KOCAELİ ÜNİVERSİTESİ * FEN BİLİMLERİ ENSTİTÜSÜ

DEĞİŞKEN KATSAYILI FOURİER SERİLERİYLE PARABOLİK DENKLEMLER İÇİN DEVİRLİ SINIR KOŞULLU KARIŞIK PROBLEMİN ANALİZİ

DOKTORA TEZİ

İrem ÇİFTÇİ

Anabilim Dalı : Matematik

Danışman: Prof. Dr. Hüseyin HALİLOV

KOCAELİ, 2007

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KOCAELİ 2007

ÖNSÖZ VE TEŞEKKÜR

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İÇİNDEKİLER

SİMGELER DİZİNİ ve KISALTMALAR

$\lVert . \rVert$	: Norm
$\lvert . \rvert$	: Mutlak Değer
$k=\overline{1,\infty}$	: k sayısı, birden sonsuza kadar tüm değerleri alır.
$C^n[a,b]$	: [a,b] aralığında n. mertebeye kadar türevleri sürekli fonksiyonlar uzayı
KTDD	: Kısmi Türevli Diferansiyel Denklem
$G(t,x,\xi)$	: Kaynak Fonksiyonu
$\overline{D}$	: D bölgesinin kapanışı

DEĞİŞKEN KATSAYILI FOURİER SERİLERİYLE PARABOLİK DENKLEMLER İÇİN DEVİRLİ SINIR KOŞULLU KARIŞIK PROBLEMİN ANALİZİ

İrem ÇİFTÇİ

Anahtar Kelimeler: Fourier Serisi, Fourier Yöntemi, Yarı Doğrusal Parabolik Denklem, Devirli Sınır Koşulu, Karışık Problem

Özet: Bu çalışmada ele alınan yarı doğrusal parabolik ve yarı doğrusal pseudo-parabolik denklemler için devirli sınır koşullu karışık problemlerin genelleşmiş çözümü ilk kez tanımlanmış ve belli koşullar dahilinde sözü edilen problemlerin genelleşmiş çözümünün varlığı ve tekliği ispatlanmıştır. Ele alınan problemlerin pratik önemi göz önünde bulundurularak, kesin çözümle ardışık yaklaşımların farkı da değerlendirilmiştir. Ayrıca, $\varepsilon \to 0$ da yarı doğrusal pseudo-parabolik denklem için incelenen problemin çözümünün, yarı doğrusal parabolik denklem için incelenen problemin çözümüne yaklaştığı gösterilmiştir. Tez, giriş ve beş bölümden oluşmaktadır. Birinci bölümde tezde sık kullanılan tanım ve kavramlarla ilgili bilgiler verilmiş, ikinci bölümde de, ele alınan yarı doğrusal parabolik denklemler için devirli sınır koşullu karışık problemin genelleşmiş çözümünün varlığı ve tekliği incelenmiştir. Tezin üçüncü bölümünde, ele alınan yarı doğrusal pseudo-parabolik denklem için devirli sınır koşullu genelleşmiş çözümünün varlığı ve tekliği incelenmiştir. İkinci ve üçüncü bölümlerde söylenenlerin yanı sıra, incelenen problemlerin çözümlerinin kesin ve yaklaşık değerlerinin farkı da değerlendirilmiştir. Dördüncü bölümde, $\varepsilon \to 0$ durumunda ikinci ve üçüncü bölümlerde ele alınan problemlerin çözümleri arasındaki ilişki incelenmiştir. Son bölümde ise sonuçlar ve öneriler verilmiştir.

VARIABLE COEFFICIENT FOURIER SERIES FOR A MIXED PROBLEM FOR QUASILINEAR PARABOLIC EQUATIONS WITH PERIODIC BOUNDARY CONDITION

İrem ÇİFTÇİ

Key Words: Fourier Series, Fourier Method, Quasilinear Parabolic Equation, Periodic Boundary Condition, Mixed Problem

Abstract: In this study, a generalized (weak) solution of mixed problems with periodic boundary conditions for quasilinear parabolic and qusilinear pseudo-parabolic equations are first considered. An existence and uniqueness of these weak solutions are proved under appropriate conditions. Taking into account an applied importance of the considered problems, the difference between exact and approximate solutions is estimated. In addition, it is shown that the solution of the problem corresponding to quasi-linear pseudo-parabolic equation approaches, as $\varepsilon \to 0$, to the solution of the problem corresponding to the quasi-linear pseudo-parabolic equation. The thesis consists of introduction and five chapters.In the first chapter necessary definitions and used notions are introduced. An existence and uniqueness of a weak(generalized) solutions of consired mixed problem for quasilinear parabolic equations with periodic boundary condition is studied in chapter 2. The same questions for quasilinear pseudo-parabolic equations with periodic boundary condition are analyzed in chapter 3. In addition to these, the difference between the corresponding exact and approximate solutions are estimated also in these two chapters. In chapter 4 the relationships between the solutions of problems consired in chapters 2 and 3 is investigated. In the final chapter 5, the obtained results and conclusions are presented.

1. GİRİŞ

1.1 Giriş

Bilindiği gibi, gerek sabit, gerekse de değişken katsayılı Fourier serilerinin yardımı ile bir takım adi ve kısmi türevli denklemler için çeşitli problemler incelenebilir.
Ele alınan tezde değişken katsayılı Fourier serilerinin yardımı ile yarı doğrusal parabolik ve yarı doğrusal pseudo-parabolik denklemler için devirli sınır koşullu karışık problemlerin incelenmesi öngörülmektedir.

İzotropik maddenin doğrusallaştırılmış ısı iletimi denkleminin,

$$c\frac{\partial u}{\partial t} = k\Delta_3 u + c\varepsilon\frac{\partial}{\partial_t}\Delta_3 u + q(t,x) \tag{1.1}$$

biçiminde olduğu bellidir [13,14,37,40,42]. Burada, u(t,x) maddenin t anında x noktasındaki sıcaklığı, q(t,x) ısı kaynağı, c maddenin ısınma ısısı, k ısı iletkenliği, ε ise küçük parametredir. Isı kaynağı u(t,x) e bağlı olduğunda (1.1) denklemi

$$c\frac{\partial u}{\partial t} = k\Delta_3 u + c\varepsilon\frac{\partial}{\partial_t}\Delta_3 u + q(t,x,u) \tag{1.2}$$

pseudo-parabolik yarı doğrusal denklemine dönüşür.

Gerek (1.1), gerekse (1.2) denklemi, $\varepsilon = 0$ durumunda çeşitli bilim adamları tarafından farklı sınır koşullarıyla ve değişik yöntemlerle incelenmiştir [4,5,6,11,17,28]. Yine bu denklemler $\varepsilon \neq 0$ durumunda da farklı sınır koşullar alınarak, Fourier yada başka yöntemlerle, çeşitli araştırmacılar tarafından incelenmiştir [1,2,20,21,22,23,24,34] . Fourier yöntemi, hiperbolik yada daha yüksek dereceli problemlere de uygulanabilir [8,19].

Ayrıca bir boyutlu durumda değişken katsayılı Fourier serilerinin yardımı ile yarı doğrusal parabolik denklem için

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t,x,u) \quad (0 < t < T < \infty) \tag{1.3}$$

$$u(t,0) = u(t,\pi) = 0 \qquad (0 \le t \le T \le \infty) \tag{1.4}$$

$$u(0,x) = \varphi(x) \qquad (0 \le x \le \pi) \tag{1.5}$$

karışık probleminin çeşitli anlamlarda çözümünün varlığı ve tekliği [28], yarı doğrusal pseudo-parabolik denklem için

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial^3 u}{\partial t \partial x^2} = f(t,x,u) \quad (0 < t < T < \infty) \tag{1.6}$$

$$u(t,0) = u(t,\pi) = 0 \qquad (0 \le t \le T \le \infty) \tag{1.7}$$

$$u(0,x) = \varphi(x) \qquad (0 \le x \le \pi) \tag{1.8}$$

karışık probleminin genelleşmiş çözümünün varlığı ve tekliğinin incelenmesinin yanı sıra, $\varepsilon \to 0$ da (1.6), (1.7), (1.8) probleminin genelleşmiş çözümünün, (1.3), (1.4), (1.5) probleminin genelleşmiş çözümüne yakınsadığı [24], gösterilmiştir.

Teknik uygulamalarda (1.4) sınırlarının yanı sıra

$$u(t,0)=u(t,\pi),\, u_x(t,0) = u_x(t,\pi) \qquad (0 \le t \le T \le \infty) \tag{1.9}$$

olmak üzere devirli sınır koşulları da sıkça raslanmaktadır [7,15]. Ayrıca, teknik açıdan (1.9) daha geniş kapsamlıdır. Ele alınan tez çalışmasında, (1.6), (1.9), (1.8) ve (1.3), (1.9), (1.5) problemlerinin incelenmesini öngörülmüştür.

Söz konusu tez, giriş ve beş bölümden ibarettir. Birinci bölümde tezde sık kullanılan tanım ve kavramlarla ilgili bilgiler verilmiştir. İkinci bölümde, ele alınan yarı doğrusal parabolik denklem için devirli sınır koşullu genelleşmiş çözümünün varlığı ve tekliği incelenmiştir, tezin üçüncü bölümünde, ele alınan pseudo-parabolik denklem için devirli sınır koşullu genelleşmiş çözümünün varlığı ve tekliği incelenmiştir. Dördüncü bölümde, $\varepsilon \to 0$ durumunda ikinci ve üçüncü bölümlerde ele alınan problemlerin çözümleri arasındaki ilişki incelenmiştir. Son olarak beşinci bölümde sonuçlar ve öneriler verilmiştir.

1.2 Genel Bilgiler

Tanım1.1 (Cauchy Eşitsizliği): f(x) ve g(x) fonksiyonları, [a,b] kapalı aralığında karesi ile integrallenebilir olduğunda,

$$\left|\int_a^b f(x)g(x)dx\right| \le \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}}\left(\int_a^b g^2(x)dx\right)^{\frac{1}{2}} \tag{1.10}$$

yani,

$$|(f,g)| \le \|f\|_{L_2} .\|g\|_{L_2}$$

eşitsizliği doğrudur. Bu eşitsizliğe Cauchy Eşitsizliği denir [26].

Tanım1.2 (Hölder Eşitsizliği): $x = \{x_1, x_2, ..., x_n, ...\} \in l_2$ ve $y = \{y_1, y_2, ..., y_n, ...\} \in l_2$ olsun. O zaman, Hölder Eşitsizliği denilen,.

$$\sum_{i=1}^{\infty}|x_i y_i| \le \left(\sum_{i=1}^{\infty}|x_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\infty}|y_i|^2\right)^{\frac{1}{2}} \tag{1.11}$$

eşitsizliği doğrudur [30,38].

Tanım1.3 (Bessel Eşitsizliği): [a,b] aralığında ortogonal sistem oluşturan $\varphi_1(t), \varphi_2(t), ..., \varphi_k(t), ...$ fonksiyonlar sistemi ve karesi ile integrallenebilir $f(t) = \sum_{k=1}^{\infty} c_k \varphi_k(t)$ fonksiyonu verilsin. Burada c_k lar Fourier katsayılarıdır.

O zaman, Bessel eşitsizliği denilen

$$\sum_{k=1}^{\infty} c_k^2 \|\varphi_k(t)\|^2 \leq \int_a^b f^2(t)dt$$

eşitsizliği doğrudur [7,15,30,38]. Bizim çalışmamızda ise, $[0,\pi]$ aralığında ortogonal sistem oluşturan, $\frac{1}{2}, \cos 2x, \sin 2x, ..., \cos 2kx, \sin 2kx, ...$ sistemi için Fourier serisi,

$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_{ck} \cos 2kx + a_{sk} \sin 2kx]$ şeklinde tanımlı olup,

Bessel eşitsizliği ise,

$$\frac{f_0^2}{2} + \sum_{k=1}^{\infty} (f_{ck}^2 + f_{sk}^2) \leq \frac{2}{\pi} \int_0^{\pi} f^2(x)dx \tag{1.12}$$

şeklindedir. Burada,

$$f_0 = \frac{2}{\pi} \int_0^{\pi} f(\xi)d\xi$$

$$f_{ck} = \frac{2}{\pi} \int_0^{\pi} f(\xi) \cos 2k\xi d\xi$$

$$f_{sk} = \frac{2}{\pi} \int_0^{\pi} f(\xi) \cos 2k\xi d\xi$$

dır.

Tanım1.4 (Gronwall Eşitsizliği): a(t), [0,T] aralığında negatif olmayan sürekli fonksiyon, b(t), c(t), [0,T] aralığında negatif olmayan integrallenebilir fonksiyonlar ve f(t) de sınırlı fonksiyon olsun. O zaman,

$$a(t) \leq \int_0^t [a(\tau)b(\tau) + c(\tau)]d\tau + f(t) \tag{1.13}$$

ise, Gronwall eşitsizliği denilen,

$$\max_{0\le t\le T} a(t) \le \left[\int_0^T c(\tau)d\tau + \sup|f(t)|\right]\exp\int_0^T b(\tau)d\tau \qquad (1.14)$$

eşitsizliği doğrudur [3,9,10,18].

Tanım1.5 (Banach Uzayı): Normlu lineer uzayda $\lim_{n\to\infty}\|x_n - x\| = 0$ ise, o zaman $\{x_n\}$ dizisi x e yakınsıyor denir ve $\lim_{n\to\infty} x_n = x$ olarak yazılır.

Normlu lineer uzay normlu yakınsamaya göre tam olduğunda, ona Banach uzayı denir [30,35,36,38,39].

Tanım1.6 (Lipshitz Koşulu): Herhangi D bölgesinde tanımlı f(x,y) fonksiyonu için, sözü edilen bölgenin her (x, y_1) ve (x, y_2) noktalarında

$$|f(x, y_1) - f(x, y_2)| \le M|y_1 - y_2|$$

eşitsizliği sağlanacak şekilde bir M>0 sayısı varsa o zaman f(x,y), D bölgesinde y ye göre Lipshitz koşulunu sağlıyor denir. M Lipshitz katsayısı diye adlandırılır [31].

Tanım1.7 (Picard Ardışık Yaklaşım Methodu): f(x,y) bir B bölgesinde sürekli olsun. $(x_0, y_0) \in B$ olsun.

Bilindiği gibi, belli koşullar dahilinde

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned} \qquad (1.15)$$

başlangıç değer probleminin çözümü,

$$y(x) = y_0 + \int_{x_0}^{x} f[\xi, y(\xi)]d\xi \qquad (1.16)$$

integral denklem sisteminin çözümüne denktir. Sözü edilen integral denklemin çözümünün varlığını göstermek için Picard ardışık yaklaşım yöntemi denilen ardışık yaklaşımları

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f[\xi, y_n(\xi)]d\xi, \quad n = \overline{1,\infty} \tag{1.17}$$

eşitlikleri ile belirlenen yöntem sıkça kullanılmaktadır [30,31,38].

Teorem 1.8: f(x,y) fonksiyonu D bölgesinde sürekli olup y ye göre Lipshitz koşulunu sağlıyorsa, o zaman $y_{n+1}(x)$ ardışık yaklaşımları $|x - x_0| \le a$ için var, sürekli ve (1.15) probleminin tek çözümüne yakınsar [12,38].

Tanım1.9 (Fourier Serisi ve Fourier Katsayıları):

[a,b] aralığında parçalı sürekli f(x) fonksiyonu, aralığın sonlu sayıda noktası hariç, diğer tüm noktalarda sürekli türeve sahip olup, sözü edilen sonlu sayıda noktada sağ ve sol türevlere sahipse, o zaman ona [a,b] aralığında parçalı pürüzsüz fonksiyon denir.

[-L,L] aralığında tanımlı f(x) fonksiyonunu ve bu aralıkta ortogonal

$$\frac{1}{2}, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, ..., \cos\frac{k\pi x}{L}, \sin\frac{k\pi x}{L}, ...$$

sistemini ele alalım.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(\xi)d\xi,$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos k\xi d\xi, \tag{1.18}$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin k\xi d\xi, (k = \overline{1,\infty})$$

olsun.

(1.18) ile belirlenen a_0, a_k ve b_k $k = \overline{1,\infty}$ lar için,

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L})$$

serisine f(x) fonksiyonunun [-L,L] aralığında Fourier serisi denir ve

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}) \qquad (1.19)$$

şeklinde yazılır. a_0, a_k ve b_k $k = \overline{1,\infty}$ ’lara, f(x)’in [-L,L] aralığında Fourier katsayıları denir [7,15,16,27].

Teorem1.10 (Dirichlet): f(x) fonksiyonu $[-L, L]$ aralığında parçalı pürüzsüz bir fonksiyon ise, f(x)’in sürekli olduğu her noktada onun Fourier serisi yakınsaktır ve

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L})$$

toplamı için aşağıdaki eşitlikler doğrudur:

a) f(x) in süreklilik noktalarında f(x)=S(x) (-L<x<L);

b) f(x) in süreksizlik noktalarında,

$$S(x) = \frac{f(x+0) + f(x-0)}{2}, (-L < x < L);$$

c) Uç noktalarında ise,

$$S(-L)=S(L)=\frac{f(-L+0)+f(L-0)}{2}$$

değerine eşittir [7,15,27].

f(x), [-L,L] aralığında çift fonksiyon olduğunda, onun Fourier serisi ve Fourier katsayıları, sırasıyla,

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{L},$$

$$a_0 = \frac{2}{L}\int_0^L f(\xi)d\xi,$$

$$a_k = \frac{2}{L}\int_0^L f(\xi)\cos k\xi d\xi\ ,(k=\overline{0,\infty}),$$

tek fonksiyon olduğunda ise,

$$f(x) \approx \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{L},$$

$$b_k = \frac{2}{L}\int_0^L f(\xi)\sin k\xi d\xi,\ (k=\overline{1,\infty}).$$

biçimini alır [7,15].

Tanım1.11 (Sturm- Liouville Problemi, Özdeğer ve Özvektörler):

$$\left(k(x)y'(x)\right)' + \left[q(x)+\lambda r(x)\right]y(x) = 0 \qquad (1.20)$$

$$a_1 y(a) + a_2 y'(a) = 0 \qquad (1.21)$$

$$b_1 y(b) + b_2 y'(b) = 0 \qquad (1.22)$$

sınır değer problemini alalım. Burada $k(x) \in C^1[a,b]$, $q(x), r(x) \in C[a,b]$ ve λ keyfi parametre ve $a_1^2 + a_2^2 \neq 0$, $b_1^2 + b_2^2 \neq 0$ olsun.

(1.20)-(1.22) sınır değer problemine sıfırdan farklı bir çözüm sağlayan λ sayısına, sözü edilen problemin öz değerleri, bunlara uygun çözümlere ise öz fonksiyonları denir. Bu durumda (1.20)-(1.22) problemine özdeğer veya Sturm- Liouville problemi denir [30,32,33,38,41,43,44].

Tanım1.12 (Periyodik Sınır Koşullu Özdeğer Problemi):

$$\left(k(x)y'(x)\right)' + \left[q(x) + \lambda r(x)\right]y(x) = 0$$
$$y(a) = y(b)$$
$$y'(a) = y'(b)$$

sınır değer problemine, devirli Sturm- Liouville problemi denir [25].

1.3 Türdeş Isı Transferi Denklemi için Devirli Sınır Koşullu Karışık Problemin Çözümü

$$U_t - a^2 U_{xx} = 0 \quad (t,x) \in D\{0 < t < T < \infty \ , \ 0 < x < \pi\} \tag{1.23}$$

$$\begin{aligned} &U(t,0) = U(t,\pi) && t \in [0,T] \\ &U_x(t,0) = U_x(t,\pi) && t \in [0,T] \end{aligned} \tag{1.24}$$

$$U(0,x) = \varphi(x) \qquad x \in [0,\pi] \tag{1.25}$$

karışık problemini ele alalım. (1.23)-(1.25) problemine, değişkenlerine ayırma yöntemini, veya diğer bir ifadeyle Fourier yöntemini uygulayalım. Bu nedenle çözümü,

U(t,x)= X(x)T(t)

şeklinde arayıp, X(x) ve T(t) yi belirleyelim.

$U_{xx}(t,x) = X''(x).T(t)$ ve $U_t(t,x) = X(x)T'(t)$ yi

(1.23) de yerlerine yazıp,

$$X(x)T'(t) = a^2 X''(x)T(t)$$

buradan da,

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} \tag{1.26}$$

eşitliğini elde ederiz. (1.26) eşitliğinin sol tarafı sadece t ye, sağ tarafı ise sadece x e bağlı olduğundan, sabit olması gerekir. Bu sabite ayırma sabiti denir ve genel olarak $-\lambda$ ile gösterilir.

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda \, .$$

Böylece, X(x) ve T(t) yi belirlemek için,

$$X''(x) = -\lambda X(x) \tag{1.27}$$
$$T'(t) = -a^2 \lambda T(t) \tag{1.28}$$

olmak üzere, iki tane adi diferansiyel denklem elde ederiz. (1.24) sınır koşullarını göz önüne aldığımızda,

$$X(0) = X(\pi)$$
$$X'(0) = X'(\pi)$$

olur. Böylece, X(x) için,

$$X''(x) + \lambda X(x) = 0, \quad x \in (0, \pi)$$
$$X(0) = X(\pi)$$
$$X'(0) = X'(\pi)$$

olmak üzere, devirli sınır koşullu Sturm-Lioville problemi elde edilir.

Bu problemin özdeğer ve özfonksiyonları

$$\lambda_0 = 0, \quad X_0(x) = \frac{1}{2}$$
$$\lambda_k = (2k)^2, \quad X_k(x) = C_{1k}\cos 2kx + C_{2k}\sin 2kx,\ k = \overline{1,\infty}$$

şeklinde bulunur. Şimdi (1.28) denkleminde λ yerine λ_k alıp, çözüm yapalım:

$$T_k'(t) + a^2\lambda_k T_k(t) = 0 \quad \Rightarrow \quad \frac{T_k'(t)}{T_k(t)} = -a^2\lambda_k \quad \Rightarrow$$
$$\text{In}\left|T_k(t)\right| = -\lambda_k a^2 t + \text{Inc}_k \Rightarrow T_k(t) = c_k e^{-\lambda_k a^2 t} \quad (k = \overline{0,\infty}).$$

Böylece $U_k(t,x) = X_k(x).T_k(t)$, yani,

$$U_0(t,x) = \frac{c_0}{2}, U_k(t,x) = (A_{ck}\cos 2kx + B_{sk}\sin 2kx)e^{-(2ak)^2 t}, (k = \overline{1,\infty})$$

çözümlerini bulmuş oluruz. Denklem, doğrusal ve türdeş olduğundan, bulunan çözümlerin toplamı da çözümdür. Yani,

$$U(t,x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(A_{ck}\cos 2kx + B_{sk}\sin 2kx)e^{-(2ak)^2 t} \tag{1.29}$$

dır.

(1.29) eşitliğinin içerdiği belirsiz katsayıları belirlemek için (1.25) başlangıç koşulunu kullanırsak,

$$\varphi(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(A_{ck}\cos 2kx + B_{sk}\sin 2kx) \tag{1.30}$$

trigonometrik serisini elde ederiz. Eşitliğin her iki tarafını sırayla 1, cos2nx, sin2nx ($n \in N^+$) ile çarpıp sözü edilen katsayıları,

$$c_0 = \varphi_0 = \frac{2}{\pi}\int_0^{\pi}\varphi(\xi)d\xi$$

$$A_{ck} = \varphi_{ck} = \frac{2}{\pi}\int_0^{\pi}\varphi(\xi)\cos 2k\xi d\xi$$

$$B_{ck} = \varphi_{sk} = \frac{2}{\pi}\int_0^{\pi}\varphi(\xi)\sin 2k\xi d\xi \qquad (k=\overline{1,\infty})$$

olarak buluruz. Böylece, (1.23)-(1.25) probleminin çözümünü biçimsel olarak,

$$U(t,x) = \frac{\varphi_0}{2} + \sum_{k=1}^{\infty}(\varphi_{ck}\cos 2kx + \varphi_{sk}\sin 2kx)e^{-(2ak)^2 t} \tag{1.31}$$

şeklinde buluruz.

Katsayılarının ifadelerini göz önüne alıp, çözümü aşağıdaki şekle dönüştürelim.

$$U(t,x) = \frac{1}{\pi}\int_0^{\pi}\varphi(\xi)d\xi + \sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^{\pi}\varphi(\xi)[\cos 2k\xi\cos 2kx + \sin 2k\xi\sin 2kx]d\xi e^{-(2ak)^2 t}$$

$$U(t,x) = \frac{2}{\pi}\int_0^{\pi}\varphi(\xi)\left[\frac{1}{2} + \sum_{k=1}^{\infty}e^{-(2ak)^2 t}\cos 2k(x-\xi)\right]d\xi$$

$$G(t,x,\xi) = \frac{2}{\pi}\left[\frac{1}{2} + \sum_{k=1}^{\infty}e^{-(2ak)^2 t}\cos 2k(x-\xi)\right]d\xi$$

kabul edersek, ele alınan problemin çözümünün,

$$U(t,x) = \int_0^{\pi}G(t,x,\xi)\varphi(\xi)d\xi$$

biçimini elde ederiz.

$G(t,x,\xi)$, kaynak fonksiyonu diye adlandırılır.

2. YARI DOĞRUSAL PARABOLİK DENKLEMLERİN GENELLEŞMİŞ ÇÖZÜMÜNÜN VARLIĞI VE TEKLİĞİNİN İNCELENMESİ

2.1 Çözümün Aranması

Yarı doğrusal parabolik denklem için, devirli sınır koşullu

$$u_t - a^2 u_{xx} = f(t,x,u) \quad (t,x) \in D\{0 < t < T < \infty \ , \ 0 < x < \pi\} \tag{2.1}$$

$$\left.\begin{aligned} &u(t,0) = u(t,\pi) && t \in [0,T] \\ &u_x(t,0) = u_x(t,\pi) && t \in [0,T] \end{aligned}\right. \tag{2.2}$$

$$u(0,x) = \varphi(x) \qquad x \in [0,\pi] \tag{2.3}$$

karışık problemi ele alınsın. Burada, $\varphi(x)$, $[0,\pi]$ üzerinde, $f(t,x,u)$ ise $\overline{D} \times (-\infty,\infty)$ üzerinde tanımlı fonksiyonlardır. Ayrıca $\varphi(x)$, $\varphi(0) = \varphi(\pi)$, $\varphi'(0) = \varphi'(\pi)$ uyum koşullarını sağlar.

Tanım 2.1: D de t ye göre birinci, x e göre ikinci mertebeden sürekli kısmi türevlere sahip, $v(t,x)$ fonksiyonu,

$$\begin{aligned} &v(t,0) = v(t,\pi) \\ &v_x(t,0) = v_x(t,\pi) \\ &v(T,x) = 0 \end{aligned}$$

koşullarını sağlıyorsa, ona test fonksiyonu denir [13,14,29,37].

Tanım 2.2: Her $v(t,x)$ test fonksiyonu için,

$$\int_0^T \int_0^\pi \left[\left(\frac{\partial v}{\partial t} + a^2 \frac{\partial^2 v}{\partial x^2} \right) u + f(t,x,u) v \right] dxdt + \int_0^\pi \varphi(x) v(0,x) dx = 0 \tag{2.4}$$

integral eşitliğini sağlayan $u(t,x)\in C(\overline{D})$ fonksiyonuna, (2.1)-(2.3) karışık probleminin genelleşmiş çözümü denir.

Teorem2.3: $\varphi(x)$ fonksiyonu $[0,\pi]$ aralığında Dirichlet teoreminin koşullarını, f(t,x,u) fonksiyonu $\overline{D}\times(-\infty,\infty)$ bölgesinde değişkenlerine göre sürekli olup her $(t,x)\in D$ için

$|f(t,x,u)-f(t,x,v)|\leq b(t,x)|u-v|$

Lipschitz koşulunu (burada $b(t,x)\geq 0,\ b(t,x)\in L_2(D)$ dir.) sağlıyorsa,

O zaman (2.1)-(2.3) probleminin D bölgesinde genelleşmiş çözümü var ve tektir.

Bu teoremi ispatlamak için çözümü,

$$u(t,x)=\frac{u_0(t)}{2}+\sum_{k=1}^{\infty}[u_{ck}(t)\cos 2kx+u_{sk}(t)\sin 2kx] \tag{2.5}$$

Fourier serisi şeklinde arayıp, $u_0(t),u_{ck}(t),u_{sk}(t)\ \ (k=\overline{1,\infty})$ bilinmeyenleri için

$$\begin{aligned}
u_0(t)&=\varphi_0+\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}[u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi])d\xi d\tau,\\
u_{ck}(t)&=\varphi_{ck}\,e^{-(2ak)^2t}+\\
&\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}f(\tau,\xi,\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}[u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi])\cos 2k\xi d\xi d\tau,\\
u_{sk}(t)&=\varphi_{ck}\,e^{-(2ak)^2t}+\\
&\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}f(\tau,\xi,\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}[u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi])\sin 2k\xi d\xi d\tau
\end{aligned} \tag{2.6}$$

doğrusal olmayan sonsuz integral denklemler sistemi elde edilir. Burada,

$$\varphi_0 = \frac{2}{\pi}\int_0^{\pi} \varphi(\xi)d\xi,$$

$$\varphi_{ck} = \frac{2}{\pi}\int_0^{\pi} \varphi(\xi)\cos 2k\xi d\xi,$$

$$\varphi_{sk} = \frac{2}{\pi}\int_0^{\pi} \varphi(\xi)\sin 2k\xi d\xi \quad (k = \overline{1,\infty})$$

dır. Elde edilen sonsuz integral denklem sistemini incelemek için,

$$\{\bar{u}(t)\} = \left\{\frac{u_0(t)}{2}, u_{c1}(t), u_{s1}(t), ..., u_{ck}(t), u_{sk}(t), ...\right\}$$

dizi elemanlarının,

$$\max_{0\le t\le T}\frac{|u_0(t)|}{2} + \sum_{k=1}^{\infty}[\max_{0\le t\le T}|u_{ck}(t)| + \max_{0\le t\le T}|u_{sk}(t)|] < \infty$$

koşulunu sağlayanlarını B ile gösterip, bu kümede,

$$\|u(t)\| = \max_{0\le t\le T}\frac{|u_0(t)|}{2} + \sum_{k=1}^{\infty}[\max_{0\le t\le T}|u_{ck}(t)| + \max_{0\le t\le T}|u_{sk}(t)|]$$

şeklinde norm tanımlandığında, Banach uzayı elde edilir.

Lemma 2.4: Teorem 2.3 ün koşulları sağlanıyorsa, B uzayında (2.6) sisteminin çözümü vardır.

İspat: B uzayında (2.6) sistemini incelemek için, aşağıdaki şekilde ardışık yaklaşımlar oluşturulsun:

$$u_0^{(N+1)}(t) = u_0^{(0)}(t) +$$

$$\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,\frac{u_0^{(N)}(\tau)}{2} + \sum_{k=1}^{\infty}[u_{ck}^{(N)}(\tau)\cos 2k\xi + u_{sk}^{(N)}(\tau)\sin 2k\xi])d\xi d\tau,$$

$$u_{ck}^{(N+1)}(t) = u_{ck}^{(0)}(t) + \tag{2.7}$$

$$\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)} f(\tau,\xi,\frac{u_0^{(N)}(\tau)}{2} + \sum_{k=1}^{\infty}[u_{ck}^{(N)}(\tau)\cos 2k\xi + u_{sk}^{(N)}(\tau)\sin 2k\xi])\cos 2k\xi d\xi d\tau,$$

$$u_{sk}^{(N+1)}(t) = u_{sk}^{(0)}(t) +$$

$$\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)} f(\tau,\xi,\frac{u_0^{(N)}(\tau)}{2} + \sum_{k=1}^{\infty}[u_{ck}^{(N)}(\tau)\cos 2k\xi + u_{sk}^{(N)}(\tau)\sin 2k\xi])\sin 2k\xi d\xi d\tau.$$

Burada,

$$u_0^{(0)}(t) = \varphi_0,$$
$$u_{ck}^{(0)}(t) = \varphi_{ck} e^{-(2ak)^2 t},$$
$$u_{sk}^{(0)}(t) = \varphi_{sk} e^{-(2ak)^2 t} \quad (k = \overline{1,\infty})$$

kabul edilmiştir.

Yazımın basitliği için,

$$Au^{(N)}(\tau,\xi) = \frac{u_0^{(N)}(\tau)}{2} + \sum_{k=1}^{\infty}[u_{ck}^{(N)}(\tau)\cos 2k\xi + u_{sk}^{(N)}(\tau)\sin 2k\xi]$$

kabul edilirse, (2.7) sistemi,

$$u_0^{(N+1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(N)}(\tau,\xi)]d\xi d\tau$$

$$u_{ck}^{(N+1)} = u_{ck}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)} f[\tau,\xi,Au^{(N)}(\tau,\xi)]\cos 2k\xi d\xi d\tau \qquad (2.8)$$

$$u_{sk}^{(N+1)} = u_{sk}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)} f[\tau,\xi,Au^{(N)}(\tau,\xi)]\sin 2k\xi d\xi d\tau$$

şeklini alır.

Önce her N için,

$$\max_{0\le t\le T}\frac{\left|u_0^{(N)}(t)\right|}{2} + \sum_{k=1}^{\infty}[\max_{0\le t\le T}\left|u_{ck}^{(N)}(t)\right| + \max_{0\le t\le T}\left|u_{sk}^{(N)}(t)\right|] \langle \infty$$

olduğunu göstermek gerekir.

Teorem2.3 ün koşullarına göre,

$$\left\|\bar{u}^{(0)}(t)\right\| = \max_{0\le t\le T}\frac{\left|u_0^{(0)}(t)\right|}{2} + \sum_{k=1}^{\infty}[\max_{0\le t\le T}\left|u_{ck}^{(0)}(t)\right| + \max_{0\le t\le T}\left|u_{sk}^{(0)}(t)\right|] < \infty$$

olduğu açıktır.

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi)]d\xi d\tau$$

eşitliği aşağıdaki şekilde dönüştürülür:

$$u_0^{(1)}(t) = \varphi_0 + \frac{2}{\pi}\int_0^t\int_0^\pi \{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\}d\xi d\tau + \frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,0)\,d\xi d\tau.$$

Her tarafın mutlak değerini alıp, sağ taraftaki integrallere Caucy eşitsizliği uygulanırsa,

$$\left|u_0^{(1)}(t)\right| \le \left|\varphi_0\right| + \left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^{\pi} \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\right\} d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^{\pi} f(\tau,\xi,0)\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

veya

$$\left|u_0^{(1)}(t)\right| \le \left|\varphi_0\right| + \sqrt{t} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^{\pi} \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\right\} d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\sqrt{t} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^{\pi} f(\tau,\xi,0)\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki birinci integral içindeki ifadeye Lipshitzs koşulunu uygulayıp, elde edilen,

$$\left|u_0^{(1)}(t)\right| \le \left|\varphi_0\right| + \sqrt{t} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^{\pi} b(\tau,\xi)\left|Au^{(0)}(\tau,\xi)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\sqrt{t} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^{\pi} f(\tau,\xi,0)\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

eşitsizliğinin sağ tarafındaki $[0,\pi]$ aralığındaki integrallere yeniden Cauchy eşitsizliği uygulanırsa,

$$\left|u_0^{(1)}(t)\right| \le \left|\varphi_0\right| + 2\sqrt{\frac{t}{\pi}} \left(\int_0^t \left\{\int_0^{\pi} b^2(\tau,\xi)\left|Au^{(0)}(\tau,\xi)\right|^2 d\xi\right\} d\tau\right)^{\frac{1}{2}} +$$

$$2\sqrt{\frac{t}{\pi}} \left(\int_0^t \int_0^{\pi} f^2(\tau,\xi,0)\, d\xi\, d\tau\right)^{\frac{1}{2}}$$

olur. Burada $\left|Au^{(0)}(\tau,\xi)\right| \leq \left|u^{(0)}(\tau)\right|$ olduğu dikkate alınırsa,

$$\left|u_0^{(1)}(t)\right| \leq \left|\varphi_0\right| + 2\sqrt{\frac{t}{\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|u^{(0)}(\tau)\right|^2\right)^{\frac{1}{2}} +$$

$$2\sqrt{\frac{t}{\pi}}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur.

Her tarafın maksimumu alınırsa,

$$\max_{0\leq t\leq T}\left|u_0^{(1)}(t)\right| \leq \left|\varphi_0\right| + 2\sqrt{\frac{T}{\pi}}\left\|b(t,\xi)\right\|_{L_2(D)}\left\|u^{(0)}(t)\right\| + 2\sqrt{\frac{T}{\pi}}\left\|f(t,\xi,0)\right\|_{L_2(D)}$$

elde edilir.

Benzer olarak,

$$u_{ck}^{(1)}(t) = u_{ck}^{(0)}(t) + \frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi)]\cos 2k\xi\ d\xi d\tau$$

eşitliğini,

$$u_{ck}^{(1)}(t) = \varphi_{ck}e^{-(2ak)^2(t-\tau)} + \frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\right\}\cos 2k\xi\, d\xi d\tau +$$

$$\frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi d\tau$$

şeklinde dönüştürüp, her iki tarafının mutlak değerini aldıktan sonra, sağ taraftaki integrallere Cauchy eşitsizliği uygulanırsa,

$$\left|u_{ck}^{(1)}(t)\right| \le \left|\varphi_{ck}\right| + \left(\int_0^t e^{-2(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0)\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\left(\int_0^t e^{-2(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Her tarafın k'ya göre $(k=\overline{1,\infty})$ toplamını alıp Hölder eşitsizliği kullanılırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| +$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^\pi\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0)\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Son eşitsizliğe Bessel eşitsizliği,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\pi}{4\sqrt{3}a}\left(\int_0^t\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0)\right\}^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\frac{\pi}{4\sqrt{3}a}\left(\int_0^t\frac{2}{\pi}\left\{\int_0^\pi f^2(\tau,\xi,0)d\xi\right\}d\tau\right)^{\frac{1}{2}},$$

daha sonra da Lipsthitz koşulu uygulanırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|Au^{(0)}(\tau,\xi)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Burada, $\left|Au^{(0)}(\tau,\xi)\right| \le \left|u^{(0)}(\tau)\right|$ olduğu göz önüne alınıp, t ye göre maksimuma geçilirse,

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|b(t,x)\right\|_{L_2(D)}\left\|\bar{u}^{(0)}(t)\right\| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|f(t,x,0)\right\|_{L_2(D)}$$

olur. Benzer işlemler

$$u_{sk}^{(1)}(t) = u_{sk}^{(0)}(t) + \frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^{\pi} f[\tau,\xi,Au^{(0)}(\tau,\xi)]\sin 2k\xi\ d\xi d\tau$$

eşitliğine uygulandığında,

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{sk}\right| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|b(t,x)\right\|_{L_2(D)}\left\|\bar{u}^{(0)}(t)\right\| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|f(t,x,0)\right\|_{L_2(D)}$$

olur.

Elde edilen eşitsizlikleri taraf tarafa toplandığında,

$$\left\|\bar{u}^{(1)}(t)\right\| = \max_{0\le t\le T}\frac{\left|u_0^{(1)}(t)\right|}{2} + \sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right| + \max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right| \le \frac{\left|\varphi_0\right|}{2} + \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \left|\varphi_{sk}\right| +$$
$$\left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi}}{2\sqrt{3}a}\right)\left(\left\|b(t,x)\right\|_{L_2(D)}\left\|u^{(0)}(t)\right\| + \left\|f(t,x,0)\right\|_{L_2(D)}\right)$$

olur.Teorem2.3 ün koşullarına göre, eşitliğin sağ tarafının sınırlı olduğu açıktır, yani,

$$\left\|\bar{u}^{(1)}(t)\right\| \langle\ \infty$$

dır.

Şimdi benzer işlemler ikinci yaklaşımlar için yapılsın:

$$u_0^{(2)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^{\pi} f[\tau,\xi,Au^{(1)}(\tau,\xi)]d\xi d\tau$$

eşitliğini,

$$u_0^{(2)}(t) = \varphi_0 + \frac{2}{\pi}\int_0^t\int_0^\pi \{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f(\tau,\xi,0)\}d\xi d\tau + \int_0^t\int_0^\pi f(\tau,\xi,0)d\xi d\tau$$

şeklinde dönüştürüp, her tarafın mutlak değerini aldıktan sonra sağ taraftaki integrallere, Caucy Eşitsizliği uygulanırsa,

$$\left|u_0^{(2)}(t)\right| \le |\varphi_0| + \left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f(\tau,\xi,0)\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

$$\left|u_0^{(2)}(t)\right| \le |\varphi_0| + \sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f(\tau,\xi,0)\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

eşitsizliği bulunur.

Sonra sağ taraftaki integral içindeki farka Lipshitzs koşulu uygulanırsa,

$$\left|u_0^{(2)}(t)\right| \le |\varphi_0| + \sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi b(\tau,\xi)\left|Au^{(1)}(\tau,\xi)\right|d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

elde edilir. Sağ taraftaki $[0,\pi]$ aralığındaki integrallere Cauchy eşitsizliği uygulanırsa,

$$\left|u_0^{(2)}(t)\right| \le |\varphi_0| + 2\sqrt{\frac{t}{\pi}}\left(\int_0^t\left\{\int_0^\pi b^2(\tau,\xi)\left|Au^{(1)}(\tau,\xi)\right|^2 d\xi\right\}d\tau\right)^{\frac{1}{2}} +$$
$$2\sqrt{\frac{t}{\pi}}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Burada, $\left|Au^{(1)}(\tau,\xi)\right| \le \left|u^{(1)}(\tau)\right|$ olduğu dikkate alınırsa,

$$\left|u_0^{(2)}(t)\right| \le |\varphi_0| + 2\sqrt{\frac{t}{\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|u^{(1)}(\tau)\right|^2\right)^{\frac{1}{2}} + 2\sqrt{\frac{t}{\pi}}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Her tarafın t ye göre maksimumu alındığında;

$$\max_{0\le t\le T}\left|u_0^{(2)}(t)\right| \le |\varphi_0| + 2\sqrt{\frac{T}{\pi}}\left\|b(\tau,\xi)\right\|_{L_2(D)}\left\|u^{(1)}(t)\right\| + 2\sqrt{\frac{T}{\pi}}\left\|f(\tau,\xi,0)\right\|_{L_2(D)}$$

elde edilir.

Yeniden,

$$u_{ck}^{(2)}(t) = u_{ck}^{(0)}(t) + \frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi f[\tau,\xi,Au^{(1)}(\tau,\xi)]\cos 2k\xi \; d\xi d\tau$$

eşitliğini

$$u_{ck}^{(2)}(t) = \varphi_{ck}e^{-(2ak)^2(t-\tau)} + \frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f(\tau,\xi,0)\right\}\cos 2k\xi \, d\xi d\tau$$
$$+\frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi d\tau$$

şekline dönüştürüp, her tarafın mutlak değerini aldıktan sonra, sağ taraftaki integrallere Cauchy eşitsizliği uygulanırsa,

$$\left|u_{ck}^{(2)}(t)\right| \le \left|\varphi_{ck}\right| + \left(\int_0^t e^{-2(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f(\tau,\xi,0)\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\left(\int_0^t e^{-2(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

veya

$$\left|u_{ck}^{(2)}(t)\right| \le \left|\varphi_{ck}\right| + \frac{1}{2\sqrt{2}ak}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f(\tau,\xi,0)\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}ak}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Her tarafın k'ya göre $(k=\overline{1,\infty})$ toplamını alıp Hölder eşitsizliği uygulanırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(2)}(t)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f(\tau,\xi,0)\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Son eşitsizliğin sağ tarafında, $\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}}$ olduğu göz önüne alınıp, Bessel eşitsizliği uygulanırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(2)}(t)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\pi}{4\sqrt{3}a}\sqrt{\frac{2}{\pi}}\left(\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f(\tau,\xi,0)\right\}^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\frac{\pi}{4\sqrt{3}a}\sqrt{\frac{2}{\pi}}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

bulunur. Burada, Lipsthitz koşulu kullanılırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(2)}(t)\right| \leq \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^{\pi} b^2(\tau,\xi)\left|Au^{(1)}(\tau,\xi)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^{\pi} f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Burada, $\left|Au^{(1)}(\tau,\xi)\right| \leq \left|u^{(1)}(\tau)\right|$ olduğunu göz önüne alınıp, maximuma geçilirse,

$$\sum_{k=1}^{\infty}\max_{0\leq t\leq T}\left|u_{ck}^{(2)}(t)\right| \leq \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|b(t,x)\right\|_{L_2(D)}\left\|u^{(1)}(t)\right\| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|f(t,x,0)\right\|_{L_2(D)}$$

elde edilir.

Benzer olarak,

$$u_{sk}^{(2)}(t) = u_{sk}^{(0)}(t) + \frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^{\pi} f[\tau,\xi,u^{(1)}(\tau)]\sin 2k\xi \; d\xi d\tau$$

eşitliğine yukarıdaki işlemler uygulanırsa,

$$\sum_{k=1}^{\infty}\max_{0\leq t\leq T}\left|u_{sk}^{(2)}(t)\right| \leq \sum_{k=1}^{\infty}\left|\varphi_{sk}\right| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|b(t,x)\right\|_{L_2(D)}\left\|u^{(1)}(t)\right\| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|f(t,x,0)\right\|_{L_2(D)}$$

olur. Elde edilen eşitsizlikler taraf tarafa toplanırsa,

$$\left\|\overline{u}^{(2)}(t)\right\| = \max_{0\leq t\leq T}\frac{\left|u_0^{(2)}(t)\right|}{2} + \sum_{k=1}^{\infty}\max_{0\leq t\leq T}\left|u_{ck}^{(2)}(t)\right| + \max_{0\leq t\leq T}\left|u_{sk}^{(2)}(t)\right| \leq \frac{\left|\varphi_0\right|}{2} + \sum_{k=1}^{\infty}\left(\left|\varphi_{ck}\right| + \left|\varphi_{sk}\right|\right) +$$
$$\left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi}}{2\sqrt{3}a}\right)\left(\left\|b(t,x)\right\|_{L_2(D)}\left\|u^{(1)}(t)\right\| + \left\|f(t,x,0)\right\|_{L_2(D)}\right)$$

olur. Teorem2.3 ün koşullarına göre eşitliğin sağ tarafının sınırlı olduğu açıktır, yani,

$$\left\|\overline{u}^{(2)}(t)\right\| \langle \; \infty$$

dır.

Buradan, tümevarım yöntemi ile

$$\left\|\overline{u}^{(N+1)}(t)\right\| = \max_{0\le t\le T}\frac{\left|u_0^{(N+1)}(t)\right|}{2} + \sum_{k=1}^{\infty}[\max_{0\le t\le T}\left|u_{ck}^{(N+1)}(t)\right| + \max_{0\le t\le T}\left|u_{sk}^{(N+1)}(t)\right|] \le \frac{|\varphi_0|}{2} + \sum_{k=1}^{\infty}(|\varphi_{ck}| + |\varphi_{sk}|) +$$
$$\left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi}}{2\sqrt{3}a}\right)\left(\|b(t,x)\|_{L_2(D)}\left\|u^{(N)}(t)\right\| + \|f(t,x,0)\|_{L_2(D)}\right)$$

yazılabilir. Teorem2.3 ün koşullarına göre eşitliğin sağ tarafı sınırlıdır, yani,

$$\left\|\overline{u}^{(N+1)}(t)\right\| \langle \infty$$

dır.

Demek ki ardışık yaklaşımlar B uzayındadır, yani,

$$\left\{\overline{u}^{(N+1)}(t)\right\} = \left\{\frac{u_0^{(N+1)}(t)}{2}, u_{c1}^{(N+1)}(t), u_{s1}^{(N+1)}(t), ..., u_{cn}^{(N+1)}(t), u_{sn}^{(N+1)}(t), ...\right\} \in B$$

dır. Böylece Lemma 2.4 ispatlanmış olur.

2.2 Ardışık Yaklaşımların Yakınsaklığı

Lemma 2.5: Teorem 2.3 ün koşulları sağlandığında, (2.8) ile tanımlanan ardışık yaklaşımları B uzayında yakınsaktır.

İspat: (2.8) ile belirlenen sisteme ardışık yaklaşımların yakınsaklığını incelemek için, aşağıdaki farklar değerlendirilir:

$$u_0^{(1)}(t) - u_0^{(0)}(t) = \frac{2}{\pi}\int_0^t\int_0^{\pi} f[\tau,\xi,Au^{(0)}(\tau,\xi)]\,d\xi d\tau\,,$$

$$u_{ck}^{(1)}(t)-u_{ck}^{(0)}(t)=\frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi)]\cos 2k\xi\ d\xi d\tau\ ,$$

$$u_{sk}^{(1)}(t)-u_{sk}^{(0)}(t)=\frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi)]\sin 2k\xi\ d\xi d\tau$$

Bu farklar aşağıdaki gibi yazılırsa,

$$\left|\overline{u}^{(1)}(t)-\overline{u}^{(0)}(t)\right|=\frac{\left|u_0^{(1)}(t)-u_0^{(0)}(t)\right|}{2}+\sum_{k=1}^{\infty}[\left|u_{ck}^{(1)}(t)-u_{ck}^{(0)}(t)\right|+\left|u_{sk}^{(1)}(t)-u_{sk}^{(0)}(t)\right|]\le$$
$$\frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi)]d\xi d\tau\right|+$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}f[\tau,\xi,Au^{(0)}(\tau,\xi)]\cos 2k\xi d\xi d\tau\right|+$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}f[\tau,\xi,Au^{(0)}(\tau,\xi)]\sin 2k\xi d\xi d\tau\right|\le$$

$$\le\frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi\{f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0)\}d\xi d\tau\right|+$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\{f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0)\}\cos 2k\xi d\xi d\tau\right|+$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\{f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0)\}\sin 2k\xi d\xi d\tau\right|+$$
$$\frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,0)d\xi d\tau\right|+$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}f(\tau,\xi,0)\cos 2k\xi d\xi d\tau\right|+$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}f(\tau,\xi,0)\sin 2k\xi d\xi d\tau\right|$$

elde edilir. Sağ taraftaki integrallere Cauchy eşitsizliğini uyguladıktan sonra, t ye göre integralleme yapıp, gereken sadeleştirmeler yapılırsa,

$$\left|\overline{u}^{(1)}(t)-\overline{u}^{(0)}(t)\right|\le \frac{1}{2}\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0))d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\left(\int_0^t e^{-(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0))\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\left(\int_0^t e^{-(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0))\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2}\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\left(\int_0^t e^{-(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\left(\int_0^t e^{-(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

yani,

$$\left|\overline{u}^{(1)}(t)-\overline{u}^{(0)}(t)\right|\le \frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0))d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0))\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0))\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur.

Sağ taraftaki sonsuz toplamlara Hölder eşitsizliği uygulanıp, $\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}=\frac{\pi}{\sqrt{6}}$ olduğu göz önüne alınırsa,

$$\left|\bar{u}^{(1)}(t)-\bar{u}^{(0)}(t)\right|\le\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi(f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0))d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi(f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0))\cos 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi(f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0))\sin 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}\le$$

$$\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\sin 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}$$

olur. Sağ taraftaki sonsuz toplamlarda, toplamla t ye göre integralin değiştirilebildiği kabul edilip, Bessel eşitsizliği uygulanırsa,

$$\left|\overline{u}^{(1)}(t)-\overline{u}^{(0)}(t)\right| \le \left(\frac{1}{2}\sqrt{T}+\frac{\pi}{4\sqrt{3}a}+\frac{\pi}{4\sqrt{3}a}\right)\left(\int_0^t \frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi)]-f(\tau,\xi,0)\right\}^2 d\xi d\tau\right)^{\frac{1}{2}}+$$

$$\left(\frac{1}{2}\sqrt{T}+\frac{\pi}{4\sqrt{3}a}+\frac{\pi}{4\sqrt{3}a}\right)\left(\int_0^t \frac{2}{\pi}\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Sonuca Lipsthitz koşulu uygulandığında ise,

$$\left|\overline{u}^{(1)}(t)-\overline{u}^{(0)}(t)\right| \le \frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}}\left\|\overline{u}^{(0)}(t)\right\|+$$

$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur.

$$A_T=\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\|b(t,x)\|_{L_2(D)}\left\|\overline{u}^{(0)}(t)\right\|+$$

$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\|f(t,x,0)\|_{L_2(D)}$$

olarak kabul edilirse, Teorem2.3 ün koşullarına göre A_T bir sayıdır denilebilir. Bu yüzden de, son eşitsizlik,

$$\left|\overline{u}^{(1)}(t)-\overline{u}^{(0)}(t)\right| \le A_T$$

şeklinde yazılabilir. Burada A_T, sınırlı bir ifadedir.

Yeniden,

$$u_0^{(2)}(t)-u_0^{(1)}(t)=\frac{2}{\pi}\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}d\xi d\tau,$$

$$u_{ck}^{(2)}(t) - u_{ck}^{(1)}(t) =$$
$$\frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\cos 2k\xi\; d\xi d\tau,$$

$$u_{sk}^{(2)}(t) - u_{sk}^{(1)}(t) =$$
$$\frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\sin 2k\xi\; d\xi d\tau$$

farkları yazılıp, art arda değerlendirilirse,

$$\left|\bar{u}^{(2)}(t) - \bar{u}^{(1)}(t)\right| = \frac{\left|u_0^{(2)}(t) - u_0^{(1)}(t)\right|}{2} + \sum_{k=1}^{\infty}\left[\left|u_{ck}^{(2)}(t) - u_{ck}^{(1)}(t)\right| + \left|u_{sk}^{(2)}(t) - u_{sk}^{(1)}(t)\right|\right] \le$$
$$\frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}d\xi d\tau\right| +$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\cos 2k\xi d\xi d\tau\right| +$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\sin 2k\xi d\xi d\tau\right|.$$

olur. Sağ taraftaki integrallere Caucy eşitsizliği uygulanıp, gereken işlemler yapılırsa,

$$\left|\bar{u}^{(2)}(t) - \bar{u}^{(1)}(t)\right| \le \frac{1}{2}\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$
$$\sum_{k=1}^{\infty}\left(\int_0^t e^{-(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$
$$\sum_{k=1}^{\infty}\left(\int_0^t e^{-(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)] - f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

$$\left|\overline{u}^{(2)}(t)-\overline{u}^{(1)}(t)\right| \le \frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\sum_{k=1}^{\infty}\left(\frac{1}{2(2ak)^2}e^{-(2ak)^2(t-\tau)}\Big|_0^t\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\sum_{k=1}^{\infty}\left(\frac{1}{2(2ak)^2}e^{-(2ak)^2(t-\tau)}\Big|_0^t\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

$$\left|\overline{u}^{(2)}(t)-\overline{u}^{(1)}(t)\right| \le \frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki ikinci ve üçüncü toplamlara Hölder eşitsizliğini uyguladığında ise,

$$\left|\overline{u}^{(2)}(t)-\overline{u}^{(1)}(t)\right| \le \frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}\sin 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}$$

elde edilir. Sağ taraftaki sonsuz toplamlarda, Teorem2.3 ün koşullarına göre toplamla t ye göre integralin yerlerinin değiştirilebildiği göz önüne alınıp, Bessel eşitsizliği uygulanırsa,

$$\left|\bar{u}^{(2)}(t)-\bar{u}^{(1)}(t)\right| \le \frac{1}{2}\sqrt{T}\left[\int_0^t \frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}^2 d\xi d\tau\right]^{\frac{1}{2}} +$$

$$\frac{\pi}{4\sqrt{3}a}\left[\int_0^t \frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}^2 d\xi d\tau\right]^{\frac{1}{2}} +$$

$$\frac{\pi}{4\sqrt{3}a}\left[\int_0^t \frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi)]-f[\tau,\xi,Au^{(0)}(\tau,\xi)]\right\}^2 d\xi d\tau\right]^{\frac{1}{2}}$$

olur.

Sağ taraftaki integraller içindeki farklara Lipsthitz koşulu uygulanırsa,

$$\left|\bar{u}^{(2)}(t)-\bar{u}^{(1)}(t)\right| \le (\frac{1}{2}\sqrt{T}+\frac{\pi}{2\sqrt{3}a})\left(\int_0^t \frac{2}{\pi}\int_0^\pi b^2(\tau,\xi)\left[Au^{(1)}(\tau,\xi)-Au^{(0)}(\tau,\xi)\right]^2 d\xi d\tau\right)^{\frac{1}{2}} \le$$

$$\left(\frac{\sqrt{T}}{2}+\frac{\pi}{2\sqrt{3}a}\right)\left(\frac{2}{\pi}\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}} \max_{0\le t\le T}\left|Au^{(1)}(\tau,\xi)-Au^{(0)}(\tau,\xi)\right| \le$$

$$\left(\frac{\sqrt{T}}{2}+\frac{\pi}{2\sqrt{3}a}\right)\left(\frac{2}{\pi}\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}} \max_{0\le t\le T}\left|\bar{u}^{(1)}(t)-\bar{u}^{(0)}(t)\right| \le$$

$$\left(\frac{\sqrt{T}}{2}+\frac{\pi}{2\sqrt{3}a}\right)\left(\frac{2}{\pi}\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}} A_T,$$

yani,

$$\left|\bar{u}^{(2)}(t)-\bar{u}^{(1)}(t)\right| \le \left(\frac{\sqrt{T}}{2}+\frac{\pi}{2\sqrt{3}a}\right)\sqrt{\frac{2}{\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}} A_T =$$

$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}} A_T$$

bulunur.

Yeniden,

$$u_0^{(3)}(t)-u_0^{(2)}(t)=\frac{2}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi)]-f[\tau,\xi,Au^{(1)}(\tau,\xi)]\right\}d\xi d\tau$$

$$u_{ck}^{(3)}(t)-u_{ck}^{(2)}(t)=$$
$$\frac{2}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi)]-f[\tau,\xi,Au^{(1)}(\tau,\xi)]\right\}\cos 2k\xi\ e^{-(2ak)^2(t-\tau)}d\xi d\tau$$

$$u_{sk}^{(3)}(t)-u_{sk}^{(2)}(t)=$$
$$\frac{2}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi)]-f[\tau,\xi,Au^{(1)}(\tau,\xi)]\right\}\sin 2k\xi\ e^{-(2ak)^2(t-\tau)}d\xi d\tau$$

farkları yazılıp, benzer yolla

$$\left|\overline{u}^{(3)}(t)-\overline{u}^{(2)}(t)\right|=\frac{\left|u_0^{(3)}(t)-u_0^{(2)}(t)\right|}{2}+\sum_{k=1}^{\infty}\left[\left|u_{ck}^{(3)}(t)-u_{ck}^{(2)}(t)\right|+\left|u_{sk}^{(3)}(t)-u_{sk}^{(2)}(t)\right|\right]\leq$$
$$\frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi)]-f[\tau,\xi,Au^{(1)}(\tau,\xi)]\right\}d\xi d\tau\right|+$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi)]-f[\tau,\xi,Au^{(1)}(\tau,\xi)]\right\}\cos 2k\xi d\xi d\tau\right|+$$
$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi)]-f[\tau,\xi,Au^{(1)}(\tau,\xi)]\right\}\sin 2k\xi d\xi d\tau\right|$$

eşitsizliği elde edilir. Burada da, yukarıdakilere benzer işlemler yapıldığında,

$$\left|\overline{u}^{(3)}(t)-\overline{u}^{(2)}(t)\right|\leq\frac{a\sqrt{3T}+\pi}{2a\sqrt{3}}\left(\int_0^t\frac{2}{\pi}\int_0^\pi b^2(\tau,\xi)\left[Au^{(2)}(\tau,\xi)-Au^{(1)}(\tau,\xi)\right]^2d\xi d\tau\right)^{\frac{1}{2}}\leq$$
$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left[\overline{u}^{(2)}(t)-\overline{u}^{(1)}(t)\right]^2d\xi d\tau\right)^{\frac{1}{2}}\leq$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 A_T\left[\int_0^t\int_0^\pi b^2(\tau,\xi)\left(\int_0^\tau\int_0^\pi b^2(\tau_1,\xi_1)d\xi_1 d\tau_1\right)d\xi d\tau\right]^{\frac{1}{2}} \leq$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 A_T\,\frac{1}{\sqrt{2}}\left[\left(\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^2\right]^{\frac{1}{2}}$$

olur.

Bu sonuçlar dikkate alınıp, matematik tümevarım yöntemi uygulanırsa,

$$\left|\overline{u}^{(N+1)}(t)-\overline{u}^{(N)}(t)\right| \leq \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N A_T\,\frac{1}{\sqrt{N!}}\left[\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right]^{\frac{N}{2}}$$

eşitsizliği elde edilir. Burada, t ye göre her iki tarafın maksimumu alınırsa,

$$\left\|\overline{u}^{(N+1)}(t)-\overline{u}^{(N)}(t)\right\| \leq \frac{A_T}{\sqrt{N!}}\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N \left\|b(t,x)\right\|_{L_2(D)}^N$$

bulunur.

Buradan da, yukarıdaki koşullar dahilinde

$$\left\{\overline{u}^{(N+1)}(t)\right\}=\left\{\frac{u_0^{(N+1)}(t)}{2}, u_{c1}^{(N+1)}(t), u_{s1}^{(N+1)}(t), \ldots, u_{cn}^{(N+1)}(t), u_{sn}^{(N+1)}(t), \ldots\right\}$$

dizisinin B de yakınsaklığı elde edilir. Böylece Lemma 2.5 ispatlanmış olur.

2.3 $\overline{u}(t)$ nin (2.6) Sistemini Sağlaması

$$\lim_{N\to\infty} \overline{u}^{(N)}(t) = \overline{u}(t)\left\{\frac{u_0(t)}{2}, u_{c1}(t), u_{s1}(t), \ldots, u_{cn}(t), u_{sn}(t), \ldots\right\}$$

olsun.

$\overline{u}(t)$ nın (2.6) sistemini sağladığını göstermek için, aşağıdaki farkları değerlendirmek yeterlidir.

Cauchy eşitsizliği kullanılırsa, sırasıyla,

1) $$\left|\frac{1}{\pi}\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}d\xi d\tau\right| \leq$$

$$\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

2) $$\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\cos 2k\xi d\xi d\tau\right| \leq$$

$$\left|\left(\int_0^t e^{-2(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}\right| \leq$$

$$\frac{1}{2\sqrt{2a}}\frac{1}{k}\left|\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}\right|,$$

3) $$\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\sin 2k\xi d\xi d\tau\right| \leq$$

$$\left|\left(\int_0^t e^{-2(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}\right|\leq$$

$$\frac{1}{2\sqrt{2a}}\frac{1}{k}\left|\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}\right|$$

bulunur.

2) ve 3)'de k ya göre $(k=\overline{1,\infty})$ toplam yapıp, elde edilen toplam ve 1) taraf tarafa toplanırsa,

$$\Delta\leq\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2a}}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2a}}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur (Burada,

$$\Delta=\left|\frac{1}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}d\xi d\tau\right|+$$

$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\cos 2k\xi d\xi d\tau\right|+$$

$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\left\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}\sin 2k\xi d\xi d\tau\right|$$

kabul edilmiştir).

Sağ taraftaki sonsuz toplamlara Hölder eşitsizliği uygulanırsa,

$$\Delta \le \frac{1}{2}\sqrt{T}\left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\cos 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\sin 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}$$

elde edilir. Teorem2.3 ün koşullarına göre, sağ taraftaki sonsuz toplamlarda, toplamla t ye göre integralin değiştirilebildiğini göz önüne alıp, Bessel eşitsizliği uygulandıktan sonra, $\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}}$ olduğu göz önüne alınırsa,

$$\Delta \le \frac{1}{2}\sqrt{T}\left[\int_0^t \frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}^2 d\xi d\tau\right]^{\frac{1}{2}} +$$

$$\frac{\pi}{4\sqrt{3}a}\left[\int_0^t \frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}^2 d\xi d\tau\right]^{\frac{1}{2}} +$$

$$\frac{\pi}{4\sqrt{3}a}\left[\int_0^t \frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}^2 d\xi d\tau\right]^{\frac{1}{2}} =$$

$$\frac{1}{2}(\sqrt{T} + \frac{\pi}{\sqrt{3}a}).\left[\int_0^t \frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\right\}^2 d\xi d\tau\right]^{\frac{1}{2}}$$

olur. Son tarafta Lipsthitz koşulu kullanılırsa,

$$\Delta \le \frac{1}{2}(\sqrt{T} + \frac{\pi}{\sqrt{3}a}).\left(\int_0^t \frac{2}{\pi}\int_0^\pi b^2(\tau,\xi)\left[Au(\tau,\xi) - Au^{(N)}(\tau,\xi)\right]^2 d\xi d\tau\right)^{\frac{1}{2}} \le$$

$$\frac{1}{2}(\sqrt{T} + \frac{\pi}{\sqrt{3}a}).\left(\frac{2}{\pi}\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau) - \bar{u}^{(N)}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} \le$$

$$\leq \frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left\|b(t,x)\right\|_{L_2(D)}\left\|\bar{u}(t)-\bar{u}^{(N)}(t)\right\|$$

olur.

$$\lim_{N\to\infty}\left\|\bar{u}(t)-\bar{u}^{(N)}(t)\right\| = 0$$

olduğundan, $\bar{u}(t)$ nın, (2.6) sistemini sağladığı görülmektedir.

2.4 Çözümün Tekliği

Lemma 2.6: Teorem 2.3 ün koşulları sağlanıyorsa, (2.6) sisteminin çözümü B uzayında tekdir.

İspat: Bunun için tersini farzedip, çözümün tek olmadığını ve

$$\bar{\bar{u}}(t) = \left\{\frac{\tilde{u}_0(t)}{2}, \tilde{u}_{c1}(t), \tilde{u}_{s1}(t), \ldots, \tilde{u}_{ck}(t), \tilde{u}_{sk}(t), \ldots\right\}$$ nin de bu sistemin diğer bir çözümü

olduğu kabul edilsin.

$$u_0(t)-\tilde{u}_0(t) = \frac{2}{\pi}\int_0^t\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,A\tilde{u}(\tau,\xi)]\}d\xi d\tau$$

eşitliğinde her iki tarafın mutlak değerini alıp, sağ taraftaki integrale Cauchy eşitsizliği uygulanırsa,

$$\left|u_0(t)-\tilde{u}_0(t)\right| \leq \left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,A\tilde{u}(\tau,\xi)]\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

$$\left|u_0(t)-\tilde{u}_0(t)\right| \le \sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,A\tilde{u}(\tau,\xi)]\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki integral içindeki ifadede Lipshitzs koşulu dikkate alınırsa,

$$\left|u_0(t)-\tilde{u}_0(t)\right| \le \sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi b(\tau,\xi)\left|Au(\tau,\xi)-A\tilde{u}(\tau,\xi)\right|d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

bulunur.

Benzer yolla,

$$u_{ck}(t)-\tilde{u}_{ck}(t)=\frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,A\tilde{u}(\tau,\xi)]\}\cos 2k\xi\, d\xi d\tau$$

farkı değerlendirilirse,

$$\left|u_{ck}(t)-\tilde{u}_{ck}(t)\right| \le \left(\int_0^t e^{-2(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}} \times$$

$$\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,A\tilde{u}(\tau,\xi)]\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

eşitsizliği elde edilir. Her tarafın k ya göre $(k=\overline{1,\infty})$ toplamını alıp, Hölder eşitsizliği uygulanırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}(t)-\tilde{u}_{ck}(t)\right| \le \frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}$$

$$\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,A\tilde{u}(\tau,\xi)]\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Bu eşitsizliğin sağ tarafında Bessel eşitsizliği uygulanırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}(t)-\tilde{u}_{ck}(t)\right| \le \frac{\pi}{4\sqrt{3}a}\left(\int_0^t \frac{2}{\pi}\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,A\tilde{u}(\tau,\xi)]\}^2 d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki integral içinde Lipsthitz koşulundan faydalanıldığında ise,

$$\sum_{k=1}^{\infty}\left|u_{ck}(t)-\tilde{u}_{ck}(t)\right| \le \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^{\pi} b^2(\tau,\xi)\left|Au(\tau,\xi)-A\tilde{u}(\tau,\xi)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

veya

$$\sum_{k=1}^{\infty}\left|u_{ck}(t)-\tilde{u}_{ck}(t)\right| \le \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^{\pi} b^2(\tau,\xi)\left|u(\tau)-\tilde{u}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

olur.

Benzer olarak,

$$u_{sk}(t)-\tilde{u}_{sk}(t)=\frac{2}{\pi}\int_0^t e^{-(2ak)^2(t-\tau)}\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,A\tilde{u}(\tau,\xi)]\}\sin 2k\xi \; d\xi d\tau$$

farkı için yukarıdaki işlemler uygulandığında,

$$\sum_{k=1}^{\infty}\left|u_{sk}(t)-\tilde{u}_{sk}(t)\right| \le \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^{\pi} b^2(\tau,\xi)\left|u(\tau)-\tilde{u}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

olur.

Elde edilen eşitsizlikler birleştirildiğinde,

$$\left|\overline{u}(t)-\overline{\overline{u}}(t)\right| = \frac{\left|u_0(t)-\widetilde{u}_0(t)\right|}{2} + \sum_{k=1}^{\infty}[\left|u_{ck}(t)-\widetilde{u}_{ck}(t)\right|' + \left|u_{sk}(t)-\overline{u}_{sk}(t)\right|] \le$$

$$\left(\sqrt{\frac{t}{\pi}} + \frac{\sqrt{2\pi}}{2\sqrt{3a}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|u(\tau)-\widetilde{u}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

elde edilir. Bu eşitliğin her tarafının karesi alınırsa,

$$\left|\overline{u}(t)-\overline{\overline{u}}(t)\right|^2 \le \left(\sqrt{\frac{t}{\pi}} + \frac{\sqrt{2\pi}}{2\sqrt{3a}}\right)^2 \int_0^t\int_0^\pi b^2(\tau,\xi)\left|u(\tau)-\widetilde{u}(\tau)\right|^2 d\xi d\tau$$

olur. Buradan, Gronwall eşitsizliğine göre,

$$\left|\overline{u}(t)-\overline{\overline{u}}(t)\right| \le 0 \times \exp\int_0^t\int_0^\pi b^2(\tau,\xi) d\xi d\tau \le 0$$

elde edilir.

Buradan da

$\overline{u}(t) = \overline{\overline{u}}(t)$, yani $u_0(t) = \widetilde{u}_0(t)$, $u_{ck}(t) = \widetilde{u}_{ck}(t)$, $u_{sk}(t) = \widetilde{u}_{sk}(t)$ $(k = \overline{1,\infty})$

olduğu görülür. Yani (2.6) sisteminin çözümü tekdir. Böylece Lemma 2.6 ispatlanmış olur.

Teorem 2.3 ün İspatı: Lemma 2.4, Lemma 2.5 ve Lemma 2.6 nın ispatından teorem ispat edilmiş olur.

2.5 Kesin Çözümle Ardışık Yaklaşımların Farkının Değerlendirilmesi

$\left|\bar{u}(\tau)-\bar{u}^{(N+1)}(\tau)\right|$ yi değerlendirmek için, önce sözü edilen fark aşağıdaki şekle dönüştürülür:

$$\left|\bar{u}(t)-\bar{u}^{(N+1)}(t)\right| = \frac{\left|u_0(t)-u_0^{(N+1)}(t)\right|}{2} +$$
$$\sum_{k=1}^{\infty}[\left|u_{ck}(t)-u_{ck}^{(N+1)}(t)\right| + \left|u_{sk}(t)-u_{sk}^{(N+1)}(t)\right|] \leq$$
$$\frac{1}{\pi}\left|\int_0^t\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}d\xi d\tau\right| +$$
$$\left|\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\cos 2k\xi d\xi d\tau\right| +$$
$$\left|\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\sin 2k\xi d\xi d\tau\right| \leq$$

$$\frac{1}{\pi}\left|\int_0^t\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]\}d\xi d\tau\right| +$$
$$\left|\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]\}\cos 2k\xi d\xi d\tau\right| +$$
$$\left|\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\{f[\tau,\xi,Au(\tau,\xi)]-f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]\}\sin 2k\xi d\xi d\tau\right| +$$
$$\frac{1}{\pi}\left|\int_0^t\int_0^\pi \{f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}d\xi d\tau\right| +$$
$$\left|\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\{f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\cos 2k\xi d\xi d\tau\right| +$$
$$\left|\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\{f[\tau,\xi,Au^{(N+1)}(\tau,\xi)]-f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\sin 2k\xi d\xi d\tau\right|.$$

Yukarıdakilere benzer olarak art arda Cauchy, Hölder, Bessel ve Lipshitz eşitsizlikleri uygulanırsa,

$$\left|\bar{u}(t)-\bar{u}^{(N+1)}(t)\right| \le \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau)-\bar{u}^{(N+1)}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}^{(N+1)}(\tau)-\bar{u}^{(N)}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} \le$$

$$\le \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau)-\bar{u}^{(N+1)}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi) d\xi d\tau\right)^{\frac{1}{2}}\left\|\bar{u}^{(N+1)}(t)-\bar{u}^{(N)}(t)\right\|$$

olur. Her iki tarafın karesini alıp, $(a+b)^2 \le 2a^2+2b^2$ eşitsizliği kullanılırsa,

$$\left|\bar{u}(t)-\bar{u}^{(N+1)}(t)\right|^2 \le 2\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau)-\bar{u}^{(N+1)}(\tau)\right|^2 d\xi d\tau\right)+$$

$$2\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 \|b(t,x)\|^2_{L_2(D)}\left\|\bar{u}^{(N+1)}(t)-\bar{u}^{(N)}(t)\right\|^2$$

olur. Gronwall eşitsizliğini ve

$$\left\|\bar{u}^{(N+1)}(t)-\bar{u}^{(N)}(t)\right\| \le \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N A_T\,\frac{1}{\sqrt{N!}}\|b(t,x)\|^N_{L_2(D)}$$

eşitsizliğini kullanıp, her tarafın karekökü alınırsa,

$$\left|\bar{u}(t)-\bar{u}^{(N+1)}(t)\right| \le \sqrt{2}\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\|b(t,x)\|_{L_2(D)}\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N \frac{A_T}{\sqrt{N!}}\|b(t,x)\|^N_{L_2(D)} \times$$

$$\exp\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 \|b(t,x)\|^2_{L_2(D)},$$

bulunur.

Son olarak da,

$$\left\|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right\| \leq \sqrt{\frac{2}{N!}}\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^{N+1} A_T \left\|b(t,x)\right\|_{L_2(D)}^{N+1} \times$$
$$\exp\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 \left\|b(t,x)\right\|_{L_2(D)}^{2}$$

bulunur.

Son eşitsizliğe dayanarak, (2.6) sisteminin yaklaşık çözümü olarak $\overline{u}^{(N+1)}(t)$ alındığında, yapılan hata N ye bağlı olarak değerlendirilebilir.

3. YARI DOĞRUSAL PSEUDO PARABOLİK DENKLEMLERİN GENELLEŞMİŞ ÇÖZÜMÜNÜN VARLIĞI VE TEKLİĞİNİN İNCELENMESİ

3.1 Çözümün Aranması

Yarı doğrusal pseudo parabolik denklem için, devirli sınır koşullu

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial^3 u}{\partial t \partial x^2} = f(t,x,u) \quad (t,x) \in D\{0 < t < T < \infty \ , \ 0 < x < \pi\} \tag{3.1}$$

$$\begin{aligned} u(t,0) &= u(t,\pi) \qquad t \in [0,T] \\ u_x(t,0) &= u_x(t,\pi) \qquad t \in [0,T] \end{aligned} \tag{3.2}$$

$$u(0,x) = \varphi(x) \qquad x \in [0,\pi] \tag{3.3}$$

karışık problemi ele alınsın. Burada, $\varphi(x)$ $[0,\pi]$ üzerinde, $f(t,x,u)$ $\overline{D} \times (-\infty,\infty)$ üzerinde tanımlı fonksiyonlar, $\varepsilon > 0$ ise küçük parametredir. Ayrıca $\varphi(x)$ fonksiyonu, $\varphi(0) = \varphi(\pi)$ ve $\varphi'(0) = \varphi'(\pi)$ uyum koşullarını sağlar.

Tanım 3.1: D de t ye göre birinci, x e göre ikinci mertebeden sürekli kısmi türevlere sahip, $v(t,x)$ fonksiyonu,

$$\begin{aligned} &v(t,0) = v(t,\pi) \\ &v_x(t,0) = v_x(t,\pi) \\ &v(T,x) = 0 \end{aligned}$$

koşullarını sağlıyorsa, ona test fonksiyonu denir.

Tanım 3.2: Her $v(t,x)$ test fonksiyonu için,

$$\int_0^T\int_0^\pi\left[\left(\frac{\partial v}{\partial t}+a^2\frac{\partial^2 v}{\partial x^2}-\varepsilon\frac{\partial^3 v}{\partial t\partial x^2}\right)u-f(t,x,u)v\right]dxdt$$
$$+\int_0^\pi \varphi(x)[v(0,x)-\varepsilon\frac{\partial^2 v(0,x)}{\partial x^2}]dx=0 \tag{3.4}$$

integral eşitliğini sağlayan $u(t,x,\varepsilon)\in C(\overline{D})\times[0,\overline{\varepsilon}]$ $(\overline{\varepsilon}>0)$ fonksiyonuna, (3.1)-(3.3) karışık probleminin genelleşmiş çözümü denir.

Teorem 3.3: $\varphi(x)$ fonksiyonu $[0,\pi]$ aralığında Dirichlet teoreminin koşullarını, f(t,x,u) fonksiyonu $\overline{D}\times(-\infty,\infty)$ bölgesinde argümanlarına göre sürekli olup her $(t,x)\in D$ için

$|f(t,x,u)-f(t,x,v)|\le b(t,x)|u-v|$

Lipschitz koşulunu (burada $b(t,x)\ge 0,\ b(t,x)\in L_2(D)$ dir.) sağlıyorsa,

o zaman, (3.1)-(3.3) probleminin D bölgesinde genelleşmiş çözümü var ve tektir.

Bu teoremi ispatlamak için çözümü,

$$u(t,x,\varepsilon)=\frac{u_0(t,\varepsilon)}{2}+\sum_{k=1}^{\infty}[u_{ck}(t,\varepsilon)\cos 2kx+u_{sk}(t,\varepsilon)\sin 2kx] \tag{3.5}$$

şeklinde arayıp, $u_0(t,\varepsilon),u_{ck}(t,\varepsilon),u_{sk}(t,\varepsilon)$ $(k=\overline{1,\infty})$ bilinmeyenleri için,

$$u_0(t,\varepsilon)=\varphi_0+\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,\frac{u_0(\tau,\varepsilon)}{2}+\sum_{k=1}^{\infty}[u_{ck}(\tau,\varepsilon)\cos 2k\xi+u_{sk}(\tau,\varepsilon)\sin 2k\xi])d\xi d\tau,$$

$$u_{ck}(t,\varepsilon)=\varphi_{ck}\,e^{\frac{-(2ak)^2t}{1+\varepsilon(2k)^2}}+\frac{1}{1+\varepsilon(2k)^2}\quad\times \tag{3.6}$$
$$\int_0^t\left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}\quad f(\tau,\xi,\frac{u_0(\tau,\varepsilon)}{2}+\sum_{k=1}^{\infty}[u_{ck}(\tau,\varepsilon)\cos 2n\xi+u_{sk}(\tau,\varepsilon)\sin 2k\xi])\cos 2k\xi d\xi\right\}d\tau,$$

$$u_{sk}(t,\varepsilon)=\varphi_{sk}\, e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}}+\frac{1}{1+\varepsilon(2k)^2}\times$$

$$\int_0^t \left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)} f(\tau,\xi,\frac{u_0(\tau,\varepsilon)}{2}+\sum_{k=1}^{\infty}[u_{ck}(\tau,\varepsilon)\cos 2n\xi+u_{sk}(\tau,\varepsilon)\sin 2k\xi])\sin 2k\xi d\xi\right\}d\tau$$

doğrusal olmayan sonsuz integral denklemler sistemi elde edilmiştir. Burada,

$$\varphi_0=\frac{2}{\pi}\int_0^\pi \varphi(\xi)\,d\xi,$$

$$\varphi_{ck}=\frac{2}{\pi}\int_0^\pi \varphi(\xi)\cos 2k\xi d\xi,$$

$$\varphi_{sk}=\frac{2}{\pi}\int_0^\pi \varphi(\xi)\sin 2k\xi d\xi$$

dır. Elde edilen sonsuz integral denklemler sistemini incelemek için,

$$\{\bar{u}(t,\varepsilon)\}=\left\{\frac{u_0(t,\varepsilon)}{2},u_{c1}(t,\varepsilon),u_{s1}(t,\varepsilon),\ldots,u_{ck}(t,\varepsilon),u_{sk}(t,\varepsilon),\ldots\right\}$$

dizi elemanlarının,

$$\max_{0\le t\le T}\frac{|u_0(t,\varepsilon)|}{2}+\sum_{k=1}^{\infty}[\max_{0\le t\le T}|u_{ck}(t,\varepsilon)|+\max_{0\le t\le T}|u_{sk}(t,\varepsilon)|\ ]<\infty$$

koşulunu sağlayanlarını B ile gösterip, bu kümede,

$$\|\bar{u}(t,\varepsilon)\|=\frac{1}{2}\max_{0\le t\le T}|u_0(t,\varepsilon)|+\sum_{k=1}^{\infty}[\max_{0\le t\le T}|u_{ck}(t,\varepsilon)|+\max_{0\le t\le T}|u_{sk}(t,\varepsilon)|]$$

şeklinde norm tanımlandığında, Banach uzayı elde edilir.

Lemma 3.4: Teorem 3.3 ün koşulları sağlanıyorsa, (3.6) sisteminin çözümü vardır.

İspat: B uzayında (3.6) sistemini incelemek için, aşağıdaki şekilde ardışık yaklaşımlar oluşturulsun:

$$u_0^{(N+1)}(t,\varepsilon) = u_0^{(0)}(t,\varepsilon) +$$

$$\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,\frac{u_0^{(N)}(\tau,\varepsilon)}{2} + \sum_{k=1}^{\infty}[u_{ck}^{(N)}(\tau,\xi)\cos 2k\xi + u_{sk}^{(N)}(\tau,\xi)\sin 2k\xi])d\xi d\tau,$$

$$u_{ck}^{(N+1)}(t,\varepsilon) = u_{ck}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2k)^2}\times \tag{3.7}$$

$$\int_0^t\left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)} f(\tau,\xi,\frac{u_0^{(N)}(\tau,\varepsilon)}{2} + \sum_{k=1}^{\infty}[u_{ck}^{(N)}(\tau,\varepsilon)\cos 2k\xi + u_{sk}^{(N)}(\tau,\varepsilon)\sin 2k\xi])\cos 2k\xi d\xi\right\}d\tau,$$

$$u_{sk}^{(N+1)}(t,\varepsilon) = u_{sk}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2k)^2}\times$$

$$\int_0^t\left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)} f(\tau,\xi,\frac{u_0^{(N)}(\tau,\varepsilon)}{2} + \sum_{k=1}^{\infty}[u_{ck}^{(N)}(\tau,\varepsilon)\cos 2k\xi + u_{sk}^{(N)}(\tau,\varepsilon)\sin 2k\xi])\sin 2k\xi d\xi\right\}d\tau.$$

Burada,

$$u_0^{(0)}(t,\varepsilon) = \varphi_0,$$

$$u_{ck}^{(0)}(t,\varepsilon) = \varphi_{ck} e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}},$$

$$u_{sk}^{(0)}(t,\varepsilon) = \varphi_{sk} e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}}$$

olarak alınmıştır.

Yazımın kolaylığı için,

$$Au^{(N)}(\tau,\xi,\varepsilon) = \frac{u_0^{(N)}(\tau,\varepsilon)}{2} + \sum_{k=1}^{\infty}[u_{ck}^{(N)}(\tau,\varepsilon)\cos 2k\xi + u_{sk}^{(N)}(\tau,\varepsilon)\sin 2k\xi]$$

kabul edilirse, (3.7) sistemi,

$$u_0^{(N+1)}(t,\varepsilon)=u_0^{(0)}(t,\varepsilon)+\frac{2}{\pi}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]d\xi d\tau,$$

$$u_{ck}^{(N+1)}(t,\varepsilon)=u_{ck}^{(0)}(t,\varepsilon)+$$
$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right\}d\tau, \tag{3.8}$$

$$u_{sk}^{(N+1)}(t,\varepsilon)=u_{sk}^{(0)}(t,\varepsilon)+$$
$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right\}d\tau$$

şeklini alır.

Önce her N için

$$\max_{0\le t\le T}\frac{\left|u_0^{(N)}(t,\varepsilon)\right|}{2}+\sum_{k=1}^{\infty}[\max_{0\le t\le T}\left|u_{ck}^{(N)}(t,\varepsilon)\right|+\max_{0\le t\le T}\left|u_{sk}^{(N)}(t,\varepsilon)\right|]\langle\infty$$

olduğu gösterilsin.

Teorem 3.3 ün koşullarına göre,

$$\left\|\bar{u}^{(0)}(t,\varepsilon)\right\|=\max_{0\le t\le T}\frac{\left|u_0^{(0)}(t,\varepsilon)\right|}{2}+\sum_{k=1}^{\infty}[\max_{0\le t\le T}\left|u_{ck}^{(0)}(t,\varepsilon)\right|+\max_{0\le t\le T}\left|u_{sk}^{(0)}(t,\varepsilon)\right|]\langle\infty$$

olduğu açıktır.

$$u_0^{(1)}(t,\varepsilon)=u_0^{(0)}(t,\varepsilon)+\frac{2}{\pi}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]d\xi d\tau$$

eşitliğinin sağ tarafına $\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,0)\,d\xi d\tau$ ifadesi ekleyip çıkarılırsa,

$$u_0^{(1)}(t,\varepsilon)=\varphi_0+\frac{2}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}d\xi d\tau+\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,0)\,d\xi d\tau.$$

olur. Her tarafın mutlak değeri alınıp, sağ taraftaki integrallere Cauchy eşitsizliği uygulanırsa,

$$\left|u_0^{(1)}(t,\varepsilon)\right|\le\left|\varphi_0\right|+\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

veya

$$\left|u_0^{(1)}(t,\varepsilon)\right|\le\left|\varphi_0\right|+\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki birinci, integral içindeki ifadeye Lipshitzs koşulunu uygulayıp, elde edilen,

$$\left|u_0^{(1)}(t,\varepsilon)\right|\le\left|\varphi_0\right|+\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi b(\tau,\xi)\left|Au^{(0)}(\tau,\xi,\varepsilon)\right|d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

eşitsizliğinin sağ tarafındaki $[0,\pi]$ aralığı üzerindeki integrallere yeniden Cauchy eşitsizliği uygulanırsa,

$$\left|u_0^{(1)}(t,\varepsilon)\right| \le |\varphi_0| + 2\sqrt{\frac{t}{\pi}}\left(\int_0^t\left\{\int_0^\pi b^2(\tau,\xi)\left|Au^{(0)}(\tau,\xi,\varepsilon)\right|^2 d\xi\right\}d\tau\right)^{\frac{1}{2}} +$$

$$2\sqrt{\frac{t}{\pi}}\left(\int_0^t\left\{\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Burada,

$\left|Au^{(0)}(\tau,\xi,\varepsilon)\right| \le \left|u^{(0)}(\tau,\varepsilon)\right|$ olduğu dikkate alınırsa,

$$\left|u_0^{(1)}(t,\varepsilon)\right| \le |\varphi_0| + 2\sqrt{\frac{t}{\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|u^{(0)}(\tau,\varepsilon)\right|^2\right)^{\frac{1}{2}} +$$

$$2\sqrt{\frac{t}{\pi}}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur.

Her tarafın maksimumu alınırsa,

$$\max_{0\le t\le T}\left|u_0{}^{(1)}(t,\varepsilon)\right| \le |\varphi_0| + 2\sqrt{\frac{T}{\pi}}\left\|b(t,\xi)\right\|_{L_2(D)}\left\|u^{(0)}(t,\varepsilon)\right\|_B + 2\sqrt{\frac{T}{\pi}}\left\|f(t,\xi,0)\right\|_{L_2(D)}$$

elde edilir.

Benzer olarak,

$$u_{ck}^{(1)}(t,\varepsilon) = u_{ck}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\cos 2k\xi\, d\xi d\tau$$

eşitliğinin sağ tarafına $\dfrac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\dfrac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\, d\xi d\tau$ ifadesi

ekleyip çıkarılırsa,

$$u_{ck}^{(1)}(t,\varepsilon)=\varphi_{ck}e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}+$$
$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\frac{2}{\pi}\int_0^\pi \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}\cos 2k\xi\, d\xi d\tau+$$
$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi d\tau$$

olur.

Elde edilen eşitliğin her iki tarafının mutlak değerini alıp, sağ taraftaki integrallere Cauchy eşitsizliği uygulanırsa,

$$\left|u_{ck}^{(1)}(t,\varepsilon)\right|\le\left|\varphi_{ck}\right|+$$
$$\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$
$$\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)]\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Her tarafın k ya göre $(k=\overline{1,\infty})$ toplamını alıp Hölder eşitsizliği kullanıldığında,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(1)}(t,\varepsilon)\right| \le \sum_{k=1}^{\infty}|\varphi_{ck}| + \frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2(\sqrt{1+\varepsilon(2k)^2})^2}\right)^{\frac{1}{2}} x$$

$$\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^{\pi}\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)] - f(\tau,\xi,0)\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2\sqrt{1+\varepsilon(2k)^2}}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^{\pi} f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur.

Son eşitsizliğe Bessel eşitsizliği uygulanıp,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(1)}(t,\varepsilon)\right| \le \sum_{k=1}^{\infty}|\varphi_{ck}| + \frac{\pi}{4\sqrt{3}a}\left(\int_0^t\frac{2}{\pi}\int_0^{\pi}\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)] - f(\tau,\xi,0)\right\}^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\frac{\pi}{4\sqrt{3}a}\left(\int_0^t\frac{2}{\pi}\left\{\int_0^{\pi} f^2(\tau,\xi,0)d\xi\right\}d\tau\right)^{\frac{1}{2}}$$

daha sonra da sağ taraftaki integral içindeki farka Lipsthitz koşulu uygulanırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(1)}(t,\varepsilon)\right| \le \sum_{k=1}^{\infty}|\varphi_{ck}| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^{\pi} b^2(\tau,\xi)\left|Au^{(0)}(\tau,\xi,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^{\pi} f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Burada,

$\left|Au^{(0)}(\tau,\xi,\varepsilon)\right| \le \left|u^{(0)}(\tau,\varepsilon)\right|$ olduğu göz önüne alınıp, t ye göre maksimuma geçilirse,

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{ck}^{(1)}(t,\varepsilon)\right| \le \sum_{k=1}^{\infty}|\varphi_{ck}| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\|b(t,x)\|_{L_2(D)}\left\|u^{(0)}(t,\varepsilon)\right\| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\|f(t,x,0)\|_{L_2(D)}$$

olur.

Benzer olarak,

$$u_{sk}^{(1)}(t,\varepsilon)=u_{sk}^{(0)}(t,\varepsilon)+\frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\sin 2k\xi\, d\xi d\tau$$

eşitliğinde yukarıdaki işlemler yapıldığında,

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{sk}^{(1)}(t,\varepsilon)\right|\le\sum_{k=1}^{\infty}|\varphi_{sk}|+\frac{\sqrt{2\pi}}{4\sqrt{3}a}\|b(t,x)\|_{L_2(D)}\left\|u^{(0)}(t,\varepsilon)\right\|+\frac{\sqrt{2\pi}}{4\sqrt{3}a}\|f(t,x,0)\|_{L_2(D)}$$

elde edilir.

Elde edilen eşitsizlikler taraf tarafa toplanırsa,

$$\left\|\bar{u}^{(1)}(t,\varepsilon)\right\|=\max_{0\le t\le T}\frac{\left|u_0^{(1)}(t,\varepsilon)\right|}{2}+\sum_{k=1}^{\infty}[\max_{0\le t\le T}\left|u_{ck}^{(1)}(t,\varepsilon)\right|+\max_{0\le t\le T}\left|u_{sk}^{(1)}(t,\varepsilon)\right|]\le$$
$$\frac{|\varphi_0|}{2}+\sum_{k=1}^{\infty}|\varphi_{ck}|+|\varphi_{sk}|+\left(\sqrt{\frac{T}{\pi}}+\frac{\sqrt{2\pi}}{2\sqrt{3}a}\right)\left(\|b(t,x)\|_{L_2(D)}\left\|u^{(0)}(t,\varepsilon)\right\|+\|f(t,x,0)\|_{L_2(D)}\right)$$

olur. Teorem2.3 ün koşullarına göre eşitliğin sağ tarafının sınırlı olduğu açıktır, yani,

$$\left\|\bar{u}^{(1)}(t,\varepsilon)\right\|\langle\ \infty$$

dır.

Benzer işlemler ikinci yaklaşımlar için uygulanır.

Önce,

$$u_0^{(2)}(t,\varepsilon)=u_0^{(0)}(t,\varepsilon)+\frac{2}{\pi}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]d\xi d\tau$$

eşitliğini,

$$u_0^{(2)}(t,\varepsilon)=\varphi_0+\frac{2}{\pi}\int_0^t\int_0^\pi\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\}d\xi d\tau+\int_0^t\int_0^\pi f(\tau,\xi,0)d\xi d\tau$$

biçiminde yazıp, her iki tarafın mutlak değerini aldıktan sonra, sağ taraftaki integrallere Caucy eşitsizliği uygulanırsa,

$$\left|u_0^{(2)}(t,\varepsilon)\right|\le|\varphi_0|+\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\}d\xi\right\}^2d\tau\right)^{\frac{1}{2}}+$$

$$\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2d\tau\right)^{\frac{1}{2}},$$

$$\left|u_0^{(2)}(t,\varepsilon)\right|\le|\varphi_0|+\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\}d\xi\right\}^2d\tau\right)^{\frac{1}{2}}+$$

$$\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2d\tau\right)^{\frac{1}{2}}$$

bulunur. Sağ taraftaki integral içindeki farka Lipshitzs koşulu uygulanırsa,

$$\left|u_0^{(2)}(t,\varepsilon)\right|\le|\varphi_0|+\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi b(\tau,\xi)\left|Au^{(1)}(\tau,\xi,\varepsilon)\right|d\xi\right\}^2d\tau\right)^{\frac{1}{2}}+$$

$$\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki $[0,\pi]$ aralığı üzerindeki integrallere Cauchy eşitsizliği uygulanırsa,

$$\left|u_0^{(2)}(t,\varepsilon)\right| \le |\varphi_0| + 2\sqrt{\frac{t}{\pi}}\left(\int_0^t \left\{\int_0^\pi b^2(\tau,\xi)\left|Au^{(1)}(\tau,\xi,\varepsilon)\right|^2 d\xi\right\} d\tau\right)^{\frac{1}{2}} +$$

$$2\sqrt{\frac{t}{\pi}}\left(\int_0^t \left\{\int_0^\pi f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur.

Burada,

$Au^{(1)}(\tau,\xi,\varepsilon) \le \left|u^{(1)}(\tau,\varepsilon)\right|$ olduğu dikkate alınırsa,

$$\left|u_0^{(2)}(t,\varepsilon)\right| \le |\varphi_0| + 2\sqrt{\frac{t}{\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|u^{(1)}(\tau,\varepsilon)\right|^2\right)^{\frac{1}{2}} +$$

$$2\sqrt{\frac{t}{\pi}}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Her tarafın t ye göre maksimumu alındığında,

$$\max_{0\le t\le T}\left|u_0^{(2)}(t,\varepsilon)\right| \le |\varphi_0| + 2\sqrt{\frac{T}{\pi}}\left\|b(\tau,\xi)\right\|_{L_2(D)}\left\|u^{(1)}(t,\varepsilon)\right\| + 2\sqrt{\frac{T}{\pi}}\left\|f(\tau,\xi,0)\right\|_{L_2(D)}$$

eşitsizliği elde edilir.

Sonra,

$$u_{ck}^{(2)}(t,\varepsilon) = u_{ck}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\ \frac{2}{\pi}\int_0^\pi f[\tau,\xi,u^{(1)}(\tau,\varepsilon)]\cos 2k\xi\ d\xi d\tau$$

eşitliğini,

$$u_{ck}^{(2)}(t,\varepsilon)=\varphi_{ck}e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}+$$

$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}\cos 2k\xi\, d\xi d\tau+$$

$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi d\tau$$

şeklinde dönüştürüp, her iki tarafın mutlak değerini aldıktan sonra sağ taraftaki integrallere Cauchy eşitsizliği uygulanırsa,

$$\left|u_{ck}^{(2)}(t,\varepsilon)\right|\le\left|\varphi_{ck}\right|+$$

$$\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)]\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Her tarafın k ya göre $(k=\overline{1,\infty})$ toplamını alıp Hölder eşitsizliği uygulanırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(2)}(t,\varepsilon)\right|\le\sum_{k=1}^{\infty}\left|\varphi_{ck}\right|+$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2\sqrt{1+\varepsilon(2k)^2}}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2\sqrt{1+\varepsilon(2k)^2}}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^\pi f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Son eşitsizliğin sağ tarafına Bessel eşitsizliği uygulanırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(2)}(t,\varepsilon)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\pi}{4\sqrt{3}a}\sqrt{\frac{2}{\pi}}\left(\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)] - f(\tau,\xi,0)\right\}^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\frac{\pi}{4\sqrt{3}a}\sqrt{\frac{2}{\pi}}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

elde edilir. Burada, Lipsthitz koşulu kullanılırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(2)}(t,\varepsilon)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|Au^{(1)}(\tau,\xi,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Burada,

$\left|Au^{(1)}(\tau,\xi,\varepsilon)\right| \le \left|u^{(1)}(\tau,\varepsilon)\right|$ olduğu göz önüne alınıp, t ye göre maksimuma geçilirse,

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{ck}^{(2)}(t,\varepsilon)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{ck}\right| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|b(t,x)\right\|_{L_2(D)}\left\|u^{(1)}(t,\varepsilon)\right\| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|f(t,x,0)\right\|_{L_2(D)}$$

elde edilir.

Benzer olarak,

$$u_{sk}^{(2)}(t,\varepsilon) = u_{sk}^{(0)}(t,\varepsilon) + \int_0^t \frac{2}{\pi}e^{-(2ak)^2(t-\tau)}\int_0^\pi f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]\sin 2k\xi \; d\xi d\tau$$

eşitliğine yukarıdaki işlemler uygulanırsa,

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{sk}^{(2)}(t,\varepsilon)\right| \le \sum_{k=1}^{\infty}\left|\varphi_{sk}\right| + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|b(t,x)\right\|_{L_2(D)}\left\|u^{(1)}(t,\varepsilon)\right\|_B + \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left\|f(t,x,0)\right\|_{L_2(D)}$$

olur.

Elde edilen eşitsizlikler taraf tarafa toplanırsa,

$$\left\|\overline{u}^{(2)}(t,\varepsilon)\right\| = \max_{0\le t\le T}\frac{\left|u_0^{(2)}(t,\varepsilon)\right|}{2} + \sum_{k=1}^{\infty}[\max_{0\le t\le T}\left|u_{ck}^{(2)}(t,\varepsilon)\right| + \max_{0\le t\le T}\left|u_{sk}^{(2)}(t,\varepsilon)\right|] \le$$
$$\frac{|\varphi_0|}{2} + \sum_{k=1}^{\infty}|\varphi_{ck}| + |\varphi_{sk}| + \left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi}}{2\sqrt{3}a}\right)\left(\|b(t,x)\|_{L_2(D)}\left\|u^{(1)}(t,\varepsilon)\right\| + \|f(t,x,0)\|_{L_2(D)}\right)$$

olur. Teorem3.3 ün koşullarına göre, eşitliğin sağ tarafının sınırlı olduğu açıktır, yani,

$$\left\|\overline{u}^{(2)}(t,\varepsilon)\right\| \langle \infty$$

dır.

Buradan tümevarım yöntemi ile,

$$\left\|\overline{u}^{(N+1)}(t,\varepsilon)\right\| = \max_{0\le t\le T}\frac{\left|u_0^{(N+1)}(t,\varepsilon)\right|}{2} + \sum_{k=1}^{\infty}[\max_{0\le t\le T}\left|u_{ck}^{(N+1)}(t,\varepsilon)\right| + \max_{0\le t\le T}\left|u_{sk}^{(N+1)}(t,\varepsilon)\right|] \le \frac{|\varphi_0|}{2} + \sum_{k=1}^{\infty}|\varphi_{ck}| + |\varphi_{sk}| +$$
$$\left(\sqrt{\frac{T}{\pi}} + \frac{\sqrt{2\pi}}{2\sqrt{3}a}\right)\left(\|b(t,x)\|_{L_2(D)}\left\|u^{(N)}(t,\varepsilon)\right\| + \|f(t,x,0)\|_{L_2(D)}\right)$$

yazılabilir. Teorem3.3 ün koşullarına göre, eşitliğin sağ tarafı sınırlıdır, yani

$$\left\|\overline{u}^{(N+1)}(t,\varepsilon)\right\| \langle \infty$$

dır.

Demek ki, ardışık yaklaşımlar B uzayındadır, yani

$$\{\overline{u}^{(N+1)}(t,\varepsilon)\} = \left\{ \frac{u_0^{(N+1)}(t,\varepsilon)}{2}, u_{c1}^{(N+1)}(t,\varepsilon), u_{s1}^{(N+1)}(t,\varepsilon), \ldots, u_{cn}^{(N+1)}(t,\varepsilon), u_{sn}^{(N+1)}(t,\varepsilon), \ldots \right\} \in B$$

dır. Böylece Lemma 3.4 ispatlanmış olur.

3.2 Ardışık Yaklaşımların Yakınsaklığı

Lemma 3.5: Teorem 3.3 ün koşulları sağlandığında, (3.8) ile tanımlanan ardışık yaklaşımları B uzayında yakınsaktır.

İspat: (3.8) ile belirlenen sisteme ardışık yaklaşımların yakınsaklığını incelemek için, aşağıdaki farklar değerlendirilir:

Bunun için,

$$u_0^{(1)}(t,\varepsilon) - u_0^{(0)}(t,\varepsilon) = \frac{2}{\pi}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]d\xi d\tau,$$

$$u_{ck}^{(1)}(t,\varepsilon) - u_{ck}^{(0)}(t,\varepsilon) = \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi} e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} \int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\cos 2k\xi \; d\xi d\tau,$$

$$u_{sk}^{(1)}(t,\varepsilon) - u_{sk}^{(0)}(t,\varepsilon) = \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi} e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} \int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\sin 2k\xi \; d\xi d\tau$$

farkları yazılıp, aşağıdaki değerlendirmeler yapılır.

$$\left|\bar{u}^{(1)}(t,\varepsilon)-\bar{u}^{(0)}(t,\varepsilon)\right| = \frac{\left|u_0^{(1)}(t,\varepsilon)-u_0^{(0)}(t,\varepsilon)\right|}{2} +$$

$$\sum_{k=1}^{\infty}\left[\left|u_{ck}^{(1)}(t,\varepsilon)-u_{ck}^{(0)}(t,\varepsilon)\right|+\left|u_{sk}^{(1)}(t,\varepsilon)-u_{sk}^{(0)}(t,\varepsilon)\right|\right] \le \frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\,d\xi d\tau\right| +$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\cos 2k\xi\, d\xi d\tau\right| +$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\sin 2k\xi\, d\xi d\tau\right| \le$$

$$\le \frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}d\xi d\tau\right| +$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}\cos 2k\xi\, d\xi d\tau\right| +$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}\sin 2k\xi\, d\xi d\tau\right| +$$

$$\frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,0)\,d\xi d\tau\right| +$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} f(\tau,\xi,0)\cos 2k\xi\, d\xi d\tau\right| +$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} f(\tau,\xi,0)\sin 2k\xi\, d\xi d\tau\right|.$$

Toplamlarda mutlak değer alıp, integrallere Cauchy eşitsizliğini uyguladıktan sonra, t ye göre integralleme yapıp, gereken sadeleştirmelerle

$$\left|\bar{u}^{(1)}(t,\varepsilon)-\bar{u}^{(0)}(t,\varepsilon)\right| \le \frac{1}{2}\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi (f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0))\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}} \times$$

$$\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi (f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0))\cos 2k\xi\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\times$$

$$\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0))\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2}\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

$$\left|\overline{u}^{(1)}(t,\varepsilon)-\overline{u}^{(0)}(t,\varepsilon)\right|\le\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0))d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0))\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0))\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

bulunur. Sağ taraftaki sonsuz toplamlara Hölder eşitsizliği uygulanıp,

$$\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}=\frac{\pi}{\sqrt{6}}$$ olduğu göz önüne alınırsa,

$$\left|\overline{u}^{(1)}(t,\varepsilon)-\overline{u}^{(0)}(t,\varepsilon)\right|\le\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0))d\xi\right\}^2d\tau\right)^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0))\cos 2k\xi d\xi\right\}^2d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}(f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0))\sin 2k\xi d\xi\right\}^2d\tau\right]^{\frac{1}{2}}\le$$

$$\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)d\xi\right\}^2d\tau\right)^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)\cos 2k\xi d\xi\right\}^2d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)\sin 2k\xi d\xi\right\}^2d\tau\right]^{\frac{1}{2}}.$$

olur. Sağ taraftaki sonsuz toplamlarda, Teorem3.3 ün koşullarına göre, toplamla t ye göre integralin yerlerinin değiştirilebildiğini göz önüne alıp, Bessel eşitsizliği uygulanırsa,

$$\left|\overline{u}^{(1)}(t,\varepsilon)-\overline{u}^{(0)}(t,\varepsilon)\right|\le(\frac{1}{2}\sqrt{T}+\frac{\pi}{4\sqrt{3}a}+\frac{\pi}{4\sqrt{3}a})\left(\int_0^t\frac{2}{\pi}\int_0^{\pi}\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\right\}^2d\xi d\tau\right)^{\frac{1}{2}}+$$

$$(\frac{1}{2}\sqrt{T}+\frac{\pi}{4\sqrt{3}a}+\frac{\pi}{4\sqrt{3}a})\left(\int_0^t\frac{2}{\pi}\int_0^{\pi}f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki integral içindeki farka Lipsthitz koşulu uygulandığında ise,

$$\left|\overline{u}^{(1)}(t,\varepsilon)-\overline{u}^{(0)}(t,\varepsilon)\right|\le\frac{a\sqrt{6T}+\pi}{a\sqrt{6\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}}\left\|\overline{u}^{(0)}(t,\varepsilon)\right\|+$$

$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left(\int_0^t\int_0^\pi f^2(\tau,\xi,0)d\xi d\tau\right)^{\frac{1}{2}}$$

olur.

$$A_T(\varepsilon)=\frac{a\sqrt{6T}+\pi}{a\sqrt{6\pi}}\left\|b(t,x)\right\|_{L_2(D)}\left\|\overline{u}^{(0)}(t,\varepsilon)\right\|+\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left\|f(t,x,0)\right\|_{L_2(D)}$$

olarak kabul edilsin. Teorem3.3 ün koşullarına göre $A_T(\varepsilon)$ sınırlıdır. Bu yüzden de, son eşitsizlik,

$$\left|\overline{u}^{(1)}(t,\varepsilon)-\overline{u}^{(0)}(t,\varepsilon)\right|\le A_T(\varepsilon)<\infty$$

şeklinde yazılabilir.

Yeniden,

$$u_0^{(2)}(t,\varepsilon)-u_0^{(1)}(t,\varepsilon)=\frac{2}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}d\xi d\tau,$$

$$u_{ck}^{(2)}(t,\varepsilon)-u_{ck}^{(1)}(t,\varepsilon)=$$
$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t\frac{2}{\pi}e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi\ d\xi d\tau,$$

$$u_{sk}^{(2)}(t,\varepsilon)-u_{sk}^{(1)}(t,\varepsilon)=$$
$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t\frac{2}{\pi}e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi\ d\xi d\tau$$

farklarını yazıp, yukarıdakilere benzer değerlendirme yapılırsa,

$$\left|\overline{u}^{(2)}(t,\varepsilon)-\overline{u}^{(1)}(t,\varepsilon)\right|=\frac{\left|u_0^{(2)}(t,\varepsilon)-u_0^{(1)}(t,\varepsilon)\right|}{2}+$$

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(2)}(t,\varepsilon)-u_{ck}^{(1)}(t,\varepsilon)\right|+\left|u_{sk}^{(2)}(t,\varepsilon)-u_{sk}^{(1)}(t,\varepsilon)\right|\le$$

$$\frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}d\xi d\tau\right|+$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi d\tau\right|+$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi d\tau\right|.$$

bulunur. Sağ taraftaki integrallere Caucy eşitsizliği uygulanıp, gereken işlemler yapıldığında,

$$\left|\overline{u}^{(2)}(t,\varepsilon)-\overline{u}^{(1)}(t,\varepsilon)\right|\le\frac{1}{2}\left(\int_0^t d\tau\right)^{\frac{1}{2}}\times$$

$$\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\times$$

$$\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\times$$

$$\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

$$\left|\overline{u}^{(2)}(t,\varepsilon)-\overline{u}^{(1)}(t,\varepsilon)\right|\le\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\left(\frac{1+\varepsilon(2k)^2}{2(2ak)^2}e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\Bigg|_0^t\right)^{\frac{1}{2}}\times$$

$$\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\left(\frac{1+\varepsilon(2k)^2}{2(2ak)^2}e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\Bigg|_0^t\right)^{\frac{1}{2}}\times$$

$$\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

$$\left|\overline{u}^{(2)}(t,\varepsilon)-\overline{u}^{(1)}(t,\varepsilon)\right|\le\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki ikinci ve üçüncü toplamlara Hölder eşitsizliği uygulandığında ise,

$$\left|\overline{u}^{(2)}(t,\varepsilon)-\overline{u}^{(1)}(t,\varepsilon)\right|\le\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}$$

elde edilir. Yine Teorem3.3 ün koşullarına göre, sağ taraftaki sonsuz toplamlarda, toplamla t ye göre integralin değiştirilebilirliği göz önüne alıp, Bessel eşitsizliği uygulanırsa,

$$\left|\overline{u}^{(2)}(t,\varepsilon)-\overline{u}^{(1)}(t,\varepsilon)\right|\le\frac{1}{2}\sqrt{T}\left[\int_0^t\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}^2d\xi d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\int_0^t\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}^2d\xi d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\int_0^t\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\right\}^2d\xi d\tau\right]^{\frac{1}{2}}$$

olur.

Sağ taraftaki integrallerin içindeki farklara Lipsthitz koşulu uygulandığında,

$$\left|\overline{u}^{(2)}(t,\varepsilon)-\overline{u}^{(1)}(t,\varepsilon)\right|\le$$

$$(\frac{1}{2}\sqrt{T}+\frac{\pi}{2\sqrt{3}a})\left(\int_0^t\frac{2}{\pi}\int_0^\pi b^2(\tau,\xi)\left[Au^{(1)}(\tau,\xi,\varepsilon)-Au^{(0)}(\tau,\xi,\varepsilon)\right]^2d\xi d\tau\right)^{\frac{1}{2}}\le$$

$$\left(\frac{\sqrt{T}}{2}+\frac{\pi}{2\sqrt{3}a}\right)\left(\frac{2}{\pi}\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}}\max_{0\le t\le T}\left|Au^{(1)}(\tau,\xi,\varepsilon)-Au^{(0)}(\tau,\xi,\varepsilon)\right|\le$$

$$\left(\frac{\sqrt{T}}{2}+\frac{\pi}{2\sqrt{3}a}\right)\left(\frac{2}{\pi}\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}}\max_{0\le t\le T}\left|\overline{u}^{(1)}(t,\varepsilon)-\overline{u}^{(0)}(t,\varepsilon)\right|\le$$

$$\left(\frac{\sqrt{T}}{2}+\frac{\pi}{2\sqrt{3}a}\right)\left(\frac{2}{\pi}\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}}A_T(\varepsilon),$$

$$\left|\overline{u}^{(2)}(t,\varepsilon)-\overline{u}^{(1)}(t,\varepsilon)\right|\le\left(\frac{\sqrt{T}}{2}+\frac{\pi}{2\sqrt{3}a}\right)\sqrt{\frac{2}{\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}}A_T(\varepsilon)=$$

$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}}A_T(\varepsilon)$$

bulunur.

Yeniden,

$$u_0^{(3)}(t,\varepsilon)-u_0^{(2)}(t,\varepsilon)=\frac{2}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]\right\}d\xi d\tau,$$

$$u_{ck}^{(3)}(t,\varepsilon)-u_{ck}^{(2)}(t,\varepsilon)=$$

$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi\ d\xi d\tau,$$

$$u_{sk}^{(2)}(t,\varepsilon)-u_{sk}^{(1)}(t,\varepsilon)=$$

$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi\ d\xi d\tau$$

farklarını yazıp, benzer yolla,

$$\left|\overline{u}^{(3)}(t,\varepsilon)-\overline{u}^{(2)}(t,\varepsilon)\right|=\frac{\left|u_0^{(3)}(t,\varepsilon)-u_0^{(2)}(t,\varepsilon)\right|}{2}+$$

$$\sum_{k=1}^{\infty}\left|u_{ck}^{(3)}(t,\varepsilon)-u_{ck}^{(2)}(t,\varepsilon)\right|+\left|u_{sk}^{(3)}(t,\varepsilon)-u_{sk}^{(2)}(t,\varepsilon)\right|\le$$

$$\frac{1}{2}\left|\frac{2}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]\right\}d\xi d\tau\right|+$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi d\tau\right|+$$

$$\sum_{k=1}^{\infty}\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au^{(2)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(1)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi d\tau\right|$$

eşitsizliği elde edilir. Burada da, yukarıdakine benzer işlemler yapıldığında,

$$\left|\overline{u}^{(3)}(t,\varepsilon)-\overline{u}^{(2)}(t,\varepsilon)\right| \le \frac{a\sqrt{3T}+\pi}{2a\sqrt{3}}\left(\int_0^t \frac{2}{\pi}\int_0^\pi b^2(\tau,\xi)\left[Au^{(2)}(\tau,\xi,\varepsilon)-Au^{(1)}(\tau,\xi,\varepsilon)\right]^2 d\xi d\tau\right)^{\frac{1}{2}} \le$$

$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left[\overline{u}^{(2)}(t,\varepsilon)-\overline{u}^{(1)}(t,\varepsilon)\right]^2 d\xi d\tau\right)^{\frac{1}{2}} \le$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 A_T(\varepsilon)\left[\int_0^t\int_0^\pi b^2(\tau,\xi)\left(\int_0^\tau\int_0^\pi b^2(\tau_1,\xi_1)d\xi_1 d\tau_1\right)d\xi d\tau\right]^{\frac{1}{2}} \le$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 A_T(\varepsilon)\frac{1}{\sqrt{2}}\left[\left(\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^2\right]^{\frac{1}{2}}$$

olur.

Bu sonuçları dikkate alıp, tümevarım yöntemiyle,

$$\left|\overline{u}^{(N+1)}(t,\varepsilon)-\overline{u}^{(N)}(t,\varepsilon)\right| \le \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N A_T(\varepsilon)\frac{1}{\sqrt{N!}}\left[\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right]^{\frac{N}{2}}$$

eşitsizliğini elde ederiz. Buradan da, t ye göre her iki tarafın maksimumumu alınırsa,

$$\left\|\overline{u}^{(N+1)}(t,\varepsilon)-\overline{u}^{(N)}(t,\varepsilon)\right\| \le \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N A_T \frac{1}{\sqrt{N!}}\left\|b(t,x)\right\|_{L_2(D)}^N$$

bulunur.

Buradan da, yukarıdaki koşullar dahilinde

$$\left\{\overline{u}^{(N+1)}(t,\varepsilon)\right\} = \left\{\frac{u_0^{(N+1)}(t,\varepsilon)}{2}, u_{c1}^{(N+1)}(t,\varepsilon), u_{s1}^{(N+1)}(t,\varepsilon),\ldots, u_{cn}^{(N+1)}(t,\varepsilon), u_{sn}^{(N+1)}(t,\varepsilon),\ldots\right\}$$

dizisinin B de yakınsaklığı elde edilir. Böylece Lemma 2.5 ispatlanmış olur.

3.3 $\overline{u}(t,\varepsilon)$ un (3.8) sistemini sağlaması

$$\lim_{N\to\infty}\overline{u}^{(N)}(t,\varepsilon)=\overline{u}(t,\varepsilon)\left\{\frac{u_0(t,\varepsilon)}{2},u_{c1}(t,\varepsilon),u_{s1}(t,\varepsilon),...,u_{cn}(t,\varepsilon),u_{sn}(t,\varepsilon),...\right\}$$

olduğunu göz önüne alıp, $\overline{u}(t,\varepsilon)$ nun

$$u_0(t,\varepsilon)=\varphi_0+\frac{2}{\pi}\int_0^t\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]d\xi d\tau,$$

$$u_{ck}(t,\varepsilon)=\varphi_{ck}\,e^{\frac{-(2ak)^2t}{1+\varepsilon(2k)^2}}+$$

$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right\}d\tau,$$

$$u_{sk}(t,\varepsilon)=\varphi_{ck}\,e^{\frac{-(2ak)^2t}{1+\varepsilon(2k)^2}}+$$

$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right\}d\tau$$

sistemini sağladığını göstermek için aşağıdaki değerlendirmeleri yapmak yeterlidir.

Cauchy eşitsizliği kullanılırsa, sırasıyla,

$$1)\ \frac{1}{\pi}\left|\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}d\xi d\tau\right|\le$$

$$\frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]d\xi\right\}^2d\tau\right)^{\frac{1}{2}},$$

$$2)\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi d\tau\right|\leq$$

$$\left|\begin{array}{l}\dfrac{1}{1+\varepsilon(2k)^2}\left(\displaystyle\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\times\\ \left(\displaystyle\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}\end{array}\right|\leq$$

$$\frac{1}{2\sqrt{2}a}\frac{1}{k}\left|\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}\right|,$$

$$3)\left|\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi d\tau\right|\leq$$

$$\left|\begin{array}{l}\dfrac{1}{1+\varepsilon(2k)^2}\left(\displaystyle\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\times\\ \left(\displaystyle\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}\end{array}\right|\leq$$

$$\frac{1}{2\sqrt{2}a}\frac{1}{k}\left|\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}\right|$$

2) ve 3) de k ya göre $(k=\overline{1,\infty})$ toplam yapıp, elde edilen toplam ve 1) taraf tarafa toplanırsa,

$$\Delta \le \frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. (Burada,

$$\Delta = \left|\frac{1}{\pi}\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}d\xi d\tau\right|+$$

$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi d\tau\right|+$$

$$\sum_{k=1}^{\infty}\left|\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi d\tau\right|$$

kabul edilmiştir).Sağ taraftaki sonsuz toplamlara Hölder eşitsizliği uygulanırsa,

$$\Delta \le \frac{1}{2}\sqrt{T}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\sum_{k=1}^{\infty}\int_0^t\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right\}^2 d\tau\right]^{\frac{1}{2}}$$

elde edilir. Teorem3.3 ün koşullarına göre sağ taraftaki sonsuz toplamlarda, toplamla t ye göre integralin değiştirilebildiğini gözöününe alıp, Bessel eşitsizliği uygulanır ve $\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}=\frac{\pi}{\sqrt{6}}$ olduğu göz önüne alınırsa,

$$\Delta\le\frac{1}{2}\sqrt{T}\left[\int_0^t\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}^2d\xi d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\int_0^t\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}^2d\xi d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4\sqrt{3}a}\left[\int_0^t\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}^2d\xi d\tau\right]^{\frac{1}{2}}=$$

$$\frac{1}{2}(\sqrt{T}+\frac{\pi}{\sqrt{3}a})\left[\int_0^t\frac{2}{\pi}\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}^2d\xi d\tau\right]^{\frac{1}{2}}$$

olur. Son tarafta Lipsthitz koşulu kullanılırsa,

$$\Delta\le\frac{1}{2}(\sqrt{T}+\frac{\pi}{\sqrt{3}a})\left(\int_0^t\frac{2}{\pi}\int_0^\pi b^2(\tau,\xi)\left[Au(\tau,\xi,\varepsilon)-Au^{(N)}(\tau,\xi,\varepsilon)\right]^2d\xi d\tau\right)^{\frac{1}{2}}\le$$

$$\left(\frac{1}{2}(\sqrt{T}+\frac{\pi}{\sqrt{3}a}\right)\left(\frac{2}{\pi}\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\overline{u}(\tau,\varepsilon)-\overline{u}^{(N)}(\tau,\varepsilon)\right|^2d\xi d\tau\right)^{\frac{1}{2}}\le$$

$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\|b(t,x)\|_{L_2(D)}\left\|\overline{u}(t,\varepsilon)-\overline{u}^{(N)}(t,\varepsilon)\right\|$$

olur.

$$\lim_{N\to\infty}\left\|\overline{u}(t)-\overline{u}^{(N)}(t)\right\|=0$$

olduğundan, $\overline{u}(t,\varepsilon)$ nın, (3.8) sistemini sağladığı görülmektedir.

3.4 Çözümün Tekliği

Lemma 3.6: Teorem 3.3 ün koşulları sağlandığında, (3.6) sisteminin çözümü tekdir.

İspat: (3.6) sisteminin tek olduğunu göstermek için tersini farzedip, çözümün tek olmadığını ve

$$\overline{\widetilde{u}}(t,\varepsilon)=\left\{\frac{\widetilde{u}_0(t,\varepsilon)}{2},\widetilde{u}_{c1}(t,\varepsilon),\widetilde{u}_{s1}(t,\varepsilon),\ldots,\widetilde{u}_{ck}(t,\varepsilon),\widetilde{u}_{sk}(t,\varepsilon),\ldots\right\}$$

nin de bu sistemin diğer bir çözümü olduğu kabul edilsin.

$$u_0(t,\varepsilon)-\widetilde{u}_0(t,\varepsilon)=\frac{2}{\pi}\int_0^t\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\widetilde{u}(\tau,\xi,\varepsilon)]\}d\xi d\tau$$

eşitliğinde her iki tarafın mutlak değerini alıp, sağ tarafına Cauchy eşitsizliği uygulanırsa,

$$\left|u_0(t,\varepsilon)-\widetilde{u}_0(t,\varepsilon)\right|\le\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\widetilde{u}(\tau,\xi,\varepsilon)]\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

$$\left|u_0(t,\varepsilon)-\widetilde{u}_0(t,\varepsilon)\right|\le\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\widetilde{u}(\tau,\xi,\varepsilon)]\}d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

bulunur. Sağ taraftaki integral içindeki ifadede Lipshitzs koşulu dikkate alınırsa,

$$\left|u_0(t,\varepsilon)-\widetilde{u}_0(t,\varepsilon)\right|\le\sqrt{t}\left(\int_0^t\left\{\frac{2}{\pi}\int_0^\pi b(\tau,\xi)\left|Au(\tau,\xi,\varepsilon)-A\widetilde{u}(\tau,\xi,\varepsilon)\right|d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur.

Benzer yolla,

$$u_{ck}(t,\varepsilon)-\widetilde{u}_{ck}(t,\varepsilon)=$$

$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi} e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)} \int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\widetilde{u}(\tau,\xi,\varepsilon)]\}\cos 2k\xi\, d\xi d\tau$$

farkı değerlendirilirse,

$$\left|u_{ck}(t,\varepsilon)-\widetilde{u}_{ck}(t,\varepsilon)\right|\le \frac{1}{1+\varepsilon(2k)^2}\left(\int_0^t e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)} d\tau\right)^{\frac{1}{2}}\times$$

$$\left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\widetilde{u}(\tau,\xi,\varepsilon)]\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

eşitsizliği elde edilir. Her tarafın k ya göre $(k=\overline{1,\infty})$ toplamını alıp, sağ tarafa Hölder eşitsizliği uygulanırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}(t,\varepsilon)-\widetilde{u}_{ck}(t,\varepsilon)\right|\le \frac{1}{2\sqrt{2}a}\left(\sum_{k=1}^{\infty}\frac{1}{k^2\sqrt{1+\varepsilon(2k)^2}}\right)^{\frac{1}{2}}\times$$

$$\left(\frac{2}{\pi}\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\widetilde{u}(\tau,\xi,\varepsilon)]\}\cos 2k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Bu eşitsizliğin sağ tarafında Bessel eşitsizliği göz önüne alınırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}(t,\varepsilon)-\widetilde{u}_{ck}(t,\varepsilon)\right|\le \frac{\pi}{4\sqrt{3}a}\left(\int_0^t \frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\widetilde{u}(\tau,\xi,\varepsilon)]\}^2 d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki integral içinde Lipsthitz koşulundan faydalanılırsa,

$$\sum_{k=1}^{\infty}\left|u_{ck}(t,\varepsilon)-\tilde{u}_{ck}(t,\varepsilon)\right| \le \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|Au(\tau,\xi,\varepsilon)-A\tilde{u}(\tau,\xi,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

veya

$$\sum_{k=1}^{\infty}\left|u_{ck}(t,\varepsilon)-\tilde{u}_{ck}(t,\varepsilon)\right| \le \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|u(\tau,\varepsilon)-\tilde{u}(\tau,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

olur.

$$u_{sk}(t,\varepsilon)-\tilde{u}_{sk}(t,\varepsilon)=$$

$$\frac{1}{1+\varepsilon(2k)^2}\int_0^t\frac{2}{\pi}e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,A\tilde{u}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi\ d\xi d\tau$$

farkı içinde benzer yolla yukarıdaki işlemler uygulandığında,

$$\sum_{k=1}^{\infty}\left|u_{sk}(t,\varepsilon)-\tilde{u}_{sk}(t,\varepsilon)\right| \le \frac{\sqrt{2\pi}}{4\sqrt{3}a}\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|u(\tau,\varepsilon)-\tilde{u}(\tau,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Elde edilen eşitsizlikler birleştirildiğinde,

$$\left|\overline{u}(t,\varepsilon)-\overline{\tilde{u}}(t,\varepsilon)\right| = \frac{\left|u_0(t,\varepsilon)-\tilde{u}_0(t,\varepsilon)\right|}{2}+\sum_{k=1}^{\infty}\left[\left|u_{ck}(t,\varepsilon)-\tilde{u}_{ck}(t,\varepsilon)\right|'+\left|u_{sk}(t,\varepsilon)-\tilde{u}_{sk}(t,\varepsilon)\right|\right] \le$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|u(\tau,\varepsilon)-\tilde{u}(\tau,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

olur. Bu eşitliğin her tarafının karesi alınırsa,

$$\left|\overline{u}(t,\varepsilon)-\overline{\tilde{u}}(t,\varepsilon)\right|^2 \le \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2\int_0^t\int_0^\pi b^2(\tau,\xi)\left|u(\tau,\varepsilon)-\tilde{u}(\tau,\varepsilon)\right|^2 d\xi d\tau$$

olur.

Buradan, Gronwall eşitsizliğine göre,

$$\left|\overline{u}(t,\varepsilon)-\overline{\overline{u}}(t,\varepsilon)\right| \le 0\times \exp\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau \le 0$$

elde edilir. Buradan da

$\overline{u}(t,\varepsilon)=\overline{\overline{u}}(t,\varepsilon)$ yani $u_0(t,\varepsilon)=\tilde{u}_0(t,\varepsilon)$, $u_{ck}(t,\varepsilon)=\tilde{u}_{ck}(t,\varepsilon)$,

$u_{sk}(t,\varepsilon)=\tilde{u}_{sk}(t,\varepsilon)$ $(k=\overline{1,\infty})$

olduğu görülür. Yani çözüm tekdir. Böylece Lemma 3.6 ispatlanmış olur.

Teorem 3.3 ün İspatı: Lemma 3.4, Lemma 3.5 ve Lemma 3.6 nın ispatından teorem ispatlanmış olur.

3.5 Kesin Çözümle Ardışık Yaklaşımların Farkının Değerlendirilmesi

$\left|\overline{u}(\tau,\varepsilon)-\overline{u}^{(N+1)}(\tau,\varepsilon)\right|$ yi değerlendirmek için, önce sözü edilen fark aşağıdaki şekle dönüştürülür:

$$\left|\overline{u}(t,\varepsilon)-\overline{u}^{(N+1)}(t,\varepsilon)\right| = \frac{\left|u_0(t,\varepsilon)-u_0^{(N+1)}(t,\varepsilon)\right|}{2}+$$

$$\sum_{k=1}^{\infty}\left[\left|u_{ck}(t,\varepsilon)-u_{ck}^{(N+1)}(t,\varepsilon)\right|+\left|u_{sk}(t,\varepsilon)-u_{sk}^{(N+1)}(t,\varepsilon)\right|\right]\le$$

$$\frac{1}{\pi}\left|\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}d\xi d\tau\right|+$$

$$\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi d\tau\right|+$$

$$\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi d\tau\right| \leq$$

$$\frac{1}{\pi}\left|\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N+1)}(\tau,\xi,\varepsilon)]\right\}d\xi d\tau\right|+$$

$$\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N+1)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi d\tau\right|+$$

$$\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N+1)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi d\tau\right|+$$

$$\frac{1}{\pi}\left|\int_0^t\int_0^\pi\left\{f[\tau,\xi,Au^{(N+1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}d\xi d\tau\right|+$$

$$\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au^{(N+1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}\cos 2k\xi d\xi d\tau\right|+$$

$$\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2k)^2}\frac{2}{\pi}\int_0^t\int_0^\pi e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{f[\tau,\xi,Au^{(N+1)}(\tau,\xi,\varepsilon)]-f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\right\}\sin 2k\xi d\xi d\tau\right|.$$

Önceki bölümdekilere benzer olarak art arda Cauchy, Hölder, Bessel ve Lipshitz eşitsizlikleri kullanıldığında,

$$\left|\bar{u}(t,\varepsilon)-\bar{u}^{(N+1)}(t,\varepsilon)\right| \leq \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau,\varepsilon)-\bar{u}^{(N+1)}(\tau,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}+$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}^{(N+1)}(\tau,\varepsilon)-\bar{u}^{(N)}(\tau,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} \leq$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau,\varepsilon)-\bar{u}^{(N+1)}(\tau,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}+$$

$$\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\left(\int_0^t\int_0^\pi b^2(\tau,\xi)d\xi d\tau\right)^{\frac{1}{2}}\left\|\bar{u}^{(N+1)}(t,\varepsilon)-\bar{u}^{(N)}(t,\varepsilon)\right\|$$

olur. Her iki tarafın karesini alıp, $(a+b)^2 \le 2a^2 + 2b^2$ eşitsizliği kullanılırsa,

$$\left|\bar{u}(t,\varepsilon) - \bar{u}^{(N+1)}(t,\varepsilon)\right|^2 \le 2\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 \left(\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau,\varepsilon) - \bar{u}^{(N+1)}(\tau,\varepsilon)\right|^2 d\xi d\tau\right) +$$
$$2\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 \|b(t,x)\|^2_{L_2(D)} \left\|\bar{u}^{(N+1)}(t,\varepsilon) - \bar{u}^{(N)}(t,\varepsilon)\right\|^2$$

olur. Gronwall eşitsizliğini ve

$$\left\|\bar{u}^{(N+1)}(t,\varepsilon) - \bar{u}^{(N)}(t,\varepsilon)\right\| \le \left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N A_T(\varepsilon)\frac{1}{\sqrt{N!}}\|b(t,x)\|^N_{L_2(D)}$$

eşitsizliğini kullanıp, her tarafın karekökü alındığında,

$$\left|\bar{u}(t,\varepsilon) - \bar{u}^{(N+1)}(t,\varepsilon)\right| \le \sqrt{2}\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)\|b(t,x)\|_{L_2(D)}\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^N \frac{A_T(\varepsilon)}{\sqrt{N!}}\|b(t,x)\|^N_{L_2(D)} \times$$
$$\exp\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 \|b(t,x)\|^2_{L_2(D)}$$

eşitsizliği elde edilir.

Buradan da, kesin çözümle (N+1) ardışık yaklaşımının farkı için

$$\left\|\bar{u}(t,\varepsilon) - \bar{u}^{(N+1)}(t,\varepsilon)\right\| \le \sqrt{\frac{2}{N!}}\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^{N+1} A_T(\varepsilon)\|b(t,x)\|^{N+1}_{L_2(D)} \times$$
$$\exp\left(\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\right)^2 \|b(t,x)\|^2_{L_2(D)}$$

değerlendirmesi bulunur.

Son eşitsizliğe dayanarak, (3.8) sisteminin yaklaşık çözümü olarak $\overline{u}^{(N+1)}(t,\varepsilon)$ alındığında, yapılan hata N ye bağlı olarak değerlendirilebilir.

4. $\varepsilon \to 0$ DURUMUNDA ASİMPTOTİK İNCELEME VE ÇÖZÜMLER ARASINDAKİ İLİŞKİNİN ANALİZİ

Bu bölümde amaç, $\varepsilon \to 0$ da

$$v_t - a^2 v_{xx} = f(t,x,v) \quad (t,x) \in D\{0 < t < T < \infty \ , \ 0 < x < \pi\} \tag{4.1}$$

$$\begin{aligned} &v(t,0) = v(t,\pi) && t \in [0,T] \\ &v_x(t,0) = v_x(t,\pi) && t \in [0,T] \end{aligned} \tag{4.2}$$

$$v(0,x) = \varphi(x) \qquad x \in [0,\pi] \tag{4.3}$$

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial^3 u}{\partial t \partial x^2} = f(t,x,u) \quad (t,x) \in D\{0 < t < T < \infty \ , \ 0 < x < \pi\} \tag{4.4}$$

$$\begin{aligned} &u(t,0) = u(t,\pi) && t \in [0,T] \\ &u_x(t,0) = u_x(t,\pi) && t \in [0,T] \end{aligned} \tag{4.5}$$

$$u(0,x) = \varphi(x) \qquad x \in [0,\pi] \tag{4.6}$$

karışık problemlerinin çözümleri arasındaki farkın değerlendirilmesidir. Bunun için, problemlerin çözümleri sırasıyla,

$$v(t,x) = \frac{v_0(t)}{2} + \sum_{k=1}^{\infty} [\, v_{ck}(t) \cos 2kx + v_{sk}(t) \sin 2kx \,] \tag{4.7}$$

$$u(t,x,\varepsilon) = \frac{u_0(t,\varepsilon)}{2} + \sum_{k=1}^{\infty} [\, u_{ck}(t,\varepsilon) \cos 2kx + u_{sk}(t,\varepsilon) \sin 2kx \,] \tag{4.8}$$

şeklinde aranıp, $v_0(t)\,, v_{ck}(t)\,, v_{sk}(t)\,, u_0(t,\varepsilon)\,, u_{ck}(t,\varepsilon)\,,\ u_{sk}(t,\varepsilon)\ \ (k = \overline{1,\infty})$ bilinmeyenleri için

$$v_0(t)=\varphi_0+\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,\frac{v_0(\tau)}{2}+\sum_{k=1}^{\infty}[v_{ck}(\tau)\cos 2k\xi+v_{sk}(\tau)\sin 2k\xi])d\xi d\tau$$

$$v_{ck}(t)=\varphi_{ck}\,e^{-(2ak)^2 t}+$$

$$\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\,f(\tau,\xi,\frac{v_0(\tau)}{2}+\sum_{k=1}^{\infty}[v_{ck}(\tau)\cos 2k\xi+v_{sk}(\tau)\sin 2k\xi])\cos 2k\xi d\xi d\tau \tag{4.9}$$

$$v_{sk}(t)=\varphi_{ck}\,e^{-(2ak)^2 t}+$$

$$\frac{2}{\pi}\int_0^t\int_0^\pi e^{-(2ak)^2(t-\tau)}\,f(\tau,\xi,\frac{v_0(\tau)}{2}+\sum_{k=1}^{\infty}[v_{ck}(\tau)\cos 2k\xi+v_{sk}(\tau)\sin 2k\xi])\sin 2k\xi d\xi d\tau$$

ve

$$u_0(t,\varepsilon)=\varphi_0+\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,\frac{u_0(\tau,\varepsilon)}{2}+\sum_{n=1}^{\infty}[u_{ck}(\tau,\varepsilon)\cos 2k\xi+u_{sk}(\tau,\varepsilon)\sin 2k\xi])\,d\xi d\tau,$$

$$u_{ck}(t,\varepsilon)=\varphi_{ck}\,e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}}+\frac{1}{1+\varepsilon(2k)^2} \tag{4.10}$$

$$\int_0^t\left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}\,f(\tau,\xi,\frac{u_0(\tau,\varepsilon)}{2}+\sum_{n=1}^{\infty}[u_{ck}(\tau,\varepsilon)\cos 2k\xi+u_{sk}(\tau,\varepsilon)\sin 2k\xi])\cos 2k\xi d\xi\right\}d\tau$$

$$u_{sk}(t,\varepsilon)=\varphi_{sk}\,e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}}+\frac{1}{1+\varepsilon(2k)^2}$$

$$\int_0^t\left\{\frac{2}{\pi}\int_0^\pi e^{\frac{-(2ak)^2}{1+\varepsilon(2k)^2}(t-\tau)}\,f(\tau,\xi,\frac{u_0(\tau,\varepsilon)}{2}+\sum_{n=1}^{\infty}[u_{ck}(\tau,\varepsilon)\cos 2k\xi+u_{sk}(\tau,\varepsilon)\sin 2k\xi])\sin 2k\xi d\xi\right\}d\tau$$

doğrusal olmayan sonsuz integral denklem sistemleri elde edilir. Belli koşullar dahilinde, bu sistemlerin çözümlerinin varlığı ve tekliği önceki bölümlerde gösterilmiştir. Yazım kolaylığı için,

$$Av(\tau,\xi)=\frac{v_0(\tau)}{2}+\sum_{k=1}^{\infty}[v_{ck}(\tau)\cos 2k\xi+v_{sk}(\tau)\sin 2k\xi]$$

$$Au(\tau,\xi,\varepsilon)=\frac{u_0(\tau,\varepsilon)}{2}+\sum_{k=1}^{\infty}[u_{ck}(\tau,\varepsilon)\cos 2k\xi+u_{sk}(\tau,\varepsilon)\sin 2k\xi]$$

kabul edilir.

$$u_0(t,\varepsilon)-v_0(t)=\frac{2}{\pi}\int_0^t\int_0^\pi[f(\tau,\xi,Au(\tau,\xi,\varepsilon))-f(\tau,\xi,Av(\tau,\xi))]d\xi d\tau$$

$$u_{ck}(t,\varepsilon)-v_{ck}(t)=\varphi_{ck}e^{\frac{-(2ak)^2t}{1+\varepsilon(2k)^2}}+\frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\frac{2}{\pi}\int_0^\pi f(\tau,\xi,Au(\tau,\xi,\varepsilon))\cos 2k\xi d\xi d\tau-$$
$$\varphi_{ck}e^{-(2ak)^2t}-\int_0^t e^{-(2ak)^2(t-\tau)}\frac{2}{\pi}\int_0^\pi f(\tau,\xi,Av(\tau,\xi))\cos 2k\xi\, d\xi d\tau$$

$$u_{sk}(t,\varepsilon)-v_{sk}(t)=\varphi_{sk}e^{\frac{-(2ak)^2t}{1+\varepsilon(2k)^2}}+\frac{1}{1+\varepsilon(2k)^2}\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\frac{2}{\pi}\int_0^\pi f(\tau,\xi,Au(\tau,\xi,\varepsilon))\sin 2k\xi d\xi d\tau-$$
$$\varphi_{sk}e^{-(2ak)^2t}-\int_0^t e^{-(2ak)^2(t-\tau)}\frac{2}{\pi}\int_0^\pi f(\tau,\xi,Av(\tau,\xi))\sin 2k\xi\, d\xi d\tau$$

farklarını aşağıdaki şekilde dönüştürüp, değerlendirilir:

$$u_0(t,\varepsilon)-v_0(t)=\frac{2}{\pi}\int_0^t\int_0^\pi[f(\tau,\xi,Au(\tau,\xi,\varepsilon))-f(\tau,\xi,Av(\tau,\xi))]d\xi d\tau,$$

$$u_{ck}(t,\varepsilon)-v_{ck}(t)=\varphi_{ck}\left(e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}}-e^{-(2ak)^2 t}\right)+$$

$$\left(\frac{1}{1+\varepsilon(2k)^2}-1\right)\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right\}d\tau+$$

$$\int_0^t\frac{2}{\pi}\int_0^\pi[e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}-e^{-(2ak)^2(t-\tau)}]f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi d\tau+$$

$$\int_0^t e^{-(2ak)^2(t-\tau)}\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}\cos 2k\xi d\xi d\tau,$$

$$u_{sk}(t,\varepsilon)-v_{sk}(t)=\varphi_{sk}\left(e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}}-e^{-(2ak)^2 t}\right)+$$

$$\left(\frac{1}{1+\varepsilon(2k)^2}-1\right)\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right\}d\tau+$$

$$\int_0^t\frac{2}{\pi}\int_0^\pi[e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}-e^{-(2ak)^2(t-\tau)}]f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi d\tau+$$

$$\int_0^t e^{-(2ak)^2(t-\tau)}\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}\sin 2k\xi d\xi d\tau.$$

Her tarafın mutlak değeri alındığında,

$$|u_0(t,\varepsilon)-v_0(t)|=\left|\frac{2}{\pi}\int_0^t\int_0^\pi[f(\tau,\xi,Au(\tau,\xi,\varepsilon))-f(\tau,\xi,Av(\tau,\xi))]d\xi d\tau\right| \tag{4.11}$$

$$\begin{aligned}|u_{ck}(t,\varepsilon)-v_{ck}(t)|=|\varphi_{ck}|&\left|e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}}-e^{-(2ak)^2 t}\right|+\\
&\left|\left(\frac{1}{1+\varepsilon(2k)^2}-1\right)\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right\}d\tau\right|+\\
&\left|\int_0^t\frac{2}{\pi}\int_0^\pi[e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}-e^{-(2ak)^2(t-\tau)}]f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi d\tau\right|+\\
&\left|\int_0^t e^{-(2ak)^2(t-\tau)}\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}\cos 2k\xi d\xi d\tau\right|\end{aligned} \tag{4.12}$$

$$
\begin{aligned}
&\left|u_{sk}(t,\varepsilon)-v_{sk}(t)\right| = \left|\varphi_{sk}\right|\left|e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}} - e^{-(2ak)^2 t}\right| + \\
&\left|\left(\frac{1}{1+\varepsilon(2k)^2}-1\right)\int_0^t e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}\left\{\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right\}d\tau\right| + \\
&\left|\int_0^t \frac{2}{\pi}\int_0^\pi [e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} - e^{-(2ak)^2(t-\tau)}]f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi d\tau\right| + \\
&\left|\int_0^t e^{-(2ak)^2(t-\tau)}\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}\sin 2k\xi d\xi d\tau\right|
\end{aligned}
\tag{4.13}
$$

Uyarı4.1: $\left|e^{\frac{-(2ak)^2 t}{1+\varepsilon(2k)^2}} - e^{-(2ak)^2 t}\right|$, $\left|\frac{1}{1+\varepsilon(2k)^2}-1\right|$, $\left|e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}} - e^{-(2ak)^2(t-\tau)}\right|$

farklarının her birinin, her k,τ ve t ($0\le\tau\le t\le T$) için sınırlı olup, $\varepsilon\to 0$ da sıfıra yaklaştığı açıktır. Onlar, sırasıyla, $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$, $\delta_3(\varepsilon)$ ile gösterilir:

$$\lim_{\varepsilon\to 0}\delta_i(\varepsilon)=0,\ (i=1,2,3).$$

Uyarı4.2: Son eşitsizliklerde Fourier katsayılarına x'e göre kısmi integralleme yapıldığında,

$$\varphi_{ck} = \frac{2}{\pi}\int_0^\pi \varphi(\xi)\cos 2k\xi d\xi = \frac{2}{\pi}\frac{1}{2k}\varphi(\xi)\sin 2k\xi\Big|_0^\pi - \frac{1}{2k}\frac{2}{\pi}\int_0^\pi \varphi'(\xi)\sin 2k\xi d\xi = -\frac{1}{2k}\varphi'_{sk},$$

benzer şekilde de,

$$\varphi_{sk} = \frac{1}{2k}\varphi'_{ck}$$

bulunur. Aynı yolla,

$$f_{ck}[\tau,\xi,Au(\tau,\xi,\varepsilon)]=\frac{2}{\pi}\int_0^{\pi} f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi =$$

$$\frac{2}{\pi}\frac{1}{2k}f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi\Big|_0^{\pi}-\frac{1}{2k}\frac{2}{\pi}\int_0^{\pi}\left(f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\right)_{\xi}\sin 2k\xi d\xi =$$

$$-\frac{1}{2k}\left\{\left(f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\right)_{\xi}\right\}_{sk},$$

ve

$$f_{sk}[\tau,\xi,Au(\tau,\xi,\varepsilon)]=\frac{1}{2k}\left\{\left(f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\right)_{\xi}\right\}_{ck}$$

elde edilir.Burada, $f[\tau,0,Au(\tau,0,\varepsilon)]=f[\tau,\pi,Au(\tau,\pi,\varepsilon)]$ koşulu dikkate alınmıştır.

(4.11),(4.12) ve (4.13) farklarının mutlak değerlerini alıp, sağ tarafındaki integrallere, önce Cauchy eşitsizliğini uygulayıp, gerekli işlemler yapıldıktan sonra uyarı 4.1 kullanılırsa,

$$\frac{1}{2}\left|u_0(t,\varepsilon)-v_0(t)\right|\le\frac{1}{2}\sqrt{T}\left[\int_0^t\left(\frac{2}{\pi}\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}d\xi\right)^2 d\tau\right]^{\frac{1}{2}},$$

$$\left|u_{ck}(t,\varepsilon)-v_{ck}(t)\right|\le\left|\varphi_{ck}\right|\delta_1(\varepsilon)+\delta_2(\varepsilon)\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\times$$

$$\left[\int_0^t\left(\frac{2}{\pi}\int_0^{\pi}f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}+$$

$$\left(\int_0^t[e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}-e^{-(2ak)^2(t-\tau)}]^2 d\tau\right)^{\frac{1}{2}}\left[\int_0^t\left(\frac{2}{\pi}\int_0^{\pi}f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}+$$

$$\left(\int_0^t e^{-2(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\times\left[\int_0^t\left(\frac{2}{\pi}\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}d\xi\right)^2 d\tau\right]^{\frac{1}{2}}\le$$

$$|\varphi_{ck}|\delta_1(\varepsilon)+\delta_2(\varepsilon)\sqrt{T}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}+$$

$$\delta_3(\varepsilon)\sqrt{T}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\cos 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2a}}\frac{1}{k}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}\cos 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}},$$

$$|u_{sk}(t,\varepsilon)-v_{sk}(t)|\le|\varphi_{sk}|\delta_1(\varepsilon)+\delta_2(\varepsilon)\left(\int_0^t e^{\frac{-2(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)^{\frac{1}{2}}\times$$

$$\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}+$$

$$\left(\int_0^t[e^{\frac{-(2ak)^2(t-\tau)}{1+\varepsilon(2k)^2}}-e^{-(2ak)^2(t-\tau)}]^2 d\tau\right)^{\frac{1}{2}}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}+$$

$$\left(\int_0^t e^{-2(2ak)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\times\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}d\xi\right)^2 d\tau\right]^{\frac{1}{2}}\le$$

$$|\varphi_{sk}|\delta_1(\varepsilon)+\delta_2(\varepsilon)\sqrt{T}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}+$$

$$\delta_3(\varepsilon)\sqrt{T}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\sin 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2a}}\frac{1}{k}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}\sin 2k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}$$

bulunur.

Son üç eşitsizliğin ikinci ve üçüncüsünde $k=\overline{1,\infty}$ olmak üzere toplam yapıp, her üç eşitsizlik taraf tarafa toplandığında,

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)| = \frac{1}{2}|u_0(t,\varepsilon)-v_0(t)| + \sum_{k=1}^{\infty}\left[|u_{ck}(t,\varepsilon)-v_{ck}(t)| + |u_{sk}(t,\varepsilon)-v_{sk}(t)|\right] \le$$

$$\frac{\sqrt{T}}{2}\left[\int_0^t \left(f[u]-f[v]\right)_0^2 d\tau\right]^{\frac{1}{2}} +$$

$$\delta_1(\varepsilon)\sum_{k=1}^{\infty}\left(|\varphi_{ck}|+|\varphi_{sk}|\right) + \delta_2(\varepsilon)\sqrt{T}\sum_{k=1}^{\infty}\left\{\left(\int_0^t f_{ck}^2[u]d\tau\right)^{\frac{1}{2}} + \left(\int_0^t f_{sk}^2[u]d\tau\right)^{\frac{1}{2}}\right\} +$$

$$\delta_3(\varepsilon)\sqrt{T}\sum_{k=1}^{\infty}\left\{\left(\int_0^t f_{ck}^2[u]d\tau\right)^{\frac{1}{2}} + \left(\int_0^t f_{sk}^2[u]d\tau\right)^{\frac{1}{2}}\right\} +$$

$$\frac{1}{2\sqrt{2a}}\sum_{k=1}^{\infty}\frac{1}{k}\left[\int_0^t \left(f[u]-f[v]\right)_{ck}^2 d\tau\right]^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2a}}\sum_{k=1}^{\infty}\frac{1}{k}\left[\int_0^t \left(f[u]-f[v]\right)_{sk}^2 d\tau\right]^{\frac{1}{2}}$$

olur. Burada,

$$\left(f[u]-f[v]\right)_0 = \frac{2}{\pi}\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] - f[\tau,\xi,Av(\tau,\xi)]\}d\xi,$$

$$\left(f[u]-f[v]\right)_{ck} = \frac{2}{\pi}\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] - f[\tau,\xi,Av(\tau,\xi)]\}\cos 2k\xi d\xi,$$

$$\left(f[u]-f[v]\right)_{sk} = \frac{2}{\pi}\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] - f[\tau,\xi,Av(\tau,\xi)]\}\sin 2k\xi d\xi$$

olarak alınmıştır. Uyarı 4.2 yi kullanıldığında,

$$|\bar{u}(t,\varepsilon)-\bar{v}(t)| = \frac{1}{2}|u_0(t,\varepsilon)-v_0(t)| + \sum_{k=1}^{\infty}\left[|u_{ck}(t,\varepsilon)-v_{ck}(t)| + |u_{sk}(t,\varepsilon)-v_{sk}(t)|\right] \leq$$

$$\frac{\delta_1(\varepsilon)}{2}\left(\sum_{k=1}^{\infty}\frac{1}{k}|\varphi'_{ck}| + \sum_{k=1}^{\infty}\frac{1}{k}|\varphi'_{sk}|\right) +$$

$$\frac{\delta_2(\varepsilon)\sqrt{T}}{2}\left\{\sum_{k=1}^{\infty}\frac{1}{k}\left[\int_0^t\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_{\xi}\right)_{sk}^2 d\tau\right]^{\frac{1}{2}} + \sum_{k=1}^{\infty}\frac{1}{k}\left[\int_0^t\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_{\xi}\right)_{ck}^2 d\tau\right]^{\frac{1}{2}}\right\} +$$

$$\frac{\delta_3(\varepsilon)\sqrt{T}}{2}\left\{\sum_{k=1}^{\infty}\frac{1}{k}\left[\int_0^t\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_{\xi}\right)_{sk}^2 d\tau\right]^{\frac{1}{2}} + \sum_{k=1}^{\infty}\frac{1}{k}\left[\int_0^t\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_{\xi}\right)_{ck}^2 d\tau\right]^{\frac{1}{2}}\right\} +$$

$$\frac{\sqrt{T}}{2}\left[\int_0^t\left(\frac{2}{\pi}\int_0^{\pi}\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}d\xi\right)^2 d\tau\right]^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\{f[\tau,\xi,u(\tau,\xi,\varepsilon)]-f[\tau,\xi,v(\tau,\xi)]\}_{ck}^2 d\tau\right)^{\frac{1}{2}} +$$

$$\frac{1}{2\sqrt{2}a}\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_0^t\{f[\tau,\xi,u(\tau,\xi,\varepsilon)]-f[\tau,\xi,v(\tau,\xi)]\}_{sk}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki sonsuz toplamlara Hölder eşitsizliğini uygulayıp, gerekli işlemler yapıldıktan sonra, $\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}}$ olduğunu göz önüne alındığında,

$$|\bar{u}(t,\varepsilon)-\bar{v}(t)| = \frac{1}{2}|u_0(t,\varepsilon)-v_0(t)| + \sum_{k=1}^{\infty}\left[|u_{ck}(t,\varepsilon)-v_{ck}(t)| + |u_{sk}(t,\varepsilon)-v_{sk}(t)|\right] \leq$$

$$\frac{\delta_1(\varepsilon)}{2}\frac{\pi}{\sqrt{6}}\left\{\left[\sum_{k=1}^{\infty}(\varphi'_{ck})^2\right]^{\frac{1}{2}} + \left[\sum_{k=1}^{\infty}(\varphi'_{sk})^2\right]^{\frac{1}{2}}\right\} +$$

$$\frac{\sqrt{T}\delta_2(\varepsilon)}{2}\frac{\pi}{\sqrt{6}}\left\{\left[\sum_{k=1}^{\infty}\int_0^t\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\}_{\xi}\right)_{sk}^2 d\tau\right]^{\frac{1}{2}} + \left[\sum_{k=1}^{\infty}\int_0^t\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]\}_{\xi}\right)_{ck}^2 d\tau\right]^{\frac{1}{2}}\right\} +$$

$$\frac{\delta_3(\varepsilon)\sqrt{T}}{2}\frac{\pi}{\sqrt{6}}\left\{\sum_{k=1}^{\infty}\left[\int_0^t\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_{\xi}\right)_{sk}^2 d\tau\right]^{\frac{1}{2}} + \sum_{k=1}^{\infty}\left[\int_0^t\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_{\xi}\right)_{ck}^2 d\tau\right]^{\frac{1}{2}}\right\} +$$

$$\frac{\sqrt{T}}{2}\left[\int_0^t\left(\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}d\xi\right)^2 d\tau\right]^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\frac{\pi}{\sqrt{6}}\sum_{k=1}^{\infty}\left(\int_0^t\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}_{ck}^2 d\tau\right)^{\frac{1}{2}}+$$

$$\frac{1}{2\sqrt{2}a}\frac{\pi}{\sqrt{6}}\sum_{k=1}^{\infty}\left(\int_0^t\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}_{sk}^2 d\tau\right)^{\frac{1}{2}}$$

olur. Sağ taraftaki ikinci ve üçüncü toplamlarda Bessel eşitsizliği uygulandığında,

$$|\overline{u}(t,\varepsilon)-\overline{v}(t)|=\frac{1}{2}|u_0(t,\varepsilon)-v_0(t)|+\sum_{k=1}^{\infty}\left[|u_{ck}(t,\varepsilon)-v_{ck}(t)|+|u_{sk}(t,\varepsilon)-v_{sk}(t)|\right]\leq$$

$$\frac{\pi}{\sqrt{6}}\delta_1(\varepsilon)\left[\frac{2}{\pi}\int_0^\pi\varphi'^2(x)dx\right]^{\frac{1}{2}}+$$

$$\frac{\pi\sqrt{T}}{\sqrt{6}}\delta_2(\varepsilon)\left[\int_0^t\frac{2}{\pi}\int_0^\pi\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_\xi\right)^2 d\xi d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi\sqrt{T}}{\sqrt{6}}\delta_3(\varepsilon)\left[\int_0^t\frac{2}{\pi}\int_0^\pi\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_\xi\right)^2 d\xi d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\sqrt{T}}{2}\left[\int_0^t\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}^2 d\xi d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4a\sqrt{3}}\left[\int_0^t\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}^2 d\xi d\tau\right]^{\frac{1}{2}}+$$

$$\frac{\pi}{4a\sqrt{3}}\left[\int_0^t\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}^2 d\xi d\tau\right]^{\frac{1}{2}}$$

olur. Gerekli gruplamalar yapıp, işaretlemelerden faydalanıldığında, son eşitsizlik aşağıdaki şekilde yazılabilir.

$$\left|\bar{u}(t,\varepsilon)-\bar{v}(t)\right| \le \sqrt{\frac{\pi}{3}}\delta_1(\varepsilon)\|\varphi'(x)\|_{L_2[0,\pi]} +$$

$$\sqrt{\frac{\pi T}{3}}[\delta_2(\varepsilon)+\delta_3(\varepsilon)]\left[\int_0^t\int_0^\pi \left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_\xi\right)^2 d\xi d\tau\right]^{\frac{1}{2}} +$$

$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left[\int_0^t\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)]-f[\tau,\xi,Av(\tau,\xi)]\}^2 d\xi d\tau\right]^{\frac{1}{2}}.$$

Son eşitsizlikteki integral içindeki farka, Lipshitz koşulunu uygulayıp,

$$\left|Au(\tau,\xi,\varepsilon)-Av(\tau,\xi)\right| \le \left|\bar{u}(\tau,\varepsilon)-\bar{v}(\tau)\right|$$

olduğu göz önüne alındığında,

$$\left|\bar{u}(t,\varepsilon)-\bar{v}(t)\right| \le \sqrt{\frac{\pi}{3}}\delta_1(\varepsilon)\|\varphi'(x)\|_{L_2[0,\pi]} +$$

$$\sqrt{\frac{\pi T}{3}}[\delta_2(\varepsilon)+\delta_3(\varepsilon)]\left[\int_0^t\int_0^\pi \left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_\xi\right)^2 d\xi d\tau\right]^{\frac{1}{2}} +$$

$$\frac{a\sqrt{3T}+\pi}{a\sqrt{6\pi}}\left[\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau,\varepsilon)-\bar{v}(\tau)\right|^2 d\xi d\tau\right]^{\frac{1}{2}}$$

olur.

Her iki tarafın karesini alıp,

$$(a+b+c)^2 \le 3\left(a^2+b^2+c^2\right)$$

eşitsizliği uygulandığında,

$$\left|\bar{u}(t,\varepsilon)-\bar{v}(t)\right|^2 \leq \pi\delta_1^2(\varepsilon)\|\varphi'(x)\|^2_{L_2[0,\pi]} +$$
$$\pi T\left[\delta_2(\varepsilon)+\delta_3(\varepsilon)\right]^2\int_0^t\int_0^\pi\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_\xi\right)^2 d\xi d\tau +$$
$$\frac{(a\sqrt{3T}+\pi)^2}{2a^2\pi}\int_0^t\int_0^\pi b^2(\tau,\xi)\left|\bar{u}(\tau,\varepsilon)-\bar{v}(\tau)\right|^2 d\xi d\tau$$

eşitsizliği elde edilir. Burada, Gronwall eşitsizliğini kullanıp, t ye göre maximum alındığında,

$$\left\|\bar{u}(t,\varepsilon)-\bar{v}(t)\right\|^2 \leq \pi\delta_1^2(\varepsilon)\|\varphi'(x)\|^2_{L_2[0,\pi]} +$$
$$\pi T\left[\delta_2(\varepsilon)+\delta_3(\varepsilon)\right]^2\int_0^t\int_0^\pi\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_\xi\right)^2 d\xi d\tau \times$$
$$\exp\frac{(a\sqrt{3T}+\pi)^2}{2a^2\pi}\|b(t,x)\|^2_{L_2(D)}$$

veya

$$\left\|\bar{u}(t,\varepsilon)-\bar{v}(t)\right\| \leq \left(\pi\delta_1^2(\varepsilon)\|\varphi'(x)\|^2_{L_2[0,\pi]} + \pi T\left[\delta_2(\varepsilon)+\delta_3(\varepsilon)\right]^2\int_0^t\int_0^\pi\left(\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)\}_\xi\right)^2 d\xi d\tau\right)^{\frac{1}{2}} \times$$
$$\exp\frac{(a\sqrt{3T}+\pi)^2}{4a^2\pi}\|b(t,x)\|^2_{L_2(D)}$$

bulunur.

Son eşitsizlik,

$$\lim_{\varepsilon\to 0}\bar{u}(t,\varepsilon)=\bar{v}(t)$$

olduğunu göstermektedir. ($\forall t \in [0,T]$)

Böylece aşağıdaki teorem ispatlanmış olur.

Teorem 4.3: a) $\varphi(x)$ ve $\varphi'(x)$ fonksiyonları $[0,\pi]$ aralığında Dirichlet teoreminin koşullarını, f(t,x,u) fonksiyonu da $\overline{D}\times(-\infty,\infty)$ bölgesinde argümanlarına göre sürekli olup her $(t,x)\in D$ için

b) $|f(t,x,u)-f(t,x,v)|\le b(t,x)|u-v|$

(Lipschitz koşulunu,(burada $b(t,x)\in L_2(D)$ dir) ve

$$\left(f[t,x,u(t,x)]\right)_x \in L_2(D)\,,\; f[t,0,u(t,0)] = f[t,\pi,u(t,\pi)]$$

koşulları sağlandığında, (4.9) ve (4.10) sonsuz integral denklemlerinin çözümleri için

$$\lim_{\varepsilon\to 0}\overline{u}(t,\varepsilon) = \overline{v}(t)$$

eşitliği doğrudur.

Başka bir deyişle de, Teorem 4.3 ün koşulları sağlandığında, (4.1)-(4.3) ve (4.4)-(4.6) karışık problemlerinin genelleşmiş çözümleri için

$$\lim_{\varepsilon\to 0} u(t,x,\varepsilon) = v(t,x)$$

eşitliği doğrudur.

5. SONUÇLAR VE ÖNERİLER

5.1 Sonuçlar

1. Yarı doğrusal parabolik denklem için devirli sınır koşullu,

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t,x,u) \quad (t,x) \in D\{0 < t < T < \infty \ , \ 0 < x < \pi\}$$

$$u(t,0) = u(t,\pi) \qquad t \in [0,T]$$

$$u_x(t,0) = u_x(t,\pi) \qquad t \in [0,T]$$

$$u(0,x) = \varphi(x) \qquad x \in [0,\pi]$$

(f(t,x,u), $\overline{D} \times (-\infty,\infty)$ da Lipschitz koşulunu ve $\varphi(x)$,$[0,\pi]$ de Dirichlet teoreminin koşullarını sağlamak üzere) karışık problemi için genelleşmiş çözümün varlığı ve tekliği ispatlanmıştır.

2. Yarı doğrusal pseudo-parabolik denklem için devirli sınır koşullu,

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial^3 u}{\partial t \partial x^2} = f(t,x,u) \quad (t,x) \in D\{0 < t < T < \infty \ , \ 0 < x < \pi\}$$

$$u(t,0) = u(t,\pi) \qquad t \in [0,T]$$

$$u_x(t,0) = u_x(t,\pi) \qquad t \in [0,T]$$

$$u(0,x) = \varphi(x) \qquad x \in [0,\pi]$$

(f(t,x,u), $\overline{D} \times (-\infty,\infty)$ da Lipschitz koşulunu ve $\varphi(x)$,$[0,\pi]$ de Dirichlet teoreminin koşullarını sağlamak üzere) karışık problemi için genelleşmiş çözümün varlığı ve tekliği ispatlanmıştır.

3. Genelde, doğrusal olmayan problemlerin kesin çözümlerinin bulunması imkansız olduğundan, tezde incelenen yarı doğrusal parabolik ve yarı doğrusal pseudo-parabolik denklemler için ele alınan problemlerin, uygulamadaki önemi göz önünde

bulundurularak, bu problemlerin kesin çözümleriyle ardışık yaklaşımların farkı değerlendirilmiştir.

4. Yarı doğrusal pseudo-parabolik denklem için devirli sınır koşullu,

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial^3 u}{\partial t \partial x^2} = f(t,x,u) \quad (t,x) \in D\{0 < t < T < \infty \ , \ 0 < x < \pi\}$$

$$u(t,0) = u(t,\pi) \qquad t \in [0,T]$$
$$u_x(t,0) = u_x(t,\pi) \qquad t \in [0,T]$$
$$u(0,x) = \varphi(x) \qquad x \in [0,\pi]$$

karışık problemin çözümünün $\varepsilon \to 0$ da yarı doğrusal parabolik denklem için devirli sınır koşullu,

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t,x,u) \quad (t,x) \in D\{0 < t < T < \infty \ , \ 0 < x < \pi\}$$

$$u(t,0) = u(t,\pi) \qquad t \in [0,T]$$
$$u_x(t,0) = u_x(t,\pi) \qquad t \in [0,T]$$
$$u(0,x) = \varphi(x) \qquad x \in [0,\pi]$$

karışık problemin çözümüne yaklaştığı gösterilmiştir.

5.2 Öneriler

1. İncelenen problemlerin daha güçlü genelleşmiş çözümü, hemen hemen çözümü ve klasik çözümü tanımlanıp, varlığı ve tekliği incelenebilir.

2. Fourier yöntemi, çubuğun salınım denklemi(4. mertebeden KTDD) için devirli sınır koşullu karışık problemin çözümüne uygulanabilir.

3. Sözü edilen problemlerin başlangıç verilerine bağlılığı yani kararlılığı ve çözümlerin diferansiyel özellikleri incelenebilir.

4. Fourier yöntemi, yarı doğrusal ve doğrusal olmayan parabolik, hiperbolik ve daha yüksek mertebeden KTDD için devirli sınır koşullu karışık problemin çözümüne de uygulanabilir.

5. $$\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}-\varepsilon\frac{\partial^3 u}{\partial t\partial x^2}=f(t,x,u,u_x,u_t) \quad (t,x)\in D\{0<t<T<\infty\ ,\ 0<x<\pi\}$$

$$u(t,0)=u(t,\pi) \qquad t\in[0,T]$$
$$u_x(t,0)=u_x(t,\pi) \qquad t\in[0,T]$$
$$u(0,x)=\varphi(x) \qquad x\in[0,\pi]$$

karışık problemi de yeni bir genelleşmiş çözüm tanımıyla incelenebilir.

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