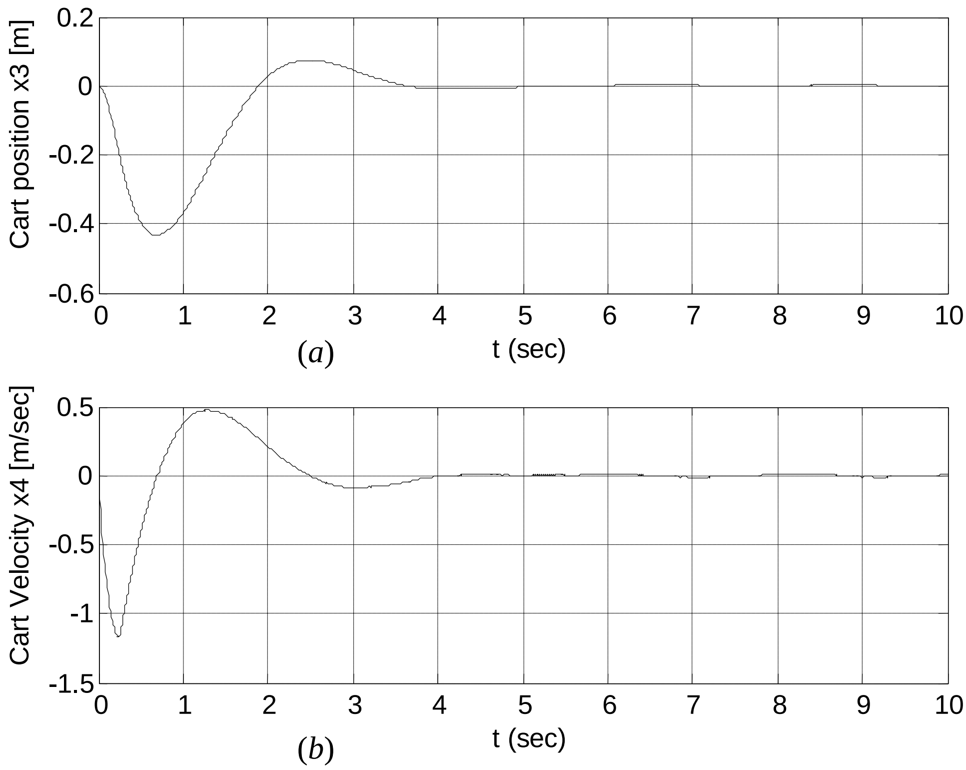

Figure 3.5: Cart position and cart velocity time response for Case 2

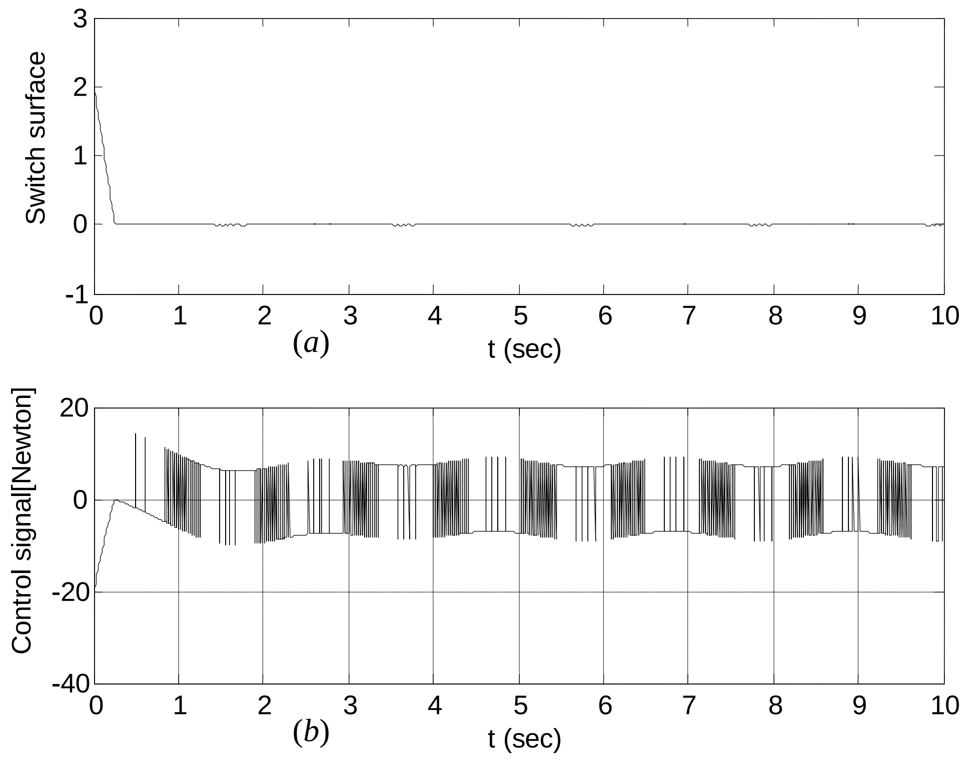

Figure 3.6: Switching surface and control signal for Case 2.

Case 3. When the following power rate reaching law (Equation (3.4.c)) is selected with $k = 8$, $\alpha = 0.5$ and $f(t)=sin3t$.

$$\dot{s}(x) = -k|s|^{\alpha} sign(s) \tag{3.4.c}$$

The control law can be found based on Equation (3.11) as:

$$u(x) = -k|s|^{\alpha}\, sign(s) + 31.4\, x_1 + 5.9569\, x_2 + 1.6488\, x_4$$

$$= -8|s|^{0.5}\, sign(s) + 31.4\, x_1 + 5.9569\, x_2 + 1.6488\, x_4$$

The closed loop performance of the system based on sliding mode in case 3 is depicted in Figures 3.7 and 3.8 where the initial condition are [$x_1(0)$, $x_2(0)$, $x_3(0)$, $x_4(0)$] = [-3.5,0,0,0]. Figures 3.7 (a) shows that the rod angle is started from $x_1(0) = -20^o$, and it approaches to zero when $t\to\infty$, and the other states also approach to zero when the $t\to\infty$, as shown by Figures 3.7 (b) and Figure 3.8 these results are shown that the closed loop system can exhibit the desired behaviors.

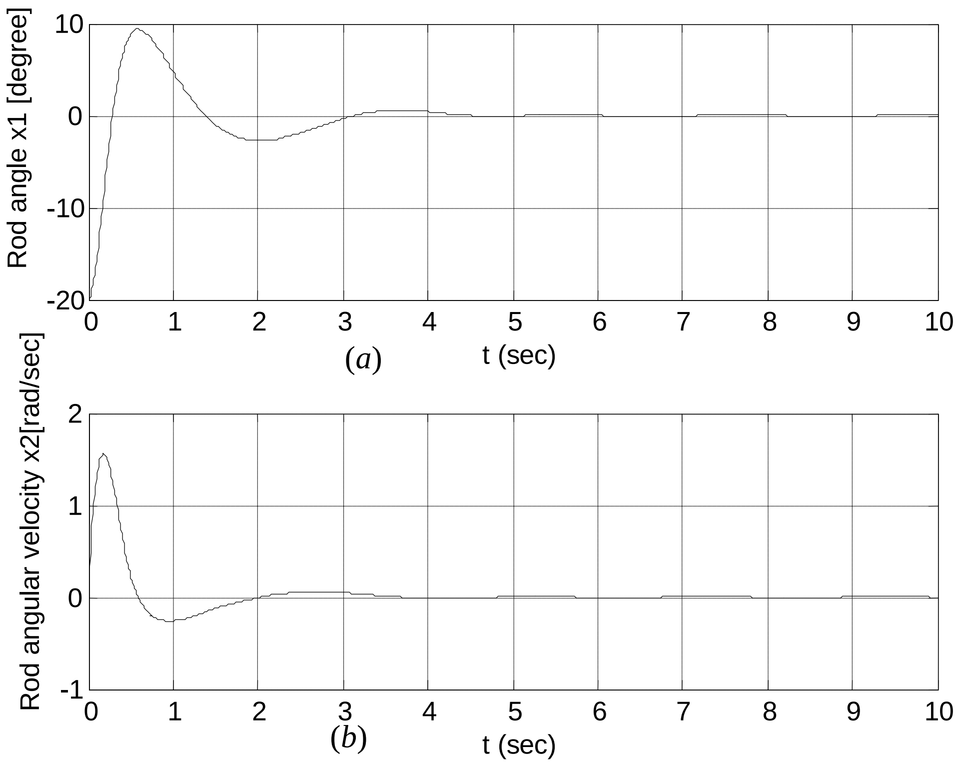

Figure 3.7: Rod angle and rod angular velocity time response for Case 3

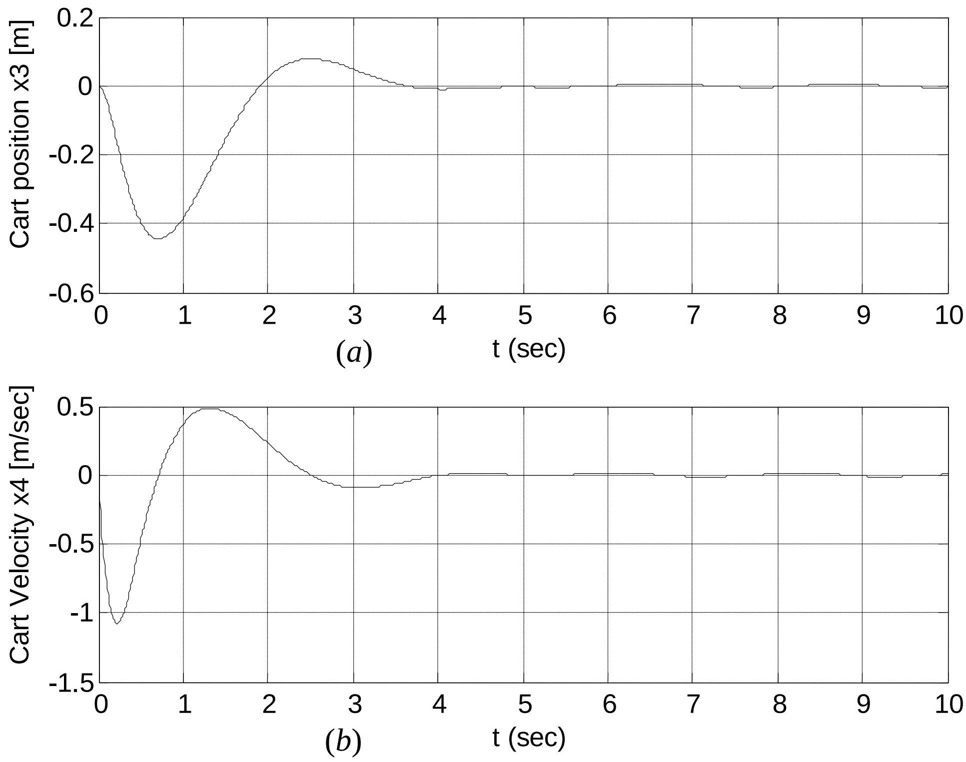

Figure 3.8: Cart position and cart velocity time response for Case 3.

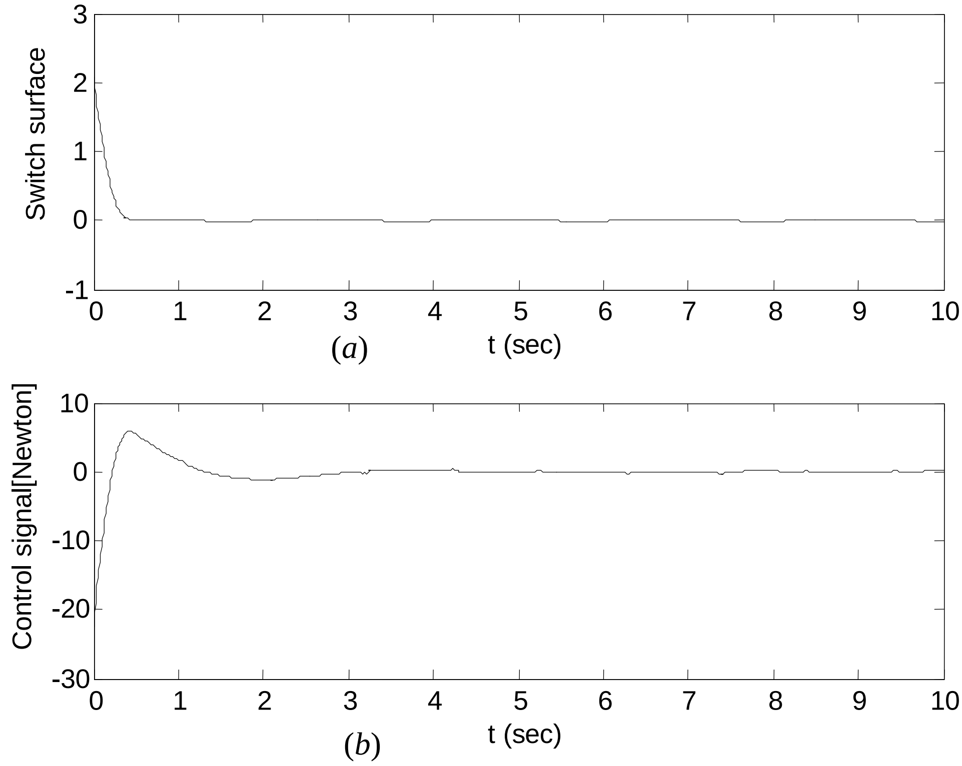

Figure 3.9: Switching surface and control signal for Case 3.

Figure 3.9 (*b*) shows the control input signal versus time for case 3, and shows that the controller based on power rate reaching law is reduced the chattering effect.

3.4 Sliding mode controller based on Linear Matrix Inequality (LMI).

Consider the linearized inverted pendulum system with bounded disturbance described by Equation (3.20), will be considered and sliding mode control will be designed. Again the problem is to keep the inverted pendulum's rod at upright position $\theta = 0$, subject to the disturbance.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 30.48 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.362 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1.48 \\ 0 \\ 0.493 \end{bmatrix} (u + f(t)) \tag{3.20}$$

where f(t) is disturbance and is $f(t) = f_0 \, sin(\omega t)$.

Designing the switching surface $s(x)$ based on LMI that proposed in [22]as follow:

$$s(x)=Cx$$

$$=B^{T}Px \tag{3.21}$$

where $P \in \Re^{n\times n}$ is positive-definet matrix and B is the input matrix.

Sliding mode control based on auxiliary feedback:

To solving the P matrix, the system controller is designed as:

$$u_t(x) = u_l(x)+u_s(x) \tag{3.22}$$

where $u_t(x)$ is total controller, $u_l(x)= -Kx$ is linear feedback controller ($K \in R^{1x4}$ vector matrix and $u_s(x)$ is sliding mode controller.

Let the following Lyapunov function is:

$$V(x)=x^{T}Px \tag{3.23}$$

Rewriting Equation (3.23) as:

$$V(x) = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= P(x_1^2 + x_2^2 + x_3^2 + x_4^2)$$

The derivative of $V(x)$ is:

$\dot{V}(x) = P(2x_1\dot{x}_1 + 2x_2\dot{x}_2 + 2x_3\dot{x}_3 + 2x_4\dot{x}_4)$

$$= 2[x_1 \;\; x_2 \;\; x_3 \;\; x_4]P\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}$$

finally the time derivative of Lyapunov function is:

$$\dot{V}(x) = 2x^T P\dot{x} \tag{3.24}$$

where the closed loop system is

$\dot{x} = Ax + Bu$

substituting Equation (3.22) in above leads to:

$\dot{x} = Ax + B(-Kx + u_s(x))$

$\dot{x} = (A - BK)x + Bu_s(x)$

$$\dot{x} = \bar{A}x + Bu_s(x) \tag{3.25}$$

where $\bar{A}$ is the closed loop matrix for u=-Kx and is:

$$\bar{A} = A - BK \tag{3.25b}$$

substituting Equation (3.25) in (3.24) yields to:

$\dot{V}(x) = 2x^T P(\bar{A}x + Bu_s(x))$

$\dot{V}(x) = 2x^T P\bar{A}x + 2x^T PBu_s(x)$

when $t \geq t_o$, there exists $s(x)= B^T Px=0$, therefore $s^T(x)= x^T PB=0$, it leads to

$\dot{V}(x) = 2x^T P\bar{A}x$

$\dot{V}(x) = x^T(P\bar{A} + \bar{A}^T P)x$

the condition to make $\dot{V}(x) < 0$ is

$$P\bar{A} + \bar{A}^T P < 0 \tag{3.26}$$

Multiplying above inequality by P^{-2}

$\bar{A}P^{-1} + P^{-1}\bar{A}^T < 0$ Let $M=P^{-1}$, then $\bar{A}M + M\bar{A}^T < 0$

Substituting Equation (3.25b) in above inequality leads to

$(A - BK)M + M(A - BK)^T < 0$

$AM - BKM + MA^T - MK^T B^T < 0$ Letting L=KM, leads to

$$AM - BL + MA^T - L^T < 0 \tag{3.27}$$

Now the problem is become to solve above matrix inequalities. The M matrix can be obtained by solving the inequality defined in Equation (3.27), and then P matrix can be obtained with *M-file* given in Table (3.2).

Table 3.2 MATLAB program for solving inequality Equation (3.27).

MATLAB *m-file*

```
A=[0 1 0 0;30.48 0 0 0;0 0 0 1;-0.362 0 0 0]
B=[0;-1.48;0;0.493]c
% LMI Var Description
setlmis([]);
M = lmivar(1, [4 1]);   % 1 -> symmetric block diagonal, then P is
symmetric
L = lmivar(2, [1 4]);   % Define L is 1 row,4 column

% LMI
%First LMI
lmiterm([1 1 1 M], A, 1, 's');   % A*M+M'*A'<0
lmiterm([-1 1 1 L], B, 1, 's');  % 0<B*L+L'*B'

%Second LMI
lmiterm([-2 1 1 M], 1, 1);       % 0<M, then P is positive matrix

lmis=getlmis;

[tmin,xfeas] = feasp(lmis);
M = dec2mat(lmis,xfeas,M);
P=inv(M);
L = dec2mat(lmis,xfeas,L);
K=L*inv(M);
```

The *M-file* given in Table (3.2) results the followings P matrix, K matrix and C gain matrix;

$$P = \begin{bmatrix} 7.4338 & 1.2471 & 1.0764 & 1.1368 \\ 1.2471 & 0.3949 & 0.2105 & 0.3250 \\ 1.0764 & 0.2105 & 0.3852 & 0.2278 \\ 1.1368 & 0.3250 & 0.2278 & 0.4285 \end{bmatrix},$$ K=[-55.6   -5.5   -4.1   -4.2] and C=$B^T P$=[-1.2853   -0.4243   -0.1993   -0.2698].

Thus the switching surface is:

$s(x)=Cx= -1.2853x_1 -0.4243\, x_2 -0.1993\, x_3 -0.2698\, x_4$

$\dot{s}(x) = C\dot{x} = CAx + CBu(x)$

Here the sliding mode controller u_s designed is based on the following exponential reaching law given in Equation (3.4.b).

$\dot{s}(x) = -ks(x) - \eta sign(s)$ where $\eta > f_0$(t)

Let f(t)=$f_o sin\omega t$, f_0(t)=1 and ω=3.

then the sliding mode controller u_s law can be found like in Equation (3.11) is:

$u_s(x) = (CB)^{-1}(-ks(x) - \eta sign(s) - CAx)$

where CB=0.4909, $(CB)^{-1}$=2.0206, CA =[-12.8336 -1.2853 0 -0.1993], η=5 and k=2

$u_s(x)$=2.02(-2s-5sgn(s)+12.83x_1 +1.2853x_2 +0.199 x_4)

$u_s(x)$=-4s-10.1sgn(s)+25.9x_1 +2.59x_2 +0.4 x_4 (3.28)

the linear controller is:

$u_l(x)$ = -kx = 55.6x_1 +5.5x_2 +4.1 x_3+4.2 x_4 (3.29)

The total controller u_t given in Equation (3.22) that obtained from Equations (3.28) and Equation (3.29) is:

$u_t(x)$ = (55.6x_1 +5.5x_2 +4.1 x_3+4.2 x_4)+(-4$s(x)$-10.1sgn(s)+25.9x_1 +2.59x_2 +0.4 x_4)

$u_t(x)$ = 81.5x_1 +8.1x_2+4.1 x_3 +4.6. x_4 -4$s(x)$-10.1sgn(s) (3.30)

Simulation results for this case where the switching surface is designed based on LMI are depicted by Figures 3.10, 3.11 and 3.12. The results show that the closed loop performance has a smaller oscillation than in the previous cases and the switch surface value is smaller in case that designed based in Ackermann's formula, finally if compared the cart displacement between last case and previous case, can be note that cart displacement in the last case has smaller deflection than the previous cases. Note that the controller rejected the disturbance.

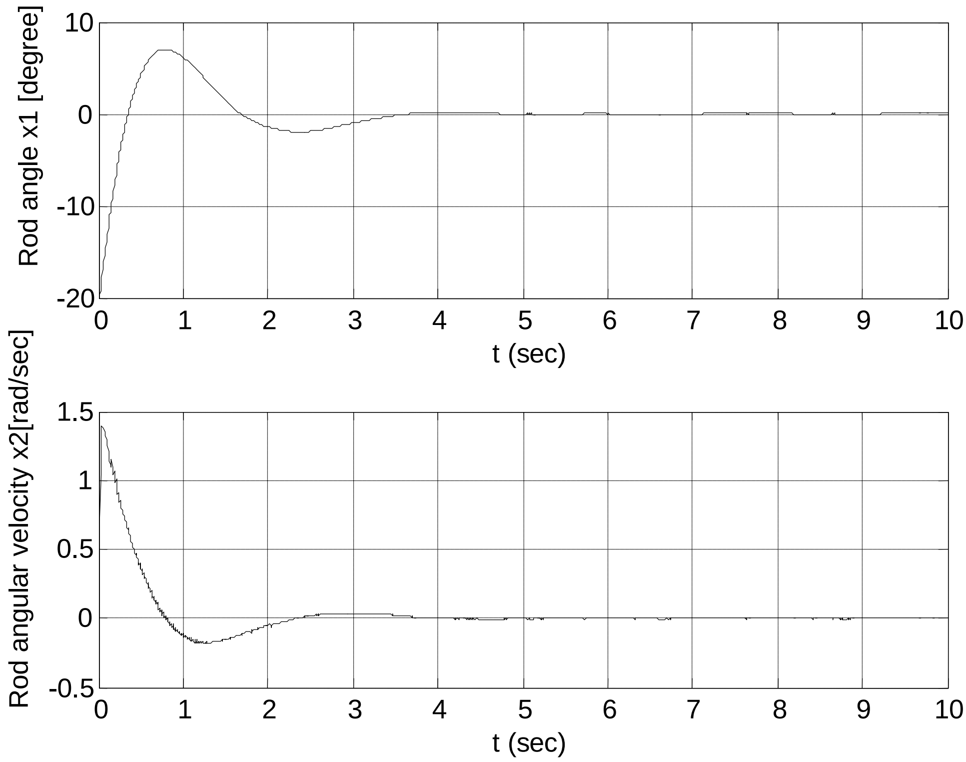

Figure 3.10: Rod angle and rod angular velocity time response for LMI.

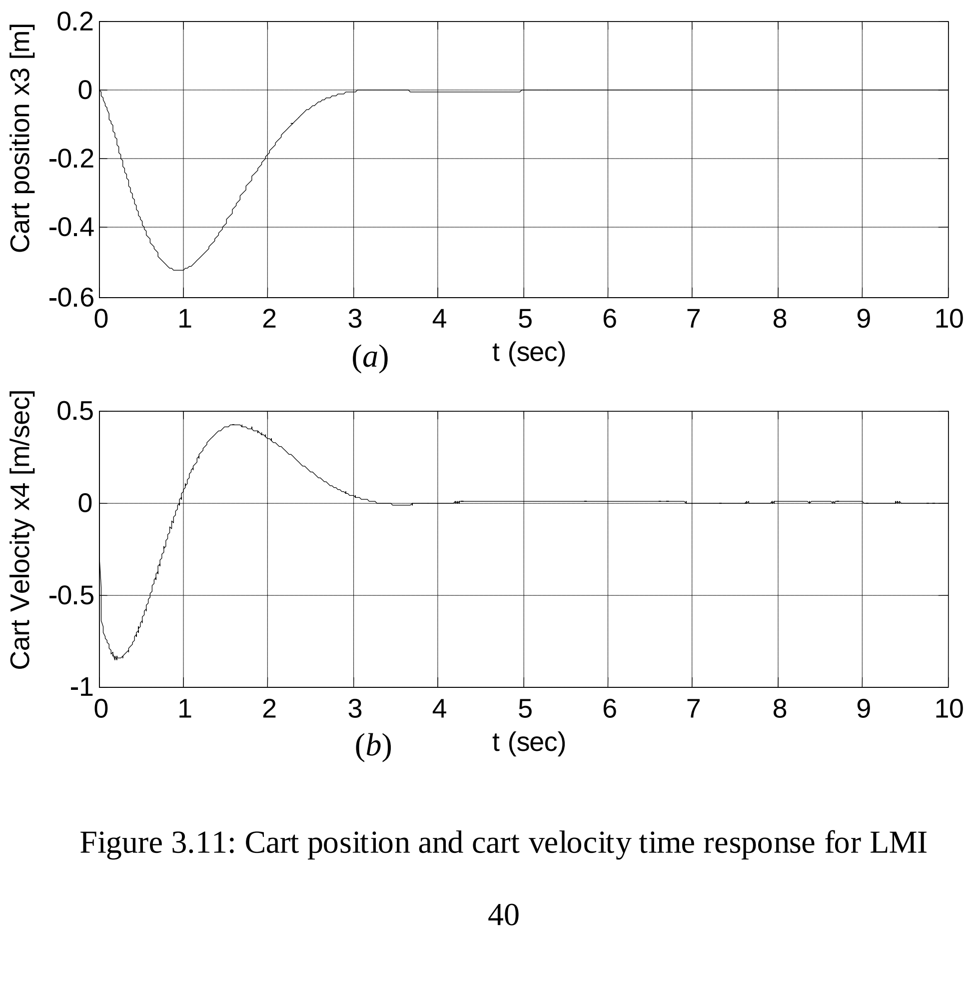

Figure 3.11: Cart position and cart velocity time response for LMI

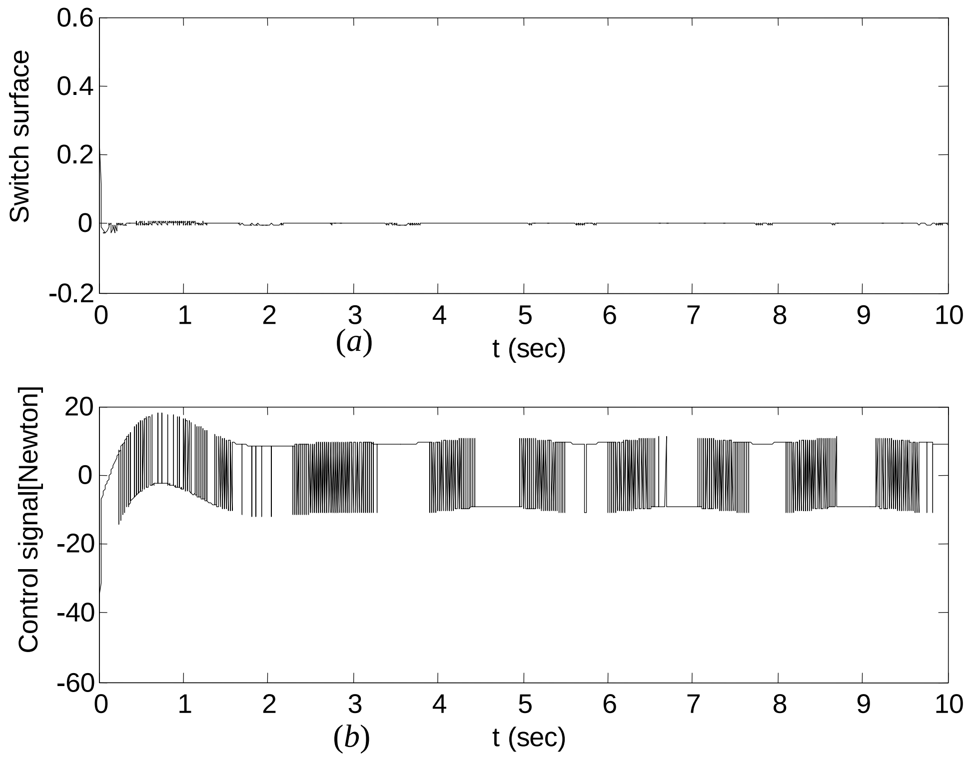

Figure 3.12: Switching surface and control signal for LMI

3.5 Sliding Mode Controller design for nonlinear systems

The most commonly nonlinear systems are given in the following form:

$$\dot{x} = f(x,t) + b(x,t)u \tag{3.31}$$

where $x \in R^n$ and $u \in R^r$.

The transient dynamics of SMC control consists of two modes, a reaching mode (non sliding mode) and sliding mode [21]. Thus the steps to design of SMC are, designing suitable switching function $s(x)$ where its dimension is r, and then design the control law for the reaching mode such that satisfy the reaching condition. The desired response for the reaching mode is to reach the switching surface (switching manifold), that described by:

$$s(x)=Cx=0 \tag{3.32}$$

Before designing a controller of nonlinear systems some point need to be made [17]:

1) The fundamental theory and basic concepts to design a controller based on SMC in nonlinear case are similar to linear case.

2) Designing the control law $u(x)$ is simple.

The switching parameters are obtained based on trial and error [21] to selecting the suitable switch, here a coupled sliding mode is considered.

The system (3.31) can be represents in another form as follow

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f_1(x) + b_1(x)u$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = f_2(x) + b_2(x)u$$

where $f_1(x)$ and $f_2(x)$ are nominal function and $b_1(x)$ and $b_1(x)$ are the nonlinear functions of the state variables [26].

In sliding mode control the controller is divided by two parts, equivalent control law and switching control law.

3.5.1 Designing the Sliding Surface (switches)

The inverted pendulum system can be divided into two subsystems with state variables (x_1, x_2) and (x_3 ,x_4) for each group a linear function is designed as a switches:

$$s_1(x_1, x_2) = c_1x_1 + x_2$$
$$s_2(x_3, x_4) = c_2x_3 + x_4$$

where c_1 and c_2 are real positive design parameters. But for the whole system it has one switching surface;

$$s(x) = \alpha_1 s_1 + \alpha_2 s_2$$
$$s(x) = \alpha_1 c_1 x_1 + \alpha_1 x_2 + \alpha_2 c_2 x_3 + \alpha_2 x_4 \tag{3.33}$$

where α_1 *and* α_2 are the sliding controller design parameters. The above approach is adopted from [27]

3.5.2 Designing the Sliding mode controller

In sliding mode by using the reaching law, the system involves two stages. The reaching stage drives the system toward the sliding surface, and sliding stage constrains the system on the sliding manifold and leads it to the origin (or equilibrium point).

Let select the exponential reaching law.

$$\dot{s} = -\varphi s(x) - \eta sign(s), \qquad \varphi > 0,\ \eta > 0 \tag{3.34}$$

According to Equation (3.33) we can drive the switching control law (u_{sw}) as follow:

$$\dot{s}(x) = \alpha_1 c_1 \dot{x}_1 + \alpha_1 \dot{x}_2 + \alpha_2 c_2 \dot{x}_3 + \alpha_2 \dot{x}_4$$

$\dot{s}(x) = \alpha_1 c_1 x_2 + \alpha_1(f_1(x) + b_1(x)u) + \alpha_2 c_2 x_4 + \alpha_2(f_2(x) + b_2(x)u)$

$\dot{s}(x) = \alpha_1 c_1 x_2 + \alpha_1 f_1(x) + \alpha_1 b_1(x)u + \alpha_2 c_2 x_4 + \alpha_2 f_2(x) + \alpha_2 b_2(x)u$

$\dot{s}(x) = \alpha_1 c_1 x_2 + \alpha_1 f_1(x) + +\alpha_2 c_2 x_4 + \alpha_2 f_2(x) + (\alpha_1 b_1(x) + \alpha_2 b_2(x))u$

By substituting $\dot{s}(x) = -\varphi s(x) - \eta sign(s(x))$

$-\varphi s(x) - \eta sign(s(x)) = \alpha_1 c_1 x_2 + \alpha_1 f_1(x) + \alpha_2 c_2 x_4 + \alpha_2 f_2(x) + (\alpha_1 b_1(x) + \alpha_2 b_2(x))u$

$$u(x) = \frac{-\varphi s(x) - \eta sign(s(x)) - \alpha_1 c_1 x_2 - \alpha_1 f_1(x) - \alpha_2 c_2 x_4 - \alpha_2 f_2(x)}{\alpha_1 b_1(x) + \alpha_2 b_2(x)} \tag{3.35}$$

To determine the sliding mode controller parameters the candidate Lyapunov function $V(s(x))$ is ($V(s) = \frac{1}{2}s^2$). The time derivative of $V(s)$ need to be negative definite.

$\dot{V}(s) = s(x)\dot{s}(x) < 0$

Substituting the exponential reaching law of (3.34) leads to.

$\dot{V}(s) = s(x)(-\varphi s(x) - \eta sign(s(x)))$

$\dot{V}(s) = -\varphi(s)^2 - \eta|s(x)|$ (*sign(s)s*=|s| is the one of *sign* function's property)

so to satisfy the Lyapunove stability condition($\dot{V}(s) < 0$) yields to $\varphi > 0$ and $\eta > 0$ The other parameters (c_1, c_2, α_1 and α_2) can be determined based on trial and error [28].

3.6 Sliding mode controller for nonlinear inverted pendulum

The numerical nonlinear system for inverted pendulum is:

$\dot{x}_1 = x_2$
$\dot{x}_2 = f_1(x_1, x_2) + b_1(x_1)u$
$\dot{x}_3 = x_4$
$\dot{x}_4 = f_2(x_1, x_2) + b_2(x_1)u$

where $f_1(x_1, x_2) = \frac{20.58 \sin x_1 - 0.025 \sin x_1 \cos x_1 \dot{x_1}^2}{0.7 - 0.025 \cos^2 x_1}$, $b_1(x_1) = \frac{\cos x_1}{0.7 - 0.025 \cos^2 x_1}$,

$f_2(x_1, x_2) = \frac{0.1 \sin x_1 x_2^2 - 2.94 \sin x_1 \cos x_1}{8.4 - 0.3 \cos^2 x_1}$ and $b_2(x_1) = \frac{4}{8.4 - 0.3 \cos^2 x_1}$

dividing system into two sub systems with state variables (x_1, x_2) and (x_3 ,x_4) for each group linear functions are derived as a switches:

$s_1(x_1, x_2) = c_1 x_1 + x_2$

$s_2(x_3, x_4) = c_2 x_3 + x_4$

The coupled switch for whole system is:

$$s(x) = \alpha_1 c_1 x_1 + \alpha_1 x_2 + \alpha_2 c_2 x_3 + \alpha_2 x_4$$

For c_1=5, c_2=0.5, α_1=-1 and α_2=-1

$$s(x) = -5x_1 - x_2 - 0.5x_3 - x_4$$

The controller law designed in Equation (3.35) is:

$$u(x) = \frac{-\varphi s(x) - \eta sign(s) - \alpha_1 c_1 x_2 - \alpha_1 f_1(x_1, x_2) - \alpha_2 c_2 x_4 - \alpha_2 f_2(x_1, x_2)}{\alpha_1 b_1(x_1) + \alpha_2 b_2(x_1)} \qquad (3.35a)$$

Let φ=1, and η=5 then.

$$u(x) = \frac{-s(x) - 5sign(s) + 5x_2 + f_1(x_1, x_2) + 0.5x_4 + f_2(x_1, x_2)}{-b_1(x_1) - b_2(x_1)}$$

Simulation results for the closed loop system are depicted in Figures 3.13, 3.14 and 3.15 where the system is stabilized based on nonlinear controller, with the system initial conditions are [$x_1(0)$, $x_2(0)$, $x_3(0)$, $x_4(0)$] = [-3.5,0,0,0]. From the results can be seen that the sliding mode controller achieves the desired performance for nonlinear system and the controller is stabilized the system.

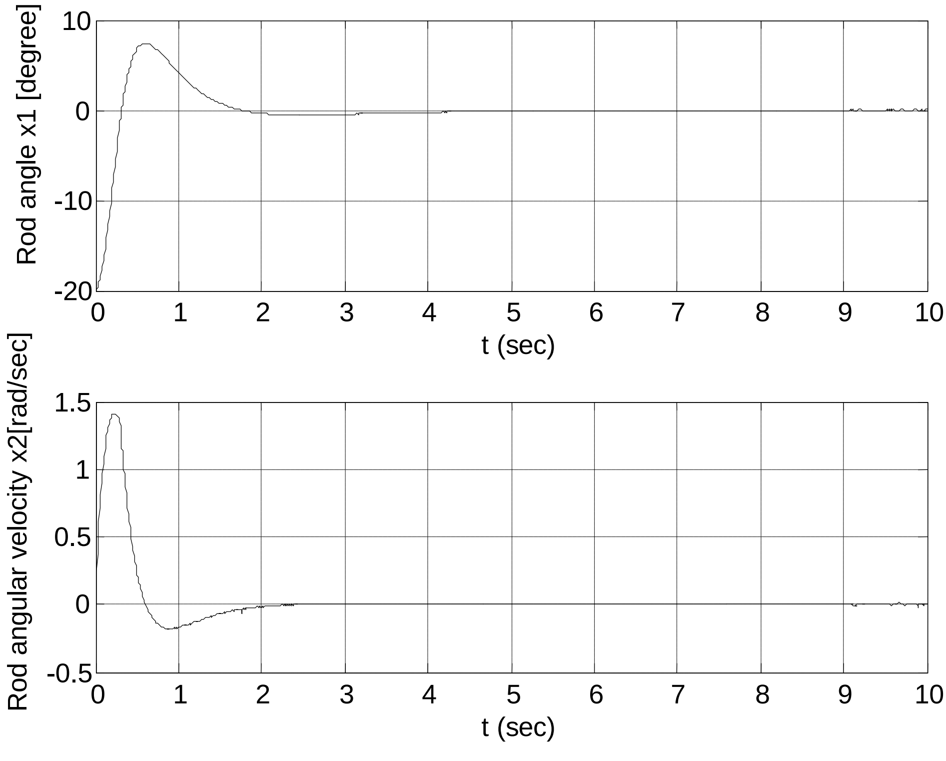

Figure 3.13: Rod angle and rod angular velocity time response for coupled SMC.

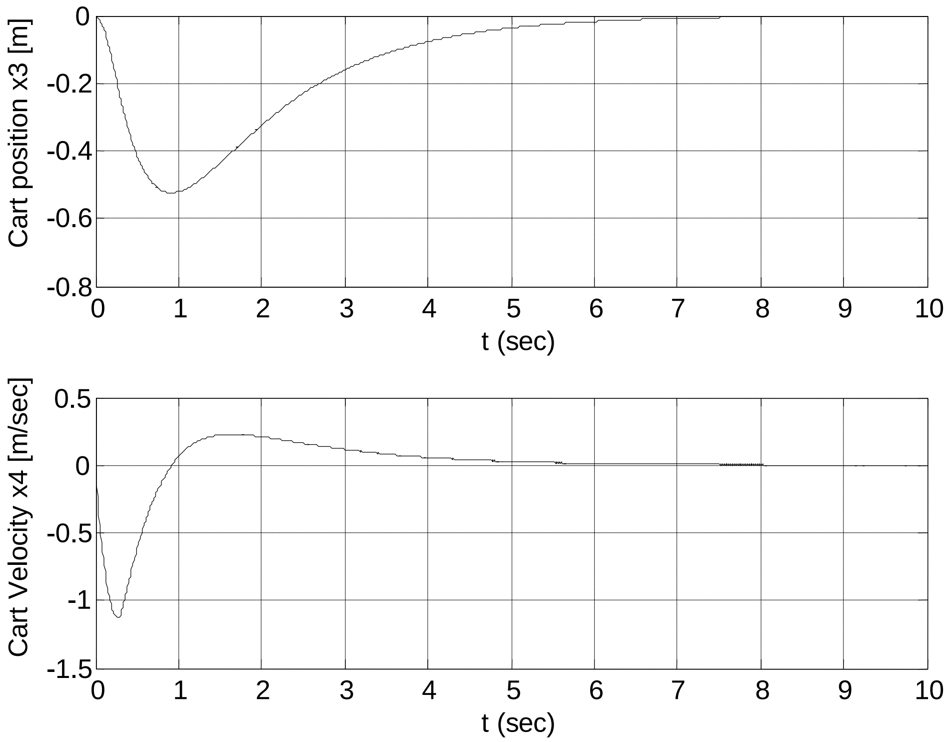

Figure 3.14: Cart position and cart velocity time response for coupled SMC.

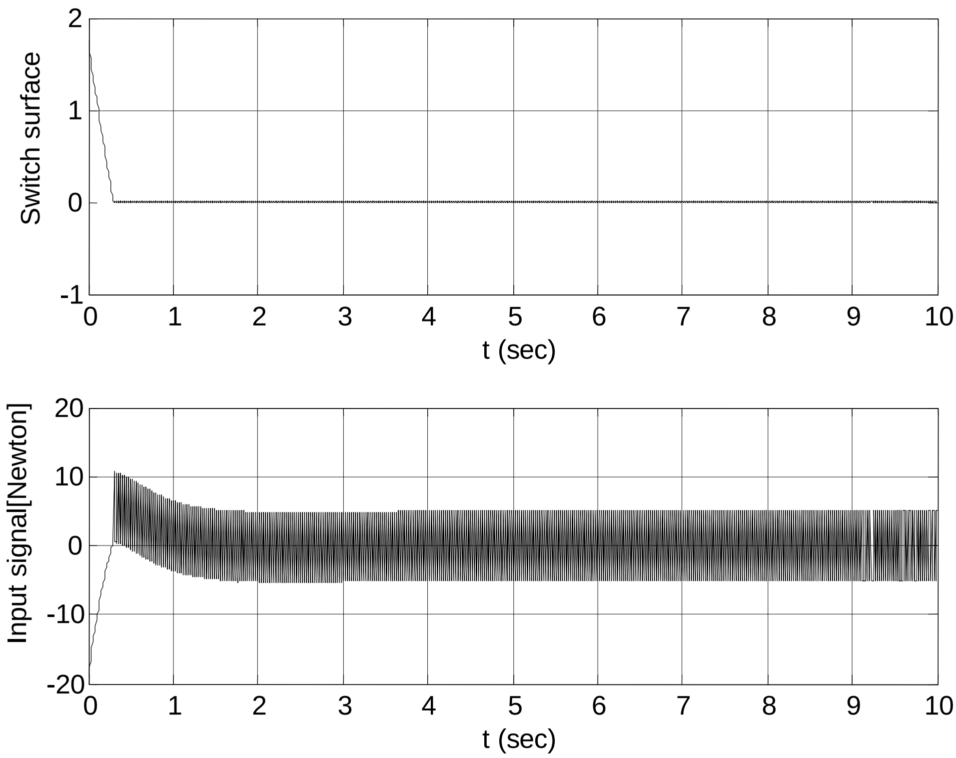

Figure 3.15: Switching surface and control signal for coupled SMC.

CHAPTER 4

SLIDING MODE CONTROLLER FOR SWING-UP AND STABILIZING INVERTED PENDULUM

In this chapter the control problem considered for two stages. Firstly the swing-up stage, $u_{sw}(\theta, \dot{\theta}, z, \dot{z})$ region is swinging up the pendulum by a nonlinear controller, to bring the pendulum from rest position, to a limited region around of upright position with the predefined desired cart position, *z*. Secondly the stabilization stage, $u_{st}(\theta, \dot{\theta}, z, \dot{z})$ region, is stabilizing pendulum at the upright position with desired cart position by employing a sliding mode controller. The swing up is achieved by energy based control combined with sliding mode control, $u_{sw}(\theta, \dot{\theta}, z, \dot{z})$. For the stabilizing the pendulum at upright position and desired card position a coupled sliding mode controller, $u_{st}(\theta, \dot{\theta}, z, \dot{z})$ is used. The closed loop stability is achieved by Lyapunov method and the performance will be discussed.

In this chapter instead of state variables [x_1, x_2, x_3, x_4], the physical state variables$\left[\theta, \dot{\theta}, z, \dot{z}\right]$ will be considered for inverted pendulum system.

Two regions are depicted in Figure 4.1 for the movement of rod angle, (angular position).

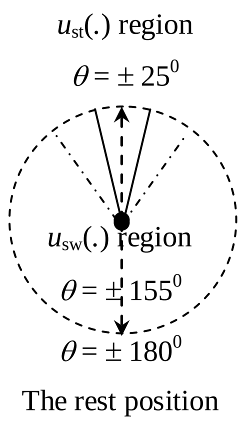

Figure 4.1: The controller parts active region: Switching process.

The controller u contains two parts;

$$u = \begin{cases} u_{sw}(\theta,\dot{\theta},z,\dot{z}) & if\ |\theta| > 25^0 \\ u_{st}(\theta,\dot{\theta},z,\dot{z}) & if\ |\theta| < 25^0 \end{cases} \tag{4.1}$$

where $u_{sw}(\theta,\dot{\theta},z,\dot{z})$ is swinging up controller which is designed to swing up the pendulum from the pendant position $\theta = \pm 180^0$and bring it to nearby upright position $\theta = 0^0$, with the desired cart position by acceleration of the cart. Once the rod entered in the region near the upright position$\theta \leq |25^0|$, the swing-up controller $u_{sw}(\theta,\dot{\theta},z,\dot{z})$ is released by a switching process and then the stabilization controller $u_{st}(\theta,\dot{\theta},z,\dot{z})$ became active to stabilize the rod at upright position $\theta = 0^0$ and with the desired cart position. The swing-up control is energy-based control combined with sliding mode control to bring the pendulum from pendant position, $\theta = \pm 180^0$ to enter in the limited region, $\theta \leq |25^0|$ away from the upright position with controlling the cart position simultaneously. The stabilizing controller for the pendulum at the upright position is designed based on coupled sliding mode controllers to stabilize pendulum and control the cart position at same time.

4.1 Swing-Up Inverted Pendulum

There are many control strategies introduced and used to swing-up of inverted pendulum. Here energy-based control combined with sliding mode control will be designed to swing up the pendulum as follows.

The nonlinear model of the inverted pendulum can be represented by the following equations:

$$ml\,\ddot{\theta}\cos\theta - ml\,\dot{\theta}^2\sin\theta + (M+m)\ddot{z} = \mathrm{u} \tag{4.2}$$

$$J\ddot{\theta} - mlg\sin\theta + ml\cos\theta\,\ddot{z} = 0 \tag{4.3}$$

where Equation (4.2) and (4.3) represent cart dynamic and pendulum dynamics respectively. Rewriting equations (4.2) and (4.3) as:

$$\ddot{\theta} = f_\theta(\theta,\dot{\theta}) + b_\theta(\theta)u \tag{4.4}$$

$$\ddot{z} = f_z(\theta,\dot{\theta}) + b_z(\theta)u \tag{4.5}$$

where $f_\theta(\theta,\dot{\theta}) = \frac{(M+m)mlg\ \sin\theta - m^2l^2\sin\theta\cos\theta\,\dot{\theta}^2}{J(M+m) - m^2l^2\cos^2\theta}$, $b_\theta(\theta) = -\frac{ml\cos\theta}{J(M+m) - m^2l^2\cos^2\theta}$, $f_z(\theta,\dot{\theta}) = \frac{Jml\ \sin\theta\,\dot{\theta}^2 - m^2l^2 g\ \sin\theta\cos\theta}{J(M+m) - m^2l^2\cos^2\theta}$ and $b_z(\theta) = \frac{J}{J(M+m) - m^2l^2\cos^2\theta}$

The swing up controller u_{sw} is consisting of two sub different controllers that can be combined together:

$u_{sw} = u_p(\theta) + u_c(z)$ (4.6)

where $u_p(\theta)$ is controller to swing up the pendulum, and $u_c(z)$ is the controller to control the cart position during the pendulum is swinging up.

4.1.1 Pendulum Swing-up controller $u_p(\theta)$.

Consider the energy equation for the pendulum subsystem is given by

The pendulum subsystem's energy equation is given by

$$E(\theta,\dot{\theta}) = \frac{1}{2}J\dot{\theta}^2 + mgl(\cos\theta - 1) \quad (4.7)$$

The time derivative of E is

$$\dot{E}(\theta,\dot{\theta}) = (J\ddot{\theta} - mlg\sin\theta)\dot{\theta} \quad (4.8)$$

Substituting Equation(4.3) in to (4.8) yields to

$$\dot{E}(\theta,\dot{\theta}) = -ml\dot{\theta}\ddot{z}\cos\theta \quad (4.9)$$

Let us consider the energy function as the candidate Lyapunov function as follows [29];

$$V(E) = \frac{1}{2}(E_0 - E)^2 \quad (4.10)$$

where E_o is desired energy and E pendulum energy.

Taking the derivative of Lyapunov function (4.10) along the system trajectory yields to

$$\dot{V}(E) = -(E_o - E)\dot{E}$$

substituting (4.9) in $\dot{V}$ yields to

$$\dot{V}(E) = ml\dot{\theta}\cos\theta\,(E_0 - E)\ddot{z} \quad (4.11)$$

substituting (4.5) in (4.11) lead to

$$\dot{V}(\theta,\dot{\theta}) = \sigma(f_z(\theta) + b_z(\theta)u) \quad (4.12)$$

where $\sigma = ml\dot{\theta}\cos\theta\,(E_0 - E)$

The nonlinear controller $u_p(E)$ is

$$u_p(E) = -\delta sgn(\sigma) \quad \text{where } \delta > 0 \quad (4.13)$$

The condition to make Equation (4.12) become semi-negative definite is:

$$\dot{V}(\theta,\dot{\theta}) = \sigma(f_z(\theta) + b_z(\theta)u_p)$$

$$\dot{V}(\theta,\dot{\theta}) = \sigma(f_z(\theta) + b_z(\theta)(-\delta sgn(\sigma)))$$

$$\dot{V}(\theta,\dot{\theta}) \le |\sigma||(f_z(\theta) + b_z(\theta)\big(-\delta sgn(\sigma)\big))|$$

$$\dot{V}(\theta,\dot{\theta}) \le |\sigma|(|f_z(\theta)| - |b_z(\theta)|\delta|sgn(\sigma)|)$$

$\dot{V}(\theta,\dot{\theta}) \leq |\sigma|(|f_z(\theta)| - |b_z(\theta)|\delta)$

where $|sgn(\sigma)|$=1, and right hand side should be smaller or equal to zero

$0 \geq |f_z(\theta)| - |b_z(\theta)|\delta$

$\delta \geq \frac{|f_z(\theta)|}{|b_z(\theta)|}$

where $|b_z(\theta)| \neq 0$.

4.1.2 Cart position controller during swinging up $u_c(z)$.

The cart position controller during swinging up is achieved by a sliding mode control $u_c(z)$ [18].

Let us define the switching surface as:

$$s_{cs} = c_{sc}e + \dot{e} \tag{4.14}$$

where c_{sc} is constant and $e= z_{rsw}$ -z where z_{rsw} is the cart desired position for pendulum during the swing up region.

The time derivative for switching surface is:

$$\dot{s}(e) = c_{sc}\dot{e} + \ddot{e}$$

$$\dot{s}(e) = c_{sc}\dot{e} + \ddot{z}_{rsw} - \ddot{z} \tag{4.15}$$

The controller u_c can be obtained by considering the candidate Lyapunov function, by select the Lyapunov function as follows [11]

$$V(s) = \frac{1}{2}s_{sc}^2 \tag{4.16}$$

Taking the derivative of Lyapunov Equation (4.16) yields to

$$\dot{V}(s) = s_{sc}\dot{s}_{sc} \tag{4.17}$$

For the reaching law

$$\dot{s}_{sc} = -\gamma s - \beta sgn(s) \tag{4.18}$$

where $\beta>0$ and $\gamma>0$ and substituting Equation(4.18) in Equation(4.15) results to.

$$-\gamma s_{sc} - \beta sgn(s_{sc}) = c_{sc}\dot{e} + \ddot{z}_{rsw} - \ddot{z} \tag{4.19}$$

then substituting Equation (4.5) in Equation (4.19) yields to the following control law;

$$-\gamma s_{sc} - \beta sgn(s_{sc}) = c_{sc}\dot{e} + \ddot{z}_{rsw} - f_z(\theta) - b_z(\theta)u_{sc}$$

$$u_{sc} = \frac{\gamma s_{sc} + \beta sgn\,(s_{sc}) c_{sc}\dot{e} + \ddot{z}_{rsw} - f_z(\theta)}{b_z(\theta)} \tag{4.20}$$

By combining the control laws given in Equations (4.13) and Equation (4.20) the swing-up controller is in the form of:

$$u_{sw}(\theta,\dot{\theta},z,\dot{z}) = -\delta sgn(\sigma) + \frac{\gamma s_{sc} + \beta sgn\,(s_{sc})c_{sc}\dot{e} + \ddot{z}_{rsw} - f_z(\theta)}{b_z(\theta)} \tag{4.21}$$

To obtain the reachability condition and study the closed loop stability let recall Equation (4.12)

$$\dot{V}(\theta,\dot{\theta}) = \sigma(f_z(\theta) + b_z(\theta)u_{sw})$$

substituting Equation (4.21) in to Equation(4.12)

$$\dot{V}(\theta,\dot{\theta}) = \sigma(f_z(\theta) + b_z(\theta)(-\delta sgn(\sigma) + \frac{\gamma s_{sc} + \beta sgn\,(s_{sc})c_{sc}\dot{e} + \ddot{z}_{rsw} - f_z(\theta)}{b_z(\theta)})) \tag{4.22}$$

$$\dot{V}(\theta,\dot{\theta}) = \sigma(f_z(\theta) - b_z(\theta)\delta sgn(\sigma) + \gamma s_{sc} + \beta sgn(s_{sc})c_{sc}\dot{e} + \ddot{z}_{rsw} - f_z(\theta))$$

Taking the norm of both side leads to.

$$\dot{V}(\theta,\dot{\theta}) \leq |\sigma||(-b_z(\theta)\delta sgn(\sigma) + \gamma s_{sc} + \beta sgn(s_{sc})c_{sc}\dot{e} + \ddot{z}_{rsw})|$$

$$\dot{V}(\theta,\dot{\theta}) \leq |\sigma|(-|b_z(\theta)|\delta|sgn(\sigma)| + \gamma|s_{sc}| + \beta|sgn(s_{sc})||c_{sc}\dot{e}| + |\ddot{z}_{rsw}|)$$

$$\dot{V}(\theta,\dot{\theta}) \leq |\sigma|(-|b_z(\theta)|\delta + \gamma|s_{sc}| + \beta|c_{sc}\dot{e}| + |\ddot{z}_{rsw}|)$$

where (|$sgn(s)$|=1.

To make Equation (4.12) to be negative semi definite, the right hand side in Equation (4.22) should be equal and smaller than zero:

$0 \geq -|b_z(\theta)|\delta + \gamma|s_{sc}| + \beta|c_{sc}\dot{e}| + |\ddot{z}_{rsw}|$ and thus,

$|b_z(\theta)|\delta \geq \gamma|s_{sc}| + \beta|c_{sc}\dot{e}| + |\ddot{z}_{rsw}|$

Finally the reachability condition is

$$\delta \geq \frac{\gamma|s_{sc}| + \beta|c_{sc}\dot{e}| + |\ddot{z}_{rsw}|}{|b_z(\theta)|}$$

The simulation results for swinging up controller $u_{sw}(\theta,\dot{\theta},z,\dot{z})$ that represented by Equation (4.21) are depicted by Figures 4.2, 4.3 and 4.4 without stabilizing controller.

Figure 4.2 (*a*) shows the pendulum starts swinging by swing up controller from θ=-180 to upright position where the swing-up parameters are: E_o=0, δ=3.5, γ=1 β=1 , c_{sc}=2 and z_{rsw}=1, and the system initial conditions are $[\theta(0),\dot{\theta}(0),z(0),\dot{z}(0)] = [-\pi,0,0,0]$. It can be noted that the swing-up controller is continues to swinging up the pendulum, because it has not any stabilizing controller to catch the pendulum at upright position. Figure 4.2 (*b*) shows the cart position time response during the pendulum swinging up. The results given in Figure 4.2 indicates that the swing-up controller push the cart from its initial position $z(0)$=0 to its desired position z_{rsw}=1 within the restricted cart deviation.

In this case the swing up parameters are: E_o=0, δ=4.5, γ=1, β=1 , c_{sc}=2 and z_{rsw}=1.

Figure 4.3 shows where the value of δ is increased (in rechability condition range) the swing up time speeds up. Note that the swing-up controller is restricted with the cart travel.

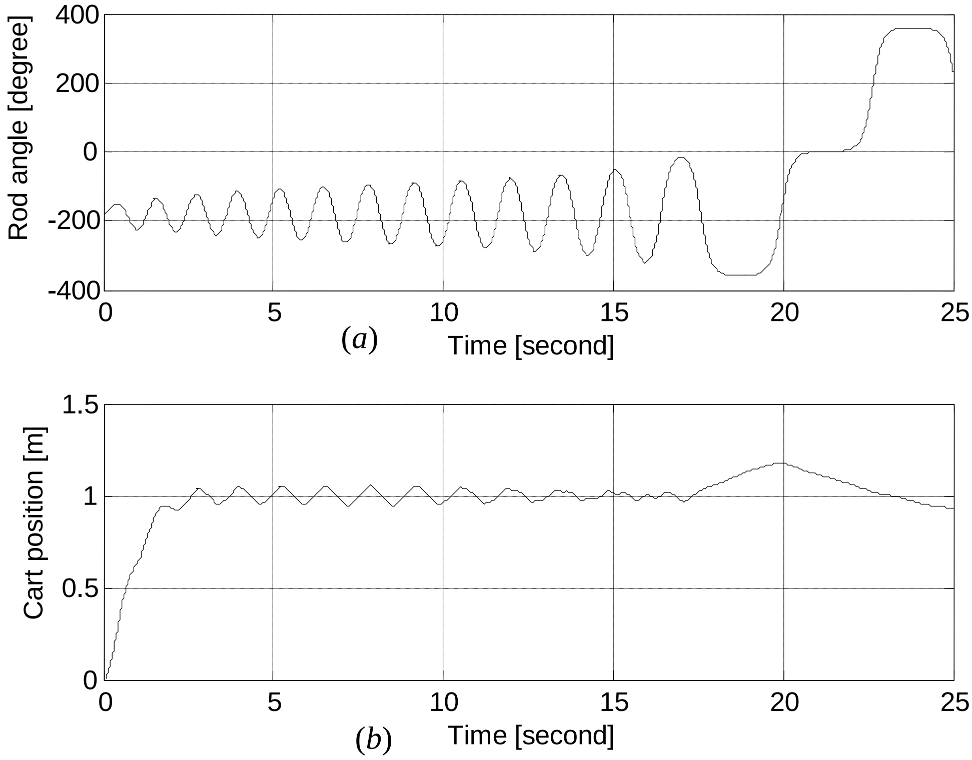

Figure 4.2: Swinging up system time response

Figure 4.4 illustrates the system time response when the selected swinging up controller's parameters are (E_o=0, δ=3.2, γ=1, β=1 , c_{sc}=2 and z_{rsw}=0_1_0), and the system initial conditions are $[\theta(0), \dot{\theta}(0), z(0), \dot{z}(0)] = [-\pi, 0, -0.5, 0]$. In this case the cart desired position three times was changed as shown by Figure 4.4(b) where dotted line represents the desired cart position and the solid line represents the actual cart position. Figure 4.4 shows that the controller is achieved the desired cart position and the closed loop performance.

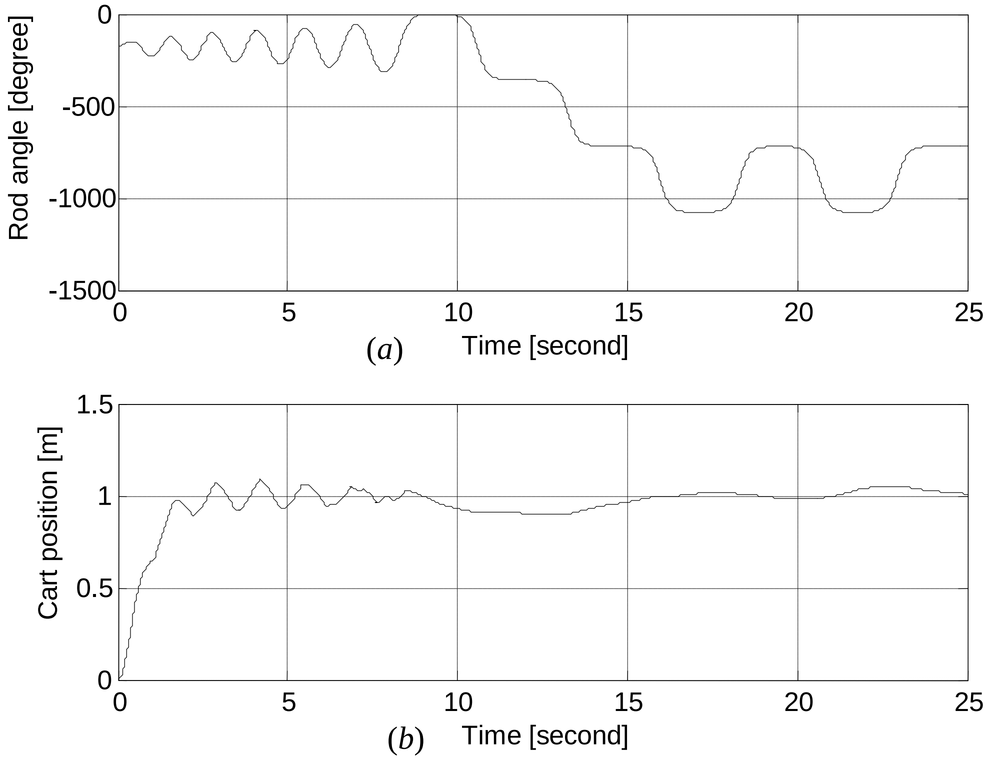

Figure 4.3: Swing up pendulum time response Case 2.

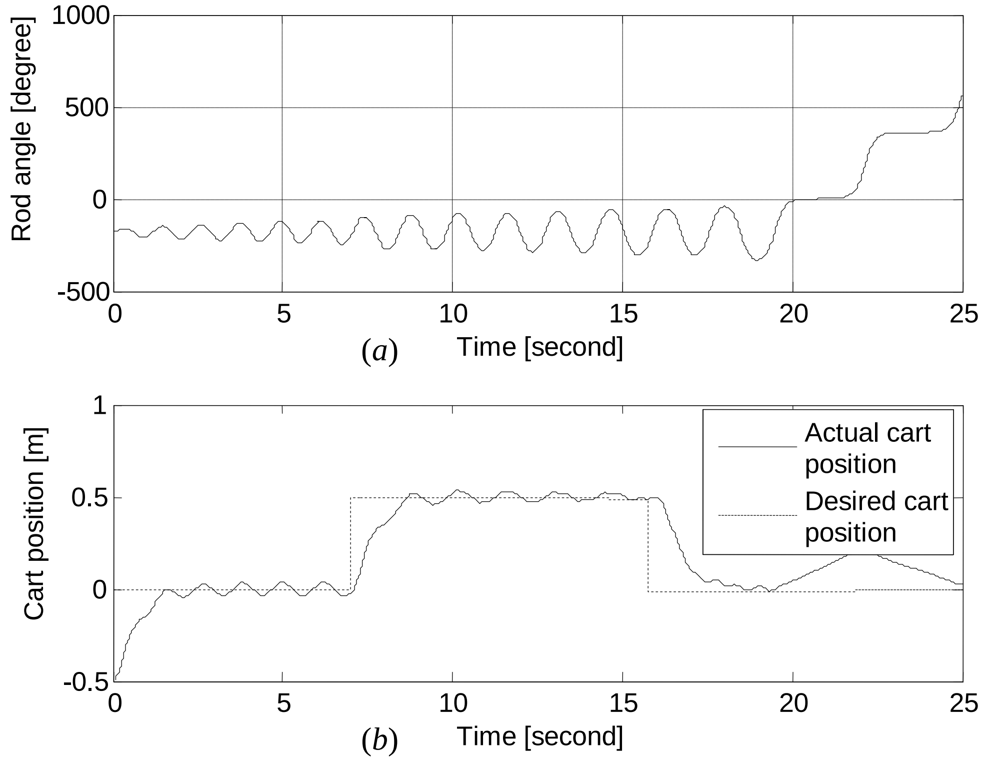

Figure 4.4: Swing up pendulum time response for Case 3.

4.2 Stabilizing the nonlinear pendulum system and controlling the cart position with the sliding mode control.

Coupled sliding mode controller is used to stabilize the pendulum at upright position where coupled switching surface is defined as[27]:

$$s_c(s_\theta(\theta,\dot{\theta}),\ s_z(z,\dot{z}))=\varepsilon_1 s_\theta+\varepsilon_2 s_z \tag{4.23}$$

where s_c is coupled switching surface for hole system, ε_1 and ε_2 are design parameter, $s_\theta(\theta,\dot{\theta})$ switching surface for pendulum subsystem and $s_z(z,\dot{z})$ switching surface for cart subsystem. For the simplicity the function will be termed without presents such $s_c(s_\theta, s_z)= s_c$.

For the pendulum subsystem the switching surface and its derivative are:

$$s_\theta = c_1 e_\theta + \dot{e}_\theta \tag{4.24}$$

and

$$\dot{s}_\theta = c_1 \dot{e}_\theta + \ddot{e}_\theta \tag{4.25}$$

where c_1 is positive constant and $e_\theta=\theta_d-\theta$ where θ_d is the desired rod angle.

Let us define the cart subsystem switching surface as:

$$s_z = c_2 e_z + \dot{e}_z \tag{4.26}$$

$$\dot{s}_z = c_2 \dot{e}_z + \ddot{e}_z \tag{4.27}$$

where c_2 is positive constant and $e_z= z_d-z$ where z_d is desired cart position.

The time derivative for coupled switching surface is:

$$\dot{s}_c = \varepsilon_1 \dot{s}_\theta + \varepsilon_2 \dot{s}_z \tag{4.28}$$

and substituting Equation(4.25), and Equation(4.27) in to Equation(4.28)

$$\dot{s}_c = \varepsilon_1 c_1 \dot{e}_\theta + \varepsilon_1 \ddot{e}_\theta + \varepsilon_2 c_2 \dot{e}_z + \varepsilon_2 \ddot{e}_z \tag{4.28b}$$

For the exponential reaching law of:

$$\dot{s}_c = -\varphi s_c - \eta sign(s_c) \qquad \text{where } \varphi > 0 \text{ and } \eta > 0 \tag{4.29}$$

then substituting Equation(4.29) in to Equation (4.28b) leads to

$$-\varphi s_c - \eta sign(s_c) = \varepsilon_1 c_1 \dot{e}_\theta + \varepsilon_1 \ddot{e}_\theta + \varepsilon_2 c_2 \dot{e}_z + \varepsilon_2 \ddot{e}_z$$

$$-\varphi s_c - \eta sign(s_c) = \varepsilon_1 c_1 \dot{e}_\theta + \varepsilon_1 (\ddot{\theta}_\mathrm{d} - \ddot{\theta}) + \varepsilon_2 c_2 \dot{e}_z + \varepsilon_2 (\ddot{z}_d - \ddot{z}) \tag{4.30}$$

and finally substituting Equation(4.4) and Equation(4.5) in to Equation (4.30)

$$-\varphi s_c - \eta sign(s_c) = \varepsilon_1 c_1 \dot{e}_\theta + \varepsilon_1 \ddot{\theta}_\mathrm{d} - \varepsilon_1 (f_\theta(\theta) + b_\theta(\theta)u) + \varepsilon_2 c_2 \dot{e}_z + \varepsilon_2 \ddot{z}_d - \varepsilon_2 (f_z(\theta) + b_z(\theta)u)$$

$$-\varphi s_c - \eta sign(s_c) = \varepsilon_1 c_1 \dot{e}_\theta - \varepsilon_1 f_\theta(\theta) - \varepsilon_1 b_\theta(\theta)u_c + \varepsilon_1 \ddot{\theta}_\mathrm{d} + \varepsilon_2 c_2 \dot{e}_z - \varepsilon_2 f_z(\theta) - \varepsilon_2 b_z(\theta)u_c + \varepsilon_2 \ddot{z}_d$$

$$\varepsilon_1 b_\theta(\theta)u_c + \varepsilon_2 b_z(\theta)u_c = \varphi s_c + \eta sign(s_c) + \varepsilon_1 c_1 \dot{e}_\theta - \varepsilon_1 f_\theta(\theta) + \varepsilon_1 \ddot{\theta}_\mathrm{d} + \varepsilon_2 c_2 \dot{e}_z -$$

$\varepsilon_2 f_z(\theta) + \varepsilon_2 \ddot{z}_d$

results the following control law.

$$u_{st}(\theta,\dot{\theta},z,\dot{z}) = \frac{\varphi s_c+\eta sign\ (s_c)+\varepsilon_1 c_1 \dot{e}_\theta - \varepsilon_1 f_\theta(\theta,\dot{\theta})+\varepsilon_1 \ddot{\theta}_{\mathrm{d}}+\varepsilon_2 c_2 \dot{e}_z - \varepsilon_2 f_z(\theta\theta,\dot{\theta})+\varepsilon_2 \ddot{z}_d}{\varepsilon_1 b_\theta(\theta)+\varepsilon_2 b_z(\theta)} \quad (4.31)$$

By using this controller the closed performance is achieved. There are several cases for desired card position is studied and the results are discussed.

Case 1: For the system state initial conditions $[\theta(0),\dot{\theta}(0),z(0),\dot{z}(0)] = [-\pi,0,0,0]$ and the desired card position z_d=0.

The parameters of the swing up controller given in Equation(4.21) are; δ=5, E_o=0, γ=1, β=1, c_{sc}=2 and z_{rsw}=0 and the parameters of the stabilizing controller given in Equation(4.31) are; ε_1=ε_2=-1, c_1=5 c_2=2, φ=1, η=3, θ_d=0 and z_d=0.

The closed loop performances of the stabilized system based on sliding mode are depicted in Figures 4.5. Figure 4.5 (*a*) shows that the swing-up controller swings up the pendulum from its rest position θ = -180^o, and brings the pendulum to near upright position θ<|25^o|. Then the stabilizing controller become active by a switch, and it converges the pendulum angle to zero when $t\to\infty$. Note that the closed loop system exhibits the desired behavior. Figure 4.5 (*b*) shows the cart position displacement where it starts from origin and it deflected from its origin approximately 12 *cm* and then back again to the desired position and it deflected in the opposite direction. Because of the bang-bang property of swing up controller the cart continues to this action till the swing up controller moves the pendulum to be entered in the stabilizing region that limited by θ<|25^o| to be replaced with the stabilizing controller. Since the swing-up controller has limited the cart deflection and it needs to move to close enough to the desired position. Figure 4.5 (*b*) shows that after 9 *sec* the stabilizing controller become active, and the cart is deflected approximately 19 *cm* away from the desired card position z_d=0. Then the stabilizing controller achieves both stabilizing the pendulum and converge the cart to the desired position simultaneously.

Figure 4.6 illustrates the control signal that have a bang-bang property and its value is large during swing-up controller and converge to zero with small charting effect after the stabilizing controller is become active.

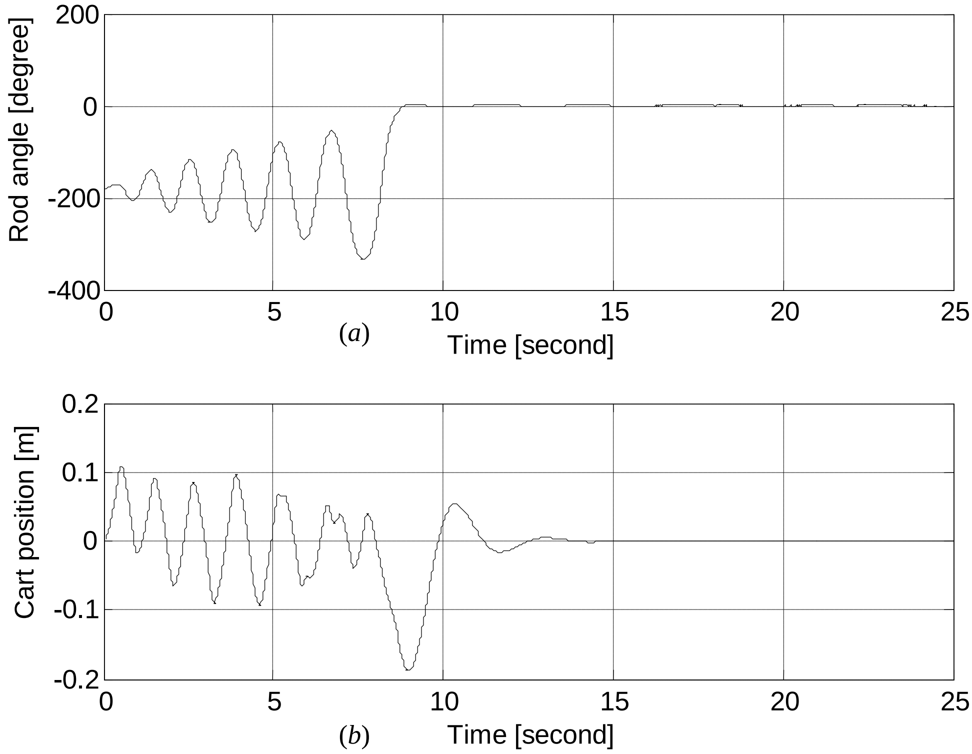

Figure 4.5: System time responses for nonlinear controller Case 1.

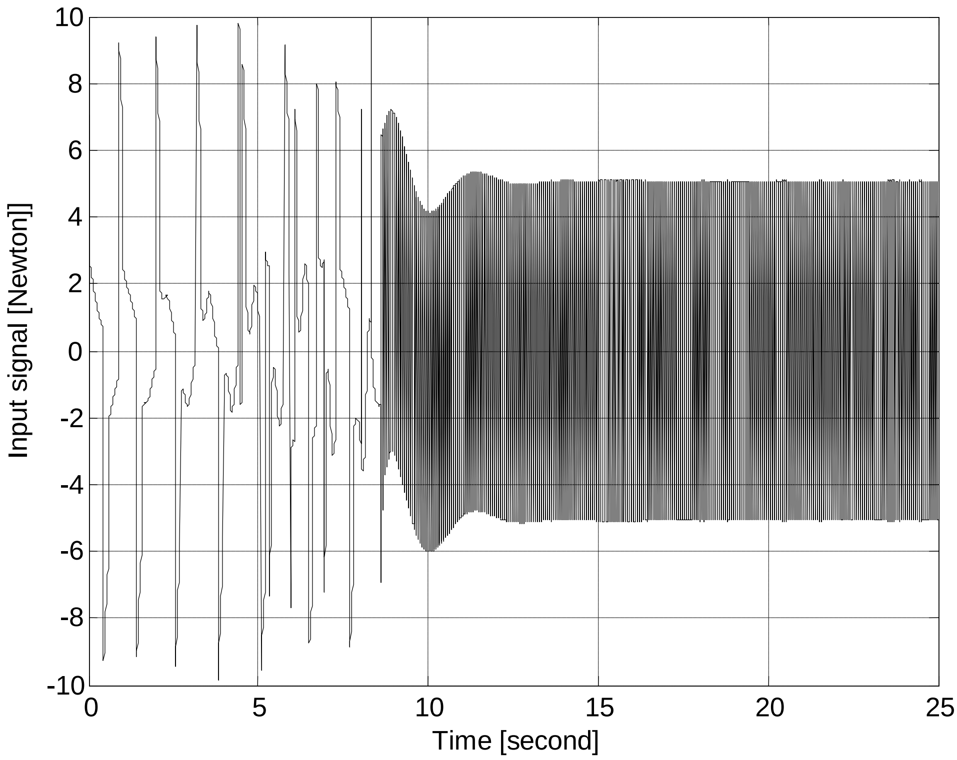

Figure 4.6: Control signal for closed loop system in Case 1.

Case 2: For the system state initial conditions $\left[\theta(0), \dot{\theta}(0), z(0), \dot{z}(0)\right] = [-\pi, 0,0,0]$ and the desired card position has a square wave function.

The parameters of the swing up controller given in (4.21) are; δ=4.5, E_o=0, γ=1, β=1.5, c_{sc}=2 and z_{rsw}=0.5-0 and the parameters of the stabilizing controller given in (4.31) are; ε_1= -1, ε_2= -2, c_1=5, c_2=2, φ=2, η=2, θ_d=(0-0.5-0)

The simulation results for case 2 are depicted in Figures 4.7 and 4.8. The pendulum time response illustrated in Figure 4.7 (*a*) and shows the swing-up controller swings up the pendulum from its initial condition and bring it to upright position after that the pendulum stabilized by the stabilizing controller. Figure 4.7 (*b*) illustrates the cart position time response where the desired cart position was changed, note that the swing-up controller and the stabilizing controller is achieved the desired cart position with limited cart deviation during swinging up the pendulum.

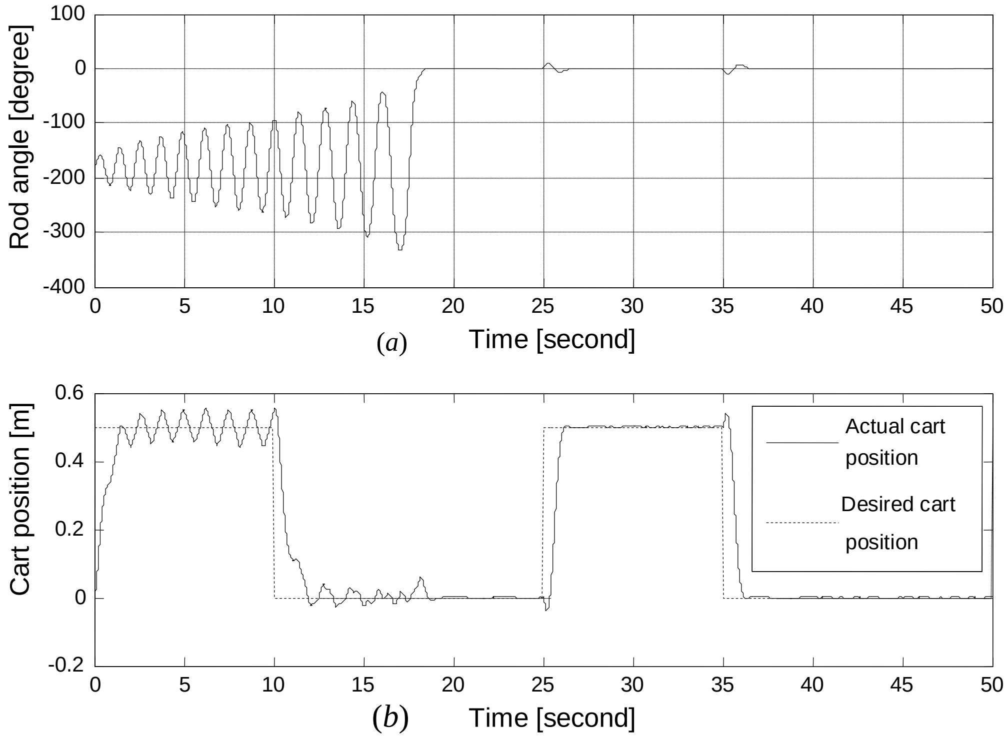

Figure 4.7: Closed loop system time response for nonlinear controller Case 2

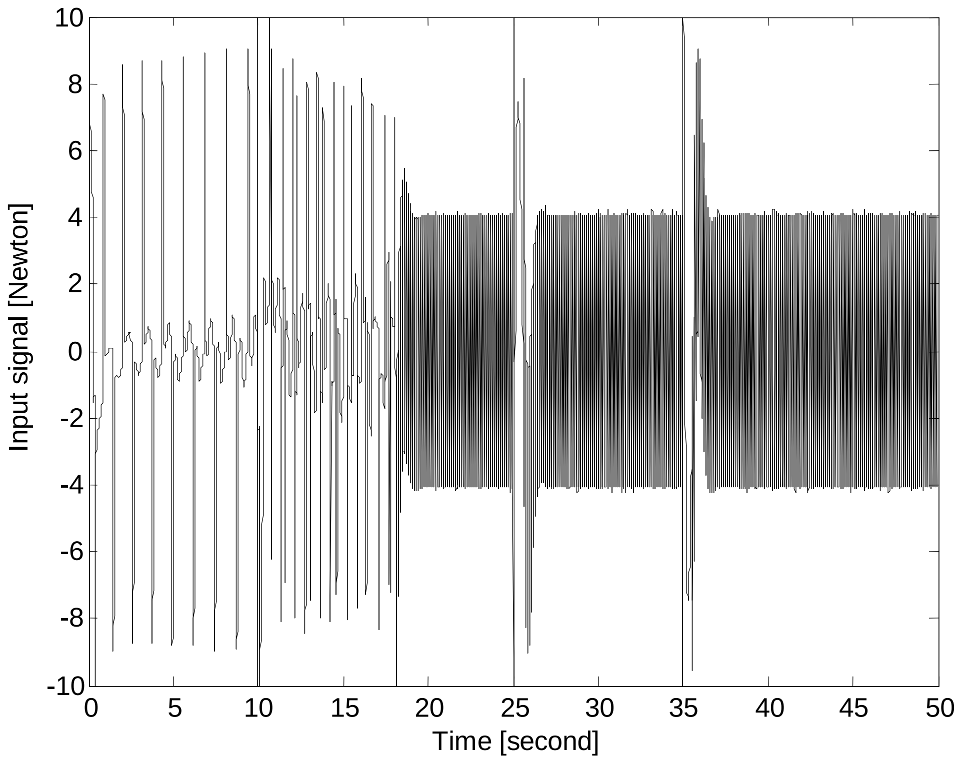

Figure 4.8: Control signal for closed loop system in Case 2

Figure 4.8 illustrates the input signal that has a bang-bang property and its value is large during swing-up controller and converges to zero with small chattering effect when the stabilizing controller become active. Note that with any changing in desired cart position the control signal was increased instantaneously, but its value was limited by a saturation level of ±10.

CHAPTER 5

COMPARISON OF RESULTS AND FUTURE WORK

5.1 Conclusion and comparison of results

Mathematical model of inverted pendulum system is obtained, nonlinear model is linearized and then three linear controllers have been applied. Linear and nonlinear models of the pendulum system are considered in this thesis.

A robust controller namely sliding mode control is rejected the disturbance, is designed and works properly when the system dynamic model has uncertainty.

From the simulation results obtained in case designing controller based on sliding mode it is noted that:

- The controller designed based on sliding mode exhibits the desired behaviors.
- Controller by sliding mode can reject the disturbance.
- The result obtained from controllers based on reaching law is more robust and automatically leads to the free-order switching scheme.
- Power rate reaching law can decrease the chattering effect.
- A new control approach is designed to swing-up from the pendent position $\theta=\pm180^o$ and stabilize the pendulum at $\theta=0^o$. Two controllers, swing up and stabilizing controller are designed. Swing-up controller is designed by combining the energy control with sliding mode controller to swing the pendulum from downward position it can enter within upright position $\theta<|25^o|$ with the restricted the cart deviation to be closed to the desired position, z_d. The swing–up's time can be selected by adjusting the controller's parameters. The controller to stabilize the pendulum at upright position and control the cart position is designed based on coupled sliding

mode controller. The stabilizing controller is not only stabilizing the pendulum at unstable equilibrium point, it also achieves the cart desired position z_d and rejects the bounded disturbance. Since stabilizing controller is designed by using Lyapunov method the closed loop stability of the system is achieved.

- Changing the desired cart position z_d is having an effect for the system stability because both controllers are a function to 2^{nd} derivative of desired cart position z_d.
- Several cases are discussed and the simulation results illustrate the effectiveness of the proposed method.

5.2. Future Work

a) Designing a one nonlinear controller for both problems (swing-up and stabilizing the inverted pendulum).

b) Designing nonlinear controller to swing-up and stabilizing the cart inverted pendulum on the slope rail, by using a sensor can measure the slope angle.

APPENDIX

Appendix A: Reduced-Order Sliding Mode

Consider a linear system (A.1) where we use the state feedback control $u=-Kx$,the desired

$$\dot{x} = Ax + bu_a \quad \text{(A.1)}$$

where $x \in R^n$ and $u \in R$

eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the (A.1) may be assigned using Ackerman formula (see pages 730-731 in [14]).

$$u_a(x) = -Kx \quad \text{(A.2)}$$

where $K=e^T P_1(A)$

where $e^T = [0\ 0\ 0\ 1][B\ \ AB\ \ A^2B \ldots\ldots\ A^nB\ ]^{-1}$ and

$$P_1(A)=(A-\lambda_1 I)(A-\lambda_2 I)\ldots(A-\lambda_{n-1} I)(A-\lambda_n I) = \alpha_1 I + \alpha_2 A + \ldots + \alpha_{n-1}A^{n-2} + \alpha_n A^{n-1} + .A^n \quad \text{(A.3)}$$

$$\neq 0$$

Here consider a controllable system described by:

$$\dot{x} = Ax + b[u + f(x,t)] \quad \text{(A4)}$$

Designing sliding mode control in system (A.4) implies a selection of a surface plain $s(x) = Cx = 0$ and then the control enforcing sliding mode in s-plain $s(x)=0$. Order of the sliding mode equation is (n-1), and does not depend on disturbance.

In this appendix the vector C will be found in an explicit form without the sliding motion equation by using Ackerman's formula, here the pole placement task is concerned.

Suppose $\lambda_1, \ldots\ \lambda_{n-1}$ are the desired eigenvalues of the sliding mode.

If $C = e^T P_1(A)$ (A5)

with

$P_1(A)=(A-\lambda_1 I)\,(A-\lambda_2 I)\,\ldots\ldots(A-\lambda_{n-1} I) = \alpha_1 I+\alpha_2 A+\ldots..+\alpha_{n-1}A^{n-2}+A^{n-1},$ (A.6)

then λ_1, ... λ_{n-1} are the eigenvalues of the sliding mode dynamics in the plane $s(x) = Cx = 0$.

Proof : According to Ackermann's formula (A.2), λ_1, ... λ_{n-1},λ_n are eigenvalues of the matrix, $\bar{A} = A - bK$ with λ_n is an arbitrary value.

Vector C is a left eigenvector of $\tilde{A}$ corresponding to λ_n . Indeed, as follows from (A.2) and (A.5):

$$C\bar{A} = CA - Cbe^T P(A)$$

Since

$$Cb= e^T P_1(A)b$$

$$= [0\ 0\ \ldots\ 1][b\ Ab\ \ldots.A^{n-1}]^{-1} P_1(A)b$$

$$=[0\ 0\ \ldots\ 1][b\ Ab\ \ldots.A^{n-1}]^{-1}(\alpha_1 I+\alpha_2 A+\ldots..+\alpha_{n-1}A^{n-2}+A^{n-1})b$$

$$=[0\ 0\ \ldots\ 1][b\ Ab\ \ldots.A^{n-1}]^{-1}(\alpha_1 Ib+\alpha_2 Ab+\ldots..+\alpha_{n-1}A^{n-2}b+A^{n-1}b)$$

$$=[0\ 0\ \ldots\ 1][b\ \ Ab\ \ldots.A^{n-1}]^{-1}\ [b\ \ Ab\ \ldots.A^{n-1}][\alpha_1\ \ \alpha_2\ \ldots..\ \alpha_{n-1}\ \ 1]^T$$

$$= 1$$

then

$$C\bar{A} = CA - e^T P(A)$$

by substituting (A.3) it become

$$C\tilde{A} = CA-e^T((A-\lambda_1 I)\,(A-\lambda_2 I)\ldots.\,(A-\lambda_{n-1} I)\,(A-\lambda_n I))$$

$$C\tilde{A} = CA-e^T P_1(A)(A-\lambda_n I) \tag{A.7}$$

where $P_1(A)=(A-\lambda_1 I)\,(A-\lambda_2 I)\ \ldots.(A-\lambda_{n-1} I)$

substituting (A.5) in (A.7):

$$C\bar{A} = CA - C(A - \lambda_n I)$$

$$C\bar{A} = C\lambda_n I \tag{A.8}$$

$$C(A-bK)= C\lambda_n I$$

$$CA- CbK= C\lambda_n I$$

$C(A-\lambda_n I)= CbK$

$C= CbK(A-\lambda_n I)^{-1}$

Note that Cb=1 as shown in above prove, and $(A-\lambda_n I)^{-1}$ because λ_n dose not an eignvalue of A [30].

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