REPUBLIC OF TURKEY

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THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

STATISTICAL CONVERGENCE OF NUMBER SEQUENCES AND SOME GENERALIZATIONS

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Master Thesis
Department of Mathematics
Program: Analysis and Functions Theory
Supervisor: Prof. Dr. Rifat ÇOLAK

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SUMMARY

STATISTICAL CONVERGENCE OF NUMBER SEQUENCES AND SOME GENERALIZATIONS

In this thesis we examine and study statistical convergence, statistical boundedness and some other notions related to these concepts for sequences of real numbers. At the first we give the statistically convergent, statistically Cauchy and statistically bounded sequences of real numbers and then we give limit inferior and limit superior of a sequence. Then we establish the relations between these concepts. After that we study the concept strong p-Cesàro summability of sequences of real numbers. In the last step we give the relationship between the sets of sequences which are statistically convergent of order α, for different α's such that $0< \alpha \leq 1$. Furthermore, we also give the relationship between the sets of sequences which are strongly p-Cesàro summable of order α for different α's such that α>0. At the end we give and study λ-statistical convergence and λ- statistical convergence of order α for number sequences, where $\lambda =(\lambda_n)$ is non-decreasing sequence of positive number such that $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$, $\lambda_n\to\infty$ as $n\to\infty$.

Key Words: Statistical Convergence, Statistical Boundedness, statistically Cauchy, Cesàro summability, Statistical Convergence of Order α, Statistical Boundedness of Order α, λ-statistical convergence, λ-statistical convergence of order α**.**

.

ÖZET

SAYI DİZİLERİ İÇİN İSTATİSTİKSEL YAKINSAKLIK VE BAZI GENELLEŞTİRMELERİ

Bu tezde istatistiksel yakınsak ve istatistiksel sınırlı reel sayı dizilerini inceleyecek ve bu kavramlarla ilişkili diğer bazı kavramlardan bahsedeceğiz. İlk olarak istatistiksel yakınsak, istatistiksel Cauchy ve istatistiksel sınırlı dizileri tanıtacak ve limit inferior ve limit superior kavramlarına yer vereceğiz. p-Cesàro toplanabilirlik kavramını verdikten sonra bu kavramlar arasındaki ilişkileri ortaya koyacağız. α yıncı dereceden istatistiksel yakınsaklık, α yıncı dereceden istatistiksel sınırlılık ve α yıncı dereceden. p-Cesàro toplanabilirlik kavramlarını verdikten sonra farklı α lar için elde edilen dizi kümeleri arasındaki kapsama bağıntılarını ve ilişkileri ortaya koyacağız. Son olarak istatistiksel sınırlılık ve α. yıncı dereceden p-Cesàro toplanabilirlik kavramlarını verdikten sonra farklı α lar için elde edilen dizi kümeleri arasındaki kapsama bağıntılarını ve ilişkileri ortaya koyacağız. Son olarak $\lambda = (\lambda_n)$ pozitif sayıların azalmayan ve her n için $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $n \to \infty$ için $\lambda_n \to \infty$ şartlarını sağlayan bir dizi olmak üzere λ -istatistiksel yakınsak ve α yıncı dereceden λ -istatistiksel yakınsak dizileri tanımlanacak ve bu kavramlara ilişkin bazı bağıntılar verilecektir.

Anahtar Kelimeler: İstatiksel yakınsaklık, istatistiksel sınırlılık, istatistiksel Cauchy, Cesàro toplanabilirlik, α yıncı dereceden istatistiksel yakınsaklık, α yıncı dereceden istatistiksel sınırlılık, λ-istatistiksel yakınsaklık, α yıncı dereceden λ -istatistiksel yakınsaklık**.**

LIST OF SYMBOLS

$\mathbb{N}$	: Natural numbers
$\mathbb{R}$	: Real numbers
w	: sequences
c	: Convergent sequences
c_0	: Null sequences
l_∞	: bounded sequences
S	: Statistically convergent sequences
S_0	: Statistically null sequences
SB	: Statistically bounded sequences
S^α	: Sequences which are statistically convergent of order α
SB^α	: Sequences which are statistically bounded of order α
S_λ	: λ-statistically convergent sequences
SB_λ	: Sequences which are λ-statistically bounded of order α
w_p	: Strongly p-Cesàro summable sequences
w_p^α	: Sequences which are strongly p-Cesàro summable of order α

1. INTRODUCTION

The notion of statistically convergence was took place in a study of Fast[1] in first time and also independently by Buck[2] and Schoenberg[3] for number sequences. One of the most recent generalizations of concept of convergence of sequences “A new type of convergence” is statistical convergence defined by Fast. Recently, it became the center of attraction for many researchers.

The statistical convergence concept for number sequences appeared in a study of Zygmund[4] and it was called it as "almost convergence ".

Statistical convergence is a type of convergence which is basically depend on the natural density of subsets of the set of positive integers.

Statistical convergence has been discussed under different names in Number theory, Fourier analyses, Trigonometric series, Measure theory and Banach spaces. Statistical convergence was further studied from the sequence space point of view and linked with ssammaubility theory by Fridy[5]· Connor[6], Savaş[7], Šalát[8], Mursaleen[9], Nuray[10], Mohiuddine et al[11], Çolak[12], [13], Çolak and Bektaş[14] and many mathematicians. In last years, the concept of statistical convergence has appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Furthermore, the concept “order α” was imported in last years and some generalizations have been given such as statistical convergence of order α.

The α -density of any subset of natural numbers, statistical convergence of order α and strong p-Cesàro summability of order α for number sequences was defined by Çolak[12]. The α -density of a subset U $\subseteq\mathbb{N}$ is defined by

$$\delta_\alpha(U) = \lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \le n : m \in U\}|,$$

if limit exists, where α∈(0,1] and $|\{m \le n : m \in U\}|$ represents the number of elements which belongs to U.

We will use the following abbreviations throughout the thesis.

“stat.” instead of “statistical”

“statly.” instead of “statistically”

2. STATISTICAL CONVERGENCE AND STATISTICAL BOUNDEDNESS OF NUMBER SEQUENCES

Definition 2.1 Let U $\subseteq\mathbb{N}=\{1,2.3.\ldots\}$ and define

$$\delta(U)=\lim_{n\to\infty}\frac{1}{n}|\{m\leq n: m\in U\}|.$$

Then the number $\delta(\mathrm{U})$ is called the naturel density of set U, if the limit exists.

Stat. convergence of a sequence is based on the density of subsets of the set $\mathbb{N}$. It can be checked that any finite subset of set $\mathbb{N}$ has zero natural density and $\delta(\mathrm{U^c})=1-\delta(\mathrm{U})$**,** where $\mathrm{U^c}=\mathbb{N}-\mathrm{U}$ for any U $\subseteq\mathbb{N}$. If $\delta(\mathrm{U})=1$ then set U is said to be statistically dense **[1]**

Definition 2.2 Let $u=(u_m)\in w$. The sequence (u_m) is said to be statly. convergent if

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n: |u_m-l|\geq\varepsilon\}|=0$$

for every *ε > 0*, for some number *l*. In this condition we say that *u* is statly. convergent to *l*. For this situation we write

$$stat-\lim_{m\to\infty}u_m=l$$

and S represents the collection of all statly. convergent sequences. **[5].**

In this study the sequences will have real entries. We recall that if a sequence $u=(u_m)$ accepts property P for all m excepting a set of density zero , then we say that $u=(u_m)$ accepts property P for "almost all m" and abbreviate this by "a.a.m".

Lemma 2.1 If $stat-lim u_m=u_0$ and $stat-lim v_m=v_0$ and c is a real constant, then

i) $stat.-lim(c.u_m)=c.u_0$

ii) $stat.-lim(u_m+v_m)=u_0+v_0$ **[5].**

Definition 2.3 A sequence $u=(u_m)$ is statly. bounded if

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n: |u_m|>M\}|=0$$

for some $M>0$ $i.e.$ $|u_m|\leq M$ $a.a.m.$ **[8].**

Theorem 2.1 Any bounded number sequence is statly. bounded **[8]**.

Proof Suppose that the sequence $u = (u_m)$ is bounded. Then for some M >0, we have $|u_m| \leq M$, for all $m \in \mathbb{N}$ and this means that

$$\{m \in \mathbb{N}: |u_m| > M\} = \emptyset .$$

Thus, we get

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq n: |u_m| > M\}| = 0$$

and it means that *u* is statly. bounded.

Remark 2.1 The reverse of Theorem 2.1 is not right in general. For example the sequence $u = (u_m)$ defined as

$$u_m = \begin{cases} m^2 , & m = n^2 \\ (-1)^m , & m \neq n^2 . \end{cases} \qquad n = 1,2,3,4, \dots ..$$

that is, the sequence $u = \{1,1,-1,16,-1,1,-1,1,81, \dots\}$ is not bounded. To show that $u = (u_m)$ is statly. bounded, let M >1 be given. Then

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq n: |u_m| > M\}| \leq \lim_{n\to\infty} \frac{\sqrt{n}}{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 ,$$

it means that *u* is statly. bounded **[8].**

Theorem 2.2 Any statly. convergent sequence is statly. bounded **[8]**.

Proof Suppose that the sequence (u_m) is statly. convergent to u_0. For any arbitrary $K > \varepsilon > 0$ we have

$\{m \leq n: |u_m - u_0| > K\} \subseteq \{m \leq n: |u_m - u_0| \geq \varepsilon\}$.

This inclusion gives the inequality

$|\{m \leq n: |u_m - u_0| > K\}| \leq |\{m \leq n: |u_m - u_0| \geq \varepsilon\}|$.

Since (u_m) is statly. convergent to u_0 we have

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq n: |u_m - u_0| \geq \varepsilon\}| = 0 ,$$

for every $\varepsilon > 0$, and from above inequality

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq n: |u_m - u_0| > K\}| = 0.$$

That is why the sequence (u_m) is statly. bounded.

Remark 2.2 The reverse of Theorem 2.2 is not right in general. For this let $u = (u_m)$ be a sequence such that

$$u_m = \begin{cases} s, & \text{if } m = 2i + 1 \\ t, & \text{if } m = 2i \end{cases} \qquad i=1,2,3,4\dots$$

where s, t∈ ℝ and s≠t. Now $u = (u_m) = \{s, t, s, t, s, t, ...\}$ is statly. bounded. Let us choose $K \geq 2max\{|s|, |t|\}$. Then $\{m \leq n: |u_m| > K\} = \emptyset$ for each n and so that

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq n: |u_m| > K\}| = 0$$

and it means that $u = (u_m)$ is statly. bounded **[8].**

Now we show that $u = (u_m)$ is not statly. convergent. Since the density of the both sets $\delta(\{m \leq n: m = 2i + 1\}) = \frac{1}{2} \neq 0$ and $\delta(\{m \leq n: m = 2i\}) = \frac{1}{2} \neq 0$ the sequence (u_m) is not statly. convergent to s and t. Therefore the sequence (u_m) is not statly. convergent.

Consequently, the sequence (u_m) is not statly. convergent however (u_m) is statly. bounded.

Note that every subsequence of a convergent sequence is also convergent but every subsequence of a statly. convergent sequence may not be convergent and may not be a statly. convergent.

For example, let us take (u_k) as

$$u_m = \begin{cases} m, & m \text{ is prime number} \\ 0, & \text{otherwise.} \end{cases}$$

Since the natural density of the collection of prime numbers is zero, that is

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq n: m \text{ is prime number}\}| = 0,$$

and so that $\delta(\{m \epsilon \mathbb{N}: m \text{ is prime number}\})$ =0. Thus (u_m) is statly. convergent to zero, but it is clear that the subsequence $(u_{m'})$ is not convergent and not statly. convergent, where $(u_{m'}) = (m) = \{1,2,3,5,7,11, \dots .\}$.

Definition 2.4 A sequence $u = (u_m)$ of real numbers is statly. Cauchy sequence if for any ε>0, there exist a number N∈ℕ such that

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq n: |u_m - u_N| \geq \varepsilon\}| = 0$$

[5].

Theorem 2.3 Every statly. Cauchy sequence is statly. bounded, but the converse is not right **[5].**

Proof. Let $u = (u_m)$ be a statly. Cauchy sequence. Then given any ε >0, there exists N=N(ε) such that $|u_m - u_N| < \varepsilon$ a.a.m.

This implies that $|u_m|<M$ a.a.m, where M= ε +$|u_N|$ i.e.

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n:|u_m|>M\}|=0$$

It means that the sequence *u* is statly. bounded.

The sequence $u=(u_m)$=(−1,1,−1,1,…) is bounded and hence statly. bounded but it is not statly. Cauchy.

Theorem 2.4 A sequence $u=(u_m)$ is statly. convergent if and only if it is satisfied the following condition

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n,m'\leq n:|u_m-u_{m'}|\geq\varepsilon\}|=0$$

where $(u_{m'})$ is a subsequence of (u_m) such that

$$\lim_{m'\to\infty}u_{m'}=l$$

for some l **[5].**

Proof Let the sequence (u_m) be statly. convergent. We will prove that

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n,m\leq n:|u_m-u_{m'}|\geq\varepsilon\}|=0 \qquad (2.1)$$

If the sequence (u_m) is statly. convergent to l, then by the Definition 2.2 we have

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n:|u_m-l|\geq\varepsilon\}|=0. \qquad (2.2)$$

for some *l*. Now, by using (2.1), we have

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n,m'\leq n:|u_m-u_{m'}|\geq\varepsilon\}|$$

$$=\lim_{n\to\infty}\frac{1}{n}|\{m\leq n,m'\leq n:|u_m-u_{m'}-l+l|\geq\varepsilon\}|$$

$$\leq\lim_{n\to\infty}\frac{1}{n}|\{m\leq n:|u_m-l|\geq\varepsilon\}|+\lim_{n\to\infty}\frac{1}{n}|\{m'\leq n:|u_{m'}-l|\geq\varepsilon\}|$$

$$\leq 0+\lim_{n\to\infty}\frac{1}{n}|\{m'\leq n:|u_{m'}-l|\geq\varepsilon\}| \qquad (2.3)$$

It is given that $\lim_{m'\to\infty}u_{m'}=l$. Since $(u_{m'})$ is convergent, it is also statly. convergent. Therefore, we can write

$$\lim_{n\to\infty}\frac{1}{n}|\{m'\leq n:|u_{m'}-l|\geq\varepsilon\}|=0 \qquad (2.4)$$

In view of the inequalities (2.3) and (2.4) we get

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n,m'\leq n:|u_m-u_{m'}|\geq\varepsilon\}|=0$$

Conversely, let

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n,m'\leq n:|u_m-u_{m'}|\geq\varepsilon\}|=0 \qquad (2.5)$$

be satisfied. To prove that the sequence (u_m) is statly. convergent let us start from the following inequality

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |u_m - l| \ge \varepsilon\}| \quad (2.6)$$

$$= \lim_{n\to\infty}\frac{1}{n}|\{m \le n, m' \le n: |u_m - u_{m'} + u_{m'} - l| \ge \varepsilon\}|$$

$$\le \lim_{n\to\infty}\frac{1}{n}|\{m \le n, m' \le n: |u_m - u_{m'}| \ge \varepsilon\}| + \lim_{n\to\infty}\frac{1}{n}|\{m' \le n: |u_{m'} - l| \ge \varepsilon\}|$$

$$\le 0 + \lim_{n\to\infty}\frac{1}{n}|\{m \le n, m': |u_m - u_{m'}| \ge \varepsilon\}| \quad (2.7)$$

by using (2.5). Since it is given that $\lim_{m'\to\infty} u_{m'} = l$ then

$$\lim_{n\to\infty}\frac{1}{n}|\{m' \le n: |u_{m'} - l| \ge \varepsilon\}|\} = 0$$

from inequality (2.7). Consequently we get $\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |u_m - l| \ge \varepsilon\}|\} = 0.$

This implies that the sequence (u_m) is statly. convergent.

Theorem 2.5 Suppose that (u_m) and (v_m) are any two sequences such that (u_m) is convergent to l and (v_m) is statly. convergent to zero. Then the sequence (u_m+v_m) is statly. convergent to l **[5].**

Proof Let

$$\lim_{m\to\infty} u_m = l \quad \text{i.e} \quad |u_m - l| \to 0 \text{ as } m \to \infty \quad (2.8)$$

and also let

$$stat - \lim_{m\to\infty} v_m = 0$$

that is,

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |v_m - 0| \ge \varepsilon\}| = 0. \quad (2.9)$$

Now, suppose that

$$stat - \lim_{m\to\infty}(u_m + v_m) = l' \quad (2.10)$$

Therefore,

$$stat - \lim_{m\to\infty}(u_m + v_m) = \lim_{n\to\infty}\frac{1}{n}|\{m \le n: |(u_m + v_m) - l'| \ge \varepsilon\}|\} = 0. \quad (2.11)$$

Now

$$\left|\lim_{m\to\infty}|u_m - l'| + \lim_{n\to\infty}\frac{1}{n}|\{m \le n: |v_m - 0| \ge \varepsilon\}|\right| = 0.$$

that is

$$\left|\lim_{m\to\infty}|u_m - l'| + 0\right| = 0$$

Hence using (2.9) we get

$$\lim_{m\to\infty}|u_m - l'| = 0$$

i.e.

$$\lim_{m\to\infty} u_m = l'. \tag{2.12}$$

But since $\lim_{m\to\infty} u_m = l$ we get $l' = l$.

From (2.10) and (2.12), it is proved that

$$stat - \lim_{m\to\infty}(u_m + v_m) = l.$$

Theorem 2.6 If a sequence (u_m) is statly. convergent to l then there are sequences (v_m), (z_m) such that $\lim_{n\to\infty}\frac{1}{n}|\{m \le n: u_m \ne v_m\}|\} = 0$ and (z_m) is a statly. null sequence where $\lim_{m\to\infty} v_m = l$, $u_m=v_m+z_m$ **[5].**

Proof Suppose that the sequence (u_m) be statly. convergent to l, that is

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |u_m - l| \ge \varepsilon\}| = 0 \tag{2.13}$$

and we have $|v_m - l| \to 0$ *as* $m \to \infty$, where, $u_m=v_m+z_m$. We should prove that

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: u_m \ne v_m\}| = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}|\{m \le n: |z_m - 0| \ge \varepsilon\}|\} = 0.$$

Since (u_m) is statly. convergent to l, we have

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |u_m - l| \ge \varepsilon\}| = 0,$$

that is $\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |u_m - l + v_m - v_m| \ge \varepsilon\}| = 0$, in another word

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |(u_m - v_m) + (v_m - l)| \ge \varepsilon\}| = 0\,.$$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |u_m - v_m| \ge \varepsilon\}| + \lim_{n\to\infty}\frac{1}{n}|\{m \le n: |v_m - l| \ge \varepsilon\}| = 0.$$

And since $\lim_{n\to\infty} v_n = l$ we get

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |u_m - v_m| \ge \varepsilon\}| + 0 = 0$$

and hence

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: |u_m - v_m| \ge \varepsilon\}| = 0$$

which implies that

$$\lim_{n\to\infty}\frac{1}{n}|\{m \le n: u_m \ne v_m\}| = 0\,.$$

Since

$$\lim_{m\to\infty} v_m = l \ \ and \ \ u_m = v_m + z_m \tag{2.14}$$

and $\lim_{n\to\infty}\frac{1}{n}|\{m \le n: u_m \ne v_m\}| = 0$ from Theorem 2.4 we have that

$$stat - \lim_{m\to\infty}(v_m + z_m) = l = \lim_{m\to\infty} v_m$$

by using (2.14) which implies that $stat - \lim_{m\to\infty} z_m$ must be equal to zero it means that (z_m) is statly. null sequence.

Definition 2.5 If $u = (u_m)$ is a sequence of real numbers, then the stat. limit superior of *u* is given by

$$stat - \limsup(u) = \begin{cases} supB_u, & if\ B_u \ne \emptyset \\ -\infty, & if\ B_u = \emptyset \end{cases}$$

where $B_u = \{b \in \mathbb{R}: \delta\{(m: u_m > b\}) \ne 0\}$.**[16]**

Definition 2.6 If $u = (u_m)$ is a sequence of real numbers, then stat. limit inferior of *u* is given by

$$stat - lim\,inf(u) = \begin{cases} infA_u, & if\ A_u \ne \emptyset \\ \infty, & if\ A_u = \emptyset \end{cases}$$

where $A_u = \{a \in \mathbb{R}: \delta(\{m: u_m < a\}) \ne 0\}$.**[16]**

Theorem 2.7 For any sequence *u*, *stat-lim inf(u)* ≤ *stat-lim sup(u).* **[16]**

Proof First suppose that *stat- limit sup(u)* = −∞. Then by Definition 2. 5 we have $B_u = \emptyset$ then for every b∈ ℝ , $\delta(\{m: u_m < b\}) = 1$ it means $\delta(\{m: u_m < b\}) = 1$. So for any a∈ ℝ , $\delta(\{m: u_m < a\}) \ne 0$. Hence *stat-lim inf(u)*= −∞.

It means that *stat- inf(u)* ≤ *stat-lim sup(u).*

In case *stat-lim sup(u)* = +∞ it is clear that *stat-lim inf(u)* ≤ *stat-lim sup(u).*

Next case assumes that *stat- lim sup(u)* = *β* and *stat-lim inf(u)*=*α*.

Since *stat-lim sup(u)* = *β*, by Definition 2.5 we have $\delta(\{m: u_m > \beta\}) \ne 0$, but β is *supB*$_u$ then for any *ε*>0, $\delta\left(\{m: u_m > \beta + \frac{\varepsilon}{2}\}\right) = 0$. Hence $\delta(\{m: u_m < \beta + \varepsilon\}) = 1$ and by **Definition 2.6** $\beta + \varepsilon \in A_u$ but *α* is *infA*$_u$ implies that $\alpha \le \beta + \varepsilon$. Since *ε* is arbitrary, we have $\alpha \le \beta$.

For example let $u = (u_m)$ be any real number sequence such that

$$u_m = \begin{cases} m, & \text{if m is square} \\ 2. & \text{if m is an odd non square} \\ 0, & \text{if m is an even non square} \end{cases}$$

Since $B_u = (-\infty, 2)$ and $A_u = (0, +\infty)$ *stat-lim sup(u)=2* and *stat-lim inf(u)=0*.

The sequence $u = (u_m)$ is statly. bounded but it is not statly. convergent, since $u = (u_m)$ have two different subsequences and the density of which are not zero.

Note: Statly. boundedness implies that *stat-lim sup* and *stat-lim inf* are finite.

Theorem.2. 8 A sequence $u = (u_m)$ of real numbers is statly. convergent if and only if it is statly. bounded and *stat-lim inf(u) = stat-lim sup(u).* **[16]**

Proof Suppose that the sequence $u = (u_m)$ is statly. convergent and $stat - \lim_{m\to\infty} u_m = l$. Then $= (u_m)$ is statly. bounded and for any *ε>0* we have

$$\delta(\{m: |u_m - l| > \varepsilon\}) = 0. \quad (2.15)$$

Suppose that *stat-lim inf(u) =α* , *stat-lim sup(u)=β.* We will show that *α =β.*

From equation (2.15) we get $(\{m: u_m > l + \varepsilon\}) = 0$. Since *stat-lim sup(u)=β* implies that

$$\beta \leq l\,. \quad (2.16)$$

Also from (2.15) we get $\delta(\{m: u_m < l - \varepsilon\}) = 0$. Since *stat-lim inf(u)=α* implies that

$$l \leq \alpha. \quad (2.17)$$

From (2.16) and (2.17) we get $\beta \leq l \leq \alpha$ i.e. $\beta \leq \alpha$.

But by Theorem 2.7 we have $\alpha \leq \beta$ thus we get that $\alpha = \beta$.

Conversely, suppose that *stat-inf u =α*, *stat-lim sup(u)=β* and $\alpha = \beta$. We will show that $u = (u_m)$ is statly. convergent.

Since *stat-lim inf(u) =α* we have

$$\delta(\{m: u_m < \alpha - \varepsilon\}) = 0 \quad (2.18)$$

And since *stat-lim sup(u)=β* we have

$$\delta(\{m: u_m > \beta + \varepsilon\}) = 0 \quad (2.19)$$

By assumption since $\alpha = \beta$ we may take $l = \alpha = \beta$ in equation (2.18) and (2.19). From this we get

$$\delta(\{m: u_m < l - \varepsilon\}) = 0 \text{ and } \delta(\{m: u_m > l + \varepsilon\}) = 0$$

This implies that $\delta(\{m: |u_m - l| > \varepsilon\}) = 0$. Hence $stat - \lim_{m\to\infty} u_m = l$.

This completes the proof.

3. CESÀRO SUMMABILITY AND STRONG p-CESÀRO SUMMABILITY

Definition 3.1 A sequence (u_n) is called Cesàro summable to l if the condition

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} u_m = l$$

is satisfied **[6]**.

Theorem 3.1 If the series $\sum_{m=1}^{\infty} u_m$ is convergent then the sequence (u_n) is Cesàro summable to zero .

Proof Suppose the series $\sum_{m=1}^{\infty} u_m$ is convergent and let l be the sum of the series. Then the sequence of its partial sum also converges to l, i.e. $S_n=\sum_{m=1}^{n} u_m \to l$ as $n\to\infty$. Now we get that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} u_m = \lim_{n\to\infty} \frac{1}{n} S_n = 0 \,.$$

Hence the sequence (u_n) is Cesàro summable to zero.

Note that the reverse of the above Theorem is not right. For example the sequence $(u_n) = \left(\frac{1}{n}\right)$ is Cesàro summable to zero, since

$\lim\limits_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} u_m = \lim\limits_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{1}{m} = 0.$

But the series $\sum_{m=1}^{\infty} \frac{1}{m}$ is divergent as known (harmonic series).

Theorem 3.2 Every convergent sequence is also Cesàro summable **[6]**.

Proof. Suppose that (u_n) is convergent and $\lim\limits_{n\to\infty} u_n = u_0$ and let $v_n = \frac{1}{n} \sum_{m=1}^{n} u_m$. We should prove that $\lim\limits_{n\to\infty} v_n = u_0$. For this given any $\varepsilon > 0$ we should find a number $N \in \mathbb{N}$ such that $|v_n - u_0| < \varepsilon$ for all $n > N$. Now, since $\lim\limits_{n\to\infty} u_n = u_0$, we know that for any $\varepsilon_1 > 0$ there exist $N_1 \in \mathbb{N}$ such that $|u_n - u_0| < \varepsilon_1$ for all $n > N_1$. Choose $\varepsilon = 2\,\varepsilon_1$. Now we may write

$$|v_n - u_0| = \left| \left(\frac{1}{n} \sum_{m=1}^{n} u_m \right) - u_0 \right|$$

$$= \left| \left(\frac{1}{n} \sum_{m=1}^{n} u_m \right) - u_0 \frac{n}{n} \right| = \left| \frac{1}{n} \sum_{m=1}^{n} (u_m - u_0) \right|$$

$$= \left|\frac{1}{n}\sum_{m=1}^{N_1}(u_m - u_0) + \frac{1}{n}\sum_{m=N_1+1}^{n}(u_m - u_0)\right|$$

$$\leq \left|\frac{1}{n}\sum_{m=1}^{N_1}(u_m - u_0)\right| + \left|\frac{1}{n}\sum_{m=N_1+1}^{n}(u_m - u_0)\right|.$$

Then

$$\left|\frac{1}{n}\sum_{m=1}^{N_1}(u_m - u_0)\right| \leq \frac{1}{n}\sum_{m=1}^{N_1}|u_m - u_0|$$

$$\leq \frac{1}{n}\sum_{m=1}^{N_1}\max_{1\leq i\leq N_1}|u_i - u_0|$$

$$= \max_{1\leq i\leq N_1}|u_i - u_0| \frac{1}{n}\sum_{m=1}^{N_1}1$$

$$= \max_{1\leq i\leq N_1}|u_i - u_0| \frac{N_1}{n}.$$

If we pick n such that

$$n > \max_{1\leq i\leq N_1}|u_i - u_0| \frac{N_1}{\varepsilon_1}.$$

we get

$$\left|\frac{1}{n}\sum_{m=1}^{N_1}(u_m - u_0)\right| < \varepsilon_1. \tag{3.1}$$

Further we may write

$$\left|\frac{1}{n}\sum_{m=N_1+1}^{n}(u_m - u_0)\right| \leq \frac{1}{n}\sum_{m=N_1+1}^{n}|u_m - u_0|$$

$$\leq \frac{1}{n}\sum_{m=N_1+1}^{n}\max_{N_1+1\leq i\leq n}|u_i - u_0|$$

$$= \max_{N_1+1\leq i\leq n}|u_i - u_0| \frac{1}{n}\sum_{m=N_1+1}^{n}1$$

$$= \max_{N_1+1\leq i\leq n}|u_i - u_0| \frac{n - N_1}{n}$$

$$\leq \max_{N_1+1\leq i\leq n}|u_i - u_0|$$

and since $u_n \to u_0$ as $n \to \infty$ then we may choose N_1 such that $|u_i - u_0| < \varepsilon_1$ for all n> N_1 so that

$$\left|\frac{1}{n}\sum_{m=N_1+1}^{n}(u_k - u_0)\right| \leq \varepsilon_1 . \qquad (3.2)$$

Therefore

$$|v_n - u_0| \leq \left|\frac{1}{n}\sum_{m=1}^{N_1}(u_m - u_0)\right| + \left|\frac{1}{n}\sum_{m=N_1+1}^{n}(u_m - u_0)\right| < \varepsilon_1 + \varepsilon_1 = 2\varepsilon_1 = \varepsilon$$

for all n> N_1 by using (3.1) and (3.2). Hence $\lim_{n\to\infty} v_n = u_0$.

Note The invers of the above theorem is not true. For example the sequence $u_m=(-1)^m$ is Cesaro summable to zero but it is not convergent.

Definition 3.2 A sequence (u_n) is called strongly Cesàro summable if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}|u_m - l| = 0.$$

For this situation, we say that u is strongly Cesàro summable to l. The collection of all strongly Cesàro summable sequences will be represented by [C,1] **[7]**.

Definition 3.3 Let $u = (u_n) \in w$ and let $p \in \mathbb{R}^+$. The sequence $u = (u_n)$ is called strongly p-Cesàro summable if there is a number u_0 such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}|u_m - u_0|^p = 0 \qquad (3.3)$$

In this situation we say that u is strongly p-Cesàro summable to u_0. The collection of all strongly p-Cesàro summable sequences is represented by w_p **[7]**.

Theorem 3.3 Let $p \in \mathbb{R}^+$. If a sequence is strongly p-Cesàro summable to u_0, then it is statly. convergent to u_0 **[7]**.

Proof Since for any sequence $u = (u_n)$ and $\varepsilon > 0$, we have

$$\sum_{m=1}^{n}|u_m - u_0|^p \geq |\{m < n : |u_m - u_0|^p \geq \varepsilon\}| . \varepsilon^p.$$

We may write

$$\frac{1}{n}\sum_{m=1}^{n}|u_m - u_0|^p \geq \frac{1}{n}|\{m < n : |u_m - u_0|^p \geq \varepsilon\}| . \varepsilon^p.$$

By taking limit at both side as $n \to \infty$ we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}|u_m-u_0|^p \geq \lim_{n\to\infty}\frac{1}{n}|\{m<n: |u_m-u_0|^p \geq \varepsilon\}|.\varepsilon^p \qquad (3.4)$$

Since the sequence u is strongly p-Cesàro summable to u_0 then from (3.3) and (3.4) we get that

$$\lim_{n\to\infty}\frac{1}{n}|\{m<n: |u_m-u_0|^p \geq \varepsilon\}| = 0.$$

It means that u is statly. convergent to u_0.

Remark 3.1 A statly. convergent sequence may not be strongly Cesàro summable. For example the sequence (u_n) defined by

$$u_n = \begin{cases} \sqrt{m}\,, & m = n^2 \\ \\ 0\,, & \text{otherwise} \end{cases} \qquad n = 1,2,3,\ldots$$

is statly. converges to zero but since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}|u_m-0| \neq 0$$

it is not strongly Cesàro summable to zero.

4. STATISTICAL CONVERGENCE OF ORDER α AND STATISTICAL BOUNDEDNESS OF ORDER α

Definition.4.1 Let $H \subseteq \mathbb{N}$ and $0 < \alpha \leq 1$, We define $\delta_\alpha(H)$ by

$$\delta_\alpha(H) = \lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : m \in H\}|. \tag{4.1}$$

If the limit exists, then the number $\delta_\alpha(H)$ is called the α-density of set H. It is easy to observe that if H is a finite subset of ℕ then $\delta_\alpha(H)=0$ but $\delta_\alpha(H^c)\neq 1-\delta_\alpha(H)$ *for* $0<\alpha<1$ in general. The equality $\delta_\alpha(H^c)=1-\delta_\alpha(H)$ is satisfied for α=1. A set H is called statistically dense if $\delta(H)=1$ **[12]**.

Lemma.4.1 Let $H\subseteq \mathbb{N}$. Then $\delta_\beta(H)\leq \delta_\alpha(H)$ if $0<\alpha<\beta\leq 1$ **[12]**.

Proof Suppose that $0<\alpha\leq\beta\leq 1$. Then $n^\alpha \leq n^\beta$ implies that $\frac{1}{n^\beta} \leq \frac{1}{n^\alpha}$ then for every $n\in\mathbb{N}$ we have

$$\lim_{n\to\infty} \frac{1}{n^\beta} |\{m \leq n : m \in H\}| \leq \lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : m \in H\}|$$

It is mean that $\delta_\beta(H)\leq\delta_\alpha(H)$.

Note that from above Lemma if $\delta_\alpha(H)=0$ then also $\delta_\beta(H)=0$ for any α, β ∈(0,1] such that α≤β.

Definition 4.2 Let α∈(0,1]. a sequence $u = (u_m)$ is said to be statly. bounded of order α if there exists some M>0, such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : |u_m| > \mathrm{M}\}| = 0.$$

The collection of all sequences which are statly. bounded of order α will be represented by SB^α.

Theorem 4.1 Any bounded number sequence is statly. bounded of order α for each α∈(0,1].

Proof Let $u = (u_m)$ be any bounded sequence. Then there is a number $M>0$ such that

$|u_m| \leq M$ for any $m \in \mathbb{N}$, so we have

$$\{m \leq n : m \in \mathbb{N} : |u_m| > M\} = \emptyset\,.$$

hence,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : |u_m| > M\}| = 0$$

for any α∈(0,1] and it means that u is statly. bounded of order α.

The inverse of above Theorem is not right. For example, let us define a sequence (u_m) as

$$u_m = \begin{cases} m, & \text{if } m = n^2 \\ \frac{1}{m}, & \text{otherwies} \end{cases} \qquad n = 1,2.3.4. \ldots.. \tag{4.2}$$

It is easily seen that the sequence $u = \left\{1, \frac{1}{2}, \frac{1}{3}, 2, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, 3, \ldots\right\}$ is not bounded. To show that $u = (u_m)$ is statly. bounded of order α, choose M=1. Then

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \le n: |u_m| > M\}| = \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{n^\alpha} = 0\,,$$

for 1>α>1/2. It means that the sequence (u_m) is statly. bounded of order α for any α∈(1/2.1].

Note that from above theorem we can say $l_\infty \subseteq SB^\alpha$

Theorem 4.2 If a sequence $u = (u_m)$ is a statly. bounded of order α then it is also statly. bounded of order β, where 0<α<β≤1.

Proof Obviously the inequality

$$\frac{1}{n^\beta} \le \frac{1}{n^\alpha} \tag{4.3}$$

is satisfied for any α, β ∈(0,1] such that 0<α<β≤1. Since the sequence $u = (u_m)$ is statly. bounded of order α, then we have

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \le n: |u_m| > M\}| = 0.$$

From this and using (4.3) we may write

$$\lim_{n\to\infty} \frac{1}{n^\beta} |\{m \le n: |u_m| > M\}| \le \lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \le n: |u_m| > M\}|$$

and so that

$$\lim_{n\to\infty} \frac{1}{n^\beta} |\{m \le n: |u_m| > M\}| = 0$$

which implies that the sequence $u = (u_m)$ is a statly. bounded of order β.

Note that the reverse of above theorem is not right. For example, the sequence $u = (u_m)$ defined in (4.2) is not statly. bounded of order β if we take 0<β<1/2.

Definition 4.3 A sequence $u = (u_m)$ is called statly. convergent of order α (where 0<α≤1) to a real number l if the following condition is satisfied for every $\varepsilon > 0$:

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{m \le n : |u_m - l| \ge \varepsilon\}| = 0.$$

In this situation we write S^{α}-$lim\ u_n = l$ or $u_n \xrightarrow{S^{\alpha}} l$. And S^{α} represents the collection of all statly. convergent sequences of order α and S_0^{α} represents the collection of all statly. null sequences of order α. We recall that if a sequence $u = (u_m)$ accepts property P for all m excepting a set of α-density zero, then we say that $u = (u_m)$ accepts property P for "almost all m according to α" and use abbreviation "a.a.m(α)" for this **[12]**.

Theorem 4.3 Let α, β∈(0,1] be given. Then ever sequence which is statly. convergent of order α is also statly. bounded of order β, that is $S^{\alpha} \subset SB^{\alpha}$ if α≤β.

Proof. Let α, β∈(0,1] be given and let α≤β. Let the sequence (u_m) be statly. convergent of order α to l. Since for any arbitrary $0<\varepsilon< M$ we have

$\{m \le n : |u_m - l| > M\} \subseteq \{m \le n : |u_m - l| \ge \varepsilon\}$.

This inclusion gives the inequality

$|\{m \le n : |u_m - l| > M\}| \ \le |\{m \le n : |u_m - l| \ge \varepsilon\}|$.

and so that

$$\lim_{n\to\infty} \frac{1}{n^{\beta}} |\{m \le n : |u_m - l| > M\}| \ \le \lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{m \le n : |u_m - l| \ge \varepsilon\}|. \tag{4.4}$$

Since (u_m) is statly. convergent of order α then

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{m \le n : |u_m - l| \ge \varepsilon\}| = 0 \tag{4.5}$$

for every $\varepsilon > 0$. Hence from (4.4) and (4.5) we get

$$\lim_{n\to\infty} \frac{1}{n^{\beta}} |\{m \le n : |u_m - l| > M\}| = 0.$$

So the sequence (u_m) is statly. bounded of order β.

Taking β= α in Theorem 4.3 we obtain the following result.

Corollary 4.1 Let α∈(0,1] be given. Then every sequence which is statly. convergent of order α is also statly. bounded of order α.

The reverse of the above Corollary 4.1 is not true. For this let $u = (u_m)$ be the sequence such that

$$u_m = \begin{cases} a, & \text{if } m = 2i+1 \\ b, & \text{if } m = 2i \end{cases} \qquad i = 1,2.3.4. \dots.. \tag{4.6}$$

where a, b∈R and a≠b. Since (u_m) is statly. bounded of order α, because if we choose $M \ge 2max\{|a|, |b|\}$. Then

$$\{m \leq n : m \in \mathbb{N} : |u_m| > M\} = \emptyset$$

so that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : |u_m| > M\}| = 0.$$

Now,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : n = 2m\}| = \lim_{n\to\infty} \frac{n}{2n^\alpha} = \infty \quad \text{for } 0 < \alpha < 1\,.$$

or

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : |u_m - b| \geq \varepsilon\}| = \lim_{n\to\infty} \frac{n}{2n^\alpha} = \infty \quad \text{for } 0 < \alpha < 1$$

and

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : n = 2m + 1\}| = \lim_{n\to\infty} \frac{n}{2n^\alpha} = \infty \quad \text{for } 0 < \alpha < 1.$$

Thus $u = (u_m)$ is not statly. convergent of order α to a or to b. Consequently (u_m) is not statly. convergent of order α.

Lemma 4.1 Let $0 < \alpha \leq 1$ be given. Then

1) $S_0^\alpha \subseteq S^\alpha$.

2) Every convergent sequence is statly. convergent of order α for every α∈(0,1] **[12]**.

Proof The proof of (1) is clear.

2) Let $u = (u_m) \to l$ as $n \to \infty$. Then for every $\varepsilon > 0$ there exists N∈ℕ such that $|u_n - l| < \varepsilon$ for all $n > N$, so that we have

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : |u_m - l| \geq \varepsilon\}| = \lim_{n\to\infty} \frac{N}{n^\alpha} = 0$$

and hence the sequence $u = (u_m)$ is statly. convergent of order α. In another word we have $c \subseteq S^\alpha$.

But the converse of this Lemma is not right. For this let us consider the sequence defined as

$$u_n = \begin{cases} 1, & k = n^5 \\ 0, & \text{otherwies} \end{cases} \qquad n = 1,2,3,4, \ldots\ldots$$

Since

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : |u_m - 0| \geq \varepsilon\}| = \lim_{n\to\infty} \frac{n^{\frac{1}{5}}}{n^\alpha} = 0$$

the sequence (u_m) is statly. convergent of order α to zero, for α>1/5. But this sequence is not convergent.

Lemma 4.2 Let α, β∈(0,1] be given. Then

1) Every statly. convergent sequence of order α is also statly. convergent of order β with same limit, i.e. $S^{\alpha} \subseteq S^{\beta}$ if $\alpha \leq \beta$ and the inclusion may remain strict if there exists a number j∈ℕ such that $\alpha < \frac{1}{j} < \beta$.

2) $S^{\alpha} \subseteq S$ for every α∈(0,1] (**[12],** [**15**]).

Proof (1) Suppose $\alpha \leq \beta$ and so that $\frac{1}{n^{\beta}} \leq \frac{1}{n^{\alpha}}$ and let the sequence (u_m) be statly. convergent of order α to the number *l*. Writing

$$\frac{1}{n^{\beta}}|\{m \leq n: |u_m - l| \geq \varepsilon\}| \leq \frac{1}{n^{\alpha}}|\{m \leq n: |u_m - l| \geq \varepsilon\}|.$$

we have $S^{\alpha} \subseteq S^{\beta}$. In order *t*o prove the inclusion is strict we consider the sequence

$u = (u_m)$ as

$$u_m = \begin{cases} 1, & \text{if } m = n^2 \\ 0, & \text{otherwies} \end{cases} \qquad n = 1,2,3,4, \ldots\ldots$$

Then, S^{β}-$lim\ u_m = 0$, that is $u \in S^{\beta}$ for $1/2 < \beta \leq 1$ but $u \notin S^{\alpha}$ for $0 < \alpha < 1/2$.

(2) If we take β=1 in (1) we have $S^{\alpha} \subseteq S$.

Remark 4.1 Stat. convergence of order α is not defined for α>1 **[12]**.

This follows from the fact that

$$\delta(U) = \lim_{n\to\infty} \frac{1}{n}|\{m \leq n: m \in U\}| = 0$$

for every subset U ⊆ℕ if α>1.

Also to see this, we may choose the following example.

Let $u = (u_m)$ be the sequence given in (4.6), where *a*, *b*∈ ℝ and $a \neq b$. If we take α>1, then we have

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}}|\{m \leq n: |u_m - a| \geq \varepsilon\}| = \lim_{n\to\infty} \frac{n}{2n^{\alpha}} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}}|\{m \leq n: |u_m - b| \geq \varepsilon\}| = \lim_{n\to\infty} \frac{n}{2n^{\alpha}} = 0.$$

It is means that (u_m) is statly. convergent to both a and b of order α, i.e.

S^{α}-$lim\ u_m = a$ *and* S^{α}-$lim\ u_m = b$ if α>1. But this is not possible.

Lemma 4.3 Let (u_m), (v_m) be any sequences such that S^{α}-$lim\ u_m = a$, S^{α}-$lim\ v_m = b$ and c be any real number. Then.

1. S^{α}-limcu_m=ca.
2. S^{α}-lim (u_m+v_m)=$a+b$ **[12].**

Proof

1) Since

$$|\{m \leq n: |cu_m - ca| \geq \varepsilon\}| = \left|\left\{m \leq n: |u_m - a| \geq \frac{\varepsilon}{|c|}\right\}\right|$$

and

$$\frac{1}{n^{\alpha}}|\{m \leq n: |cu_m - ca| \geq \varepsilon\}| = \frac{1}{n^{\alpha}}\left|\left\{m \leq n: |u_m - a| \geq \frac{\varepsilon}{|c|}\right\}\right|$$

we have

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}|\{m \leq n: |cu_m - ca| \geq \varepsilon\}| = \lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\left\{m \leq n: |u_m - a| \geq \frac{\varepsilon}{|c|}\right\}\right| = 0$$

so that S^{α}-lim cu_m=ca.

2) Since

$$|\{m \leq n: |(u_m + v_m) - (a+b)| \geq \varepsilon\}| = |\{m \leq n: |(u_m - a) + (v_m - b)| \geq \varepsilon\}|$$
$$\leq \left|\left\{m \leq n: |u_m - a| \geq \frac{\varepsilon}{2}\right\}\right| + \left|\left\{m \leq n: |v_m - b| \geq \frac{\varepsilon}{2}\right\}\right|$$

by taking limit to both side where n goes to infinity then we get

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}|\{m \leq n: |(u_m + v_m) - (a+b)| \geq \varepsilon\}| \leq \lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\left\{m \leq n: |u_m - a| \geq \frac{\varepsilon}{2}\right\}\right|$$
$$+ \lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\left\{m \leq n: |v_m - b| \geq \frac{\varepsilon}{2}\right\}\right|$$

which implies that

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}|\{m \leq n: |(u_m + v_m) - (a+b)| \geq \varepsilon\}| = 0.$$

so that S^{α}-lim (u_m+v_m)=$a+b$.

Definition 4.4 Let $\alpha \in \mathbb{R}^{+}$ such that $0<\alpha \leq 1$ and Let $u = (u_m)$ be any real number sequence. The sequence $u = (u_m)$ is said to be statly. Cauchy sequence of order α if there exists a number N∈ℕ such that

$$|u_m - u_N| \leq \varepsilon$$

for a. a. m(α)., i.e.

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}|\{m \leq n: |u_m - u_N| \geq \varepsilon\}| = 0\,,$$

for every $\varepsilon > 0$ [**11**].

Theorem 4.4 Let 0 < α ≤ 1 be given. Then the sequence $u = (u_m)$ of real numbers is statly. convergent of order α if and only if it is statly. Cauchy sequence of order α.

Proof Suppose that the sequence $u = (u_m)$ is statly. convergent of order α to l. Then we have

$$|u_n - l| < \frac{\varepsilon}{2} \text{a.a.m}(\alpha) \Rightarrow \lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\left\{m \le n: |u_m - l| \ge \frac{\varepsilon}{2}\right\}\right| = 0. \qquad (4.7)$$

Given $\varepsilon > 0$, choose a number N=N(ε) so that

$$|x_N - l| \le \frac{\varepsilon}{2} \Rightarrow \lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\left\{m \le n: |u_N - l| \ge \frac{\varepsilon}{2}\right\}\right| = 0. \qquad (4.8)$$

We need to show that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{m \le n: |u_m - u_N| \ge \varepsilon\}| = 0$$

for every $\varepsilon > 0$.

Now

$$|\{m \le n: |u_m - u_N| \ge \varepsilon\}| = |\{m \le n: |u_m - u_N + l - l| \ge \varepsilon\}|$$
$$\le \left|\left\{m \le n: |u_m - l| \ge \frac{\varepsilon}{2}\right\}\right| + \left|\left\{m \le n: |u_N - l| \ge \frac{\varepsilon}{2}\right\}\right|$$

implies that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{m \le n: |u_m - u_N| \ge \varepsilon\}| \le \lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\left\{m \le n: |u_m - l| \ge \frac{\varepsilon}{2}\right\}\right| +$$
$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\left\{m \le n: |u_N - l| \ge \frac{\varepsilon}{2}\right\}\right| \qquad (4.9)$$

and by using (4.7) and (4.8) in (4.9) we get

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{m \le n: |u_m - u_N| \ge \varepsilon\}| = 0\,. \qquad (4.10)$$

Hence the sequence (u_m) is statly. Cauchy of order α.

Conversely suppose that the sequence $u = (u_m)$ is statly. Cauchy of order α, i.e.

$|u_m - u_N| \le \varepsilon$ a.a.m(α).

Given ε=1 choose N_1 such that

$$|u_m - u_{N_1}| \le 1 \quad \text{a.a.m}(\alpha) \Rightarrow u_m \in [u_{N_1} - 1, u_{N_1} + 1] \text{ a.a.m}(\alpha)$$

Also Given ε=$\frac{1}{2}$ choose M such that

$$|u_m - u_M| \le \frac{1}{2} \quad \text{a.a.m}(\alpha) \Rightarrow x_k \in \left[u_M - \frac{1}{2}, u_M + \frac{1}{2}\right] \text{ a.a.m}(\alpha)$$

Let $\text{I} = [u_{N_1} - 1, u_{N_1} + 1]$ and $\text{I}' = [u_M - 1/2. u_M + 1/2]$ and consider $\text{I}_1 = \text{I} \cap \text{I}'$. Then $u_m \in \text{I}_1$ a.a.m(α). Because we have$\{m \le n: u_m \notin \text{I} \cap \text{I}'\} = \{m \le n: u_m \notin \text{I}\} \cup$

$\{k \leq n : u_m \notin \mathrm{I}'\}$. Therefore

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : u_m \notin \mathrm{I} \cap \mathrm{I}'\}| \leq \lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : u_m \notin \mathrm{I}\}| + \lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : u_m \notin \mathrm{I}'\}|$$

and this implies that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : u_m \notin \mathrm{I} \cap \mathrm{I}'\}| = 0.$$

Therefore I_1 is a closed interval of length less than or equal to 1 that contains u_m a.a.m(α). Now continuing in this way by choosing N(2) so that $I'' = [u_{N(2)} - 1/4. u_{N(2)} + 1/4]$, $u_m \in I''\ a.a.m(\alpha)$ and from the previous argument. Let $I_2 = I_1 \cap I''$ then $u_m \in I_2\ \mathrm{a.a.m}(\alpha)$ and I_2 has length less than or equal to 1/2 . We obtain a sequence (I_m) of closed interval such that for every m, $I_{m+1} \subseteq I_m$ the length of I_m is not larger than 2^{1-m}, and $u_m \in I_m\ \ a.a.m(\alpha)$. By the nested Intervals Theorem there is a number λ such that $\cap I_m = \{\lambda\}$. Using the fact that $u_m \in I_m\ \ a.a.m(\alpha)$ we take an increasing sequence (T_m) of positive integers such that

$$\frac{1}{n^\alpha} |\{m \leq n : u_m \notin I_m\}| < \frac{1}{m} \text{ if } n > T_m \tag{4.11}$$

Now define a subsequence z of u consisting of all terms u_m such that $m>T_1$ and if $T_m < m \leq T_{m+1}$ then $u_m \notin I_m$. Next we define a sequence as follows

$$v_m = \begin{cases} \lambda, & \text{if } u_m \text{ is a term of } z \\ u_m, & \text{otherwies} \end{cases}.$$

Then $\lim_{m\to\infty} v_m = \lambda$ if $\varepsilon > \frac{1}{m} > 0$ and $k > T_m$. Then if u_m is a term of z, then $v_m = \lambda$

else $v_n = u_k \in I_m$ and $|v_m - \lambda| \leq$ lenght of $I_m \leq 2^{1-m}$. We also assert that $v_m = u_m\ \mathrm{a.a.m}(\alpha)$. To confirm this we note that if $T_m < m \leq T_{m+1}$then

$\{m \leq n : v_m \neq u_m\} \subseteq \{m \leq n : u_m \notin I_m\}$. So by (4.11) we may write

$$\{m \leq n : v_m \neq u_m\} \subseteq \{m \leq n : u_m \notin I_m\}$$

$$\Rightarrow \frac{1}{n^\alpha} |\{m \leq n : v_m \neq u_m\}| \leq \frac{1}{n^\alpha} |\{m \leq n : u_m \notin I_m\}| < \frac{1}{m}.$$

From this inequality we get

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{m \leq n : v_m \neq u_m\}| = 0$$

and hence the sequence $u = (u_m)$ is statly. convergent of order α, since $v_m = u_m\ \ \mathrm{a.a.m}(\alpha)$ and $\lim_{m\to\infty} v_m = \lambda$.

5. CESÀRO SUMMABILITY AND STRONG p-CESÀRO SUMMABILITY OF ORDER α

Definition 5.1 Let *α >0.* A sequence $u = (u_n)$ is called Cesàro summable of order α to *l* if the following condition satisfies

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{m=1}^{n} u_m = l.$$

Obviously that the sequence $u = (u_m)$ is Cesàro summable of order α to zero if the series $\sum_{m=1}^{\infty} u_m$ is convergent. And it is same with Cesàro summability for *α*=1 **[12]**.

Note: Let *α* >0. Then

1) The set of Cesàro summable sequences is a subset of the sequences which are Cesàro summable of order α.

2) Every convergent sequence is Cesàro summable but this is not true for Cesàro summable of order α, in case *α* ≠1. For example, the sequence $u = (u_n) = (3)$ is convergent but since

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{m=1}^{n} u_m = \lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{m=1}^{n} 3 = \lim_{n\to\infty} \frac{3n}{n^\alpha} = \infty$$

if *α* <1. So that $(u_n) = (3)$ is not Cesàro summable of order α in case *α* <1.

Definition 5.2 Let a real number *α >0* be given. A sequence $u = (u_n)$ is called strongly Cesàro summable of order α if

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{m=1}^{n} |u_m - l| = 0$$

for some real number *l*. The collection of all sequences which are strongly Cesàro summable of order α is represented by $[C^\alpha,1]$ **[12]**.

Definition 5.3 Let $p \in \mathbb{R}^+$ and *α >0.* A sequence $u = (u_n)$ is strongly *p-Cesàro summable* of order α if the following condition is satisfied for some *l*:

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{m=1}^{n} |u_m - l|^p = 0.$$

The strong p-Cesàro summability of order α reduces to strong p-Cesàro summability for α=1. The set of all strongly p-Cesàro summable sequences of order α will be represented by w_p^α and w_{0p}^α *is* the collection of strongly p-Cesàro summable null sequences of order α. It is clear that $w_{0p}^{\alpha} \subseteq w_p^{\alpha}$. In case p=1 w_p^{α} will be [C^{α},1] and w_p^{α} will be [C,1] ıf p=1 and α=1 **[12]**.

Theorem 5.1 If $0 < \alpha \leq \beta$ then $w_p^{\alpha} \subseteq w_p^{\beta}$ and the inclusion may remain strict for some $\alpha < \beta$ **[12]**.

Proof Suppose $p>0$ and $u \in w_p^{\alpha}$. Since $0 < \alpha \leq \beta$, then we may write $\frac{1}{n^{\beta}} \leq \frac{1}{n^{\alpha}}$ which implies that

$$\frac{1}{n^{\beta}} \sum_{m=1}^{n} |u_m - l|^p \leq \frac{1}{n^{\alpha}} \sum_{m=1}^{n} |u_m - l|^p$$

It means that $w_p^{\alpha} \subseteq w_p^{\beta}$. To prove that the inclusion is strict let us define a sequence (u_m) as follows:

$$u_m = \begin{cases} 1, & m = i^2 \\ 0, & \text{otherwies} \end{cases} \qquad i=1,2.3\ldots.$$

Since

$$\frac{1}{n^{\beta}} \sum_{m=1}^{n} |u_m - 0|^p \leq \frac{\sqrt{n}}{n^{\beta}} = \frac{1}{n^{\beta - \frac{1}{2}}} \to 0$$

$as\ n \to \infty$ for ½< β, then $u = (u_n)$ is strongly p-Cesàro summable of order β i.e. $u \in w_p^{\beta}$. Now since

$$\frac{\sqrt{n} - 1}{n^{\alpha}} \leq \frac{1}{n^{\alpha}} \sum_{m=1}^{n} |u_m - 0|^p$$

and since $\frac{\sqrt{n}-1}{n^{\alpha}} \to \infty \quad as\ n \to \infty$, then $u \notin w_p^{\alpha}$ for $\alpha < 1/2$.

Corollary 5.1 Let $\alpha > 0$ be given and p>0, then we have $w_p^{\alpha} \subseteq w_p$, for every $0 < \alpha \leq 1$ **[12]**.

Proof By take β=1 in above theorem we obtain the proof..

Theorem 5.2 Let $0 < \alpha \leq \beta \leq 1$, p>0 and the sequence u be strongly p-Cesàro summable of order α to l, then it is statly. convergent of or.der β to l **[12]**.

Proof Since for any sequence $u = (u_n)$ and $\varepsilon > 0$, we have

$$\sum_{m=1}^{n} |u_m - l|^p \geq |\{m < n: |u_m - l|^p \geq \varepsilon\}|.\, \varepsilon^p$$

We may write

$$\frac{1}{n^{\alpha}}\sum_{m=1}^{n}|u_m-l|^p \geq \frac{1}{n^{\alpha}}|\{m<n: |u_m-l|^p \geq \varepsilon\}|.\,\varepsilon^p$$

$$\geq \frac{1}{n^{\beta}}|\{m<n: |u_m-l|^p \geq \varepsilon\}|.\,\varepsilon^p \tag{5.1}$$

Since the sequence $u=(u_n)$ is strongly p-Cesàro summable of order α to *l* then we have

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\sum_{m=1}^{n}|u_m-l|^p=0,$$

and from inequality (5.1) we get

$$\lim_{n\to\infty}\frac{1}{n^{\beta}}|\{m<n: |u_m-l|^p \geq \varepsilon\}|=0.$$

It means that $u=(u_n)$ is statly. convergent of order β to l.

Remark 5.1 Note that the reverse of the above Theorem is not true in general **[12]**.

For example the sequence (u_n) defined by

$$u_n=\begin{cases} 1, & m=n^3 \\ \frac{1}{\sqrt{n}}, & \text{otherwiese}\end{cases}$$

is statly. convergent of order α to zero for $1/3<\alpha\leq 1$.

For every $n\geq 2$ we have the inequality

$$\sum_{m=1}^{n}\frac{1}{\sqrt{m}}>\sqrt{n}\,.$$

Define $E_n=\{m\leq n: m\neq n^{3}.\ n=1,2.3.\ldots\}$ and take p=1. Since

$$\sum_{m=1}^{n}|u_m|^p=\sum_{k=1}^{n}|u_m|=\sum_{\substack{k\in E_n\\1\leq k\leq n}}|u_m|+\sum_{\substack{k\notin E_n\\1\leq k\leq n}}|u_m|$$

$$=\sum_{\substack{m\in E_n\\1\leq m\leq n}}\frac{1}{\sqrt{m}}+\sum_{\substack{m\notin E_n\\1\leq m\leq n}}1>\sum_{m=1}^{n}\frac{1}{\sqrt{m}}>\sqrt{n}$$

$$\frac{1}{n^{\alpha}}\sum_{m=1}^{n}|u_m|^p=\frac{1}{n^{\alpha}}\sum_{m=1}^{n}|u_m|>\frac{1}{n^{\alpha}}\sum_{m=1}^{n}\frac{1}{\sqrt{m}}>\frac{1}{n^{\alpha}}\sqrt{n}=\frac{1}{n^{\alpha-\frac{1}{2}}}$$

and $\frac{1}{n^{\alpha-\frac{1}{2}}}\to\infty$ as $n\to\infty$ we have $u\notin w_p^{\alpha}$ for $0<\alpha\leq 1/2$. But $u\in S^{\alpha}$ for $1/3<\alpha\leq 1/2$.

6. λ - STATISTICAL CONVERGENCE AND λ - STATISTICAL CONVERGENCE OF ORDER α

Let $\lambda = (\lambda_n)$ be a non- decreasing sequence of positive real numbers such that $\lambda_1=1$, $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_n \to \infty$ as $n \to \infty$.The collection of all such sequences will be denoted by Λ.

Definition 6.1 A sequence $u = (u_m)$ is said to be λ-statly. bounded if there exists some $K > 0$, such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{m \in I_n : |u_m| > K\}| = 0$$

where $I_n = [n - \lambda_n + 1, n]$. The collection of all λ-statly. bounded sequences is represented by SB_λ.

Definition 6.2 Let a sequence $\lambda = (\lambda_n) \in \Lambda$ be given. A sequence $u = (u_m)$ of real numbers is said to be λ-statly. convergent to l if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{m \in I_n : |u_m - l| \geq \varepsilon\}| = 0.$$

In this status we write $S_\lambda - \lim_{n\to\infty} u = l$. The collection of all λ-statly. convergent sequences will be denoted by S_λ and $S_{\lambda,0}$ is the set of all λ-statly. null sequences. Note that if we take $\lambda_n = n$ then $S_\lambda = S$ **[9]** .

Theorem 6.1 Every bounded number sequence is λ -statly. bounded, that is $l_\infty \subseteq SB_\lambda$ for each $\lambda \in \Lambda$.

Proof Let $u = (u_m)$ be a bounded sequence. Then there is a real number $M > 0$ such that

$$|u_m| < M$$

for all $n \in \mathbb{N}$ and this implies that

$$\{m \in I_n :\ m \in \mathbb{N}, |u_m| \geq M\} = \emptyset$$

$$\Rightarrow |\{m \in I_n :\ m \in \mathbb{N}, |u_m| \geq M\}| = 0$$

Theorem 6.2 Every λ-statly. convergent sequence is λ-statly. bounded.

Proof Suppose that the sequence $u = (u_m)$ is λ-statly. convergent. For any arbitrary $\varepsilon > 0$ and a large number $M > 0$ we have

$$|\{m \in I_n : |u_m - l| \geq M\}| \leq |\{m \in I_n : |u_m - l| > \varepsilon\}|$$

and so that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{m\in I_n: |u_m-l|\geq M\}|\leq \lim_{n\to\infty}\frac{1}{\lambda_n}|\{m\in I_n: |u_m-l|\geq \varepsilon\}| \qquad (6.1)$$

for any $\lambda=(\lambda_n)\in \Lambda$. Since (u_m) is λ-statly. convergent sequence we have

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{m\in I_n: |u_m-l|\geq \varepsilon\}|=0$$

And from (6.1) we get that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{m\in I_n: |u_m-l|\geq M\}|=0.$$

Since $|u_m-l|\geq$M implies $|u_m|\geq$M+$|l|$ we conclude that the sequence $u=(u_m)$ is λ-statly. bounded.

Note reverse of the above Theorem is not true. For example, the sequence $u=(u_m)$ defined as

$$u_m=\begin{cases} a\,, & \text{if } m \text{ is even}\\ b\,, & \text{if } m \text{ is odd}\end{cases}$$

obviously is λ-statly. bounded where a≠b. Since

$$0\neq\frac{1}{2}=\lim_{n\to\infty}\frac{\lambda_n-1}{2\lambda_n}\leq \lim_{n\to\infty}\frac{1}{\lambda_n}|\{m\in I_n: |u_m-a|>\varepsilon\}|$$

and

$$0\neq\frac{1}{2}=\lim_{n\to\infty}\frac{\lambda_n-1}{2\lambda_n}\leq \lim_{n\to\infty}\frac{1}{\lambda_n}|\{m\in I_n: |u_m-b|>\varepsilon\}|$$

The sequence $u=(u_m)$ is not λ-statly. convergent.

Theorem 6.3 Let (u_m), (v_m) be any sequences of real numbers such that $S_\lambda\text{-}lim\ u_m=a$, $S_\lambda\text{-}lim\ v_m=b$ and c be any real number. Then

i) $S_\lambda\text{-}lim\ cu_m=ca$.

ii) $S_\lambda\text{-}lim\ (u_m+v_m)=a+b$ **[14]**.

Proof *i)* Since

$$|\{m\leq n: |cu_m-ca|\geq\varepsilon\}|=\left|\left\{m\leq n: |u_m-a|\geq\frac{\varepsilon}{|c|}\right\}\right|$$

$$\frac{1}{\lambda_n}|\{m\leq n: |cu_m-ca|\geq\varepsilon\}|=\frac{1}{\lambda_n}\left|\left\{m\leq n: |u_m-a|\geq\frac{\varepsilon}{|c|}\right\}\right|$$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{m\leq n: |cu_m-ca|\geq\varepsilon\}|=\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{m\leq n: |u_m-a|\geq\frac{\varepsilon}{|c|}\right\}\right|=0$$

It means that $S_\lambda\text{-}lim\ cu_m=ca$.

iii)

$$|\{m\leq n: |(u_m+v_m)-(a+b)|\geq\varepsilon\}|=|\{m\leq n: |(u_m-a)+(v_m-b)|\geq\varepsilon\}|$$

$$\leq \left|\left\{m \leq n: |u_m - a| \geq \frac{\varepsilon}{2}\right\}\right| + \left|\left\{m \leq n: |v_m - b| \geq \frac{\varepsilon}{2}\right\}\right|$$

implies that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{m \leq n: |(u_m + \ v_m\ ) - (a+b)| \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{m \leq n: |u_m - a| \geq \frac{\varepsilon}{2}\right\}\right|$$

$$+ \lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{m \leq n: |v_m - b| \geq \frac{\varepsilon}{2}\right\}\right|$$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{m \leq n: |(u_m + \ v_m\ ) - (a+b)| \geq \varepsilon\}| = 0.$$

It means that S^{α}-$lim\ (u_m + v_m) = a+b$.

Definition 6.4 Let a sequence $\lambda = (\lambda_n) \in \Lambda$ and a real number $\alpha \in (0,1]$ be given. Then a sequence $u = (u_m)$ is said to be λ-statly. bounded of order α if there exists some $K > 0$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |\{m \in I_n: |u_m| > K\}| = 0.$$

The set of all λ-statly. bounded sequences denoted by SB_{λ}^{α}.

Definition 6.5 Let a sequence $\lambda = (\lambda_n) \in \Lambda$ and a real number $\alpha \in (0,1]$ be given and define $\lambda^{\alpha} = (\lambda_n^{\alpha}) = \{\lambda_1^{\alpha}, \lambda_2^{\alpha}, \lambda_3^{\alpha}, \dots\}$. Then the sequence $u = (u_m)$ is said to be λ-statly. convergent of order α if

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |\{m \in I_n: |u_m - l| \geq \varepsilon\}| = 0.$$

In this situation we write $S_{\lambda}^{\alpha} - \lim_{n\to\infty} u = l$. The set of all λ-statly. convergent sequences of order α will be represented by S_{λ}^{α} and $S_{\lambda,0}^{\alpha}$ is the set of all λ-statly. null sequences of order α **[14]** .

Note: λ-statly. convergence of order α is not meaningful for α>1 in general.

Theorem 6.4

1) $l_{\infty} \subset SB_{\lambda}^{\alpha}$.

2) $S_{\lambda}^{\alpha} \subset SB_{\lambda}^{\alpha}$.

3) $S_{\lambda}^{\alpha} \subseteq S_{\lambda}$ **[14]**.

Theorem 6.5 Let (u_m), (v_m) be any sequences of real numbers such that

$S_{\lambda}^{\alpha} - \lim_{m\to\infty} u_m = a$, $S_{\lambda}^{\alpha} - \lim_{m\to\infty} v_m = b$ and c be any real number. Then

i) $S_{\lambda}^{\alpha} - \lim_{m\to\infty} cu_m = ca$.

ii) $S_{\lambda}^{\alpha} - \lim_{m\to\infty} (u_m + v_m) = a + b$ **[14]**.

Theorem 6.6 Let $0 < \alpha \leq \beta \leq 1$. Now if $S_\lambda^\alpha - \lim_{m\to\infty} u_m = l$ then we have $S_\lambda^\beta - \lim_{m\to\infty} u_m = l$, that is $S_\lambda^\alpha \subseteq S_\lambda^\beta$ **[14]**.

Proof Suppose that $0 < \alpha \leq \beta \leq 1$, then we have

$$\frac{1}{\lambda_n^\beta} \leq \frac{1}{\lambda_n^\alpha}.$$

and so that

$$\frac{1}{\lambda_n^\beta}|\{m \in I_n: |u_m - 1| > \varepsilon\}| \leq \frac{1}{\lambda_n^\alpha}|\{m \in I_n: |u_m - 1| > \varepsilon\}|.$$

By taking limit in both side as n goes to infinity we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\beta}|\{m \in I_n: |u_m - 1| > \varepsilon\}| \leq \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|\{m \in I_n: |u_m - 1| > \varepsilon\}| = 0.$$

Therefore $S_\lambda^\alpha \subseteq S_\lambda^\beta$.

If we take $\beta = 1$ in Theorem 6.6 we get the following result.

Corollary 6.1 For each $\lambda = (\lambda_n) \in \Lambda$ and each $\alpha \in (0,1]$ we have $S_\lambda^\alpha \subseteq S_\lambda$ inclusion. $S_\lambda = S_\lambda^\alpha$ if $\alpha = 1$ **[14]** .

Theorem 6.7 Let $\alpha \in (0,1]$ be given. Then $S \subseteq S_\lambda^\alpha$ if

$$\liminf_{n\to\infty} \frac{\lambda_n^\alpha}{n} > 0$$

[14].

Proof For any $\varepsilon>0$, we have

$$\{m \leq n: |u_m - l| > \varepsilon\} \supset \{m \in I_n: |u_m - l| > \varepsilon\}$$

Therefore,

$$\frac{1}{n}|\{m \leq n: |u_m - l| > \varepsilon\}| \geq \frac{1}{n}|\{m \in I_n: |u_m - l| > \varepsilon\}|$$

$$= \frac{\lambda_n^\alpha}{n} \cdot \frac{1}{\lambda_n^\alpha}|\{m \in I_n: |u_m - l| > \varepsilon\}|$$

After taking limit in both side as $n \to \infty$ we get that

$$u_n \to l(s) \Rightarrow u_n \to l(\lambda_n^\alpha).$$

CONCLUSION

We have examined the notion of stat-convergence, stat-convergence of order α, strong p-Cesàro summability, strong p-Cesàro summability of order α, λ- stat-convergence, λ- stat- convergence of order α and boundedness for real number sequences. Also we showed the special case of each types of statly-convergent sequences and the relationship between the sets of sequences which are statly-convergent of order α, for different α's such that $0< \alpha \leq 1$. Furthermore, we also have shown the relationship between the sets of sequences which are strongly p-Cesàro summable of order α for different α's such that α>0.

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