JANUARY 2014 M. Sc. THESIS IN PHYSICS YUSUF YETER

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SCATTERING AND BOUND STATES OF DIRAC PARTICLES IN AN EXTERNAL FIELD

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YUSUF YETER

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Scattering and Bound States of Dirac Particles in an External Field

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Supervisor

Prof. Dr. Ramazan KOÇ

By

Yusuf YETER

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Yusuf Yeter

ABSTRACT

SCATTERING AND BOUND STATES OF DIRAC PARTICLES IN AN EXTERNAL FIELD

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M. Sc. In Engineering Physics

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In order to understand the nature of basic particles, the Dirac equation has been studied for many years. The Dirac equation plays an important role in well known modelling physical problem. Additionally, to describe properties of the spin $\frac{1}{2}$ particles, the Dirac equation is a useful one, so one can consider the solution of Dirac equation is important in physics. Furthermore, the study of Dirac equation including various potentials like harmonic oscillator potential, Coulomb potential, Rosen-Morse potential etc. has recently attracted interest on physical systems.

In this thesis, in the presence of exactly solvable potential and an external magnetic field, the Dirac equation has been studied for scattering and bound states. Because of that this work consists of three parts. In the first part the Dirac equation is generally solved for free particles and existing various potentials. The scattering of a Dirac particle is studied in the external magnetic field in the second part. As in the last part, the bound state of Dirac particle is investigated. The ultimate goal of the study in this thesis is defining the scattering and bound states of Dirac particles in an external field.

Key words: Dirac equation, exactly solvable potentials, scattering and bound states

ÖZ

DIRAC PARÇACIKLARININ BİR DIŞ ALANDA SAÇILMA VE BAĞLI DURUMLARI

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Temel parçacıkların doğasını anlamak için yıllardır Dirac denklemi çalışılmaktadır ve bu denklem iyi bilinen fizik problemlerini modellemek için önemli bir rol oynar. Ayrıca spini $\frac{1}{2}$ olan parçacıkların özelliklerini açıklamak için Dirac denklemi faydalı olan bir denklemdir. Bundan dolayı Dirac denkleminin çözümünün fiziksel problemlerde önemli bir yere sahip olduğu düşünülür.

Başka bir deyişle harmonik osilatör potansiyeli, Coulomb potansiyeli, Rosen-Morse potensiyali gibi potansiyeller içeren Dirac denkleminin çalışmaları son zamanlarda fiziksel sistemlerin çözümü üzerine ilgi çekmektedir.

Bu tezde bazı tam çözülebilen potansiyeller ve bir dış manyetik alan içeren Dirac denkleminin çözümü Dirac parçacıklarının saçılma ve bağlı durumları çalışıldı. Bundan dolayı bu çalışma üç bölümden oluşmaktadır. Birinci bölümde genel olarak Dirac denklemi serbest parçacık ve çeşitli potansiyellerin varlığında çözüldü. Ikinc bölümde bir Dirac parçacığının bir dış manyetik alanda saçılma durumu çalışıldı. Son bölüm ise bir Dirac parçacığının bir dış alandaki bağlı durumu araştırıldı. Bu tezin son amacı ise Dirac parçacıklarının bir dış alandaki saçılma ve bağlı durumlarının tayını yapıldı.

Anahtar kelimler: Dirac denklemi, tam çözülebilir potansiyeller, saçılma ve bağlı durumlar.

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LIST OF SYMBOLS

[.,.]	Commutator
$\sigma_{\pm}$	Pauli martices
m	Mass of particle
E	Energy of particle
$\hbar$	Planck constant
c	Speed of light
p	Momentum operator
A	Vector potential
B	Magnetic field
α, β	The 4x4 Dirac matrices
W(r)	Super potential
$V_{\pm}$	Partner potential
ω_L	Larmor frequency
ψ, φ, ϕ	Wave functions
β_{γ_i}	Pauli spin matrices
Ω	Hermitian operator
ω_n	Eigenvalue
q_i, s_i	Coordinates
$p^{\mu}, p_{\mu}, \partial_{\mu}$	Four covariant vectors
ψ^*	Complex conjugate
ρ	Probability density
N,c,k_i	Normalization constants
MR	Manning-Rosen potential
RM	Rosen-Morse potential
PT	Poschl-Teller potential
κ	Quantum number
H	Relativistic hamiltonian
$\upsilon(r)$	Superpotential
$A^{\pm}$	Supersymmetric operator

ω_T	Frequency
$L_n{}^{\pm}$	Laguerre polynomials
$P_n^{(\mu,\nu)}$	Jacobi polynomials
NU	Nikiforov-Uvarov method
Y_m^l	Spherical harmonic function
F(a,b,c,z)	Hypergeometric functions
$f(\theta)$	Scattering amplitude
$\sigma(\theta)$	Cross section
τ	Parameter
$l, \tilde{l}$	Angular quantum number
S	Scalar potential
V	Repulsive vector potential
Σ	Sum of potential
I	2X2 unit matrx
τ	Parameter of potential
$C_\Delta, \Delta, C_\Sigma$	Constants

CHAPTER 1

INTRODUCTION

In this thesis, we have studied the Dirac equation for behavior of elementary particles. We know that the Dirac equation is an relativistic quantum equation and this equation is consistent with the results of the classical mechanics. Moreover, in order to describe spin $-\frac{1}{2}$ particles, the Dirac equation is a useful equation. If we are supposed to give a few examples, they are electron, positron, neutrions and the like. Furthermore, to describe the spins of these particles we need to the principles of quantum mechanics and the theory of special relativity [1]. Also, one can explain the probability amplitude for a single particle by using the Dirac equation with the theory of special relativity and the principles of quantum mechanics. Many of the fine structures in atomic lines are explained by using this single particle theory [2]. Also, some features of electrons like the spin and the magnetic moment are given by that theory [3]. Additionally, the peculiar prediction is taken by the theory of an infinite set of quantum states, so the electron may possess negative energy [1]. The Dirac equation firstly was considered for describtion of electrons, but it can be applied to the other elementary particles too. Also, describtion of protons and neutrons can be explained by a modified Dirac equation. We know that these particles are not elementary particles that are made of smaller particles called quarks. Now, we should recall some features of relativistic quantum mechanics.

Non-relativistic principles should be agreeable with the relativistic quantum theory. These principles [2]: We can define a wave function ψ for every physical system including all information about the system. For instance, with a wave function ψ a particle's motion is defined by coordinates $(q_1 \ldots q_n, s_1 \ldots s_n)$ at any time t; the wave function of particle has a physical meaning only by choice $|\psi(q_1 \ldots q_n, s_1 \ldots s_n)|^2 \geq 0$ [4]. This explains the particle's probability. For all acceptable wave functions the sum of positive contributions $|\psi|^2$ given for all values of coordinates (q1…qn, s1…sn) must be defined as finite values [5].

An observable physical system is represented by a linear Hamiltonian operator. For example, ($\hat{p}_i \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial q_i}$) represents the momentum operator. When we apply this operator to a physical system, we can obtain the eigenvalues (ω_n) of the system. Because of the hermitian operator, the eigenvalues are taken as real values [1].

$$\Omega\psi_n = \omega_n\psi_n \tag{1.1}$$

The functions (ψ_n) in the left side of the equation (1.1) represent the eigenstates of the physical system which can be obtained after some measurements. A state functions of a physical system can be written in terms of a complete orthonormal set of eigenfunctions ψ_n as [4]

$$\psi_n = \sum a_n\psi_n \tag{1.2}$$

and $\sum\int(dq_1 \dots)\,\psi_n^*(\mathrm{q}\dots,\mathrm{s}\dots,\mathrm{t})\,\psi_m(\mathrm{q}\dots,\mathrm{s}\dots,\mathrm{t}) = \delta_{nm}$ is the explanation of orthonormality [5]. Also, we can write $\psi = |\psi\rangle$ and $\psi^+ = \langle\psi|$, and than; $\psi\psi^+ = \langle\psi|\psi\rangle = \langle a_n\psi|a_m\psi\rangle = a_n a_m\delta_{nm}|a_n|^2$ [1]. In order to understand the physical systems, probability is another feature. Any of eigenvalue ω_n is measured by a physical observable Ω with a probability $|a_n|^2$. The value of probablity of physical observability is measured by $\langle\hat{\Omega}\rangle$ [4].

$$\langle\hat{\Omega}\rangle_\psi = \sum\int\psi^*(q_1 \dots, s_1 \dots, t)\hat{\Omega}\ \psi(q_1 \dots, s_1 \dots, t)(dq_1 \dots) = \sum|a_n|^2\,\omega_n. \tag{1.3}$$

One of the best equation is the Schrödinger equation in order to explain the features of the physical systems or particles. The Schrödinger equation explains a physical system depending on the time. That equation is written as [1]

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\,\psi \tag{1.4}$$

Here $\hat{H}$ is a linear hermitian operator and it has suspense time dependence. If we consider a closed physical system, the hamiltonian in that system depends on the time ($\frac{\partial\hat{H}}{\partial t}$ = 0). Now, by using the properties of conservation and linearity, we can conclude conservation of probability and linearity of the hamiltonian that form [5];

$$\frac{d}{dt}\sum\int\psi^*\psi(dq_1 \dots) = \frac{i}{\hbar}\sum\int(dq_1 \dots)[(\hat{H}\,\psi)\psi^* - \psi^*(\hat{H}\,\psi)] = 0 \tag{1.5}$$

We can obtain the relativistic Dirac equation in terms of these principles of non-relativistic theory.

1.1 Formulation of Relativistic Quantum Theory

We try to obtain the Schrödinger equation in terms of relativistic principles. A non-relativistic hamiltonian for an isolated free particle is written as [4]

$$\hat{H} = \frac{P^2}{2m} \tag{1.6}$$

and we use that hamiltonian as an operator in quantum mechanic as follows

$$\hat{H} = i\hbar\frac{\partial}{\partial t}, \;\; \hat{P} = \frac{\hbar}{i}\nabla \tag{1.7}$$

By using these operators the non-relativistic Schrödinger equation takes form [5]

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = -\frac{\hbar^2\nabla^2}{2m}\,\partial\psi(q,t) \tag{1.8}$$

Also, the momentum operator (p_x, p_y, p_z) and the total energy E are written in terms of components of a contravariant four vector in the special relativistic quantum mechanics as follows [6];

$$p^\mu = (p^0, p^x, p^y, p^z) = (\frac{E}{c}, p_x, p_y, p_z) \tag{1.9}$$

Invariant length of four momentum [6] is

$$\sum_\mu p_\mu p^\mu = \frac{E^2}{c^2} - \hat{P}^2 = m^2c^2 \tag{1.10}$$

From here the relativistic hamiltonian is concluded as [6]

$$\hat{H} = \sqrt{p^2c^2 + m^2c^4} \tag{1.11}$$

Now we can write the relativistic form of the Schrödinger equation in terms of this hamiltonian as [6]

$$i\hbar\frac{\partial\psi}{\partial t} = \sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}\;\psi \tag{1.12}$$

The right-side of the equation (1.12) represents an operator [4] which is expanded as follow,

$$\sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4} = \mathrm{m}c^2 (1 - \frac{1}{2} \frac{\hbar^2 \nabla^2}{m^2 c^2} + \cdots) \tag{1.13}$$

An equation is obtained for all powers of the derivative operators $\hat{\nabla}$ and this equation can not be solved to handle which is an nonlocal equation, so if we take the square $\hat{H}$ in equation (1.7), we find an equation which is relativistic form of the Schrödinger equation [6].

$$(i\hbar \frac{\partial}{\partial t})\,(i\hbar \frac{\partial}{\partial t}) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4)\,\psi \tag{1.14}$$

$$[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + (\frac{mc}{\hbar})^2]\,\psi = 0 \tag{1.15}$$

This equation is named Klein-Gordon equation [7,8]. $\hat{H} = \pm\sqrt{p^2 c^2 + m^2 c^4}$ presents a negative energy root appearing with a minus sign which gives the negative energy solutions of antiparticles. Now, we try to find a conserved current from the Klain-Gordon equation. Firstly, multiplying left side of equation (1.15) by ψ^*; secondly, taking hermitian conjugate of this equation and then multiplying this from left-side by ψ , finally, we can obtain the equation after some calculations [6];

$$\partial^\mu [\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^*] = 0 \tag{1.16}$$

Or in explict form

$$\frac{\partial}{\partial t} [\frac{i\hbar}{2mc^2} (\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t})] + \hat{\nabla} \frac{\hbar}{2im} [\psi^* \hat{\nabla} \psi - \psi \hat{\nabla} \psi^*] = 0 \tag{1.17}$$

Obviously, the term in the first parentheses in equation (1.17) looks like the probability density ρ. But, it is not a definite expression. On the other hand, equation (1.9) gives negative energy solutions in physical interpretation [6].

Furthermore, when we study in quantum mechanics, we use wave functions which represent a special function space in mathematics. This space is also an infinitive-dimensional linear vector space. Now we try to define the space as three-dimension as follow; Firstly, a wave function (a vector) is defined with each given $\vec{r}$ position in the space. Because each position $\vec{r}$ takes infinitive values, the wave function is dimensionless [9].

$$\{u_i\} \rightarrow \psi(\vec{r}) \quad (\mathrm{i} = 1,2,3\ldots) \tag{1.18}$$

Secondly, it can be defined scalar product for wave functions $\psi(\hat{r})$, $\phi(\hat{r})$ as

$$c_i = \int d^3r\psi^*(\vec{r})\,\phi(\vec{r}) \tag{1.19}$$

here ψ^* is complex conjugate. Also ;

$$\|\psi\|^2 = (\psi, \psi) = \int d^3r\psi^*(\vec{r})\,\psi(\vec{r}) \tag{1.20}$$

represents norm of wave functions [9].

Finally, we consider presence of a function group $\{u_i(\vec{r})\}$ in the wave functions space (i=1,2,3,…N), where the numbers of N can take limited or unlimited values. The function group $\{u_i(\vec{r})\}$ is an orthonormal group [9], when multiplying member of these functions take that forms;

$$(u_i, u_j) = \int d^3ru_i{}^*(\vec{r})\,u_j(\vec{r}) = \begin{cases}1, & i=j\\ 0, & i\neq j\end{cases} \tag{1.21}$$

Also, we can define each wave function $\psi(\vec{r})$ in this wave functions space in terms of member of the space as;

$$\psi(\vec{r}) = \sum_i^N c_i\,u_i(\vec{r}) \tag{1.22}$$

the group of $\{u_i(\vec{r})\}$ creates an orthonormal phase in the wave function space. Here, the functions $\{u_i\}$ and constants of c_i represent the phase vectors and componenets of wave function ψ, respectively. The components are defined as

$$c_i = (u_i, \psi) = \int d^3ru_i{}^*(\vec{r})\,\psi(\vec{r}) \qquad \text{(i=1,2,3…)}$$

Now one can define a wave function completely by selecting phase $\{u_i\}$ and given components of c_i. We know that the space which has these features is called the Hilbert space in mathematics [9]. It is easy to present the three-dimensional complex number problem, definition of spin and the behavior of particles in quantum mechanics in this space. As a result, the solution of wave functions in the quantum mechanics is obtained without the complex integrates, complex derivatives and their solutions are easy to find by defining algebraic constants c_i, [9] written as

$$(\psi, \varphi) = (\sum_i c_i u_{i,} \sum_j d_j u_j) = \sum_i \sum_j c_i^* d_j \delta_{ij} = \sum_i c_i^* d_i \tag{1.23}$$

And explanation of norm can be written as [9]

$$(\psi,\psi) = \sum_i |c_i|^2 \tag{1.24}$$

Additionally, in this thesis we would like to try to solve the (2+1) Dirac equation for a free particle whose mass is m with the Coulomb scalar potential. We can write the (2+1) Dirac equation for a free particle in terms of two-component spinors ψ as [3]

$$\mathrm{E}\,\psi = [\ \sum_{n=1}^{2} \mathrm{c}\beta\gamma_i p_i + \beta mc^2]\,\psi \tag{1.25}$$

Furthermore, we can see solution of Dirac equation including various solvable potentials in this study. These potentials are harmonic oscillator potential, Coulomb potential and Morse potential [10]. We can define the Dirac equation in the presence appropriate potentials to analyze relativistic effect on the spectrum of such physical systems and to obtain the solution of the Dirac equation including these potentials. Two dimensional Dirac oscillator can be used in order to determine the spectrum and properties of such systems in the relativistic physics. In this study, we see only exactly solvable potentials, relativistic extensions of these potentials have also turned out to be important in the description of (2+1)-dimensional phonemena. Spectrum of different condensed matter physics phonemaena existence (2+1) dimensional is determine by the Dirac equation including various potentials. We know that the Dirac equation is important to understand the relativistic particles that have spin-1/2. Meantime, we can state that the Hamiltonian obtained from the Klein-Gordon equation, called Feshbach-Villars equation [11], has been builded by two different states. One of them is spinless particles. The Hamiltonian has been constructed in a two-component for this particles and other one has been constructed in a eight-component for spin-1/2 particles. Unfortunately, the Dirac equation is not exactly solvable in all species of potentials, it is exactly solvable only in a very limited potentials. In this chapter we propose the solution of the Dirac equation including a class of potential whose spectrum can accurately be determined. The Dirac equation is transformed into the Schrödinger-like equation for this purpose. For example, physical systems which have few electrons, stuck on the problem in the building of the Schrödinger-like equations. A Dirac equation was considered previously with an interaction linear in coordinates, and recently, it is reinvented in the case of the relativistic theories. If we add a very strong spin-orbit coupling term into the Dirac oscillator, the equation becomes a harmonic oscillator in the non-relativistic limit. Dirac oscillator has attracted much attention. Analogous to the

Dirac equation, with a modified momentum operator, in the non-relativistic limit turns out to the usual Schrödinger equation. As we have already noted the Dirac equation including various potentials might attract much attention because it may have some physical applications, particularly in the condensed matter physics. It seems that one can present more realistic modals for the artificial atoms using the procedure given here.

In this study, we will see another section which is scattering of Dirac particles by an external potential field. Here, for the solution of Dirac particles, we will use a mixed potential including the Coulomb scalar potential. The exact solution of the Dirac equation with the coulomb potential plays a very important role in relativistic and non-relativistic quantum mechanics. For instance, for a harmonic oscillator in three-dimension and a hydrogen atom the exact solution of the Schrödinger equation is important for the beginning stage of quantum mechanics and solutions provided a strong evidence for quantum theory. In order to understand a given quantum system, the studying of the bound states then that of the scattering states can be paid more attention. To understand the quantum systems completely, we can study both bound states and scattering states. Because of establishment of scattering and bound states particles in quantum mechanics, the applying of the Schrödinger equation to quantum systems has been known well. For example, the bouth systems of quantum mechanics and classical mechanic can be understood well by studying the scattering of particles by the Coulomb potential field. However, when we consider the relativistic effect on a quantum system, the Dirac equation has to be employed to study the electrons scattering by a nucleus potential. Also, one can carry out the exact solution of the Dirac equation including Coulomb plus scalar potential in different dimensions which are two-dimension, three-dimension and higher-dimensions. One can study scattering phase shifts in the two-dimension Dirac equation including the Coulomb potential given by the second order differential equation and by algebraic method, for the condensed matter physics and interest of the lower-dimensional quantum field theory. Our aim in this chapter is to study the scattering of the two-dimensional Dirac equation including the Coulomb plus scalar potential. Finally, in the last section the bound states of Dirac particles have been studied. In this chapter we will study the bound states solutions of Dirac particles under an external potential field. Let's consider a particle which is exposed to a strong potential field. For this particle the relativistic effects must be considered. It should be noted that when a spinless particle is exposed to an external potential field, the relativistic effects are expressed with the Klein-Gordon equation [6]. For some fields of physics like nuclear and atomic and molecular physics, the solution of Dirac equation is also important [12,13]. We consider the Dirac equation in terms of scalar potential S and the repulsive vector potential V. If the potentials are nearly equal such $S \sim V$ in the nuclei (as an arbitrary constant C_{Δ}), the potentials take that relation; $\Delta = V - c = C_{-} = C_{\Delta}$) [14]. The spin symmetry arises within the framework of the Dirac equation. However, the pseudospin (pspin) symmetry occurs if $S \sim -V$ are nearly equal (as an arbitrary constant C_{Δ}, the sum potential $\sum = V + S = C_{+} = C_{\sum}$) [12,13]. The spin symmetry is available for mesons [15]. Recently, many authors have extensively applied the spin and pspin symmetries on various physical potentials [16,17]. For example, we can see solution of Dirac equation for some potentials which are deformed generalized Poschl-Theller (PT) potential [18], well potential [19], Manning-Rosen potential (MR) [20], modified PT potential [21,22], modified Rosen-Morse (RM) potential [23] and class of potentials including harmonic oscillator, Hulthen, trigonometric RM potential, Scarf Eckert, Morse, MR, and others [24]. By using the properties of quantization rules, algebraic methods, we can obtain the solution of Dirac equation including these potentials in the framework of the spin-orbit centrifugal term of the approximation to the spin-orbit. If we want to obtain the exact solutions of the Dirac equation for the exponential-type potentials, we should choose the s-wave ($\kappa = \pm 1$ case) [25]. The Dirac equation has exact solution for these potentials only in the case of the wave. However, for the spin-orbit and pseudospin centrifugal, $\frac{\kappa(\kappa+1)}{r^2}$ and $\frac{\kappa(\kappa-1)}{r^2}$), respectively, terms (1.1) state an approximation which can be used to deal with that spin-orbit and psedospin-orbit values [25]. To this end, the Dirac equation including large number of potential has been solved to obtain the two-component spinor wave functions and to obtain the energy by many works. The values of the energy spectra do not depend on the structure of the particle, but they depend on the spin $-\frac{1}{2}$ or spin-0 particle [25]. Also, by choosing spin $-\frac{1}{2}$ or spin-0 particle which have the same mass and depend on potential of equal magnitude ($S = \pm V$ ($\Delta = \sum = 0$ or $C_{\pm} = 0$) [12-14] which are scalar potential S and vector potential V, the spectrum of energy (isospectrality) including bound and scattering states will be the same. When we solve the Dirac equation with the harmonic oscillator potential for the massless particles (or the ultra-relativistic particles), we see that the spin and pspin spectrum of that particles are the same [10]. For example, the spin symmetric and pspin bound state solutions of Dirac equation including the standard Rosen-Morse well potential model [23] can be obtained as

$$V(r) = -V_1\, sech^2 \alpha r\, V_2\, tanh \alpha r$$

Where V_1 and V_2 constants represent the depth of the potential and α is arranged the potential. Our aim in the next paper is to extend the s-wave solution by using the Dirac equation including some physical potential and using the Nikiforov-Uvarov method [26,27] adding an approximation with the presence of centrifugal (pseudo-centrifugal) potential term. The spin-orbit centrifugal barrier $\frac{\kappa(\kappa-1)}{r^2}$ is arranged for κ which is the value of spin-orbit coupling quantum number with the using of approximation scheme [25]. Where the quantum numbers were given is not large and vibrations of the small amplitude. By using the definition of spin symmetry $S{\sim}V$ and pspin symmetry $S \sim -V$ the bound state energy eigenvalues and their corresponding upper and lower spinor wave functions can be calculated [12]. Also, the spin and pspin symmetric Dirac solutions can be shown, when pspin symmetry limitation is chosen $\Delta = 0$ and $\sum = 0$, respectively and for spin symmetry $\Delta = C_-$ and $\sum = C_+$ [13] can be reduced to the exact spin symmetry. Furthermore, if we use suitable matching of parameters, by using the non-relativistic limit of the Dirac equation, the bound state solutions of Schrödinger equation can be obtained. In the following sections, we mainly will define the basic spin and pspin Dirac equation. Then, the (2+1)-dimensional Dirac equation including the Rosen-Morse potential and reflectionless-type potentials that will be approached for analytical bound state solution [25]. Also, we use a parametric generalization of the Nikiforov-Uvarov method [26,27], in order to obtain these potentials that are also obtained in the presence of the spin and pspin limits. After that, we will study non-relativistic limit and special case of the s-wave $\kappa = \pm 1 (l = \tilde{l} = 0)$. Finally, the suitable conclusion will be given.

CHAPTER 2

THE DIRAC EQUATION

After two results of the Klein-Gordon equation that are negative energy states and negative probability density, Dirac tried to explain these unphysical cases, so In 1928, Dirac found a relativistic equation. The equation includes the form of time dependence in the Schrödinger equation with positive definite probability density. Due to the linearity such a formation in the time-derivative, it is essential to solve an equation of linear in space derivatives. This equation can be written as [3]

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \left[\alpha_1 \frac{\partial \psi}{\partial x^1} + \alpha_2 \frac{\partial \psi}{\partial x^2} + \alpha_3 \frac{\partial \psi}{\partial x^3} \right] + \beta m c^2 = \hat{H}\psi \qquad (2.1)$$

This equation has some properties that are;

1- The coefficient α_i should be matrices from the invariance of spatial rotations.

2- The wave function ψ is represented N-dimensional spinors in general that is not a simple scalar.

3- The probability density $\rho = \psi^{*}\psi$ should be the time component of a conserved four-vector, if ρ is integrated over all space, it must be an invariant. Dirac put forwarded that the equation (2.1) is a matrix equation. Also, he thought that the wave function ψ must be coressponded to spin wave functions of non-relativistic quantum mechanics. The spin wave function ψ [4] is

$$\psi = \begin{pmatrix} \psi_1 \\ \cdot \\ \cdot \\ \cdot \\ \psi_N \end{pmatrix} \qquad (2.2)$$

In equation (2.1) α_i and β are NxN matrices, so the N-coupled first order equations can be written as in the following [5]

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\sum_{j=1}^{N}\left(\alpha_j\frac{\partial\psi}{\partial x^j} + \beta mc^2\right)\psi = \sum_{j=1}^{N}\widehat{H}\,\psi_j \tag{2.3}$$

This N-coupled first order equation must be provided the correct relation of energy-momentum for a free particle as [3]

$$E'^2 - p'^2c^2 = E^2 - p^2c^2 \tag{2.4}$$

Moreover, Lorentz transformation can not be allowed to a variant system of an equation, it must allow a continuity and probability interpretation for the wave function ψ. The Dirac equation gives correct energy-momentum relation, so each component of ψ must satisfy the Klein-Gordon equation. Now, we can write the Dirac equation in terms of the Klein-Gordon equation as follows [6]

$$i\hbar\frac{\partial\psi}{\partial t}\left(i\hbar\frac{\partial\psi}{\partial t}\right) = \left[\frac{\hbar c}{i}\sum_{j=1}^{3}\alpha_i\frac{\partial}{\partial x^j} + \beta mc^2\right]\left[\frac{\hbar c}{i}\sum_{i=1}^{3}\alpha_i + \beta mc^2\right]\psi \tag{2.5}$$

After some rearrangement we get the Dirac equation of the form [3]

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = -\hbar^2c^2\sum_{i,j=1}^{3}\left(\frac{\alpha_i\alpha_{j\,+}\,\alpha_j\alpha_i}{2}\,\frac{\partial^2\psi}{\partial x^i\partial x^j}\right)$$

$$+\frac{\hbar mc^3}{i}\sum_{i=1}^{3}(\alpha_i\beta + \beta\alpha_i)\frac{\partial\psi}{\partial x^i} + \beta^2m^2c^4\psi \tag{2.6}$$

Where we don't know that α_i and α_j are commutative or anti-commutative, so we take $\frac{\alpha_i\alpha_{j+}\,\alpha_j\alpha_i}{2}$ instead $\alpha_i\alpha_j$. If we compare equation (2.5) with equation (2.6), we can write these relations [6].

$$\alpha_i\alpha_{j+}\,\alpha_j\alpha_i = 2\delta_{ij} \tag{2.7a}$$

$$\alpha_i\beta + \beta\alpha_i = 0 \tag{2.7b}$$

$$\alpha_i{}^2 = \beta^2 = 1 \tag{2.7c}$$

Quantum mechanics postulates notify that Hamiltonian, can be easily seen in equations (2.7), must be a hermitical operator, so α_j and β have to be hermitical matrices. α_i and β have the matrix form [6];

$$\alpha_i = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \tag{2.8}$$

Here σ_i represented the Pauli 2x2 matrices, I is a 2x2 unit matrix. They are [6]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.9}$$

By using equation (2.2), a current conversation equation can be found out and then we can take hermitical conjugate wave functions [3]

$$\psi^* = ({\psi_1}^*, \dots, {\psi_4}^*) \tag{2.10}$$

And then by multiplying equation (2.5) ψ^+ from left and it is fond [3]

$$i\hbar\, \psi^+ \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \sum_{j=1}^{3} \psi^+ \alpha_j \frac{\partial \psi}{\partial x^j} + mc^2 \psi^+ \beta \psi \tag{2.11}$$

And than the Hamiltonian and conjugate of wave are taken into the equation (2.5) and then multiplying it by ψ from right-side, we obtain that equation [10]

$$-i\hbar \; \frac{\partial \psi^+}{\partial t} \psi = -\frac{\hbar c}{i} \sum_{j=1}^{3} \frac{\partial \psi^+}{\partial x^j} {\alpha_j}^* \psi + mc^2 \psi^+ \beta \psi \tag{2.12}$$

Where ${\alpha_j}^* = \alpha,\ \beta^* = \beta$. Subtracting equation (2.12) from equation (2.11), we get [10]

$$i\hbar \left[\psi^+ \frac{\partial \psi}{\partial t} + \frac{\partial \psi^+}{\partial t} \psi \right] = \frac{\hbar c}{i} \sum_{j=1}^{3} \left[\psi^* \alpha_j \frac{\partial \psi}{\partial x^j} + \frac{\partial \psi^*}{\partial x^j} \; \alpha_j \psi \right] \tag{2.13}$$

Now we understand that the terms on the left hand-side in brackets are probability density ρ, and the other terms on right-side in brackets represent current density, $J^k = \psi^* c \alpha^k \psi$. By putting these into the equation (2.13) in their place, we obtain the continuity equation [6]

$$\frac{\partial \rho}{\partial t} + \nabla . J = 0 \tag{2.14}$$

If we integrate equation (2.14) over all space, we can write [3]

$$\frac{\partial}{\partial t} \int \rho d^3 x + \sum_{j=1}^{3} \int \psi^* c \alpha_j \psi d^3 x = 0 \tag{2.15}$$

It is obviously understood in the last equation that the second term integration gives zero. Then the first term is also zero, so we can see ρ has positive definite probability density. The density and current in equation (2.13) must be invariant under Lorentz transformation. Therefore, the probability density ρ and current J from a four-vector under this transformation.

2.2. Non-relativistic Approach to the Dirac Equation

Dirac equation, which is a relativistic equation, is related to quantum theory as we stated before. Therefore, results are accompanied to non-relativistic results, so that let us show them. Now, we try to solve the Dirac equation for a free electron which is at rest. In this case, the Dirac equation [6] is written as;

$$i\hbar\frac{\partial\psi}{\partial t} = \beta mc^2\psi \tag{2.16}$$

Where β is a matrix and ψ is spinor. By using ψ and matrix form of β, the equation (2.16) is written in the form [4]

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = mc^2\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \tag{2.17}$$

All of the components of the wave functions are found as [6]

$$\psi_1 = e^{\frac{-imc^2t}{\hbar}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \psi_2 = e^{\frac{-imc^2t}{\hbar}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\psi_3 = e^{\frac{-imc^2t}{\hbar}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \psi_4 = e^{\frac{-imc^2t}{\hbar}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{2.18}$$

Where ψ_1 and ψ_2 correspond positive energy solutions and ψ_3 and ψ_4 correspond negative energy solutions [3]. We try to find out an equation of Pauli spin theory by reduction of the Dirac equation. To do this we describe a four-potential A^{μ} and a four-momentum $\mathrm{P}^{\mu} - \left(\frac{e}{c}\right)\mathrm{A}^{\mu}$ of gauge invariant form replacing P^{μ} to introduce the interaction of point charge by appling an external electromagnetic field. Now, the Dirac equation [6] takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c.\vec{\alpha}\,\left(\vec{p} - \frac{e}{c}\right)\vec{A} + \beta mc^2 + e\Phi\right]\psi \tag{2.19}$$

This last equation express the interaction of a point particle with an applied electromagnetic field. The classical partner of the hamiltonian in this equation is $H = H_0 + H'$. Where $H' = -e\vec{\alpha}.\vec{A}$ +e Φ and we note that classical hamiltonian is $H'_{classical} = -\frac{e}{c}\,\vec{v}.\vec{A} + e\Phi$ [6].

We can write a wave function in terms of two component column matrices for the function [6] in the Dirac equation as

$$\psi = e^{\frac{imc^2t}{\hbar}}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \tag{2.20}$$

By using this two-compenent column matrices; the equation (2.20) is rewritten as an equation depending on matrix form after some calculations as follows; [6]

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = c.\vec{\sigma}.\vec{\pi}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \chi\end{pmatrix} + e\varphi\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \tag{2.21}$$

Where $\pi = p - \frac{e}{c}A$. Components of χ can be taken in terms of large one φ as [6]

$$\chi = \frac{\vec{\sigma}.\vec{\pi}}{2mc}\,\varphi \tag{2.22}$$

And putting this instead of χ as seen in equation (1.21) and using the definition [6]

$$\vec{\sigma}.a\vec{\sigma}.b = \vec{a}.\vec{b} + i\vec{\sigma}.\vec{a}\,\mathrm{x}\,\vec{b} \tag{2.23}$$

To find out $\vec{\sigma}.\vec{\pi}\vec{\sigma}.\vec{\pi} = \pi^2 + i\vec{\sigma}.\vec{\pi}\,\mathrm{x}\,\vec{\pi}$ and putting its value into the equation, we carry out two-component spinor equation [6]; that is

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m} - \frac{e\hbar}{2mc}\vec{\sigma}.\vec{B} + e\varphi\right]\psi \tag{2.24}$$

This equation is known as Pauli equation.

We can reduce the Pauli equation to a general form by using some changing which is the electron spin momentum $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$, we can omit the field interaction because it has a small effect, as expansion of π^2 $\vec{A} = \frac{1}{2}\,\vec{r}\mathrm{x}\vec{B}$, $\vec{L} = \frac{1}{2}\,\vec{r}\mathrm{x}\vec{p}$ and neglecting higher order terms in equation (2.24) after these transforms the Pauli equation takes [3]

$$i\hbar\frac{\partial\varphi}{\partial t} = \left[\frac{p^2}{2m} - \frac{e}{2mc}\left(2\vec{S} + \vec{L}\right).\vec{B}\right]\varphi \qquad (2.25)$$

the founded last equation is the corresponding non-relativistic equation to the Dirac equation which gives convenient values.

2.3. Dirac Equation for Free Particle

The Dirac equation is a relativistic wave equation. It is a useful equation to describe elementary spin -1/2 particles which is fully consistent with the principles of quantum mechanics and mainly consistent with the theory of special relativity, such as electrons, positrons, neutrons and the like. Additionally, this equation is required for the nature of particle spin and existence of antiparticle. It also describes the probability amplitudes for a single electrons. This single particle theory gives a fairly good prediction of the spin and magnetic moment of the electron and explains much of the fine structure observed in atomic spectral lines. Additionally, it makes the peculiar prediction that exists an infinite set of quantum states in which the electron possesses negative energy because the Dirac equation was originally invented to describe the electron, here we will mention about electrons. Actually, the equation applies to other types of elementary spin -1/2 particles, such as neutrinos. Also, a modified Dirac equation can be used to describe protons and neutrinos approximately, which are made of smaller particles called quarks and they are not elementary particles.

2.3.1. Structure of Dirac Equation for free particle

We try to derive a relativistic equation which describes the electron, by using a non-relativistic equation and then we add perturbation. The non-relativistic equation can be non-relativistic Schrödinger equation, written in the form;

$$i\hbar\frac{\partial}{\partial t}\,\psi = \frac{1}{2m}p^2\psi + \frac{e\hbar}{2m}(\sigma.B)\psi \qquad (2.26)$$

Here, the notation used like $p \rightarrow p + eA$. There is an easy way to rewrite down the two magnetic terms. Consider the following combination, in which we have studied to simplify [3]

$$[\vec{\sigma}\,.\,\vec{p}]^2 = \sum \sigma_i p_i \ \sum \sigma_j p_j \ = \sum\left(\delta_{ij} + i\sum \varepsilon_{ijk}\sigma_k\right) p_i p_j = \vec{p}^2 + i\sum \varepsilon_{ijk}\,\sigma_k p_i p_j$$

$$=\vec{p}^2 + i\sigma_x\,[p_y, p_z] + i\sigma_y\,[p_z, p_x] + \mathrm{i}\sigma_x\,[p_x, p_y] \tag{2.27}$$

Where

$$\delta_{ij} = \begin{cases} 0 & for \quad i \neq j \\ 1 & for \quad i = j \end{cases} \tag{2.28a}$$

$$\varepsilon_{ijk} = \begin{cases} 0 & for \quad i = j = k \\ +1 & for \quad (i,j,k) \ \in \ \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & for \quad (i,j,k) \ \in \end{cases} \tag{2.28b}$$

Also, we use Pauli matrices σ again which satisfies the following identity

$\sigma_i \sigma_j = \delta_{ij} + i\sum_k \varepsilon_{ijk}\sigma_k$. The matrices forms are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.29}$$

Now, we try to find an equation like-Schrödinger equation instead of magnetic field so the commutative of p_x, p_y, p_z depends on the relations which are written as

$$[p_x, p_y] = -ie\hbar B_z \ \Rightarrow i[p_x, p_y] = ie\hbar B_z \tag{2.30a}$$

$$[p_y, p_z] = -ie\hbar B_x \ \Rightarrow i[p_y, p_z] = ie\hbar B_x \tag{2.30b}$$

$$[p_z, p_x] = -ie\hbar B_y \ \Rightarrow i[p_z, p_x] = ie\hbar B_y \tag{2.30c}$$

We can obtain the results by using upward identities with simple processes [3].

$$[\vec{\sigma}\,.\,\vec{p}]^2 = p^2 + e\hbar\sigma B \tag{2.31}$$

This allows us to rewrite the non-relativistic Schrödinger equation in the form of the electromagnetic terms that dropped to help simplify our understanding. If we look at the remaining equation, we see that [3]

$$i\hbar\frac{\partial}{\partial t}\,\psi = \frac{1}{2m}(p.\sigma)^2\psi \tag{2.32}$$

If we infer this equation for the plane wave, we obtain that equation

$$E = \frac{1}{2m}(p.\sigma)^2 = \frac{p^2}{2m} \qquad (2.33)$$

We easily see that result. This is certainly a non-relativistic equation. We would prefer to have something like the corresponding relativistic equation namely [3]

$$E^2 = p^2c^2 + m^2c^4 \qquad (2.34)$$

This suggests the following form for our equation [3].

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \phi = c^2(p.\sigma)^2\phi + m^2c^4 \qquad (2.35)$$

We have changed the wave function ϕ from ψ temporarily because the left side of the equation includes second order time, this equation does not exactly transform what we need. This violates certain basic assumption of quantum mechanics; for example, that the future of the wave function $\phi(r,t)$ depends only on the value $\phi(r,t=0)$ of the wave function at $t = 0$, and not on its time derivative $\phi(r,t=0)$. To attempt to remedy this problem, we first bring the first operator on the right to the left side, and then the difference of squares gives [3];

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \phi = c^2(p.\sigma)^2\phi + m^2c^4\phi \qquad (2.36)$$

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \phi - c^2(p.\sigma)^2\phi = m^2c^4\phi \qquad (2.37)$$

$$\left(i\hbar\frac{\partial}{\partial t} + cp.\sigma\right)\left(i\hbar\frac{\partial}{\partial t} - cp.\sigma\right)\phi = m^2c^4\phi \qquad (2.38)$$

To eliminate the second order equation, we define a second field φ as [3]

$$\left(i\hbar\frac{\partial}{\partial t} - cp.\sigma\right)\phi = -mc^2\varphi \qquad (2.39)$$

Plugging this into our second order differential equation [3], we see that

$$\left(i\hbar\frac{\partial}{\partial t} + cp.\sigma\right)\varphi = mc^2\phi \qquad (2.40)$$

These two equations are treated as if they were the first a definition and the second derived. Thinking of both φ and ϕ as the given wave function, then we have two coupled differential equations for φ and ϕ because each of these equations have only first order in time, specifying the initial conditions for both φ and ϕ will completely determine both wave functions in future times. We now use the combination of these two equations for writing a single equation for finding the Dirac equation as follows, also because the constants of the Dirac equation are not real numbers, we write them as matrices and so together like this [3];

$$\left[i\hbar\frac{\partial}{\partial t} - c\begin{pmatrix}\sigma & 0\\ -\sigma & 0\end{pmatrix}.p\right]\begin{pmatrix}\phi\\ \varphi\end{pmatrix} = mc^2\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\phi\\ \varphi\end{pmatrix} \tag{2.41}$$

Where $\psi, \alpha, and\ \beta$ are four-component wave function and some new matrices and they are written as follows, respectively.

$$\psi = \begin{pmatrix}\phi\\ \varphi\end{pmatrix}, \quad \alpha = \begin{pmatrix}\sigma & 0\\ -\sigma & 0\end{pmatrix}, \quad \beta = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} \tag{2.42}$$

After all these, the Dirac equation is written [3]

$$i\hbar\frac{\partial}{\partial t}\psi = c\alpha.p\psi + mc^2\beta\psi \tag{2.43}$$

Now we have a relativistic Hamiltonian and it is written

$$\mathrm{H} = c\alpha.p + mc^2\beta \tag{2.44}$$

Here we look at shortly the properties of the matrices α and β. They are all 4x4 Hermitical matrices, and the square of any of them is equal to one. They anti-commute with each other as well [3]. So we have

$$\beta^\dagger = \beta, \quad \alpha^\dagger = \alpha, \quad \beta^2 = \alpha_i^2 = 1 \tag{2.45a}$$

$$\beta\alpha_i + \alpha_i\beta = \alpha_i\alpha_j + \alpha_j\alpha_i = 0 \quad \text{if} \quad \mathrm{i} \neq j \tag{2.45b}$$

It can be denoted that these properties of the α and β matrices all we barely ever might to enforce calculation. Indeed, any set of 4x4 matrices with these properities may be written tending to the same solution. They apart from the components of ψ and will be caught up with each other [3].

2.3.2. Solution of Dirac Equation for Free Particle

In order to solve the Dirac equation for free particle the plane wave function was chosen [3]. The wave function is written as in the following,

$$\psi = \psi_0 e^{i(\frac{p.r-Et}{\hbar})} \tag{2.46}$$

If we substitute this equation into the Dirac equation, we obtain

$$E\psi_0 = (c\alpha.p + mc^2\beta) \tag{2.47}$$

when we expand each side of equation, we simply care the values of equation.

$$E^2\psi_0 = (c\alpha.p + mc^2\beta)^2\,\psi_0 \tag{2.48}$$

Now, the right side of equaiton is more complex according to the other side because all four matrices anti-commute, the cross terms will all vanish. Since all the matrices are squared, this simplifies to

$$E^2\psi_0 = (c^2p^2 + m^2c^4)^2\,\psi_0 \tag{2.49}$$

This leads to two different values which are positive and negative values;

$$\mathrm{E} = \pm\sqrt{c^2p^2 + m^2c^4} \tag{2.50}$$

This result is definitely the equation we want to except the $\pm$ symbol. Actually, one can find that two different solutions for each momentum p which have values of negative energy and positive energy, If the equation is viewed more carefully. We know that, the states of positive energy correspond to the electrons spin states, however, we have not got any idea for the negative energy states. Dirac tried to solve this problem. By the time he came up with this theory, the Pauli exclusion principle was already understood, and Dirac proposed that negative energy solutions, since they had negative energy, were already filled up with negative energy electrons (figure 2.1) [3]. As a consequence, any electron can not fall into these negative energy states. The assumption here is that the empty space is filled with countless negative electron states, and we don’t notice them because that is the naturel state of space.

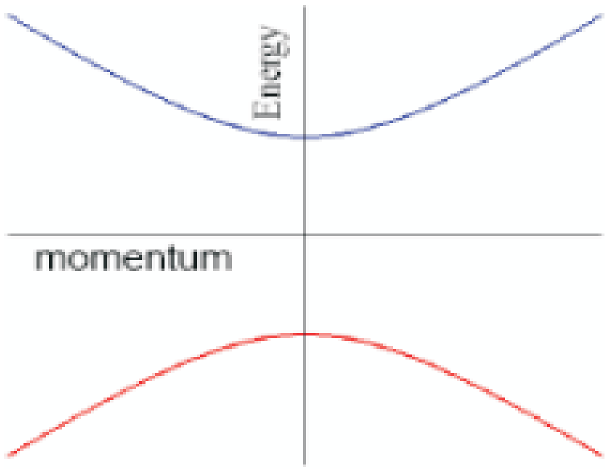


Figure 2.1: The energy of momentum functions for electron. The negative energy states defined by the lower curve[3].

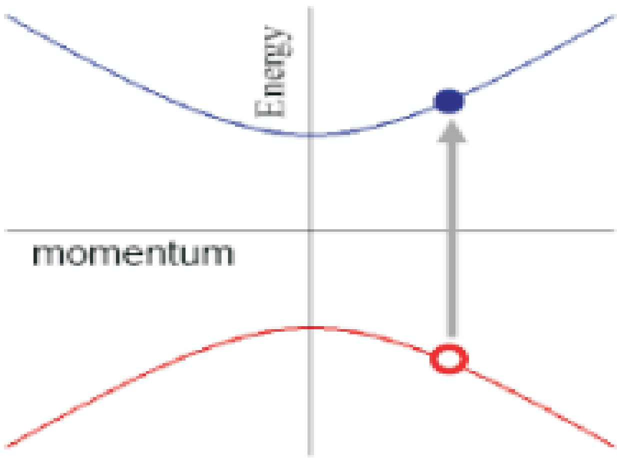


Figure 2.2: When a negative energy electron is bumben up to a positive energy state, it creates an electron and leaves behind a hole with positive charge and positive energy, which are called positron [3]

Now, some of the electrons have negative energy and positive energy state. But missing electrons (Fig.2.2) are left over from the hole. We have a negative charge (which means positive charge) by the absence of a negative energy (which means positive energy) would have accepted this hole. The electron mass is at rest as we know. Finally, in this chapter we have given the basic properties of the Dirac equation for free particle. In the following chapters, we will discuss the solution of the Dirac equation including various potential and then scattering of Dirac particles and bound state of Dirac particles.

CHAPTER 3
THE DIRAC EQUATION INCLUDING VARIOUS EXACTLY SOLVABLE POTENTIALS

In order to analyze relativistic effects on the spectrum of such physical systems, one should construct the Dirac equation including adequate potentials and obtain its solutions. Two dimensional Dirac oscillator [28,29] can be used in order to determine the spectrum and properties of such systems in the relativistic physics. In this study, we have studied only exactly solvable potentials, relativistic extensions of these potentials have also turned out to be of importance in the description of (2+1)-dimensional phonemena [30,31]. The Dirac equation Hamiltonian presence various potential can be used to find the spectrum of (2+1) dimensional different condensed matter physical systems. We know that the Dirac equation is important to understand the relativistic particles that have spin-1/2. Meantime, we can say that the Hamiltonian obtained from the Klein-Gordon equation, called Feshbach-Villars equation, has been built by two different states. One of them is spinless particles [11]. The Hamiltonian has been constructed in a two-component spinors for this particles and other one has been constructed in an eight-component for spin-1/2 particles[32,33]. Unfortunately, the Dirac equation is not exactly solvable all species of potentials, it exactly is solvable only in a very limited potentials. In this chapter we propose that the solution of the Dirac equation including a class of potential whose spectrum can accurately be determined. The Dirac equation is transformed into the Schrödinger-like equation for this purpose. For example, physical systems which have few electrons, stuck on the problem in the building of the Schrödinger-like equations. A Dirac equation was considered previously with an interaction linear in coordinates, and recently, it is reinvented in the case of the relativistic theories [34]. If we add a very strong spin-orbit coupling term into the Dirac oscillator, the equation becomes a harmonic oscillator in the non-relativistic limit [35,36,37].

As we have already noted, the Dirac equation including various potentials might attract much attention, because it may have some physical applications, particularly in the condensed matter physics. It seems that one can present more realistic models for the artificial atoms using the procedure given here.

3.1. Structure of Potential

We can write the (2+1)-dimensional Dirac equation for a particle without any potential or magnetic field, that the particle has mass m, in terms of two-component spinors ψ, as follows [38]

$$E\psi = [\sum_{i=1}^{2} c\beta\gamma_i p_i + \beta mc^2]\psi \tag{3.1}$$

Where by using the Pauli spin matrices, we have defined two component spinors β and $\beta\gamma_i$ and we have used only that two components. Also, the relation of $\sigma_i\sigma_j = \delta_{ij} + \varepsilon_{ijk}\sigma_k$ gives the components β and $\beta\gamma_i$ and they written as [38]

$$\beta\gamma_1 = \sigma_1, \quad \beta\gamma_2 = \sigma_2, \quad \beta\gamma_3 = \sigma_3 \tag{3.2}$$

We have taken the equation without any potential, so the momentum operator p_i is a differantial operator written by two component $p = -i\hbar(\partial_x, \partial_y)$ in (2+1)-dimension. Now, we consider we have a magnetic field. Because of the magnetic field in our equation, the momentum operator is replaced to $p \to p - eA$, where A is vector potential. And then, we add a parameter into the momentum operator as $p \to p - im\omega\sigma_3 r\hat{r}$ and we use it into the Dirac oscillator [38]. We are seeking for a certain form of the momentum operator that can be interpreted as exactly solveable Schrödinger equation in the non-relativistic limit. For this purpose, we introduce the following momentum operator.

$$p \to p - eA + i\sigma_3 v(r)\hat{r} \tag{3.3}$$

When we substitute the equation (3.3) into the (3.1) the momentum operator takes that form [38];

$$\begin{aligned}[E - \sigma_0 mc^2]\psi &= c\sigma_+[p_x + ip_y - e(A_x + iA_y) - i(v_x + iv_y)]\psi \\ &c\sigma_-[p_x - ip_y - e(A_x - iA_y) + i(v_x - iv_y)]\psi\end{aligned} \tag{3.4}$$

Where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where, in the equation (3.4) p_i represents momentum components, A_i represents vector potential and finally ν_i represents the potential in the cartesian coordinate system. We try to writte the (2+1) dimensional Dirac equation in terms of spherical coordinates under some suitable transformations [39]. Now, we can define transformations as vector potential; $A_x = -A(r)sin\phi$ and $A_y = -A(r)cos\phi$, scalar potential; $\nu_x = -\nu(r)cos\phi$ ve $\nu_y = -\nu(r)sin\phi$ and $x = rcos\phi$, $y = rsin\phi$. Under these transformations, the equation takes that form [38];

$$E_-\psi_+ = ce^{i\phi}[-i\hbar\frac{\partial}{\partial r} + \frac{\hbar}{r}\frac{\partial}{\partial\phi} - i(eA(r) + \nu(r))]\psi \tag{3.5}$$

$$E_+\psi_- = ce^{-i\phi}[-i\hbar\frac{\partial}{\partial r} - \frac{\hbar}{r}\frac{\partial}{\partial\phi} + i(eA(r) + \nu(r))]\psi \tag{3.6}$$

Where $E_\pm = E \pm mc^2$ and $\psi_\pm = \psi_\pm(r,\phi)$ represents upper and lower components of the spinor ψ. The substitution of the wave function [38]

$$\psi_\pm(r,\phi) = \frac{e^{-i\left(l \pm \frac{1}{2}\right)\phi}}{\sqrt{r}} f_\pm(r) \tag{3.7}$$

We can write the following set of coupled differantial equations [38]

$$E_-f_+ = -ic[\hbar\frac{\partial}{\partial r} + \frac{\hbar l}{r} + (eA(r) + \nu(r))]f_-(\mathrm{r}) \tag{3.8}$$

$$E_+f_- = -ic[\hbar\frac{\partial}{\partial r} - \frac{\hbar l}{r} - (eA(r) + \nu(r))]f_+(r) \tag{3.9}$$

Our aim is now to write the equation (3.8) and (3.9) in terms of the Schrödinger-like equation, so if we substitute the above terms into the equation, we get the formulain the following [38]

$$W(r) = \frac{l}{r} + \frac{e(A(r)+\nu(r))}{\hbar},\ E_\pm = \pm ic\hbar\varepsilon_\pm \text{ and } \varepsilon^2 = \varepsilon_+\varepsilon_- \tag{3.10}$$

leads to that defination [38]

$$\varepsilon_- \, f_+(r) = [\frac{\partial}{\partial r} + W(r)]f_-(r) \tag{3.11}$$

$$\varepsilon_+ \, f_-(r) = [-\frac{\partial}{\partial r} + W(r)]f_+(r) \tag{3.12}$$

We can see that a supersymmetric treatment of the Dirac equation is hidden into the results of equations (3.11) and (3.12) [39], and the supersymmetric operators are written as $A^{\pm} = \pm\frac{\partial}{\partial r} + W(r)$ [39]. It is clear that the functional form of the W(r) and $\nu(r)$, which are superpotential, have the same exception for the radial function $\frac{\hbar l}{r}$ and because of the superpotential of the non-relativistic quantum mechanics. Now we can see that the equations (3.11) and (3.12) are written as [38]

$$-\frac{\partial^2}{\partial r^2} - W^2(r) + w'(r) + \varepsilon^2)f_-(r) = 0 \tag{3.13}$$

$$-\frac{\partial^2}{\partial r^2} + W^2(r) + w'(r) - \varepsilon^2)f_+(r) = 0 \tag{3.14}$$

This gives an interesting result for the potential. $V_{\pm} = W^2 \pm W'$ leads to a common spectrum. Thus, an isospectral system can be formed. In the following study we analyze the solutions and the energy spectrum of the (2+1)-dimensional Dirac equation including various exactly solvable potentials [39].

3.2. Exactly Solvable Potentials

We can obtain the construction of the equations in presence of exactly solvable potentials with appropriate choices of the superpotential W(r) [38]. For the a large class potential, the equations (3.13) and (3.14) have exact solutions which are the Dirac equation in form of Schrödinger-like equations. Now, we try to solve the Dirac equation for three of these exactly solvable potential.

3.2.1. Harmonic Oscillator Potential

In order to solve the Dirac equation including the harmonic oscillator potential let's use the Dirac oscillator. The Dirac oscillator constructs with choices of the superpotential $W(r) = \frac{m}{\hbar}\omega_T r - \frac{l+1}{r}$. Adding this superpotential into the Schrödinger-type equations (3.13) and (3.14), they are taking the following form; [38]

$$\left[-\frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2} + \left(\frac{m}{\hbar}\omega_T\right)^2 r^2 + \frac{m}{\hbar}\omega_T(2l+3) - \varepsilon^2\right] f_-(r) = 0 \tag{3.15}$$

$$\left[-\frac{\partial^2}{\partial r^2} + \frac{l(l+1)(l+2)}{r^2} + \left(\frac{m}{\hbar}\omega_T\right)^2 r^2 - \frac{m}{\hbar}\omega_T(2l+3) - \varepsilon^2\right] f_+(r) = 0 \tag{3.16}$$

In these equations, the scalar potential $\nu(r)$ and the vector potential $A(r)$ are defined by [38]

$$\nu(r) = \frac{m}{\hbar}\omega_r - \frac{(2l+1)}{r}, \quad A(r) = \frac{1}{2}Br \tag{3.17}$$

And the frequency ω_T can be denoted in terms of the Larmor frequency that form [38]

$$\omega_T = \omega + \omega_L = \omega + \frac{eB}{2m} \tag{3.18}$$

To solve the equation (3.15), we familiarize wave function [39]

$$f_-(z) = C\, z^{\frac{l+1}{2}}\, e^{\frac{z}{2}}\, g_-(z) \tag{3.19}$$

Where we have changed the variable $z = \frac{m}{\hbar}\omega_T r^2$. By using these change, the equation (3.15) takes; [38]

$$\left[z\frac{\partial^2}{\partial z^2} + \left(l + \frac{3}{2} - z\right) + n\right] g_-(z) = 0 \tag{3.20}$$

Where C is the normalization constant. The relation of energy and natural number n is written in the form [3]

$$\omega^2 = \frac{E^2 - m^2c^4}{\hbar^2 c^2} = 4n\frac{m}{\hbar}\omega_T \tag{3.21}$$

By choices of $E = E_{nr} + mc^2$ we obtain the non-relativistic limit of energy and by choice of the relation of terms $E_{nr} \ll mc^2$, the non-relativistic energy is obtained as [38]

$$E_{nr} = 4n\hbar\omega_T$$

Now we add the effect of spin energy into the Hamiltonian. It is worth to obtain non-relativistic form of the equation (3.15) to analyze spin effect [38]

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{2mr^2} + \frac{1}{2}m\omega_T^2 r^2 + \hbar\omega_T\left(l + \frac{3}{2}\right) - E_{nr}\right] f_-(r) = 0 \tag{3.22}$$

The corresponding Hamiltonian is the Hamiltonian of a harmonic oscillator with an additional spin dependent form of $\hbar\omega_T\left(l + \frac{3}{2}\right)$. We turn our attention to the normalization of the wave function for couplet our analysis. We can see that the solution of equation (3.20) associates with the Laguerre polynomials, $L_n^{l+\frac{1}{2}}(z)$, then we can write $f(z)$ as [38]

$$f_-(z) = C\, z^{\frac{l+1}{2}}\, e^{-\frac{z}{2}}\, L_n^{l+\frac{1}{2}}(z) \tag{3.23}$$

Now we can define the upper component $f_+(z)$ of spinor $\psi(r)$ from equation (3.11) and it is written as [38]

$$f_+(z) = \frac{2C\sqrt{\omega_T}}{\varepsilon_-}\, z^{\frac{l+2}{2}}\, e^{-\frac{z}{2}}\, L_{n-1}^{l+\frac{3}{2}}(z)$$

Expression of normalization condition in polar coordinates is

$$\langle\psi|\psi\rangle = \int_0^{\infty}\left(|f_+(r)|^2 + |f_-(r)|^2\right) dr = 1 \tag{3.24}$$

Laguerre polynomials satisfy the orthogonally conditions [38]

$$\int_0^{\infty} z^l \ e^{-z} \ L_n^l(\text{z}) \, dr = \frac{\Gamma(n+l+1)}{n!} \ \ \delta_{nk} \tag{3.25}$$

Finally, for the normalization constant C, an expression is obtained [38]

$$C = \left[\frac{\sqrt{\omega_T} \ \varepsilon_-^2 \ \Gamma(n+1)}{n\omega_T + \varepsilon_-^2 \Gamma(n) \ \Gamma\left(n+l+\frac{3}{2}\right)}\right]^{\frac{1}{2}} \tag{3.26}$$

The Dirac oscillator in two-dimensional space has been solved which permits series of interesting result. The Dirac oscillator has various physical applications particularly in semiconductor physics [40].

3.2.2. Coulomb Potential

We can give another example such as relativistic hydrogen atom for the exactly solvable Dirac equation including its potential. If we choose that the magnetic field equals to zero, $A(r) = 0$ [41,42] and in the asset of the Coulomb potential $v(r) = \frac{me^2}{4\pi\varepsilon_0 \hbar(l+1)} - \frac{\hbar(2l+1)}{r}$, the Dirac equation including that potential can be solved exactly. In addition, superpotential $W(r) = \frac{me^2}{4\pi\varepsilon_0 \hbar^2(l+1)} - \frac{(l+1)}{r}$, by using the change these three terms the Schrödinger-like equations (3.13) and (3.14) can be written as follows [38]

$$\left[-\frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2} - \frac{me^2}{2\pi\varepsilon_0 \hbar^2 r} + \left(\frac{me^2}{4\pi\varepsilon_0 \hbar^2(l+1)}\right)^2 - \varepsilon^2\right] f_-(r) = 0 \tag{3.27}$$

$$\left[-\frac{\partial^2}{\partial r^2} + \frac{(l+1)(l+2)}{r^2} - \frac{e^2}{r} + \left(\frac{me^2}{4\pi\varepsilon_0 \hbar^2(l+1)}\right)^2 - \varepsilon^2\right] f_+(r) = 0 \tag{3.28}$$

By using the similar developments in the construction of the harmonic oscillator problem, we transform the equation (3.27) in the form [38]

$$\left[z\frac{\partial^2}{\partial z^2} + (2l + 2 - z) + n\right] g_-(z) = 0 \tag{3.29}$$

After changing the variable $r = 2\pi\varepsilon_0\hbar^2(n+l+1)\frac{z}{me^2}$ and the wave function [38]

$$f_{-}(z) = C\, z^{l+1}\, e^{-\frac{z}{2}}\, g_{-}(z) \tag{3.30}$$

The energy of the Hamiltonian is written by the natural number n as [38]

$$\varepsilon^2 = \frac{E^2-m^2c^4}{\hbar^2c^2} = \left(\frac{me^2}{4\pi\varepsilon_0\hbar^2(l+1)}\right)^2 - \left(\frac{me^2}{4\pi\varepsilon_0\hbar^2(n+l+1)}\right)^2 \tag{3.31}$$

In order to express the solution of the wave function, it is written by solution of the equation (3.29) as [38]

$$f_{-}(z) = C\, z^{l+1}\, e^{-\frac{z}{2}}\, L_n^{2l+1}(z) \tag{3.32}$$

And the wave function $f_{+}(z)$ is written from the equation (3.11) as follows [38]

$$f_{+}(z) = \frac{-C}{\varepsilon_{-}}\, z^{l+1}\, e^{-\frac{z}{2}}\left[2(l+1)L_{n-1}^{2l+2}(z) + \left(l+1-\frac{me^2}{2\pi\varepsilon_0\hbar^2}(2l+1)\right)L_{n-1}^{2l+2}(z)\right] \tag{3.33}$$

Now If we use the identity, the equation takes the form [38]

$$\int_0^\infty z^{l+1}\, e^{-z} L_n^l(z) L_n^l(z)\, dz = (2n+l+1)^{l+2}\frac{\Gamma(n+l+1)}{n!} \tag{3.34}$$

We can obtain the normalization constant after some straight forward calculation [38]

$$C = \left[\frac{2\pi\varepsilon_0\hbar^2}{me^2}\left(\frac{4(l+1)^2}{\varepsilon_{-}^2\Gamma(n)} + (2n+2l+2)^{2l+3}K\right)\Gamma(n+2l+2)\right]^{-\frac{1}{2}} \tag{3.35}$$

Where K is given by

$$K = \left[1 + \frac{1}{\varepsilon_{-}^2\, n!}\left(l+1-\frac{me^2}{2\pi\varepsilon_0\hbar^2}(2l+1)\right)^2\right] \tag{3.36}$$

For the resent interest of the two-dimensional field theory in the condensed matter physics, as physically the two-dimensional Coulomb potential is relevant and we can see that some new features after this result of the solution [43,44].

3.2.3. Morse Potential

One of the quantum mechanical problem is the exactly solvable Morse oscillator. In order to understand the interaction of the atoms in the diatomic molecules, the exactly solvable Morse oscillator can be used as a model [20]. One can be provided by using the parameters $v(r) = -\frac{\hbar l}{r} - \hbar a e^{\alpha r} + \hbar b,\ A(r) = 0$ and $W(r) = b - ae^{-\alpha r}$ as the following potential [38].

$$\left[-\frac{\partial^2}{\partial r^2} + a^2 e^{-2\alpha r} - a(a+2b)e^{-\alpha r} + b^2 - \varepsilon^2\right] f_-(r) = 0 \tag{3.37}$$

$$\left[-\frac{\partial^2}{\partial r^2} + a^2 e^{-2\alpha r} + a(a-2b)e^{-\alpha r} + b^2 - \varepsilon^2\right] f_+(r) = 0 \tag{3.38}$$

We take the variable $e^{-\alpha r} = \frac{\alpha}{2a} z$ for solving of the equation (3.37) and the wave function is chosen as in the following [39];

$$f_-(r) = z^{\frac{b}{\alpha} - n}\, e^{-\frac{z}{2}}\, g_-(z) \tag{3.39}$$

After these changes, the equation is obtained in the form of [38]

$$\left[z\frac{\partial^2}{\partial z^2} + \left(\frac{2b}{\alpha} - 2n + 1 - z\right) + n\right] g_-(z) = 0 \tag{3.40}$$

Also, we can write the energy expression of Morse potential; [38]

$$\varepsilon^2 = \alpha n(\alpha n - 2b)$$

Now, by using the equations (3.40) and (3.11) we can define the wave function [38]

$$f_-(r) = C\, z^{\frac{b}{\alpha} - n}\, e^{-\frac{z}{2}}\, L_n^{\frac{2b}{\alpha} - 2n}(z) \tag{3.41}$$

$$f_+(r) = \frac{C\alpha}{\varepsilon_-} z^{\frac{b}{\alpha} - n} e^{-\frac{z}{2}} \left[n L_n^{\frac{2b}{\alpha} - 2n}(z) + z L_{n-1}^{\frac{2b}{\alpha} - 2n+1}(z)\right] \tag{3.42}$$

For the normalization constant, normalization condition is appropriate with the following expressions [38]

$$C = \left[\frac{\varepsilon^2\Gamma(n)\Gamma(n-1)}{\Gamma(\frac{2b}{\alpha}+1-n)(\varepsilon^2\Gamma(n)+2\alpha^2)\Gamma(n+1)}\right]^{-\frac{1}{2}} \tag{3.43}$$

The potentials are useful for considering of both mathematical and physical problem which we have derived in this section. For example, the solution of the Dirac equation including single quark potential can be solved by the relativistic quark model. Murci has recently treated the behavior of a large distance r and short distance $\frac{1}{r}$ [45]. Also, we can use the Morse oscillator potential for large distance potential in some various physical systems [25].

CHAPTER 4

SOLUTION OF SCATTERING STATES OF DIRAC PARTICLES INCLUDING AN EXTERNAL POTENTIAL FIELD

The exact solution of the Dirac equation with the Coulomb potential plays a very important role in relativistic and non-relativistic quantum mechanics. For instance, for a harmonic oscillator in three-dimension [46] and a hydrogen atom the exact solution of Schrödinger equation is important for beginning the stage of quantum mechanics and that solution provides a strong evidence for quantum theory. In order to understand a given quantum system, studying of the bound states then that of the scattering states can be paid more attention, so that for understanding quantun systems completely, we can study both of them because of the scattering establishment and bound states particles in quantum mechanics. Appling of the Schrödinger equation [47] to quantum systems has been known well. For example, the both systems of quantum mechanics and classical mechanic can be understood well by studying of the scattering of particles by Coulomb potential field. However, when we consider the relativistic effect on a quantum system, the Dirac equation has to be employed to study the electrons scattering by a nucleus potential. Also, one can carry out the exact solution of the Dirac equation including Coulomb plus scalar potential in different dimensions which are two-dimension [48,49], three-dimension[50] and higher-dimensions [51]. One can study scattering phase shifts in the two-dimension Dirac equation including the Coulomb potential given by the second order differential equation and by algebraic method [52], for the condensed matter physics and interest of the lower-dimensional quantum field theory. The aim of us in this chapter is to study the scattering of the two-dimensional Dirac equation including plus scalar potential. The Dirac equation with a given mixed potential including the Coulomb plus scalar potential [48,49] can be considered and it can be written as [48]

$$[\alpha.p+\ \beta(m+V_s)-(E-V_c)]\psi_{jE}(t,r)=0 \tag{4.1}$$

In this equation m represents the mass of the particle and

$$V_c=\ -\frac{A_1}{r},\qquad V_s=-\frac{A_2}{r} \tag{4.2}$$

Also, A_1 and A_2 as constants, A_1 is electrostatic and A_2 is scalar coupling, respectively. The others terms in equation (4.1) V_c and V_s correspond electrostatic and scalar potentials, respectively. When adding this scalar potential into the mass term, the Dirac equation turns into effective position-dependent mass equation.

Following the previous study [48], $p=\ -i\frac{\partial}{\partial x}$, $r=\ |x|$ are taken. We consider In 2+1 dimensions, by choosing $\alpha^1=\ -\sigma^2$, $\alpha^2=\ -\sigma^1$ and $\beta=\ \sigma^3$, where

$\sigma^\mu(\mu=1{,}2{,}3)$ are the Pauli matrices, the equation written as [39]

$$\psi_{jE}(t,r)=\ (2\pi)^{-\frac{1}{2}}\,e^{-iEt}\,r^{-\frac{1}{2}}\begin{pmatrix}F_{jE}(r)\,e^{i(j-\frac{1}{2})\varphi}\\ G_{jE}(r)\,e^{i(j-\frac{1}{2})\varphi}\end{pmatrix} \tag{4.3}$$

We know that *j* represents the angular momentum $j=\ \pm\frac{1}{2},\pm\frac{3}{2},\ldots$. Also $F_{jE}(r)$ and $G_{jE}(r)$ denote the radial components which are adequate for first order differential equations [39].

$$\frac{d}{dr}\,G_{jE}(r)+\,\frac{j}{r}G_{jE}(r)\ =\ \left(E-m+\ \frac{A_1+A_2}{r}\right)F_{jE}(r) \tag{4.4a}$$

$$-\frac{d}{dr}\,F_{jE}(r)+\,\frac{j}{r}F_{jE}(r)\ =\ \left(E+m+\ \frac{A_1-A_2}{r}\right)G_{jE}(r) \tag{4.4b}$$

In order to introduce the scattering state this case is appropriate [39].

$$\rho=kr,\ \ k=\ \sqrt{E^2-m^2},\quad m<|E| \tag{4.5}$$

By using the equations (4.5), (4.4a), and (4.4b) [39] turn out

$$\frac{d}{d\rho}\,G_{jE}(\rho)+\,\frac{j}{\rho}G_{jE}(\rho)\ =\ \left(\varepsilon+\ \frac{A_1+A_2}{\rho}\right)F_{jE}(\rho) \tag{4.6a}$$

$$-\frac{d}{d\rho}\,F_{jE}(\rho)+\,\frac{j}{\rho}F_{jE}(\rho)\ =\ \left(\frac{1}{\varepsilon}+\ \frac{A_1-A_2}{\rho}\right)G_{jE}(\rho) \tag{4.6b}$$

Where ε is defined $\sqrt{\frac{E-m}{E+m}}$

Also, the wave functions $\Phi_{\pm}(\rho)$ are defined with the forms [39]

$$F_{jE}(\rho) = \frac{1}{2}\, e^{i\rho}\, [\Phi_{+}(\rho) + \Phi_{-}(\rho)] \tag{4.7a}$$

$$G_{jE}(\rho) = -\frac{i}{2}\, \varepsilon e^{i\rho}\, [\Phi_{+}(\rho) - \Phi_{-}(\rho)] \tag{4.7b}$$

One can obtain following equations by substitution of equations (4.7a) and (4.7b) into the equations (4.6a) and (4.6b) [39].

$$\frac{d\Phi_{+}}{d\rho} - \frac{i\alpha}{\rho}\,\Phi_{+} - \frac{i\alpha'+j}{\rho}\,\Phi_{-} = 0 \tag{4.8a}$$

$$\frac{d\Phi_{-}}{d\rho} + 2i\,\Phi_{-} + \frac{i\alpha}{\rho}\,\Phi_{-} + \frac{i\alpha'-j}{\rho}\,\Phi_{+} = 0 \tag{4.8b}$$

Where α and α' are defined [39] as follow;

$$\alpha = \frac{A_1 + A_2}{2\varepsilon} + \frac{(A_1 - A_2)\varepsilon}{2} \tag{4.9a}$$

$$\alpha' = \frac{A_1 + A_2}{2\varepsilon} - \frac{(A_1 - A_2)\varepsilon}{2} \tag{4.9b}$$

Also we have a property between these equations [39] that is

$$\alpha^2 - (\alpha')^2 = {A_1}^2 - {A_2}^2 \tag{4.10}$$

we can obtain an equation for Φ_{+} by eliminating Φ_{-} [39],

$$\rho\frac{d\,\Phi_{+}}{d\rho^2} + (1+2\mathrm{i}\,\rho)\frac{d\,\Phi_{+}}{d\rho} + \left(2\alpha - \frac{j^2 + A_1^2 - {A_2}^2}{\rho}\right)\Phi_{+} = 0 \tag{4.11}$$

Equation (4.11) looks like a special case of Tricomi equation [53]. A tricomi equation [39] is written as in the following

$$\frac{d^2y}{dx^2} + (\mathrm{a}+\frac{b}{x})\,\frac{dy}{dx} + (c + \frac{d}{x} + \frac{f}{x^2})y = 0 \tag{4.12}$$

Where a,b,c,d and f are constants. By using behaviors of the asymptotic wave functions at the origin, one can define

$$\Phi_{+} = \rho^{\tau}\omega(\rho), \qquad \tau = \sqrt{j^2 + A_2^2 - A_1^2} \tag{4.13}$$

By using the equation (4.13) into the equation (4.12) leads to [39]

$$\rho \frac{d^2\omega}{d\rho^2} + (1 + 2\tau + 2i\rho)\frac{d\omega}{d\rho} + 2(\alpha + i\tau)\omega = 0 \tag{4.14}$$

Changing another term which is $z = -2i\rho$, the equation (4.14) takes a new form [39]

$$z\frac{d^2\omega}{dz^2} + (1 + 2\tau - z)\frac{d\omega}{dz} - \alpha(\tau - i\alpha)\omega = 0 \tag{4.15}$$

We have not any solution for this equation, but turning the confluent hypergeometric functions [54] that are written as

$$\omega(\rho) = {}_1F_1(\tau, -i\alpha,\ 1 + 2\tau; -2i\alpha) \tag{4.16}$$

So we have an equation for Φ_+ that is

$$\Phi_+ = N\rho^\tau\ {}_1F_1(\tau - i\alpha, 1 + 2\tau;\ -2i\rho) \tag{4.17}$$

One can obtain a useful formula by using the following recursive relations between the confluent hypergeometric functions [54].

$$\gamma \frac{d}{dz}\ {}_1F_1(\alpha, \gamma; z) = \alpha_1F_1(\alpha + 1, \gamma + 1; z) \tag{4.18}$$

$$z_1F_1(\alpha + 1, \gamma + 1; z) = \gamma_1F_1(\alpha + 1, \gamma; z) - \gamma_1F_1(\alpha, \gamma; z) \tag{4.19}$$

We may obtain a useful formula [39]

$$(z\frac{d}{dz} + \alpha)\ {}_1F_1(\alpha, \gamma; z) = \alpha_1F_1(\alpha + 1, \gamma; z) \tag{4.20}$$

By using the equation (4.17) into the equation (4.8a) and helping of the equation (4.20) it yields [39]

$$\Phi_-(\rho) = N\frac{\tau - i\alpha}{j + i\alpha'}\ \rho^\tau \times {}_1F_1(\tau - i\alpha + 1, 1 + 2\tau;\ -2i\rho) \tag{4.21}$$

Finally, we can express $G_{jE}(\rho)$ and $F_{jE}(\rho)$ directly by the behaviors of the radial function in the combinations of the confluent hypergeometric functions. And for the value of $r \to \infty$, We now study the asymptotic behaviors of the radial wave functions. For $r \to \infty$, we have [39]

$$F_{out} = \frac{N}{2^{\tau+1}}\, e^{-\frac{1}{2}\alpha\pi}\, e^{ikr+i\alpha\ln(2kr)} \times \frac{\Gamma(1+2\tau)}{\Gamma(1+\tau+i\alpha)} \tag{4.22a}$$

$$F_{in} = \frac{N}{2^{\tau+1}}\, e^{-\frac{1}{2}\alpha\pi}\, e^{-ikr-i\alpha\ln(2kr)} \times \frac{\Gamma(1+2\tau)}{\Gamma(1+\tau-i\alpha)}\, e^{i\pi\tau}\, \frac{\tau-\mathrm{i}\alpha}{j+i\alpha'} \tag{4.22b}$$

we can obtain the following relation by using the equations [39] (4.22a) and (4.22b)

$$\frac{F_{out}}{F_{in}} = e^{2i[kr+\alpha\ln(2kr)]}\, \frac{\Gamma(1+\tau-i\alpha)}{\Gamma(1+\tau+i\alpha)}\, e^{-i\pi\tau}\, \frac{j+i\alpha'}{\tau-i\alpha} \tag{4.23}$$

so, we can obtain the phase shifts δ_j as [39]

$$e^{2i\delta_j} = e^{i\pi j}\frac{F_{out}}{F_{in}} = \frac{\Gamma(1+\tau-i\alpha)}{\Gamma(1+\tau+i\alpha)}\, e^{i\pi(j-\tau)}\, \frac{j+i\alpha'}{\tau-i\alpha} \tag{4.24}$$

It is shown that, the Dirac equation was discussed [54] only by Coulomb potential that is similar to this exact expression. However, the problem discussed in this work includes the scalar coupling potential,i.e, the parameter τ is related to two potential parameters A_1 and A_2.

As we know, once the phase shifts are obtained we can study the scattering amplitude and the differantial cross section. For the sake of simplicity, in the following we can obtain the scattering amplitude as [54]

$$f(\theta) = -\frac{i}{\sqrt{2\pi k}} \sum_j \left[e^{(2i\delta_j-1)}\right] e^{im\theta} \tag{4.25}$$

because of $\sum_j e^{im\theta} = 2\pi\delta(\theta)$, we can write for $\theta \neq 0$

$$f(\theta) = -\frac{i}{\sqrt{2\pi k}} \sum_j e^{(2i\delta_j)}\, e^{im\theta} \tag{4.26}$$

From which the cross section may be calculated as [54]

$$\sigma(\theta) = |f(\theta)|^2 \tag{4.27}$$

It should be noted that the equation (4.26) is very difficult to obtain an analytical expression [39]. Nevertheless, It can be expressed by choice of $Lin\ (Z < s), (A_1^2 \approx \frac{Z^2}{137^2}) \ll 1$ [54] in the special case $A_2 = 0$ for the light nucleus [39]. However, if we choose the parameters τ in terms of the parameters j, A_1 and A_2which are given in equation (4.13), they turn into a very complicated term [39]. We now write the equation in special case $A_1 = A_2$. Substituting of the relation $A_1 = A_2$ into the equation allows us to obtain $\tau = |j|$ [39]. On the other hand, we have a relation depending on α and α' given in equation (4.9).

$$\alpha = \alpha' = \frac{A_1}{\varepsilon} \tag{4.28}$$

so equation (24) can be written as [39]

$$e^{2i\delta_j} = (j + i\alpha)\frac{\Gamma(\tau - i\alpha)}{\Gamma(1+\tau+i\alpha)} e^{i\pi(j-\tau)} \tag{4.29}$$

From which [39]

$$e^{2i\delta_j} = \begin{cases} \frac{\Gamma(m+\frac{1}{2}-i\alpha)}{\Gamma(m+\frac{1}{2}+i\alpha)}, & for\ \ j > 0 \\ \frac{\Gamma(|m|+\frac{1}{2}-i\alpha)}{\Gamma(|m|+\frac{1}{2}+i\alpha)}, & for\ \ j < 0 \end{cases} \tag{4.30}$$

By substitution of this result into the equation (4.26) we can obtain these relations [39]

$$f(\theta) = -\frac{i}{\sqrt{2\pi k}}\left[\frac{\Gamma(\frac{1}{2} - i\alpha)}{\Gamma(\frac{1}{2} + i\alpha)}\ F(1, \frac{1}{2} - i\alpha, \frac{1}{2} + i\alpha, e^{i\theta})\right]$$

$$-\frac{i}{\sqrt{2\pi k}}\left[\frac{\Gamma(\frac{3}{2}-i\alpha)}{\Gamma(\frac{3}{2}+i\alpha)} e^{i\theta} F(1, \frac{3}{2} - i\alpha, \frac{3}{2} + i\alpha, e^{-i\theta})\ \right] \tag{4.31}$$

Where F(a,b,c,z) represents the hypergeometric functions. We can write following important equations [46].

$$F(\alpha, \beta, \gamma, z) = (1 - z)^{\alpha-\beta-\gamma} F(\gamma - \alpha, \gamma - \beta, \gamma, z) \tag{4.32}$$

And

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\beta)\Gamma(\gamma-\beta)}(-z)^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \frac{1}{z})$$

$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\beta)\Gamma(\gamma-\beta)}(-z)^{-\beta} F(\beta, \beta - \gamma + 1, \beta - \alpha + 1, \frac{1}{z}) \tag{4.33}$$

Now the scattering amplitude $f(\theta)$ can be obtained by equation (4.32) and (4.33) substituting into equation (4.31) [39].

$$f(\theta) = -i\frac{\Gamma(\frac{1}{2}-i\alpha)e^{i\alpha\ln(sin^2\frac{\theta}{2})}}{\sqrt{2k}\Gamma(i\alpha)\sin(\frac{\theta}{2})} \tag{4.34}$$

From these equations the following formulas can be written [39]

$$\Gamma(iy)\Gamma(-iy) = |\Gamma(iy)|^2 = \frac{\theta}{ysin(\pi y)} \tag{4.35a}$$

$$\Gamma\left(\frac{1}{2}+iy\right)\Gamma\left(\frac{1}{2}-iy\right) = \left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2 = \frac{\pi}{ycosh(\pi y)} \tag{4.35b}$$

As a result we have [39]

$$\sigma(\theta) = |\mathrm{f}(\theta)|^2 = \frac{\alpha\tanh(\alpha\pi)}{2ksin^2(\frac{\theta}{2})} \tag{4.36}$$

Where It is shown that respect to the scattering angle $\theta = \pi$ is symmetry with the cross section $\sigma(\theta)$ as shown in fig.4.1[39].

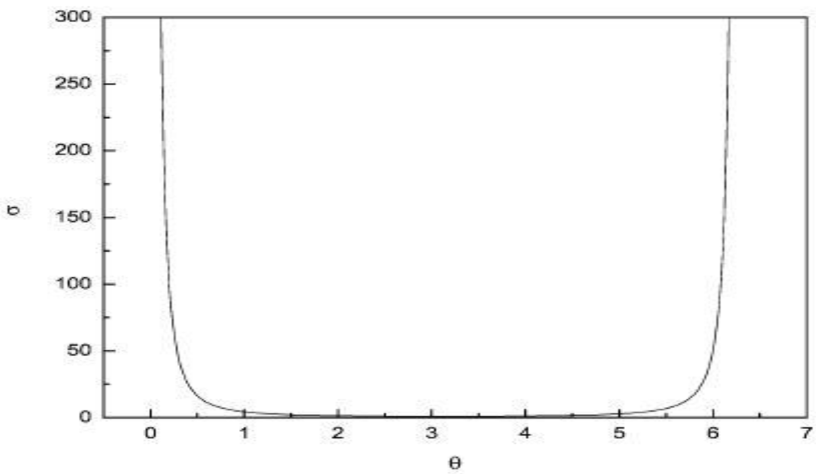


Figure 4.1: It is shown that respect to the scattering angle $\theta = \pi$ is symmetry with the cross section section $\sigma(\theta)$ $where\ A_1 = 1, k = \sqrt{3}$ and $\alpha = \sqrt{3}$ [39]

Phase shifts of Dirac equation has solved as a second order trigonometric differential equation. Then It showed that scattering angle $\theta = \pi$ is symmetry with cross section $\sigma(\theta)$ [39].

CHAPTER 5

SOLUTION OF BOUND STATES OF DIRAC PARTICLES WITH SOME PHYSICAL QUANTUM POTENTIALS

In this chapter we will study the bound states solutions of Dirac particles under an external potential field. Let's consider a particle which is exposed to a strong potential field. For that particle the relativistic effects must be considered. It should be noted that when a spinless particle is exposed to an external potential field, the relativistic effects are expressed with the Klein-Gordon equation. For some fields of physics like nuclear and atomic and molecular physics, the solution of Dirac equation is also important [12,13]. We can consider the Dirac equation in terms of the scalar potential S and the repulsive vector potential V. Within the framework the Dirac equation the spin symmetry arises if the magnitude of the spherical attractive scalar potential S and repulsive vector V potential nearly equal such that $S \sim V$ in the nuclei. However, the pseudospin symmetry occurs if $S \sim -V$ are nearly equal [14]. Recently, many authors have been extensively applied on various physical potentials by helping of spin and pspin symmetries [16,17]. For example, we can see solution of Dirac equation for some potentials which are deformed generalized Poschl-Theller (PT) potential [18], well potential [19], Manning-Rosen potential (MR) [20], modified PT potential [21,22], modified Rosen-Morse (RM) potential [23] and class of potentials including harmonic oscillator, Hulthen, trigonometric RM potential, Scarf Eckert, Morse, MR [24], and others. By using the properties of quantization rules, algebraic methods, we have the solution of Dirac equation including those potentials in the framework of the spin-orbit centrifugal term of the approximation to the spin-orbit.

If we want to obtain the exact solutions of the Dirac equation for the exponential-type potentials, we should choose the s-wave ($\kappa = \pm 1$ case). The Dirac equation has exact solution for those potentials only this case of the wave [55].

However, for the spin-orbit and psedospin centrifugal, $\frac{\kappa(\kappa+1)}{r^2}$ and $\frac{\kappa(\kappa-1)}{r^2}$, respectively, the terms states approximation can be used to deal with that spin-orbit and psedospin-orbit values [56,57]. To this end, the Dirac equation including large number of potential has been solved to obtain the two-component spinor wave functions on the energy by many works. The values of the energy spectra do not depend on the structure of the particle [58], whether they depend on the spin $-\frac{1}{2}$ or spin-0 particle. Also, when by choose spin $-\frac{1}{2}$ or spin-0 particle which have same mass and depend on potential of equal magnitude ($S = \pm V$ ($\Delta = \sum\ = 0$ or $C_{\pm} = 0$) which are scalar potential S and vector potential V, the spectrum of energy (isospectrality) including bound and scattering states will have been same [58]. When we solve the Dirac equation with the harmonic oscillator potential for the massless particles (or the ultra-relativistic particles), we see that the spin and pspin spectra of that particles are the same [59].

The spin symmetric and pspin bound state solutions of Dirac equation including the standard Rosen-Morse well potential model can be obtained as [25,60]

$$V(r) = -V_1\, sech^2\alpha r\; V_2\, tanh\alpha r \qquad (5.1)$$

Where V_1 and V_2 constants represent the depth of the potential and α has an inverse of length dimension which is the range of the potential. For three different values of parameters V_1 and V_2 the potential (5.1) is plotted in figure.4 [25].

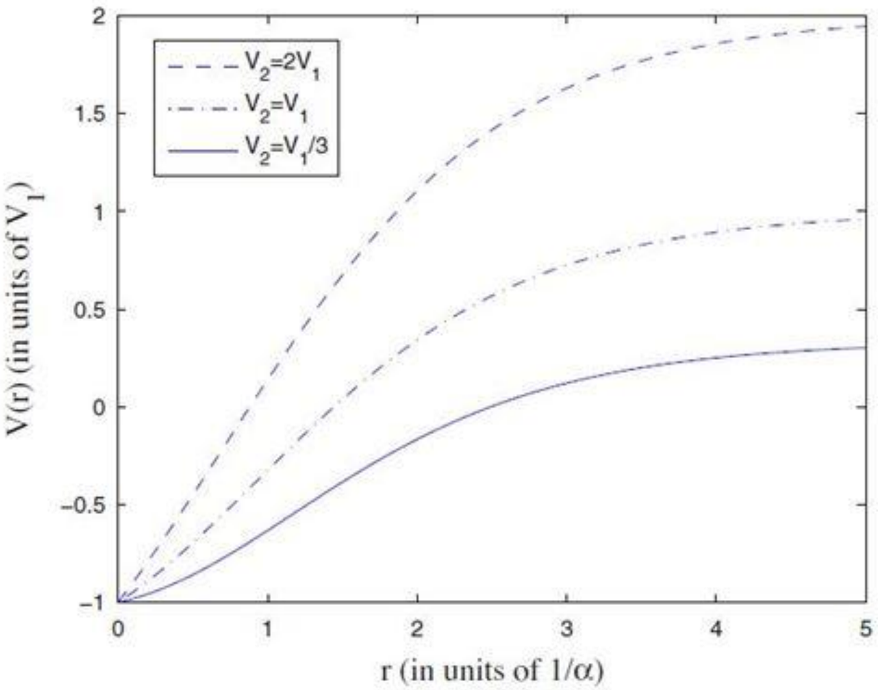


Figure 5.1: For three cases $V_2 = 2V_1,\ V_2 = 2V_1\ and\ V_2 = \frac{V_1}{3}\ a\ plot\ of\ the\ Rosen-Morse\ potential$

Our aim in the next papers is to extend of the s-wave solution by using the Dirac equation including some physical potential [50] and using the Nikiforov-Uvarov method [26,27] adding an approximation to look after with the presence of centrifugal (pseudo-centrifugal) potential term [61,62]. The spin-orbit centrifugal barrier $\frac{\kappa(\kappa-1)}{r^2}$ is arranged for κ which are the values of spin-orbit coupling quantum numbers with the use of approximation scheme. Where the quantum numbers have not got large values and they are vibrations of the small amplitude [62]. By using the definition of spin symmetry $S{\sim}V$ and pspin symmetry $S \sim -V$ the bound state energy eigenvalues and their corresponding upper and lower spinor wave functions can be calculated. Also, the spin and pspin symmetry Dirac solutions can be shown when pspin symmetry limitation is chosen $\Delta = 0$ and $\sum = 0$, respectively. And for spin symmetry $\Delta = C_-$ and $\sum = C_+$ can be reduced to the exact spin symmetry. Furthermore, if we use suitable matching of parameters, by using the non-relativistic limit of the Dirac equation the bound state solutions of Schrödinger equation can be obtained.

In the following sections, we mainly will define the basic spin and pspin Dirac equation. Then, the (3+1)-dimensional Dirac equation with the Rosen-Morse potential and reflectionless-type potentials are approached for analytical bound state. Also, we use a parametric generalization of the Nikiforov-Uvarov method [26,27] in order to obtain those potentials that are also obtained in the presence of the spin and pspin limits. After that, we will study non-relativistic limit and special case of the s-wave $\kappa = \pm 1 (l = \tilde{l} = 0)$.

5.1. Basic Spin and Pspin Dirac Equations

Let's write the Dirac equation for fermion massive particles which have spin $-\frac{1}{2}$. And the equation depends on vector and scalar potential is written as [12]

$$\left[c\alpha\vec{p} + \beta\left(mc^2 + S(r)\right) + V(r) - E \right]\psi_{n\kappa}(\vec{r}) = 0 \tag{5.2}$$

Where $\psi_{n\kappa}(\vec{r})$ is written in terms of r, θ, ϕ parameters as in the following [25]

$$\psi_{n\kappa}(\vec{r}) = \psi(r, \theta, \phi\ )$$

And E is the connecting relativistic binding energy of the system, m is corresponds the mass of particle, $\vec{p} = -i\hbar\nabla$ is momentum operator and α and β are 4x4 Dirac matrices [14,61,63]. The spinor wave functions are defined as

$$\psi_{n\kappa}(\vec{r}) = \begin{pmatrix} F_{n\kappa}(\vec{r})\, Y_{jm}^{l}(\theta,\phi) \\ iG_{n\kappa}(\vec{r})\, Y_{jm}^{\tilde{l}}\,(\theta,\phi) \end{pmatrix} \tag{5.3}$$

In the equation (5.3) $F_{n\kappa}(\vec{r})$ and $G_{n\kappa}(\vec{r})$ demonstrate the radial wave functions of upper-spinor and lower-spinor ingredients, respectively. And $Y_{jm}^{l}(\theta, \phi)$ and $Y_{jm}^{\tilde{l}}\,(\theta, \phi)$ are represented the spherical harmonic functions coupled to the total angular momentum j and its projection m on the z axis [25].

We can obtain for the upper-spinor ingredient a second differential equation [61,63,64,65] in form

$$F''_{n\kappa}(r) - \left(\frac{\kappa(\kappa+1)}{r^2} + A_s^2 + B_s\Sigma\right) F_{n\kappa}(\vec{r}) = 0 \tag{5.4}$$

Where

$$A_s^2 = \frac{1}{\hbar^2 c^2}\left[m^2c^4 - E_{n\kappa}^2 - (mc^2 - E_{n\kappa})C_-\right] \tag{5.5a}$$

$$B_s = \frac{1}{\hbar^2 c^2}(mc^2 + E_{n\kappa} - C_-) \tag{5.5b}$$

And for choices of two values κ, for $\kappa < 0$, $\kappa(\kappa + 1) = l(l + 1)$ and for $\kappa > 0$, $\kappa = -l(l + 1)$. Furthermore, the lower-spinor ingredient is obtained as [25]

$$G_{n\kappa}(r) = \frac{1}{mc^2 + E_{n\kappa} - C_-}\left(\frac{d}{dr} + \frac{\kappa}{r}\right) F_{n\kappa}(r) \tag{5.6}$$

Where, if we choose exact spin symmetric case $C_- = C_\Delta = 0$, $E_{n\kappa}$ is non-zero ($E_{n\kappa} \neq 0$). This means that we have only positive energy spectrum [25].Now, considering the same things for the existence of pspin symmetry ($\sum = C_+ = C_\sum$), we can obtain a second-order differential equation again for the lower-spinor [25] ingredient,

$$G''_{n\kappa}(r) - \left(\frac{\kappa(\kappa+1)}{r^2} + A_{ps}^2 - B_{ps}\Delta\right) G_{n\kappa}(r)=0 \tag{5.7}$$

Where [25]

$$A_{ps}^2 = \frac{1}{\hbar^2 c^2}[m^2 c^4 - E_{n\kappa}^2 + (mc^2 + E_{n\kappa})C_+] \tag{5.8a}$$

$$B_{ps} = \frac{1}{\hbar^2 c^2}(mc^2 - E_{n\kappa} + C_+) \tag{5.8b}$$

We can obtain the upper-spinor [25] component $F_{n\kappa}(r)$ as in the following

$$F_{n\kappa}(r) = \frac{1}{mc^2 + E_{n\kappa} + C_+}\left(\frac{d}{dr} - \frac{\kappa}{r}\right) G_{n\kappa}(r) \tag{5.9}$$

Where if the exact pspin symmetric case is chosen ($C_+ = C_\sum = 0$), $E_{n\kappa}$ is not be equal mc^2 ($E_{n\kappa} \neq mc^2$) [25]. This means that we have only negative energy spectrum. From the overhead equations, the energy eigenvalues depend on the quantum numbers n and κ, and also the pseudo-orbital angular quantum number l according to $\kappa(\kappa - 1) = \tilde{l}(\tilde{l} + 1)$, which implies that $j = \tilde{l} \pm \frac{1}{2}$ are degenerate for $\tilde{l} \neq 0$. The quantum condition for bound states demands the finiteness of the solution at infinity and at the origin point [25].

It is known that the equations (5.4) and (5.7) can be solved exactly only for the case of $\kappa = -1$ ($l = 0$) and $\kappa = 1$ ($\tilde{l} = 0$), respectively, when the spin-orbit coupling centrifugal and pseudo-centrifugal terms will get suppressed. In the case of non-zero

l and $\tilde{l}$ values, we can use the approximation scheme to deal with the spin-orbit centrifugal (psedo-centrifugal) term when κ is not large and when vibrations of the small amplitude near the minimum point $r = r_e$ [62,63].

$$\frac{1}{r^2} \approx \frac{1}{r_e^2}\left[D_0 + D_1 \frac{-e^{-2\alpha r}}{1+e^{-2\alpha r}} + D_2 \left(\frac{-e^{-2\alpha r}}{1+e^{-2\alpha r}}\right)^2\right] \tag{5.10}$$

At this equation D_i corresponding the parameters of coefficient $(i = 0{,}1{,}2)$ are obtained by [25]

$$D_0 = 1 - \left(\frac{1+e^{-2\alpha r_e}}{2\alpha r_e}\right)^2 \left(\frac{8\alpha r_e}{1+e^{-2\alpha r_e}} - (3 + 2\alpha r_e)\right) \tag{5.11a}$$

$$D_1 = -2(1 + e^{2\alpha r_e})\left[3\left(\frac{1+e^{-2\alpha r_e}}{2\alpha r_e}\right) - (3 + 2\alpha r_e)\left(\frac{1+e^{-2\alpha r_e}}{2\alpha r_e}\right)\right] \tag{5.11b}$$

$$D_2 = (1 + e^{2\alpha r_e})^2 \left(\frac{1+e^{2\alpha r_e}}{2\alpha r_e}\right)^2 \left(3 + 2\alpha r_e - \frac{4\alpha r_e}{1+e^{2\alpha r_e}}\right) \tag{5.11c}$$

And at this point, we have neglected higher order terms.

5.1.1 Spin Symmetric Solution

A total of potential is taken in equation (5.4) in terms of standard Rosen-Morse well potential [25] given in equation (5.1).

$$\sum = \mathrm{V(r)} = -4\mathrm{V}_1 \frac{\mathrm{e}^{-2\alpha\mathrm{r}}}{(1+\mathrm{e}^{-2\alpha\mathrm{r}})^2} + V_2\left(\frac{1-\mathrm{e}^{-2\alpha\mathrm{r}}}{1+\mathrm{e}^{-2\alpha\mathrm{r}}}\right) \tag{5.12}$$

As a new variable z(r) $=\mathrm{e}^{-2\alpha\mathrm{r}}$ substituting into the sum potential given by equation (5.12) and which is written into the equation (5.4) can be in such form that [25],

$$F''_{n\kappa}(z) + \frac{1+z}{z(z+1)} F'_{n\kappa}(z) + \frac{-a_2 z^2 + a_1 z - a_0^2}{z^2(1+z)^2} F_{n\kappa}(z) = 0 \tag{5.13}$$

Where $F_{n\kappa}(0) = F_{n\kappa}(-1)$ requires that at boundaries we can show the parameters a_i $(i = 0{,}1{,}2)$ whose forms are [25]

$$a_0 = \frac{1}{2\alpha}\sqrt{\frac{\kappa(\kappa+1)}{r_e^2} D_0 + B_s V_2 + A_s^2} > 0 \tag{5.14a}$$

$$a_1 = \frac{1}{4\alpha^2}\left(\frac{\kappa(\kappa+1)}{r_e^2}(D_1 - 2D_0) + 4B_s V_1 - 2A_s^2\right) \quad (5.14b)$$

$$a_2 = \frac{1}{4\alpha^2}\left(\frac{\kappa(\kappa+1)}{r_e^2}(D_0 - D_1 + D_2) + A_s^2 - B_s V_2\right) \quad (5.14c)$$

We can start with the NU method's [26,27] applications by comparing the hypergeometric differential equation [25] with the equation (5.13)

$$\psi_n''(r) + \frac{\tilde{\tau}(r)}{\sigma(r)}\psi_n'(r) + \frac{\tilde{\tau}(r)}{\sigma^2(r)}\psi_n(r) = 0 \quad (5.15)$$

Where

$$\psi_n(r) = \phi(r)\, y(r) \quad (5.16)$$

To determine the parameters [25]

$$\tilde{\tau}(z) = 1 + z,\ \ \sigma(z) = z(z+1),\ \ \tilde{\sigma}(z) = -a_2 z^2 + a_1 z - a_0^2 \quad (5.17)$$

And more calculate the function $\pi(z)$ as [25]

$$\pi(z) = \frac{1}{2}[\sigma'(r) - \tilde{\tau}(r)] \pm \sqrt{\frac{1}{4}[\sigma'(r) - \tilde{\tau}(r)]^2 - \tilde{\sigma}(r) + k\sigma(r)}$$

$$= \frac{z}{2} \pm \frac{1}{2}\sqrt{[1 + 4(\alpha_2 + k)]z^2 + 4(k - a_1)z + 4a_0^2} \quad (5.18)$$

In order to make the discriminant of the expression under square root, we are also looking for a physical value of k by using the equation (5.18) which is equal to zero [25]

$$\text{k} = a_1 + 2a_0^2 \pm 2a_0 q,\ \text{q}= \sqrt{1 + \frac{\kappa(\kappa+1)D_2}{\alpha^2 r_e^2} + \frac{4V_1 B_s}{\alpha^2}} \quad (5.19)$$

By using the value of k into the equation (5.18), the following appropriate solutions can be obtained [25]

$$\pi(z) = -a_0 - \frac{1}{2}(2a_0 + q - 1)z \quad (5.20)$$

And

$$k = a_1 + 2a_0^2 + a_0 q \tag{5.21}$$

The function $\tau(z) = \tilde{\tau}(z) + 2\,\pi(z)$ can be calculated by regarded to equations (5.17) and (5.20). If we use this equation in the bound state condition, we can establish the solution when $\tau'(z) > 0$ as [25]

$$\tau(z) = 1 - 2a_0 - (2a_0 + q - 2)\pi, \quad \tau'(z) = -(2a_0 + q - 2) < 0 \tag{5.22}$$

Where prime corresponds to the derivative of variable z. The energy eigenvalues can be calculated from the finding energy equation according to the method [26,66], so we need to obtain the values of the parameters: $\bar{\lambda} = k + \pi'(s)$ and $\bar{\lambda}_n = -n\,\tau'(s) - \frac{1}{2}n(n-1)\sigma''(s), n = 0,1,2, ...$, as [25]

$$\bar{\lambda} = \frac{1}{2} + a_1 + 2a_0^2 - a_0 + \left(a_0 - \frac{1}{2}\right) q \tag{5.23}$$

and

$$\bar{\lambda}_n = -n^2 - n + n(2a_0 + q), n = 0,1,2, \dots \tag{5.24}$$

The supernatural energy equation, including vector and scalar potential of particles which has spin $-\frac{1}{2}$ can be obtained by the definitions of variables in equations (5.14) and (5.19) and the relation of $\bar{\lambda} = \bar{\lambda}_n$ [25].

$$\frac{1}{\hbar^2 c^2}\left[m^2c^4 - E_{n\kappa}^2 - (mc^2 - E_{n\kappa})C_-\right] = -\frac{\kappa(\kappa+1)D_0}{r_e^2} - B_s V_2$$

$$+\alpha^2\left[n + \frac{1}{2} - \frac{q}{2}\frac{\kappa(\kappa-1)\frac{D_1 - D_2}{r_e^2} + 2B_s V_2}{4\alpha^2\left(n + \frac{1}{2} - \frac{q}{2}\right)}\right]^2 \tag{5.25}$$

In addition, according to the Klein-Gordon theory in units of $\hbar = c = 1$ equally mixed Rosen-Morse-type potentials, we can obtain the arbitrary l −wave energy equation in the case of exact spin symmetric. $V = s,\ \Delta = 0,\ C_- \to 0$ [25].

$$m^2 - E_{nl}^2 = -\frac{l(l+1)D_0}{r_e^2} - (m + E_{nl})V_2$$

$$+\alpha^2\left[n + \delta + \frac{l(l+1)\frac{D_1 - D_2}{r_e^2} + 2(m + E_{nl})V_2}{4\alpha^2(n+\delta)}\right]^2 \tag{5.26}$$

With

$$\delta = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{l(l+1)D_2}{\alpha^2 r_e^2} + \frac{4}{\alpha^2}(m + E_{nl})V_1} \tag{5.27}$$

Where the values of n $(n = 0,1,2, ...)$ and the values of l $(l = 0,1,2, ...)$ are corresponding to the quantum number and the orbital quantum number, respectively. By using the present potential model we try to find the correspond wave function. Firstly, the weight function can be defined like in ref. [66]. The function is written as

$$\rho(z) = \frac{1}{\sigma(z)} e^{\int \frac{\tau(z)}{\sigma(z)} dz} = z^{-2a_0}(1+z)^{-q} \tag{5.28}$$

And by using the first part of the equation (16) as [25]

$$\phi(z) = e^{\int \frac{\tau(z)}{\sigma(z)} dz} = z^{-a_0}(1+z)^{\frac{1-q}{2}} \tag{5.29}$$

Again by using the second part of the equation (5.16) a wave function can be obtained called Rodrigues representation [25]

$$y_n(z) = \frac{K_n}{\rho(r)} \frac{d^n}{dr^n}[\sigma^n(r)\rho(r)] = K_n z^{2a_0}(1+z)^q \frac{d^n}{dz^n}[z^{n-2a_0}(1+z)^{n-q}]$$

$$\sim P_n^{(-2a_0,q)}(1+2z),\ z \in [0,1] \tag{5.30}$$

Where K_n is the normalization constant and the Jacobi polynomials [25] $P_n^{(\mu,\upsilon)}(x)$ are defined for $\Re(\mu) > -1$ and $Re(\upsilon) > -1$ for the $x \in [-1,+1]$.

The upper-spinor wave function can be written by using $F_{n\kappa} = \phi(z)\, y_n(z)$ as [25]

$$F_{n\kappa} = K_{n\kappa}(e^{-2\alpha r})^{a_0}(1+e^{-2\alpha r})^{\frac{1-q}{2}} P_n^{(-2a_0,-q)}(1+2e^{-2\alpha r})$$

$$= N_{n\kappa}(e^{-2\alpha r})^{a_0}(1+e^{-2\alpha r})^{\frac{1-q}{2}}\ {}_2F_1(-n, n+1-2a_0-q; -2a_0+1; e^{-2\alpha r}) \tag{5.31}$$

Where $a_0 > 0$ and $q > -1$. We calculate the normalization constant $K_{n\kappa}$ for the upper-spinor component are [25]

$$N_{n\kappa} = \left[\frac{\Gamma(-q+2)\Gamma(-2a_0+1)}{2\alpha\Gamma(n)} \sum_{m=0}^{\infty} \frac{(-1)^m (n-2a_0+1-q)_m \Gamma(n+m)}{m!(m-2a_0)\Gamma(m-2a_0-q+2)} f_{n\kappa}\right]^2 \tag{5.32}$$

With

$$f_{n\kappa} = {}_3F_2(-2a_0 + m, -n, n + 1 - 2a_0; m - 2a_0 - q + 2; 1 - 2a_0; 1 \quad (5.33)$$

Furthermore, we can obtain the corresponding lower component $G_{n\kappa}$ [25] as in the following

$$G_{n\kappa}(r) = C_{n\kappa} \frac{(e^{-2\alpha r})^{a_0}((1 + e^{-2\alpha r})^{\frac{1-q}{2}})}{mc^2 + E_{n\kappa-} - C_-} \left[-2\alpha a_0 - \frac{\alpha(1-q)e^{-2\alpha r}}{1 + e^{-2\alpha r}} + \frac{\kappa}{r} \right]$$

$$\times\ {}_2F_1(-n, n - a_0 - q + 1; -2a_0 + 1; e^{-2\alpha r})$$

$$+C_{n\kappa} \left[\frac{2\alpha n[n-2a_0+1-q](e^{-2\alpha r})^{a_0+1}(1+e^{-2\alpha r})^{\frac{1-q}{2}}}{(2a_0+1)(mc^2+E_{n\kappa-}-C_-)} \right]$$

$$\times\ {}_2F_1(-n + 1;\ n - 2a_0 - q + 2; -2a_0 + 2; e^{-2\alpha r}), a_0 > 0 \quad (5.34)$$

for the exact spin symmetry $E_{n\kappa} \neq -mc^2$. Also, we end the hypergeometric series ${}_2F_1(-n, n - 2a_0 - q + 1; -2a_0 + 1; e^{-2\alpha r})$ for $n = 0$ and hence, by choosing the all values of real parameters $q > 0$ and $a_0 > 0$ it is converged [25].

5.1.2. Pspin Symmetric Solution

Now we take a difference potential in equation (5.7) in the same way as stated before as [25]

$$\Delta = V(r) = -4V_1 \frac{e^{-2\alpha r}}{(1+e^{-2\alpha r})^2} + V_2 \left(\frac{1-e^{-2\alpha r}}{1+e^{-2\alpha r}} \right) \quad (5.35)$$

And by defining a new variable $Z(r) = e^{-2\alpha r}$, we can obtain a Schrödinger-like equation for $G_{nr}(r)$ which is the lower-spinor component [25],

$$G_{n\kappa}''(z) + \frac{z+1}{z(z+1)} G_{n\kappa}'(z) + \frac{(b_2 z^2 + b_1 z - b_0^2)}{z^2(z^2+1)} G_{n\kappa}(z) = 0 \quad (5.36)$$

Where the parameters [25] b_j $(j = 0,1,2)$ are showed by

$$b_0 = \frac{1}{2\alpha} \sqrt{\frac{\kappa(\kappa-1)D_0}{r_e^2} + B_{ps}V_2 + A_{ps}^2} > 0 \quad (5.37a)$$

$$b_1 = \frac{1}{4\alpha^2} \left(\frac{\kappa(\kappa-1)}{r_e^2} (D_1 - 2D_0) + 4B_{ps}V_1 - 2A_{ps}^2 \right) \quad (5.37b)$$

$$b_2 = \frac{1}{4\alpha^2} \left(\frac{\kappa(\kappa-1)}{r_e^2} (D_0 - D_1 + D_2) + A_{ps}^2 - B_{ps}V_2 \right) \quad (5.37c)$$

In order to avoid the repetition of the solution of equation (5.36), we understand from the relation between the previous set of parameters (a_0, a_1, a_2) and the present set of (b_0, b_1, b_2), for the negative energy solution of the pseudo-spin symmetry such $\sum = C_+ = C_\sum$. Also, we can understand the positive energy solution for the spin-symmetry by performing changes [61,63].

$$F_{n\kappa}(r) \leftrightarrow G_{n\kappa}(r),\ V(r) \to -V(r) \text{ (or } V_1 \to -V_1, V_2 \to -V_2)$$

$$E_{n\kappa} \to -E_{n\kappa} \text{ and } C_- \to -C_+ \tag{5.38}$$

Considering the previous result in equation (5.25) and applying the above transformations, we finally arrive at the pspin symmetric energy equation [25].

$$[m^2c^4 - E_{n\kappa}^2 + (mc^2 + E_{n\kappa})C_+] = \frac{\hbar^2c^2\kappa(\kappa-1)D_0}{r_e^2} + (mc^2 - E_{n\kappa} + C_+)V_2 +$$

$$\frac{\hbar^2c^2\alpha^2}{4}\left[2n + 1 - p + \frac{\frac{\hbar^2c^2\kappa(\kappa-1)(D_1-D_2)-2(mc^2+E_{n\kappa}+C_+)V_2}{r_e^2}}{\hbar^2c^2\alpha^2(2n+1-p)}\right] \tag{5.39}$$

Where

$$p = \sqrt{1 + \frac{\kappa(\kappa-1)D_2}{\alpha^2r_e^2} - \frac{4V_1B_{ps}}{\alpha^2}} \tag{5.40}$$

Again the radial lower-spinor wave function [25] in equation (5.31) become

$$G_{n\kappa}(r) = d_{n\kappa}(e^{-2\alpha r})^{-b_0}(1 + e^{-2\alpha r})^{\frac{1}{2}(1-p)}P_n^{(-2b_0,-p)}(1 + 2e^{-2\alpha r})$$

$$= d_{n\kappa}(e^{-2\alpha r})^{-b_0}(1 + e^{-2\alpha r})^{\frac{1}{2}(1-p)}\ {}_2F_1(-n, n - 2b_0 - p + 1, -2b_0 + 1; e^{-2\alpha r}) \tag{5.41}$$

This equation satisfies for the bound states, $p > 0$ and $b_0 > 0$ and the normalization constant [25] is

$$d_{n\kappa} = \left[\frac{\Gamma(-p+2)\Gamma(-2b_0+1)}{2\alpha\Gamma(n)}\sum_{m=0}^{\infty}\frac{(-1)^m(n+1-2b_0-p)\ {}_m\Gamma(\mathrm{n}+1)}{m!(m-2b_0)\Gamma(m-2b_0-p+2)}g_{n\kappa}\right]^{-\frac{1}{2}} \tag{5.42}$$

With

$$g_{n\kappa} = {}_3F_2(-2b_0 + m, -n, n + 1 - 2b_0 - p); 1 - 2b_0; 1) \tag{5.43}$$

5.2. Applications to some physical potential models

We accept two physical potential cases as the reflectionless-type potential and the Rosen –Morse potential [25].

5.2.1. The reflectionless-type potential

Now, we consider the Rosen-Morse potential. In this potential, when we choose the coefficient of $tan\alpha r$ equals to zero, this can be achieved. Also, this type potential is the special case of the symmetrical double-well potential. The reflectionless-type potential takes the form [67]

$$V(r) = -a^2 sech^2 \alpha r,\ a^2 = \frac{\lambda(\lambda+1)}{2},\ \lambda = 1,2,3,\dots \tag{5.44}$$

For three values $\lambda = 1,2\ and\ 3$ the plot of the potential can be shown in figure. 5 [25] as

Now, we consider the Rosen-Morse potential. In this potential, when we choose the coefficient of $tan\alpha r$ equals to zero, this can be achieved. Also, this type potential is the special case of the symmetrical double-well potential. The reflectionless-type potential takes the form [67]

$$V(r) = -a^2 sech^2 \alpha r,\ a^2 = \frac{\lambda(\lambda+1)}{2},\ \lambda = 1,2,3,\dots \tag{5.44}$$

For three values $\lambda = 1,2\ and\ 3$ the plot of the potential can be showed in figure. 5 [25] as

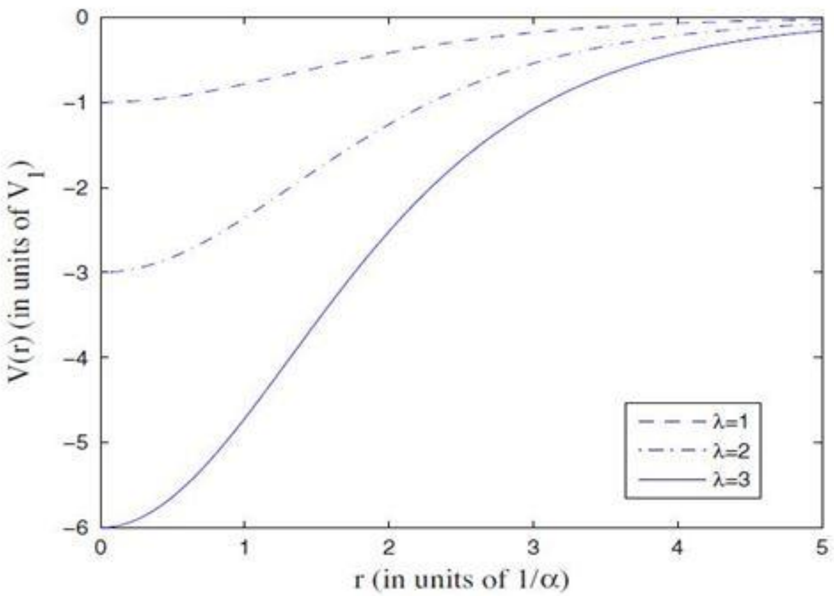


Figure 5.2: For Three values of $\lambda = 1{,}2{,}3$, a plot of reflectionless potential.

By using the potential in equation (25), the energy equation becomes [25]

$$m^2 - E_{r\kappa}^2 - C_-(m - E_{n\kappa}) = -\frac{\kappa(\kappa+1)D_0}{r_e^2} + \frac{\alpha^2}{4}\left[2n + 1 - q_0 + \frac{\kappa(\kappa+1)\frac{D_1 - D_2}{r_e^2}}{\alpha^2(2n+1-q_0)}\right]^2 \tag{5.45}$$

And again this potential substituting into the equation (5.31) for obtaining the upper-spinor wave functions [25], it turns into

$$F_{n\kappa}(r) = N_{n\kappa}(e^{-2\alpha r})^{-s_0}(1 + e^{-2\alpha r})^{\frac{1}{2}(1-q_0)}(1 + 2e^{-2\alpha r}) \tag{5.46}$$

Where

$$S_0 = \frac{1}{2\alpha}\sqrt{\frac{\kappa(\kappa+1)D_0}{r_e^2} + A_s{}^2} > 0, \quad q_0 = \sqrt{\frac{\kappa(\kappa+1)D_2}{\alpha^2 r_e^2} + \frac{4a^2 B_s}{\alpha^2}} \tag{5.47}$$

These results which are given in equations (45) and (46) are identical [18] for s-wave case ($\kappa = -1$). If we have the pspin case, the spectrum of the energy [25] becomes

$$m^2 - E_{n\kappa}^2 + C_+(m + E_{n\kappa}) = -\frac{\kappa(\kappa-1)D_0}{r_e^2} + \frac{\alpha^2}{4}\left[2n + 1 - p_0 + \frac{\kappa(\kappa-1)\frac{D_1 - D_2}{r_e^2}}{\alpha^2(2n+1-p_0)}\right]^2 \tag{5.48}$$

And the lower-spinor component of pspin symmetry wave function [25]

$$G_{n\kappa}(r) = \widetilde{N}_{n\kappa}(e^{-2\alpha r})^{-\omega_0}(1+e^{-2\alpha r})^{\frac{1}{2}(1-p_0)}P_n(2\omega_0, -P_0)(1+2e^{-2\alpha r}) \tag{5.49}$$

Where

$$\omega_0 = \frac{1}{2\alpha}\sqrt{\frac{\kappa(\kappa-1)D_0}{r_e^2} + {A_{ps}}^2} > 0, \quad P_0 = \sqrt{1+\frac{\kappa(\kappa-1)D_2}{\alpha^2 r_e^2} - \frac{4a^2 B_{ps}}{\alpha^2}} > 0 \tag{5.50}$$

For the bound state $\frac{4a^2 B_{ps}}{\alpha^2} \leq 1$ when $\kappa = 1$.

In the next step, the non-relativistic limit of the energy eigenvalues and wave functions of the solutions are discussed. If one consider the transformation $E_{n\kappa} + m \cong 2\mu$ and $E_{n\kappa} - m \cong E_{nl}$ and take $C_- = 0\ (\Delta = 0)$, the expression for the equation (5.45) and wave function equation (5.46) (in $\hbar = c = 1$) can be written as in the following [25]

$$E_{nl} = \frac{l(l+1)D_0}{2\mu r_e^2} - \frac{\alpha^2}{2\mu}\left[n+\frac{1}{2}-\frac{1}{2}q_0+\frac{\hbar^2 l(l+1)(D_1-D_2)}{4\alpha^2 r_e^2(n+\frac{1}{2}-\frac{1}{2}q_0)}\right]^2 \tag{5.51}$$

And the wave function [25]

$$R_{nl}(r) = N_{nl}(e^{-2\alpha r})^{-s_0}(1+e^{-2\alpha r})^{\frac{1}{2}(1-q_0)}P_n^{(2s_0,q_0)}(1+2e^{-2\alpha r}) \tag{5.52}$$

Where

$$S_0 = \frac{1}{2\alpha}\sqrt{\frac{l(l+1)D_0}{r_e^2} - \frac{2\mu}{\hbar^2}E_{nl}} > 0, \quad q_0 = \sqrt{\frac{l(l+1)D_2}{\alpha^2 r_e^2} + \frac{8\mu\alpha^2}{\alpha^2\hbar^2}} \tag{5.53}$$

To conclude, it is necessary to mention that reflectionless-type potential here reminds one of the modified PT potential in the one-dimensional case [22]. However, for the present case, it is in the three-dimensional case. Thus, the original symmetry is broken. The energy levels could be obtained readily [22].

5.2.2 The Rosen-Morse Potential

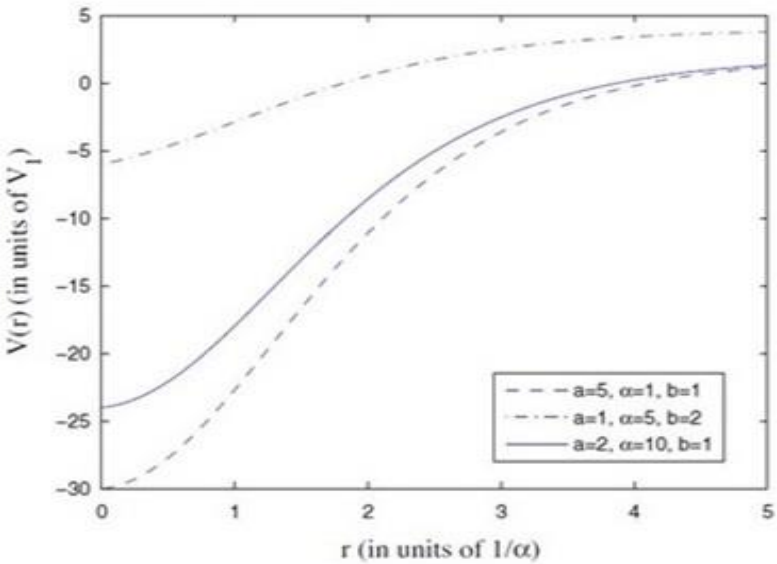
The standard Rosen-Morse potential is introduced by Rosen and Morse [60] which is helpful for discussing polyatomic vibrational energies and useful to describe interatomic interaction of the linear molecules. If we are supposed to give an example for this case, application of the NH_3 molecule's vibrational states can be considered [25]. One can achieve, when

$$V(r) = -a(a+\alpha)\, sech^2\alpha r + 2b\, tanh\alpha r \tag{5.54}$$

Where a and b represent real dimensionless parameters [25]. This potential can be plotted like in figure 3 [25]. The potential is plotted for the set of three parameter values. Now, let us obtain the spin symmetry energy spectrum for the Rosen-Morse well potential in equation (25) and (31). It takes the form [25],

$$m^2 - E_{n\kappa}^2 - C_-(m - E_{n\kappa}) = -\frac{\kappa(\kappa+1)D_0}{r_e^2} - 2b(m + E_{n\kappa} - C_-) +$$

$$\frac{\alpha^2}{4}\left[2n+1-q_1+\frac{\frac{\kappa(\kappa+1)(D_1-D_2)}{r_e^2}+4b(m+E_{n\kappa}-C_-)}{\alpha^2(2n+1-q_1)}\right]^2 \tag{5.55}$$



Figere 5.3: A plot of the Rosen – Morse potential for three different sets of parameters.

And the upper spinor component of the wave functions $F_{n\kappa}$ [25] obtained as,

$$F_{n\kappa}(r) = N_{n\kappa}(e^{-2\alpha r})^{-s_1}(1+e^{-2\alpha r})^{\frac{1}{2}(1-q_1)}P_n^{\ (-2s_{1,}-q)}(1+2e^{-2\alpha r}) \tag{5.56}$$

Where

$$S_1 = \frac{1}{2\alpha}\sqrt{\frac{\kappa(\kappa+1)D_0}{r_e^2} + 2bB_s + A_s^2} > 0, \quad q_1 = \sqrt{\frac{\kappa(\kappa+1)D_2}{\alpha^2 r_e^2} + \frac{4a(a+\alpha)B_s}{\alpha^2}} \tag{5.57}$$

Furthermore, the spectrum of energy for the Rosen-Morse well potential is defined in the presence of the spin symmetric as [25]

$$m^2 - E_{n\kappa}^2 + C_+(m+E_{n\kappa}) = -\frac{\kappa(\kappa+1)D_0}{r_e^2} + 2b(m - E_{n\kappa} + C_+) +$$

$$\frac{\alpha^2}{4}\left[2n+1-p_1+\frac{\frac{\kappa(\kappa-1)(D_1-D_2)}{r_e^2} - 4b(m-E_{n\kappa}+C_+)}{\alpha^2(2n+1-p_1)}\right]^2 \tag{5.58}$$

And the lower-spinor wave function can be written as [25]

$$G_{n\kappa}(r) = d_{n\kappa}(e^{-2\alpha r})^{-\omega_1}(1+e^{-2\alpha r})^{\frac{1}{2}(1-q_1)}P_n^{\ (-2\omega_{1,}-p_1)}(1+2e^{-2\alpha r}) \tag{5.59}$$

Where

$$w_1 = \frac{1}{2\alpha}\sqrt{\frac{\kappa(\kappa-1)D_0}{r_e^2} + 2bB_{ps} + A_{ps}^2} > 0, \quad p_1 = \sqrt{\frac{\kappa(\kappa-1)D_2}{\alpha^2 r_e^2} - \frac{4a(a+\alpha)B_{ps}}{\alpha^2}} \tag{5.60}$$

For the bound state $\frac{4a(a+\alpha)B_{ps}}{\alpha^2} \leq 1$ when $\kappa = 1$. In the next step, the non-relativistic limit [24,55] of the energy eigenvalues and wave functions of the our solutions are discussed. By taking $C_- = 0$ $(\Delta = 0)$, and one considers the expression for the equation (5.55) and wave function equation (5.56) (in $\hbar = c = 1$) can ben shown as in the following [25]

$$E_{nl} = \frac{l(l+1)D_0}{2\mu r_e^2} - \frac{\alpha^2}{2\mu}\left[n+\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{l(l+1)D_2}{\alpha^2 r_e^2}+\frac{8\mu a(a+\alpha)}{\alpha^2}} + \frac{\frac{l(l+1)(D_1-D_2)}{r_e^2}+8\mu b}{\alpha^2\left(n+\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{l(l+1)D_2}{\alpha^2 r_e^2}+\frac{8\mu a(a+\alpha)}{\alpha^2}}\right)}\right]^2 \tag{5.61}$$

And

$$R_{nl}(r) = N_{nl}(e^{-2\alpha r})^{-s_1}(1+e^{-2\alpha r})^{\frac{1}{2}(1-q_1)}P_n^{\ (-2s_{1,}-q_1)}(1+2e^{-2\alpha r}) \tag{5.62}$$

Where

$$s_1 = \frac{1}{2\alpha}\sqrt{\frac{l(l+1)D_0}{r_e^2} + 4\mu b - 2\mu E_{nl}} > 0, \quad q_1 = \sqrt{1 + \frac{l(l+1)D_2}{\alpha^2 r_e^2} + \frac{8a(a+\alpha)}{\alpha^2}} \tag{5.63}$$

Also, N_{nl} represents the normalization constant [24].

5.3. Discussion

In this chapter two special cases of the energy eigenvalues have been studied which are written in equations (5.25) and (5.39) for spin and pspin symmetry, respectively.

I. We have introduced the energy spectrum and the wave function equations in the presence of the reflectionless-type potential for the s-wave spin symmetric case ($\kappa = -1, l = 0$) and choosing the units parameters ($\hbar = c = 1$) [25] we have

$$m^2 - E_{n,-1}^2 - (m - E_{n,-1})C_- = \alpha^2\left[n + \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4a^2}{\alpha^2}(m + E_{n,-1} - C_-)}\right]^2 \tag{5.64}$$

And

$$F_{n,-1}(r) = C_{n,-1}(e^{-2\alpha r})^{\frac{-A_{-1}}{2\alpha}}$$

$$(1 + e^{-2\alpha r})^{\frac{1}{2}\left(1 - \sqrt{1 + \frac{4a^2(m + E_{n,-1} - C_-)}{\alpha^2}}\right)} P_n^{\frac{-A_{-1}}{2\alpha} - \sqrt{1 + \frac{4a^2(m + E_{n,-1} - C_-)}{\alpha^2}}}(1 + 2e^{-2\alpha r}) \tag{5.65}$$

Where, ${A_{-1}}^2 = m^2 - E_{n,-1}^2 - (m - E_{n,-1})C_-$. And for another case

($\kappa = -1, \tilde{l} = 0$) the $\tilde{s}$-wave pspin symmetric case

$$m^2 - E_{n,1}^2 + (m + E_{n,1})C_+ = \alpha^2\left[n + \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4a^2}{\alpha^2}(m - E_{n,1} + C_+)}\right]^2 \tag{5.66}$$

Where $\frac{4a^2}{\alpha^2}(m - E_{n,1} + C_+) \leq 1$ and we can write the lower-spinor component of pspin symmetric wave functions [25].

$$G_{n,1}(r) = d_{n,1}(e^{-2\alpha r})^{\frac{A_1}{2\alpha}}(1 + e^{-2\alpha r})^{\frac{1}{2}\left(1 + \sqrt{1 - \frac{4a^2(m - E_{n,1} + C_+)}{\alpha^2}}\right)} P_n^{\frac{-A_1}{\alpha} - \sqrt{1 - \frac{4a^2(m - E_{n,1} + C_+)}{\alpha^2}}}(1 + 2e^{-2\alpha r}) \tag{5.67}$$

Here, $A_1^2 = m^2 - E_{n,1}^2 + (m + E_{n,1})C_+$. For the spin symmetric case the Rosen-Morse potential model we can write that equation,

$$m^2 - E_{n,-1}^2 - (m - E_{n,-1})C_- = -2b(m + E_{n,-1} - C_-) + \alpha^2\left[n + \frac{1}{2} - \frac{\beta_{-1}}{2} + \frac{b(m + E_{n\kappa} - C_-)}{\alpha^2(n + \frac{1}{2} - \frac{\beta_{-1}}{2})}\right]^2 \tag{5.68}$$

And the wave functions $F_{n\kappa}(r)$ have that upper-spinor component [25]

$$F_{n,-1}(r) = N_{n,-1}(e^{-2\alpha r})^{-\gamma_{-\frac{1}{2}}} (1 + e^{-2\alpha r})^{\frac{1}{2}(1-\beta_{-1})} P_n^{(-\gamma_{-1},-\beta_{-1})}(1 + 2e^{-2\alpha r})$$

Where

$\beta_1 = \sqrt{1 + \frac{4a(a+\alpha)}{\alpha^2}(m + E_{n,-1} - C_-)}$ and $\gamma_{-1} = \sqrt{\frac{2b(m+E_{n,-1}-C_-)+A_{-1}^2}{2\alpha}}$ we can write the energy equation for the pseudospin case as in the following [25]

$$m^2 - E_{n,1}^2 + (m + E_{n,1})C_+ = 2b(m - E_{n,1} + C_+) + \alpha^2 \left[(n + \frac{1}{2} - \frac{\beta_1}{2} - \frac{b(m-E_{n,1}+C_+)}{\alpha^2(n+\frac{1}{2}-\frac{\beta_1}{2})}\right]^2 \tag{5.69}$$

And the lower-spinor wave function is showed for this case as [25]

$$G_{n,1}(r) = d_{n,1}(e^{-2\alpha r})^{-\frac{\gamma_1}{2}} (1 + e^{-2\alpha r})^{\frac{1-\beta_1)}{2}} P_n^{(-\gamma_1,-\beta_1)}(1 + 2e^{-2\alpha r}) \tag{5.70}$$

Where

$\beta_1 = \sqrt{1 - \frac{4a(a+\alpha)}{\alpha^2}(m - E_{n,1} + C_+)}$ and $\gamma_1 = \sqrt{2b(m - E_{n,1} + C_+) + \frac{A_1^2}{\alpha}}$

II. We have introduced the other potential forms by the transformation of the potential (5.1). We consider a potential depending on variable x which is $V(x)$. Now we choose the transformations: $x \to -x$, $\alpha \to i\alpha$ and $V_2 \to iV_2$. These are complex transform parameters, the equation transforms into a trigonometric Rosen-Morse-type form [25].

$$V(x) = -V_1 \, sec^2\alpha x + V_2 \, tan\alpha x \tag{5.71}$$

Where $\alpha = \frac{\pi}{2a}$ and $x = [0, a]$. Where $\Re(V_1) > 0$ when $x \to -x$ and $i \to -i$, if relation $V(-x) \to V^*$ exists, the potential $V(x)$ is said to be PT-symmetric, where P denotes parity operators (space reflection) and T denotes time reversal [68, 69]. This PT-symmetric potential is plotted in figure 4 [25] for various sets of parameters V_1 and V_2. Thus the spin-symmetric energy equation ($\kappa = -1$) can be obtained from equations (5.19) and (5.25) as

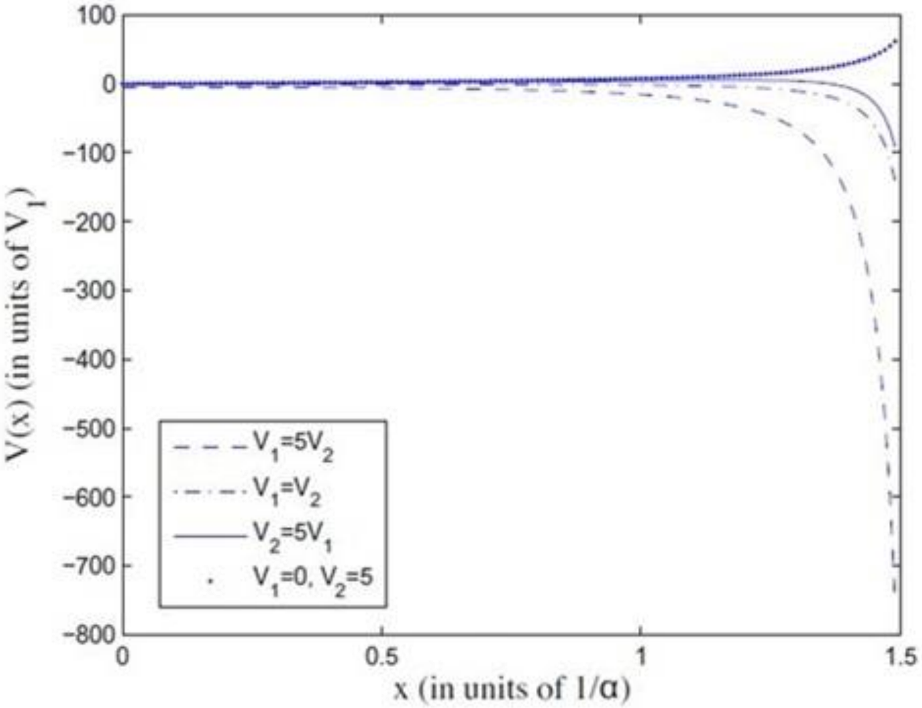


Figure 5.4: Plot of the trigonometric Rosen – Morse type potential [see eq.(71)] for various of parameters.

$$m^2 - E_{n,-1}^2 - (m - E_{n,-1})C_- =$$

$$-\alpha^2\left[n+\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4V_1}{\alpha^2}(m+E_{n,-1}-C_-)}\right]^2 + \left(\frac{V_2}{2\alpha}\right)^2\left(\frac{m+E_{n,-1}-C_-}{\left[n+\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4V_1}{\alpha^2}(m+E_{n,-1}-C_-)}\right]}\right)^2 \qquad (5.72)$$

Here, $4V_1(m+E_{n,-1}-C_-) \leq \alpha^2$ [25].

CHAPTER 6

CONCLUSION

As a result, firstly we have obtained the Dirac equation from the Klein-Gordon equation. Then, the solution of Dirac equation has been defined for the free particle and obtained the eigenfunctions and eigenvalues energy of the particle. It is obvious that the particles have negative solutions which can be seen in Figure 2.1 and Figure 2.2 (it can be considered positron = anti-particle = anti-matter). And then, we have compared these results with results of non-relativistic studies. Another study in this thesis is the solution of Dirac equation for three (2+1) dimension exactly solvable potentials which are harmonic oscillator potential, Coulomb potential and Morse potential.

Furthermore, we have studied the scattering solution of Dirac particle including the (2+1) dimension Coulomb plus scalar potential. Phase shifts of Dirac equation has been solved as a second order trigonometric differential equation. From which the scattering angle $\theta = \pi$ has symmetry with cross section $\sigma(\theta)$.

Finally, we have studied bound state of Dirac particle under an external potential field. For this solution, we have defined a relation of upper and lower spinor wave function in the presence of reflectionless spherical scalar and vector potential and approach of analytical Rosen-Morse type potential under the condition of relativistic spectrum of energy. The solution of wave functions has been expressed in terms of generalized Jacobi polynomials and hypergeometric functions. Also, we have used the recently introduced exponential approach of centrifugal spin-orbit (psedospin-centrifugal) potential terms. One of the most interesting result is the spin (pspin) energy spectrum of Dirac equation that has been the same as the solution of energy spectrum of the Klein-Gordon equation by choices of $V = \pm S$ $(\sum = \Delta = 0,\ C_{\pm} = 0)$. We can state that this result is possible. It has been seen that spin and pspin symmetry of the Dirac equation is similar to the relativistic spin $-\frac{1}{2}$ and spin-0 particles in the presence of spherical scalar and vector potentials. Obviously, the limit of non-relative theory can be reduced with suitable transformations [61]. Also, the problem turns to s-wave solution in case of quantum value $\kappa = \pm 1$ $(l = \tilde{l} = 0)$ [70,71].

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