REPUBLIC OF TURKEY
FIRAT UNIVERSITY
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STATISTICAL CONVERGENCE OF NUMBER SEQUENCES AND SOME GENERALIZATIONS WITH RESPECT TO MODULUS FUNCTIONS

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Ibrahim S. IBRAHIM

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SUMMARY

Statistical Convergence of Number Sequences and Some Generalizations With Respect to Modulus Functions

In this thesis, we examine and study f-statistical convergence, f-statistical boundedness, f-strong Cesàro summability, f-strong lacunary summability and some other notions related to these concepts for sequences of real or complex numbers, where f is a modulus function. At the first, we give f-statistically convergent and f-statistically bounded sequences and then we study the concepts of f-strong Cesàro summability of sequences of real or complex numbers with some concepts. Then we provide the relations between these concepts. After that, we establish the relations between the sets w^f and w^g, w^f and S^g, for different modulus functions f and g under some conditions, which is the original part of this thesis. Furthermore, for some special modulus functions, we obtain the relations between the sets w^f and w, S^f and S. Then we study the concepts of f-statistical convergence of order α such that $0 < \alpha \leq 1$ and f-strong Cesàro summability of order α such that $0 < \alpha \leq 1$, and we also give the relations between them. Finally, we give and study f-strong lacunary summability of order α, and we establish the relations between the sets $N_\theta^\beta(f)$ and $N_\theta^\alpha(g)$, $N_\theta^\alpha(f)$ and $N_\theta^\beta(g)$, $N_\theta^\alpha(f)$ and $S_\theta^\beta(g)$, $\ell_\infty \cap S_\theta^\alpha(f)$ and $N_{\theta'}^\beta(g)$, where f and g are different modulus functions under some conditions and $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$, which is another original part of this thesis. Furthermore, for some special modulus functions, we obtain the relations between the sets $N_\theta(f)$ and N_θ, $N_\theta^\alpha(f)$ and N_θ^α for $\alpha \in (0, 1]$.

Key Words: f-statistical convergence, f-statistical boundedness, f-strong summability, f-statistical convergence of order α, f-strong summability of order α, f-lacunary statistical convergence of order α, f-strong lacunary summability of order α.

ÖZET

Sayı Dizilerinin İstatistiksel Yakınsaklığı ve Modulus Fonksiyonlarına Göre Bazı Genelleştirmeleri

Bu tezde f bir modülüs fonksiyonu olmak üzere, f-istatistiksel yakınsaklık, f-istatistiksel sınırlılık, f-kuvvetli Cesàro toplanabilme, kuvvetli f-lacunary toplanabilme ve bu kavramlarla ilgili diğer bazı kavramlar reel sayı dizileri için incelenmektedir. İlk önce reel sayı dizileri için f-istatistiksel yakınsaklık, f-istatistiksel sınırlılık ve ardından f-kuvvetli Cesàro toplanabilirlik kavramı verilmekte ve ilişkili bazı kavramlar incelenmektedir. Sonra bu kavramlar arasındaki ilişkiler ortaya konulmaktadır. Bundan sonra, bazı şartlara sahip farklı modülüs fonksiyonları için w^f ve w^g, w^f ve S^g kümeleri arasındaki kapsama bağıntıları elde edilmektedir. Ayrıca f üzerindeki bazı özel şartlar altında w^f ve w, S^f ve S sınıfları arasındaki ilişkiler elde edilmektedir. Daha sonra $0 < \alpha \leq 1$ şartına sahip herhangi bir α için $\alpha.$ dereceden f-istatistiksel yakınsaklık ve $\alpha.$ dereceden f-kuvvetli Cesàro toplanabilirlik üzerinde çalışılıp, bu iki kavram arasındaki ilişkiler de verilmektedir. Son olarak, $\alpha.$ dereceden kuvvetli f-lacunary toplanabilme incelenip, $0 < \alpha \leq \beta \leq 1$, f ve g modülüs fonksiyonları için $N_\theta^\beta(f)$ ve $N_\theta^\alpha(g)$, $N_\theta^\alpha(f)$ ve $S_\theta^\beta(g)$, $\ell_\infty \cap S_\theta^\alpha(f)$ ve $N_{\theta'}^\beta(g)$ kümeleri arasındaki ilişkiler ortaya konulmaktadır. Ayrıca, bazı özel modülüs fonksiyonları için $N_\theta(f)$ ve N_θ, $N_\theta^\alpha(f)$ ve N_θ^α kümeleri arasındaki ilişkiler elde edilmektedir.

Anahtar Kelimeler: f-istatistiksel yakınsaklık, f-istatistiksel sınırlılık, f-kuvvetli toplanabilirlik, $\alpha.$ dereceden istatistiksel yakınsaklık, $\alpha.$ dereceden f-kuvvetli toplanabilirlik, $\alpha.$ dereceden f-lacunary istatistiksel yakınsaklık, $\alpha.$ dereceden f-lacunary kuvvetli toplanabilirlik.

LIST OF SYMBOLS

$\mathbb{N}$	: Natural numbers.
$\mathbb{R}$	: Real numbers.
$\mathbb{C}$	: Complex numbers.
s	: All sequences.
c	: Convergent sequences.
ℓ_∞	: Bounded sequences.
S	: Statistically convergent sequences.
$S(b)$	: Statistically bounded sequences.
S^f	: f-Statistically convergent sequences.
S_0^f	: f-Statistically null sequences.
$S^f(b)$	: f-Statistically bounded sequences.
w	: Strongly Cesàro summable sequences.
w^f	: f-Strongly Cesàro summable sequences.
w_0^f	: f-Strongly Cesàro summable null sequences.
w_∞^f	: f-Strongly Cesàro summable bounded sequences.
S_f^α	: f-Statistically convergent sequences of order α.
w_α^f	: f-Strongly Cesàro summable sequences of order α.
S_θ	: Lacunary statistically convergent sequences.
N_θ	: Strongly lacunary summable sequences.
$S_\theta^\alpha(f)$	: f-Lacunary statistically convergent sequences of order α.
$N_\theta^\alpha(f)$	: f-Strongly lacunary summable sequences of order α.

1. INTRODUCTION

The principle of statistical convergence was performed first in the survey of Fast [1] and also severally for number sequences by Buck [2] and Schoenberg [3]. Regarding the subsequent work of Fridy [4] and Salát [5], this thinking has arisen as one of the qualified fields of research in summability theory. Statistical convergence, as implemented by Buck [2], acts as an instance of convergence in density. In recent decades, statistical convergence has been mentioned in many several fields and under different names, such as measure theory, approximation theory, Banach spaces, hopfield neural network, locally convex spaces, trigonometric series, number theory, summability theory, ergodic theory, turnpike theory, Fourier analysis and optimization. Further details and applications of this principle are available in [6–11].

Statistical convergence is a kind of convergence that depends technically on the natural density of subsets of the natural numbers.

Let $U \subset \mathbb{N}$. Then the number $\delta(U)$ is called a natural density of U and is identified via

$$\delta(U) = \lim_{n\to\infty} \frac{1}{n} |\{u \leq n : u \in U\}|$$

where $|\{u \leq n : u \in U\}|$ is the number of elements of U which are less than or equal to n.

In 1953, Nakano [12] presented the thought of a modulus function for the first time. By using a modulus function Bhardwaj and Singh [13], Connor [14], Çolak [15], Gosh and Srivastava [16], Maddox [17], Ruckle [18], Altin and Et [19] and others have constructed and discussed some sequence spaces.

In 2014, with the benefit of an unbounded modulus function, Aizpuru et al. [20] characterized another density's idea, as an outcome, a new nonmatrix convergence principle was acquired. By using the way of Aizpuru et al. [20], Bhardwaj et al. [21] has recently identified and concentrated a new concept of f-statistical boundedness, which is actually a generalization of the statistical convergence's principle. It has demonstrated that the concept of f-statistical boundedness is intermediate between the ordinary boundedness and the statistical boundedness, and it has demonstrated that bounded sequences are definitely those sequences which are f-statistically bounded for every unbounded modulus.

2. THE SETS OF f-STATISTICALLY CONVERGENT SEQUENCES AND f-STATISTICALLY BOUNDED SEQUENCES

2.1. f-Statistical Convergence

Since the concepts and results given in this section are known and widely used in the literature, their sources, that is references are not given.

We provide the definition of statistical convergence at the beginning of this section, as it will be needed in this study.

Definition 2.1.1 Let (x_k) be any sequence in $\mathbb{R}$ (or $\mathbb{C}$). The sequence (x_k) is named statistically convergent (or $S-$convergent) to the number l if

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. We write $S - \lim x_k = l$ or $x_k \to l(S)$ in this particular instance. The class of all S-convergent sequences will be symbolized by S throughout the study.

Definition 2.1.2 A function $f: [0, \infty) \to [0, \infty)$ is named modulus function, or modulus, if

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for every $x, y \in [0, \infty)$,
3. f is increasing,
4. f is continuous from the right at 0.

Within these characteristics, it is obvious that wherever on $[0, \infty)$ a modulus f is continuous. A modulus could be either unbounded or bounded. As an example, $f(x) = \log(x + 1)$ is an unbounded modulus, but $f(x) = \frac{x}{x+1}$ is a bounded modulus. Furthermore, for every modulus f and each positive integer n we have $f(nx) \leq nf(x)$ from condition 2.

Definition 2.1.3 Let f be an unbounded modulus and $A \subset \mathbb{N}$. The number $\delta^f(A)$ of a set A is named the $f-$density of a set A and is identified by

$$\delta^f(A) = \lim_{n\to\infty} \frac{f(|\{k \leq n : k \in A\}|)}{f(n)}$$

in the case this limit exists.

Remark 2.1.1 The f $-$density becomes the natural density if we take $f(x) = x$. It is obvious for the case of natural density for any $A \subset \mathbb{N}$ we have $\delta(A) + \delta(\mathbb{N} \setminus A) = 1$. But this conclusion is different for f $-$density, i.e., $\delta^f(A) + \delta^f(\mathbb{N} \setminus A) = 1$ does not have to be true, in general. This fact is shown in the example below.

Example 2.1.1 Let us take $f(x) = \log(x+1)$ and $E = \{2,4,6,\dots\}$. Then $\delta^f(E) = \delta^f(\mathbb{N} \setminus E) = 1$. Indeed since $\frac{n}{2} - 1 \leq |\{k \leq n : k \in E\}| \leq \frac{n}{2}$ for each $n \in \mathbb{N}$ and f is a modulus, then we may write

$$\lim_{n\to\infty} \frac{1}{f(n)} f\left(\frac{n}{2} - 1\right) \leq \lim_{n\to\infty} \frac{1}{f(n)} f|\{k \leq n : k \in E\}| \leq \lim_{n\to\infty} \frac{1}{f(n)} f\left(\frac{n}{2}\right)$$

and hence

$$\lim_{n\to\infty} \frac{1}{\log(n+1)} \log\left(\frac{n}{2}\right) \leq \lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : k \in E\}|) \leq \lim_{n\to\infty} \frac{1}{\log(n+1)} \log\left(\frac{n}{2} + 1\right)$$

$$1 \leq \lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : k \in E\}|) \leq 1.$$

Thus, $\delta^f(E) = 1$. Furthermore, using the fact $\frac{n+1}{2} - 1 \leq |\{k \leq n : k \in \mathbb{N} \setminus E\}| \leq \frac{n+1}{2}$ for every $n \in \mathbb{N}$, we have $\delta^f(\mathbb{N} \setminus E) = 1$.

Remark 2.1.2 From the previous remark, the situation $\delta^f(A) + \delta^f(\mathbb{N} \setminus A) = 1$ is not true for every unbounded modulus, in general. But it happens for any unbounded modulus function when $\delta^f(A) = 0$. Indeed, suppose $A \subset \mathbb{N}$ and $\delta^f(A) = 0$. Then for any $n \in \mathbb{N}$, we have

$$f(n) \leq f(|\{k \leq n : k \in A\}|) + f(|\{k \leq n : k \in \mathbb{N} \setminus A\}|)$$

and so,

$$1 \leq \frac{f(|\{k \leq n : k \in A\}|)}{f(n)} + \frac{f(|\{k \leq n : k \in \mathbb{N} \setminus A\}|)}{f(n)} \leq \frac{f(|\{k \leq n : k \in A\}|)}{f(n)} + 1$$

By taking limits as $n \to \infty$, we get that $\delta^f(\mathbb{N} \setminus A) = 1$.

Remark 2.1.3 f $-$density is similar to the natural density for any finite $A \subset \mathbb{N}$, so $\delta^f(A) = 0$ and $\delta^f(A) + \delta^f(\mathbb{N} \setminus A) = 1$.

Remark 2.1.4 For any unbounded modulus f and $A \subset \mathbb{N}$, if $\delta^f(A) = 0$, then $\delta(A) = 0$. Indeed, if $\delta^f(A) = 0$ then $\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : k \in A\}|) = 0$. Now for any $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then we have

$$f(|\{k \leq n : k \in A\}|) \leq \frac{1}{p} f(n) \leq \frac{1}{p} p f\left(\frac{1}{p} n\right) = f\left(\frac{1}{p} n\right),$$

which implies,

$$|\{k \leq n : k \in A\}| \leq \frac{1}{p} n.$$

Thus $\delta(A) = 0$. But the opposite does not have to be true in general (see [22]). In any case, according to Remark 2.1.3, $\delta(A) = 0$ implies $\delta^f(A) = 0$ for any finite $A \subset \mathbb{N}$.

Definition 2.1.4 Let f be an unbounded modulus. A sequence (x_k) in $\mathbb{R}$ (or $\mathbb{C}$) is called f −statistically convergent (or S^f-convergent) to l if the set $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ has f −density zero, for every $\varepsilon > 0$, i.e.,

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k - l| \geq \varepsilon\}|) = 0,$$

and we write this as $S^f - \lim x_k = l$ or $x_k \to l(S^f)$. In this study, S^f denotes the class of all S^f-convergent sequences. We also symbolize the set of all f −statistically null sequences by S_0^f. Certainly for any unbounded modulus f we have $S_0^f \subset S^f$.

According to Remark 2.1.4 and Definition 2.1.4, we have $S^f \subset S$ for every unbounded modulus f. But the reverse, in particular, does not need to be true (for detail see [22]).

Theorem 2.1.1 Every convergent sequence is f-statistically convergent, that is, $c \subset S^f$ for any unbounded modulus f.

Proof Suppose (x_k) is convergent and $\lim_{k\to\infty} x_k = l$. Given any $\varepsilon > 0$, then there is $n_0 \in \mathbb{N}$ such that

$$|x_k - l| < \varepsilon \text{ for all } k \geq n_0.$$

So, the set $\{k \in \mathbb{N}: |x_k - l| \geq \varepsilon\}$ is finite and thus $\delta^f(\{k \in \mathbb{N}: |x_k - l| \geq \varepsilon\}) = 0$, this fulfills the proof.

Remark 2.1.5 The contrary to Theorem 2.1.1, in general, does not have to be correct as of the example below.

Example 2.1.2 Define the sequence (x_k) as

$$x_k = \begin{cases} k, & k = n^2 \\ 0, & k \neq n^2 \end{cases} \qquad n = 1,2,3, \ldots$$

and take $f(x) = x^p, 0 < p \leq 1$. Apparently $(x_k) \notin c$, but for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{f(|\{k \leq n: |x_k - 0| \geq \varepsilon\}|)}{f(n)} \leq \lim_{n\to\infty} \frac{f(\sqrt{n})}{f(n)} = 0.$$

Therefore, (x_k) is S^f-convergent.

During this study, we remember [21] that let (x_k) be any number sequence and $A \subset \mathbb{N}$ with $\delta^f(A) = 0$, if for each $k \in \mathbb{N} \setminus A$ the member x_k of (x_k) satisfies property P, then we say that (x_k) satisfies P for "almost all k with respect to f," where f is any unbounded modulus and this is often referred to as "a.a. k w.r.t. f."

The definition of f −statistical convergence can be rewritten using this idea as follows.

Definition 2.1.5 The sequence (x_k) in $\mathbb{R}$ (or $\mathbb{C}$) is called S^f-convergent to l, if for every $\varepsilon > 0$,

$$|x_k - l| < \varepsilon \quad \text{a.a. } k \text{ w.r.t. } f.$$

Theorem 2.1.2 If $x = (x_k) \in S^f$ and $S^f - \lim x_k = l$, then there is a sequence $y = (y_k)$ and $z = (z_k) \in S_0^f$ such that $\lim_{k\to\infty} y_k = l$ and $x = y + z$. Furthermore, if the sequence x is bounded then the sequence z is also bounded and $\|z\|_\infty \leq \|x\|_\infty + |l|$.

Proof As $x \in S^f$ and $S^f - \lim x_k = l$, there is some $A \subset \mathbb{N}$ with $\delta^f(A) = 0$ such that $\lim_{k\in\mathbb{N}\setminus A} x_k = l$. We define $y = (y_k)$ and $z = (z_k)$ as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in \mathbb{N} \setminus A; \\ l, & \text{if } k \in A, \end{cases}$$

$$z_k = \begin{cases} 0, & \text{if } k \in \mathbb{N} \setminus A; \\ x_k - l, & \text{if } k \in A. \end{cases}$$

Obviously from our construction, we have $x = y + z$. Since $\{k \in \mathbb{N}: |z_k - 0| > \varepsilon\} \subset A$ for every $\varepsilon > 0$, then $\delta^f(\{k \in \mathbb{N}: |z_k - 0| > \varepsilon\}) = 0$ for every $\varepsilon > 0$. Hence $z = (z_k) \in S_0^f$ and $\|z\|_\infty \leq \|x\|_\infty + |l|$, if x is bounded. For $k \in \mathbb{N}$, we have

$$\|y_k - l\| = \begin{cases} \|x_k - l\|, & \text{if } k \in \mathbb{N} \setminus A; \\ 0, & \text{if } k \in A, \end{cases}$$

and so

$$\{k \in \mathbb{N}: \|y_k - l\| > \varepsilon\} \subset \{k \in \mathbb{N}: \|x_k - l\| > \varepsilon\} \cap (\mathbb{N} \setminus A).$$

As $\lim_{k \in \mathbb{N} \setminus A} x_k = l$, then the set $\{k \in \mathbb{N}: \|x_k - l\| > \varepsilon\} \cap (\mathbb{N} \setminus A)$ is finite for each $\varepsilon > 0$ and thus $\lim_{k \to \infty} y_k = l$.

Using the idea of Theorem 2.1.2, we obtain the result below.

Corollary 2.1.1 If $(x_k) \in s$ and $S^f - \lim x_k = l$, then the sequence (x_k) has a subsequence $y = (y_k)$ such that $\lim_{k \to \infty} y_k = l$.

Remark 2.1.6 We are sure that if a sequence is convergent, then each of its subsequences is convergent, but this case does not have to be true for f −statistical convergence, this says that an f −statistically convergent sequence may have a subsequence which is not f −statistically convergent. For instance, if we take $f(x) = x$, then for the sequence $(x_k) = \left(1, \frac{1}{2}, \frac{1}{3}, 4, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, 9, \ldots\right)$ we have $S^f - \lim x_k = 0$, although its subsequence $(1,4,9, \ldots)$ is not S^f-convergent.

Theorem 2.1.3 Suppose f is an unbounded modulus and $x = (x_k) \in s$. Then $(x_k) \in S^f$ if and only if there exists $y = (y_k) \in c$ such that $x_k = y_k$ a.a. k w.r.t. f.

Proof Assuming f is an unbounded modulus and $(x_k) \in S^f$. The technique follows the lines held out in Theorem 2.1.2, we get such a convergent sequence (y_k) such that

$$\delta^f(\{k \in \mathbb{N} : x_k \neq y_k\}) \leq \delta^f(A) = 0.$$

Therefore, $x_k = y_k$ a.a. k w.r.t. f.

Conversely, we have

$$\{k \in \mathbb{N}: |x_k - l| > \varepsilon\} \subset \{k \in \mathbb{N}: x_k \neq y_k\} \cup \{k \in \mathbb{N}: |y_k - l| > \varepsilon\}$$

Since $\lim_{k\to\infty} y_k = l$, the set $\{k \in \mathbb{N}: |y_k - l| > \varepsilon\}$ is finite. Therefore, $\delta^f(\{k \in \mathbb{N}: |x_k - l| > \varepsilon\}) = 0$ for any $\varepsilon > 0$. Hence (x_k) is S^f-convergent.

2.2. f-Statistical Boundedness

Definition 2.2.1 Suppose f is an unbounded modulus and (x_k) is any sequence in $\mathbb{R}$ (or $\mathbb{C}$). Then (x_k) is said to be f −statistically bounded (or S^f-bounded) if there is $M > 0$ such that $\delta^f(\{k \in \mathbb{N}: |x_k| > M\}) = 0$, that is, $|x_k| \leq M$ a.a. k w.r.t. f. We write $S^f(b)$ to symbolize the class of all S^f-bounded sequences.

Theorem 2.2.1 Every bounded sequence is f-statistically bounded, that is, $\ell_\infty \subset S^f(b)$ for every unbounded modulus f, even so, the converse does not have to be true.

Proof Suppose f is an unbounded modulus and $(x_k) \in \ell_\infty$. Then there is a number $M > 0$ such that $|x_k| \leq M$, for all $k \in \mathbb{N}$, that is, $\delta^f(\{k \in \mathbb{N}: |x_k| > M\}) = \delta^f(\emptyset) = 0$. So (x_k) is f −statistically bounded. For the opposite part, the sequence of Example 2.2.1 serves the purpose if we take $f(x) = x$.

We will write $S(b)$ instead of $S^f(b)$ in case $f(x) = x$, where $S(b)$ denotes the class of all statistically bounded sequences.

Theorem 2.2.2 Every f-statistically bounded sequence is statistically bounded, that is, $S^f(b) \subset S(b)$ for every unbounded modulus f.

The proof is due to the assertion that $\delta^f(A) = 0$ implies $\delta(A) = 0$ for any $A \subset \mathbb{N}$ and any unbounded modulus f. Using this fact if (x_k) is f −statistically bounded, then we have $\delta^f(\{k \in \mathbb{N}: |x_k| > M\}) = 0$ for some enough large number $M > 0$. Now $\delta^f(\{k \in \mathbb{N}: |x_k| > M\})$=0 implies $\delta(\{k \in \mathbb{N}: |x_k| > M\}) = 0$ and this means that (x_k) is statistically bounded.

Remark 2.2.1 Note that the contrary to the above theorem is not true, in general. The following example verifies this.

Example 2.2.1 Let $f(x) = \log(x+1)$ and $(x_k) = (1,0,0,4,0,0,0,0,9,\dots)$. Then for any number $M > 0$, we have $\{k \in \mathbb{N}: |x_k| > M\} = \{1,4,9,\dots\} \setminus$ a finite subset of $\mathbb{N}$.

Since $\delta^f(\{1,4,9,\dots\}) = 1/2 \neq 0$ and $\delta(\{1,4,9,\dots\}) = 0$, then $(x_k) \notin S^f(b)$ and $(x_k) \in S(b)$. As a result, $S^f(b) \subsetneq S(b)$.

Theorem 2.2.3 Suppose f is an unbounded modulus and $(x_k) \in s$. Then $(x_k) \in S^f(b)$ if and only if there exists a subset $A \subset \mathbb{N}$ such that $\delta^f(A) = 0$ and $(x_k)_{k\in\mathbb{N}\setminus A} \in \ell_\infty$.

Proof Assume that (x_k) is S^f-bounded. So we can take a number $M > 0$ such that $\delta^f(\{k \in \mathbb{N}: |x_k| > M\}) = 0$. Now take $A = \{k \in \mathbb{N}: |x_k| > M\}$. Then $\delta^f(A) = 0$ and $|x_k| \leq M$ for $k \in \mathbb{N} \setminus A$; that is, $(x_k)_{k\in\mathbb{N}\setminus A} \in \ell_\infty$.

Conversely, since $(x_k)_{k\in\mathbb{N}\setminus A} \in \ell_\infty$, then for some enough large number $M > 0$ we have $|x_k| \leq M$ for all $k \in \mathbb{N} \setminus A$. This means that $\{k \in \mathbb{N}: |x_k| > M\} \subset A$ and so $\delta^f(\{k \in \mathbb{N}: |x_k| > M\}) = 0$. Therefore, (x_k) is S^f-bounded.

Remark 2.2.2 It is obvious that if a sequence is bounded, then all of its subsequences are bounded. Even so, for f −statistically boundedness this situation is not true in general; that is, an f −statistically bounded sequence may have a subsequence which is not f −statistically bounded. The example below illustrates this.

Example 2.2.2 Consider $(x_k) = (1,0,0,4,0,0,0,0,9,\dots)$ and $f(x) = x^p, 0 < p \leq 1$. Now we have $\delta^f\left(\left\{k \in \mathbb{N}: |x_k| > \frac{1}{2}\right\}\right) = \delta^f(\{1,4,9,\dots\})$. Since $|k \leq n : k \in \{1,4,9,\dots\}| \leq \sqrt{n}$ for every $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \frac{1}{f(n)} f\left(\left\{\left|k \leq n : |x_k| > \frac{1}{2}\right|\right\}\right) \leq \lim_{n\to\infty} \frac{1}{f(n)} f(\sqrt{n}) = 0.$$

So, $\delta^f\left(\left\{k \in \mathbb{N} : |x_k| > \frac{1}{2}\right\}\right) = 0$. Thus $(x_k) \in S^f(b)$. However, $(1,4,9,\dots)$ is a subsequence of (x_k) and $(1,4,9,\dots) \notin S^f(b)$.

Theorem 2.2.4 $S^f \subset S^f(b)$ for any unbounded modulus f, although the converse is not true in general.

Proof Suppose $(x_k) \in S^f$ and S^f-lim $x_k = l$. Then for every $\varepsilon > 0$, we have that $\delta^f(\{k \in \mathbb{N}: |x_k - l| > \varepsilon\}) = 0$. As $\{k \in \mathbb{N}: |x_k| > |l| + \varepsilon\} \subset \{k \in \mathbb{N}: |x_k - l| > \varepsilon\}$, so $|x_k| \leq |l| + \varepsilon$ a.a. k w.r.t. f. Thus, $(x_k) \in S^f(b)$.

For the converse part, let us take the identity map $f(x) = x$ and $(x_k) = (1,2,1,2, \ldots)$, then $(x_k) \in S^f(b)$, but $(x_k) \notin S^f$.

Theorem 2.2.5 A sequence (x_k) is f −statistically bounded if and only if there exists a bounded sequence (y_k) such that $x_k = y_k$ a.a. k w.r.t. f.

Proof Suppose (x_k) is f −statistically bounded. Then for some enough large number $M > 0$ we have $\delta^f(\{k \in \mathbb{N}: |x_k| > M\}) = 0$. Now take $A = \{k \in \mathbb{N}: |x_k| > M\}$ and define (y_k) as follows: if $k \notin A$ we set $y_k = x_k$ and otherwise, we set $y_k = 0$. Then $(y_k) \in \ell_\infty$ and we can say that $x_k = y_k$ a.a. k w.r.t. f. Conversely, since $(y_k) \in \ell_\infty$ so there is a number $M > 0$ such that $|y_k| \leq M$ for all $k \in \mathbb{N}$. Let $H = \{k \in \mathbb{N}: x_k \neq y_k\}$. Since $x_k = y_k$ a.a. k w.r.t. f, then $\delta^f(H) = 0$. This gives that $|x_k| \leq M$ a.a. k w.r.t. f, because $\{k \in \mathbb{N}: |x_k| > M\} \subset H$.

Lemma 2.2.1 [20] For every infinite $H \subset \mathbb{N}$ there exists an unbounded modulus function f such that $\delta^f(H) = 1$.

Theorem 2.2.6 If $(x_k) \in S^f(b)$ for every unbounded modulus f, then $(x_k) \in \ell_\infty$.

Proof We assume the contrary; that is, we assume that $(x_k) \notin \ell_\infty$. Then the set $\{k \in \mathbb{N}: |x_k| > M\}$ is infinite for every number $M > 0$. So by Lemma 2.2.1, there is an unbounded modulus f such that $\delta^f(\{k \in \mathbb{N}: |x_k| > M\}) = 1$ which is completely contrary to the assumption that $(x_k) \in S^f(b)$ for every unbounded modulus.

3. THE SETS OF f-STRONGLY CESÀRO SUMMABLE SEQUENCES

3.1. f-Strong Cesàro Summability

Definition 3.1.1 [23] Let (x_k) be any sequence in $\mathbb{R}$ (or $\mathbb{C}$) and f be any modulus. Then the sequence (x_k) is called f −strongly Cesàro summable (or w^f-summable) to l, if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|) = 0.$$

In this situation, we write $x_k \to l(w^f)$ and the class of all f −strongly Cesàro summable sequences will be symbolized by w^f.

Here, the spaces w_0^f, w^f and w_∞^f are defined as follows:

$$w_0^f = \left\{(x_k) : \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(|x_k|) = 0\right\},$$

$$w^f = \left\{(x_k) : \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|) = 0 \text{ for some number } l\right\},$$

$$w_\infty^f = \left\{(x_k) : \sup_n \frac{1}{n}\sum_{k=1}^{n} f(|x_k|) < \infty\right\}.$$

Remark 3.1.1 In case $f(x) = x$, the f −strong Cesàro summability reduces to the strong Cesàro summability and the sequence spaces w_0^f, w^f and w_∞^f becomes w_0, w and w_∞ of strongly Cesàro summable sequences, respectively.

Theorem 3.1.1 [22] Suppose f is any modulus. Then

(i) $w_0^f \subset w_\infty^f$,

(ii) $w^f \subset w_\infty^f$.

Proof The first inclusion is perfectly clear, we here only prove the second inclusion. Assume that $(x_k) \in w^f$. Then by the properties of a modulus function f, we have that

$$\frac{1}{n}\sum_{k=1}^{n} f(|x_k|) \leq \frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|) + f(|l|)\frac{1}{n}\sum_{k=1}^{n} 1.$$

Since $(x_k) \in w^f$, we get $(x_k) \in w_\infty^f$, this fulfills the proof.

Theorem 3.1.2 [22] Suppose f is any modulus. Then

(i) $w \subset w^f$,

(ii) $w_0 \subset w_0^f$,

(iii) $w_\infty \subset w_\infty^f$.

Proof

(i) To prove that $w \subset w^f$. Let $x = (x_k) \in w$ so that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_k - l| = 0$$

for some l. Given $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $t \in (0, \delta]$.

Let $y_k = |x_k - l|$ and consider

$$\sum_{k=1}^{n} f(y_k) = \sum_{\substack{k=1 \\ y_k \leq \delta}}^{n} f(y_k) + \sum_{\substack{k=1 \\ y_k > \delta}}^{n} f(y_k).$$

Since $f(y_k) < \varepsilon$ for $y_k \leq \delta$, then

$$\sum_{\substack{k=1 \\ y_k \leq \delta}}^{n} f(y_k) < \varepsilon n,$$

and also for $y_k > \delta$, we have

$$y_k < \frac{y_k}{\delta} < 1 + \left[\frac{y_k}{\delta}\right],$$

where $[t]$ denotes the integer part of real number t. Since f is a modulus, then

$$f(y_k) \leq f\left(1+\left[\frac{y_k}{\delta}\right]\right) \leq f(1)\left(1+\left[\frac{y_k}{\delta}\right]\right) \leq 2f(1)\frac{y_k}{\delta},$$

so, we get

$$\sum_{\substack{k=1\\ y_k>\delta}}^{n} f(y_k) \leq \frac{2f(1)}{\delta}\sum_{\substack{k=1\\ y_k>\delta}}^{n} y_k.$$

Thus,

$$\frac{1}{n}\sum_{k=1}^{n} f(|x_k-l|) \leq \varepsilon + \frac{2f(1)}{\delta}\frac{1}{n}\sum_{\substack{k=1\\ |x_k-l|>\delta}}^{n} |x_k-l|$$

$$\leq \varepsilon + \frac{2f(1)}{\delta}\frac{1}{n}\sum_{k=1}^{n}|x_k-l|.$$

Since $x \in w$, then we get $x \in w^f$.

(ii) The proof of this part is similar to the first part when $l = 0$ and therefore is omitted.

(iii) Let $(x_k) \in w_\infty$ so that

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}|x_k| < \infty.$$

Given any $\varepsilon > 0$ and choose $\delta \in (0,1)$ such that $f(t) < \varepsilon$ for $t \in (0,\delta]$. Now let us consider

$$\sum_{k=1}^{n} f(|x_k|) = \sum_{\substack{k=1\\ |x_k|\leq\delta}}^{n} f(|x_k|) + \sum_{\substack{k=1\\ |x_k|>\delta}}^{n} f(|x_k|).$$

Since $f(|x_k|) < \varepsilon$ for $|x_k| \leq \delta$, then we have

$$\sum_{\substack{k=1\\ |x_k|\leq\delta}}^{n} f(|x_k|) < \varepsilon n,$$

and also for $|x_k| > \delta$ we have

$$|x_k| < \frac{|x_k|}{\delta} < 1 + \left[\frac{|x_k|}{\delta}\right].$$

Since f is a modulus, for $|x_k| > \delta$ we have

$$f(|x_k|) \leq f\left(1 + \left[\frac{|x_k|}{\delta}\right]\right) \leq \left(1 + \left[\frac{|x_k|}{\delta}\right]\right) f(1) \leq 2f(1)\frac{|x_k|}{\delta}.$$

So, we get

$$\sum_{\substack{k=1 \\ |x_k|>\delta}}^{n} f(|x_k|) \leq \frac{2f(1)}{\delta} \sum_{\substack{k=1 \\ |x_k|>\delta}}^{n} |x_k| \leq \frac{2f(1)}{\delta} \sum_{k=1}^{n} |x_k|,$$

and therefore,

$$\frac{1}{n}\sum_{k=1}^{n} f(|x_k|) \leq \varepsilon + \frac{2f(1)}{\delta}\frac{1}{n}\sum_{k=1}^{n} |x_k|.$$

Since $x \in w_\infty$, we have $x \in w_\infty^f$ and this fulfills the proof.

Lemma 3.1.1 [24] The limit $\lim_{t\to\infty} \frac{f(t)}{t} = \beta$ exists for any modulus f and $\lim_{t\to\infty} \frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t} : t > 0\right\}$.

Theorem 3.1.3 [22] Suppose f is any modulus. If $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, then $w^f \subset w$.

Proof On the basis of Lemma 3.1.1, we have that $\beta = \lim_{t\to\infty} \frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t} : t > 0\right\}$ exists and from this we have $f(t) \geq \beta t$ for all $t \geq 0$. Since $\beta > 0$, we also have $t \leq \frac{1}{\beta} f(t)$ for all $t \geq 0$ and so

$$\frac{1}{n}\sum_{k=1}^{n} |x_k - l| \leq \frac{1}{\beta}\frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|)$$

from where it follows that $x \in w$ whenever $x \in w^f$.

Through combining Theorem 3.1.3 and Theorem 3.1.2 we obtain the result below.

Corollary 3.1.1 [22] Let f be any modulus. If $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, then $w^f = w$.

3.2. Some Relations Between the Sets of f-Strongly Cesàro Summable Sequences

In this study, we establish the relations between w^f and w^g, w^f and S^g, for different modulus functions f and g under some conditions on the considered modulus functions. However the relations between the sets w^f and w, S^f and S are known already for a modulus f (see [22] and [20]).

Theorem 3.2.1 Suppose f and g are modulus functions. If

$$\sup_{t\in(0,\infty)} \frac{f(t)}{g(t)} < \infty,$$

then we have $w^g \subset w^f$ and the inclusion may be strict.

Proof Suppose that $\alpha = \sup_{t\in(0,\infty)} \frac{f(t)}{g(t)} < \infty$. Then we have $\frac{f(t)}{g(t)} \leq \alpha$ and so that $f(t) \leq \alpha g(t)$ for every $t \in [0,\infty)$. Now it is clear that $\alpha > 0$ and if $x = (x_k)$ is g −strongly Cesàro summable to l, we may write

$$\frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|) \leq \frac{1}{n}\sum_{k=1}^{n} \alpha g(|x_k - l|).$$

Taking limit on both sides as $n \to \infty$, we obtain that $x \in w^g$ implies $x \in w^f$.

The example below shows that the inclusion $w^g \subset w^f$ is strict at least for some special modulus functions f and g.

Example 3.2.1 Define the sequence $x = (x_k)$ as

$$x_k = \begin{cases} k, & k = m^3 \\ 0, & k \neq m^3 \end{cases} \quad m \in \mathbb{N},$$

and the modulus $f(t) = \frac{t}{t+1}$ and $g(t) = t$. Now $\sup_{t\in(0,\infty)} \frac{f(t)}{g(t)} = 1 < \infty$ and so that $w^g \subset w^f$. Using the $f(0) = 0$ equality, we have

$$\frac{1}{n}\sum_{k=1}^{n} f(|x_k|) = \frac{1}{n}\sum_{\substack{k=1\\k=m^3}}^{n} f(k) + \frac{1}{n}\sum_{\substack{k=1\\k\neq m^3}}^{n} f(0)$$

$$= \frac{1}{n}\sum_{\substack{k=1\\k=m^3}}^{n} \frac{k}{1+k} < \frac{1}{n}\sum_{\substack{k=1\\k=m^3}}^{n} 1 \leq \frac{\sqrt[3]{n}}{n}.$$

Since $\frac{\sqrt[3]{n}}{n}$ tends to 0 as $n \to \infty$, we get $x \in w^f$. But since

$$\frac{1}{n}\sum_{k=1}^{n} g(|x_k|) = \frac{1}{n}\sum_{k=1}^{n} g(x_k) = \frac{1}{n}\sum_{\substack{k=1\\k=m^3}}^{n} k + \frac{1}{n}\sum_{\substack{k=1\\k\neq m^3}}^{n} g(0)$$

$$= \frac{1}{n}(1^3 + 2^3 + 3^3 + \cdots + i^3),\ \max_{i\in\mathbb{N}} i^3 \leq n$$

$$= \frac{1}{n}\left[\frac{i(i+1)}{2}\right]^2$$

$$\geq \frac{1}{n}\left[\frac{\left(\left[\sqrt[3]{n}\right]-1\right)\left(\left[\sqrt[3]{n}\right]\right)}{2}\right]^2$$

and right-hand side tends to ∞ as $n \to \infty$ we get $x \notin w^g$, where $[r]$ denotes the integral part of the real number r. Hence $x \in w_f - w_g$ and the inclusion $w^g \subset w^f$ is strict.

Theorem 3.2.2 Suppose f and g are modulus functions. If

$$\inf_{t\in(0,\infty)} \frac{f(t)}{g(t)} > 0,$$

then $w^f \subset w^g$.

Proof Suppose that $\beta = \inf_{t\in(0,\infty)} \frac{f(t)}{g(t)} > 0$. Then we have $\frac{f(t)}{g(t)} \geq \beta$ and so that $\beta g(t) \leq f(t)$ for every $t \in [0,\infty)$. Now if $x = (x_k)$ is f −strongly Cesàro summable to l we may write

$$\frac{1}{n}\sum_{k=1}^{n} g(|x_k - l|) \leq \frac{1}{\beta}\frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|).$$

Taking limit on both sides as $n \to \infty$, we obtain that $x \in w^f$ implies $x \in w^g$ and this fulfills the proof.

Taking $g(t) = t$ in Theorem 3.2.2 we get Corollary 3.1.1 by Lemma 3.1.1.

We get the following result from Theorem 3.2.1 and Theorem 3.2.2.

Corollary 3.2.1 Suppose f and g are modulus functions. If

$$0 < \inf_{t\in(0,\infty)} \frac{f(t)}{g(t)} \leq \sup_{t\in(0,\infty)} \frac{f(t)}{g(t)} < \infty,$$

then $w^f = w^g$.

Theorem 3.2.3 Assume that f and g are unbounded modulus functions. If $\inf_{t\in(0,\infty)} \frac{f(t)}{g(t)} > 0$ and $\lim_{t\to\infty} \frac{g(t)}{t} > 0$, then every $f-$strongly Cesàro summable sequence is $g-$statistically convergent.

Proof Suppose that $\beta = \inf_{t\in(0,\infty)} \frac{f(t)}{g(t)} > 0$. Then we have $\frac{f(t)}{g(t)} \geq \beta$ and so that $\beta g(t) \leq f(t)$ for every $t \in [0,\infty)$. Now if $x = (x_k)$ is $f-$strongly Cesàro summable to l, we may write

$$\frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|) \geq \beta \frac{1}{n}\sum_{k=1}^{n} g(|x_k - l|) \geq \beta \frac{1}{n} \sum_{\substack{k=1 \\ |x_k - l|\geq\varepsilon}}^{n} g(|x_k - l|)$$

$$\geq \beta \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}|\, g(\varepsilon)$$

$$\geq \beta \frac{1}{n} g(|\{k \leq n : |x_k - l| \geq \varepsilon\}|)\, \frac{g(\varepsilon)}{g(1)}$$

$$= \frac{g(|\{k \leq n : |x_k - l| \geq \varepsilon\}|)}{g(n)}\, \frac{g(n)}{n} \frac{g(\varepsilon)}{g(1)} \beta.$$

Taking the limit on both sides as $n \to \infty$, we obtain that $x \in w^f$ implies $x \in S^g$, since $\lim_{t\to\infty} \frac{g(t)}{t} > 0$ and $\beta > 0$. Here is the proof.

The following result is obtained by taking $g(t) = f(t)$ from Theorem 3.2.3.

Corollary 3.2.2 Assume that f is an unbounded modulus. If $\lim_{t\to\infty}\frac{f(t)}{t}>0$, then every f −strongly Cesàro summable sequence is f −statistically convergent.

Remark 3.2.1 Corollary 3.2.2 was given with the extra condition "$f(xy)\geq cf(x)f(y)$ for all $x\geq 0$, $y\geq 0$ and some positive number c" in [22]. It seems that this extra condition is unnecessary, so it should be removed from Corollary 4.3 in [22].

The following result is obtained by taking $g(t)=t$ from Theorem 3.2.3 (see also in [22]).

Corollary 3.2.3 Assume that f is an unbounded modulus function. If $\inf_{t\in(0,\infty)}\frac{f(t)}{t}>0$, then every f −strongly Cesàro summable sequence is statistically convergent.

Taking $f(t)=t$ in Corollary 3.2.3, we get the statement below, which is the first part of Theorem 2.1 of Connor [25], for the case $q=1$.

Corollary 3.2.4 A strongly Cesàro summable sequence is statistically convergent.

Theorem 3.2.4 For any unbounded modulus functions f and g we have

$$\ell_\infty\cap S^f\subset w^g$$

and this inclusion may be strict.

Proof Suppose that f and g are unbounded modulus functions. Since $S^f\subset S$ by part 1 of Corollary 2.2 of [20], and since $\ell_\infty\cap S\subset w$ by the second part of Theorem 2.1 of [25], then we have $\ell_\infty\cap S^f\subset\ell_\infty\cap S\subset w$, that is $\ell_\infty\cap S^f\subset w$. On the other hand since $w\subset w^g$ for any modulus g by the first part of Theorem 3.1.2, it follows that $\ell_\infty\cap S^f\subset w^g$.

The following example shows that the inclusion $\ell_\infty\cap S^f\subset w^g$ is strict.

Example 3.2.2 As an example, let us define the sequence $x=(x_k)$ as

$$x_k=\begin{cases}1, & k=n^2\\ 0, & k\neq n^2\end{cases}\quad n\in\mathbb{N},$$

and consider the modulus function $g(x)=f(x)=\log(x+1)$. Using the $g(0)=0$ equality, we have

$$\frac{1}{n}\sum_{k=1}^{n} g(|x_k - 0|) = \frac{1}{n}\sum_{k=1}^{n} g(x_k) = \frac{1}{n}\sum_{\substack{k=1\\k=n^2}}^{n} g(1) + \frac{1}{n}\sum_{\substack{k=1\\k\neq n^2}}^{n} g(0)$$

$$= \frac{1}{n}\sum_{\substack{k=1\\k=n^2}}^{n} \log 2 \leq \frac{\sqrt{n}}{n}\log 2 \to 0 \text{ as } n \to \infty,$$

we get $x \in w^g$, but since

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k| \geq \varepsilon\}|)$$

$$\geq \lim_{n\to\infty} \frac{1}{f(n)} f\left(\sqrt{n} - 1\right) = \lim_{n\to\infty} \frac{\log\left(\sqrt{n}\right)}{\log(n+1)} = \frac{1}{2} \neq 0,$$

we have $x \notin S^f$. Therefore, the inclusion $\ell_\infty \cap S^f \subset w^g$ is strict.

We get the statement below of strict inclusions from Theorem 3.2.4.

Corollary 3.2.7 For any unbounded modulus function f we have

(i) $\ell_\infty \cap S^f \subset w^f$,

(ii) $\ell_\infty \cap S^f \subset w$,

(iii) $\ell_\infty \cap S \subset w^f$.

Taking $f(t) = g(t) = t$ in Theorem 3.2.4, we get the statement below, which is Theorem 3.1 of [25], for the case $q = 1$.

Corollary 3.2.8 A bounded and statistically convergent sequence is strongly Cesàro summable.

4. THE SETS OF f-STATISTICALLY CONVERGENT SEQUENCES OF ORDER α AND f-STRONGLY CESÀRO SUMMABLE SEQUENCES OF ORDER α

4.1. f-Statistical Convergence of Order α

Gadjiev and Orhan [26] provided the order of statistical convergence for a sequence of operators and then Çolak [27] examined statistical convergence of order α for a sequence of numbers.

Definition 4.1.1 [22] Suppose $A \subset \mathbb{N}$, f is an unbounded modulus and $0 < \alpha \leq 1$. Then we define $\delta_\alpha^f(A)$ (or the δ_α^f-density of A) by

$$\delta_\alpha^f(A) = \lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(|\{k \leq n : k \in A\}|),$$

in the case this limit exists.

It can be easily observed that if A is a finite subset of $\mathbb{N}$, then $\delta_\alpha^f(A) = 0$ but $\delta_\alpha^f(\mathbb{N} - A) \neq 1 - \delta_\alpha^f(A)$ for $0 < \alpha < 1$ and any modulus f, in general. For example, take $f(x) = x^p$, $0 < p \leq 1$, $0 < \alpha < 1$ and $A = \{2n : n \in \mathbb{N}\}$, then $\delta_\alpha^f(\mathbb{N} - A) = \infty = \delta_\alpha^f(A)$.

Remark 4.1.1 For $\alpha = 1$, the δ_α^f-density becomes the f-density and for $f(x) = x$, the δ_α^f-density becomes α-density. For the special case $\alpha = 1$ and $f(x) = x$, the δ_α^f-density becomes natural density.

Remark 4.1.2 Let $A \subset \mathbb{N}$, f be a modulus and $0 < \alpha \leq 1$. If $\delta_\alpha^f(A) = 0$, then $\delta_\alpha(A) = 0$ and hence $\delta(A) = 0$. Indeed, assume that $\delta_\alpha^f(A) = 0$, then for each $p \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have

$$f(|\{k \leq n : k \in A\}|) \leq \frac{1}{p} f(n^\alpha) \leq \frac{1}{p} p f\left(\frac{n^\alpha}{p}\right) = f\left(\frac{n^\alpha}{p}\right),$$

and since f is a modulus, then it is obviously increasing, and also since $0 < \alpha \leq 1$, then

$$\frac{1}{n}|\{k \leq n : k \in A\}| \leq \frac{1}{n^\alpha}|\{k \leq n : k \in A\}| \leq \frac{1}{p}.$$

Thus $\delta_\alpha(A) = 0$ and so $\delta(A) = 0$.

Remark 4.1.3 The converse of Remark 4.1.2 does not have to be true in general. It could be confirmed by the example below.

Example 4.1.1 Let $f(x) = \log(x+1)$ and $A = \{1,4,9,...\}$. Then $\delta(A) = 0$ and $\delta_\alpha(A) = 0$ for $1/2 < \alpha \leq 1$ but $\delta_\alpha^f(A) \geq \delta^f(A) = 1/2$. Therefore, $\delta_\alpha^f(A) \neq 0$.

Lemma 4.1.1 [22] Let $A \subset \mathbb{N}$ and f be any unbounded modulus. Then $\delta_\beta^f(A) \leq \delta_\alpha^f(A)$ if $0 < \alpha \leq \beta \leq 1$.

Proof Suppose that $0 < \alpha \leq \beta \leq 1$ and f is an unbounded modulus. Then $n^\alpha \leq n^\beta$ and this implies that $\frac{1}{f(n^\beta)} \leq \frac{1}{f(n^\alpha)}$ for every $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} \frac{1}{f(n^\beta)} f(|\{k \leq n : k \in A\}|) \leq \lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(|\{k \leq n : k \in A\}|).$$

So that $\delta_\beta^f(A) \leq \delta_\alpha^f(A)$.

Note From above Lemma if $\delta_\alpha^f(A) = 0$, then $\delta_\beta^f(A) = 0$ for any $\alpha, \beta \in (0,1]$ such that $\alpha \leq \beta$.

Definition 4.1.2 [22] Suppose f is an unbounded modulus and $0 < \alpha \leq 1$. A sequence (x_k) in $\mathbb{R}$ (or $\mathbb{C}$) is called f-statistically convergent of order α to l (or S_α^f-convergent to l) if the following condition is satisfied for every $\varepsilon > 0$:

$$\lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(|\{k \leq n : |x_k - l| \geq \varepsilon\}|) = 0.$$

In this situation, we write $S_\alpha^f - \lim x_k = l$ or $x_k \to (S_\alpha^f)$. We symbolize the class of all S_α^f-convergent sequences by S_α^f, and the class of all f-statistically null sequences of order α will be denoted by $S_{\alpha,0}^f$. That is

$$S_\alpha^f = \left\{(x_k) : \lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(|\{k \leq n : |x_k - l| \geq \varepsilon\}|) = 0 \text{ for every } \varepsilon > 0\right\},$$

$$S_{\alpha,0}^f = \left\{(x_k) : \lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(|\{k \leq n : |x_k| \geq \varepsilon\}|) = 0 \text{ for every } \varepsilon > 0\right\}.$$

Note that if $\alpha = 1$, then the class S_α^f will reduce to the class S^f and if we take $f(x) = x$, then the class S_α^f will reduce to the class S_α, and also in the particular case $\alpha = 1$ and $f(x) = x$, the class S_α^f will reduce to the class S.

Lemma 4.1.2 [22] Let $0 < \alpha \leq 1$ be given and f be an unbounded modulus. Then

(i) $S_{\alpha,0}^f \subset S_\alpha^f$ and the inclusion is strict.

(ii) $c \subset S_\alpha^f$ and the inclusion is strict.

Proof The first inclusion is clear, so it is omitted.

(ii) Let $x = (x_k) \to l$. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_k - l| < \varepsilon$ for all $n > N$, so that we have

$$\lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(|\{k \leq n: |x_k - l| \geq \varepsilon\}|) = \lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(N) = 0$$

and hence $x_k \to l(S_\alpha^f)$. For strict inclusion, let us define the sequence (x_k) as

$$x_k = \begin{cases} 1, & k = n^7 \\ 0, & k \neq n^7 \end{cases} \quad n = 1,2,3, \dots$$

and take $f(x) = x^p$, $0 < p \leq 1$. Then the sequence $(x_k) \in S_\alpha^f$ for $\alpha \in \left(\frac{1}{7}, 1\right]$. But this sequence is not convergent.

Remark 4.1.4 f-statistical convergence of order α is not well defined for $\alpha > 1$. To see this situation, we may choose the sequence $x = (x_k)$ as

$$x_k = \begin{cases} a_1, & k = 2n \\ a_2, & k \neq 2n \end{cases} \quad n = 1,2,3, \dots$$

where $a_1, a_2 \in \mathbb{R}$ and $a_1 \neq a_2$, and take $f(x) = x$. Now if we take any $\alpha > 1$, then we have

$$\lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(|\{k \leq n: |x_k - a_1| \geq \varepsilon\}|) \leq \lim_{n\to\infty} \frac{n}{2n^\alpha} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{f(n^{\alpha})} f(|\{k \le n: |x_k - a_2| \ge \varepsilon\}|) \le \lim_{n\to\infty} \frac{n}{2n^{\alpha}} = 0.$$

It means that $x_k \to a_1(S_\alpha^f)$ and $x_k \to a_2(S_\alpha^f)$ if $\alpha > 1$. But this is not possible.

Theorem 4.1.1 [22] Suppose $\alpha, \beta \in (0,1]$ such that $\alpha \le \beta$ and f is an unbounded modulus. Then $S_\alpha^f \subset S_\beta^f$ and this inclusion is strict.

Proof Suppose $\alpha \le \beta$ so that $n^\alpha \le n^\beta$ and hence $\frac{1}{f(n^\beta)} \le \frac{1}{f(n^\alpha)}$ and let the sequence $x = (x_k)$ be S_α^f-convergent to the number l, i.e. $x_k \to l(S_\alpha^f)$. Then

$$\frac{1}{f(n^{\beta})} f(|\{k \le n: |x_k - l| \ge \varepsilon\}|) \le \frac{1}{f(n^{\alpha})} f(|\{k \le n: |x_k - l| \ge \varepsilon\}|).$$

Taking the limit as $n \to \infty$, we have $x \in S_\alpha^f$ implies $x \in S_\beta^f$. In order to prove the inclusion is strict, we may consider the sequence $x = (x_k)$ as

$$x_k = \begin{cases} 1, & k = n^3 \\ 0, & k \ne n^3 \end{cases} \quad n = 1,2,3,\ldots$$

and take $f(x) = x^p$, $0 < p \le 1$. Then $S_\beta^f - \lim x_k = 0$ for $\frac{1}{3} < \beta \le 1$ and so that $x = (x_k) \in S_\beta^f$, but $x = (x_k) \notin S_\alpha^f$ for $0 < \alpha \le \frac{1}{3}$.

The outcome below is a result of Theorem 4.1.1.

Corollary 4.1.1 [22]

(i) $S_\alpha^f \subset S^f$ for every $\alpha \in (0,1]$ and modulus f,

(ii) $S_\alpha \subset S_\beta$ for every $\alpha, \beta \in (0,1]$ such that $\alpha \le \beta$,

(iii) $S_\alpha \subset S$ for every $\alpha \in (0,1]$.

From Remark 4.1.2 and Definition 4.1.2, we have the following.

Corollary 4.1.2 [22]

(i) $S_\alpha^f \subset S_\alpha$ for every $\alpha \in (0,1]$ and modulus f,

(ii) $S_\alpha^f \subset S$ for every $\alpha \in (0,1]$ and modulus f.

4.2. f-Strong Cesàro Summability of Order α

Definition 4.2.1 [22] Let $\alpha \in (0,1]$ and f be any modulus. A sequence (x_k) in $\mathbb{R}$ (or $\mathbb{C}$) is called f-strongly Cesàro summable of order α if

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k=1}^{n} f(|x_k - l|) = 0$$

for some real number l. In this case, we write $x_k \to l(w_\alpha^f)$. We write w_α^f to symbolize the class of all f-strongly Cesàro summable sequences of order α, and the class of all f-strongly Cesàro summable null sequences of order α is represented by $w_{\alpha,0}^f$. That is

$$w_\alpha^f = \left\{(x_k) : \lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k=1}^{n} f(|x_k - l|) = 0 \text{ for some number } l\right\},$$

$$w_{\alpha,0}^f = \left\{(x_k) : \lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k=1}^{n} f(|x_k|) = 0\right\}.$$

If we take $\alpha = 1$, then the class w_α^f reduces to the class w^f of all f-strongly Cesàro summable sequences. In the case $f(x) = x$, the class w_α^f reduces to the class w_α of all strongly Cesàro summable sequences of order α, and also in the special case $\alpha = 1$ and $f(x) = x$, then then the class w_α^f reduces to the class w of all strongly Cesàro summable sequences. It is clear that $w_{\alpha,0}^f \subset w_\alpha^f$ for every $0 < \alpha \leq 1$ and modulus f.

Theorem 4.2.1 [22] Suppose f is any modulus and $0 < \alpha \leq \beta \leq 1$. Then $w_\alpha^f \subset w_\beta^f$ and the inclusion may remain strict for some $\alpha < \beta$.

Proof Suppose f is a modulus and $x = (x_k) \in w_\alpha^f$. Since $\alpha \leq \beta$, then we may write $\frac{1}{n^\beta} \leq \frac{1}{n^\alpha}$ which implies that

$$\frac{1}{n^\beta}\sum_{k=1}^{n} f(|x_k - l|) = 0 \leq \frac{1}{n^\alpha}\sum_{k=1}^{n} f(|x_k - l|).$$

It means that $w_\alpha^f \subset w_\beta^f$. To prove that $w_\alpha^f \subset w_\beta^f$ is a strict inclusion, consider the sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} 1, & k = n^2 \\ 0, & k \neq n^2 \end{cases} \quad n = 1,2,3, \ldots$$

and take $f(x) = x$. Since

$$\frac{1}{n^\beta}\sum_{k=1}^{n} f(|x_k - 0|) = \frac{1}{n^\beta}\sum_{k=1}^{n} f(x_k) = \frac{1}{n^\beta}\sum_{\substack{k=1 \\ k=n^2}}^{n} f(1) + \frac{1}{n^\beta}\sum_{\substack{k=1 \\ k\neq n^2}}^{n} f(0)$$

$$= \frac{1}{n^\beta}\sum_{\substack{k=1 \\ k=n^2}}^{n} 1 \leq \frac{\sqrt{n}}{n^\beta} \to 0$$

as $n \to \infty$ for $\frac{1}{2} < \beta$, then $x_k \to 0 \left(w_\beta^f\right)$ i.e. $x \in w_\beta^f$. But since

$$\frac{\sqrt{n}-1}{n^\alpha} \leq \frac{1}{n^\alpha}\sum_{k=1}^{n} f(|x_k - 0|)$$

and $\frac{\sqrt{n}-1}{n^\alpha} \to \infty$ as $n \to \infty$ for $\alpha < \frac{1}{2}$, then $x \notin w_\alpha^f$. Therefore, the inclusion $w_\alpha^f \subset w_\beta^f$ is strict.

Taking $\beta = 1$ in Theorem 4.2.1, we get the statement below.

Corollary 4.2.1 [22] $w_\alpha^f \subset w^f$ for every modulus f and every α such that $0 < \alpha \leq 1$.

Taking $f(x) = x$ in Corollary 4.2.1, we get the statement below.

Corollary 4.2.2 [27] $w_\alpha \subset w$ for every $0 < \alpha \leq 1$.

The following theorem was given in [22] with the extra condition "$f(xy) \geq cf(x)f(y)$ for all $x \geq 0, y \geq 0$ and some positive number c" but we will prove it without using this extra condition as follows.

Theorem 4.2.2 Assume that f is an unbounded modulus and $0 < \alpha \leq 1$. If $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, then $w_\alpha^f \subset S_\alpha^f$.

Proof Suppose that $\beta = \lim_{t\to\infty} \frac{f(t)}{t} > 0$. Then by Lemma 3.1.1, we have $\beta = \inf_{t\in(0,\infty)} \frac{f(t)}{t} > 0$ and so that $\beta t \leq f(t)$ for every $t \in [0,\infty)$. Now if $x = (x_k) \in w_\alpha^f$, then we may write

$$\frac{1}{n^\alpha}\sum_{k=1}^{n} f(|x_k - l|) \geq \beta\frac{1}{n^\alpha}\sum_{k=1}^{n}|x_k - l| \geq \beta\frac{1}{n^\alpha}\sum_{\substack{k=1 \\ |x_k - l|\geq\varepsilon}}^{n} |x_k - l|$$

$$\geq \beta\frac{1}{n^\alpha}|\{k \leq n : |x_k - l| \geq \varepsilon\}|\,\varepsilon$$

$$\geq \beta\frac{1}{n^\alpha} f(|\{k \leq n : |x_k - l| \geq \varepsilon\}|)\,\frac{\varepsilon}{f(1)}$$

$$= \frac{f(|\{k \leq n : |x_k - l| \geq \varepsilon\}|)}{f(n^\alpha)}\,\frac{f(n^\alpha)}{n^\alpha}\frac{\varepsilon}{f(1)}\beta.$$

Taking the limit on both sides as $n \to \infty$, we obtain that $x \in w_\alpha^f$ implies $x \in S_\alpha^f$, since $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. Here is the proof.

Taking $\alpha = 1$ in Theorem 4.2.2, we get the statement below.

Corollary 4.2.3 [22] Assume that f is an unbounded modulus. If $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, then $w^f \subset S^f$.

Taking $f(x) = x$ in Theorem 4.2.2, we get the statement below.

Corollary 4.2.4 [27] $w_\alpha \subset S_\alpha$ for every α such that $0 < \alpha \leq 1$.

5. ON STRONG LACUNARY SUMMABILITY OF ORDER α WITH RESPECT TO MODULUS FUNCTIONS

In this section, we establish the relations between $N_\theta^\beta(f)$ and $N_\theta^\alpha(g)$, $N_\theta^\alpha(f)$ and $N_\theta^\beta(g)$, $N_\theta^\beta(f)$ and $S_\theta^\alpha(g)$, $\ell_\infty \cap S_\theta^\alpha(f)$ and $N_{\theta'}^\beta(g)$, where f and g are different modulus functions under some conditions on the considered modulus functions and $\alpha, \beta \in (0,1]$ such that $\alpha \leq \beta$. Furthermore, for some special modulus functions, we obtain the relations between the sets $N_\theta(f)$ and N_θ, $N_\theta^\alpha(f)$ and N_θ^α for $\alpha \in (0,1]$. However the relations between the sets N_θ and S_θ, S_θ and S are known already (see [28]).

5.1. f-Lacunary Statistical Convergence of Order α

We imply an increasing sequence $\theta = (k_r)$ of nonnegative integer numbers with $k_0 = 0$ by a lacunary sequence such that $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals put by θ shall be represented by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ can be shortened by q_r (see [28]).

Fridy and Orhan [28] have defined lacunary statistical convergence as the following expression.

Definition 5.1.1 [28] Suppose $\theta = (k_r)$ is a lacunary sequence. A sequence (x_k) in $\mathbb{R}$ (or $\mathbb{C}$) is called lacunary statistically convergent to l, or simply S_θ-convergent to l, if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - l| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$. For this situation, we write $S_\theta - \lim x_k = l$. Throughout the paper, the class of S_θ-convergent sequences would be symbolized by S_θ.

Definition 5.1.2 [29] Suppose $\theta = (k_r)$ is a lacunary sequence, $0 < \alpha \leq 1$ and f is an unbounded modulus. Then the sequence (x_k) in $\mathbb{R}$ (or $\mathbb{C}$) is called f-lacunary statistically convergent of order α to l, or simply $S_\theta^\alpha(f)$-convergent to l, if

$$\lim_{r\to\infty} \frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |x_k - l| \geq \varepsilon\}|) = 0$$

for every $\varepsilon > 0$. We write $S_\theta^\alpha(f)-\lim x_k = l$ if (x_k) is $S_\theta^\alpha(f)$-convergent to l. Throughout this study, $S_\theta^\alpha(f)$ represents the class of $S_\theta^\alpha(f)$-convergent sequences. That is,

$$S_\theta^\alpha(f) = \left\{(x_k): \lim_{r\to\infty} \frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |x_k - l| \geq \varepsilon\}|) = 0 \text{ for every } \varepsilon > 0 \right\}.$$

We write S_θ^α instead of $S_\theta^\alpha(f)$ in case $f(x) = x$. For $\alpha = 1$, we write $S_\theta(f)$ instead of $S_\theta^\alpha(f)$ and also in the particular case $\alpha = 1$ and $f(x) = x$, we write S_θ instead of $S_\theta^\alpha(f)$.

Remark 5.1.1 It is easy to illustrate that the f-lacunary statistical convergence of order α is not well defined for $\alpha > 1$.

Lemma 5.1.1 The $S_\theta^\alpha(f)$ $-$limit of an $S_\theta^\alpha(f)$-convergent sequence is unique.

Theorem 5.1.1 Suppose $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq 1$. Then

(i) $S_\theta^\alpha(f) \subset S_\theta(f)$ for every unbounded modulus f,

(ii) $S_\theta^\alpha \subset S_\theta$.

The proof is clear, so it is omitted.

5.2. f-Strong Lacunary Summability of Order α

Definition 5.2.1 Suppose $\theta = (k_r)$ is a lacunary sequence, $\alpha \in (0,1]$, and suppose f is a modulus function. Then a sequence (x_k) in $\mathbb{R}$ (or $\mathbb{C}$) is called f-strongly lacunary summable of order α to l, or simply strongly $N_\theta^\alpha(f)$-summable to l, if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} f(|x_k - l|) = 0.$$

If the sequence (x_k) is strongly $N_\theta^\alpha(f)$-summable to l, we write $N_\theta^\alpha(f)\text{-}\lim x_k = l$. The class of strongly $N_\theta^\alpha(f)$-summable sequences will be symbolized by $N_\theta^\alpha(f)$. That is,

$$N_\theta^\alpha(f) = \left\{(x_k) : \lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} f(|x_k - l|) = 0 \text{ for some number } l \right\}.$$

Note that this definition does not require the modulus function f to be unbounded.

The strong $N_\theta^\alpha(f)$-summability will reduce to the strong N_θ^α-summability if we take $f(x) = x$, and in the particular case $\alpha = 1$ and $f(x) = x$, the strong $N_\theta^\alpha(f)$-summability will reduce to the strongly N_θ-summability.

Theorem 5.2.1 Suppose $\theta = (k_r)$ is a lacunary sequence and suppose f and g are modulus functions. If

$$\sup_{x\in(0,\infty)} \frac{f(x)}{g(x)} < \infty,$$

then $N_\theta^\alpha(g) \subset N_\theta^\beta(f)$ for $0 < \alpha \le \beta \le 1$ and the inclusion may be strict.

Proof Suppose $p = \sup_{x\in(0,\infty)} \frac{f(x)}{g(x)} < \infty$. Then we get $0 < \frac{f(x)}{g(x)} \le p$ and so that $f(x) \le pg(x)$ for every $x \in [0,\infty)$. Now it is clear that if (x_k) is strongly $N_\theta^\alpha(g)$-summable to l, we may write

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} f(|x_k - l|) \le \frac{1}{h_r^\alpha}\sum_{k\in I_r} pg(|x_k - l|).$$

Since $\alpha \le \beta$, then

$$\frac{1}{h_r^\beta}\sum_{k\in I_r} f(|x_k - l|) \le p\frac{1}{h_r^\alpha}\sum_{k\in I_r} g(|x_k - l|).$$

Taking limit on both sides as $r \to \infty$, we obtain that $(x_k) \in N_\theta^\alpha(g)$ implies $(x_k) \in N_\theta^\beta(f)$. The following example shows that the inclusion $N_\theta^\alpha(g) \subset N_\theta^\beta(f)$ is strict at least for some $\alpha, \beta \in (0, 1]$ such that $\alpha \le \beta$ and special modulus functions f and g.

Example 5.2.1 Let the lacunary sequence $\theta = (k_r)$ be given and choose $\alpha = \beta = 1$ and also consider the sequence (x_k) such that x_k to be $[\sqrt{h_r}]$ at the first $[\sqrt{h_r}]$ integers in I_r, and $x_k = 0$ otherwise. Now if we take the modulus functions $f(x) = \frac{x}{x+1}$ and $g(x) = x$, then $\sup_{x\in(0,\infty)} \frac{f(x)}{g(x)} = 1 < \infty$. By using the $f(0) = 0$ equality, we have

$$\frac{1}{h_r^\beta}\sum_{k\in I_r} f(|x_k|) = \frac{1}{h_r}[\sqrt{h_r}]f([\sqrt{h_r}]) = \frac{[\sqrt{h_r}][\sqrt{h_r}]}{h_r([\sqrt{h_r}] + 1)}.$$

By taking limits as $r \to \infty$ we get that $N_\theta^\beta(f)$-$\lim x_k = 0$, so $(x_k) \in N_\theta^\beta(f)$. But since

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} g(|x_k|) = \frac{1}{h_r}\left[\sqrt{h_r}\right]g\left(\left[\sqrt{h_r}\right]\right) = \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r}$$

and since $\frac{[\sqrt{h_r}][\sqrt{h_r}]}{h_r} \to 1$ as $r \to \infty$, we get $(x_k) \notin N_\theta^\alpha(g)$. Hence $(x_k) \in N_\theta^\beta(f) - N_\theta^\alpha(g)$ and the inclusion $N_\theta^\alpha(g) \subset N_\theta^\beta(f)$ is being strict.

The outcome below of inclusions is obtained from Theorem 5.2.1.

Corollary 5.2.1 Suppose f and g are modulus functions, $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq \beta \leq 1$.

(i) If $\sup_{x\in(0,\infty)} \frac{f(x)}{g(x)} < \infty$, then $N_\theta^\alpha(g) \subset N_\theta^\alpha(f)$.

(ii) If $\sup_{x\in(0,\infty)} \frac{f(x)}{g(x)} < \infty$, then $N_\theta(g) \subset N_\theta(f)$.

(iii) $N_\theta^\alpha(f) \subset N_\theta^\beta(f)$.

(iv) $N_\theta^\alpha \subset N_\theta^\beta$.

Theorem 5.2.2 Suppose $\theta = (k_r)$ is a lacunary sequence, and suppose f and g are modulus functions. If

$$\inf_{x\in(0,\infty)} \frac{f(x)}{g(x)} > 0,$$

then $N_\theta^\alpha(f) \subset N_\theta^\beta(g)$ for $0 < \alpha \leq \beta \leq 1$ and the inclusion may be strict.

Proof Assuming that $q = \inf_{x\in(0,\infty)} \frac{f(x)}{g(x)} > 0$. Then $\frac{f(x)}{g(x)} \geq q$ and so that $qg(x) \leq f(x)$ for every $x \in [0,\infty)$. Now if (x_k) is strongly $N_\theta^\alpha(f)$-summable to l, we may write

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} g(|x_k - l|) \leq \frac{1}{h_r^\alpha}\sum_{k\in I_r} \frac{1}{q} f(|x_k - l|).$$

Since $\alpha \leq \beta$, then

$$\frac{1}{h_r^{\beta}}\sum_{k\in I_r} g(|x_k - l|) \leq \frac{1}{h_r^{\alpha}}\sum_{k\in I_r}\frac{1}{q} f(|x_k - l|).$$

Taking limit on both sides as $r \to \infty$, we obtain that $(x_k) \in N_\theta^\alpha(f)$ implies $(x_k) \in N_\theta^\beta(g)$. For the strict inclusion, the sequence of Example 5.2.1 with modulus functions $g(x) = \frac{x}{x+1}$ and $f(x) = x$ serve the purpose in the case $\alpha = \beta = 1$.

The outcome below of strict inclusions is a result of Theorem 5.2.2.

Corollary 5.2.2 Suppose f and g are modulus functions, $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq \beta \leq 1$.

(i) If $\inf_{x\in(0,\infty)} \frac{f(x)}{g(x)} > 0$, then $N_\theta^\alpha(f) \subset N_\theta^\alpha(g)$.

(ii) If $\inf_{x\in(0,\infty)} \frac{f(x)}{g(x)} > 0$, then $N_\theta(f) \subset N_\theta(g)$.

(iii) $N_\theta^\alpha(f) \subset N_\theta^\beta(f)$.

The following result is obtained from Theorem 5.2.1 and Theorem 5.2.2.

Corollary 5.2.3 Suppose f and g are modulus functions, $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq \beta \leq 1$. If

$$0 < \inf_{x\in(0,\infty)} \frac{f(x)}{g(x)} \leq \sup_{x\in(0,\infty)} \frac{f(x)}{g(x)} < \infty,$$

then $N_\theta^\alpha(f) = N_\theta^\beta(g)$.

Corollary 5.2.4 Suppose $\theta = (k_r)$ is a lacunary sequence, and suppose f is any modulus function. If $\sup_{x\in(0,\infty)} \frac{f(x)}{x} < \infty$, then $N_\theta^\alpha \subset N_\theta^\beta(f)$ for any $\alpha, \beta \in (0,1]$ such that $\alpha \leq \beta$.

Since $\sup_{x\in(0,\infty)} \frac{f(x)}{x} < \infty$, taking $g(x) = x$ in Theorem 5.2.1 the proof follows directly.

The following result is obtained by taking $\beta = \alpha$ in the above corollary.

Corollary 5.2.5 Suppose $\theta = (k_r)$ is a lacunary sequence, and suppose f is any modulus function. If $\sup_{x\in(0,\infty)} \frac{f(x)}{x} < \infty$, then $N_\theta^\alpha \subset N_\theta^\alpha(f)$ for any $\alpha \in (0,1]$.

Corollary 5.2.6 Suppose $\theta = (k_r)$ is a lacunary sequence, and suppose f is any modulus function. If $\inf_{x\in(0,\infty)} \frac{f(x)}{x} > 0$, then $N_\theta^\alpha(f) \subset N_\theta^\beta$ for any $\alpha, \beta \in (0,1]$ such that $\alpha \le \beta$.

Since $\inf_{x\in(0,\infty)} \frac{f(x)}{x} > 0$, taking $g(x) = x$ in Theorem 5.2.2 the proof follows directly.

The following result is obtained by taking $\beta = \alpha$ in the above corollary.

Corollary 5.2.7 Suppose $\theta = (k_r)$ is a lacunary sequence, and suppose f is any modulus function. If $\inf_{x\in(0,\infty)} \frac{f(x)}{x} > 0$, then $N_\theta^\alpha(f) \subset N_\theta^\alpha$ for any $\alpha \in (0,1]$.

From Corollary 5.2.5 and Corollary 5.2.7 we get the following result.

Corollary 5.2.8 Suppose $\theta = (k_r)$ is a lacunary sequence and f is a modulus function. If $0 < \inf_{x\in(0,\infty)} \frac{f(x)}{x} \le \sup_{x\in(0,\infty)} \frac{f(x)}{x} < \infty$, then $N_\theta^\alpha(f) = N_\theta^\alpha$ for any $\alpha \in (0,1]$.

Theorem 5.2.3 For any modulus function f we have $N_\theta \subset N_\theta(f)$.

Proof Assume that $(x_k) \in N_\theta$ so that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - l| = 0$$

for some l. Given $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $t \in (0,\delta]$. Now consider

$$\sum_{k\in I_r} f(|x_k - l|) = \sum_{\substack{k\in I_r \\ |x_k - l| \le \delta}} f(|x_k - l|) + \sum_{\substack{k\in I_r \\ |x_k - l| > \delta}} f(|x_k - l|)$$

Since $f(|x_k - l|) < \varepsilon$ for $|x_k - l| \le \delta$, then

$$\sum_{\substack{k\in I_r \\ |x_k - l| \le \delta}} f(|x_k - l|) < \varepsilon h_r,$$

and also for $|x_k - l| > \delta$, we have

$$|x_k - l| < \frac{|x_k - l|}{\delta} < 1 + \left[\frac{|x_k - l|}{\delta}\right].$$

Since f is a modulus, so that

$$f(|x_k - l|) \leq f\left(1 + \left[\frac{|x_k - l|}{\delta}\right]\right) \leq f(1)\left(1 + \left[\frac{|x_k - l|}{\delta}\right]\right) \leq 2f(1)\frac{|x_k - l|}{\delta}.$$

So,

$$\sum_{\substack{k\in I_r \\ |x_k-l|>\delta}} f(|x_k - l|) \leq \frac{2f(1)}{\delta} \sum_{\substack{k\in I_r \\ |x_k-l|>\delta}} |x_k - l|$$

Thus,

$$\frac{1}{h_r}\sum_{k\in I_r} f(|x_k - l|) \leq \varepsilon + \frac{2f(1)}{\delta}\frac{1}{h_r} \sum_{\substack{k\in I_r \\ |x_k-l|>\delta}} |x_k - l|$$

$$\leq \varepsilon + \frac{2f(1)}{\delta}\frac{1}{h_r}\sum_{k\in I_r}|x_k - l|.$$

Since $(x_k) \in N_\theta$, then we get $(x_k) \in N_\theta(f)$.

Corollary 5.2.9 Suppose $\theta = (k_r)$ is a lacunary sequence and f is a modulus function. If $\inf_{x\in(0,\infty)} \frac{f(x)}{x} > 0$, then $N_\theta(f) = N_\theta$.

Since $N_\theta \subset N_\theta(f)$ for any modulus f by Theorem 5.2.3, taking $g(x) = x$ and $\alpha = \beta = 1$ in Theorem 5.2.2 we get $N_\theta(f) \subset N_\theta$. Therefore $N_\theta(f) = N_\theta$ if $\inf_{x\in(0,\infty)} \frac{f(x)}{x} > 0$.

Theorem 5.2.4 Assume that f and g are unbounded modulus functions, $\theta = (k_r)$ is a lacunary sequence and $0 < \alpha \leq \beta \leq 1$. If $\inf_{x\in(0,\infty)} \frac{f(x)}{g(x)} > 0$ and $\lim_{x\to\infty} \frac{g(x)}{x} > 0$, then every strongly $N_\theta^\alpha(f)$-summable sequence is $S_\theta^\beta(g)$-statistically convergent.

Proof Suppose that $q = \inf_{x\in(0,\infty)} \frac{f(x)}{g(x)} > 0$. Then $\frac{f(x)}{g(x)} \geq q$ and so $qg(x) \leq f(x)$ for every $x \in [0,\infty)$. Now if (x_k) is strongly $N_\theta^\beta(f)$-summable to l and $0 < \alpha \leq \beta \leq 1$, we may write

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} f(|x_k - l|) \geq q\frac{1}{h_r^\alpha}\sum_{k\in I_r} g(|x_k - l|)$$

$$\geq q\frac{1}{h_r^{\beta}}\sum_{k\in I_r} g(|x_k-l|)$$

$$= q\frac{1}{h_r^{\beta}}\sum_{\substack{k\in I_r\\ |x_k-l|\geq\varepsilon}} g(|x_k-l|) + q\frac{1}{h_r^{\beta}}\sum_{\substack{k\in I_r\\ |x_k-l|<\varepsilon}} g(|x_k-l|)$$

$$\geq q\frac{1}{h_r^{\beta}}\sum_{\substack{k\in I_r\\ |x_k-l|\geq\varepsilon}} g(|x_k-l|)$$

$$\geq q\frac{1}{h_r^{\beta}}|\{k\in I_r : |x_k-l|\geq\varepsilon\}|\, g(\varepsilon).$$

Now since $|\{k\in I_r : |x_k-l|\geq\varepsilon\}|$ is a positive integer, then we have

$$\frac{1}{h_r^{\alpha}}\sum_{k\in I_r} f(|x_k-l|) \geq \frac{1}{h_r^{\beta}} g(|\{k\in I_r : |x_k-l|\geq\varepsilon\}|)\,\frac{g(\varepsilon)}{g(1)}q$$

$$= \frac{g(|\{k\in I_r : |x_k-l|\geq\varepsilon\}|)}{g\left(h_r^{\beta}\right)}\,\frac{g\left(h_r^{\beta}\right)}{h_r^{\beta}}\frac{g(\varepsilon)}{g(1)}q.$$

Taking the limit on both sides as $r\to\infty$, we obtain that $(x_k)\in N_\theta^{\alpha}(f)$ implies $(x_k)\in S_\theta^{\beta}(g)$ since $\lim_{x\to\infty}\frac{g(x)}{x} > 0$. Here is the proof.

Remark 5.2.1 In general, contrary to the above theorem could not be possible. This fact can be seen in the illustration below.

Example 5.2.2 Let θ be given and select the sequence (x_k) as in Example 5.2.1 and also consider the modulus functions $f(x) = x = g(x)$. Then, $\inf_{x\in(0,\infty)}\frac{f(x)}{g(x)} > 0$ and $\lim_{x\to\infty}\frac{g(x)}{x} > 0$. Now if we take $0 < \alpha \leq \frac{1}{2} < \beta \leq 1$, then for every $\varepsilon > 0$, we have

$$\lim_{r\to\infty}\frac{1}{g\left(h_r^{\beta}\right)} g(|\{k\in I_r : |x_k-0|\geq\varepsilon\}|) = \lim_{r\to\infty}\frac{\left[\sqrt{h_r}\right]}{h_r^{\beta}} = 0.$$

So, $(x_k)\in S_\theta^{\beta}(g)$. On the other hand, we have

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r} f(|x_k-0|)=\lim_{r\to\infty}\frac{[\sqrt{h_r}][\sqrt{h_r}]}{h_r^\alpha}=\infty .$$

So, $(x_k)\notin N_\theta^\alpha(f)$.

The following result is obtained by taking $g(x)=f(x)$ in Theorem 5.2.4.

Corollary 5.2.10 Assume that f is an unbounded modulus function, $\theta=(k_r)$ is a lacunary sequence and $0<\alpha\le\beta\le 1$. If $\lim_{x\to\infty}\frac{f(x)}{x}>0$, every strongly $N_\theta^\alpha(f)$-summable sequence is $S_\theta^\beta(f)$-statistically convergent.

The following result is obtained by taking $\beta=\alpha$ in Theorem 5.2.4.

Corollary 5.2.11 Assume that f and g are unbounded modulus functions, $\theta=(k_r)$ is a lacunary sequence and $0<\alpha\le 1$. If $\inf_{x\in(0,\infty)}\frac{f(x)}{g(x)}>0$ and $\lim_{x\to\infty}\frac{g(x)}{x}>0$, every strongly $N_\theta^\alpha(f)$-summable sequence is $S_\theta^\alpha(g)$-statistically convergent.

The following result is obtained by taking $g(x)=x$ in Corollary 5.2.11, which is also Theorem 2.9 of [30], for the case $p=1$.

Corollary 5.2.12 Assume that f is an unbounded modulus function and $\theta=(k_r)$ is a lacunary sequence. If $\inf_{x\in(0,\infty)}\frac{f(x)}{x}>0$, then every strongly $N_\theta^\alpha(f)$-summable sequence is S_θ^α-statistically convergent.

We obtain the result below by taking $\alpha=1$ in Corollary 5.2.12.

Corollary 5.2.13 Assume that f is an unbounded modulus function and $\theta=(k_r)$ is a lacunary sequence. If $\inf_{x\in(0,\infty)}\frac{f(x)}{x}>0$, then every strongly $N_\theta(f)$-summable sequence is S_θ-statistically convergent.

The following result is obtained by taking $f(x)=x$ in Corollary 5.2.13, which is also the first part of Theorem 1 of [28].

Corollary 5.2.14 $N_\theta\subset S_\theta$ for any lacunary sequence $\theta=(k_r)$.

Theorem 5.2.5 Suppose f and g are any unbounded modulus functions, $0 < \alpha \leq \beta \leq 1$, and suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are lacunary sequences such that $I_r \subset J_r$ for each $r \in \mathbb{N}$. If $\lim_{r\to\infty} \frac{v_r}{h_r^\beta} = 1$ and $\sup_{x\in(0,\infty)} \frac{g(x)}{x} < \infty$, then every bounded and $S_\theta^\alpha(f)$-convergent sequence is strongly $N_{\theta'}^\beta(g)$-summable, i.e.

$$\ell_\infty \cap S_\theta^\alpha(f) \subset N_{\theta'}^\beta(g)$$

and the inclusion may be strict.

Proof Let f and g be unbounded modulus functions, $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$, $v_r = s_r - s_{r-1}$ and $0 < \alpha \leq \beta \leq 1$. Suppose that $(x_k) \in \ell_\infty \cap S_\theta^\alpha(f)$ and $S_\theta^\alpha(f)$-$\lim x_k = l$. In order to verify that $(x_k) \in N_{\theta'}^\beta(g)$, we shall first prove that $S_\theta^\alpha(f) \subset S_\theta^\alpha$. Since f is a modulus and $S_\theta^\alpha(f)$-$\lim x_k = l$, then for each $p \in \mathbb{N}$, there exists $r_0 \in \mathbb{N}$ such that, if $r > r_0$, we have

$$f(|\{k \in I_r : |x_k - l| \geq \varepsilon\}|) \leq \frac{1}{p} f(h_r^\alpha) \leq \frac{1}{p} p f\left(\frac{h_r^\alpha}{p}\right) = f\left(\frac{h_r^\alpha}{p}\right)$$

for every $\varepsilon > 0$. So,

$$\frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - l| \geq \varepsilon\}| \leq \frac{1}{p}.$$

It follows that $S_\theta^\alpha(f) \subset S_\theta^\alpha$ and so that $\ell_\infty \cap S_\theta^\alpha(f) \subset \ell_\infty \cap S_\theta^\alpha$. Since $\lim_{r\to\infty} \frac{v_r}{h_r^\beta} = 1$, so we have $\ell_\infty \cap S_\theta^\alpha \subset N_{\theta'}^\beta$ by the second part of Theorem 2.14 of [31]. On the other hand, since $\sup_{x\in(0,\infty)} \frac{g(x)}{x} < \infty$, we have $N_{\theta'}^\beta \subset N_{\theta'}^\beta(g)$ by Corollary 5.2.5. Thus, $\ell_\infty \cap S_\theta^\alpha(f) \subset N_{\theta'}^\beta(g)$.

The following example shows that the inclusion $\ell_\infty \cap S_\theta^\alpha(f) \subset N_\theta^\beta(g)$ is strict at least for some $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$ and some special modulus functions f and g.

Example 5.2.3 As an example, let the lacunary sequence $\theta = (k_r)$ be provided and $\theta' = \theta$. Consider the sequence (x_k) such that x_k to be $\left[\sqrt[3]{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $x_k = 0$ otherwise and also consider the modulus functions $f(x) = g(x) = x$. Now if we take

$0 < \alpha \leq \frac{1}{2}$ and $\beta = 1$, then $\lim_{r\to\infty} \frac{v_r}{h_r^\beta} = \lim_{r\to\infty} \frac{h_r}{h_r} = 1$ and $\sup_{x\in(0,\infty)} \frac{g(x)}{x} = 1 < \infty$. Also for every $r \in \mathbb{N}$, we have

$$\frac{1}{h_r^\beta} \sum_{k\in I_r} g(|x_k - 0|) = \frac{1}{h_r^\beta} \sum_{k\in I_r} g\left(\left[\sqrt[3]{h_r}\right]\right) = \frac{\left[\sqrt{h_r}\right]\left[\sqrt[3]{h_r}\right]}{h_r}.$$

Since $\frac{[\sqrt{h_r}][\sqrt[3]{h_r}]}{h_r} \to 0$ as $r \to \infty$, then $(x_k) \in N_{\theta'}^\beta(g)$. But for every $\varepsilon > 0$, we have

$$\frac{1}{h_r^\alpha} f(|\{k \in I_r : |x_k - 0| \geq \varepsilon\}|) = \frac{1}{h_r^\alpha} f\left(\left[\sqrt{h_r}\right]\right) = \frac{\left[\sqrt{h_r}\right]}{h_r^\alpha}.$$

So, $(x_k) \notin S_\theta^\alpha(f)$ since $\frac{[\sqrt{h_r}]}{h_r^\alpha} \to \infty$ as $r \to \infty$ for $0 < \alpha < \frac{1}{2}$ and $\frac{[\sqrt{h_r}]}{h_r^\alpha} \to 1$ as $r \to \infty$ for $\alpha = \frac{1}{2}$. Therefore, the inclusion $\ell_\infty \cap S_\theta^\alpha(f) \subset N_{\theta'}^\beta(g)$ is strict.

The outcome below of inclusions is a result of Theorem 5.2.5.

Corollary 5.2.15 Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are lacunary sequences, $0 < \alpha \leq \beta \leq 1$, and suppose f is any unbounded modulus function. If $\lim_{r\to\infty} \frac{v_r}{h_r^\beta} = 1$ and $\sup_{x\in(0,\infty)} \frac{f(x)}{x} < \infty$, then

(i) $\ell_\infty \cap S_\theta^\alpha(f) \subset N_{\theta'}^\beta(f)$,

(ii) $\ell_\infty \cap S_\theta^\alpha(f) \subset N_{\theta'}^\alpha(f)$,

(iii) $\ell_\infty \cap S_\theta^\alpha \subset N_{\theta'}^\alpha(f)$.

CONCLUSION

In this study, by using a modulus function f, we have given f-statistical convergence, f-statistical boundedness, f-strong Cesàro summability, f-statistical convergence of order α for $0 < \alpha \leq 1$ and f-strong Cesàro summability of order α for $0 < \alpha \leq 1$. We have also given some relations between the sets of f-statistically convergent sequences and f-statistically bounded sequences, f-statistically convergent sequences and f-strongly Cesàro summable sequences.

Furthermore, we have established the relations between w^f and w^g, w^f and S^g, for different modulus functions f and g under some conditions on the considered modulus functions. Also for some special modulus functions, we have obtained the relations between the sets w^f and w, S^f and S.

Finally, we have established the relations between $N_\theta^\beta(f)$ and $N_\theta^\alpha(g)$, $N_\theta^\alpha(f)$ and $N_\theta^\beta(g)$, $N_\theta^\beta(f)$ and $S_\theta^\alpha(g)$, $\ell_\infty \cap S_\theta^\alpha(f)$ and $N_{\theta'}^\beta(g)$, where f and g are different modulus functions under some conditions and $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$. Also, for some special modulus functions, we have obtained the relations between the sets $N_\theta(f)$ and N_θ, $N_\theta^\alpha(f)$ and N_θ^α for $\alpha \in (0, 1]$.

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BACKGROUND

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