JANUARY 2018 M.Sc. in Electrical and Electronics Engineering IBRAHIM ISMAIL IBRAHIM AL-NUAIMI

UNIVERSITY OF GAZİANTEP

GRADUATE SCHOOL OF NATURAL & APPLIED SCIENCES

A COMPARATIVE STUDY ON SWITCHING SURFACE DESIGN FOR SLIDING MODE CONTROL

M. Sc. THESIS

IN

ELECTRICAL AND ELECTRONICS ENGINEERING

By

IBRAHIM ISMAIL IBRAHIM AL-NUAIMI

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Supervisor

Assist. Prof. Dr. Tolgay KARA

By

Ibrahim Ismail Ibrahim AL-NUAIMI

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Ibrahim Ismail Ibrahim AL-NUAIMI

ABSTRACT

A COMPARATIVE STUDY ON SWITCHING SURFACE DESIGN FOR SLIDING MODE CONTROL

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M.Sc. in Electrical and Electronics Engineering
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One of the important steps to build the sliding mode control system is to design the switching surface. There are many methods that have been developed in order to obtain the switching surface. The aim of this thesis is to study and analyze different switching surface design methods for sliding mode control of the uncertain systems. The main problem to be solved is how to design the switching surface with different methods. The design methods are applied to the same systems which are operating under the uncertainty conditions. The required objective is to attract all the state vectors to the switching surface so as to obtain the desired performance. The final step will be designing the sliding mode controller to enforce the state vectors towards the designed switching surface and to remain on the manifold until reaching the equilibrium point. Three different approaches of designing switching surface are studied here, the variable structure controllers are presented according to each method through the reaching condition of sliding mode.

Key Words: Variable structure systems, sliding mode control, switching surface.

ÖZET

KAYAN KIPLI KONTROL ANAHTARLAMA YÜZEYİ TASARIMI ÜZERINE KIYASLAMALI ÇALIŞMASI

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50 sayfa

Kayan kipli kontrol sistemini tasarlamanın önemli adımlarından biri de anahtarlama yüzeyi tasarlamaktır. Anahtarlama yüzeyini tasarlamak için geliştirilmiş birçok yöntem vardır. Bu tezin amacı, belirsiz sistemlerin kayan kipli kontrolü için farklı anahtarlama yüzey tasarımı yöntemlerini araştrmak ve analiz etmektir. Çözülmesi gereken ana problem, anahtarlama düzeyinin farklı yöntemlerle nasıl tasarlanacağıdır. Tasarım yöntemleri, belirsizlik şartlarında çalışan aynı sistemlere uygulanmıştır. Beklenen sonuç, tüm durum vektörlerini istenen başarımı verecek şekilde anahtarlama yüzeyine çekmektir. Son aşamada, durum vektörlerini tasarlanan anahtarlama yüzeyine çekecek ve denge noktasına ulaşıncaya kadar manifold üzerinde tutacak kayan kipli denetleyici tasarlanacaktır. Anahtarlama yüzeyini tasarlamak için üç farklı yaklaşım önerilmiştir, değişken yapılı denetleyiciler kayar kipli kontrolün ulaşılma koşuluyla her bir yönteme göre sunulmuştur.

Anahtar Kelimeler: Değişken yapı sistemleri, kayan kipli kontrol, anahtarlama yüzeyi.

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LIST OF SYMBOLS/ABBREVIATIONS

A	Controller system matrix
B	Controller input matrix
u	Controller
x_0	Initial state/condition
x_{eq}	Equilibrium point
K	Controller gain
VSS	Variable structure system
VSCS	Variable structure control systems
SMC	Sliding mode control
S(x)	Switching surface
$u_1(x)$	Linear part of the sliding mode controller
$u_s(x)$	Nonlinear part of the sliding mode controller
$u_t(x)$	Total sliding mode controller
LMI	Linear matrix inequality
MIMO	Multi Input-Multi Output

CHAPTER ONE
INTRODUCTION

1.1 Motivation

A variable structure system (VSS) is a functional system whose frame changes as per the present estimation. VSS is a system composed of independent frames jointly with a switching logic amidst each of the frames. VSS composed of a group of continual subsystems with the switching logic so that control actions are discontinuous functions of the system state, disturbances and reference input. VSS was presented and studied in the beginning of 1950's in the former Soviet Union by a group of Russian scientists. Their first case of the study was a linear second-order system and then the VSS has been developed to be applied in a wide range of systems such as nonlinear, MIMO, discrete time, large-scale and infinite dimensional, and stochastic systems. The most important feature of Variable structure control system (VSCS) is the ability to result in very robust control plants. Nowadays the improvement of VSCS is continuing to apply it to a wide spectrum of plants [9]. The main purpose of VSCS is to collect the useful properties of each structure [21], [22]. This feature of flexibility allows the designer to get the valuable and good dynamic properties of the structures which cannot be used in a wide continuous period of time [1]. The principle operation mode in VSS became the Sliding mode control (SMC) which makes it independent to parameters changes, uncertainty and outer disorders. The mathematical models of any physical systems are always given in an approximate way. SMC is one of the most efficient approaches to building robust controllers for complex nonlinear dynamic systems which operating under the uncertainty conditions. The connotation of SMC saw the light in the Russian scientific research in the late of 1950s. However, some earlier research can be considered as SMC such as V. Kulebakin (1932) who made a search about the vibration of DC generators of an aircraft and Nikolski (1934) who use the relays to control the course of a ship [28], but Emelyanov was the first scientist who finds that the properties could be achieved according to change the structure of controlling process even if the properties are not available in each structure. Then the work was developed and became more popular in the English world because of the research was done by Utkin (1977) [22]. The sliding mode turn into the precept process mode in seeming VSS. The principle of SMC is to drive the system state towards the manifold and then constrain it to stay there on which the system will offer eligible features. The controlled system in sliding mode is settled from the effects of the uncertainties so that the SMC is the appropriate option for robust control. The main subject of this thesis is to give a key point to studying and understanding the VSCS with SMC. The designing of SMC system is considered based on different methods to see the benefits and flexibility of each method as well as to achieve one similar performance for the same system. The entire idea of designing the switching surface was presented with theoretical and mathematical aspects including the simulation results for each method. Since the Lyapunov stability will be a tool to design sliding mode control to achieve the required performance, the definition and stability of feedback system in the sense of Lyapunov is considered. The theory of stability has a distinctive feature with its different types in control systems. It's necessary to obtain the desired plant performance, despite its different types which the researchers face in their work. This study focuses on the stability of equilibrium points in the sense of Lyapunov and illustrates the different stability definitions.

1.2 Statement of the Problem

One of the phases to design sliding mode control for the uncertain system is designing the switching surface. There are different ways and types of switching surface have been introduced. In this study, the design approach introduced for the switching surface design will be considered. Special attention will be given to Linear matrix inequality (LMI) to design the switching surface. A comparative study on designing of sliding mode control will be performed for the different class of dynamical systems.

1.3 Thesis contribution

Since the Lyapunov stability will be a tool to design sliding mode control to achieve the desired performance, this thesis deals with the definitions and stability of feedback system in the sense of Lyapunov. Literature survey is given on VSS and SMC.

- **Mathematical contribution**

    1. Different switching surface design methods are studied with mathematical and simulation aspects.

1.4 Thesis structure

This thesis is built as follows:

1. Chapter one presents the thesis in general.
2. Chapter two introduces Lyapunov stability.
3. Chapter three gives an introduction to the VSS and a literature review with the principles and main parts of SMC.
4. Chapter four gives the different methods to design switching surface for systems, the studied approaches are illustrated with mathematical and simulation aspects. A comparison between the obtained results are also given in this chapter.
5. Finally, a conclusion and possible future studies are given in chapter five.

CHAPTER TWO
LYAPUNOV STABILITY OF THE DYNAMICAL SYSTEMS

2.1 Lyapunov Stability

The stability of Lyapunov became well known when the Russian mathematician Aleksandr Lyapunov issued his book "The general problem of stability of motion" in 1892. He was the first scientist who considers that the alterations are needful for nonlinear plants to the linear theory of stability based on linearizing near a point of equilibrium. Lyapunov research at first issued in Russian and then translated into French. [2]. The interest in this work arises in the period between (1953-1962) when the " Lyapunov second method " was considered to be usable in the aerospace guidance systems which are usually contain robust nonlinearities. which cannot be studied within the other methods. The theory of stability has a very important function in the theory of control systems and engineering. In the study of dynamical systems there are different types of stability problems which appear, such that stability of equilibrium points, input-output stability and stability of periodic orbits. Here a detailed explanation of stability of equilibrium points will be presented which is generally described in the sense of Lyapunov.

2.1.1 The stability in the sense of Lyapunov

Consider the nonlinear system given in Equation 2.1

$$\dot{x} = f(x) \tag{2.1}$$

Where $x \in R^n$ is state variables and $\dot{x}$ is their derivatives

The system at the equilibrium point when

$$f\left(x_{eq}\right) = 0 \tag{2.2}$$

For a given state variable initial condition $x_o = x(t_o)$ the system trajectory will be

$\Phi(t; x, x_o)\,, t_o \leq t < \infty$ (2.3)

An equilibrium state (x_{eq}) of the system is said to be stable in the sense of Lyapunov if corresponding to each $S(\varepsilon)$ there is an $S(\delta)$ such that trajectories starting in $S(\delta)$ do not leave $S(\varepsilon)$ as $t \geq t_o$. In general, the region for the system solution $S(\varepsilon)$, is determined. Then the system trajectory start within the region of $S(\delta)$, which do not leave $S(\varepsilon)$, as $t \rightarrow \infty$ is determined [15]. To determine the stability of an equilibrium point in Equation 2.2 a region is illustrated for $n = 2$ phase plane in Figure 2.1.

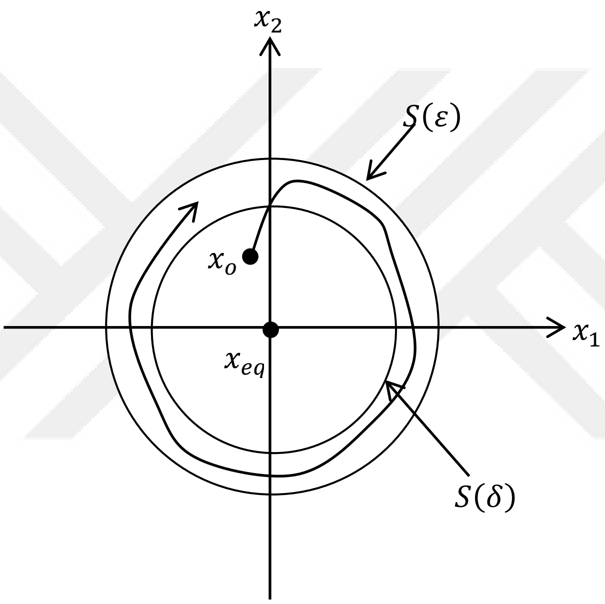

Figure 2.1 An equilibrium point and stability in the sense of Lyapunov.

2.1.2 Asymptotic stability in the sense of Lyapunov

Consider the nonlinear system defined in Equation 2.1 with the equilibrium point in Equation 2.2. Let the radius of $S(\delta) = \|x_o - x_{eq}\| \leq \delta$ and the radius of $S(\varepsilon) = \|\Phi(t; x, x_o) - x_{eq}\| \leq \varepsilon$**.** Again the system solution for the state initial condition $x_o = x(t_o)$ is $\Phi\,(t; x\,, x_o)\,, t_o \leq t \leq \infty$**.** The equilibrium point x_{eq} is asymptotically stable if all solutions starting with $S(\delta)$ region and do not leave $S(\varepsilon)$ and tend to the equilibrium point x_{eq} as time approaches infinity, $t \rightarrow \infty$. The equilibrium of the above system (x_{eq}) is said to be asymptotically stable if it is Lyapunov stable and if there exists $\delta > 0$ such $\|x_o - x_{eq}\| \leq \delta$, then

$\lim_{t\to\infty}\|\Phi(t; x, x_o) - x_{eq}\| = 0$. In another word, an equilibrium point, (x_{eq}) of the system is said to be asymptotically stable if it is stable in the sense of Lyapunov and if every solution starting within $S(\delta)$ converges, without leaving $S(\varepsilon)$, to (x_{eq}) as $t \to \infty$. Establishing asymptotic stability does not mean that the plant will work duly. The important thing to be known is the size of the largest region of asymptotic stability and this region is called the domain of attraction [15]. The asymptotical stability is illustrated in Figure 2.2.

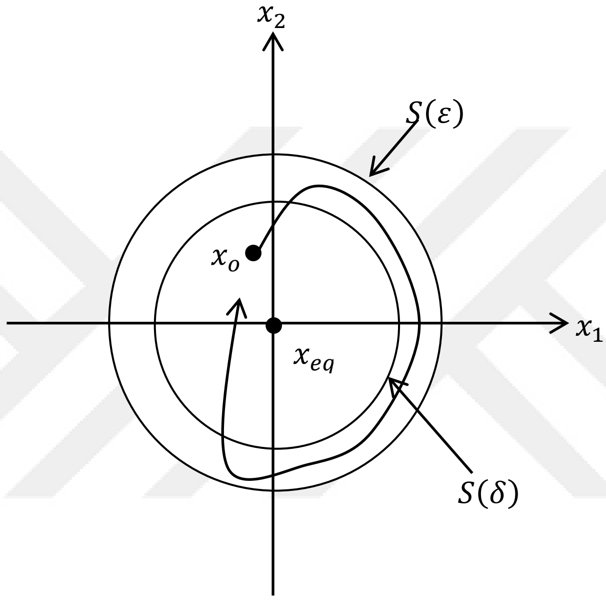

Figure 2.2 An equilibrium point and asymptotic stability in the sense of Lyapunov.

2.1.3 Asymptotic stability in the large

When asymptotic stability holds for all states from which the trajectories originate, the equilibrium state is said to be asymptotically stable in the large. The equilibrium state (x_{eq}) of the system is said to be asymptotically stable in large if it is stable and if every solution converges to (x_{eq}), as $t \to \infty$. Obviously, a necessary condition for asymptotic stability in the large is that there be only one equilibrium state in the whole state space. In control engineering problems, a desirable feature is an asymptotic stability in the large. When the equilibrium state (x_{eq}) is not asymptotically in large, then the problems becomes that of determining the largest region of asymptotic stability and this is usually very difficult to obtain [15]. In the actual applications however, it is appropriate to define a region of asymptotic stability large enough so that no disturbance will exceed it [30]. Note that in linear control theory, the only system that are asymptotically stable are called stable systems, and those systems that are stable in sense of Lyapunov, but are not asymptotically stable, are called unstable systems. The asymptotic stability in the large is illustrated in Figure 2.3.

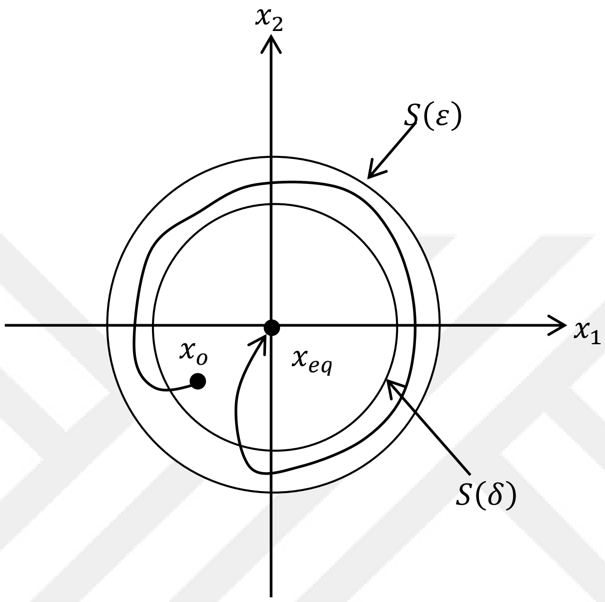

Figure 2.3 Asymptotic stability in the large in the sense of Lyapunov.

2.1.4 Exponentially stable in the sense of Lyapunov

Consider the nonlinear system defined in Equation 2.1 with the equilibrium point in Equation 2.2. Let the radius of $S(\delta) = \left\|x_o - x_{eq}\right\| \leq \delta$ and the radius of $S(\varepsilon) = \left\|\Phi(t; x, x_o) - x_{eq}\right\| \leq \varepsilon$. Again the system solution for the state initial condition $x_o = x(t_o)$ is $\Phi\,(t; x\,, x_o)\,, t_o \leq t \leq \infty$. The equilibrium point of the above system, (x_{eq}) is said to be exponentially stable if it is asymptotically stable and if there exist $\alpha\,, \beta\,, \delta > 0$ such that if $\left\|x_o - x_{eq}\right\| \leq \delta$, then $\left\|\Phi(t; x, x_o) - x_{eq}\right\| \leq \alpha \left\|x_o - x_{eq}\right\| e^{-\beta t}$ for every $t \geq 0$. Exponential stability means that solutions do not only converge but in fact, converges faster than asymptotically stable system or at least as fast as a particularly known rate $\alpha \left\|x_o - x_{eq}\right\| e^{-\beta t}$ [15]. The exponential stability is illustrated in Figure 2.4.

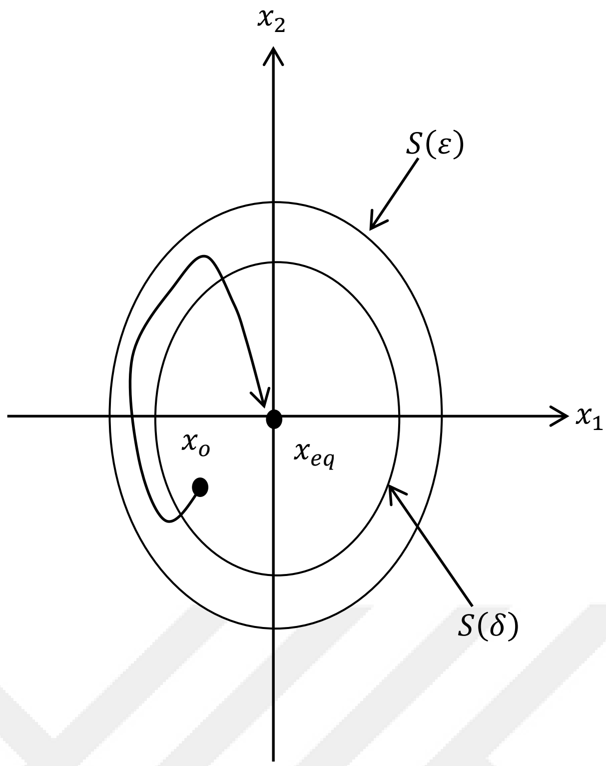

Figure 2.4 Exponential stability in the sense of Lyapunov.

2.2 Instability in the sense of Lyapunov

Consider the nonlinear system defined in Equation 2.1 with the equilibrium point in Equation 2.2. Let the radius of $S(\delta) = \|x_o - x_{eq}\| \leq \delta$ and the radius of $S(\varepsilon) = \|\Phi(t; x, x_o) - x_{eq}\| \leq \varepsilon$. Again the system solution for the state initial condition $x_o = x(t_o)$ is $\Phi\,(t; x\,, x_o)\,, t_o \;\leq t \;\leq \infty$. The equilibrium point of the above system, (x_{eq}) is said to be unstable if for some real numbers $(\varepsilon > 0)$ any real number $(\delta > 0)$, no matter how small, there is always initial state (x_o) in $S(\delta)$ such that the trajectory starting at this states leaves $S(\varepsilon)$ [15]. The Instability is illustrated in Figure 2.5.

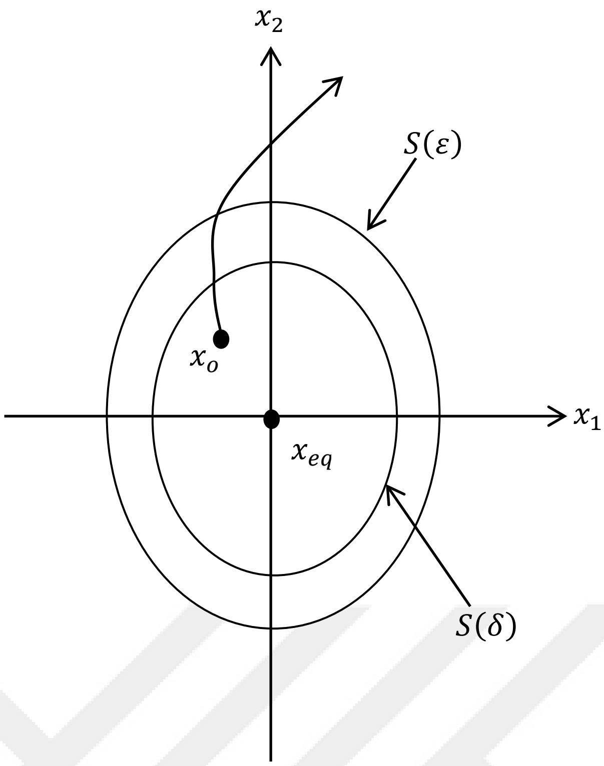

Figure 2.5 Instability in the sense of Lyapunov.

CHAPTER THREE

VARIABLE STRUCTURE SYSTEM AND SLIDING MODE CONTROL

3.1 Variable Structure System

The VSS is a functional system whose frame changes as per the present estimation. The state plane of VSS is illustrated in Figure 3.1.

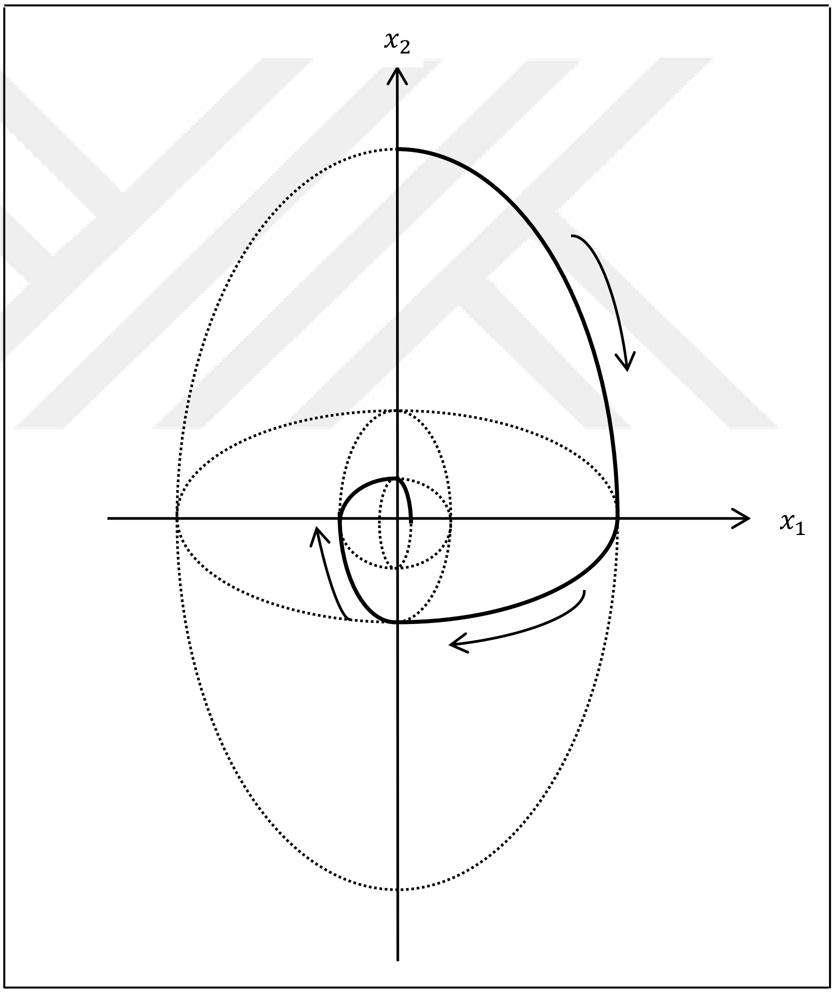

Figure 3.1 State plane of variable structure system.

VSS is a system collected of separate structures jointly with an appropriate logical surface amidst each of the structures and within this surface the VSS can take advantage of the required features of each of the structures which forms the system itself, also the VSS can own a feature which isn't available in any of its frames [30].

VSS was presented and studied in the beginning of 1950's in the former Soviet Union by a group of Russian scientists. Their first case of the study was a linear second-order system and then the VSS has been developed to be applied in a broad field of systems such as nonlinear, MIMO, discrete time, large-scale and infinite dimensional, and stochastic systems. The most important merit of VSCS is the power to achieve very robust control plants. Nowadays the improvement of VSCS is continuing to apply it to a wide spectrum of plants [9]. The main purpose of VSCS is to collect the useful properties of each structure [21], [22]. This feature of flexibility allows the designer to get the valuable and good dynamic properties of the structures which cannot be used in a wide continuous period of time [1]. The principle operation mode in VSS became the SMC which makes it insensitive to parameter variations, uncertainty and external disturbances.

3.2 Motivation for Sliding Mode Control

For any actual system, its mathematical representation will be estimated. Each designer aims to maintain the desired function and stability of the system should take in mind the uncertainties so that the designing of feedback control system is necessary to be robust to the uncertainties. From a wide range of techniques used for controlling the uncertain plants, the SMC is a basic and powerful strategy. The SMC is a specific sort of variable structure control. The main function of SMC is to drive the system state towards the manifold and after that constrain it to stay there on which the system will offers eligible features. The SMC transforms the N-dimensional single-input single-output tracking problem into the stabilization problem of the first order which is simple to be controlled. The controlled system in the SMC has the insensitivity to parameters changes, uncertainty and outer disorders which exclude the need of exact modeling so that the desired performance and stability are kept up. Because of this feature, the SMC is the proper choice for the robust control.

3.3 Literature Review of Sliding Mode Control

The SMC strategy is one of the most effective approaches to build robust controllers for complex nonlinear dynamic systems which affected by the uncertainties [27]. The concept of SMC showed up for the first time in the case of VSS especially in the relay plants, and then became the key working mode for such a plants. The research in this field was first done in the former Soviet Union in the late of 1950’s. Some earlier research can be considered as SMC such as V. Kulebakin (1932) who made a search about the vibration of DC generators of an aircraft and Nikolski (1934) who use the relays to control the course of a ship [28], but Emelyanov was the first scientist who finds that the properties could be achieved according to change the structure of controlling process even if the properties are not available in each structure [26]. Then the work was developed and became more popular in the English world because of the research was done by Utkin (1977) [22]. The theory of VSS with SMC was applied to controller design for the manipulator by Kar-keung d. Young (1978) [11]. Thereafter the SMC has been used for adaptive control, chemical processes, electrical motors and underwater vehicles [17]. Ryan and Corless [16] proposed the extreme limitations and asymptotic steadiness of a group of uncertain plants (1984). Burton and Zinober [3] used the Continuous approximation of variable structure control for the smoothness of the control scheme (1986). Spurgeon and Davies [19] proposed the robust sliding mode for the plants which operating under the unmatched uncertainty (1993). The full state information is required for sliding mode controller which is always not easy to obtain so that an observer is needed for the prediction of the system states [4,18,31]. Furat and Eker (2014) proposed Second-order integral SMC and the stability and robustness properties of the proposed controller are proved by means of Lyapunov stability theorem [13]. In (2016) Furat and Eker proposed the chattering-eliminated adaptive SMC and The proposed controller is compared experimentally using an electromechanical system with five different conventional sliding-mode controllers presented in the literature, their experimental results are presented to show the effectiveness of the proposed controller particularly regarding the accuracy of control input, disturbance rejection, and being an alternative controller to use in industrial applications [14]. SMC is characteristically a nonlinear strategy, so that SMC usage is not limited to linear plants. For controller design, the SMC gives a free area for a broad field of nonlinear systems. Any controller based on a nonlinear model can be expected to perform in an efficient way more than the controller based on a linear approximation. The applications of this theory have been expanded in different directions. The SMC became well known because of its ability to be applied to a broad field of plants.

3.4 Principles of Sliding Mode

SMC system is a group of continuous subsystems together by utilizing a high speed switching control which forced the system state to be oriented towards a certain surface called switching surface.

Consider the following second order nonlinear system, $n = 2$.

$$\dot{x}_1 = f_1(x_1, x_2, u) \tag{3.1}$$

$$\dot{x}_2 = f_2(x_1, x_2, u) \tag{3.2}$$

where $x = (x_{1,} x_2)^T \in$ Rn is state variable. If a scalar control u is regarded, then the function $S(x)$ is scalar as well and its points of discontinuity,

$S(x) = \{x \in R^n \mid S(x) = 0\}$ are a line in the state space.

When a control action is designed according to

$$u = \begin{cases} u^+ (x,t) & s(x) > 0 \\ u^- (x,t) & s(x) < 0 \end{cases} \tag{3.3}$$

The control input $u\ (x)$ should be designed in a way which guarantees that the state track is oriented to the manifold. When the state track reaches the manifold it will be forced to stay there by the control action and the system will move on the manifold until reaching the equilibrium point and this motion will call sliding mode. The sliding mode control scheme guarantee that the control action has the ability to keep the system state on the manifold on which the system will get the desirable properties and this is the basic principle of sliding mode.

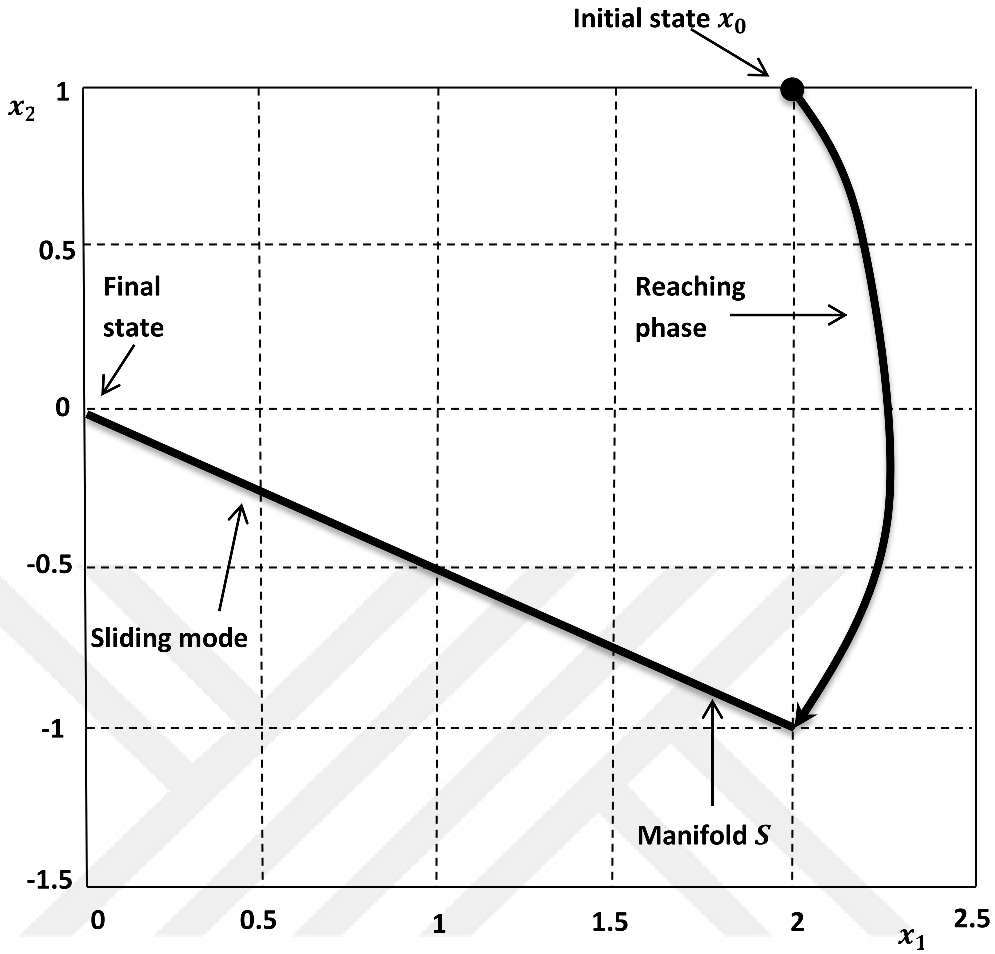

Figure 3.2 State plane of an actual movement of the system state.

An actual movement of the system state plane for the second-order nonlinear system when the control action is applied, the movement is consist of two parts:

- First part is the reaching phase (the system state moves towards the manifold for any initial condition).
- Second part is the sliding phase (the system state moves along the manifold until reaching the equilibrium point in a finite time).

3.5 Sliding Mode Controller Design

The designing of sliding mode controller is divided into two main parts:

- First part is designing the sliding surface to obtain the wanted dynamic conduct (stability to the equilibrium point).

- Second part is the designing of discontinuous control law which provides the attractivity to the system state when it is in the vicinity of the sliding surface and provides the stability to the closed-loop system when it's on the sliding surface.

The main advantages which the designer obtain from the system when the system is operating in the sliding mode are as follows [30]:

- The system is robust to the effect of matched uncertainties.
- The system performance is dominated by reduced set differential equations.

3.5.1 Sliding Surface

The sliding surface $S(x)$ is always characterized a linear surface $S(x) = 0$, which is considered as the switching surface. The sliding surface is designed to obtain the desired behavior from the plant so that the system state is governed by the sliding surface when it is sliding on the sliding surface. For example, if any system is maintained to be stable at zero, the sliding surface will be designed to be a stable differential equation of the system itself. When the system state which is in the vicinity of the sliding surface oriented towards the surface, sliding mode will exist and the surface will pull in all the system states which lie in its vicinity. The sliding surface will be attractive in two cases, the state trajectory starts on the surface will stay there and the state trajectory starts outside the surface will head for the surface asymptotically. In general, the sliding surface can be defined as :

$$S(x) = Cx \tag{3.4}$$

Where $S(x)$ is m-dimensional and C is the n-dimensional constant row vector. The main condition to the sliding mode to have occurred is

$$\lim_{S\to 0^+} \dot{S} < 0 \text{ and } \lim_{S\to 0^-} \dot{S} > 0 \tag{3.5}$$

The previous conditions in Equation 3.5 [30] gave a guarantee that the system states on each side of the sliding surface are head for the sliding surface as it will be seen in the Figure 3.3.

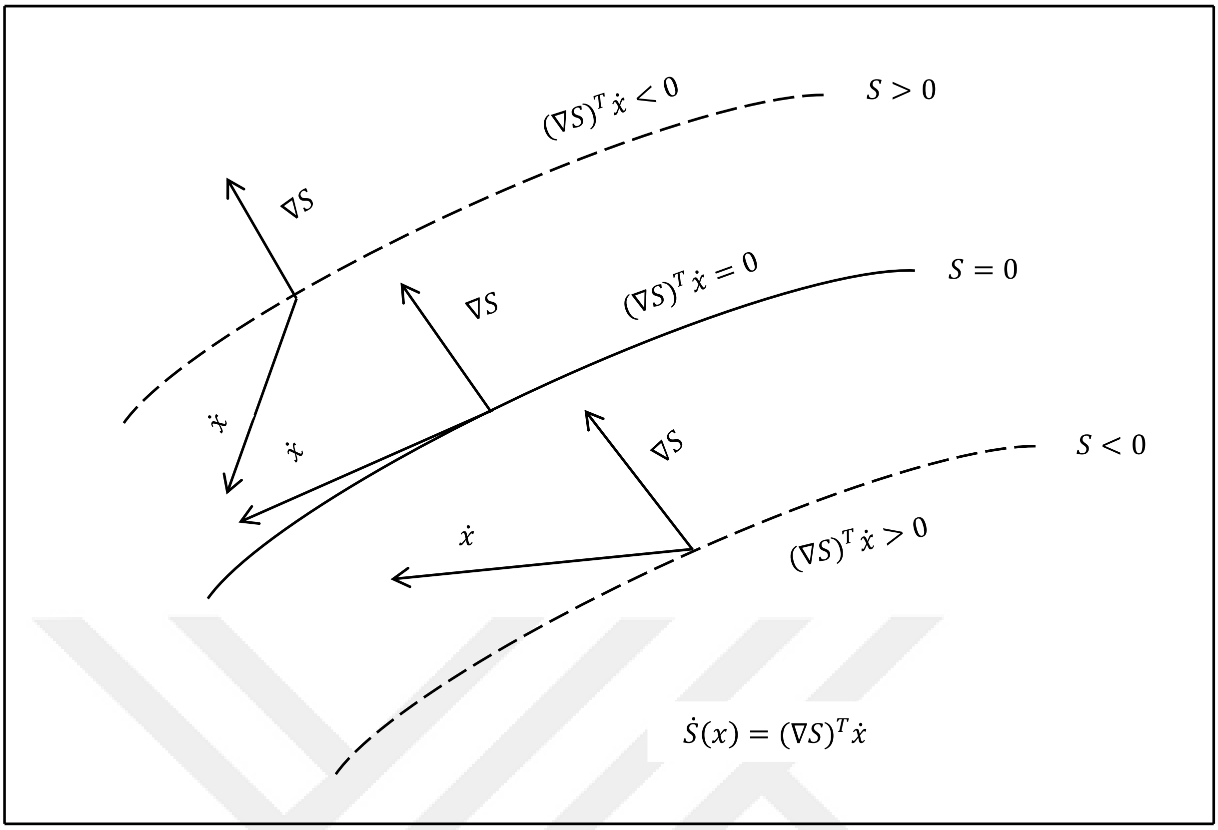

Figure 3.3 Sufficient condition of the attractivity to the sliding surface.

In the Figure 3.3 we notice that:

1. When $S > 0$ the state derivative must be negative definite so that the state can reach the sliding surface.
2. When $S = 0$ the state remain on the sliding surface.
3. When $S < 0$ the state derivative must be positive definite so that the state can reach the sliding surface.

3.5.2 Reaching Conditions and Reaching Law

The motion of the system states toward the sliding surface is subject to a certain condition which called the reaching condition. And the system states under the reaching conditions is called the reaching law or the reaching mode. According to [9], there are three main approaches to determine the reaching conditions.

1. The Lyapunov function approach: $V(x,t) = S^T S$ according to [21] the general reaching condition ensures the finite reaching time which is $\dot{V}(x,t) < 0$ when $S \neq 0$. In another word, the derivative of the Lyapunov function

should be negative definite. So that this approach leading to the final sliding mode switch scheme.

2. The straight switching function approach: which is proposed by [5],[23]

$$\begin{Bmatrix} \dot{S}(x) > 0 \ \ when \ \ S(x) < 0 \\ \dot{S}(x) < \ 0 \ \ when \ \ S(x) > 0 \end{Bmatrix} \tag{3.6}$$

In general, the reaching condition does not ensure a finite reaching time and also its difficult to be applying for the multi-input variable structure system.

3. The reaching law approach: the base of this approach is specifying the characteristics of the plant when the system states are moving towards the sliding surface and also set up the reaching conditions. And according to [29] this approach can be used for the facilitation of variable structure systems problems and also for measurement for the lowering of chattering.

The reaching law approach general formula is:

$$\dot{S}\,(x) = -\eta \, sign\left(\, S(x)\right) - kf(s) \tag{3.7}$$

Where η, k are positive gains and different values of these gains will give different rates for S and product different structure in the reaching law.

3.5.3 Control Law

The control law $u(x)$ provide the attractivity to the system state when it is in the vicinity of the sliding surface and provides the stability to the closed-loop system when it's on the sliding surface. In another word, the control law $u(x)$ is the responsible for driving the system state towards the sliding surface. The control law $u(x)$ can be obtained through two approaches:

1. Using the formula in equation 3.3 and the control action will be:

$$u(x) = -\eta \, sign \, S(x) \tag{3.8}$$

2. Substituting the state Equation $[\dot{x} = Ax + Bu]$ in the time derivative of the Equation 3.4 $\left[\dot{S}(x) = C\dot{x}\right]$ and the control action will be:

$$\dot{S}\,(x) = C\,(Ax + Bu) \tag{3.9}$$

CHAPTER FOUR
SWITCHING SURFACE DESIGN

4.1 Motivation

VSS is a system collected of separate structures jointly with an appropriate logical surface amidst each of the structures, and this surface is known as sliding surface or switching surface [30]. The switching surface is considered as the first main part of the SMC system design. The proper switching surface allows the designer to achieve the desirable dynamics features of the system and maintain the system stability in the sliding mode. For the multi-input system (m-input) there is multi-switching surface ($2^m - 1$). The first one may call the basic sliding surface since each one of the others is related with a single switching surface, and the last one is the surface where all the states should reach finally so that it may call the eventual sliding surface [29]. According to [9], there are some possible plans to force the system states to enter different sliding surfaces, for example:

1. Fixed order sliding surface in which the sliding mode happen in a particular order while the system state is crossing its space. It seems to be simple but for the VSS it has a long period of transient time as shown in Figure 4.1.a.

2. Eventual sliding surface in which the sliding mode happen where the system state is directed toward the final surface. It provides a faster transient time than the fixed order for VSS as shown in Figure 4.1.b.

3. Free order sliding surface in which the sliding mode happen where the system state reaches any sliding surface (the first surface where the state intersects with). The motion of the system state is faster in the sliding mode than the previous types as shown in Figure 4.1.c. Also, it provides a very fast transient time with less saturation to happen.

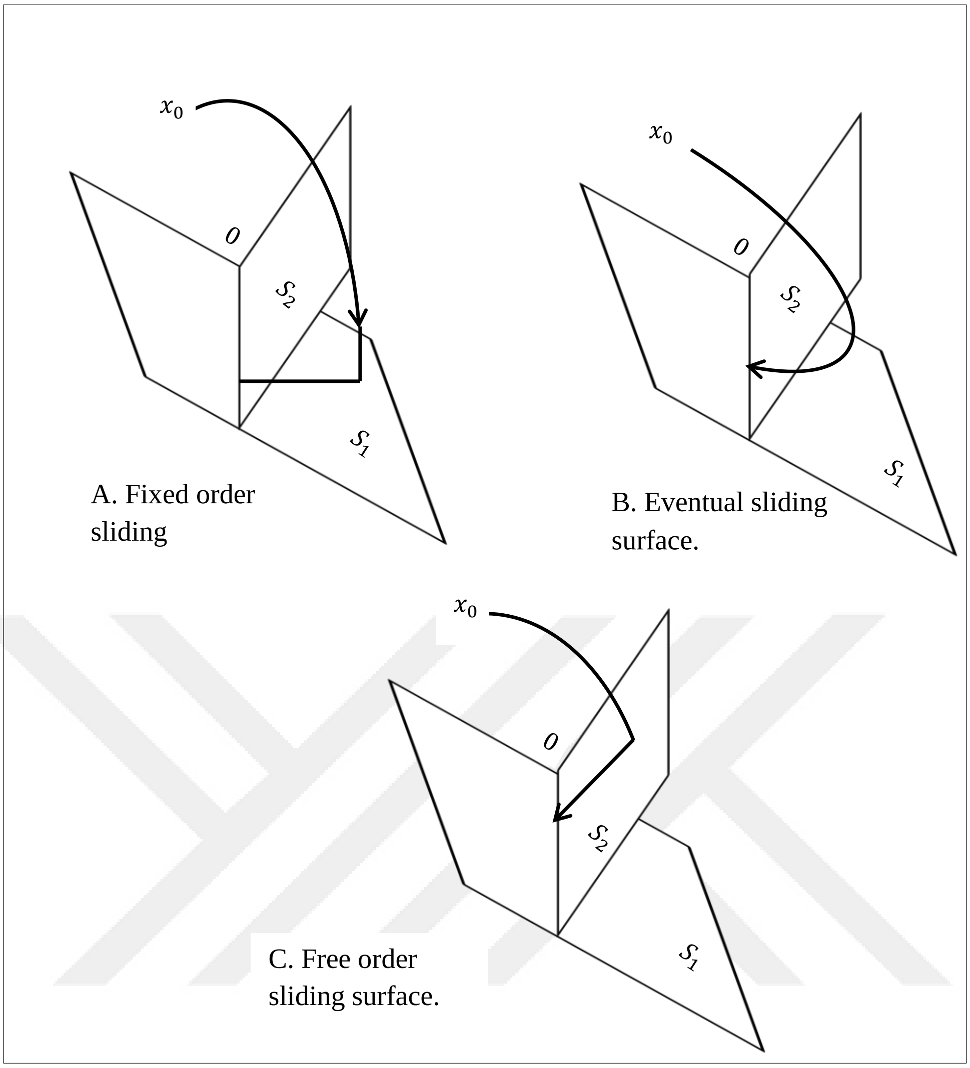

Figure 4.1 The switching surface schemes.

4.2 Switching Surface Design Methods

In the last decades, the designing of switching surface took much attention from the international control engineering community. All known results about switching surface design are founded according to restrictive appropriate assumption [9,6]. According to [7,12] all the existing design methods in their time do not ensure that the sliding mode dynamics features are fully constant with the matched and mismatched uncertainties. This section presents the idea of designing the switching surface in different approaches. The design of switching surface can be achieved via different approaches according to the system dynamics behavior and the desired features from such a design. Different methods are considered to design the switching surface with a special attention about the LMI method.

4.2.1 Switching surface design based on LMI

In the dynamical systems analysis, the usage of LMI returns back to the 1890's when the Russian scientist Lyapunov presented his main idea in this field which nowadays known as the Lyapunov theory. Lyapunov theory shows that the differential equation $\frac{d}{d_t}x(t) = Ax(t)$ is stable if there is a positive definite matrix P such that $PA + A^T P < 0$ so that the two conditions $P > 0$ & $PA + A^T P < 0$ are the Lyapunov inequality on P, which is a particular shape of an LMI. In another word the first attempts of analysing the stability of dynamical systems in terms of LMI was done by solving a set of linear equations in order to solve the Lyapunov inequality. The second step in this field was done by a group of Russian scientists Lur'e, Postnikov and others in the former Soviet Union in the 1940's. These scientists used the Lyapunov's method to solve the problem of stability of a control system with a nonlinearity in the actuator, but the resulting LMI were solved analytically by hand and this limited their work only for the small systems. The major contribution in this field starts in the 1960's when Kalman, Yakubovich, Popov and other scientists achieved the desired aim in reduction the solution of the LMIs which appeared in the problem of Lur'e to be simple schematic criteria as possible using what nowadays known as a positive-real lemma PR and this point leads to many criterion which applied to the higher order systems but could not applied to the systems with more than one nonlinearity. In 1971 a group of scientists finds out some methods to solve special forms of LMIs for the small systems by solving the Lyapunov or Riccati equations and these methods were already an analytic solution to the special forms of LMIs. In the early 1980's a group of researchers finds out that the solution of many LMIs can be achieved according to convex programming and the late of 1980's was the time for the evolution of interior-point algorithms for LMIs [20]. According to the work done in [6], many researchers have presented the switching surface design based on LMI for uncertain systems with matched and mismatched uncertainties. This section proposes an unrivaled design method of the switching surface which portray a linear sliding surface in terms of the LMI's. This approach ensures that the sliding mode dynamics features are stable and fully independent of the matched and mismatched uncertainties [6]. The main advantages of this approach are offering additional design flexibility and some ease in the computational aspect so that the switching surface for extensive systems can be readily achieved [8].

Example 4.1 Consider the following uncertain second-order system described by the Equation 4.1. The sliding mode control system will be designed according to the Table 4.1.

Table 4.1 Information table

States	Initial states	Desired states
$x_1(0)$	5	0
$x_2(0)$	0	0

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} (u + \emptyset) \tag{4.1}$$

Where $\emptyset = sin(t)$ is measurable and matched disturbance.

Designing the switching surface based on LMI as following:

$$S(x) = Cx$$

$$S(x) = B^T P x \tag{4.2}$$

Where $P \in R^{nxn}$ is positive matrix and B is the input matrix. To solve the matrix P, the system controller is designed as :

$$u_t(x) = u_1(x) + u_s(x) \tag{4.3}$$

Where $u_t(x)$ is the total controller, $u_1(x) = -K\,x$ is the linear feedback controller *(*$K \in R^{1x4}$) vector matrix and $u_s(x)$ is the sliding controller.

Consider the following Lyapunov function:

$$V(x) = X^T P x \tag{4.4}$$

$$V(x) = [x_1 \quad x_2] P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$V(x) = P({x^2}_1 + {x^2}_2)$$

the derivative of $V(x)$ will be:

$$\dot{V}(x) = P(2x_1\dot{x}_1 + 2x_2\dot{x}_2)$$

$$\dot{V}(x) = 2[x_1 \quad x_2] P \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$\dot{V}(x) = 2X^T P \dot{X} \tag{4.5}$$

The closed loop system is:

$$\dot{x} = Ax + Bu \tag{4.6}$$

Substituting Equation 4.3 in the Equation 4.6

$$\dot{x} = Ax + B\big(-Kx + u_s(x)\big)$$

$$\dot{x} = (A - BK)x + Bu_s(x)$$

Let $(\bar{A} = A - BK)$

$$\dot{x} = \bar{A}x + Bu_s(x) \tag{4.7}$$

Now subsituting Equation 4.7 in the Equation 4.5

$$\dot{V}(x) = 2X^T P\big(\bar{A}x + Bu_s(x)\big)$$

$$\dot{V}\,(x) = 2\,X^T P\,\overline{A}x \;\; + 2\,X^T PB\,u_s(x)$$

When $t \geq \; t_0$, there exists $S(x) = \; B^T\,Px = 0$

therefore $s^T(x) = x^T\,PB = 0$, it leads to : $\dot{V}\,(x) = 2\,X^T P\,\overline{A}x$

$$\dot{V}(x) = x^T \left(P\,\overline{A} + \overline{A}^T P\right) x$$

The condition to make $\dot{V}(x) < 0$ is

$$\left(P\,\overline{A} + \overline{A}^T P\right) < 0$$

Multiplying above inequality by P^{-2}

$$\overline{A}\,P^{-1} + P^{-1}\overline{A}^T < 0 \;, Let \quad P^{-1} = M$$

$$\overline{A}\,M + M\overline{A}^T < 0 \tag{4.8}$$

$$(A - BK)M + M(A - BK)^T < 0$$

$$AM - BKM + MA^T - B^T K^T M < 0$$

Let $L = KM$

$$AM - BL + MA^T - B^T L^T \tag{4.9}$$

The matrix M can be obtained by solving the inequality defined in Equation 4.9 by using MATLAB M-File given in the Table 4.2.

Table 4.2 MATLAB M-File for LMI

```
Calculation M* matrix %%%%%
A=[0 1;1 0]
B=[0;2]
setlmis([]);
M=lmivar(1,[2 1]);
L=lmivar(2,[1 2]);
lmiterm([1 1 1 M],A,1,'S');
lmiterm([-1 1 1 L],B,1,'S');
lmiterm([-2 1 1 M],1,1);
lmis=getlmis;
[tmin,xfeas]=feasp(lmis);
M=dec2mat(lmis,xfeas,M)
P=inv(M)
L=dec2mat(lmis,xfeas,L)
k=L*inv(M)
```

$$M = \begin{bmatrix} 90.9732 & -30.3244 \\ -30.3244 & 90.9732 \end{bmatrix}$$

$P = \begin{bmatrix} 0.0124 & 0.0041 \\ 0.0041 & 0.0124 \end{bmatrix}$

$L = [90.9732 \quad 7.5811]$

$K = [1.1562 \quad 0.4687]$

$C = B^T P = [0.0082 \quad 0.0247]$

Now the switching surface will be:

$S(x) = Cx = 0.0082x_1 + 0.0247x_2$

Equating the Equation 3.9 with the Equation: $\dot{S}\,(x) = -\eta\, sign\left(\, S(x)\right) - k\, s(x)$

See the Equations A1, A2 and A3 in the Appendix A.

$$-\eta\, sign\left(\, S(x)\right) - k\, s(x) \; = C\,(Ax + Bu) \tag{4.10}$$

$$-\eta\, sign(\, S(x)) - k\, s(x) \; = CAx + CBu$$

Then the nonlinear sliding mode controller based on exponential reaching law can be obtained as follows:

$$u_s(x) \; = \left(-\eta\, sign\left(\, S(x)\right) - k\, s(x) \; - CAx\right) \,(CB)^{-1} \tag{4.11}$$

And the linear sliding mode controller can be obtained as follows:

$$u_1(x) \; = -Kx \tag{4.12}$$

The total sliding mode controller can be obtained as follows:

$$u_t(x) = \; u_1(x) + \; u_s(x) \tag{4.13}$$

$CB = 0.0495$, $CB^{-1} = 20.2163$, $CA = [0.247 \quad 0.0082]$

The positive gains of the nonlinear sliding mode controller are selected as follows:

$\eta = 5 \;\; , \;\; k = 2$

$u_1(x) \; = -Kx = -1.1562x_1 - 0.4687x_2$

$u_s(x) \; = (-\eta\, sign(\, S(x)) - k\, s(x) \; - CAx)\,(CB)^{-1}$

$$u_s(x) = (-5\ sign\ S(x) - 2S(x)\ -\ 0.247x_1 - 0.0082\ x_2\ )\ 20.2163$$

$$u_s(x) = (-101.0815\ sign\ S(x) - 40.3226S(x)\ -\ 4.9934x_1 - 0.1657\ x_2\ )$$

$$u_t(x) = (-1.1562x_1 - 0.4687x_2) + (-101.0815\ sign\ S(x) - 40.3226S(x) - 4.9934x_1 - 0.1657\ x_2)$$

$$u_t(x) = -101.0815\ sign\ S(x) - 40.3226S(x) - 6.1496x_1 - 0.6344x_2$$

The design of the SMC system in its two main stages (Switching surface design and control action design) is done for the previous second order system. Our main point is to design the switching surface $S(x)$ and its achieved through the LMI method. The procedure and the results of this design method are clarified what we said earlier about the advantages of the LMI in the ease of the computational aspect so that the desired function and stability of the system are ensured. The design of switching surface based on LMI method will be applied to linearized inverted pendulum system in order to see the system required function and stability in a much clear way than what is done earlier.

Example 4.2 Consider the linearized inverted pendulum system described by the Equation 4.14. The cart and the unstable inverted pendulum shown in the Figure 4.2. The sliding mode control system will be designed according to the Table 4.3.

Table 4.3 Information table

States	States type	Initial states	Desired states
$x_1(0) = \theta$	Angle response	-15°	0
$x_2(0) = \dot{\theta}$	Angle speed response	0	0
$x_3(0) = x$	Cart position response	5°	0
$x_4(0) = \dot{x}$	Cart speed response	0	0

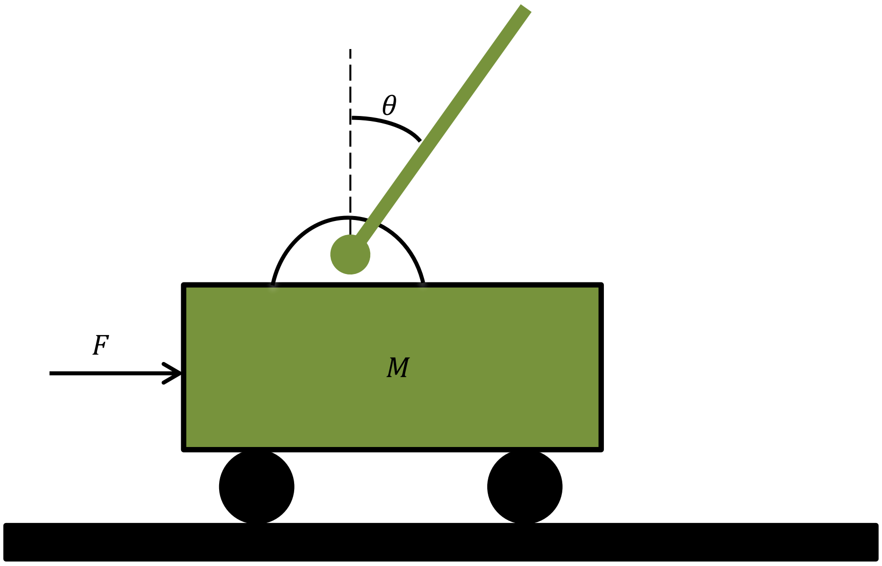

Figure 4.2 A cart and an inverted pendulum.

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} (u + \emptyset) \tag{4.14}$$

Where $\emptyset = 0.3sin(t)$ is measurable and matched disturbance.

Designing the switching surface based on LMI as following:

$$S(x) = Cx$$

$$S(x) = B^T Px \tag{4.15}$$

Where $P \in R^{nxn}$ is positive matrix and B is the input matrix. To solve the matrix P, the system controller is designed as :

$$u_t(x) = u_1(x) + u_s(x) \tag{4.16}$$

Where $u_t(x)$ is the total controller, $u_1(x) = -K\,x$ is the linear feedback controller $(K \in R^{1x4})$ vector matrix and $u_s(x)$ is the sliding controller.

Consider the following Lyapunov function:

$$V(x) = X^T Px \tag{4.17}$$

$$V(x) = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \end{bmatrix} P \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

$$V(x) = P\ (x^2{}_1 + \ x^2{}_2 + \ x^2{}_3 + \ x^2{}_4)$$

the derivative of $V(x)$ will be:

$$\dot{V}(x) = P(2x_1\dot{x}_1 + 2x_2\dot{x}_2 + 2x_3\dot{x}_3 + 2x_4\dot{x}_4)$$

$$\dot{V}\ (x) = 2\ x^T\ P\ \dot{x} \tag{4.18}$$

The closed loop system is

$$\dot{x} = A\,x\ +\ \ B\,u \tag{4.19}$$

Substituting Equation 4.16 in the Equation 4.19

$$\dot{x}\ = Ax + B\big(-Kx + u_s(x)\big)$$

$$\dot{x}\ = (A - BK)x + Bu_s(x)$$

Let $(\bar{A} = A - BK)$

$$\dot{x}\ = \bar{A}x + Bu_s(x) \tag{4.20}$$

Now subsituting Equation 4.20 in the Equation 4.18

$$\dot{V}(x) = 2X^TP\big(\bar{A}x + Bu_s(x)\big)$$

$$\dot{V}\ (x) = 2\ X^TP\ \overline{A}x\ \ + 2\ X^TPB\ u_s(x)$$

When $t \geq\ t_0$, there exists $S(x) =\ B^T\ Px = 0$

therefore $s^T(x) = x^T\ PB\ = 0$, it leads to : $\dot{V}\ (x)\ = 2\ X^TP\ \overline{A}x$

$$\dot{V}\ (x) = x^T\ \left(P\,\overline{A} +\ \overline{A}^T\ P\right)x$$

The condition to make $\dot{V}\ (x)\ < 0$ is

$$\left(P\,\overline{A} +\ \overline{A}^T\ P\right)\ < 0$$

Multiplying above inequality by P^{-2}

$\overline{A}\,P^{-1} + P^{-1}\overline{A}^{T} < 0$, $Let \quad P^{-1} = M$

$$\overline{A}\,M + M\overline{A}^{T} < 0 \tag{4.21}$$

$(A - BK)M + M(A - BK)^{T} < 0$

$AM - BKM + MA^{T} - B^{T}K^{T}M < 0$

Let $L = KM$

$$AM - BL + MA^{T} - B^{T}L^{T} \tag{4.22}$$

The matrix M can be obtained by solving the inequality defined in Equation 4.22 by using MATLAB M-File given in the Table 4.4.

Table 4.4 MATLAB M-File for LMI

```
Calculation M* matrix %%%%%
A=[0 1 0 0;0 -0.1818 2.6727 0;0 0 0 1;0 -0.4545 31.1818 0]
B=[0;1.8182;0;4.5455]
setlmis([]);
M=lmivar(1,[4 1]);
L=lmivar(2,[1 4]);
lmiterm([1 1 1 M],A,1,'S');
lmiterm([-1 1 1 L],B,1,'S');
lmiterm([-2 1 1 M],1,1);
lmis=getlmis;
[tmin,xfeas]=feasp(lmis);
M=dec2mat(lmis,xfeas,M)
P=inv(M)
L=dec2mat(lmis,xfeas,L)
k=L*inv(M)
```

$$M = \begin{bmatrix} 4.2526 & -1.1725 & 0.4474 & -0.1400 \\ -1.1725 & 6.0546 & 0.0387 & 4.0537 \\ 0.4474 & 0.0387 & 0.2850 & -0.7053 \\ -0.1400 & 4.0537 & -0.7053 & 7.3276 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.4768 & 0.3140 & -1.5733 & -0.3160 \\ 0.3140 & 0.5440 & -1.7023 & -0.4588 \\ -1.5733 & -1.7023 & 11.1131 & 1.9813 \\ -0.3160 & -0.4588 & 1.9813 & 0.5750 \end{bmatrix}$$

$L = [4.0820 \quad 0.9239 \quad 3.4366 \quad -4.6933]$

$K = [-1.6871 \quad -1.9123 \quad 20.8971 \quad 2.3966]$

$C = B^T P = [-0.8656 \quad -1.0964 \quad 5.9110 \quad 1.7793]$

Now the switching surface will be:

$S(x) = Cx = -0.8656x_1 - 1.0964x_2 + 5.9110x_3 + 1.7793x_4$

Equating the Equation 3.9 with the equation: $\dot{S}(x) = -\eta \, sign\left(S(x)\right) - k \, s(x)$

See the Equations A1, A2 and A3 in the Appendix A.

$$-\eta \, sign\left(S(x)\right) - k \, s(x) \ = C\,(Ax + Bu) \tag{4.23}$$

$$-\eta \, sign(S(x)) - k \, s(x) \ = CAx + CBu$$

Then the nonlinear sliding mode controller based on exponential reaching law can be obtained as follows:

$$u_s(x) \ = \left(-\eta \, sign\left(S(x)\right) - k \, s(x) \ - CAx\right) \, (CB)^{-1} \tag{4.24}$$

And the linear sliding mode controller can be obtained as follows:

$$u_1(x) \ = -Kx \tag{4.25}$$

The total sliding mode controller can be obtained as follows:

$$u_t(x) = \ u_1(x) + \ u_s(x) \tag{4.26}$$

$CB = 6.0942$, $CB^{-1} = 0.1641$

$CA = [0 \quad -1.4750 \quad 52.5506 \quad 5.9110]$

The positive gains of the nonlinear sliding mode controller are selected as follows:

$\eta = 5 \ \ , \ \ k = 2$

$u_1(x) \ = -Kx = 1.6871x_1 + 1.9123x_2 - 20.8971x_3 - 2.3966x_4$

$u_s(x) \ = (-\eta \, sign(S(x)) - k \, s(x) \ - CAx) \, (CB)^{-1}$

$u_s(x) = (-5 \, sign \, S(x) - 2S(x) \ + 1.4750x_2 - 52.5506 \, x_3 - 5.9110x_4 \,) \; 0.1641$

$$u_s(x) = (-0.8205\, sign\, S(x) - 0.3282S(x)\ + 0.2420x_2 - 8.6235\, x_3 - 0.9699x_4\ )$$

$$u_t(x) = (1.6871x_1 + 1.9123x_2 - 20.8971x_3 - 2.3966x_4) + (-0.8205\, sign\, S(x) - 0.3282S(x)\ + 0.2420x_2 - 8.6235\, x_3 - 0.9699x_4)$$

$$u_t(x) = -0.8205\, sign\, S(x) - 0.3282S(x) + 1.6871x_1 + 2.1543x_2 - 29.5206x_3 - 3.3665x_4$$

4.2.2 Switching surface design based on Ackermann's formula

In the control system design Ackermann's formula is an approach used to solving the problem of pole allocation, and it's also a proper method to mark a linear state-feedback control law in a certain feature which results in the feedback system with desired eigenvalues [10]. The equation of SMC is linear and depends on the coefficients of the surface equation a similar task emerge in the design of SMC for the linear system with a linear surface [24]. The design of scalar SMC based on Ackermann's formula to obtain a discontinuity plane equation in explicit form as well as in terms of the original system.

Example 4.3 Consider the linearized inverted pendulum system described by the Equation 4.27. the cart and the unstable inverted pendulum shown in the Figure 4.3. The sliding mode control system will be designed according to the Table 4.5.

Table 4.5 Information table

States	States type	Initial states	Desired states
$x_1(0) = \theta$	Angle response	-15°	0
$x_2(0) = \dot{\theta}$	Angle speed response	0	0
$x_3(0) = x$	Cart position response	5°	0
$x_4(0) = \dot{x}$	Cart speed response	0	0

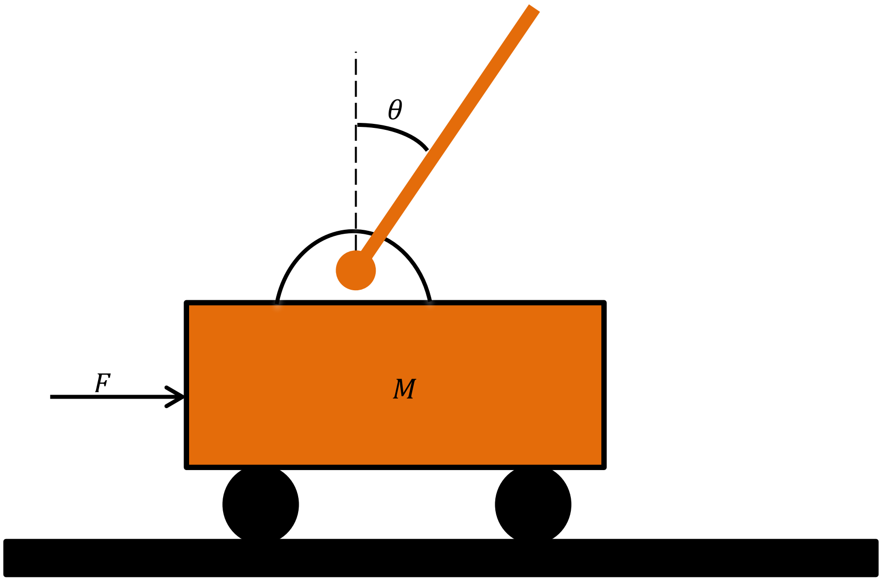

Figure 4.3 A cart and an inverted pendulum.

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -0.98 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \left(u + f(t)\right) \tag{4.27}$$

Where $f(t) = 0.5sin(3t)$ is measurable and matched disturbance.

Designing the switching surface $S(x)$ based on Ackermann's as follows:

The desired eigenvalues of sliding motion are: $[r_1 = -1,\ r_2 = -2,\ r_3 = -3]$

$$S(x) = Cx = 0 \tag{4.28}$$

$$C = e^T P_1(A) \tag{4.29}$$

$$e^T = [0 \quad 0 \quad 0 \quad 1][B \quad AB \quad A^2B \quad A^3B]^{-1} \tag{4.30}$$

$$P_1(A) = (A - r_1 I)(A - r_2 I)(A - r_3 I) \tag{4.31}$$

$$C = [0 \quad 0 \quad 0 \quad 1][B \quad AB \quad A^2B \quad A^3B]^{-1}(A - r_1 I)(A - r_2 I)(A - r_3 I) \tag{4.32}$$

The matrix C can be obtained by solving the Equation 4.32 by using MATLAB M-File given in the Table 4.6.

Table 4.6 MATLAB M-File for Ackermann's formula

```
Calculation C* matrix %%%%%
A=[0 1 0 0;0 0 -0.98 0;0 0 0 1;0 0 1 0]
B=[0;1;0;1]
r1=-1;r2=-2;r3=-3;
M=inv([B A*B A*A*B A*A*A*B])
E=[0 0 0 1]*M
I=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1]
P=(A-(r1*I))*(A-(r2*I))*(A-(r3*I))
C=E*P
```

$C = [-3.0303 \quad -5.5556 \quad 9.0303 \quad 6.5556]$

Now the switching surface $S(x) = Cx = 0$ will be:

$S(x) = -3.0303x_1 - 5.5556x_2 + 9.0303x_3 + 6.5556x_4$

Equating the Equation 3.9 with the equation: $\dot{S}(x) = -\eta\, sign\left(S(x)\right) - k\, s(x)$

See the Equations A1, A2 and A3 in the Appendix A.

$$-\eta\, sign\left(S(x)\right) - k\, s(x) = C\,(Ax + Bu) \tag{4.33}$$

$$-\eta\, sign(S(x)) - k\, s(x) = CAx + CBu$$

Then the sliding mode controller based on exponential reaching law can be obtained as follows:

$$u_s(x) = \left(-\eta\, sign\left(S(x)\right) - k\, s(x) - CAx\right)\,(CB)^{-1} \tag{4.34}$$

$(CB) = 1\,,\ (CB)^{-1} = 1$

$CA = [0 \quad -3.0303 \quad 12 \quad 9.0303]$

$k = 2, \eta = 5$

$u(x) = (-5signS(x) - 2S(x) + 3.0303x_2 - 12x_3 - 9.0303x_4)$

4.2.3 Switching surface design based on Classical method

This section presents the idea of designing switching surface as well as the design of SMC system based on the classical approach to be considered as a basic stage among the previous two approaches (Switching surface design based on LMI and Switching surface design based on Ackermann's formula). To clarify this approach we consider the single-dimensional motion of a unit mass as shown in the Figure 4.4.

Example 4.4 Consider the system described in the Equation 4.35. The sliding mode control system will be designed according to the Table 4.7.

Table 4.7 Information table

States	States type	Initial states	Desired states
$x_1(0)$	Position	5	0
$x_2(0)$	Velocity	-10	0

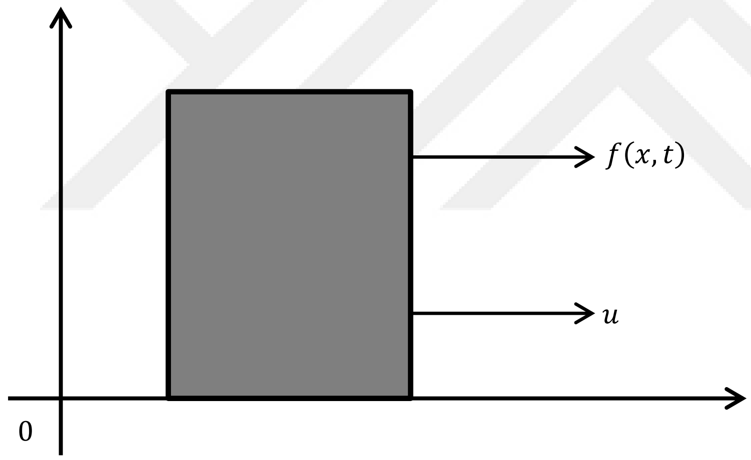

Figure 4.4 Single-dimensional motion of a unit mass.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \big(u + f(x_1, x_2, t)\big) \tag{4.35}$$

$f(x_1, x_2, t) = sin(2t)$ which is disturbance force.

Designing the switching surface $S(x)$ based on this method as follows:

$$S(x) = Cx_1 + x_2,\ C > 0 \tag{4.36}$$

The sliding mode controller will be considered as:

$$u(x) = -Cx_2 - \rho sign\, S(x) \tag{4.37}$$

The positive gains are selected as follows: $C = 1.5$ and $\rho = 2$

$S(x) = 1.5x_1 + x_2$

$u(x) = -1.5x_2 - 2sign\, S(x)$

4.3 Simulation results

This section presents the simulation results of all previous examples.

1. Simulation results for the design method applied in Example 4.1.

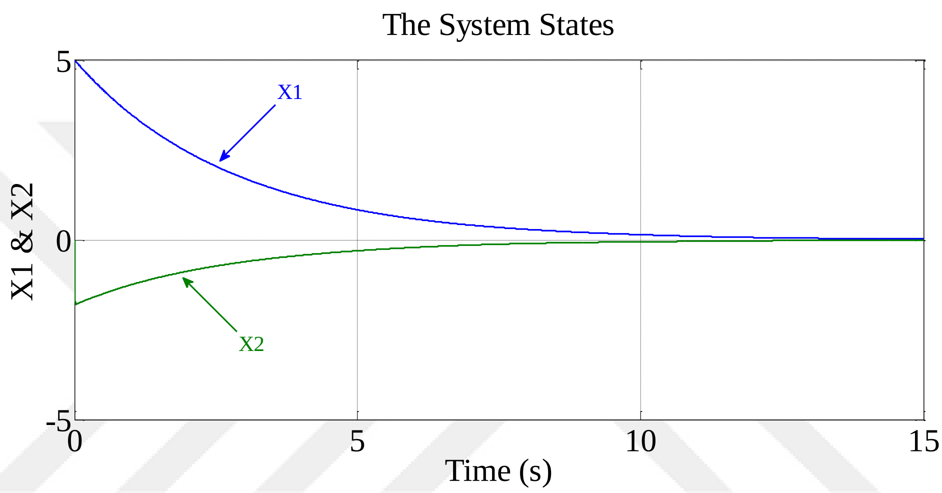

Figure 4.5a The system states.

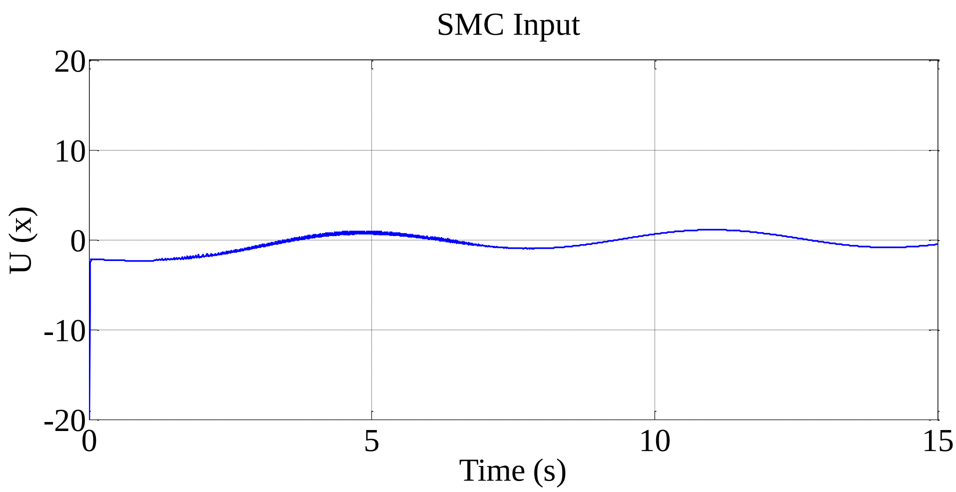

Figure 4.5b The sliding mode controller.

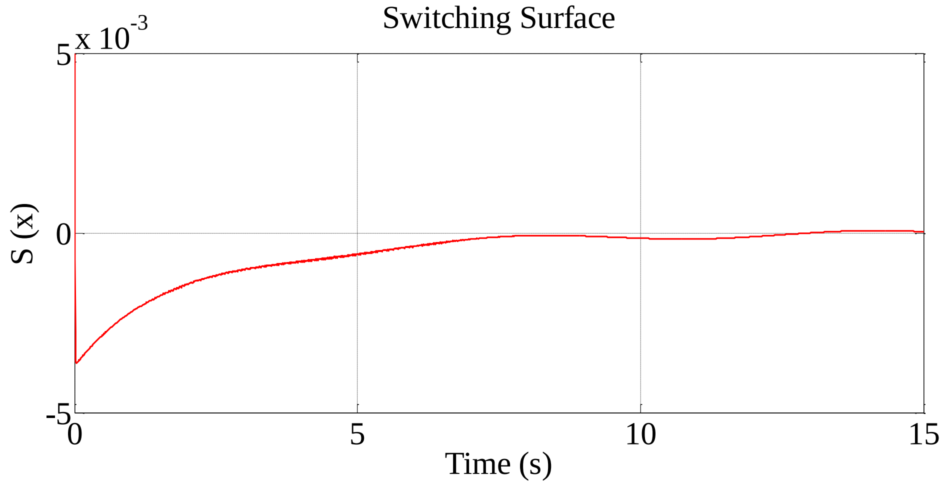

Figure 4.5c The switching surface.

In the Figure 4.5a we notice the movement of the system states from its initial conditions $[x_1(0) = 5, x_2(0) = 0]$ to the desired destination (switching surface) and it approaches to zero as $t \to \infty$ with approximately 12 to 13 seconds as settling time for x_1 and 9 to 10 seconds as settling time for x_2. Figure 4.5b shows the sliding mode controller signal which has less of chattering effect. In the Figure 4.5c we notice the switching surface curve goes to zero as $t \to \infty$ with settling time approximately between 12 to 13 seconds.

2. Simulation results for the design method applied in Example 4.2.

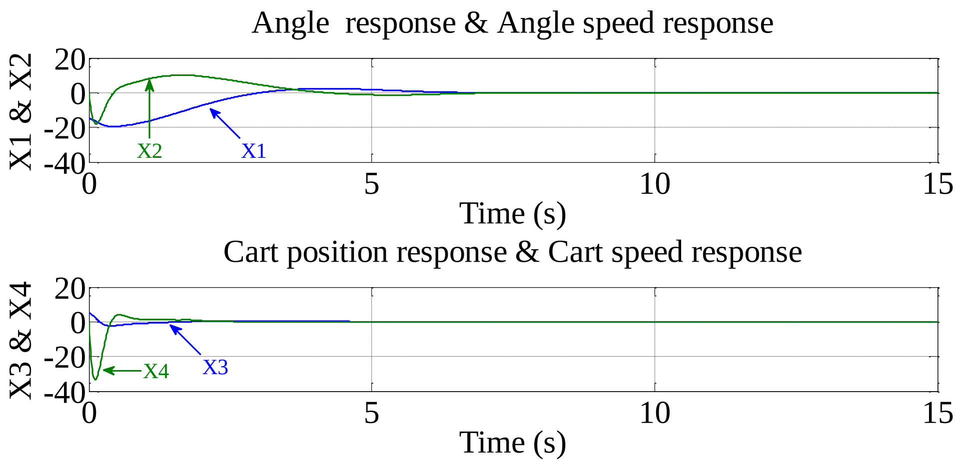

Figure 4.6a The system states.

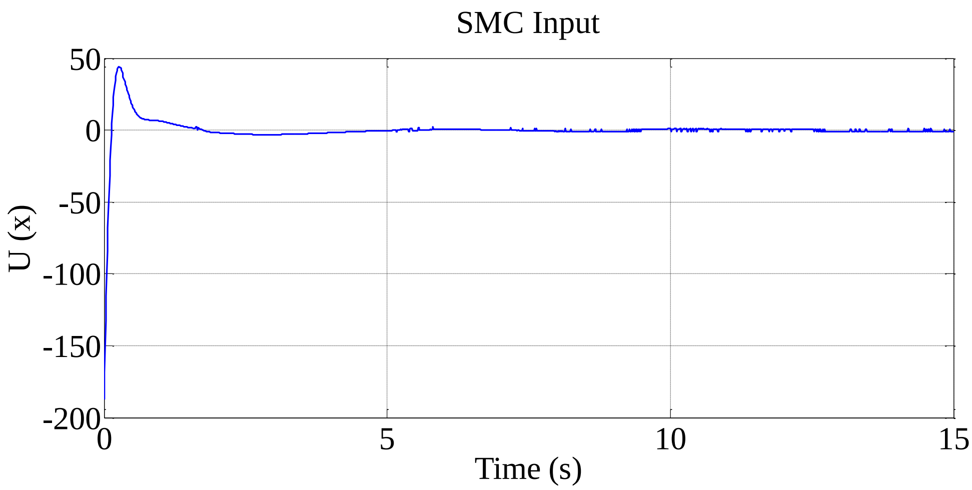

Figure 4.6b The sliding mode controller.

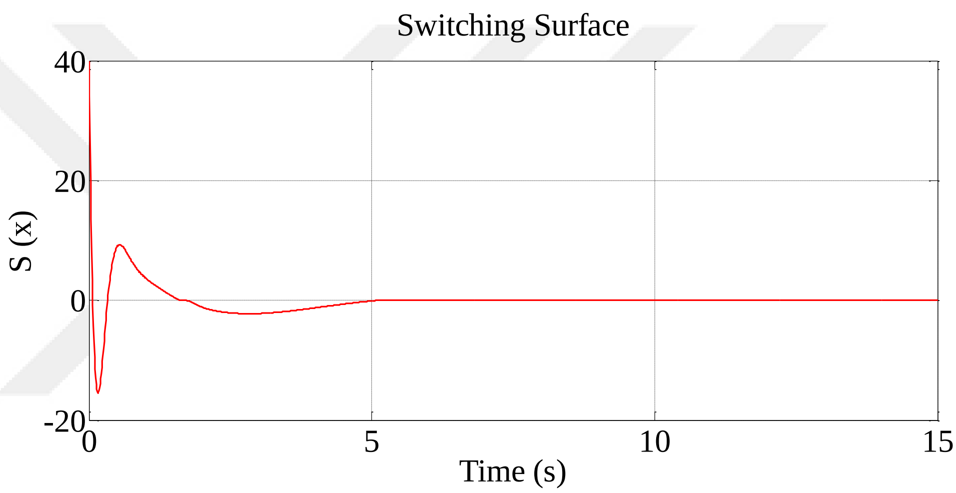

Figure 4.6c The switching surface.

In the Figure 4.6a we notice the movement of the system states from its initial conditions $[x_1(0) = -15^{\circ}, x_2(0) = 0, x_3(0) = 5^{\circ} \text{and } x_4(0) = 0]$ to the desired destination (switching surface) and it approaches to zero as $t \to \infty$ with settling time approximately 6 seconds for x_1 and x_2, 1.5 to 2 seconds for x_3 and 2.5 seconds for x_4. Figure 4.6b shows the sliding mode controller signal which has less of chattering effect. In the Figure 4.6c we notice the switching surface curve goes to zero as $t \to \infty$ with settling time about 5 seconds.

3. Simulation results for the design method applied in Example 4.3.

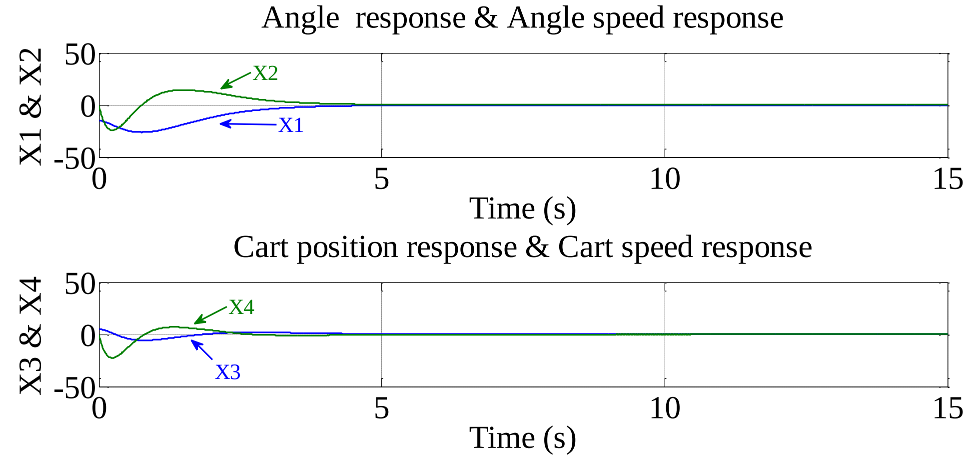

Figure 4.7a The system states.

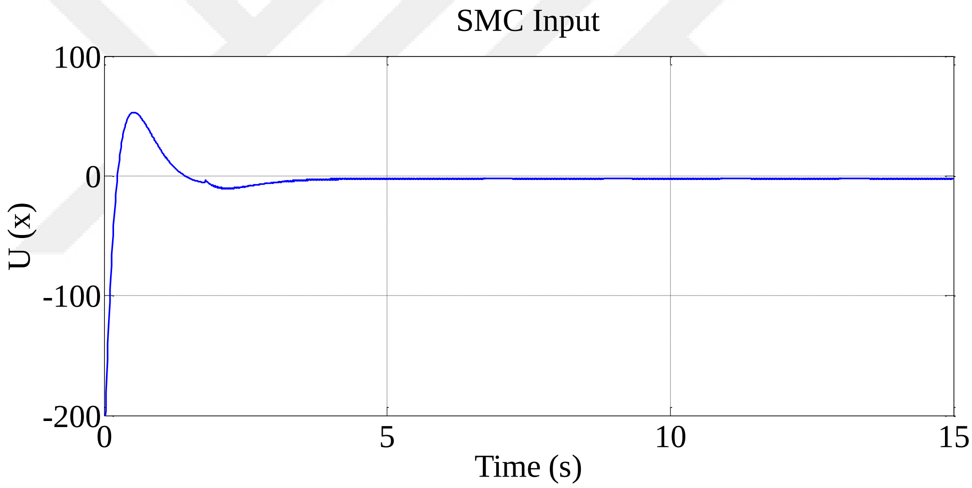

Figure 4.7b The sliding mode controller.

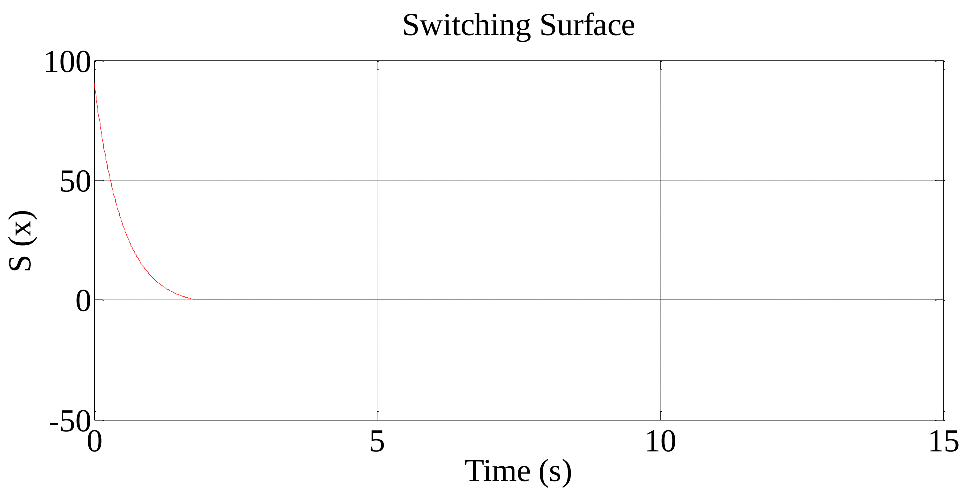

Figure 4.7c The switching surface.

In the Figure 4.7a we notice the movement of the system states from its initial conditions $[x_1(0) = -15^{\circ}, x_2(0) = 0, x_3(0) = 5^{\circ} \text{and } x_4(0) = 0]$ to the desired destination (switching surface) and it approaches to zero as $t \to \infty$ with settling time approximately 5 seconds for x_1 and x_2, 5 seconds for x_3 and x_4. Figure 4.7b shows the sliding mode controller signal which has less of chattering effect. In the Figure 4.7c we notice the switching surface curve goes to zero as $t \to \infty$ with settling time approximately between 1.5 to 2 seconds.

4. Simulation results for the design method applied in Example 4.4.

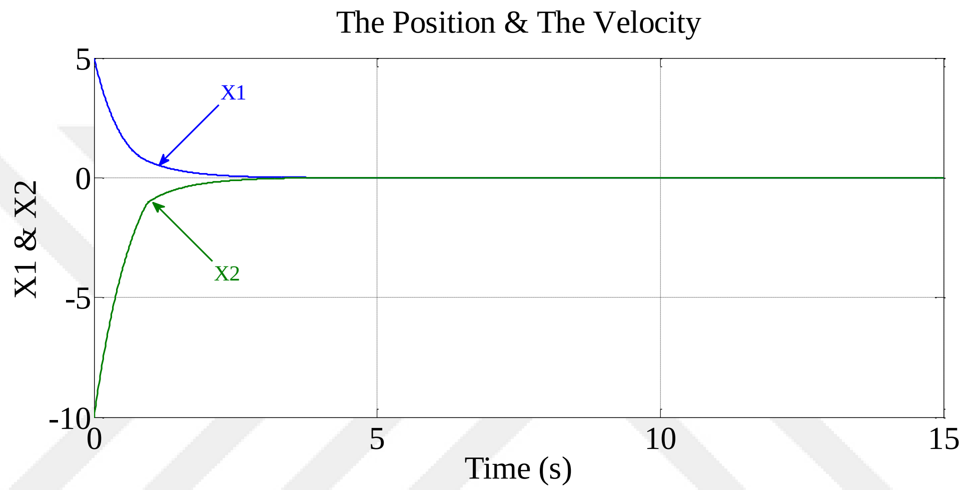

Figure 4.8a The system states.

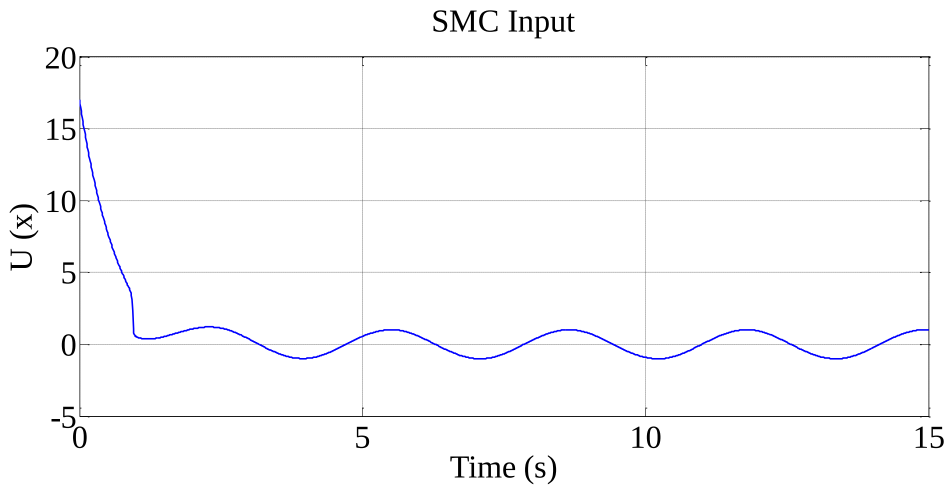

Figure 4.8b The sliding mode controller.

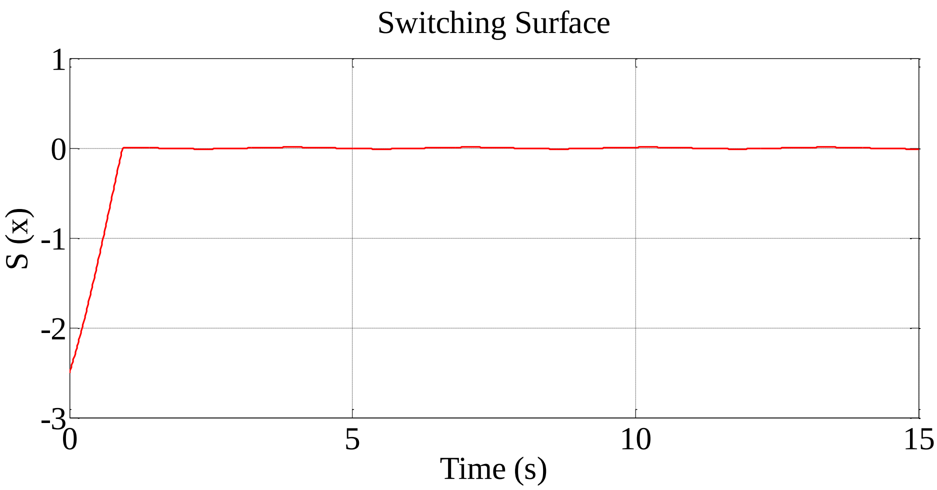

Figure 4.8c The switching surface.

In the Figure 4.8a we notice the movement of the system states from its initial conditions $[x_1(0) = 5, x_2(0) = -10]$ to the desired destination (switching surface) and it approaches to zero as $t \to \infty$ with approximately 3 seconds as settling time. Figure 4.8b shows the sliding mode controller signal which has less of chattering effect. In the Figure 4.8c we noticed the switching surface curve goes to zero as $t \to \infty$ with settling time approximately about 1 seconds.

4.4 Comparison of results

This part submits a comparison of the results which obtained in the previous three approaches which are LMI, Ackermann's formula and the Classical method in order to see the differences between them and also to point to the advantages and disadvantages of each method. The comparison will be given in figures and words to increase the clarity and emphasizes the comparison.

4.4.1 Comparison between LMI and Ackermann's formula

In the following figures the dash line representing the LMI method while the continuous line representing the Ackermann's formula method.

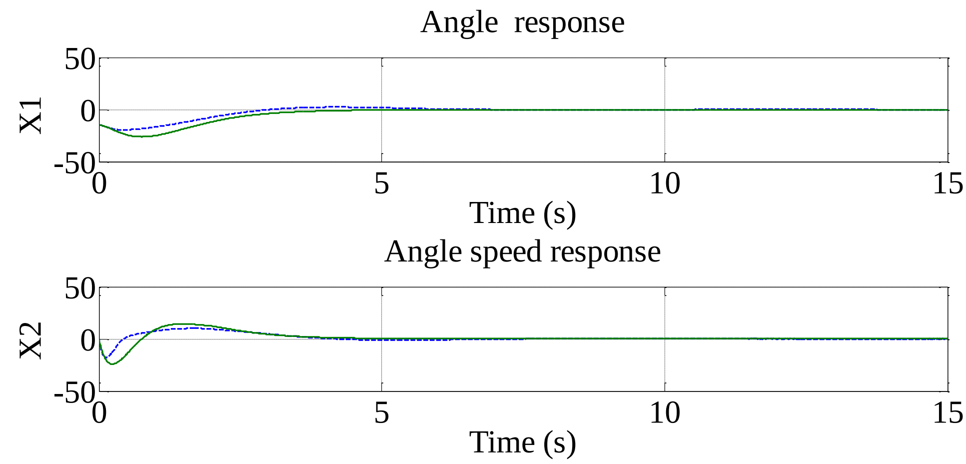

Figure 4.9a The system states.

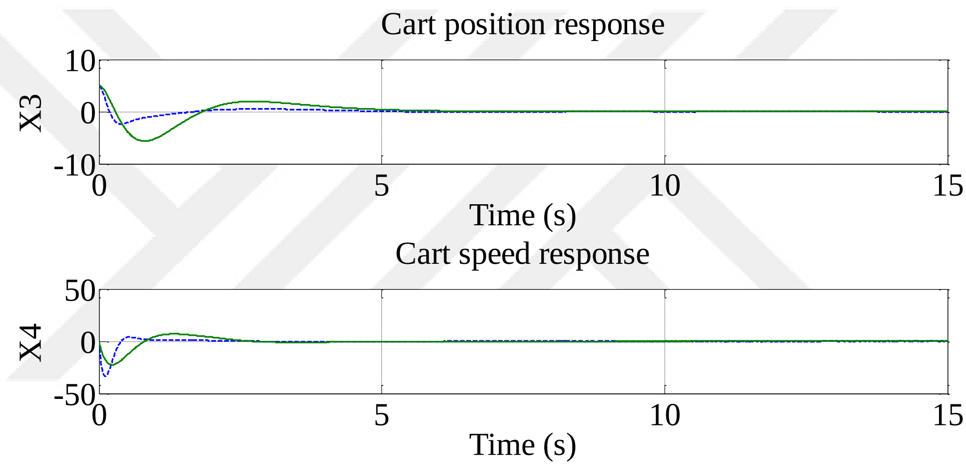

Figure 4.9b The system states.

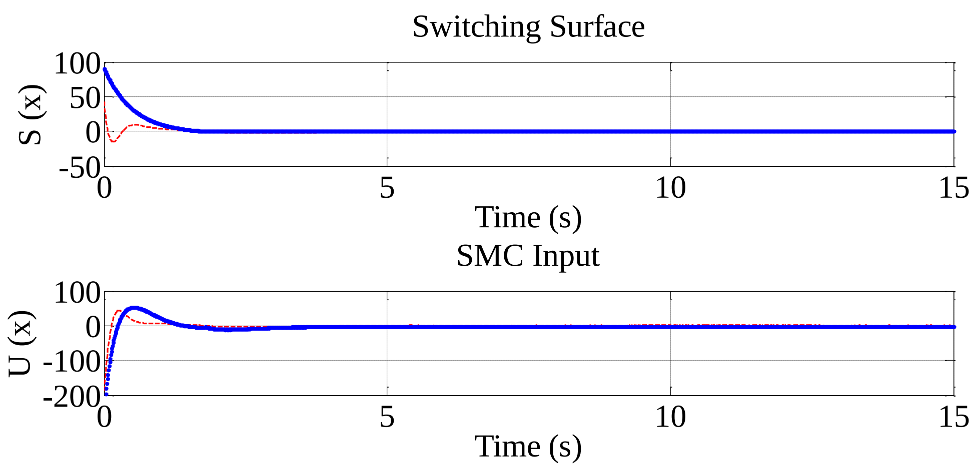

Figure 4.9c Switching surface and Sliding mode controller.

In the Figure 4.9a and 4.9b we notice the motion of the system states in the two methods from its initial values $[x_1(0) = -15^{\circ}, x_2(0) = 0, x_3(0) = 5^{\circ}$ and $x_4(0) = 0]$ to the desired destination (switching surface) and it approaches to zero as $t \to \infty$ with settling time approximately as follows:

1. LMI method
   - 6 seconds for the x_1 and x_2.
   - 1.5 to 2 seconds for the x_3.
   - 2.5 seconds for the x_4.
2. Ackermann's formula method
   - 5 seconds for the x_1, x_2,x_3 and x_4.

So that for the given inverted pendulum system the cart position response and the cart speed response reaches the stability in the LMI method in less time than the time taken in the Ackermann's formula method, while the angle response and the angle speed response reaches the stability in the LMI method in one second after it reaches the stability in the Ackermann's formula method. Figure 4.9c shows the switching surface curve of the two methods starting from different quantities and goes to zero as $t \to \infty$ with settling time approximately about 1.5 to 2 seconds for the Ackermann's formula method and about 5 seconds for the LMI method. In the same figure, we notice that the sliding mode control signal for the LMI method seems to more convenient than the sliding mode control signal for the Ackermann's formula method. From the previous results we can see that the Ackermann's formula gives the result with simple way according to its procedure but its faces some difficulties compared with the LMI method which enables the designer to easily attack various interesting problems and also for its ability of robustness against matched disturbances.

4.4.2 Comparison between LMI and Classical method

In the following figures the dash line representing the LMI method while the continuous line representing the Classical method.

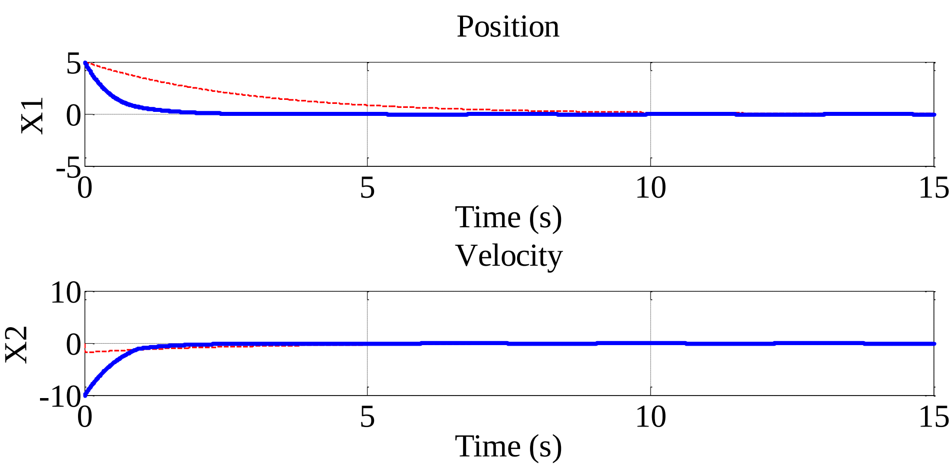

Figure 4.10a The system states.

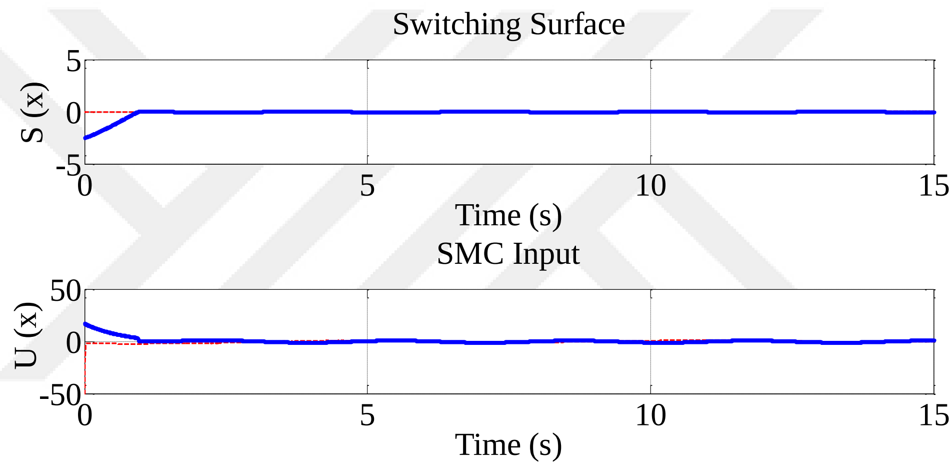

Figure 4.10b Switching surface and Sliding mode controller.

In the Figure 4.10a we notice the motion of the system states in the two methods from its initial values $[x_1(0) = 5, x_2(0) = 0\,]$ for the LMI and $[x_1(0) = 5, x_2(0) = -10\,]$ for the Classical method to the desired destination (switching surface) and it approaches to zero as $t \to \infty$ with settling time approximately as follows:

1. LMI method
    - 12 to 13 seconds for x_1
    - 9 to 10 seconds as settling time for x_2
2. Classical method
    - 3 seconds for x_1 and x_2

So that for the given second-order system the state position vector x_1 and the state velocity vector x_2 reaches the stability in the Classical method in less time than the time taken in the LMI method. Figure 4.10b shows the switching surface curve of the two methods starting from different quantities and goes to zero as $t \rightarrow \infty$ with settling time approximately about 1 second for the Classical method and about 0.1 second for the LMI method. In the same figure, we notice that the sliding mode control signal for the LMI method seems to more convenient than the sliding mode control signal for the Classical method. Despite the fact that the previous results show the ease of the Classical method to be applied but the LMI method have the advantage that its results seem more logical especially in the acquired switching surface and the sliding mode controller. This chapter displayed what the switching surface is and presented different methods to design the proper switching surface. In the next chapter, a conclusion about the achieved results will be given as well as the possible future studies in the area of SMC.

CHAPTER FIVE
CONCLUSION AND FUTURE STUDIES

5.1 Conclusion

The stability of the dynamical system is an important subject in the control systems theory so that one of the most common approaches has been studied within this thesis which is the Lyapunov method. The stability of equilibrium points is given with different cases for stability and the case of instability in the sense of Lyapunov. Since the prime core of this study is the designing of switching surface, it was necessary to enter this topic by presenting an introduction about VSS with a historical view and the main concepts of SMC to create some ideas which can help any researcher to give an answers to the following questions:

1. What is the meaning of SMC?
2. What are the main parts of SMC system?
3. Where did this topic come from?
4. What is the major contribution of this topic in the field of control systems?
5. What are the main developments which took place on this topic?

After studying the main points which mentioned earlier, it was important to clarify the implicit details of this subject, which can be considered as the cornerstone of understanding how this system work and thus the ability to build a solid control system against the matched disturbances so as to reach the stability and obtain the desired performance of such a system. From the obtained switching surface through different design methods the following main points can be inferred:

1. The LMI method has a unique procedure depends on the positive Lyapunov function and some mathematical aspects in order to achieve the desired formula in terms of LMIs to describing linear sliding surfaces and obtain the desired properties from such a design.

2. The Ackermann’s formula method has proposing a scalar sliding mode control design depends on the desired eigenvalues and the controllability matrix to achieve the desired SMC performance with respect to its flexibility of solution when its applied to special types of systems.
3. The Classical method has the easiest procedure to obtain the switching surface but it can be only applied to specific types of system to achieve the desired results.

As a result of the previous three conclusions, it’s possible to consider the best approach to design the switching surface according to the ability of the selected approach to give the most accurate results with respect to the capability of such a method dealing with a various types of systems with the existence of the matched disturbances. The LMI method gave the designer the ability to attack interesting problem for even higher order system and this is the main reason to consider the LMI method as a comprehensive design approach compared to the other two methods. The simulation results of each method clarify the system states behavior under the matched disturbances conditions as well as displays the normal path which is taken by the switching surface curve and the SMC signal until reaching the stability point in a specific period of time.

5.2 Future studies

According to the fast evolution taking place in the scientific field, it’s necessary to keep up with this huge progress by developing the previous methods and planning to find a new approaches which allow us to achieve the desired aims in order to present the valuable things to the humanity and society. In the area of SMC it will be a great contribution to develop and find the switching surface design based on feedback linearization besides designing the SMC for aircraft systems.

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APPENDIX

Appendix A: Exponential reaching law

In general, the reaching law determines the dynamic characteristics of the system during the reaching phase and establishes the reaching condition in addition, it contains a simplification of the solution for (VSC) and providing a measure for the reduction of chattering. Reaching law consist of two main parts which are :

- Reaching stage: it drives the plant toward the sliding surface
- Sliding stage: it can constrain the system on the sliding manifold and lead it to the origin (Equilibrium Point)

The general formula for exponential reaching law is:

$$\dot{S}(x) = -\eta \, sign\left(S(x)\right) - k\, S(x) \tag{A1}$$

Where $\eta > 0$, $k > 0$ and $\dot{S}(x) = -k\, S(x)$ is the exponential term, can be solved as:

$$\frac{dS(x)}{dt} = -k\, s(x) \Rightarrow \frac{dS(x)}{S(x)} = -kdt$$

$$\int \frac{dS(x)}{S(x)} = -k \int dt \quad \Rightarrow \quad ln\, S(x) = -kt$$

Leads to $S(x) = S(0)\, e^{-kt}$

By adding the proportional rate term $-k\, S(x)$ to this reaching law forces the state to approach the switching surface faster when $S(x)$ is enough large.

It can be shown that the reaching time for X state is move from an initial state $x(0)$ to the switching surface $S(x) = Cx = 0$ is finite and given by T = $\frac{1}{k}\; ln \frac{k\,|s|+\eta}{\eta}$

To obtain the sliding mode controller we have the direct switching function approach

$\dot{S}(x) > 0 \quad when\, S(x) < 0$

$\dot{S}(x) < 0 \; when\, S(x) > 0$

Where

$$\dot{S}(x) = C\dot{x} \tag{A2}$$

From equating the two equations (A1) and (A2) we obtain:

$$-\eta\, sign\big(S(x)\big) - k\, S(x) = C\dot{x}$$

Where $\dot{x} = Ax + Bu$

$$-\eta\, sign\big(S(x)\big) - k\, S(x) = C\,(Ax + Bu)$$

$$-\eta\, sign\big(S(x)\big) - k\, S(x) = CAx + CBu$$

Then the nonlinear sliding mode controller can be obtained as :

$$u_s(x) = \big(-\eta\, sign\big(S(x)\big) - k\, S(x) - CAx\big)\,(CB)^{-1} \tag{A3}$$