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HAYDER DHAHIR

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Supervisor

Assoc. Prof. Dr. Eser OLĞAR

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Hayder DHAHIR

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Assoc. Prof. Dr. Metin BEDİR
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. A. Necmeddin YAZICI
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Eser OLĞAR
Supervisor

Examining Committee Members	Signature
Prof. Dr. Hayriye TÜTÜNCÜLER	
Assoc. Prof. Dr. Eser OLĞAR	
Assist. Prof. Dr. Eser ÖZKEKLİKÇİ	

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Hayder DHAHIR

ABSTRACT

DIRAC EQUATION WITH POSITION-DEPENDENT MASS AND ITS APPLICATION

DHAHIT, HAYDER
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The formalism of Dirac equation for constant mass and position dependent mass is outlined in this study. During formalism, the equation is transformed into first order and second order differential equation forms respectively to solve for various physical potential.

The eigenfunctions and eigenvalues of corresponding physical potentials are obtained in the framework of Asymptotic Iteration Method for both first order and second order form of Dirac equation. This method is applied to Coulomb potential, Harmonic Oscillator problem, Manning-Rosen potential, Eckart potential and Hulthen potential. The results are in a good agreement with the corresponding values found in the literature.

Keywords: Dirac equation, position dependent mass, eigenvalues, eigenfunctions, asymptotic iteration method, physical potentials.

ÖZ

SABİT-POZISYONA BAĞLİ KUTLE İÇİN DİRAC DENKLEMİ VE UYGULAMALAR

DHAHIT, HAYDER
YüksekLisansTezi, FizikMühendisliği, Gaziantep Üniversitesi

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Bu çalışmada, Dirac denkleminin sabit ve pozisyona bağlı kütle için formülasyonu özetlendi. Formülasyonda, Dirac denklemi farklı fiziksel potansiyellerin çözümü için sırasıyla birinci ve ikinci dereceden diferansiyel denklem formuna dönüştürüldü.
Bazı fiziksel potansiyellerin enerji özdeğerleri ve özfonksiyonları Asimtotik iterasyon metodu kullanılarak Dirac denkleminin her iki formunda elde edildi. Bu metot Coulomb potansiyeli, Harmonic Oscillator problemi, Manning-Rosen potansiyeli, Eckart potansiyeli and Hulthen potansiyeli için uygulandı. Elde edilen sonuçlar literatürdeki sonuçlarla iyi bir uyum sergiledi.

Anahtar Kelimeler: Dirac denklemi, konuma bağlı kütle, özdeğer, özfonksiyon, asimtotik iterasyon metodu, fiziksel potansiyel.

To my mother

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I would like to express the deepest appreciation to my supervisor Assoc. Prof. Eser OLĞAR. His help and encouragement have taught me to believe myself and not to give up .Without his guidance and persistent this work would not have been possible.

I would like to say a special thank you to my mother and father who always believed in me, and to all of my brothers. Without each one of these people, I would not be where I am today. Furthermore, I would like to dedicate this work to the spirit of my friend, Sinan Abdullah, who passed away here in Turkey before he completes his thesis.

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LIST OF SYMBOLS

c	Speed of light
$\hbar$	The Plank constant
$M(r)$	Variable Mass of particle
m_0	Constant Mass
∇	Gradient operator
$\mathbf{p}$	Momentum operator
$\psi(r,t)$	Wavefunction
$S(r)$	Scalar potential
$V(r)$	Vector potential
E	Energy eigenvalues
F_{nk}	Radial wavefunction
G_{nk}	Radial wavefunction
$\dagger$	The conjugate transpose of a matrix.
$\Delta_k(x)$	The termination condition of AIM
L_{ij}	Orbital angular momentum operators
X	Hypergeometric function.

CHAPTER ONE

1 INTRODUCTION

The solution of one of the relativistic differential equation, Dirac equation for quantum mechanical systems in both case of spatially dependent mass and constant mass plays an important role [1–29].These physical systems are very useful in modeling the physical electronic properties of semiconductors[3], quantum wells ,quantum dots [3,4], quantum liquids [5]and semiconductors hetero structures [6].In addition to the modeling systems, the relativistic effects are important in the field of heavy ion doping and heavy atoms [7–9].

Since the important of investigation of the Dirac equation with position dependent mass, there has been increased a great interest on it [7–18]. The solution of Dirac equation for Coulomb potential in different dimensions [19-24], Dirac particle in a central potential [25, 26] have been discussed. Additionally, different multidimensional non-relativistic and relativistic equations have been studied ,For example ,the Schrodinger equation in D-dimensional has been studied with the pseudo harmonic potential [28], Coulomb potential [27], Hulthen potential[29]and Poschl-Teller potential [30]. Also, different potentials with D-dimensional for Klein-Gordon equation [31] and Dirac equations [32-35] has been studied.

In the science of quantum mechanics, physicists searched for methods to solve the Schrödinger and the Dirac equations. For this task, researchers have developed different access to get the exact or numerical solutions of both Schrödinger and the Dirac equation for many of potentials.

From these methods, there are many techniques, the more commonly used is the algebraic method [38], the Laplace transformation [39, 40], the power series expansion[36, 37], the factorization method [41], supersymmetry, shape invariance method [44],the path integral method [42, 43], WKB as well as supersymmetric WKB (SWKB) methods [45].The developing history of quantum mechanics for non-relativistic and relativistic shown that the new method promoted us to research in this field.

In few years ago, Ciftci *et al* [46, 47] proposed an asymptoticiteration method (AIM) it's very efficient to establish the eigenvalues, which draws the attention of a many researchers [48–50] and gives a number of numerical or exact solutions to theSchrödinger equation for a lot of interesting potentials.After successfully using asymptotic iteration method to non-relativisticquantum mechanics, researchers moved to applythismethodto relativistic quantum mechanics to solve the Dirac equation [51].

In this thesis, we choose AIM method to solve the spectrum of some physical potentials for Dirac equation.

The organization of thesis is as follow: the fundamental of Dirac equation for d-dimension [52] is introduced in Chapter 2. In chapter (3), the ideas which make the Dirac equation persuasiveand the properties of certain Dirac matrices with relationship α and β pointed out .In subsequent chapter, the formulism of Dirac equation for constant mass and variable are derived both for first order and second order differential equation form. In chapter 4, the method used in the calculation of eigenvalues and eigenfunctions is presented. The applications of method are outlined in Chapter 5. The last chapter devotes to main conclusion of this thesis.

CHAPTER 2

2 FORMULASIMOF THE DIRAC EQUATION

One of the relativistic equation, Dirac equation was presented by the Paul Adrien Maurice Dirac [1] in 1928, who constructed an essential contribution to the development of both quantum electrodynamics and quantum mechanics. Dirac equation was constructed to explain the behavior of fermions and predicted to appearance of the antimatter [2]. Dirac searched for relativistic thatequation which was consistent with special theory of relativity without a sureness negative results of probability density. Toconstruct the relation of the energyand momentum, the Dirac equation must be linear in time derivative. And also to be preserved the Lorentztransformation, it also to be relativistic covariant.Therefore, Dirac proposed the Dirac equation which led to experimental discovery, the discovery of the positron, the anti-particle of the electron thatis one of greatest theoretical physics [3].

2.1 Derivation ofDirac Equation

The Dirac equation is derived withseveral methods by considering the fundamental of the relativistic relation between the energy and momentum. This relation in the form of

$$\mathrm{E}^2 = \mathbf{p}^2\mathrm{c}^2 + m^2\mathrm{c}^4 \tag{2.1}$$

Where c is speed of light, E the energy, **p** the momentum and m the mass of particlesubsequently, the operator form of quantummechanical observables E and **p**is

$$\mathrm{E} \rightarrow \mathrm{i}\hbar\frac{\partial}{\partial \mathrm{t}} \quad \text{and} \quad \mathbf{p} \rightarrow \frac{\hbar}{\mathrm{i}}\nabla$$

where ∇ is donates the gradient and $\hbar$ is given in terms of constant $h$as$\hbar = \frac{h}{2\pi}$.Substituting this operator in Eq.(2.1) and acting on the quantum mechanical wave function $\boldsymbol{\psi}(r, t)$, yields

$$-\hbar\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} = (-\hbar\,\mathrm{c}^2\nabla^2 + m^2\mathrm{c}^4)\boldsymbol{\psi} \tag{2.2}$$

this equation represents the Klein-Gordon equation for free particle. The time dependent second order differential equation (namely (2.2)) is solved by using

boundary conditions for $\psi(r,t)$and $\psi'(r,t)$.

The relativistic solution of Klein-Gordon gives negative energy solution due toE^2. To get rid of these problems, some physicists tried to write the Eq. (2.2) in different form which combines the quantum mechanics and special relativity. Also, the transformed form of Klein-Gordon equation is first order differential equation depending on time and it can be invariant under Lorentz transformations.

Through the studies, Paul Dirac [3, 57] considered square root of Eq. (2.1) as

$$E = \sqrt{\mathbf{p}^2\mathrm{c}^2 + m^2\mathrm{c}^4} \tag{2.3}$$

This energy equation is reduced to in the limit of $\boldsymbol{p} \to 0$ equation (2.3) become

$$\sqrt{\mathbf{p}^2\mathrm{c}^2 + m^2\mathrm{c}^4} \quad \to mc^2$$

And in limit of $m \to 0$

$$\sqrt{\mathbf{p}^2\mathrm{c}^2 + m^2\mathrm{c}^4} \quad \to pc\ .$$

From these limits, Dirac generalized the energy equation in

$$\sqrt{\mathbf{p}^2\mathrm{c}^2 + m^2\mathrm{c}^4} \ = \alpha\ \mathbf{p}\ c + \beta\ m\ \mathrm{c}^2$$

where α and $\beta$are matrices which they are defined in following section. Substitution this assumptions to the Eq. (2.2) yields

$$i\hbar\frac{\partial\boldsymbol{\psi}}{\partial t} = (\alpha\ \mathbf{p}\ c + \beta\ m\ c^2\ )\ \boldsymbol{\psi} \tag{2.4}$$

Taking the natural units ($\hbar = \ c = 1$), the Eq. (2.4) transform to

$$i\frac{\partial\boldsymbol{\psi}}{\partial t} = \sum_{j=1}^{d}\alpha_j\,p_j + \beta\ m\ \ )\ \boldsymbol{\psi} \tag{2.5}$$

where the momentum operatorp_j represents

$$p_j = -i\frac{\partial}{\partial x^j}\ , 1 \le j \le d$$

The Eq. (2.5) is called the famous Dirac equation for free particle in (d + 1) dimensions.

2.2 Matrix Representations of α and β

From equation (2.5) it is clear that the coefficients α_j and βshould investigate the condition of invariant with respect to spatial rotations. In order to satisfied this condition, these parameters can't be numbers, they should be in matrix form which are called Dirac matrices. In addition, the wave function $\boldsymbol{\psi}$ must be column vector form.

Thus, the time dependent wavefunction in column matrices form for *n* rows can be written as

$$\boldsymbol{\psi} = \begin{pmatrix} \psi_{\mathbf{1}}(\boldsymbol{r},t) \\ \psi_{\mathbf{2}}(\boldsymbol{r},t) \\ \psi_{\mathbf{3}}(\boldsymbol{r},t) \\ \vdots \\ \vdots \\ \psi_{\boldsymbol{n}}(\boldsymbol{r},t) \end{pmatrix} \tag{2.6}$$

The matrices α_j and β must be a square matrix for obeying the matrix formalism of Dirac equation [1, 2].

The equation (2.5) is rearranged by considering

$$\left(\sum_{j=1}^{d} \alpha_j\, p_j + \beta\, m \; - i\frac{\partial}{\partial t}\right)\boldsymbol{\psi} = 0 \tag{2.7}$$

By multiplying the Eq. (2.7) with the operator

$$\sum_{j=1}^{d} \alpha_j\, p_j + \beta\, m + i\frac{\partial}{\partial t}$$

from left , it causes the Eq. (2.7) to reduce in

$$\left(\sum_{i,j=1}^{d} \alpha_i\, \alpha_j p_i p_j + m\sum_{j=1}^{d} (\,\alpha_j\, \beta + \beta\, \alpha_j) p_j + m^2\beta^2 + \frac{\partial^2}{\partial t^2}\right)\boldsymbol{\psi} = 0 \tag{2.8}$$

where $\sum_{i,j=1}^{d} \alpha_i\, \alpha_j p_i p_j = (\,\sum_{i,j=1}^{d} \alpha_i\, \alpha_j p_i p_j + \sum_{i,j=1}^{d} \alpha_j \alpha_i\, p_j p_i)/2$

After some algebraic calculations, the Eq. (2.8) is obtained as

$$-\frac{\partial^2}{\partial t^2} = \left(\sum_{i,j=1}^{d} \frac{\alpha_i \alpha_j + \alpha_j \, \alpha_i}{2} p_i p_j + m \sum_{j=1}^{d} (\, \alpha_j \, \beta + \beta \, \alpha_j) p_j + m^2 \beta^2 \right) \boldsymbol{\psi} = 0 \qquad (2.9)$$

In a similar way, by substituting the matrix form of $\boldsymbol{\psi}$ from (2.9) into Klein-Gordon with the natural units Eq. (2.6), the Eq. (2.2) is written as

$$-\frac{\partial^2 \psi_i}{\partial t^2} = (\mathbf{p}^2 + m^2 \,) \psi_i \qquad (2.10)$$

It can be seen that from Eq. (2.10), the each components of $\psi_i(\boldsymbol{r}, t)$ satisfy the Klein-Gordon equation.

By comparing the equation (2.9) and (2.10), it is clear that the requirement for matrices α_j and β is

$$\beta^2 = {\alpha_i}^2 = 1 \qquad (2.11)$$

$$\alpha_i \alpha_j + \alpha_j \, \alpha_i = 2 \, \delta_{ij} (2.12), \qquad \alpha_j \, \beta + \beta \, \alpha_j = 0 \qquad (2.13)$$

where δ_{ij} is called kroneccker delta function which is

$$\delta_{ij} = \begin{Bmatrix} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{Bmatrix}$$

In order to satisfy requirements of α_j and β matrices, as written in equation Eq. (2.11),Eq. (2.12) and Eq. (2.13) the corresponding matrices are anti-commute rather than commute with each other. Because the Hamiltonian in Eq. (2.5),is hermitian (self-adjoint), as

$$H = \sum_{j=1}^{d} \alpha_j \, p_j + \beta \, m$$

Therefore if *H* is hermitian, Dirac matrices also have to be hermitian.This means that the elements on the main diagonal are real and the matrices are symmetric .This condition results to be their eigenvalues real, and:

$$\beta^\dagger = \beta \qquad \text{and} \qquad {\alpha_j}^\dagger = \alpha_j \qquad (2.14)$$

α_j in its eigen-representation with eigenvalues (α_1 , α_2, ...) has considered to be in form of

$$\alpha_j = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix} \tag{2.16}$$

Then with this representation, $\alpha_i{}^2$ becomes from Eq. (2.16) and Eq. (2.11) we obtain

$$\alpha_i{}^2 = \begin{pmatrix} \alpha_1{}^2 & 0 & \cdots & 0 \\ 0 & \alpha_2{}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n{}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.17}$$

From these equations, generalizing forα, it can be found that $\alpha_k = \pm 1$ where k=1, 2,…n . Similarly, if the same procedure is applied to matrixβ, the eigenvalues for α_j and β are ± 1 .

Multiplying Eq. (2.13) from the left with constant β , and with the aid of Eq. (2.11) we get

$$\beta\ \alpha_j\ \beta = -\ \alpha_j \tag{2.18}$$

Using the algebraic properties of matrices in [3], the trace of A B is written as

$$tr\ \mathrm{AB} = tr\ \mathrm{BA}$$

where the operator tr A , trace of a square matrix A, is the sum of the elements on the diagonal of matrix A. Now from equation (2.11) and Eq. (2.18),

$$tr\ \alpha_j = tr\ (\beta^2\ \alpha_j) = tr\ (\beta\ \ \alpha_j\ \beta) = -tr\ (\ \alpha_j)$$

Thus

$$tr\ \alpha_j = 0 \tag{2.19}$$

By using the same argument of α_j and applying the some algebraic procedure to matrix β , we get

$$-\ \beta = \ \alpha_j\ \beta\ \alpha_j$$

So

$$tr\beta = tr\ \alpha_j{}^2\ \beta = tr\ \ \alpha_j\ \beta\ \ \alpha_j\ \ = -tr\ \beta$$

And similarly,

$$tr\ \beta = 0 \tag{2.20}$$

Again from Eq. (2.13) with a new arrangement:

$$\alpha_j \, \beta = (-1\,)\beta \, \alpha_j \tag{2.21}$$

Taking determinant of Eq. (2.21) for both sides,

$$\det \, \alpha_j \det \beta = (-1)^n \det \, \alpha_j \det \beta \tag{2.22}$$

From the factor of $(-1)^n$, n should be even to satisfy the corresponding equation. Consider a matrix R with dimensions $(n \, \times n)$ that transform the matrix α_j into its diagonal form as

$$R \, \alpha_j \, R^{-1} = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix} \tag{2.23}$$

From equation (2.23) and the relation$tr(\mathrm{AB}) = tr(\mathrm{BA})$, it is found

$$tr \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix} = \sum_{\boldsymbol{k=1}}^{\boldsymbol{n}} \alpha_k = tr \, R \, \alpha_j \, R^{-1} \tag{2.24}$$

$$= tr \, \alpha_j \, R \, R^{-1} = tr \, \alpha_j$$

From properties of Dirac matrices, their trace are equal to zero, and the corresponding eigenvalues are ± 1. But, from Eq. (2.24), the trace has the same value with its eigenvalues. So, it is concluded the number of negative eigenvalues has to be the same like the number of positive eigenvalues.

From equation Eq. (2.11), Eq. (2.12) and Eq. (2.13) and as shown in [17] there are only three matrices satisfy anticommuting properties which are called the Pauli. These are

$\sigma_m$where m =1,2,3

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.25}$$

Therefor it is useful to construct the Dirac matrices in terms of Pauli matrices as

$$\alpha_j \; = \begin{pmatrix} 0 & \sigma_m \\ \sigma_m & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{2.26}$$

where $I$is the unit matrix $(2 \, \times 2), I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

CHAPTER 3

3 TRANSFORMATION OF DIRAC EQUATION

In literature, the transformation of the Schrödinger equation from 3-dimensions to arbitrary *d*-dimensions has been done by using the eigenvalues of the generalized orbital angular momentumL^2, in place of the three-dimensional ones [53,54].

The transformation method For Dirac equation is different. It must treat with spin angular momentum and generalized orbital operators, and its agreement with element of a Lie group and a Lie algebra. The authors in reference [55] were derived The Diracequation in arbitrary *d*-dimensionsby using the self-adjoint ladder operator method.

In this chapter, the derivation of the radial Dirac equation *d*-dimensions is discussed in simple algebraicderivation and the similar structure of the Dirac equations in arbitrary spatial dimensions in a central field is pointed out.

3.1 First-Order coupled differential equations form

The derivation of Dirac equation in first order linear form can be obtained for two different cases of mass function when it is constant mass and variable mass function.

3.1.1 Constant mass Case,

The Dirac equation for a central field in (d + 1)dimensions is written for spherically symmetric vector V(r) and S(r) spherically symmetric scalar potential by using Eq. (2.5)

$$i\frac{\partial \boldsymbol{\psi}}{\partial t} = H\ \boldsymbol{\psi} \qquad H = \sum_{j=1}^{d} \alpha_j\, p_j + \beta\left(m + S(r)\right) + V(r) \tag{3.1}$$

where (d + 1) dimensional matrices also satisfy the anti-commutative relations.

Without any approximation, the Dirac equation for a central fieldin spherical coordinates can be separated into the variables. Thus, it has mean the eigenfunction of the orbital and spin angular momentum can be found.

From Eq. (3.1), the radial velocity operators and radial momentum are reduced as

$$p_r = r^{-1}\left(\sum_{j=1}^{d} x_j\, p_j - i\frac{d-1}{2}\right) \qquad , \qquad \alpha_j = r^{-1}\left(\sum_{j=1}^{d} x_j\, \alpha_j\right) \tag{3.2}$$

Now, the orbital angular momentum operators is defined as

$$L_{ij} = x_i p_j - x_j p_i = -i\left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right) \tag{3.3}$$

According to reference [55], Eq. (3.3) satisfy the following algebraic relations

$L_{ij} = -L_{ji}$(3.4), $L_{ij} = L_{ij}{}^{\dagger}$(3.5), $[L_{ij}, L_{ik}] = i\, L_{ik}$(3.7)

(3.7)

$$[L_{ij}, L_{kl}] = 0\,, \qquad for\ i \neq j \neq k \neq l$$

$$L_{ij}L_{kl} + L_{ki}, L_{jl} + L_{jk}, L_{il} = 0\,, \qquad for\ i \neq j \neq k \neq l \tag{3.8}$$

Where the indices (*i*, *j*, *k*, *l*)take the values (1, 2, . . . *d*) and *d*being the dimension of the space.

The total orbital angular momentumL^2 and generalized spin angular momentum σ_{ij} are defined as

$$L^2 = \sum_{i<j}^{d} L_{ij}^2 \qquad (3.9) \quad , \qquad \sigma_{ij} = -\frac{i}{2}[\alpha_i, \alpha_j] \tag{3.10}$$

The σ_{ij} are satisfy the following relations which are introduced in [56] as

$\sigma_{ij} = -\sigma_{ji}$(3.11), $\sigma_{ij} = \sigma_{ji}{}^{\dagger}$(3.12), ${\sigma_{ij}}^2 = 1$(3.13)

$$[\sigma_{ij}, \sigma_{ik}] = i\, \sigma_{jk}\,, \qquad for\ \ i \neq j \neq k \tag{3.14}$$

$$[\sigma_{ij}, \sigma_{kl}] = 0\,, \qquad for\ \ i \neq j \neq k \neq l \tag{3.15}$$

Introducing the operator$k_d$which is related to the total angular momentum and commute with corresponding potential

$$k_d = \beta\left(\sum_{i<j}^{d} \sigma_{ij} L_{ij} + i\frac{d-1}{2}\right) \tag{3.16}$$

by substituting Eq. (3.2) and Eq.(3.3) into Eq.(3.1) ,it gives

$$\sum_{j=1}^{d} (x^j)^2 = r^2$$

Thus, the Hamiltonian turns to the

$$H = \alpha_r p_r + \frac{i}{r}\alpha_r\, \beta\, k_d + \beta\big(m + S(r)\big) + V(r) \tag{3.17}$$

It can be see that, the corresponding Hamiltonian Eq. (3.17) is depends on three operators α_r , p_r , k_d . If the eigenvalues of matrix or operator, k_d the eigenvalues of Hamiltonian H can be calculated.

For this aim, let introduce a new operator as

$$\mathcal{L} = \left(\sum_{i<j}^{d} \sigma_{ij} L_{ij}\right) \tag{3.18}$$

Eq. (3.16) with the introducing operator transform to:

$$k_d = \beta\left(\mathcal{L} + i\frac{d-1}{2}\right) \tag{3.19}$$

Since , $[H\, , \mathcal{L}] = 0$ and $[L^2\, , \mathcal{L}] = 0$

Where $H, L^2 and\ \mathcal{L}$ have common eigenfunction .The eigenvalues of L can be found by establishing a relation between L and L^2 which introduce in reference [57] .The power squares of L^2 , can be written from Eq. (3.18) as following

$$\mathcal{L}^2 = \sum_{i<j}^{d}\sum_{k<l}^{d} \sigma_{ij} L_{ij}\sigma_{kl}L_{kl} \tag{3.20}$$

The solution is get by rewritten Eq. (3.20),

$$\mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{L}_3^2$$

Where

$\mathcal{L}_1^2$ $\rightarrow$ is the content of equal indices (*i*=*j* and *j*=*l*)

$\mathcal{L}_2^2$ → is the content of unequal indices ($i \neq j \neq k \neq l$)

$\mathcal{L}_3^2$ → is the can be contracted to $(\sigma_{ij} L_{ij})$

Therefore, we have

$$\mathcal{L}_1^2 = \sum_{i<j}^{d} (\sigma_{ij} L_{ij})^2 = L^2 \tag{3.20}$$

and

$$\mathcal{L}_2^2 = \sum_{i<j}^{d} \sum_{k<l}^{d} \sigma_{ij} L_{ij} \sigma_{kl} L_{kl} \ , \qquad i \neq j \neq k \neq l \tag{3.21}$$

In similar way the last part of Hamiltonian $\mathcal{L}_3^2$ has a definition of:

$$\mathcal{L}_3^2 = \sum_{i<j}^{d} \left(\sum_{k=1}^{d-1} \sigma_{ki} L_{ki} \sigma_{kj} L_{kj} + \sum_{k=j+1}^{d} \sigma_{ki} L_{ki} \sigma_{jk} L_{jk} \sum_{k=i+1}^{j-1} \sigma_{kj} L_{kj} \sigma_{ik} L_{ik} \right) \tag{3.22}$$

$$= -(d-2)L$$

The addition of terms in Eq. (3.18) is symmetric, therefore terms appeared in Eq. (3.20) can be covered by summing up only the following triple terms.

$$\sigma_{ij} L_{ij} \sigma_{kl} L_{kl} + \sigma_{ik} L_{ik} \sigma_{jl} L_{jl} + \sigma_{jk} L_{jk} \sigma_{il} L_{il}$$

$$= \sigma_{ij} L_{kl} \left(L_{ij} L_{kl} + L_{ik} L_{jl} + L_{jk} L_{il} \right) = 0 \, , i < j < k < l \tag{3.23}$$

By using the equations Eq. (3.6) and Eq. (3.13), it can be concluded that the number of terms involved in each partial sum are:

$$\mathrm{N}_1 = \frac{\mathrm{d}(\mathrm{d}-1)}{2} \ , \qquad \text{in } \mathcal{L}_1^2 \tag{3.24}$$

$$N_2 = 6 \sum_{i<j<k<l}^{d} 1 = \frac{d(d-1)(d-2)(d-3)}{4} \ , \qquad in \ \mathcal{L}_2^2 \tag{3.25}$$

$$N_3 = d(d-1)(d-2) \, , \qquad in \ \mathcal{L}_3^2 \tag{3.26}$$

Then it easy to see that

$$N = N_1 + N_2 + N_3 = \left(\frac{d(d-1)}{2} \right)^2 \tag{3.27}$$

By comparing the equations Eq. (3.20), Eq. (3.21) and Eq. (3.23), L^2 is found as

$$L^2 = \mathcal{L}(\mathcal{L} + d - 2) \tag{3.28}$$

where the eigenfunctions ofL^2 are doubly degenerate we may write

$$\mathcal{L}\,\psi_{\mathbf{1}} = l\,\psi_{\mathbf{1}} \qquad (3.29), \quad \mathcal{L}\,\psi_{\mathbf{2}} = -(l+d-2)\,\psi_{\mathbf{2}} \qquad (3.30)$$

These above equations lead to

$$L^2\psi_{\boldsymbol{i}} = l(l+d-2)\psi_{\boldsymbol{i}}\ , i = 1,2 \qquad (3.31)$$

For that the eigenvalue of operator k_d will be

$$k_d = \mp\left(j + \frac{d-2}{2}\right), \qquad j = l \mp \frac{1}{2} \qquad (3.32)$$

which are the same as derived in [1],by introducing two-component wavefunction.

$$\boldsymbol{\psi} = r^{-\frac{d-1}{2}}\begin{pmatrix} F \\ i\,G \end{pmatrix} \qquad (3.33)$$

Theradial Dirac equations in d-dimensions are derived in following form

$$\frac{dF(r)}{dr} = -\frac{k_d}{r}F(r) + \big(E + m - V(r) + S(r)\big)G(r) \qquad (3.34)$$

$$\frac{dG(r)}{dr} = \frac{k_d}{r}G(r) - \big(E - m - V(r) - S(r)\big)\,F(r) \qquad (3.35)$$

These two equations are called the first-order linear coupled Dirac equations. Thus, by using these equations Dirac equations are ready to apply to physical systems.

3.1.2 Variable Mass Case

As we introduced the first-order coupled Dirac equations with variable mass, can be using the last equations Eq. (3.34) and Eq. (3.35) for a constant mass as following

$$\begin{aligned} \frac{dF(r)}{dr} &= -\frac{k_d}{r}F(r) + \big(E + M(r) - V(r) + S(r)\big)G(r) \\ \frac{dG(r)}{dr} &= \frac{k_d}{r}G(r) - \big(E - M(r) - V(r) - S(r)\big)\,F(r) \end{aligned} \qquad (3.36)$$

3.2 Second Order-Differential Equations Form

3.2.1 Constant Mass Case

Similar to the calculations done is section 3.1, here the second-order differential equations for the lower and upper componentsof the Dirac equation ispresented for constant mass and spatially mass by using Dirac wave equation in first order and Dirac spinorwavefunction.

With the aid of Eq. (3.33), the Dirac spinor can be written due to upper $F_{nk}(r)$ and lower $G_{nk}(r)$ wavefunctions as

$$\boldsymbol{\psi} = r^{-\frac{d-1}{2}}\begin{pmatrix} S \\ iQ \end{pmatrix} = r^{-\frac{d-1}{2}}\begin{pmatrix} S_{nk}(r)\, Y_{jm}^{l}(\theta,\phi) \\ iQ_{nk}(r) Y_{jm}^{\tilde{l}}(\theta,\phi) \end{pmatrix} \tag{3.37}$$

where $Y_{jm}^{l}(\theta,\phi)$represents the spin spherical harmonic and $Y_{jm}^{\tilde{l}}(\theta,\phi)$represents the pseudospin spherical harmonic. m, is the projection angular momentum on the z axis, n is the radial quantum number, and l and $\tilde{l}$ orbital angular momentum quantum numbers to the spin and pseudospin respectively.

Rearranging the two coupled differential equations in equations (3.35) and (3.36) for upper and lower radial wave functions in, as yields

$$\left(\frac{d}{dr} - \frac{k_d}{r}\right) G(r) = \left(E + m - \sum(r)\right) F(r) \tag{3.38}$$

$$\left(\frac{d}{dr} + \frac{k_d}{r}\right) F(r) = -\left(E - m - \Delta(r)\right) G(r) \tag{3.39}$$

where $\sum(r) = V(r) + S(r)$, and$\Delta(r) = V(r) - S(r)$.

The general form of two second-order differential equations for corresponding eigenfunctions are obtain by eliminating wavefunction$F(r)$ in Eq. (3.38) and $G(r)$ in Eq.(3.39) ,as following

$$\left[\frac{d^2}{dr^2} - \frac{k_d(k_d+1)}{r^2} - \left(m + E - \Delta(r)\right)\left(m - E + \sum(r)\right) - \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr} + \frac{k_d}{r}\right)}{m - E - \Delta(r)}\right] F_{nk}(r) = 0 \tag{3.40}$$

$$\left[\frac{d^2}{dr^2} + \frac{k_d(k_d-1)}{r^2} - \left(m + E - \Delta(r)\right)\left(m - E + \sum(r)\right) - \frac{\frac{d\sum(r)}{dr}\left(\frac{d}{dr} - \frac{k_d}{r}\right)}{m - E + \sum(r)}\right] G_{nk}(r) = 0 \tag{3.41}$$

3.2.2 Variable Mass Case

From section 3.2.1, by using the equations Eq. (3.38) and Eq. (3.39), we can write coupled radial Dirac equation for the spinor component with central potential $V(r)$ and position-dependent mass $M(r)$ as

$$\frac{dF_{nk}(r)}{dr} + \frac{k}{r}F_{nk}(r) = \big(E_{nl} + M(r) - V(r) + S(r)\big)G_{nk}(r) \tag{3.42}$$

$$\frac{dG_{nk}(r)}{dr} - \frac{k}{r}G_{nk}(r) = -\big(E_{nl} - M(r) - V(r) - S(r)\big)F_{nk}(r) \tag{3.43}$$

where *k = - (l + 1)* for the total angular momentum *j = l + 1/2*, and *l* is angular momentum quantum number. $F_{nk}(r)$and $G_{nk}(r)$are the radial wave function of the upper and the lower-spinor components respectively, and the general form of two second-order differential equations for corresponding eigenfunctions are obtain by eliminating wavefunction$F_{nk}(r)$ in Eq. (3.42) and $G_{nk}(r)$in Eq.(3.43) we get

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2}\right]F_{nk}(r) - \frac{\left(\frac{dM(r)}{dr} - \frac{d\Delta(r)}{dr}\right)\left(\frac{d}{dr} + \frac{k}{r}\right)F_{nk}(r)}{M(r) + E_{nl} - \Delta(r)}$$
$$= \left[\big(M(r) + E_{nl} - \Delta(r)\big)\big(M(r) - E_{nl} + \textstyle\sum(r)\big)\right]F_{nk}(r) \tag{3.44}$$

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2}\right]G_{nk}(r) - \frac{\left(\frac{dM(r)}{dr} + \frac{d\Delta(r)}{dr}\right)\left(\frac{d}{dr} - \frac{k}{r}\right)G_{nk}(r)}{M(r) - E_{nl} + \Delta(r)}$$
$$= \left[\big(M(r) + E_{nl} - \Delta(r)\big)\big(M(r) - E_{nl} + \textstyle\sum(r)\big)\right]G_{nk}(r) \tag{3.45}$$

CHAPTER 4

4 ASYMTOTIC ITERATION METHOD (AIM)

It is well known that the second-order homogeneous linear differential equations play an important role in many branches of physics to explain the physical systems .The solution of these type of differential equations with boundary conditions has been solved by various techniques. One of method is called Asymptotic Iteration Method. By finding a suitable algorithm in AIM, the spectrum of corresponding system is obtained rapidly and more correctly by using computer programming.

In this chapter, the formulism of AIM is given for both first [46] and second-order linear differential equations form [47].

4.1 AIM for the First Order Linear Differential Equations.

Starting from the first–order linear coupled equations

$$y_1' = \lambda_0(x) y_1 + s_0(x) y_2 \qquad (4.1), \qquad y_2' = w_0(x) y_1 + p_0(x) y_2 \qquad (4.2)$$

where "$'$" represent the operator d/dx , and $\lambda_0(x)$, $s_0(x), w_0(x)$and$p_0(x)$ are sufficiently differentiable.Taking the derivation of equations (2.1) and Eq. (2.2) yields to

$$y_1'' = \lambda_0'(x) y_1 + \lambda_0(x) y_1' + s_0'(x) y_2 + s_0(x) y_2' \qquad (4.3)$$

$$y_2'' = w_0'(x) y_1 + w_0(x) y_1' + p_0'(x) y_2 + p_0(x) y_2' \qquad (4.4)$$

By rearranging these equations, it can be founds

$$y_1'' = \lambda_1(x) y_1 + s_1(x) y_2 \qquad (4.5), \qquad y_2'' = w_1(x) y_1 + p_1(x) y_2 \qquad (4.6)$$

where

$$\lambda_1(x) = \lambda_0' + \lambda_0^2 + s_0 w_0 \qquad (4.7)$$

$$s_1(x) = s_0' + \lambda_0 s_0 + s_0 p_0 \qquad (4.8)$$

$$w_1(x) = w_0' + \lambda_0 w_0 + p_0 w_0 \tag{4.9}$$

$$p_1(x) = p_0' + p_0^2 + s_0 w_0 \tag{4.10}$$

For (n) of differentiate to equations (4.5) and Eq. (4.6) we have

$$y_1^{(n+2)} = \lambda_{n+1}(x) y_1 + s_{n+1}(x) y_2 \tag{4.11}$$

$$y_2^{(n+2)} = w_{n+1}(x) y_1 + p_{n+1}(x) y_2 \,. \tag{4.12}$$

.

The functions $\lambda(x)\,, s(x), w(x)$and $p(x)$ are

$$\lambda_{n+1}(x) = \lambda_n' + \lambda_n \lambda_0 + s_n w_0 \quad , \quad s_{n+1}(x) = s_n' + \lambda_n s_0 + s_n p_0$$
$$w_{n+1}(x) = w_n' + \lambda_0 w_n + p_n w_0 \quad , \quad p_{n+1}(x) = p_n' + p_n p_0 + s_0 w_n \tag{4.13}$$

By taking the ratio to equation (4.11) we get

$$\frac{y_1^{(n+2)}}{y_1^{(n+1)}} = \frac{\lambda_{n+1}}{\lambda_n} \left(\frac{y_1 + \left(\frac{s_{n+1}}{\lambda_{n+1}}\right)\lambda_2}{y_1 + \left(\frac{s_n}{\lambda_n}\right)\lambda_2} \right) \tag{4.14}$$

And we can write

$$\frac{y_1^{(n+2)}}{y_1^{(n+1)}} = \frac{d}{dx}\left(\ln(y_1^{(n+1)}\right) \tag{4.15}$$

According to the method with large sufficiently of (n) the ratio of $\frac{s_n}{\lambda_n}$ becomes

$$\frac{s_{n+1}}{\lambda_{n+1}} = \frac{s_n}{\lambda_n} = \alpha \tag{4.16}$$

By using equations (4.14), (4.15) and (4.16) we get

$$\frac{d}{dx}\left(\ln(y_1^{(n+1)}\right) = \frac{\lambda_{n+1}}{\lambda_n} \tag{4.17}$$

From equation (4.13) and (4.16), we can write equation (4.17) in the form

$$\frac{\lambda_{n+1}}{\lambda_n} = \frac{d}{dx}(\ln(\lambda_n)) + \lambda_0 + \alpha\, w_0 \tag{4.18}$$

now from equations (4.17) and (4.18) we have

$$\ln(y_1^{(n+1)}) = \int^x \left(\frac{d}{dx}(\ln(\lambda_n(t))) + \lambda_0(t) + \alpha(t)\, w_0(t)\right)$$

or

$$y_1^{(n+1)}(x) = C_1 \lambda_n exp\left(\left(\int^x (\lambda_0(t) + \alpha(t)\, w_0(t))\right) dt\right. \tag{4.19}$$

where C_1 is the integral constant.

From equation (4.11) we can write

$$y_1^{(n+1)} = \lambda_n(x)y_1 + s_n(x)y_2 \tag{4.20}$$

Substituting equation (2.19) into equation (4.20) with using equation (2.16) we get

$$\lambda_n(x)y_1 + s_n(x)y_2 = C_1\lambda_n exp\left(\int^x (\lambda_0(t) + \alpha(t)\, w_0(t))\right)dt$$

or

$$y_1(x) = C_1 exp\left(\int^x (\lambda_0(t) + \alpha(t)\, w_0(t))\right)dt - \alpha(x)y_2(x) \tag{4.21}$$

After some algebraic calculations Eq. (4.2) results to

$$y_1(x) = y_2'(x) - p_0(x)y_2(x)/\ w_0(x) \tag{4.22}$$

combining equations (2.21) and (2.22) we obtain

$$C_1\, w_0(x) exp\left(\int^x \big(\lambda_0(t) + \alpha(t)\, w_0(t)\big)\right)dt - \alpha(x)y_2(x)\, w_0(x) = y_2'(x) - p_0(x)y_2(x) \tag{4.23}$$

or

$$y_2' + \varphi(x)y_2(x) = \Theta(x) \tag{4.24}$$

where

$$\varphi(x) = \alpha(x)\, w_0(x) - p_0(x)\,.$$

and

$$\Theta(x) = C_1\, w_0(x) exp\left(\int^x \big(\lambda_0(t) + \alpha(t)\, w_0(t)\big)\right)dt$$

Comparing with Eq. (2.24), the solution of this differential equation is given in [58] as

$$y_2(x) = exp^{-1}\left(\int^x \varphi(t)\, dt\right)\left(\int^x \Theta(t)\, dt\ \ exp\left(\int^x \varphi(t)\, dt\right)dt + C_2\right) \tag{4.25}$$

Now, by Substitution of fuctions ($\varphi(x)$and$\Theta(x)$)in the Eq. (2.25), the general solution of $y_2(x)$becomes

$$y_2(x) = exp\left(\int^x (p_0 - \alpha\, w_0)\, dt\right)\left[C_2 + C_1 \int^x w_0\ exp\left(\int^t (\lambda_0 - p_0 + 2\,\alpha\, w_0) d\tau\right) dt\right] \quad (4.26)$$

In similar procedure, by using Eq. (2.26) and Eq. (4.21), the solution of $y_1(x)$ is obtained in form of

$$y_1(x) = C_1 exp\left(\int^x (\lambda_0(t) + \alpha(t) w_0(t))\, dt\right) - \alpha(x)\, exp\left(\int^x (p_0 - \alpha\, w_0)\, dt\right)\left[C_1 \int^x w_0\, exp\left(\int^t (\lambda_0 - p_0 + 2\,\alpha\, w_0) d\tau\right) dt + C_2\right] \quad (4.27)$$

4.2 Second -Order Formalism of AIM

Initially, when AIM is proposed by Ciftcietal [46, 47], it is constructed for the solution of second-order linear differential equation in the form of

$$y'' = \lambda_0(x) y' + s_0(x) y \quad (4.28)$$

The second order formulism is obtained by using the similar procedure with the first order.

First of all, taking the derivation of Eq. (4.28) with respect to time gives.

$$y''' = \lambda_1(x) y' + s_1(x) y \quad (4.29)$$

With sufficiently differentiable

$$\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2(x)$$
$$s_1(x) = s_0'(x) + s_0(x)\lambda_0(x)$$

again, taking the derivative of Eq. (4.29) yields

$$y'''' = \lambda_2(x) y' + s_2(x) y \quad (4.30)$$

with functions

$$\lambda_2(x) = \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x)$$
$$s_2(x) = s_1'(x) + s_0(x)\lambda_1(x)\,.$$

Generalizing for the$(n+1)^{th}$and$(n+2)^{th}$ derivatives of Eq. (4.28), for $n = 1, 2, 3, \ldots$, it gives

$$y^{(n+1)} = \lambda_{n-1}(x)y' + s_{n-1}(x)y \quad (4.31)$$

$$y^{(n+2)} = \lambda_n(x)y' + s_n(x)y \quad (4.32)$$

with arbitrary function

$$\lambda_n(x) = \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x) \quad (4.33)$$

$$s_n(x) = s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x). \quad (4.34)$$

Taking the ratio of the $(n+2)^{th}$and$(n+1)^{th}$ and from derivations of Eq. (4.31) and Eq. (4.32), give a results

$$\frac{d}{dx} ln\left(y^{(n+1)}\right) = \frac{y^{(n+2)}}{y^{(n+1)}} = \frac{\lambda_n(x)(y' + \frac{s_n(x)}{\lambda_n(x)}y)}{\lambda_{n-1}(x)(y' + \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}y)} \quad (4.35)$$

for the large limit cases for *n*, there is asymptotic expression

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = \alpha(x) \quad (4.36)$$

And the termination condition is in the form of

$$\Delta_k(x) = \begin{vmatrix} s_n(x) & \lambda_n(x) \\ s_{n-1}(x) & \lambda_{n-1}(x) \end{vmatrix} = \lambda_{n-1}(x)s_n(x) - \lambda_n(x)s_{n-1}(x) \quad (4.37)$$

$k = 1,2,3, \ldots n$. Which gives the solution of physical systems .By using the relation in Eq. (4.36), Eq. (4.36) is reduced to

$$\frac{d}{dx} ln(y^{n+1}) = \frac{\lambda_n(x)}{\lambda_{n-1}(x)} \quad (4.38)$$

which yields

$$y^{(n+1)}(x) = C_1 \text{ Exp}\left(\int \frac{\lambda_n(t)}{\lambda_{n-1}} dt\right) = C_1\lambda_{n-1} \text{ Exp } (\int (\alpha + \lambda_0)dt) \quad (4.39)$$

whereC_1 is the constant integration. Substituting Eq. (4.39) into Eq. (4.31), as

$$y' + \alpha y = C_1 \text{Exp}(\int \alpha + \lambda_0)dt) \quad (4.40)$$

At this point,the general solution to Eq. (4.31) can be easily calculated by using Eq. (4.40) as

$$y(x) = \text{Exp}\left(-\int \alpha\, dt\right)\left[C_2 + C_1 \int \text{Exp}\left(\int \left(\lambda_0(\tau) + 2\alpha(\tau)\right)d\tau\right) dt\right] \quad (4.41)$$

This is the solution of wavefunctions of AIM method.

CHAPTER 5

5 APPLICATION OF DIRAC EQUATION

The application of AIM on Dirac equation is classified into two cases as for constant mass and variable mass function.

5.1 Constant Mass Applications

5.1.1 Dirac-Coulomb Problem

In chapter three from section 3.1 we obtained the radial Dirac equation as the first–order differential equation with spherically symmetric vector V(r)) and spherically symmetric scalar S(r) .In case of coulomb problem we will substitute the scalar potential by zero, considering that the particle is moving in a pure vector coulomb filed [47].

Let now replace the $S(r)$ by zero and $V(r)$ by ($-\frac{A}{r}$) into equations (3.34) and (3.35), if we discuss the operator k_d from equation (3.32) in 3-dimensions becomes

$$k = \mp\left(j + \frac{1}{2}\right), \qquad j = l \mp \frac{1}{2}$$

if we namely: $\varpi = \mp 1$, then $j = l - \frac{\varpi}{2}$ and the operator k will be

$$k = \varpi\left(l + \frac{1}{2}\right) - \frac{1}{2}$$

where the simple ϖ will take the negative sign for aligned spin $\left(l + \frac{1}{2}\right)$and the positive sign for unaligned spin $(l - \frac{1}{2})$ .So, Eq. (3.35) and (3.36) reduce to

$$\frac{dG}{dr} = -\frac{k}{r}\,G(r) + (E + m + \frac{A}{r}\,)\,S(r) \tag{5.1}$$

$$\frac{dF}{dr} = -\frac{k}{r}\,F(r) - \left(E - m - \frac{A}{r}\right)G(r) \tag{5.2}$$

We should get some asymptotic forms for functions $G(r)$and $F(r)$, for that treat with (r) when it a small and large.

First, at (r) near to the origin its approach to zero .So, all the terms with (r) will be very large than the term $(E \pm m)$ .In this case equations (5.1) and (5.2) become

$$\frac{dG}{dr} = \frac{A}{r} F(r) - \frac{k}{r} G(r) \tag{5.3}$$

$$\frac{dF}{dr} = \frac{k}{r} F(r) - \frac{A}{r} G(r) \tag{5.4}$$

if we derivate the equations (5.3) and (5.4) with respect to (r) we have :

$$\frac{d^2G}{dr^2} = \frac{A}{r}\frac{dF}{dr} - F(r)\frac{A}{r^2} + \frac{k}{r^2} G(r) - \frac{k}{r}\frac{dG}{dr} \tag{5.5}$$

$$\frac{d^2F}{dr^2} = \frac{k}{r}\frac{dF}{dr} - F(r)\frac{k}{r^2} + \frac{A}{r^2} G(r) - \frac{F}{r}\frac{dG}{dr} \tag{5.6}$$

Firstly, we substitute equations (5.3) and (5.4) into (5.5) and then (5.6) with deleting the similar terms we get

$$\frac{d^2G}{dr^2} = \frac{k^2}{r^2} G(r) - G(r)\frac{A^2}{r^2} - F(r)\frac{A}{r^2} + -G(r)\frac{k}{r^2} \tag{5.7}$$

$$\frac{d^2F}{dr^2} = \frac{k^2}{r^2} F(r) - F(r)\frac{A^2}{r^2} - F(r)\frac{k}{r^2} - G(r)\frac{A}{r^2} \tag{5.8}$$

if we multiply the equations (5.3) and (5.4) by $(-\frac{1}{r})$ we find :

$$-\frac{1}{r}\frac{dG}{dr} = -\frac{A}{r^2} F(r) + \frac{k}{r^2} G(r) \tag{5.9}$$

$$-\frac{1}{r}\frac{dF}{dr} = -\frac{k}{r^2} F(r) + \frac{A}{r^2} Q(r) \tag{5.10}$$

now we replace the last two terms in equations (5.7) and (5.8) by equations (5.9) and (5.10) we obtain

$$\frac{d^2G}{dr^2} = G(r)\frac{k^2 - A^2}{r^2} - \frac{1}{r}\frac{dG}{dr} \tag{5.11}$$

$$\frac{d^2F}{dr^2} = F(r)\frac{k^2 - A^2}{r^2} - \frac{1}{r}\frac{dF}{dr} \tag{5.12}$$

If we look at for equations (5.11) and (5.12) ,by following [59] we find they are Euler equation and the solution of them will be :

$$Q(r) = r^{\delta} \tag{5.13}$$

Where, $\delta = \sqrt{k^2 - F^2}$, and $Q(r)$ can be furcation $F(r)$ or $G(r)$

The equation (5.13) represent the solution of equation (5.1) and (5.2) at (r) very small .Now , for a second case at (r) is very large ,its approach to infinity .So, the terms with (r) are approaching to zero .we can write equation (5.1) and (5.2) as following

$$\frac{dG}{dr} = (E + m) F(r) \tag{5.14}$$

$$\frac{dF}{dr} = (m - E) G(r) \tag{5.15}$$

Again by taking the derivative of equations (5.14) and (5.15) respect to (r) we get

$$\frac{d^2G}{dr^2} = (E + m) \frac{dF}{dr} \tag{5.16}$$

$$\frac{d^2F}{dr^2} = (m - E) \frac{dG}{dr} \tag{5.17}$$

By using equations (5.14) and (5.15) into last equations we see:

$$\frac{d^2G}{dr^2} = (m^2 - E^2) G \tag{5.18}$$

$$\frac{d^2F}{dr^2} = (m^2 - E^2) F \tag{5.19}$$

The solution to these equations is given by :

$$M(r) = Ce^{-r\sqrt{m^2-E^2}} \tag{5.20}$$

Where C is the constant of integration, and $M(r)$ is a function of $G(r)$ or (r) .

After we got the solutions of equations (5.1) and (5.2) at (r) small and large ,we adopt representations for radial functions$G(r)$ and$F(r)$ by using equations (5.13) and (5.20) in the following form :

$$G(r) = r^\delta \sqrt{m+E}\, e^{-r\sqrt{m^2-E^2}} \left(\theta_1(r) + \theta_2(r)\right) \tag{5.21}$$

$$F(r) = r^\delta \sqrt{m-E}\, e^{-r\sqrt{m^2-E^2}} \left(\theta_1(r) - \theta_2(r)\right) \tag{5.22}$$

Now, let us make the following notation:

$$r = \eta\, r_1 \;, \qquad \frac{d}{dr} = \frac{1}{r_1}\frac{d}{d\eta} \tag{5.23}$$

Where: $r_1 = \frac{1}{2\sqrt{m^2-E^2}}$. By substituting these notations into equations (5.21) and (5.22) to find

$$G(\eta) = {r_1}^{\delta}\eta^{\delta}e^{-\frac{\eta}{2}}\sqrt{m+E}\;(\,\theta_1(\eta) + \theta_2\,(\eta)) \tag{5.24}$$

$$F(\eta) = {r_1}^{\delta}\eta^{\delta}e^{-\frac{\eta}{2}}\sqrt{m-E}\;(\,\theta_1(\eta) - \theta_2\,(\eta)) \tag{5.25}$$

And also we substitute equation (5.23) into equations (5.1) and (5.2) we get :

$$\frac{dG}{d\eta} = \left[r_1(\,m+E) + \frac{A}{\eta}\,)\right]\,F(\eta) - \frac{k}{\eta}\,G(\eta) \tag{5.26}$$

$$\frac{dF}{d\eta} = \left[-r_1(\,m+E) + \frac{A}{\eta}\,)\right]\,G(\eta) + \frac{k}{\eta}\,F(\eta) \tag{5.27}$$

The next step we will use equation (5.24) into equation (5.26) with multiplying by the following term $\left(\frac{1}{{r_1}^{\delta}\eta^{\delta}e^{-\frac{\eta}{2}}\sqrt{m+E}}\right)$, we find

$$(\theta_1+\theta_2\,)\left(\frac{\delta}{\eta}-\frac{1}{2}\right)+\frac{d\theta_1}{d\eta}+\frac{d\theta_2}{d\eta} = -\frac{k}{\eta}(\theta_1+\theta_2\,)+\left[r_1(\,m+E)+\frac{A}{\eta}\,)\right]\sqrt{\frac{m-E}{m+E}}(\theta_1-\theta_2\,) \tag{5.28}$$

By similar steps to find equation (5.28) ,we use equation (5.25) into equation (5.27) then get :

$$(\theta_1-\theta_2\,)\left(\frac{\delta}{\eta}-\frac{1}{2}\right)+\frac{d\theta_1}{d\eta}-\frac{d\theta_2}{d\eta} = \frac{k}{\eta}(\theta_1-\theta_2\,)+\left[r_1(\,m-E)-\frac{A}{\eta}\,)\right]\sqrt{\frac{m+E}{m-E}}(\theta_1+\theta_2\,) \tag{5.29}$$

Let us add equation (5.28) to equation (5.29), with deleting all the similar terms and then use the following notation :

$$\sqrt{\frac{m-E}{m+E}} = 2\,r_1\,(m-E) \qquad\qquad \sqrt{\frac{m+E}{m-E}} = 2\,r_1\,(m+E)$$

We obtain :

$$(\theta_1 - \theta_2)[2\, r_1^2 (m^2 - E^2) - 2\, r_1 (\, m - E)\,)] + \left[2\, r_1 (m^2 - E^2) - 2\, r_1^2 (\, m + E) + \frac{A}{\eta}\,)\right] (\theta_1 + \theta_2\,) = \frac{2\,\delta}{\eta}\theta_1 - \theta_1 + 2\frac{d\theta_1}{d\eta} + 2\frac{\varpi\left(l + \frac{1}{2}\right) + \frac{1}{2}}{\eta}\theta_2 \tag{5.30}$$

Also we remember that $r_1 = \frac{1}{2\sqrt{m^2 - E^2}}$ .So, of course $(\,2\, r_1^2 (m^2 - E^2) = \frac{1}{2}\,)$ and equation (5.30) becomes

$$\frac{\delta}{\eta}\theta_1 - \theta_1 + \frac{d\theta_1}{d\eta} = -\frac{k}{\eta}\theta_2 \; - 2\, r_1 \frac{A}{\eta}\, E\, \theta_1 - 2\, r_1 \frac{A}{\eta}\, m\, \theta_2 \tag{5.31}$$

If we make new notation as following

$$C = 2\, r_1\; E\;, \qquad D = 2\, r_1\; m$$

Finally, equation (5.31) becomes

$$\frac{d\theta_1}{d\eta} = \left(1 - \frac{C - \delta}{\eta}\right)\, \theta_1(\eta) - \left(\frac{D + k}{\eta}\right)\theta_2(\eta) \tag{5.32}$$

Also by subtracting equations (2.28) and (5.29) with using the same steps to get equation (5.32) we find

$$\frac{d\theta_2}{d\eta} = \left(\frac{D - k}{\eta}\right)\, \theta_1(\eta) + \left(\frac{C - \delta}{\eta}\right)\theta_2(\eta) \tag{5.33}$$

Equations (5.32) and (5.33) are first order linear differential equations and they can be solved by power series method .At this point, these equations are amenable to solve with AIM.

By comparing the equations Eq. (4.1) and Eq. (4.2) with Eq. (5.32) and (5.33), we get

$$\lambda_0(\eta) = \left(1 - \frac{C - \delta}{\eta}\right) \qquad s_0(\eta) = -\left(\frac{D + k}{\eta}\right)$$

$$w_0(\eta) = \left(\frac{D - k}{\eta}\right) \qquad p_0(\eta) = \left(\frac{C - \delta}{\eta}\right)$$

then we calculate the sufficiently differentiable $\lambda_{n+1}(\eta), s_{n+1}(\eta), w_{n+1}\; and\;, p_{n+1}(\eta)$ from equations (4.13) .With using iteration condition according to AIM in equation (4.16) as following :

$$\lambda_n(\eta) s_{n+1}(\eta) - s_n(\eta)\lambda_{n+1}(\eta) = 0 \tag{5.34}$$

we progressed the calculations above ,get the following result, in Table [1]

n	C	D
0	δ	$-k$
1	$1+\delta$	$-k, \mp - \sqrt{1+k^2+2\delta}$
2	$2+\delta$	$-k, \mp\sqrt{3+k^2+2\delta}\,, \mp\sqrt{4+k^2+4\delta},$
3	$3+\delta$	$-k, \mp\sqrt{5+k^2+2\delta}, \mp\sqrt{8+k^2+4\delta}, \mp\sqrt{9+k^2+6\delta}$
4	$4+\delta$	$-k, \mp\sqrt{7+k^2+2\delta}, \mp\sqrt{k^2+4(3+\delta)}, \mp\sqrt{15+k^2+6\delta},$ $\mp\sqrt{k^2+8(2+\delta)}$
5	$5+\delta$	$-k, \mp\sqrt{9+k^2+2\delta}, \mp\sqrt{k^2+8(3+\delta)}, \mp\sqrt{k^2+4(4+\delta)},$ $\mp\sqrt{21+k^2+6\delta}, \mp\sqrt{25+k^2+10\delta}$
6	$6+\delta$	$-k, \mp\sqrt{11+k^2+2\delta}, \mp\sqrt{k^2+4(5+\delta)},$ $\mp\sqrt{k^2+8(4+\delta)}, \mp\sqrt{27+k^2+6,}$ $\mp\sqrt{35+k^2+10\delta}, \mp\sqrt{k^2+12(3+\delta)}$

Table[1]:the results of *C* and *D* .

as we see from results in Table 1 we conclude the formulas of (*C* and *D*) for arbitrary n :

$$C = 1+\delta\ ,D = \mp\sqrt{\epsilon(2n-\epsilon)+2\epsilon\delta+k^2} \tag{5.35}$$

we introduce the symbol ϵ to be : $\epsilon = 0,1,2,3 \dots n$. As we know from notation of :

$$C = 2\,r_1 E\ , \qquad (D = 2\,r_1\ m), and\ \ r_1 = \frac{1}{2\sqrt{m^2-E^2}}$$

By using formulas from (5.35) we find two different equations of energy as following:

$$\frac{A\,E}{\sqrt{m^2-E^2}} = n+\delta$$

So ,

$$E = \mp\frac{m}{\sqrt{1+\left(\frac{A}{n+\delta}\right)^2}} \tag{5.36}$$

And

$$\frac{A\,E}{\sqrt{m^2-E^2}} = \mp\sqrt{\epsilon(2n-\epsilon)+2\epsilon\delta+k^2}$$

$$E = \mp m\sqrt{1-\frac{A^2}{\epsilon(2n-\epsilon)+2\epsilon\delta+k^2}} \tag{5.37}$$

If we look at the equations (5.36) and (5.37) we find that the energy will be equal, at a (ϵ) take the (n or $n + \delta$). But δ is not integer, that is mean ϵ and n are integers .therefore we find ($\epsilon = n$) .if we use the conclusion of ($\epsilon = n$) with equation (5.35) for Dwe get

$$C = 1 + \delta \,, D = \mp\sqrt{(2\,n\,\delta) + n^2 + k^2} \tag{5.38}$$

And at n=0equation above become

$$D = -\,k^2$$

Thus the energy formula has the form

$$E = \mp \frac{m}{\sqrt{1 + \left(\frac{A}{n+\delta}\right)^2}} \tag{5.39}$$

If we define the principal quantum number as following

$$n_r = n + |k| - \frac{d-3}{2} = 1,2,3\,... \tag{5.40}$$

In 3-dimensions quantum number with using our definition of operator k we see

$$n_r = n + \left|\varpi\left(j + \frac{1}{2}\right) - \frac{1}{2}\right| = 1,2,3\,... \tag{5.41}$$

We know that $\delta = \sqrt{k^2 - A^2}$ for $n = 0,1,2,3\,....$. So, we rewrite equation (5.39) and get:

$$E = \mp \frac{m}{\sqrt{1 + \left(\frac{A}{n_r - \left|\varpi\left(j+\frac{1}{2}\right) - \frac{1}{2}\right| + \sqrt{\left(\varpi\left(j+\frac{1}{2}\right) - \frac{1}{2}\right)^2 - F^2}}\right)^2}} \tag{5.42}$$

Equation (5.42) represent the well-known formula for coulomb energy in three dimensions.

5.1.1.1 Eigenfunctions

In this section we will solve the wave functions and obtain the general formula of functions $\theta_1(\eta)$ and $\theta_2(\eta)$ in equations (5.35) and (5.36) to range of (n) ,by using our introducing of AIM .

If we turn back to chapter four and discuss the equation (4.26) we see that equation includes two parts ,where the first part in general represent the physical meaning and the second part go for infinity solution. Therefore we use the first part as a factor for the wave function generator as following

$$\theta_2(\eta) = C_2 exp\left(\int^{\eta} (p_0 - \alpha\, w_0)\, dt\right) \tag{5.43}$$

The constant integration C_2 , determined by normalization .We follow the same iteration procedures to find sufficiently $\lambda_{n+1}(\eta), s_{n+1}(\eta), w_{n+1}\ and\ , p_{n+1}(\eta)$ in section 5.2 and then applied into equation (5.43) . We obtain the results of $\theta_2(\eta)$ in the Table [2], as following:

n	$\theta_2(\eta)$
0	1
1	$-(2\ \delta + 1) + (1 - \frac{\eta}{2\ \delta + 1})$
2	$(2\ \delta + 1)(2\ \delta + 2)(1 - \frac{2\,\eta}{2\ \delta + 1} - \frac{\eta^2}{(2\ \delta + 1)(2\ \delta + 2)})$
3	$-(2\ \delta + 1)(2\ \delta + 2)(2\ \delta + 3)(\,1 - \frac{3\,\eta}{2\ \delta + 1} + \frac{3\,\eta^2}{(2\ \delta + 1)(2\ \delta + 2)} - \frac{\eta^3}{(2\ \delta + 1)(2\ \delta + 2)(2\ \delta + 3)})$

Table[2]: the results of $\theta_2(\eta)$.

From results in Table[2], we can write general formula of $\theta_2(\eta)$as :

$$\theta_2(\eta) = (-1)^n \frac{(2\ \delta + n)!}{(2\ \delta)!} C_2\, X(-n, 2\,\delta + 1, \eta) \tag{5.44}$$

Where X is the hypergeometric function.

We will calculate the $\theta_1(\eta)$,for this case we again go back to chapter four and discuss the equation (4.21) ,we find its content two parts .where the part [$C_1 exp\left(\int^x (\lambda_0(t) + \alpha(t)\, w_0(t))\right)$] is an infinite series and the second part is a polynomial. So we choose a polynomial part ,we have

$$\theta_1(\eta) = -\alpha(x)\theta_2(\eta) \tag{5.45}$$

Now we will use the results we have obtained for $\theta_2(\eta)$ into equation (5.45) and by following same iteration procedure, we get the results for function $\theta_1(\eta)$ in Table [3]

n	$\theta_1(\eta)$
0	0

1	$(k+D)$
2	$-(k+D)(2\ \delta+1)\ (1-\frac{\eta}{2\ \delta+1})$
3	$(k+D)(2\ \delta+1)(2\ \delta+2)\ (\ 1-\frac{2\,\eta}{2\ \delta+1}+\frac{\eta^2}{(2\ \delta+1)(2\ \delta+2)})$

Table [3]: the results of $\theta_1(\eta)$

as we found the general formula for $\theta_2(\eta)$, we conclude the general formula for $\theta_1(\eta)$. Depending on the results in Table[3], like a following

$$\theta_1(\eta)=(-1)^{n+1}(k+D)\frac{(2\ \delta+n-1)!}{(2\ \delta)!}C_2\,X(1-n,2\,\delta+1,\eta) \tag{5.46}$$

5.1.2 Eckart Potential

The Eckart potential [60] plays a fundamental role in molecular physics,itused to describe molecular vibrations and to find theenergy spectra of nonlinear and linear systems. It is very useful for describing the interatomic interaction of the molecules and for describingpolyatomic vibration energies within the vibration states of the NH3 molecule.

To investigate the relativistic behavior of spin-1/2 particles in order to understand the nuclear structure, theresearch of the spin and pseudospin symmetric solutions of the Dirac equation has been an interesting area of study in nuclear physics, the technique ofpseudospin and spin symmetry withthe nuclear shell model has been widely used in solving a number of phenomena in nuclear physics.

In this section we solve the Dirac equation for the Eckart potential in any κ-state by using a asymptotic iteration method AIM, in framework an approximation to spin–orbit coupling potential in arrangement to find the relativisticbound state eigenvalues and the identical Dirac spinors by pseudospin symmetry and spin symmetry concept.

By what we introduced in section 3.2 for Dirac spinors from equation(3.37) with orbital angular momentum numbers, l to spin and $\tilde{l}$ to pseudospin, and what we obtained for the second-order differential equations. We will use equations (3.40) and (3.41) with helping of operator $k=\varpi\left(l+\frac{1}{2}\right)-\frac{1}{2}$from section 5.1 to rewrite second-order differential Dirac equations in 3-dimension.

5.1.2.1 Eckart potential with Spin symmetric

From our knowledge from section 3.1.2, the difference between the vector potential and scalar potential is $\Delta(r) = V(r) - S(r)$.In case of spin symmetric this a difference be constant, i.e.$\Delta(r) = C$, $\frac{d\Delta(r)}{dr} = 0$. Therefore we can write equation(3.40) by following form

$$\left[\frac{d^2}{dr^2} + \frac{k(k+1)}{r^2} - (m + E_{nk} - C)\left(m - E_{nk} + \sum(r)\right)\right] F_{nk}(r) = 0 \tag{5.47}$$

where $k = l$ for negative k and $k = -(l+1)$for positive k.

The eigenvalues depend on n and$k$ or $E_{nk} = E\ (n, l(l+1))$ .We replace the summation of vector potential and scalar potential$\sum(r)$ byEckart potential , where the Eckart potential defined according to [29], as

$$V(r) = V_1 cosech^2\,(\alpha\, r) - V_2\, coth(\alpha\, r)$$

here α is a screening parameter , we can write the Ecart potential in the exponential form :

$$V(r) = 4V_1 \frac{e^{-2\,\alpha\, r}}{(1 - e^{-2\,\alpha\, r})^2} - V_2 \frac{1 + e^{-2\,\alpha\, r}}{1 - e^{-2\,\alpha\, r}} \tag{5.48}$$

similar to [61] when α and k are small ,we can replace the spin orbit coupling potential $V_k = \frac{k(k+1)}{r^2}$,by take an approximate spin-orbit coupling as

$$V(r) = 4\alpha^2\, k(k+1) \frac{e^{-2\,\alpha\, r}}{(1 - e^{-2\,\alpha\, r})^2} \tag{5.49}$$

if we insert the Ecart potential from equation (5.48) and approximate spin-orbit from equation (5.49) into first equation (5.47) ,then make the following notations

$$z = e^{-2\,\alpha\, r}\,, \varepsilon = \frac{m + E_{nk} - C(m - E_{nk})}{4\alpha^2}\,, \beta = \frac{V_2\ (m + E_{nk} - C)}{4\alpha^2}, \tag{5.50}$$

$$\gamma(\gamma+1) = k(k+1)\ \frac{V_1}{\alpha^2}(m + E_{nk} - C)$$

the Dirac equation (5.47) will be reduced to the formula

$$\frac{d^2 F_{nk}(z)}{dz^2} + \frac{1}{z}\frac{d\,F(z)}{dz} + \left[-\frac{\varepsilon}{z^2} + \frac{\gamma(\gamma+1)}{z(1-z)^2} - \frac{\beta(1+z)}{z^2(1-z)^2}\right] F_{nk}(z) = 0 \tag{5.51}$$

respect to the boundary conditions to the wavefunction we see for :

$r \to \infty$ then $F_{nk}(0) = 0$ $at\ z = 0$, and for $r \to 0$ then $F_{nk}(1) = 0$ $at\ z = 1$

with this conditions we suppose the reasonable wavefunction as following

$$F_{nk}(\mathrm{z}) = \mathrm{z}^{\sqrt{\varepsilon+\beta}}(1-\mathrm{z})^{\gamma}\, \mathrm{S}(\mathrm{z}) \tag{5.52}$$

lets insert equation(5.52) into Dirac equation(5.51) we obtain the second-order homogeneous linear differential equation in new formula

$$\frac{\mathrm{d}^2\mathrm{F(z)}}{\mathrm{dz}^2} = \left[\frac{\left(1+2\gamma+2\sqrt{\varepsilon+\beta}\right)\mathrm{z}-\left(2\sqrt{\varepsilon+\beta}\right)+1)}{\mathrm{z}(1-\mathrm{z})}\right]\times\frac{\mathrm{d\,F(z)}}{\mathrm{dz}} + \left[\frac{2\sqrt{\varepsilon+\beta}\gamma-\gamma^2+2\beta)}{\mathrm{z}(1-\mathrm{z})}\right]\mathrm{F(z)} \tag{5.53}$$

after obtained equation(5.53),which is amenable to asymptotic iteration method .So, let's refer to section 4.2 and compare it with equation(4.28). Also we can calculate sufficiently differentiable of values of k ($\lambda_k(\mathrm{z})$ and $\mathrm{s}_k(\mathrm{z})$) by means of equations ((4.31)-(4.34)) , we get

$$\lambda_0(\mathrm{z}) = \frac{\left(1+2\gamma+2\sqrt{\varepsilon+\beta}\right)\mathrm{z}-\left(2\sqrt{\varepsilon+\beta}+1\right)}{\mathrm{z}(1-\mathrm{z})} \tag{5.54a}$$

$$\mathrm{s}_0(\mathrm{z}) = \frac{2\sqrt{\varepsilon+\beta}\gamma+\gamma^2+2\beta}{\mathrm{z}(1-\mathrm{z})} \tag{5.54b}$$

$$\lambda_1(\mathrm{z})\frac{\left(6\sqrt{\varepsilon+\beta}\gamma+2+6\sqrt{\varepsilon+\beta}\gamma+3\gamma^2+6\gamma+4\varepsilon+2\beta\right)z^2+2+6\sqrt{\varepsilon+\beta}+4\varepsilon+4\beta}{z^2(z-1)^2} + \frac{\left(-12\sqrt{\varepsilon+\beta}-4\gamma+\gamma^2-6\sqrt{\varepsilon+\beta}\gamma-6\beta-8\varepsilon-4\right)\mathrm{z}}{\mathrm{z}^2(\mathrm{z}-1)^2}$$

$$\mathrm{s}_1(\mathrm{z}) = \frac{\left(\gamma^2+2\beta+2\sqrt{\varepsilon+\beta}\gamma\right)\left(-2+3\mathrm{z}+2\gamma\mathrm{z}+2\sqrt{\varepsilon+\beta}\mathrm{z}-2\sqrt{\varepsilon+\beta}\right)}{\mathrm{z}^2(\mathrm{z}-1)^2}$$

…. ……..etc

from the a termination condition for AIM in equation(4.37), we can write

$$\xi_1 = \lambda_1 s_0 - \lambda_0 s_1 = 0 \tag{5.55}$$

by using sufficiently differentiable above into equation(5.55) we calculate the first value of ξ_1 as

$$\xi_1(\mathrm{z}) = \frac{\left(2\beta+2\sqrt{\varepsilon+\beta}+2\sqrt{\varepsilon+\beta}\gamma+1+2\gamma+\gamma^2\right)\left(2\sqrt{\varepsilon+\beta}\gamma+2\beta+\gamma^2\right)}{\mathrm{z}^2(\mathrm{z}-1)^2} \tag{5.56}$$

the solution of equation(5.56) give the roots and we see the first value of ε_0 as

$$\varepsilon_0 = \frac{4\beta^2 + \gamma^4}{4\gamma^2}$$

and inserting the value ε_0 into equation(5.56) given the term $\sqrt{\varepsilon + \beta}$,its equal to the term $\sqrt{\frac{4\beta^2+\gamma^4}{4\gamma^2}}$ .For that the roots of equation will be $\pm\frac{4\beta^2+\gamma^4}{4\gamma^2}$. we take only the negative root because it satisfies equation(5.56) and which is also valid for $\varepsilon_1, \varepsilon_2 \ldots\ldots$ etc. So, if we continue using the condition in Eq.(5.55),yields to other ξ and ε as

$$\varepsilon_1 = \frac{4\beta^2 + \gamma^4 + 6\gamma^2 + 4\gamma^3 + 4\gamma + 1}{4(1+\gamma)^2} , \qquad at \quad \xi_2 = \lambda_2 s_1 - \lambda_1 s_2 = 0$$

$$\varepsilon_2 = \frac{(4\beta^2 + 16 + 32\gamma + 24\gamma^2 + 8\gamma^3 + \gamma^4)}{4(2+\gamma)^2}, at \quad \xi_3 = \lambda_3 s_2 - \lambda_2 s_3 = 0$$

………………… etc. (5.57)

when the formulas above are generalized by induction, we can write the eigenvalues of ε as

$$\varepsilon_{nk} = \frac{4\beta^2 + (n+\gamma)^4}{4(n+\gamma)^2} \quad , \qquad n = 0,1,2,3 \tag{5.58}$$

now from equations (5.50)and (3.14) we calculate the energy eigenvalues E_{nk} as

$$(m - E_{nk})(m + E_{nk} - C) = \alpha^2 \left[(n+\gamma)^2 + \frac{4\beta^2}{(n+\gamma)^2}\right] \tag{5.59}$$

with

$$\gamma = \frac{1}{2} \pm \frac{1}{2}\left[1 + 4k(k+1) + 4\frac{V_1}{\alpha^2}(m + E_{nk} - C)\right]^{1/2}$$

To obtain the eigenfunctions we again return to the section 4.2, by using the exponential part in equation(4.41) $y(x) = C_2 \text{Exp}(-\int \alpha \, dt)$as a wavefunction generator. So, we get

$$F(z) = (-1)^2 C_2 \frac{\Gamma\,(n + 2\sqrt{\varepsilon_{nk}+\beta} + 1}{\Gamma\,(2\sqrt{\varepsilon_{nk}+\beta} + 1)} \, \text{X}\big(-n\,, 2\big(\sqrt{\varepsilon_{nk}+\beta} + \gamma\big) + n, 1 + 2\sqrt{\varepsilon_{nk}+\beta}; z\big) \tag{5.60}$$

where Γ gamma function and X the Gauss hypergeometric function.

Now by helping from equation (5.52) and using equation above, we find the total radial wavefuction as following

$$F_{nk}(z) = N\, z^{\sqrt{\varepsilon_{nk}+\beta}}(1-z)^{\gamma} \text{X}\big(-n\,, 2\big(\sqrt{\varepsilon_{nk}+\beta} + \gamma\big) + n, 1 + 2\sqrt{\varepsilon_{nk}+\beta}; z\big)$$

(5.61)

where N is a normalization constant.

5.1.2.2 Eckart potential with pseudospin symmetric

For the case of exact pseudospin symmetry the summation term be constant,i.e.$\sum(r) = C$,$\frac{d\sum(r)}{dr} = 0$. Therefore, we can write equation(3.41) by following form

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - \left(m + E_{nk} - \Delta(r)\right)(m - E_{nk} + C)\right] G_{nk}(r) = 0 \tag{5.62}$$

where $k = -\tilde{l}$ for negative k and, $k = \left(\tilde{l} + 1\right)$for positive k and we replace The term $\Delta(r)$ by the Eckart potential .The energy eigenvalues depend on n and$\tilde{l}$ or $E_{nk} = E\ (n, \tilde{l}(\tilde{l}+1))$,if we follow [62] see the eigenstates for $j = \tilde{l} \pm \frac{1}{2}$ are degenerate to $\tilde{l} \neq 0$. So, the Dirac equation didn't have exactly solution for Eckart potential in $k \neq 0$ by way of standard method .Such we did in previous section in order to obtain an approximation for a term $\frac{k(k+1)}{r^2}$, by replacing the term $\frac{1}{r^2}$ with the approximation $4\alpha^2 \frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}$.

If we substitute equations (5.48) and (5.49) from previous section into equation (5.62) above and make following notations

$$z = e^{-2\alpha r}, \varepsilon = \frac{m^2 - {E_{nk}}^2 + C(m + E_{nk})}{4\alpha^2}, \beta = \frac{V_2\ (m - E_{nk} + C)}{4\alpha^2},$$

$$\gamma(\gamma - 1) = k(k-1) - \frac{V_1}{\alpha^2}(m - E_{nk} + C)$$

then we can rewrite the Dirac equation in (5.62) can by the following way

$$\frac{d^2G_{nk}(z)}{dz^2} + \frac{1}{z}\frac{dG_{nk}}{dz} + \left[-\frac{\varepsilon}{z^2} - \frac{\gamma(\gamma-1)}{z(1-z)^2} - \frac{\beta(1+z)}{z^2(1-z)}\right] G_{nk}(z) = 0 \tag{5.63}$$

by the similar to boundary conditions of the wavefunction in previous section ,we propose the wavefunction $G_{nk}(z)$ as

$$G_{nk}(z) = z^{\sqrt{\varepsilon+\beta}}(1-z)^{\gamma}\ G(z)$$

with similar steps in the previous section to obtain the equation ((5.53)-(5.58)) and get the sufficiently differentiable .We can instantly get the energy eigenvalues for pseudospin case ,as like following formula

$$(m + E_{nk})(m - E_{nk} + C) = \alpha^2 \left[(n+\gamma)^2 + \frac{4\beta^2}{(n+\gamma)^2} \right] \quad (5.64)$$

with $$\gamma = \frac{1}{2} \pm \frac{1}{2} \left[1 + 4k(k-1) + 4\frac{V_1}{\alpha^2}(m + E_{nk} - C) \right]^{1/2}$$

To calculate total radial wavefunction with case of pseudospin , again we indicated in section 5.2.1 and find corresponding wavefunctions with help from equation (4.41),we have

$$G(z) = (-1)^{\mathrm{n}} C_2 \frac{\Gamma\,(n + 2\sqrt{\varepsilon_{nk} + \beta} + 1}{\Gamma\,(2\sqrt{\varepsilon_{nk} + \beta} + 1)} \mathrm{X}\big(-n\,, 2\big(\sqrt{\varepsilon_{nk} + \beta} + \gamma\big) + n, 1 + 2\sqrt{\varepsilon_{nk} + \beta}; z\big)$$

Finally, we write the radial wavefunction as following

$$G_{nk}(z) = N\, z^{\sqrt{\varepsilon_{nk} + \beta}} (1 - z)^{\gamma} \mathrm{X}\big(-n\,, 2\big(\sqrt{\varepsilon_{nk} + \beta} + \gamma\big) + n, 1 + 2\sqrt{\varepsilon_{nk} + \beta}; z\big)$$

5.2 Variable Mass

5.2.1 Manning-Rosen Potential

The Manning–Rosen potential is given as [62,63],

$$V(r) = \frac{1}{\Omega b^2} \left(\frac{\alpha(\alpha-1)e^{-2\,r/b}}{(1 - e^{-\,r/b})^2} - \frac{A\, e^{-\,r/b}}{1 - e^{-\,r/b}} \right)$$

where$\Omega = \frac{2\,m(r)}{\hbar^2}$, A and α are dimensionless parameters ,b has dimension of length. Also, b isscreening parameter for the potential[48].we refer to this potential is remains invariant for $\alpha \leftrightarrow (1-\alpha)$ and it has a minimum value for$\alpha > 1$, as

$$V(r_0) = \frac{A^2}{4b^2\,\alpha(\alpha-1)} \mathrm{at} r_0 = b \ln \left[1 + \frac{2\alpha(\alpha-1)A^2}{A} \right]$$

From Equation (3.44) in section 3.2.2 for variable mass,this equation can't be solve analytically because of the last term in the equation. So, to eliminate this term we use the following condition.

$$\frac{dM(r)}{dr} - \frac{dV(r)}{dr} = 0 \quad (5.65a)$$

By introducing the mass, function as

$$M(r) = \mu_0 + \frac{1}{\Omega\, b^2}\left(\frac{\alpha(\alpha-1)e^{-2\,r/b}}{(1-e^{-r/b})^2} - \frac{Ae^{-r/b}}{1-e^{-r/b}}\right) \tag{5.65b}$$

where μ_0 is the integral constant with relating the rest mass of the Dirac particle.Now by inserting the Manning-Rosen potential from and variable mass from Eq. (5.65b) into Eq.(3.44), and using the condition in Eq.(5.65a) we get

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (\mu_0 + E_{nk}) \left(\mu_0 + E_{nk} + \frac{2}{\Omega b^2}\left(\frac{\alpha(\alpha-1)e^{-2\,r/b}}{(1-e^{-\,r/b})^2} - \frac{A\, e^{-\,r/b}}{1-e^{-\,r/b}}\right) F_{nk}(r)\right)\right] = 0 \tag{5.66}$$

lets using the exponential approximation for the centrifugal term as

$$\frac{1}{r^2} \approx \frac{e^{-\,r/b}}{b^2(1-e^{-\,r/b})^2}$$

where this approximation is valid for large values of the parameter b. By defined $z = e^{-2\,\alpha r}$ and following transformation

$\xi_0 = k(k+1)$, $\xi_1 = b^2(\mu_0{}^2 - E_{nk}{}^2)$, $\xi_2 = \dfrac{2\,(\mu_0 + E_{nk})\alpha(\alpha-1)}{\Omega}$,
$\xi_3 = \dfrac{2\,(\mu_0 + E_{nk})\,A}{\Omega}$

Now, we can rewrite the Eq.(5.66) as

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - \frac{\xi_0}{z(1-z)^2} - \frac{\xi_1}{z^2} + \frac{\xi_2}{(1-z)^2} + \frac{\xi_3}{z(1-z)}\right] S_{nk}(z) = 0 \tag{5.66a}$$

Respect to the boundary conditions for the wavefunction, i.e.

for $r \to \infty$ then $F_{nk}(0) = 0\ \ at\ z = 0$,and for $r \to 0$ then $F_{nk}(1) = 0\ at\ z = 1$

with this conditions we suppose the reasonable wavefunction as following

$$F_{nk}(\mathrm{z}) = \mathrm{z}^{\sqrt{\xi_1}}(1-\mathrm{z})^{\frac{1}{2}(1-\gamma)} F(\mathrm{z}) \tag{5.66b}$$

where $\gamma = \sqrt{1 + 4\xi_0 + 4\xi_2}$.

Lets insert Eq.(5.66b) into Eq.(5.66a) we obtain the second-order homogeneous linear differential equation as

$$\frac{d^2F(z)}{dz^2} = \left[\frac{\left(-1-2\sqrt{\xi_1}\right)+z\left(2+2\sqrt{\xi_1}+\gamma\right)}{z(z-1)}\right]\times \frac{d\,F(z)}{dz} - \left[\frac{1+2\xi_0+\gamma+2\sqrt{\xi_1}(1+\gamma)-2\xi_3}{2\,z(z-1)}\right] F(z) \tag{5.67}$$

By comparing Eq. (5.67) with the aid of the AIM from Eq. (4.28) we find

$$\lambda_0(z) = -\frac{-1+z+2(z-1)\sqrt{\xi_1}+z(3+\gamma)}{z\,(z-1)},\; s_0(z) = \frac{1+2\xi_0+2\sqrt{\xi_1}(1+\gamma)-2\xi_3}{2\,z\,(z-1)}$$

by means of equations ((4.33) and Eq.(4.34)) ,we can calculate sufficiently differentiable of ($\lambda_n(z)$ and $s_n(z)$) as

$$\lambda_1(z) = \frac{4+8(z-1)^2\xi_1+6(z-1)\sqrt{\xi_1}(-2+z(3+\gamma))}{2\,z^2(z-1)^2} +$$

$$z(8-2\xi_0+3(1+\gamma)+2\xi_3)+z^2(4-2\xi_0+5(1+\gamma)+2(1+\gamma)^2+2\xi_3)$$

$$s_1(z) = \frac{(-2+2(z-1)\sqrt{\xi_1}+z(4+\gamma))\,(1+2\xi_0+\gamma+2\sqrt{\xi_1}(1+\gamma)-2\xi_3)}{2\,z^2(z-1)^2}$$

…… etc.

Again with help AIM, combining these results quantization condition given by Eq. (4.37) yields

$$\lambda_1 s_0 - \lambda_0 s_1 = 0 \Rightarrow \xi_1 = \frac{(1+2\xi_0+\gamma-2\xi_3)^2}{4(1+\gamma)^2} \tag{5.68a}$$

$$\lambda_2 s_1 - \lambda_1 s_2 = 0 \Rightarrow \xi_1 = \frac{(2+2\xi_0+3(1+\gamma)-2\xi_3)^2}{4(3+\gamma)^2} \tag{5.68b}$$

$$\lambda_3 s_2 - \lambda_2 s_3 = 0 \Rightarrow \xi_1 = \frac{(8+2\xi_0+5(1+\gamma)-2\xi_3)^2}{4(5+\gamma)^2} \tag{5.68c}$$

………………… etc.

when the Eq.(5.68) are generalized by induction, we can write the eigenvalues of ξ as

$$\xi_{1n} = \frac{1}{4}\left(\frac{2n^2+(1+2n)(1+\gamma)+2\xi_0-2\xi_3}{2n+1+\gamma}\right) \tag{5.69}$$

If $\xi_0$and $\xi_3$are inserted into Eq. (5.69)and is compared with ξ_1, the energy eigenvalue of the for the Manning–Rosen potential position with dependent mass can be obtained by the following expression

$$E_{nk}^2 = \mu_0^2 - \frac{1}{4b^2}$$

$$\left[\frac{2n^2+(1+2n)\left(1+\sqrt{1+4k(k+1)+\frac{8(m_0+E_n)\,\alpha(\alpha-1)}{\Omega}}+4k(k+1)\right)-\frac{4(m_0+E_n)A}{\Omega}}{2n+1+\sqrt{1+4k(k+1)+\frac{8(m_0+E_n)\,\alpha(\alpha-1)}{\Omega}}}\right]^2$$

(5.70)

5.2.1.1 Eigenfunctions

By using the wave function generator given in the section 4.2, from using the exponential part in Eq. (4.41)$y(x)=C_2\mathrm{Exp}(-\int\alpha\,dt)$, the eigenfunctions can be obtained as following

$S_0(z)=-1$

$$S_1(z)=-\,C_2\ (1+2\sqrt{\xi_1})\left(1-\frac{2+2\sqrt{\xi_1}+\gamma}{1+2\sqrt{\xi_1}}\ \mathrm{z}\right)$$

$$S_2(z)=\ C_2\ (1+2\sqrt{\xi_1})(2+2\sqrt{\xi_1})\left(1-\frac{2(3+2\gamma\sqrt{\xi_1})}{1+2\sqrt{\xi_1}}\ \mathrm{z}+\frac{2(3+2\gamma\sqrt{\xi_1})(4+2\gamma\sqrt{\xi_1})}{(1+2\sqrt{\xi_1})(2+2\sqrt{\xi_1})}z^2\right)$$

$$S_3(z)=\ C_2\ (1+2\sqrt{\xi_1})(2+2\sqrt{\xi_1})(3+2\sqrt{\xi_1})$$

$$\left(1-\frac{6(2+\gamma\sqrt{\xi_1})}{1+2\sqrt{\xi_1}}\mathrm{z}+\frac{(2+2\gamma\sqrt{\xi_1})(5+2\gamma\sqrt{\xi_1})}{(1+2\sqrt{\xi_1})(1+\sqrt{\xi_1})}z^2-\frac{(4+2\gamma\sqrt{\xi_1})(5+2\gamma\sqrt{\xi_1})(3+2\gamma\sqrt{\xi_1})}{(1+2\sqrt{\xi_1})(1+\sqrt{\xi_1})(3+2\gamma\sqrt{\xi_1})}z^3\right)$$

…..etc.

Therefore, the wave function $F(z)$ in general form can be written as

$$F(z)=(-1)^{\mathrm{n}}C_2\ \frac{\Gamma\,(1+2\sqrt{\xi_1}+n}{\Gamma\,(1+2\sqrt{\xi_1})}\ \mathrm{X}(-n\,,1+2\sqrt{\xi_1}+\gamma+n,1+2\sqrt{\xi_1}\,;z)$$

From Eq. (5.66b) the total radial wavefunction can be given as follow

$$F_{nk}(z)=N\,z^{\sqrt{\xi_1}}(1-z)^{\frac{1}{2}(1+\gamma)}\ \mathrm{X}(-n\,,1+2\sqrt{\xi_1}+\gamma+n,1+2\sqrt{\xi_1}\,;z)$$

5.2.2 Dirac Coulomb Potential

By considering the coulomb like potential as

$$V(r)=\frac{V_0}{r},\text{and } S(r)=\frac{S_0}{r}. \qquad (5.71)$$

Form section 3.2.2,the Eq. (3.44) can’t be solved analytically because of the last term in the equation$\left(\frac{dM(r)}{dr}-\frac{dV(r)}{dr}=0\right)$, so, we use the equality to eliminate this term. Thus, using this equality condition, the mass function is obtained as the following

$$M(r) = \frac{(V_0 - S_0)}{r} + m_0 \tag{5.72}$$

By inserting the coulomb potential from Eq.(5.71) and variable mass into Eq.(3.44) we get

$$-\frac{k(k+1)}{r^2}F_{nk}(r) - (E_{nl} + m_0)\left(-E_{nl} + \frac{2(V_0 - S_0)}{r} + m_0\right)F_{nk}(r) + \frac{d^2}{dr^2}F_{nk}(r) = 0 \tag{5.73}$$

By making the following notations

$$E_{nl}^2 - m_0^2 = \varepsilon_{nl}^2 \;,\; k(k+1) = A(A+1),\; 2(E_{nl} + \mathrm{m}_0)(S_0 - V_0) = B$$

The eigenvalues equation transforms to

$$\left(-\varepsilon_{nl}^2 - \frac{A(A+1)}{r^2} + \frac{B}{r}\right)F_{nk}(r) + \frac{d^2}{dr^2}F_{nk}(r) = 0 \tag{5.74}$$

Propose to the wavefunction by using AIM as

$$F_{nk}(r) = r^{A+1}e^{(-\varepsilon_{nl}r)}\chi(r)$$

by inserting the wavefuction above into Eq.(5.73) we obtain

$$(B - 2(1+A)\varepsilon_{nl})\chi[r] + 2(1 + A - \varepsilon_{nl}r)\chi'[r] + r\chi''[r] = 0 \tag{5.75}$$

after rearrange Eq.(5.75) we have

$$\chi''[r] = -\frac{2(1 + A - \varepsilon_{nl}r)}{r}\chi'[r] + \frac{-B + 2(1+A)\varepsilon_{nl}}{r}\chi[r] \tag{5.76}$$

By comparing Eq.(5.76) with second-order diffraction equation Eq.(4.28) by mean of AIM , we obtain

$$\lambda_0(\mathrm{r}) = -\frac{2(1 + A - \varepsilon_{nl}r)}{r}, \qquad \mathrm{s}_0(\mathrm{r}) = \frac{-B + 2(1+A)\varepsilon_{nl}}{r}$$

By using termination condition for energy, we get the general form of eigenvalues is

$$\varepsilon_{nl} = \frac{B}{2(n + 1 + A)}$$

Go back to the parameters definition, we get

$$E_{nl}^2 - m_0^2 = \varepsilon_{nl}^2 = \left(\frac{B}{2(n + 1 + A)}\right)^2$$

which yields

$$E_{nl}^2 = m_0^2 + \left(\frac{B}{2(n + 1 + A)}\right)^2$$

In order to find corresponding the energy eigenfunctions with AIM, we may use the following energy eigenfunction generator for

$$\chi(r) = \exp\left(-\int^{r} \frac{s_k(r)}{\lambda_k(r)} dr\right)$$

By applying the function generator, the $f_n(r)$ functions can be written in series expansion by hypergeometric functions with constant $(B+n+1)^n$ and $\prod_{k=0}^{(n-1)}(B+2+k)$. Generalizing these expansions, we get

$$\chi(r) = (B+n+1)^n \left[\prod_{0}^{n-1}(2B+2+k)\right] x\ {}_1F_1(-n, 2B+2; 2\varepsilon_{nl} r)$$

Thus, we can write the upper spinor component of the radial wave function as

$$F_{nk}(r) = r^{A+1} e^{(-\varepsilon_{nl} r)} (B+n+1)^n \left[\prod_{0}^{n-1}(2B+2+k)\right] x\ {}_1F_1(-n, 2B+2; 2\varepsilon_{nl} r)$$

The lower spinor wave function can be obtained in a similar algebraic calculation. It gives the same results as

$$G_{nk}(r) = r^{A+1} e^{(-\varepsilon_{nl} r)} (B+n+1)^n \left[\prod_{0}^{n-1}(2B+2+k)\right] x\ {}_1F_1(-n, 2B+2; 2\varepsilon_{nl} r)$$

where

$$E_{nl}^2 - m_0^2 = -\varepsilon_{nl}^2\ ,\ \ k(k-1) + 4S_0(S_0+V_0) = A(A+1),$$

$$-2E_{nl}V_0 + 2\mathrm{m}_0)(2S_0+V_0) = B$$

are the corresponding parameters gives the results with those of in [34] after transforming the parameters.

5.2.3 Dirac Harmonic Oscillator

By considering the Harmonic Oscillator potential as

$$V(r) = V_0 r^2 \tag{5.77}$$

By following similar steps in section 5.2.2, again by using this equality condition, the mass function is obtained as the following

$$M(r) = m_0(1 + V_0 r^2) \tag{5.78}$$

By inserting the Harmonic Oscillator potential from Eq. (5.77) and variable mass into Eq.(3.44) we get

$$-\frac{k(k+1)}{r^2}F_{nk}(r)-(E_n+m_0)(-E_n+2\,m_0V_0r^2+m_0)F_{nk}(r) + \frac{d^2}{dr^2}F_{nk}(r)=0 \tag{5.79}$$

By making the following notations

$E_n^2-m_0^2=\varepsilon_n^2$, $k(k+1)\to A(A+1)$, $2\varepsilon_n m_0 V_0+2V_0 m_0^2\to B^2$

The equation Eq.(5.79) becomes

$$\left(-\varepsilon_n-\frac{A(A+1)}{r^2}+B^2r^2\right)F_{nk}(r)-\frac{d^2}{dr^2}F_{nk}(r)=0 \tag{5.80}$$

Propose to the wavefunction by using AIM as

$$F_{nk}(r)=r^{A+1}e^{\left(\frac{-Br^2}{2}\right)}\chi(r)$$

by substituting the wavefunction above into Eq.(5.80) we obtain

$$\frac{d^2}{dr^2}\chi(r)=\frac{3Br\chi(r)+2ABr\chi(r)-\varepsilon_n r\chi(r)}{y}+\frac{(-2-2A+2Br^2)}{r}\frac{d}{dr}\chi(r) \tag{5.81}$$

By comparing Eq.(5.81) with second-order diffraction equation Eq.(4.28) by mean of AIM , we obtain

$$\lambda_0(r)=-\frac{2(1+A-Br^2)}{r},\quad s_0(r)=(3+2A)B-\varepsilon_n$$

at this step, by using termination condition for energy given by Eq. (4.37),we get the following results

$$\varepsilon_0=(3+2A)B$$

$$\varepsilon_1=(7+2A)B$$

$$\varepsilon_2=(11+2A)B$$

Generalizing for energy ε_n , it can be found as

$$\varepsilon_n=(3+4n+2A)B.$$

If go back to the parameters A, and B, the ε_n relation transforms to

$$E_n^2-m_0^2=\big((3+4n+2A)B\big)^2$$

and

$$E_n = \sqrt{m_0^2 + \left((3 + 4n + 2A)B\right)^2}$$

The corresponding eigenfunction is constructed by using wave function generator as [65]

$$\chi(r) = (-1)^n C_2 2^n (\sigma)_n x\ {}_1F_1(-n, \sigma; Br^2)$$

Thus, we can write the upper spinor as

$$F_{nk}(r) = r^{A+1} e^{\left(\frac{-Br^2}{2}\right)} (-1)^n C_2 2^n (\sigma)_n x\ {}_1F_1(-n, \sigma; Br^2)$$

where

$$\sigma = \frac{2A + 3}{2} \text{and} (\sigma)_n = \frac{\Gamma(\sigma + n)}{\Gamma(\sigma)}$$

CHAPTER 6

6 CONCLUSION

In this thesis, the spectrum of the relativistic Dirac equation is obtained for some physical potentials by using AIM method.

Through the corresponding calculations, the mass of the relativistic particle is considered as positron dependent mass case and constant mass case. In spatially dependent case, the mass function is considered as in the form of function satisfying the equality condition, $\frac{\boldsymbol{dM(r)}}{\boldsymbol{dr}} - \frac{\boldsymbol{dV(r)}}{\boldsymbol{dr}} = \mathbf{0}$.

By substituting the mass function in the differential equation of S(r) and Q(r) ,the AIM is applied and the calculated results satisfy the exact results .This method has an advantages with respect to other methods :These are ;

i) Without using complex calculations, it gives the simple way by obtaining $\lambda(y)$ and $s(x)$.

ii) It reproduce the corresponding the energy eigenvalues and eigenfunction in a good accuracy.

In this thesis, the spectrum of Coulomb and harmonic Potentials in Dirac equation are calculated for constant mass cases in addition to the applications for constant mass.

It is possible to applied this method for first order and second order Dirac equation by choosing appropriate asymptotic wavefunction form.

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