# Development Of A Solvable Class Of Linear Time Varying Systems

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**Supervisor**

**Prof. Dr. Arif NACAROĞLU**

**by**

**İhsan DAL**

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Prof. Dr. Metin BEDİR

Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Ergün ERÇELEBİ

Head of Department

This is to certify that we have read this thesis and that in our consensus/majority opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master.

Prof. Dr. Arif NACAROĞLU

Supervisor

Examining Committee Members

Y. Doç. Dr. Sabri BİRLİK ……………………

Prof. Dr. Arif NACAROĞLU ……………………

Prof. Dr. Rauf MİRZABABAYEV ……………………

Y. Doç. Dr. Nurdal WATSUJİ ……………………

Y. Doç. Dr. Taner İNCE ……………………

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İhsan DAL

## ABSTRACT

## DEVELOPMENT OF A SOLVABLE CLASS OF LINEAR TIME VARYING SYSTEMS

**DAL, İhsan**
**M.Sc. in Electrical and Electronics Engineering**
**Superviser: Prof. Dr. Arif NACAROĞLU**
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There have been various solution methods including Laplace transform developed for Time Invariant Linear Systems (TILS). However, there isn’t a general analytical solution method for Time Varying Linear Systems (TVLS). To solve these systems, we should find suitable transfer matrices to transform TVLS into TILS so that analytical solution becomes possible. In this study, we have added a new class to known solution classes ($A_1$Class, $A_h$Class, HG Class) to transform TVLS into TILS and the proposed solution class is presented. The solvability conditions for the linear system is reviewed and the relation between solvability conditions and the eigenvalues are given. Transformation method of the new class of linear systems is derived and supported by some examples.

**Keywords:** Time varying linear system, Solvable linear system

## ÖZET

## ZAMANLA DEĞİŞEN LİNEER SİSTEMLER İCIN YENİ BİR ÇÖZÜLEBİLİR YAPININ GELİŞTİRİLMESİ

**DAL, İhsan**
**Yüksek Lisans Tezi, Elektrik ve Elektronik Mühendisliği Bölümü**
**Tez Yöneticisi: Prof. Dr. Arif NACAROĞLU**
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Zamanla değişmeyen doğrusal sistemlerin çözümü için Laplace dönüşümünün de içinde bulunduğu birçok çözüm yöntemi geliştirilmiştir. Ancak zamanla değişen doğrusal sistemlerin genel bir analitik çözüm metodu bulunmamaktadır. Bu sistemlerin çözülebilmesi, sistemlerin uygun transfer matrisleri ile zamanla değişmeyen doğrusal sistemlere dönüştürülmesi ile mümkündür. Bu çalışmada zamanla değişen doğrusal sistemlerin, zamanla değişmeyen doğrusal sistemlere dönüştürülmesini sağlayan sınırlı sayıdaki sınıflandırmalara ($A_1$Class, $A_h$Class, HG Class) yeni bir sınıf eklenmiş ve belirtilen yeni sınıf için çözüm metodu açıklanmıştır. Sistemin bu var olan sınıflandırma sistemlerinin hangisine ait olduğunun belirlenmesi, sistemin öz değerlerinin yapısı ile doğrudan ilişkilidir ve bu çalışmada yine öz değerler dikkate alınarak, yukarıda belirtilen sınıflandırmalara ait olmayan yeni bir çözüm yöntemi açıklanmıştır.

**Anahtar Kelimeler:** Zamanla değişen doğrusal sistemler, Çözülebilir doğrusal sistemler

## ACKNOWLEDGEMENTS

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**TABLE OF CONTENTS**

**Page**

## LIST OF FIGURES

**Page**

# LIST OF SYMBOLS

| | |
|---|---|
| $x(t)$ | Vector of State Variables |
| $x(t_0)$ | Vector of State Variables at Initial Time |
| $y(t)$ | Output |
| $y(t+\delta)$ | Time Shifted Output |
| $a_n(t)$ | A known Continuous Function of Time |
| $b_n(t)$ | A known Continuous Function of Time |
| $z(t)$ | A New Vector of State Variables of Constant System Matrix |
| $z(t_0)$ | A New Vector of State Variables of Constant System at Initial Time |
| $T(t)$ | Transfer Matrix |
| $T^{-1}(t)$ | Inverse of Transfer Matrix |
| $h(t, t_0)$ | Time Varying Function |
| $H(s, t)$ | Time Variable System Function |
| $A_1$ | A Class of Solvable Time Varying Linear System |
| $A_h$ | A Class of Solvable Time Varying Linear System |
| $\bar{A}$ | A Constant System Matrix which is produced |
| $A$ | Constant System Matrix |
| $A(t)$ | Time Varying System Matrix |
| $B(t)$ | Time Varying Input Matrix |
| $u(t)$ | Input Vectors |
| $U(s)$ | Laplace Transform of $u(t)$ |
| $u(t+\delta)$ | Time Shifted Input |
| $\phi(t, t_0)$ | State Transition Matrix |

$\bar{\phi}(t, t_0)$ State Transition Matrix of Constant System Matrix which is produced

λ Eigenvalue of System Matrix

$g(t)$ Scalar Time Function

TVNL Time Varying Non-Linear System

TINL Time Invariant Non-Linear System

TV Time Varying System

TVL Time Varying Linear System

TI Time Invariant System

TIL Time Invariant Linear System

IC Integrated Circuit

LTVS Linear Time Varying System

LTIS Linear Time Invariant System

RF Radio Frequency

# CHAPTER 1

# INTRODUCTION

The solution methods of dynamical systems have attracted many scientist, as it is very important for the exact or approximate analysis of the physical phenomenon. The mathematical models of physical events can be described in the form of differential equation in time domain. Differential equation is a type of a mathematical equation which includes some function of one or more variable with its derivative.

Almost all physical dynamic events show non-linear behavior making them very difficult to solve, but the linearization of the problem helps us comprehend and solve the problems. The difficulty of the mathematical model is proportional to its degree of derivative of the system. If coefficients of the variables or its derivatives are constant, the solution is relatively easier. If not, the analytical solution of the problem is more difficult. To solve differential equations we generally put it into the form of state space. By doing several transformations higher order (nth order) differential equations are converted to first order nth dimensional state space form. The degree of the derivative reflects in the size of matrices in state space form.

The n-dimensional linear time varying or constant is shown by

$$\frac{\mathrm{dx(t)}}{\mathrm{dt}} = A(t)x(t) + B(t)u(t) \qquad (1.1)$$

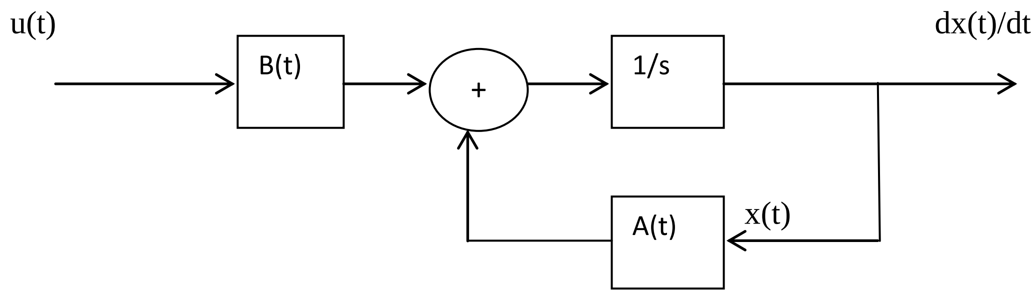


Figure 1.1 Block diagram representation of the state space equations

where x(t) is nx1 vector of state variables, u(t) is 1x1 input vectors, A(t) is nxn time-varying system matrix, B(t) is nx1 time-varying input matrix. A(t) and B(t) are matrices with the elements directly related to the circuit elements and if the elements of circuit are time-dependent system is time-dependent. Equations (1.1) can be written compactly in a matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \dots & \vdots \\ a_{n1(t)} & \dots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix} u(t) \tag{1.2}$$

If excitation u(t) is zero, the linear equations becomes

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{1.3}$$

and it is called as homogeneous systems.

There is a general and easy solution of linear time-invariant systems. If A(t) is constant matrix A or it is commutative with it is functions,

$$\frac{dx(t)}{dt} = Ax(t) \tag{1.4}$$

then the homogeneous solution of (1.4) is [1-3]

$$\begin{aligned} x(t) &= \phi(t, t_0)\, x(t_0) \\ &= L^{-1}[sI - A]^{-1} x(t_0) \\ &= e^{A(t-t0)} x(t_0) \end{aligned} \tag{1.5}$$

If A(t) is a function of time in homogeneous time varying systems defined in (1.3), a general solution method has not been defined yet. However the solution of the system is possible for some limited time-varying linear systems [4]. In paper [4] transformation matrices are proposed to change the linear time varying system into the linear time invariant system to make the analytical solution possible.

There are generalized solutions for three type of time varying systems which are not commutative depending on eigenvalues $(A_1, A_{h,} and\ HG)$. The possible transformation matrices are given in many works $(A_1, A_{h,} and\ HG)$ to change linear time varying systems into linear time invariant system. For example if eigenvalues are constant ($\lambda_1$=$k_1$, $\lambda_2$=$k_2$, ……., $\lambda_n$=$k_n$), then the system is defined $A_1$ class. If eigenvalues are the multiple of the same time varying function h(t) ($\lambda_1$=$k_1$h(t), $\lambda_2$=$k_2$h(t), ……., $\lambda_n$=$k_n$h(t)) , then the system belongs to $A_h$ class [3]. In HG class; if the system eigenvalue are $\lambda_1$=$t^{k-1}$-1 and $\lambda_2$=$t^{k-1}$+1, for integer k , the transfer matrix converting the system into time invariant system is possible [9].

In this thesis, we are presenting a new solution method for time varying linear system class. If A(t) is defined as in (1.6) and if its components meet requirements, transformation matrix which converts the system into time invariant system can be found easily.

$$A(t)=\begin{bmatrix} f(t)-g(t) & 1 \\ \frac{dg(t)}{dt}-g(t)^2 & f(t)+g(t) \end{bmatrix} \qquad (1.6)$$

where, f(t) and g(t) are function of t with condition of g(t) meeting derivability. The eigenvalues can be resulted as

$$\boldsymbol{\lambda}_1=f(t)-\sqrt{\frac{dg(t)}{dt}} \text{ and } \boldsymbol{\lambda}_2=f(t)+\sqrt{\frac{dg(t)}{dt}} \text{ for } A(t). \qquad (1.7)$$

# CHAPTER 2

# DYNAMIC SYSTEMS

## 2.1 Introduction

Dynamic Systems are a vast topic; they refer anything which change or evolve with time. Dynamics systems are available in all sciences. A communication network and vehicles (aircraft, spacecraft, motorcycles, cars) are examples of dynamical systems. In fact, all dynamic systems are nonlinear time varying systems. However, with some assumptions and approximations, the system is simplified into linear and modeled as a simple and understandable. That way the system is linearized to be easy enough for comprehending and solving the system. For instance suppose we are dealing with a car's location speeds etc. In order to exactly calculate car's location speed acceleration, we have to take all dynamic parameters into account. But this makes problem very difficult and complicated to solve. That is why we do some assumptions for linearization. For example car's mass is generally considered to be constant. On the other hand mass of the car varies in time. As it moves its weight decreases because of cars fuel consumption. If we ignore this and similar conditions which affect the mass, system becomes linear and result is still very close to precise result.

The general dynamic systems are as these four matrices completely specify the state-space model.

| **System** | **Type** |
|---|---|
| $\frac{\mathrm{dx(t)}}{\mathrm{dt}} = \alpha(t)x(t)$ | TVL(Time Varying Linear System) |
| $\frac{\mathrm{dx(t)}}{\mathrm{dt}} = \alpha x(t)$ | TIL(Time Invariant Linear System) |
| $\frac{\mathrm{dx(t)}}{\mathrm{dt}} = \alpha(t)x^2(t)$ | TVNL(Time Varying Non-linear System) |
| $\frac{\mathrm{dx(t)}}{\mathrm{dt}} = \alpha x^2(t)$ | TINL(Time Invariant Non-linear System) |

In this work, only linear problems will be considered and they will be modeled as n dimensional problem for n-chosen state variables as $\frac{\mathrm{dx(t)}}{\mathrm{dt}} = A(t)x(t)$ and there for the first order coupled differential equations representing the j-th state becomes

$$\frac{\mathrm{d}x_j(t)}{\mathrm{dt}} = a_{j1}x_1(t) + a_{j2}x_2(t) + \cdots + a_{jn}x_n(t) \tag{2.1}$$

In system representation,

$$\mathrm{A(t)}=\begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \dots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix} \tag{2.2}$$

$$\mathrm{x(t)}= \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{2.3}$$

$$\frac{\mathrm{dx(t)}}{\mathrm{dt}} = \begin{bmatrix} \frac{\mathrm{d}x_1\mathrm{(t)}}{\mathrm{dt}} \\ \vdots \\ \frac{\mathrm{d}x_n\mathrm{(t)}}{\mathrm{dt}} \end{bmatrix} \tag{2.4}$$

Then,

$$\begin{bmatrix} \frac{\mathrm{d}x_1\mathrm{(t)}}{\mathrm{dt}} \\ \vdots \\ \frac{\mathrm{d}x_n\mathrm{(t)}}{\mathrm{dt}} \end{bmatrix} = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \dots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{2.5}$$

which can be summarized as

$$\frac{\mathrm{dx(t)}}{\mathrm{dt}} = A(t)x(t) \tag{2.6}$$

## 2.2 Time Invariant System (TI)

Systems can be classified in a various ways depending on their different properties. However, the more important modes of classification in system theory are dichotomies in the sense that they involve but two categories, say linear time invariant systems and linear time varying systems. In reality all physical systems are time varying. Central Heating Boiler thermo dynamic system of a building, the launching of the dynamics of an object is moving, dynamic system of a moving car, an operating system of electronic circuit etc. are just several example of time varying system. In short, no system is really time invariant and linear. But the linearization of systems makes the analysis easier. For example, sometimes a problem is solved with the assumptions the system is considered linear and time invariant.

Let an input u(t) applied to a system produces an output y(t). The delay of the application instant of the input signal causes the delay of the output in same amount of time. If the waveform of the output remains unchanged than the system is said to be time invariant. Time invariance is defined for systems, not for signals. Signals are mostly time varying. If a signal is time invariant such as *u (t)* = 1 for all t, then it is a very simple or a tri vial signal. The characteristics of time-invariant systems must be independent of time.

Some physical systems must be modeled as time-varying systems. For example, a burning rocket is a time-varying system because its mass decreases rapidly with time. Although the performance of an automobile or a TV set may deteriorate over a long period of time, its characteristics do not change appreciable in the first couple of years. Thus a large number of physical systems can be modeled as time-invariant systems over a limited time period.

The mathematical model of the homogeneous time invariant system is given as

$$\begin{bmatrix} \frac{\mathrm{d}x_1(\mathrm{t})}{\mathrm{dt}} \\ \vdots \\ \frac{\mathrm{d}x_m(\mathrm{t})}{\mathrm{dt}} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} \tag{2.7}$$

### 2.2.1 Time Invariant Linear System (TILS)

Time Invariant Linear (TIL) systems are systems that are both linear and time invariant. Let $x_1(t)$ and $x_2(t)$ be any two signals. Suppose that the output of a system to $x_1(t)$ is $y_1(t)$ and the output of the system to $x_2(t)$ is $y_2(t)$. If this always implies that the output of the system to $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is $\alpha_1 y_1(t) + \alpha_2 y_2(t)$, then the system is linear and the superposition principle is said to hold. A system is said to be time invariant if when y(t) is the output that corresponds to x(t), than any time shifted δ , y(t - δ ) is the output that corresponds to x(t- δ ).

Mathematically, TIL system is expressed by a state equation given as

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \qquad (2.8)$$

Multiply by $e^{-At}$on both sides

$$e^{-At}\frac{dx(t)}{dt} - e^{-At}Ax(t) = Bu(t) \qquad (2.9)$$

where

$$\frac{d}{dt}[e^{-At}x(t)] = e^{-At}Bu(t) \qquad (2.10)$$

and the solution is

$$x(t) = e^{At}x(t_0)+\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \qquad (2.11)$$

#### 2.2.1.1 Example

Consider the equation of the system is given as below

$$\begin{bmatrix} dx_1(t)/dt \\ dx_2(t)/dt \end{bmatrix} = Ax(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad (2.12)$$

Its solution is

$$x(t) = e^{At}x(t_0)+\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \qquad (2.13)$$

the matrix function $e^{At}$ is the inverse Laplace transform of $[sI - A]^{-1}$, which can be found as below

$$[sI - A]^{-1} = \begin{bmatrix} s/(s^2+1) & 1/(s^2+1) \\ -1/(s^2+1) & s/(s^2+1) \end{bmatrix} \qquad (2.14)$$

Thus we have

$$e^{At} = L^{-1}[\mathrm{sI - A}]^{-1} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & cost \end{bmatrix} \quad (2.15)$$

The solution of the system is found to be

$$\mathrm{x(t)} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & cost \end{bmatrix} x(t_0) \quad (2.16)$$

### 2.3 Time Varying System (TV)

Time Varying System is a system in which certain quantities governing the system's behavior change with time, so that the system will respond differently to the same input at different times. A time varying system is a system that is not time invariant and take an important place in modern technology. They are widely used as in the human vocal tract, communication systems, power electronic circuits, electrical machinery and electronics.

In communication system, the communication channels are time varying due to movement of the source, receiver or scatters. Therefore, the channel is acting like time varying filters. Besides that, parametric converters, time varying filters, switched capacitor networks, mixers and RF circuits are also different types of time varying systems.

In power electronic circuits high power semiconductors devices such as thristors, diacs, triacs are used and these devices are either triggered by the response signals; in either case the controlling signal is periodic and these devices behave as periodically time varying components. Because of time varying nature of power systems, time varying system analysis methods are used in the systems as, power system protection, power system transient, partial discharges, load forecasting, power system measurement [4].

In integrated circuits (IC) area, due to the heat generated by IC, circuit parameters are changing. The parameter variations need to be quantified in order to ensure a robust circuit.

Time-variant state equation is given as below:

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) \quad (2.17)$$

We can say that the general solution to time-variant state-equation is defined as:

$$x(t) = \phi(t, t_0)\, x(t_0) + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau \quad (2.18)$$

The function $\phi$ is called the state-transition matrix, because it (like the matrix exponential from the time-invariant case) controls the change for states in the state equation.

In a time-variant system, the general solution is obtained when the state-transition matrix is determined. For that reason, the first thing (and the most important thing) that we need to do here is find that matrix. We will discuss the solution to that matrix below.

The state transition matrix $\phi$ is not completely unknown, it must always satisfy the following relationships:

$$\frac{d\phi\,(t,t_0)}{dt} = A(t)\phi\,(t,t_0) \quad (2.19)$$

$$\phi\,(\tau, \tau) = I \quad (2.20)$$

And also must have the following properties:

$$\phi(t_2,t_1)\phi(t_1,t_0) = \phi(t_2,t_0) \quad (2.21)$$

$$\phi^{-1}(t,\tau) = \phi(\tau,t) \quad (2.22)$$

$$\phi^{-1}(t,\tau)\phi(\tau,t) = I \quad (2.23)$$

$$\frac{d\phi\,(t,t_0)}{dt} = A(t) \quad (2.24)$$

If the system is time-invariant, we can define $\phi$ as

$$\phi(t,t_0) = e^{A(t-t_0)} \quad (2.25)$$

### 2.3.1 Time Varying Linear System (TVL)

Any linear system represented with the time dependent operator A(t) demonstrates different properties at least at two different time instances. A system of this type is called time varying linear system (TVL) or time variant.

The system approach is a widely used in modeling electronic and mechanical systems. Linear systems are highly popular models due to their simplicity and convenience for mathematical analysis. Thus, many systems can be modeled as linear time varying systems at least for a limited range of operation. Figure 2.1 describes the general notion of an input-output system in a block diagram. The input is u and the output is y to describe physical quantities and their relations.

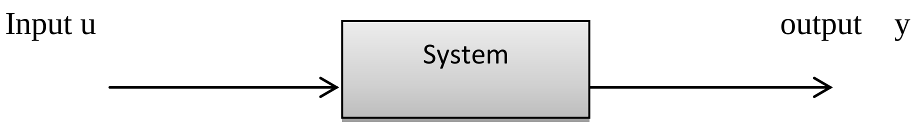


Figure 2.1 Input-Output System

A system is linear if it satisfies the property of superposition, that is, for any couple of inputs and outputs $y_1 = f(u_1)$ and $y_2 = f(u_2)$, the equation $ay_1 + by_2 = af(u_1) + bf(u_2)$ must be satisfied for any couple of scalars a and b.

The relation between the input and output of a time varying can be expressed in a variety of ways. This forms ‘‘ characterization’’ (representation) of the system. Basically, the input output relation of a linear time varying system may be expressed as

$$a_n(t)\frac{d^n y}{dt^n} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y(t) = b_m(t)\frac{d^m u}{dt^m} + \cdots + b_1(t)\frac{du}{dt} + b_0(t)u(t) \tag{2.26}$$

where $a_n(t)$ and $b_m(t)$ are known continuous functions of time. This equation is referred to as the fundamental equation of the system [5]. If there are more than one input and or output in the system then, in general, we have more than one high order simultaneous differential equations containing multi-input, multi-output variables.

The classical differential equation solution techniques can be applied successfully to a small class of systems and corresponding basis functions can be found in. This small class contains the systems, which are characterized by the following equations: Bessel equations, Weber equations, Hyper geometric equations, Airy equations and other [5].

The equation (2.26) defines a periodically time varying linear system if the coefficients of functions $a_n(t)$ and $b_m(t)$ are periodic with the system's fundamental period $T_0$. For periodically time varying systems, the periodicity makes it possible to apply some special techniques such as Floquet theory and spectral analysis [5]. In spectral analysis fundamental differential equation of linear LTV system is expressed in terms of algebraic matrix-vector relation by defining operational matrices for derivative, integral and any time varying component behavior. The system equations are transferred to spectral domain. Thus, solution of the system equation can be easily obtained by using the matrix operations. The solution is computed in spectral domain in term of Fourier coefficients. Then it is carried to the time domain by applying inverse Fourier Transform. This method gives the steady-state analysis of periodically time varying system. However the general analysis methods of LTV systems are still continuing to investigate.

Due to above mentioned difficulties of system representation by a single high order differential equation; state-space representation, has been developed. In modern system theory, it is preferred and found very convenient methods especially for computer simulations to use set of N first order linear differential equations of the form Eq. (2.27) together with the expression Eq. (2.28) for the output.

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) \qquad (2.27)$$

$$y(t) = C(t)x(t) + D(t)u(t) \qquad (2.28)$$

In these equations x(t) $\in R^n$, u(t) $\in R^n$, y(t) $\in R^n$ are the state, input and output respectively, at time t $\in R^n$; A(t),B(t),C(t),D(t) are matrices of order compatible with x(t), u(t) and y(t), and their elements are known and they are piece-wise continues functions defined on $R^+$. It is well known that the state solution is

$$x(t)=\phi(t, t_0)\, x(t_0)+\int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau \qquad (2.29)$$

where $\phi(t, t_0)$ is called the state transition matrix [3,5]. The $\phi(t, t_0)$ is the key to the solution of Eq.(2.29). Some solution techniques are given in [3] for different classes of linear system equations. The common one is commutative class. State-space representation of LTV system can be transformed into time invariant representation through the commutative class by using transformation as,

$$x(t) = T(t)z(t) \qquad (2.30)$$

Here, T(t) is the transformation matrix, which transforms the system representation into commutative or even a linear time invariant system representation.

Although, It is concluded in [3], [6] that, the commutative property is not an inherent property of a dynamic system, but rather is just a system representation property it is difficult to find transformation matrix T(t) given in Equation.(2.30). Therefore it is not easy to get the solution of system if it is not commutative. Spectral analysis method [5] can be applied efficiently for this representation, if A(t), B(t), C(t), D(t) matrices are periodically time varying.

A LTV system is excited by an impulse function, that is the delta function, δ(t) and the system's response to the impulse function is called ''impulse response'' and denoted as $h(t, t_0)$. The system response y(t) to the input u(t) applied at $t= t_0$ is given by the superposition integral

$$y(t) = \int_{t_0}^{t} h(t,\tau)u(\tau)d\tau \qquad (2.31)$$

This superposition is expressed as convolution of input-output, that is

$$y(t) = h(t, t_0) * u(\tau) \qquad (2.32)$$

However, a method for analytic expression of $h(t, t_0)$ is generally unknown and same difficulties mentioned in differential equation are valid for this representation.

Frequency domain approach for analysis of LTV is first developed by L.A. Zadeh, Zadeh's approach is essentially an extension of the frequency analysis techniques commonly used in LTI systems. He defines a time variable system function H(s,t), for a variable linear network. This function possesses most of the fundamental properties of the transfer function of fixed network. For this reason it is conveniently used to interpret the frequency domain behavior of systems and to realize the given frequency domain requirements in design problem. Further, once H(s,t) has been determined, the response to any given input can be obtained by treating H(s,t) as if it were the transfer function of a fixed network.

For a single-input, single-output time varying linear system, which is initially, relaxed, the time varying system function is defined by the relation

$$H(s,t) = \int_{-\infty}^{\infty} h(t,\tau)e^{-s(t-\tau)}\mathrm{d}\tau \tag{2.33}$$

The response of linear system to any input $u(t), t \geq t_0 \geq 0$, can be derived by

$$y(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} H(s,t)U(s)e^{-st}\mathrm{ds} \tag{2.34}$$

where U(s) is the Laplace transform of u(t).

However, there are similar difficulties to determine H(s,t) involved in solving the fundamental equation or the state equations of the system. To overcome some of difficulties the system equations transformed to spectral domain to use the spectral analysis techniques. The spectral analysis method basically uses Fourier series expansion of variables in linear periodically time varying systems.

Shortly, time varying system is that the same system in different times and for different results. If the input signal u(t) produces an output y(t) than any time shifted input, $u(t+\delta_1)$, results in a time shifted output $y(t+\delta_2)$,

where $\delta_1 \neq \delta_2$

Consider the nth-order time linear state-space description

$$\frac{\mathrm{dx(t)}}{\mathrm{dt}} = A(t)x(t) + B(t)u(t) \tag{2.35}$$

The system matrix A(t) is not constant (A(t)≠A). It is changing with time.

# CHAPTER 3

# SOLVABLE CLASS LTV SYSTEMS

## 3.1 Introduction

All kinds of differential equations can be numerically solved with acceptable deviations. But analytical solutions are still important especially for some of the limit problems. The difficult level of analytical solutions depends on the chosen mathematical model. Before deciding to do analytical solution, there are some solvability tests which help us known whether there is analytical solution. For instance, eigenvalues of the n dimensional dynamical systems give enough information about solvability of the system. There either in n'th order differential equation form or n dimensional state space form the characteristic equation plays important role in the termination of solvability.

The characteristic equation of the eigenvalues determines the solvability of the system and the system's ''solvable class'' is classified with their eigenvalues. In this chapter we will introduce three types of solvable classes, which are $\mathrm{A}_1$ Class, $\mathrm{A}_h$ Class and HG Class, of the systems.

## 3.2 $\mathrm{A}_1$ Class

$\mathrm{A}_1$ class is one of the solvable linear time varying systems. The eigenvalues of this class systems must be constant. In this case it is always possible to a constant matrix $\mathrm{A}_1$ that satisfies [6]

$$\mathrm{A}_1 A(t) - A(t)\mathrm{A}_1 = \frac{\mathrm{dA(t)}}{\mathrm{dt}} \tag{3.1}$$

The linear time varying system considered is governed by

$$\frac{\mathrm{dx(t)}}{\mathrm{dt}} = A(t)x(t) + B(t)u(t) \quad x(\mathrm{t}_0) = \mathrm{x}_0 \tag{3.2}$$

that solution of (3.2) is given by

$$x(t)=\phi(t, t_0)\, x(t_0)+\int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau \tag{3.3}$$

for example, system is $\frac{dx(t)}{dt} = A(t)x(t)$ and A(t) has $\frac{dA(t)}{dt}$ and if there exist a constant matrix $A_1$ that satisfies [3]

$$A_1 A(t) - A(t)A_1 = \frac{dA(t)}{dt} \tag{3.4}$$

Then the system is in $A_1$ class.

Considering nxn matrix A(t), $f_A(\lambda) = \det(\lambda I - A(t))$ is a polynominal of degree n of the form

$$|\lambda I - A(t)| = 0$$

$$\left|\begin{bmatrix} \lambda_{11} & \dots & 0 \\ \vdots & \lambda_{ij} & \vdots \\ 0 & \dots & \lambda_{nn} \end{bmatrix} - \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \dots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix}\right| = 0$$

$$\begin{bmatrix} \lambda_{11} - a_{11}(t) & \dots & -a_{1n}(t) \\ \vdots & \dots & \vdots \\ -a_{n1}(t) & \dots & \lambda_{nn} - a_{nn}(t) \end{bmatrix} = 0 \tag{3.5}$$

$$f_A(\lambda) = \lambda^n - tr\big(A(t)\big)\lambda^{n-1} + \dots + (-1)^n \det\big(A(t)\big) \tag{3.6}$$

$f_A(\lambda)$ is called the characteristic polynomial of A(t).

For $A_1$ class system, the system matrix must have constant eigenvalues.

$$\lambda_1 = k_1, \lambda_2 = k_2, \dots \dots \dots, \lambda_n = k_n \tag{3.7}$$

If the system matrix is as below

$$A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \dots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix} \tag{3.8}$$

The eigenvalues are found equating the determinant of $(\lambda I - A(t))$ to zero.

The transformation matrix

$$T(t) = e^{A_1 t} \tag{3.9}$$

is used to transform

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{3.10}$$

into

$$\frac{dz(t)}{dt} = \bar{A}(t)z(t) \tag{3.11}$$

In new state representation,

$$\bar{A} = T^{-1}(t)A(t)T(t) - T^{-1}(t)\frac{dT(t)}{dt} \tag{3.12}$$

and

$$x(t) = T(t)z(t) \tag{3.13}$$

### 3.2.1 Example

Consider the system (3.2) with A(t) being

$$A(t)=\begin{bmatrix} -1+\alpha\cos^2 t & 1-\alpha sintcost \\ -1-\alpha sintcost & -1+\alpha\sin^2 t \end{bmatrix} \tag{3.14}$$

It can be checked that A(t) in (3.14) belongs to the $A_1$ class because the eigenvalues of the system matrix A(t) are

$$\lambda_1 = \frac{(\alpha-2)+\sqrt{\alpha^2-4}}{2} \qquad \lambda_2 = \frac{(\alpha-2)-\sqrt{\alpha^2-4}}{2} \tag{3.15}$$

As shown in (3.15), eigenvalues are constant. One simple constant matrix $A_1$ that satisfies (3.4) is

$$A_1 A(t) - A(t)A_1 = \frac{dA(t)}{dt}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} -1+\alpha\cos^2 t & 1-\alpha sintcost \\ -1-\alpha sintcost & -1+\alpha\sin^2 t \end{bmatrix}$$

$$-\begin{bmatrix} -1+\alpha\cos^2 t & 1-\alpha sintcost \\ -1-\alpha sintcost & -1+\alpha\sin^2 t \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -2\alpha sint cost & -\alpha(\cos^2 t - \sin^2 t) \\ -\alpha(\cos^2 t - \sin^2 t) & 2\alpha sint cost \end{bmatrix} \tag{3.16}$$

$$\mathrm{A}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.17}$$

and related transformation matrix and state space matrix are found as

$$T(t) = e^{\mathrm{A}_1 t} = \mathrm{L}^{-1}[(SI - A)^{-1}] = \begin{bmatrix} cost & sint \\ -sint & cost \end{bmatrix} \tag{3.18}$$

Hence the new state representation becomes

$$\bar{A} = T^{-1}(t)A(t)T(t) - T^{-1}(t)\frac{\mathrm{dT(t)}}{\mathrm{dt}}$$

$$\bar{A} = \begin{bmatrix} (\alpha - 1) & 0 \\ 0 & -1 \end{bmatrix} \tag{3.19}$$

and

$$z(t) = e^{\bar{A}t} z(t_0) = \mathrm{L}^{-1}[(SI - \bar{A})^{-1}] = \begin{bmatrix} \mathrm{e}^{(\alpha-1)t} & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix} z(t_0) \tag{3.20}$$

Therefore the transformation of the solution into the system A(t) results as

$$x(t) = T(t)z(t) = \begin{bmatrix} \mathrm{e}^{(\alpha-1)t} cost & \mathrm{e}^{-t} sint \\ -\mathrm{e}^{(\alpha-1)t} sint & \mathrm{e}^{-t} cost \end{bmatrix} z(t_0) \tag{3.21}$$

### 3.3 $\mathrm{A}_h$ Class

$\mathrm{A}_h$ class is also one of the solvable linear time varying systems. The solution of the $\mathrm{A}_h$ class and the $\mathrm{A}_1$class are approximately the same. The only difference is to find transformation matrix. To classify the class of the system if it belongs to $\mathrm{A}_h$ class or not, the eigenvalues of the system A(t) must be found and checked if they are the multiple of some differentiable function or not. Let us this function is h(t) and $\frac{\mathrm{dh(t)}}{\mathrm{dt}}$ exists. The solution of [6]

$$\mathrm{A}_1 A(t) - A(t)\mathrm{A}_1 = \frac{\frac{\mathrm{dA(t)}}{\mathrm{dt}}}{\mathrm{h(t)}} - \frac{\frac{\mathrm{dh(t)}}{\mathrm{dt}}}{\mathrm{h^2(t)}} A(t) \tag{3.22}$$

gives us many possible $\mathrm{A}_1$ and then the best $\mathrm{A}_1$ which makes the solution of the system $\bar{A}$ easier is chosen.

Since the eigenvalues of the system A is multiple of function as

$$\lambda_1 = k_1 h(t), \lambda_2 = k_2 h(t), \dots \dots \dots\dots, \lambda_n = k_n h(t) \tag{3.23}$$

and since

$$g(t) = \int_{t_0}^{t} h(\tau) d\tau \tag{3.24}$$

is satisfied then the transformation matrix is written as

$$T(t) = e^{A_1 g(t)} \tag{3.25}$$

and $A_2$ is a constant matrix given by

$$A_2 = A_h(t_0) - A_1 \tag{3.26}$$

where

$$A_h(t_0) = \lim_{t \to t_0} \frac{A(t)}{h(t)} \tag{3.27}$$

The solution for system A can be easily performed as

$$x(t) = T(t)z(t) = \phi(t, t_0)\, x(t_0)$$

$$x(t) = \exp[A_1 g(t)] \exp[A_2 g(t)] \exp[-A_1 g(t_0)] x(t_0) \tag{3.28}$$

**3.3.1 Example**

Consider the system with A(t) being

$$A(t) = \begin{bmatrix} -3t^2 & 0 \\ -3t^5 & -6t^2 \end{bmatrix} \tag{3.29}$$

The eigenvalues of the system are obtain as

$$\lambda_1 = -3t^2 \text{ and } \lambda_2 = -6t^2 \tag{3.30}$$

As it can be seen easily both eigenvalues are the multiple of $h(t) = 3t^2$ and it means the system belongs to $A_h$ class where

$$A_1 A(t) - A(t)A_1 = \frac{\frac{dA(t)}{dt}}{h(t)} - \frac{\frac{dh(t)}{dt}}{h^2(t)} A(t)$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -3t^2 & 0 \\ -3t^5 & -6t^2 \end{bmatrix} - \begin{bmatrix} -3t^2 & 0 \\ -3t^5 & -6t^2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$$

$$\begin{bmatrix} -2t^{-1} & 0 \\ 5t^2 & -4t^{-1} \end{bmatrix} \begin{bmatrix} -2t^{-1} & 0 \\ 2t^2 & -4t^{-1} \end{bmatrix} \tag{3.31}$$

$$A_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \tag{3.32}$$

Satisfies equation (3.22) and the transformation matrix is written as

$$T(t) = \begin{bmatrix} e^{-t^3} & 0 \\ t^3 e^{-t^3} & e^{-t^3} \end{bmatrix} \tag{3.33}$$

which transforms the system into time invariant form and the solution of the system is found to be

$x(t)=\phi\,(t, t_0)\; x(t_0)=$

$$\begin{bmatrix} e^{-(t-t_0)^3} & 0 \\ ((t-t_0)^3 - 1)e^{-(t-t_0)^3} + e^{-2(t-t_0)^3} & e^{-(t-t_0)^3} \end{bmatrix} x(t_0) \tag{3.34}$$

**3.4 HG Class**

Another solvable class of system is called HG class. In the solvable class of system presented in this class is second order time varying system with in the form of $\lambda_1 = t^{k-1} - 1 \; and \;\; \lambda_2 = t^{k-1} + 1$ . Here, k is any integer and therefore eigenvalues are any order of t polynomials with conjugates.

Let us consider the eigenvalues of the second order dynamical system are as below

$$\lambda_1 = t^{k-1} - 1 \; and \;\; \lambda_2 = t^{k-1} + 1 \tag{3.35}$$

The dynamical system represented by these eigenvalues can be represented in the state space form as

$$\frac{\mathrm{dx(t)}}{\mathrm{dt}} = A(t)x(t)$$

$$= \begin{bmatrix} t^{k-1} - t & 1 \\ -(t^2 - 1) & t^{k-1} + t \end{bmatrix} x(t) \tag{3.36}$$

The transformation matrix which transforms the system into the time invariant form is in the form of [7]

$$T(t) = \mathrm{e}^{\frac{t^k}{k}} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \tag{3.37}$$

The transformation matrix T(t) convert the system A(t) into $\overline{A}$ for any k as

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.38}$$

Hence the solution of the new system

$$\frac{\mathrm{dz(t)}}{\mathrm{dt}} = \bar{A}z(t) \tag{3.39}$$

becomes

$$x(t) = \mathrm{e}^{\frac{(t-t_0)^k}{k}} \begin{bmatrix} 1 & (t - t_0) \\ (t - t_0) & (t - t_0)^2 + 1 \end{bmatrix} x(t_0) \tag{3.40}$$

Applying the invers transformation to that solution including initial conditions, the solution of the system is found as

$$\mathrm{x(t)=T(t)}\bar{\phi}\ (t, t_0)T^{-1}(t_0)\mathrm{x}(t_0) \tag{3.41}$$

### 3.4.1 Example

Second order TVL mathematical equation is given as

$$\frac{d^2y(\mathrm{t})}{\mathrm{d}t^2} - 2(t^2)\frac{dy(\mathrm{t})}{\mathrm{d}t} + (t^4 - 1)y(t) = f(t) \tag{3.42}$$

State – space form homogeneous system (f(t)=0) can be written as

$$\begin{bmatrix} dx_1(t)/dt \\ dx_2(t)/dt \end{bmatrix} = \mathrm{A(t)x(t)} =$$

$$\begin{bmatrix} t^2 - t & 1 \\ -(t^2 - 1) & t^2 + t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.43}$$

The eigenvalues of the system are obtain as

$$\lambda_1 = t^2 - 1 \ and \ \ \lambda_2 = t^2 + 1 \tag{3.44}$$

We use the following transformation matrix

$$T(t) = \mathrm{e}^{\frac{t^3}{3}} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \tag{3.45}$$

a new time invariant system can be written as

$$\frac{\mathrm{dz(t)}}{\mathrm{dt}} = \bar{A}z(t)$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z(t) \tag{3.46}$$

which transforms the system into time invariant form and the solution of the system is found to be

$$\mathrm{x(t)} = e^{\mathrm{e}^{\frac{(t-t_0)^3}{3}}} \begin{bmatrix} 1 & (t - t_0) \\ (t - t_0) & (t - t_0)^2 + 1 \end{bmatrix} e^{\mathrm{e}^{\frac{t_0{}^3}{3}}} \begin{bmatrix} 1 & 0 \\ -t_0 & 1 \end{bmatrix} x(t_0) \tag{3.47}$$

if the initial time is $t_0 = 0$, then the homogeneous solution becomes

$$\mathrm{x(t)} = \phi\ (t, 0)\ \mathrm{x}(t_0) = e^{\mathrm{e}^{\frac{(t)^3}{3}}} \begin{bmatrix} 1 & t \\ t & t^2 + 1 \end{bmatrix} x(0) \tag{3.48}$$

# CHAPTER 4

# A NEW SOLVABLE CLASS OF TIME VARYING SYSTEMS

## 4.1 Definition

In this chapter, we propose a new solvable class of linear time varying systems. In the previous chapter we explain how to solve $A_1$, $A_h$ and $HG$ classes by using proper transformation operation. The eigenvalues are important parameter for the solution of the dynamic system such as $A_1$, $A_h$ and $HG$ classes. The new solvable class of system which is presented in this work is second order time varying system with eigenvalues in the form of $\lambda_1 = f(t) + \sqrt{\frac{dg(t)}{dt}}$ *and* $\lambda_2 = f(t) - \sqrt{\frac{dg(t)}{dt}}$ where f(t) is integrable function of t, and g(t) is differentiable function of t. Any system which can be put in the form of (4.3) has analytical solution with using transformation matrix. In our study we define a suitable transformation matrix (4.4) which changes time varying system that meets requirements (4.3) into time invariant system.

Let us assume the eigenvalues of the second order dynamical system are as

$$\lambda_1 = f(t) + \sqrt{\frac{dg(t)}{dt}} \quad and \quad \lambda_2 = f(t) - \sqrt{\frac{dg(t)}{dt}} \tag{4.1}$$

Clearly, the system with these eigenvalues (4.1) don't belong to $A_1$ class, $A_2$ class and HG class.

A dynamical system which has these eigenvalues (4.1) can be put on the state space form as below.

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{4.2}$$

$$\frac{dx(t)}{dt} = \begin{bmatrix} f(t) - g(t) & 1 \\ \frac{dg(t)}{dt} - g(t)^2 & f(t) + g(t) \end{bmatrix} x(t) \quad (4.3)$$

$$T(t)=e^{\int_{t_0}^{t} f(t)dt} \begin{bmatrix} 1 & 0 \\ g(t) & 1 \end{bmatrix} \quad (4.4)$$

The transformation matrix T(t) transforms the system A(t) into $\bar{A}$ for any f(t) and g(t) as below

$$\bar{A}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (4.5)$$

As a result, the solution of the new system can be found as below

$$\frac{dz(t)}{dt} = \bar{A}z(t) \quad (4.6)$$

where

$$z(t)=e^{\bar{A}(t-t_0)}z(t_0) \quad (4.7)$$

After we find z(t), the system must be transformed in original domain with transformation matrix T(t) and finally the result is obtained as

$$x(t)=T(t)\bar{\phi}\,(t, t_0)T^{-1}(t_0)x(t_0) \quad (4.8)$$

Thus, the general solution for x(t) will be obtain as below

$$x(t)= e^{\int_{t_0}^{t} f(t)dt} \begin{bmatrix} 1 & t \\ g(t) & t\,g(t)+1 \end{bmatrix} x(t_0) \quad (4.9)$$

### 4.1.1 Example

Let us consider a time varying linear system that has state-space form (4.10)

$$\begin{bmatrix} dx_1(t)/dt \\ dx_2(t)/dt \end{bmatrix} = A(t)x(t)=$$

$$\begin{bmatrix} 2\sin(t)\cos(t) - 2\sin(t) - 1 & 1 \\ 2\cos(t) - (2\sin(t)+1)^2 & 2\sin(t)\cos(t) + 2\sin(t) + 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (4.10)$$

The eigenvalues of the system can be calculated as below

$$\lambda_1 = 2\sin(t)\cos(t) + \sqrt{2\cos(t)} \text{ and } \lambda_2 = 2\sin(t)\cos(t) - \sqrt{2\cos(t)}. \qquad (4.11)$$

Obviously, the system with this these eigenvalues values are not related to eigenvalues $of$ $A_1$ class, $A_h$class or HG class. Thus, we propose new solution method for this kind of systems.

By using transformation matrix below

$$T(t)=e^{(\sin(t))^2}\begin{bmatrix} 1 & 0 \\ 2\sin(t)+1 & 1 \end{bmatrix} \qquad (4.12)$$

and after transformation applied, a new time invariant system can be represented as

$$\frac{dz(t)}{dt} = \bar{A}z(t)$$

$$\frac{dz(t)}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z(t) \qquad (4.13)$$

In this equation (4.13) $\bar{A}$ is equal to

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (4.14)$$

Laplace transform is used to find z(t) as below

$$z(t)=e^{\bar{A}(t-t_0)}z(t_0) \qquad (4.15)$$

After finding z(t), it must be transformed in original domain with transformation matrix T(t) and finally the result is obtained as below

$$x(t)=T(t)z(t)T^{-1}(t_0)x(t_0)$$

$$x(t) = e^{(\sin t - t_0))^2}\begin{bmatrix} 1 & (t-t_0) \\ 2\sin(t-t_0)+1 & 2(t-t_0)sin(t-t_0)+2 \end{bmatrix} e^{(\sin(t_0))^{-2}}\begin{bmatrix} 1 & 0 \\ -2\sin(t\,t_0)+1 & 1 \end{bmatrix} x(t_0) \qquad (4.16)$$

if the initial time is $t_0 = 0$, then the homogenous solution becomes

$$x(t) = e^{(\sin t))^2}\begin{bmatrix} 1 & t \\ 2\sin(t)+1 & 2(t)sin(t)+2 \end{bmatrix} \quad (4.17)$$

For input u(t), the general solution for x(t) can be found as below

$$x(t)=\phi(t, t_0)\, x(t_0)+\int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau \quad (4.18)$$

**4.1.2 Example**

Another example for a time varying linear system which has state-space form (4.19)

$$\begin{bmatrix} dx_1(t)/dt \\ dx_2(t)/dt \end{bmatrix}=A(t)x(t)=$$

$$\begin{bmatrix} \ln(t)-t^3/3 & 1 \\ t^2-t^6/9 & \ln(t)+t^3/3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (4.19)$$

The eigenvalues of the system are obtain as below

$$\lambda_1 = \ln(t) - t \text{ and } \lambda_2 = \ln(t) + t \quad (4.20)$$

As it can be seen these values are not similar to eigenvalues $of$ $A_1$ class, $A_h$class or HG class.

By using transformation matrix below

$$T(t)=e^{t\ln(t)-t}\begin{bmatrix} 1 & 0 \\ t^3/3 & 1 \end{bmatrix} \quad (4.21)$$

and after transformation applied, a new time invariant system can be represented as

$$\frac{dz(t)}{dt} = \bar{A}z(t)$$

$$\frac{dz(t)}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z(t) \quad (4.22)$$

Laplace Transform is used to find z(t) as below

$$z(t)=e^{A(t-t_0)}z(t_0) \tag{4.23}$$

After we find z(t), it must be transformed in original domain with transformation matrix T(t) and finally the result is obtained as

$$x(t)=T(t)z(t)T^{-1}(t_0)x(t_0) \tag{4.24}$$

$$x(t) = e^{(t-t_0)\ln(t-t_0)-(t-t_0)}\begin{bmatrix} 1 & (t-t_0) \\ (t-t_0)^3/3 & (t-t_0)^4/3 \end{bmatrix} e^{((t-t_0)\ln(t-t_0)-(t-t_0))^{-1}}\begin{bmatrix} 1 & 0 \\ -(t-t_0)^3/3 & 1 \end{bmatrix} x(t_0) \tag{4.25}$$

if the initial time is $t_0 = 0$, then the homogenous solution becomes

$$x(t) = e^{t\ln(t)-t}\begin{bmatrix} 1 & t \\ (t)^3/3 & (t)^4/3 \end{bmatrix} \tag{4.26}$$

### 4.1.3 Example

Consider the series RLC circuit given in Fig. 4.1.

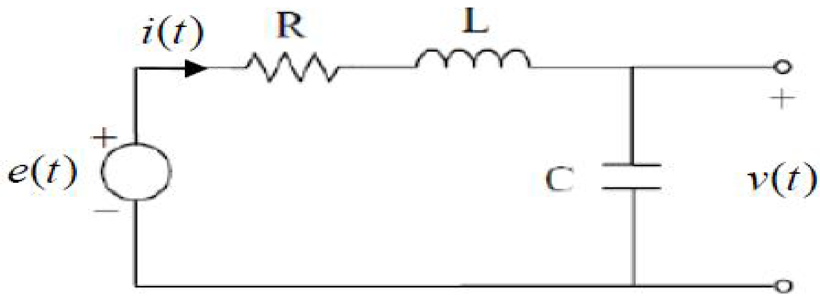


Figure 4.1 Series RLC circuit

where

$$R = \frac{2t}{t^2+1}, \qquad L = \frac{1}{t^2+1}, \qquad C = 1 \tag{4.27}$$

Using the terminal equations of the capacitor and inductor, and KVL,

$$V_L = L\frac{di(t)}{dt} \text{ and } V_C = \frac{1}{C}\int i(t)dt \quad e(t)=i(t)R+L\frac{di(t)}{dt}+\frac{1}{C}\int i(t)dt \tag{4.28}$$

resulting differential equations will be found as

$$\frac{d^2\mathrm{v(t)}}{\mathrm{dt}^2} + \frac{R}{L}\frac{\mathrm{dv(t)}}{\mathrm{dt}} + \frac{1}{LC}v(t) = \frac{1}{LC}e(t) \tag{4.29}$$

$$\mathrm{x}_1(t) = v(t) \tag{4.30}$$

$$\mathrm{x}_2(t) = \frac{\mathrm{dv(t)}}{\mathrm{dt}} \tag{4.31}$$

$$\frac{d\mathrm{x}_1(t)}{\mathrm{dt}} = \mathrm{x}_2(t) = \frac{\mathrm{dv(t)}}{\mathrm{dt}} \tag{4.32}$$

$$\frac{d\mathrm{x}_2(t)}{\mathrm{dt}} = \frac{d^2\mathrm{v(t)}}{\mathrm{dt}^2} \tag{4.33}$$

State-space form of equation will be found as

$$\begin{bmatrix} dx_1(t)/dt \\ dx_2(t)/dt \end{bmatrix} = \mathrm{A(t)x(t)} =$$

$$\begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1-t^2 & -2t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{4.34}$$

From equation (4.34) f(t) and g(t) can be found as below

$$\mathrm{f(t)} = -\mathrm{t} \text{ and } \mathrm{g(t)} = -\mathrm{t} \tag{4.35}$$

The eigenvalues of the system are obtain as below

$$\lambda_1 = -\mathrm{t} - i \text{and } \lambda_2 = -\mathrm{t} + i \tag{4.36}$$

By using transformation matrix below

$$\mathrm{T(t)} = e^{e^{-\frac{t^2}{2}}} \begin{bmatrix} 1 & 0 \\ -\mathrm{t} & 1 \end{bmatrix} \tag{4.37}$$

a new time invariant system can be represented as

$$\frac{\mathrm{dz(t)}}{\mathrm{dt}} = \bar{A}z(t)$$

$$\frac{\mathrm{dz(t)}}{\mathrm{dt}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z(t) \tag{4.38}$$

Laplace Transform is used to find z(t) as below

$$z(t)=e^{\bar{A}(t-t_0)}z(t_0) \tag{4.39}$$

If the initial time is $t_0 = 0$, then the homogenous solution becomes

$$x(t) = e^{-\frac{t^2}{2}}\begin{bmatrix} 1 & t \\ -t & -t^2+1 \end{bmatrix} \tag{4.40}$$

# CHAPTER 5

# RESULT AND CONCLUSION

Dynamic Systems are a very broad subject; they refer anything which change or evolve with time, but it is not easy to comprehend dynamic behavior of a linear time varying system is due mostly to we are not able to find the exact solution of dynamic systems explicitly. The result of this study enables us to solve a new solvable time varying linear system class. As we know all kinds of differential equations can be numerically solved with acceptable deviations. But analytical solutions are still important especially for some of the limit problems and the problems that require very fine accuracy and no margin of error. General analytical solutions of linear time invariant systems are already available, but the general analytical solution of linear time varying system is still lacking. However, there are some kind of classes of linear time varying systems that can be solved by $A_1$ Class, $A_h$ Class and HG Class solution methods. The common feature of these classes is to use a suitable transformation matrix which transforms the time varying linear systems into the time invariant linear systems which makes the analytical solution possible The characteristic equation of the eigenvalues determines the solvability of the system. Our results show that in addition to the well known transformation matrices due to the special values of the eigenvalues, another group of the time varying linear systems having another specific forms of eigenvalues may be possibly transformed into the time invariant case and this new group of differential equations can be analytically solved. Obviously, transformation matrix method appears to be the key to the solution of linear time-varying systems in general.

# REFERENCES


[1] L.A. Zadeh & C.A. Desoer. (1963). Linear System Theory, McGraw-Hill, pp 300-326.

[2] Veli Dündar. (2007). Dynamical Systems On Time Scales, Engineering and Sciences of Izmir Institute of Technology, p 18.

[3] Min-Yen Wu. (1980). Solution of certain classes of linear time-varying systems. *International Journal of Control*. Vol. **31**, No.1, 11-20.

[4] Hasari Karcı and Gülay Tohumoğlu. (2005). Zamanla Değişen Lineer Sistemlerin Dalgacik Analizi Ile Çözümü. University of Gaziantep.

[5] Hasari Karcı, Assoc. Prof. Dr. Gülay Tohumoğlu. (2005). Analysis of Linear Time Varying Systems in Wavelet Domain. M.Sc. Thesis in Electrical and Electronics Engineering . University of Gaziantep.

[6] Min-Yen Wu. (1980). Solvability and representation of linear time-varying systems. *International Journal of Control*, Vol. **31**, No.5, 937-945.

[7] ] Hasan Güneyli, Prof. Dr. ArifNacaroğlu. (2011). A New Solvable Class Linear Time Varying System. EMO, Fırat Üniversitesi, Elektrik Elektronik Bilgisayar Sempozyumu, Elazığ, Turkey.

**CURRICULUM VITAE**

**PERSONEL INFORMATION**

Surname, Name : DAL, İhsan

Nationality : TC

Birth place and date: Bismil,1 Jan 1982,

Marital Status : Maried

Phone number: +90506 3231921

Email : idal@tpao.gov.tr

**EDUCATION**

| Degree | Institution | Year of Graduation |
|---|---|---|
| Bachelor | University of Dicle | 2004 |
| High School | Marmaris Sabancı Lisesi | 1999 |

**WORK EXPERIENCE**

| Year | Place | Enrollment |
|---|---|---|
| 2007-Present | TPAO Anonim Ortaklığı Adıyaman Bölge Müdürlüğü | Chief |
| 2005-2007 | Eltem-Tek Anonim Şti. | Manager |

**FOREIGN LANGUAGE**

English