T.C.
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THE GRADUATE SCHOOL OF NATURAL AND
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SOME CHARACTERIZATIONS OF AW(k)-TYPE CURVES

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MASTERS THESIS

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Muhammad ABUBAKAR ISAH

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TABLE OF CONTENTS

Page No

SUMMARY

Some characterizations of AW(k)-type curves

In curve theory, we can obtain alot of special curves by using Frenet frame. In this thesis, we examined AW(k)-type curves, heix and slant helices.

AW(k)-type curves have been studied by many mathematicians. Özgür and Gezgin have studied AW(k)-type curves in E^3 [7] . Külahcı, Bektaş and Ergüt have studied harmonic curvatures of null curves of the AW(k)-type curves in Lorentzian space [9] ..Kula and Et Al gave the characterizations of slant helices in Euclidean 3-space [8] . Yıldırım Yılmaz and Et Al studied slant helix in Riemann-otsuki space [10] . Ahmad gave general helices in Euclidean 3-space [13] .

In this thesis, we studied AW(k)-type curves of Pseudo null and Null cartan curves in Minkowski 3-space E_1^3. Furthermore, we defined helix and slant helix according to Bishop frame in E_1^3. Additionally, we gave some necessary and sufficient conditions for the slant helix and helix in Minkowski 3-space.

Key words: Pseudo null curve, null cartan curve, Bishop frame, helix, slant helix, Minkowski 3-space.

ÖZET

AW(k)-tipinden Eğrilerin Bazı Karakterizasyonları

Eğriler teoresinde, Frenet çatısını kullanarak birçok özel eğri elde edebiliriz. Bu tezde, AW(k)-tipinden eğriler, helis ve slant helisleri inceledik.

AW(k)-tipinden eğriler birçok matematikçi tarafından çalışıldı. Özgür ve Gezgin E^3 de AW(k)-tipinden eğrileri çalıştı [7] . Külahcı, Bektaş ve Ergüt Lorentz uzayında AW(3)-tipinden null eğrilerin harmonik eğriliklerini çalıştı [9]. Kula ve diğerleri Öklid 3-uzayında slant helislerin Karakterizasyonlarını verdiler [8]. Yıldırım Yılmaz ve diğerleri Riemann-otsuki uzayında slant helis çalıştılar [10]. Ahmad, Öklid 3-uzayında genel helisleri verdi [13].

Bu tezde, Minkowski 3-uzayı E_1^3 de Bishop çatısına göre Pseudo null ve Null cartan eğrilerin AW(k)-tipinden eğrileri çalıştık. Bundan başka, E^3 de Bishop çatısına göre helis ve slant helis tanımladık. Ayrıca, Minkowski 3-uzayında slant helis ve helis için gerekli ve yeter şartları verdik.

Anahtar Kelimeler: Pseudo null eğri, null cartan eğri, Bishop çatı, helis, slant helis, Minkowski 3-uzayı.

1. INTRODUCTION

It would be a great asset for mathematicians to be able to "walk" along a three-dimensional space curve and demonstrate the curve's properties, such as curvature and torsion. This ability is provided by the classic Serret-Frenet frame. The tangent, normal and binormal vector fields can be called TNB frame. But at some points, the curve may not be continuous, which is undefined when the curve's second derivative vanishes [1]. In 1975, Richard Lawrence Bishop first presented the parallel frame as a new frame that is well defined even if the curve has a second derivative disappearing, then in the research the parallel frame came to be called the Bishop frame [1, 2, 3]. Bishop frame includes the tangential vector field T and two normal vector fields N_1 and N_2, that is found by rotating the Frenet vectors N and B in the normal curve plane $T^{\perp}$ so that they are relatively parallel [3]. The Bishop frames are used in the field of Biology and Computer Graphics. For instance, data on the shape of DNA sequences can be determined by applying a curve defined by the Bishop frame. It also provides a new way of controlling virtual cameras in computer animation [4]. Some Bishop frame apps can be found in Minkowski spaces [5, 6].

The pseudo null and null Cartan curve in Minkowski 3-space are known as the curves with two null vector fields in the Frenet and Cartan frames respectively [3].

In differential geometry, a general helix in Euclidean 3-space R^3 is defined in such a way that the tangent makes a constant angle with a fixed direction [7, 8, 9, 10]. In nature, helical structures are found in nano-springs, helices, DNA, carbon nano-tubes, bacterial flagella in salmonella and escherichia coli, tendrils, stems, screws, helical staircases, bacterial shape in spirochets, and sea shells [11, 12, 13]. In fractal geometry, for instance hyper-helices, helical structures are used.

If the principal normal N makes a constant angle with a fixed direction, this curve can be called a slant helix in Euclidean 3-space [8] Furthermore, Izumiya and Takeuchi [14] have shown that γ is a slant helix in E^3 necessary and sufficient the geodesic curvature of the principal normal of γ is a constant function.

In curves theory, J. Bertrand examined curves in Euclidean 3-space E^3, the principal normals of which are the principal normals of other curve. Thus, the Bertrand curve is obtained. These curves have a distinctive feature that first and second curvatures are in linear relationship [7, 15].

2. THEOREMS AND DEFINITIONS

Definition 2.1

For a given function F of two real variables x, y. $F(x,y) = 0$ describes a "*level curve*" whenever the gradient of F does not vanish, that is to say if $\frac{\partial F}{\partial x} \neq 0$ or $\frac{\partial F}{\partial y} \neq 0$ at every point satisfying $F(x,y) = 0$ [16].

Definition 2.2

For a given function F of x, y, z the equation $F(x,y,z) = 0$ describes a "*level surface*" whenever the gradient of F does not vanish, i.e, if $\frac{\partial F}{\partial x} \neq 0$ or $\frac{\partial F}{\partial y} \neq 0$ or $\frac{\partial F}{\partial z} \neq 0$ at every point satisfying $F(x,y,z) = 0$ [16].

Definition 2.3

A *regular curve* is a continuously differentiable immersion $\gamma : I \to E^n$, defined on a real interval $I \subseteq R$. Thus, $\dot{\gamma} = \frac{d\gamma}{dt} \neq 0$ holds everywhere [3].

Definition 2.4

Let $\gamma : I \to E^n$ be a curve. γ can be called regular if $\gamma'(t)$ is always non-zero $(|\gamma'(t)| = 0 \ for \ all \ t \in I)$. It can be called *unit-speed* if $|\gamma'(t)| = 1 \ \ for \ all \ t \in I$ [17].

Definition 2.5

Euclidean n-space E^n is defined as the set of $P = (p^1, ..., p^n)$, where $p^i \in R$, for each $i = 1, ..., n$. Given any two n-tuples $P = (p^1, ..., p^n)$, $q = (q^1, ..., q^n)$ and any real number c, we define two operations

$$\begin{aligned} p + q &= (p^1 + q^1, ..., p^n + q^n) \\ cp &= (cp^1, ..., cp^n), \end{aligned}$$

as *sum* and *scalar multiplication* of vectors respectively, Euclidean space requires the structure of a vector space of n dimensions [18].

Definition 2.6

The angle between vectors $x \neq 0$ and $y \neq 0$ is defined to be the number $\theta \in [0, \pi]$ for which

$$\theta = \arccos\left(\frac{\langle x, y\rangle}{\|x\| \, \|y\|}\right) \quad [17].$$

Definition 2.7

(A, V, f) is a real affine space where A is a set of points, V is a real vector space and

$f : A \times A \to V$ is a map endorsing:

1. $\forall P \in A$ and $\forall u \in V$ there is a unique $Q \in A$ such that

$$f(P, Q) = u$$

2. $f(P, Q) + f(Q, R) = f(P, R)$ for every $P, Q, R \in A$.

Notation We can show $(P, Q) = \overline{PQ}$. The elements contained on the set A are called points of A and we will say that V is the vector space associated to the affine space (A, V, f) [19].

Definition 2.8

A set of vectors in V can be called *linearly dependent* if there exist $a_1, ..., a_m \in F$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$ [20].

Definition 2.9

A set of vectors is called *orthonormal* if

$\langle e_j, e_k \rangle = 0$ when $j \neq k$ and $\langle e_j, e_k \rangle = 1$ when $j = k$ $(for \ \ j, k = 1, ..., m)$ [20].

Definition 2.10

Let $\gamma(s)$ be a regular curve in E^n, which is parametrized by arc length and $n-$times continuously differentiable. Then γ is called a *Frenet curve*, if at every point the vectors $\gamma', \gamma'', ..., \gamma^{(n-1)}$ are linearly independent. The Frenet $n-$frame $e_1, e_2, ..., e_n$ is then uniquely defined by the following statements:

(i) $e_1, e_2, ..., e_n$ are orthonormal and positively oriented.

(ii) For every $k = 1, ..., n-1$ one has $sp(e_1, e_2, ..., e_n) = sp(\gamma', \gamma'', ..., \gamma^{(k)})$, where sp represents the linear span.

(iii) $\langle \gamma^{(k)}, e_k \rangle > 0$ for $k = 1, ..., n-1$ [16].

Definition 2.11

Let $\gamma : I \to E^3$ be a regular space curve, let $t \in I$ with $k(t) \neq 0$, the Frenet frame at t is the basis $\{T(s), N(s), B(s)\}$ of E^3 defined as

$$\begin{aligned} T(t) &= \frac{v(t)}{|v(t)|} \quad (tangent) \text{ where } v(t) = \gamma'(t) \\ N(t) &= \frac{T'(t)}{|T'(t)|} \quad (normal) \\ B(t) &= T(t) \times N(t) \quad (binormal) \end{aligned}$$

The triple $\{T(s), N(s), B(s)\}$ is called the *Frenet-Serret Frame* of R^3 at the point $\gamma(s)$ of the curve. The important of this basis over the fixed basis (i, j, k) is that the Frenet frame is naturally adapted to the curve. It spreads along with the curve with the *tangent vector* always pointing in the *direction of motion*, and the *normal and binormal vectors* pointing in the directions in which the curve is *tending to curve* [17, 18] .

Definition 2.12

Let γ be a Frenet curve in E^n with Frenet n-frame $e_1, ..., e_n$. Then there are functions $k_1, ..., k_{n-1}$ defined on that curve with $k_1, ..., k_{n-2} > 0$, so that every k_i is $(n-1-i)$-times continuously differentiable and

$$\begin{bmatrix} e_1' \\ e_2' \\ \vdots \\ \vdots \\ \vdots \\ e_{n-1}' \\ e_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & 0 & \cdots & 0 \\ -k_1 & 0 & k_2 & 0 & 0 & \ddots & \vdots \\ 0 & -k_2 & 0 & 0 & & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & k_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ \vdots \\ \vdots \\ e_{n-1} \\ e_n \end{bmatrix}$$

k_i is called the $i-th$ Frenet curvature and the equations are called the *Frenet equations* [16] .

Definition 2.13

Let $\gamma(s) : I \to E^n$ be an arc-length parametrization of a path γ in E^n. The curvature of γ is a function $k : I \to R$ defined as follows:

(1) For a plane curve $\gamma(s)$ let

$$T'(s) = k(s)N(s)$$

and thus

$$k(s) = \pm|T^{'}(s)|$$

(2) For a space curve $\gamma(s)$

$$k(s) = |T^{'}(s)| \geq 0 \qquad [12]\,.$$

Definition 2.14

Let $t \in I$ and assume that $k\,(s) \neq 0$ (non zero curvature) the number

$$\tau\,(s) = \frac{\det\left[\gamma'\,(t)\,,\gamma''\,(t)\,,\gamma'''\,(t)\right]}{\|\gamma'\,(t) \times \gamma''\,(t)\|^2}$$

is called the torsion of γ at t [22].

Definition 2.15

A differential equation is an ordinary differential equation if it involves an unknown function of only one variable The simplest differential equations are first order equations of the form

$$\frac{dy}{dx} = f\,(x)$$

or, equivalently,

$$y^{'} = f\,(x)\,,$$

where f is a defined function of x [23] .

Definition 2.16.

The Minkowski 3-space E_1^3 is the real vector space E^3 which is included with the standard indefinite flat metric $\langle .,. \rangle$ defined by

$$\langle u, v\rangle = -u_1 v_1 + u_2 v_2 + u_3 v_3, \tag{2.1}$$

for any two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in E_1^3. Since $\langle .,. \rangle$ is an indefinite metric, an arbitrary vector $u \in E_1^3 \backslash \{0\}$ can have one of three causal characters:

i) it can be space-like, if $\langle u, u\rangle_1 > 0$,

ii) time-like, if $\langle u, u\rangle_1 < 0$ or

iii) light-like or isotropic or null vector, if $\langle u, u\rangle_1 = 0$, but $u \neq 0$.

In particular, the norm (length) of a non lightlike vector $u \in E_1^3$ is given by

$$\|u\| = \sqrt{|\langle u, u \rangle|}.$$

Given a regular curve $\gamma : I \to E_1^3$ can locally be spacelike, timelike or null (lightlike), if $\gamma'(t)$ fulfills $\langle \gamma'(t), \gamma'(t) \rangle_1 > 0$, $\langle \gamma'(t), \gamma'(t) \rangle_1 < 0$, *or* $\langle \gamma'(t), \gamma'(t) \rangle_1 = 0$, respectively, at any $t \epsilon I$, where $\gamma'(t) = \frac{d\gamma}{dt}$ [3].

Definition 2.17.

The curve $\gamma(t)$ is said to be a geodesic if $k_g(t) = |\gamma''(t)| = 0$ for all t [24].

Definition 2.18.

A spacelike curve $\gamma : I \to E_1^3$ can be called a pseudo null curve if $\langle N, B \rangle = 1$ is satisfied, where N is principal normal vector field and B is binormal vector field and they are null vector fields. The Frenet formulae is as the following

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ 0 & \tau & 0 \\ -k & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.2}$$

where the torsion $\tau(s)$ is an arbitrary parametrized arc-length function of γ in s and the curvature $k(s) = 1$, satisfying the equations

$$\begin{aligned} \langle N, B \rangle &= 1, \langle T, N \rangle = \langle T, B \rangle = 0, \\ \langle T, T \rangle &= 1, \langle N, N \rangle = \langle B, B \rangle = 0, \end{aligned} \tag{2.3}$$

and

$$T \times N = N, N \times B = T, B \times T = B. \tag{2.4}$$

if $det(T, N, B) = [T, N, B] = 1$, then the frame is called a positively oriented frame [3].

Definition 2.19.

Let T be a tangential vector field, N_1 and N_2 be two normal vector fields in Bishop frame. By rotating N and B in the normal plane $T^{\perp}$ of the curve, Bishop frame can be obtained, in this way they are relatively parallel. Furthermore, their derivatives N_1' and N_2' with respect to s of the curve are collinear with the tangential vector field [3].

Definition 2.20.

Let $\{T_1, N_1, N_2\}$ be Bishop frame of a pseudo null curve in E_1^3. Bishop frame is positively oriented pseudo orthonormal frame and T_1 is a tangential vector field, N_1 and N_2 are two relatively parallel lightlike normal vector fields. By using the hyperbolic rotation to the principal normal vector field N, Bishop vector N_1 can be obtained. In addition to this, by using the composition of three rotations about two lightlike and one spacelike axis to the binormal vector B, normal Bishop vector N_2 can be found [3] .

Definition 2.21.

If the tangent vector $\gamma' = T$ is a null vector, a curve $\gamma : I \to E_1^3$ can be called a null curve. If *null curve* $\gamma = \gamma(s)$ is parametrized by the pseudo-arc function s defined by

$$s(t) = \int_0^t \sqrt{\|\gamma''(u)\|}du, \tag{2.5}$$

a null curve can be called a null cartan curve. $\{T, N, B\}$ is a unique cartan frame along a non-geodesic null cartan curve verifying the following cartan equations

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -\tau & 0 & k \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.6}$$

where the curvature $k(s) = 1$ and the tursion $\tau(s)$ is an arbitrary function with pseudo-arc parameter s. Null cartan curve is said to be *null cartan cubic* if $\tau(s) = 0$, the following equations must satisfy the cartan's frame

$$\begin{aligned} \langle N, N \rangle &= 1, \langle T, T \rangle = \langle B, B \rangle = 0, \\ \langle T, B \rangle &= -1, \langle T, N \rangle = \langle N, B \rangle = 0 \end{aligned} \tag{2.7}$$

and

$$T \times N = -T,\ N \times B = -B,\ B \times T = N, \tag{2.8}$$

if $det(T, N, B) = [T, N, B] = 1$ then the cartan frame is called a positively oriented frame [3] .

Definition 2.22.

A regular curve $\gamma : I \subseteq R \to E^n$ is said to be a $W-$curve of rank d , if γ is a Frenet curve of osculating order d, curvatures k_i , $1 \leq i \leq d-1$ are non-zero constants. $AW(k)$ curves of rank 3 is a right circular helix [7] .

Definition 2.23.

A curve $\gamma : I \to E^3$ with $k_1 \neq 0$ is said to be cylindrical helix if the tangent lines of γ make a constant angle with a fixed direction. The curve $\gamma(s)$ is a cylindrical helix if and only if $(\frac{k_1}{k_2})(s)$ =constant. If both $k_1(s) \neq 0$ and $k_2(s)$ are constants, it is of course a cylindrical helix but it is called circular helix [7].

Definition 2.24.

A curve γ with $k_1(s) \neq 0$ is called a slant helix if the principal normal lines of γ make a constant angle with a fixed direction [7].

Definition 2.25.

For two vector fields X, Y on S, we can define the Levi-Civita covariant derivative $\triangledown_X Y$. (This derivation $\triangledown$ is called the *Levi-Civita connection* or *Levi-Civita covariant derivative.*) [24].

3. CURVES OF AW(k)-TYPE IN E^3

Let $\gamma : I \subseteq E \rightarrow E^n$ be a unit speed curve in E^n. The curve γ is called a frenet curve of osculating order d if its higher derivatives $\gamma'(s)$, $\gamma''(s)$, ..., $\gamma^{(d)}(s)$ are linearly dependent and $\gamma'(s)$, $\gamma''(s)$, ..., $\gamma^{(d+1)}(s)$ are no longer linearly independent for all $s \in I$. Each Frenet curve of order d can be related with an orthonormal d-frame v_1, v_2, ..., v_d along γ (such that $\gamma'(s) = v_1$) called the Frenet frame and $d-1$ functions k_1, k_2, ... , $k_{d-1} : I \rightarrow R$, called the Frenet curvatures, such that the Frenet formulas are defined in the usual way

$$\begin{aligned}
D_{v_1}\gamma'(s) &= k_1(s)v_2(s) \\
D_{v_1}v_2(s) &= -k_1(s)v_1(s) + k_2(s)v_3(s) \\
&\;\;\cdots \qquad\qquad \cdots \\
D_{v_1}v_i(s) &= -k_{i-1}(s)v_{i-1}(s) + k_i(s)v_{i+1} \\
D_{v_1}v_{i+1}(s) &= -k_i(s)v_i(s)
\end{aligned}$$

where D is called the Levi-civita connection of E^n. After this, we consider Frenet curves of osculating order 3 of E^n. Firstly, let's use the results in [7].

Proposition 3.1. Suppose that γ is a Frenet curve of osculating order 3. The following can be obtained [7]

$$\begin{aligned}
\gamma'(s) &= v_1, \\
\gamma''(s) &= D_{v_1}\gamma'(s) = k_1(s)v_2, \\
\gamma'''(s) &= D_{v_1}D_{v_1}\gamma' = (k_1(s)v_2)' = k_1'(s)v_2 + k_1(s)v_2' \\
&= k_1'(s)v_2 + k_1(s)(-k_1(s)v_1 + k_2(s)v_3) \\
&= -k_1^2(s)v_1 + k_1'(s)v_2 + k_1(s)k_2(s)v_3, \\
\gamma''''(s) &= -3k_1(s)k_1'(s)v_1 + (k_1''(s) - k_1(s)k_2^2(s) \\
&\quad -k_1^3(s))v_2 + (2k_1'(s)k_2(s) + k_1(s)k_2'(s))v_3.
\end{aligned}$$

Notation 3.2. Let us write

$$N_1(s) = k_1(s)v_2 \tag{3.1}$$

$$N_2(s) = k_1'(s)v_2 + k_1(s)k_2(s)v_3 \tag{3.2}$$

$$N_3(s) = (k_1''(s) - k_1(s)k_2^2(s) - k_1^3(s))v_2 + (2k_1'(s)k_2(s) + k_1(s)k_2'(s))v_3 \tag{3.3}$$

Corollary 3.3. $\gamma'(s)$, $\gamma''(s)$, $\gamma'''(s)$ and $\gamma''''(s)$ are linearly dependent if and only if $N_1(s)$, $N_2(s)$ and $N_3(s)$ are linearly dependent [7] .

Theorem 3.4. Let $\gamma : I \to E^3$ be a curve with $k_1 \neq 0$, then γ is a slant helix if and only if

$$\sigma(s) = \left(\frac{k_1^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}}\left(\frac{k_2}{k_1}\right)'\right)(s), \tag{3.4}$$

is a constant function [7] .

Theorem 3.5. Let $\gamma : I \to E^3$ be a curve.

i) Suppose that $k_2(s) \neq 0$. Then γ is a Bertrand curve if and only if there are non-zero real numbers A , B such that $Ak_1(s) + Bk_2(s) = 1$ for any $s \in I$. It follows from the fact that a circular helix is a Bertrand curve.

ii) Suppose that $k_1(s) \neq 0$ and $k_2(s) \neq 0$. Then γ is a Bertrand curve if and only if there exist a non-zero real number A such that

$$A(\, k_2'(s)k_1(s) - k_1'(s)k_2(s)) - k_2'(s) = 0. \tag{3.5}$$

In this situation, the Bertrand mate of γ can be written as

$$\bar{\gamma}(s) = \gamma(s) + Av_2(s) \quad [7]. \tag{3.6}$$

Theorem 3.6. Let $\gamma : I \to E^3$ be a curve with $k_1 \neq 0$, then γ is a slant helix if and only if there is a real number C such that

$$k_2'(s)k_1(s) - k_1'(s)k_2(s) = C\left(k_1^2(s) + k_2^2(s)\right)^{\frac{3}{2}} \quad [7]. \tag{3.7}$$

Theorem 3.7. Let $\gamma : I \to E^3$ be a Bertrand curve with $k_1(s) \neq 0$ and $k_2(s) \neq 0$. If γ is not a cylindrical helix and if there exists a real number $C \neq 0$ such that

$$k_2'(s) = C\left(k_1^2(s) + k_2^2(s)\right)^{\frac{3}{2}} \quad [7]. \tag{3.8}$$

Definition 3.8. Frenet curves (of osculating order 3) are

i) of type weak $AW(2)$ if

$$N_3(s) = \langle N_3(s), N_2^{\star}(s)\rangle N_2^{\star}(s), \tag{3.9}$$

ii) of type weak $AW(3)$ if

$$N_3(s) = \langle N_3(s), N_1^{\star}(s)\rangle N_1^{\star}(s), \tag{3.10}$$

where

$$N_1^{\star}(s) = \frac{N_1(s)}{\|N_1(s)\|}, \tag{3.11}$$

$$N_2^{\star}(s) = \frac{N_2(s) - \langle N_2(s), N_1^{\star}(s)\rangle N_1^{\star}(s)}{\left\|N_2(s) - \langle N_2(s), N_1^{\star}(s)\rangle N_1^{\star}(s)\right\|} \tag{3.12}$$

Definition 3.9. Frenet curves are

i) of type $AW(1)$ if

$$N_3(s) = 0, \tag{3.13}$$

ii) of type $AW(2)$ if

$$\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s)\rangle N_2(s), \tag{3.14}$$

iii) of type $AW(3)$ if

$$\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s)\rangle N_1(s) \qquad [25], \tag{3.15}$$

(see [25] for the general case).

By using the above definitions, we have the following theorems.

Theorem 3.10. Suppose that γ is a Frenet curve of osculating order 3. Then γ is $AW(1)-$type if and only if

$$k_1''(s) - k_1^3(s) - k_2^2(s)\, k_1(s) = 0 \quad ; \quad k_2(s) = \frac{c}{k_1^2(s)} \tag{3.16}$$

c is a constant [7].

Theorem 3.11. Suppose that γ is a Frenet curve of osculating order 3. Then γ is $AW(2)-$type if and only if

$$\begin{aligned} & 2(k_1'(s))^2 + k_1(s)\, k_1'(s)\, k_2(s) \\ = \; & k_1''(s)\, k_1(s)\, k_2(s) - k_1^4(s)\, k_2(s) - k_1^2(s)\, k_2^3(s) \quad [7]. \end{aligned} \tag{3.17}$$

Theorem 3.12. Suppose that γ is a Frenet curve of osculating order 3. Then γ is $AW(3)-$type if and only if

$$2k_2(s)k_1'(s) + k_2'(s)k_1(s) = 0 \tag{3.18}$$

and the solution of this differential equation is $k_2(s) = \frac{c}{k_1^2(s)}$, c is a constant [7].

Theorem 3.13. Suppose that γ is a Frenet curve of osculating order 3. Then γ is of weak $AW(2)-type$ if and only if

$$k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s) = 0 \quad [7] \tag{3.19}$$

Theorem 3.14. Any Bertrand curve $\gamma : I \to E^3$ of $AW(1)-type$ with $k_1(s) \neq 0$ and $k_2(s) \neq 0$ doesn't exist [7].

Theorem 3.15. Let $\gamma : I \to E^3$ be a Bertrand curve with $k_1(s) \neq 0$ and $k_2(s) \neq 0$, then γ of $AW(2)-type$ if and only if there is a non-zero real number A such that

$$\begin{aligned} & \left(k_1'(s)\right)^2 k_2(s)\left(2 - Ak_1(s)\right) + Ak_1^2(s)k_1'(s)k_2'(s) \\ = \; & k_1''(s)k_1(s)k_2(s) - k_1^4(s)k_2(s) - k_1^2(s)k_2^3(s) \quad [7]. \end{aligned} \tag{3.20}$$

Theorem 3.16. Let $\gamma : I \to E^3$ be a Bertrand curve with $k_1(s) \neq 0$ and $k_2(s) \neq 0$, then γ of $AW(3)-type$ if and only if γ is a right circular helix [7].

Theorem 3.17. Let $\gamma : I \to E^3$ be a conical geodesic curve with $k_2(s) \neq 0$. Then γ is of weak $AW(2)-type$ if and only if there is a real number C such that

$$k_2''(s) - k_1^2(s)k_2(s) - k_2^3(s) = ck_1'(s) \quad [7]. \tag{3.21}$$

Theorem 3.18. Let $\gamma : I \to E^3$ be a weak $AW(2)-type$ conical geodesic curve, if $k_2(s)$ is a non-zero constant then

$$k_1(s) = \tan\left(\frac{c_1(s+c)}{c_2}\right)c_1, \tag{3.22}$$

where c_1, c_2 and c are real constants [7].

Theorem 3.19. Let $\gamma : I \to E^3$ be a conical geodesic curve. Then γ is of $AW(3)-type$ if and only if the curvatures of γ are as the following

$$k_1(s) = -(c_1 s + c_2)^{-\frac{1}{3}} \tag{3.23}$$

$$k_2(s) = c(c_1 s + c_2)^{\frac{2}{3}}, \tag{3.24}$$

where c_1, c_2 and c are real constants [7].

4. THE BISHOP FRAME IN MINKOWSKI 3-SPACE E_1^3

4.1. Pseudo null curve according to Bishop frame in E_1^3

Theorem 4.1.1. Let γ be a pseudo null curve in E_1^3 with the curvature $k(s) = 1$ and the torsion $\tau(s)$ (s is arc-length parameter) So, the Bishop frame $\{T_1, N_1, N_2\}$ and the Frenet frame $\{T, N, B\}$ of γ are associated with

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{4.1}$$

and the Frenet equations of γ according to the Bishop frame is

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_2 & k_1 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{4.2}$$

where $k_1(s) = 0$ and $k_2(s) = c_0 e^{\int \tau(s)ds}$, $c_0 \in R_0^+$, satisfying the conditions [3] .

$$\begin{aligned} \langle T_1, T_1 \rangle &= 1, \langle N_1, N_1 \rangle = \langle N_2, N_2 \rangle = 0, \\ \langle N_1, N_2 \rangle &= 1, \langle T_1, N_2 \rangle = \langle T_1, N_1 \rangle = 0. \end{aligned}$$

4.2. Pseudo null curves of AW(k)- type in E_1^3

Theorem 4.2.1. Suppose that γ is a Frenet curve of osculating order 3 in E_1^3, so we get

$$\begin{aligned} \gamma'(s) &= T_1(s) \\ \gamma''(s) &= k_2 N_1 + k_1 N_2 \\ \gamma'''(s) &= (k_2 N_1 + k_1 N_2)' = k_2' N_1 + k_2 N_1' + k_1' N_2 + k_1 N_2' \\ &= k_2' N_1 + k_2(-k_1 T_1) + k_1' N_2 + k_1(-k_2 T_1) \\ &= k_2' N_1 - k_2 k_1 T_1 + k_1' N_2 - k_1 k_2 T_1 \\ &= -2k_1 k_2 T_1 + k_2' N_1 + k_1' N_2, \\ \gamma''''(s) &= \left(-2k_1 k_2 T_1 + k_2' N_1 + k_1' N_2\right)' \\ &= -2k_1' k_2 T_1 - 2k_1 k_2' T_1 - 2k_1 k_2 T_1' + k_2'' N_1 + k_2' N_1' + k_1'' N_2 + k_1' N_2' \\ &= -2k_1' k_2 T_1 - 2k_1 k_2' T_1 - 2k_1 k_2 (k_2 N_1 + k_1 N_2) \\ &\quad + k_2'' N_1 + k_2'(-k_1 T_1) + k_1'' N_2 + k_1'(-k_2 T_1) \\ &= \left(-3k_1' k_2 - 3k_1 k_2'\right) T_1 + \left(k_2'' - 2k_1 k_2^2\right) N_1 + \left(k_1'' - 2k_1^2 k_2\right) N_2 \end{aligned}$$

Notation 4.2.2. Let us write :

$$\begin{aligned} M_1(s) &= k_2 N_1 + k_1 N_2, && (4.3) \\ M_2(s) &= k_2^{'} N_1 + k_1^{'} N_2, && (4.4) \\ M_3(s) &= \left(k_2^{''} - 2k_1k_2^2\right) N_1 + \left(k_1^{''} - 2k_1^2k_2\right) N_2. && (4.5) \end{aligned}$$

Corollary 4.2.3. $\gamma^{'}(s), \gamma^{''}(s), \gamma^{'''}(s)$ and $\gamma^{''''}(s)$ are linearly dependent if and only if $M_1(s), M_2(s)$ and $M_3(s)$ are linearly dependent.

Theorem 4.2.4. Suppose that γ is a Frenet curve of osculating order 3 in E_1^3, so γ is $AW(1)-$type if and only if

$$\begin{aligned} k_1^{''} &= 2k_1^2k_2 && (4.6) \\ k_2^{''} &= 2k_1k_2^2. && (4.7) \end{aligned}$$

Proof Since γ is a curve of type $AW(1)$, then γ must satisfy (3.13)

$$\begin{aligned} M_3(s) &= \left(k_2^{''} - 2k_1k_2^2\right) N_1 + \left(k_1^{''} - 2k_1^2k_2\right) N_2 \\ 0 &= \left(k_2^{''} - 2k_1k_2^2\right) N_1 + \left(k_1^{''} - 2k_1^2k_2\right) N_2 \end{aligned}$$

Since N_1 and N_2 are linearly independent , then we have

$$\begin{aligned} k_2^{''} - 2k_1k_2^2 &= 0 \\ k_2^{''} &= 2k_1k_2^2 \end{aligned}$$

and

$$\begin{aligned} k_1^{''} - 2k_1^2k_2 &= 0 \\ k_1^{''} &= 2k_1^2k_2, \end{aligned}$$

which completes the proof of the theorem.

Theorem 4.2.5. Suppose that γ is a Frenet curve of osculating order 3 in E_1^3, so γ is $AW(2)$-type if and only if

$$k_1^{''}k_2^{'} - 2k_1^2k_2k_2^{'} = k_1^{'}k_2^{''} - 2k_1k_1^{'}k_2^2 \qquad (4.8)$$

Proof Assume that γ is a Frenet curve of order 3. From (3.2) and (3.3),the following equations can be obtained

$$\begin{aligned} M_2(s) &= \beta(s) N_1 + \alpha(s) N_2, \\ M_3(s) &= \delta(s) N_1 + \eta(s) N_2, \end{aligned}$$

where $\beta(s)$, $\alpha(s)$, $\delta(s)$ and $\eta(s)$ are differential functions. Since $M_2(s)$ and $M_3(s)$ are linearly dependent , determinant of the coefficients is equal to zero and so we can have

$$\begin{vmatrix} \beta(s) & \alpha(s) \\ \delta(s) & \eta(s) \end{vmatrix} = 0 \tag{4.9}$$

where

$$\begin{aligned} \beta(s) &= k_2^{'}\,, \quad \alpha(s) = k_1^{'} \\ \delta(s) &= k_2^{''} - 2k_1k_2^2, \\ \eta(s) &= k_1^{''} - 2k_1^2k_2 \end{aligned} \tag{4.10}$$

$$\begin{aligned} \beta(s)\eta(s) - \alpha(s)\delta(s) &= 0 \\ k_2^{'}\left(k_1^{''} - 2k_1^2k_2\right) - k_1^{'}\left(k_2^{''} - 2k_1k_2^2\right) &= 0 \\ k_2^{'}\left(k_1^{''} - 2k_1^2k_2\right) &= k_1^{'}\left(k_2^{''} - 2k_1k_2^2\right) \\ k_1^{''}k_2^{'} - 2k_1^2k_2k_2^{'} &= k_1^{'}k_2^{''} - 2k_1k_1^{'}k_2^2, \end{aligned}$$

equation (4.8) is proved.

Theorem 4.2.6. Suppose that γ is a Frenet curve of osculating order 3 in E_1^3, so γ is $AW(3)$-type if and only if

$$k_1^{''}k_2 = k_1k_2^{''}. \tag{4.11}$$

Proof Assume that γ is a Frenet curve of order 3. We can write

$$\begin{aligned} M_1(s) &= \beta(s)N_1 + \alpha(s)N_2, \\ M_3(s) &= \delta(s)N_1 + \eta(s)N_2, \end{aligned}$$

where $\beta(s)$, $\alpha(s)$, $\delta(s)$ and $\eta(s)$ are differential functions. Since $M_1(s)$ and $M_3(s)$ are linearly dependent , determinant of the coefficients is equal to zero and hence one can write

$$\begin{aligned} \begin{vmatrix} \beta(s) & \alpha(s) \\ \delta(s) & \eta(s) \end{vmatrix} &= 0 \\ \beta(s)\eta(s) - \alpha(s)\delta(s) &= 0 \end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
\beta(s) &= k_2 \quad \alpha(s) = k_1 \\
\delta(s) &= k_2^{''} - 2k_1 k_2^2, \\
\eta(s) &= k_1^{''} - 2k_1^2 k_2
\end{aligned}$$

$$\begin{aligned}
\beta(s)\eta(s) - \alpha(s)\delta(s) &= 0 \\
k_2\left(k_1^{''} - 2k_1^2 k_2\right) - k_1\left(k_2^{''} - 2k_1 k_2^2\right) &= 0 \\
k_1^{''} k_2 - 2k_1^2 k_2^2 - k_1 k_2^{''} + 2k_1^2 k_2^2 &= 0 \\
k_1^{''} k_2 &= k_1 k_2^{''},
\end{aligned}$$

equation (4.11) is proved..

Theorem 4.2.7. Suppose that γ is a Frenet curve of osculating order 3 in E_1^3, so γ is of weak $AW(2)-$type if and only if

$$\left(k_2^{''} - 2k_1 k_2^2\right) = \frac{p}{p^2+q^2}\left[\left(k_2^{''} - 2k_1 k_2^2\right) q + \left(k_1^{''} - 2k_1^2 k_2\right) p\right] \tag{4.13}$$

$$\left(k_1^{''} - 2k_1^2 k_2\right) = \frac{q}{p^2+q^2}\left[\left(k_2^{''} - 2k_1 k_2^2\right) q + \left(k_1^{''} - 2k_1^2 k_2\right) p\right], \tag{4.14}$$

where

$$\begin{aligned}
p &= \left(k_2^2 + k_1^2\right) k_2^{'} - \left(k_1 k_2\right)^{'} k_2 \ and \\
q &= \left(k_2^2 + k_1^2\right) k_1^{'} - \left(k_1 k_2\right)^{'} k_1.
\end{aligned}$$

Proof Since γ is of weak $AW(2)-$type, by using (3.9), (3.11) and (3.12)

$$\begin{aligned}
M_1^{\star}(s) &= \frac{M_1(s)}{\|M_1(s)\|} = \frac{k_2 N_1 + k_1 N_2}{\sqrt{k_2^2 + k_1^2}}, \\
M_2^{\star}(s) &= \frac{M_2(s) - \langle M_2(s), M_1^{\star}(s)\rangle M_1^{\star}(s)}{\|M_2(s) - \langle M_2(s), M_1^{\star}(s)\rangle M_1^{\star}(s)\|},
\end{aligned} \tag{4.15}$$

by simplifying the numerator

$$M_2(s) - \langle M_2(s), M_1^{\star}(s)\rangle M_1^{\star}(s) = \left(\begin{array}{c} k_2^{'} N_1 + k_1^{'} N_2 \\ -\left\langle k_2^{'} N_1 + k_1^{'} N_2, \frac{k_2 N_1 + k_1 N_2}{\sqrt{k_2^2+k_1^2}} \right\rangle \\ \cdot \left(\frac{k_2 N_1 + k_1 N_2}{\sqrt{k_2^2+k_1^2}}\right) \end{array}\right)$$

$$
\begin{aligned}
&= k_2' N_1 + k_1' N_2 - \left\langle k_2' N_1 + k_1' N_2, k_2 N_1 + k_1 N_2 \right\rangle \frac{k_2 N_1 + k_1 N_2}{k_2^2 + k_1^2} \\
&= k_2' N_1 + k_1' N_2 - \left(k_1 k_2' + k_1' k_2 \right) \frac{k_2 N_1 + k_1 N_2}{\left(k_2^2 + k_1^2\right)} \\
&= \frac{\left(k_2' N_1 + k_1' N_2 \right) \left(k_2^2 + k_1^2\right) - \left(k_1 k_2' + k_1' k_2 \right) \left(k_2 N_1 + k_1 N_2 \right)}{\left(k_2^2 + k_1^2\right)} \\
&= \frac{\begin{pmatrix} \left(k_2^2 + k_1^2\right) k_2' N_1 + \left(k_2^2 + k_1^2\right) k_1' N_2 \\ - \left(k_1 k_2' + k_1' k_2 \right) k_2 N_1 - \left(k_1 k_2' + k_1' k_2 \right) k_1 N_2 \end{pmatrix}}{\left(k_2^2 + k_1^2\right)} \\
&= \frac{\begin{pmatrix} \left(\left(k_2^2 + k_1^2\right) k_2' - \left(k_1 k_2' + k_1' k_2 \right) k_2 \right) N_1 \\ + \left(\left(k_2^2 + k_1^2\right) k_1' - \left(k_1 k_2' + k_1' k_2 \right) k_1 \right) N_2 \end{pmatrix}}{\left(k_2^2 + k_1^2\right)} \\
&= \frac{1}{\left(k_2^2 + k_1^2\right)} \left(\left(k_2^2 + k_1^2\right) k_2' - \left(k_1 k_2' + k_1' k_2 \right) k_2 \right) N_1 \\
&\quad + \frac{1}{\left(k_2^2 + k_1^2\right)} \left(\left(k_2^2 + k_1^2\right) k_1' - \left(k_1 k_2' + k_1' k_2 \right) k_1 \right) N_2 \\
&= \frac{1}{\left(k_2^2 + k_1^2\right)} \left(\left(k_2^2 + k_1^2\right) k_2' - \left(k_1 k_2 \right)' k_2 \right) N_1 \\
&\quad + \frac{1}{\left(k_2^2 + k_1^2\right)} \left(\left(k_2^2 + k_1^2\right) k_1' - \left(k_1 k_2 \right)' k_1 \right) N_2,
\end{aligned}
$$

therefore

$$
\begin{aligned}
M_2^\star(s) &= \frac{\begin{pmatrix} \frac{1}{\left(k_2^2+k_1^2\right)} \left(\left(k_2^2 + k_1^2\right) k_2' - \left(k_1 k_2 \right)' k_2 \right) N_1 \\ + \frac{1}{\left(k_2^2+k_1^2\right)} \left(\left(k_2^2 + k_1^2\right) k_1' - \left(k_1 k_2 \right)' k_1 \right) N_2 \end{pmatrix}}{\left\| \begin{pmatrix} \frac{1}{\left(k_2^2+k_1^2\right)} \left(\left(k_2^2 + k_1^2\right) k_2' - \left(k_1 k_2 \right)' k_2 \right) N_1 \\ + \frac{1}{\left(k_2^2+k_1^2\right)} \left(\left(k_2^2 + k_1^2\right) k_1' - \left(k_1 k_2 \right)' k_1 \right) N_2 \end{pmatrix} \right\|} \\
&= \frac{\frac{1}{\left(k_2^2+k_1^2\right)} \begin{bmatrix} \left(\left(k_2^2 + k_1^2\right) k_2' - \left(k_1 k_2 \right)' k_2 \right) N_1 \\ + \left(\left(k_2^2 + k_1^2\right) k_1' - \left(k_1 k_2 \right)' k_1 \right) N_2 \end{bmatrix}}{\sqrt{\left(\frac{\left(k_2^2+k_1^2\right) k_2' - \left(k_1 k_2\right)' k_2}{\left(k_2^2+k_1^2\right)} \right)^2 + \left(\frac{\left(k_2^2+k_1^2\right) k_1' - \left(k_1 k_2\right)' k_1}{\left(k_2^2+k_1^2\right)} \right)^2}} \\
&= \frac{\frac{1}{\left(k_2^2+k_1^2\right)} \begin{bmatrix} \left(\left(k_2^2 + k_1^2\right) k_2' - \left(k_1 k_2 \right)' k_2 \right) N_1 \\ + \left(\left(k_2^2 + k_1^2\right) k_1' - \left(k_1 k_2 \right)' k_1 \right) N_2 \end{bmatrix}}{\frac{1}{\left(k_2^2+k_1^2\right)} \sqrt{\begin{array}{c} \left(\left(k_2^2 + k_1^2\right) k_2' - \left(k_1 k_2 \right)' k_2 \right)^2 \\ + \left(\left(k_2^2 + k_1^2\right) k_1' - \left(k_1 k_2 \right)' k_1 \right)^2 \end{array}}}
\end{aligned}
$$

$$
= \frac{\begin{array}{c}\left(\left(k_2^2+k_1^2\right)k_2^{'}-\left(k_1k_2\right)^{'}k_2\right)N_1\\ +\left(\left(k_2^2+k_1^2\right)k_1^{'}-\left(k_1k_2\right)^{'}k_1\right)N_2\end{array}}{\sqrt{\begin{array}{c}\left(\left(k_2^2+k_1^2\right)k_2^{'}-\left(k_1k_2\right)^{'}k_2\right)^2\\ +\left(\left(k_2^2+k_1^2\right)k_1^{'}-\left(k_1k_2\right)^{'}k_1\right)^2\end{array}}}.
$$

Let

$$
\begin{aligned}
p &= \left(k_2^2+k_1^2\right)k_2^{'}-\left(k_1k_2\right)^{'}k_2,\\
q &= \left(k_2^2+k_1^2\right)k_1^{'}-\left(k_1k_2\right)^{'}k_1,
\end{aligned}
$$

therefore

$$
M_2^{\star}(s)=\frac{pN_1+qN_2}{\sqrt{p^2+q^2}}. \tag{4.16}
$$

Since γ is of weak $AW(2)-$type, then it must satisfy

$$
\begin{aligned}
M_3(s) &= \left\langle M_3(s), M_2^{\star}(s)\right\rangle M_2^{\star}(s)\\
&= \left\langle \begin{pmatrix}\left(k_2^{''}-2k_1k_2^2\right)N_1\\ +\left(k_1^{''}-2k_1^2k_2\right)N_2\end{pmatrix}, \frac{pN_1+qN_2}{\sqrt{p^2+q^2}}\right\rangle \frac{pN_1+qN_2}{\sqrt{p^2+q^2}}\\
&= \left\langle \begin{array}{c}\left(k_2^{''}-2k_1k_2^2\right)N_1+\left(k_1^{''}-2k_1^2k_2\right)N_2,\\ pN_1+qN_2\end{array}\right\rangle \frac{pN_1+qN_2}{p^2+q^2}\\
&= \left[\left(k_2^{''}-2k_1k_2^2\right)q+\left(k_1^{''}-2k_1^2k_2\right)p\right]\frac{pN_1+qN_2}{p^2+q^2}
\end{aligned}
$$

$$
\begin{aligned}
M_3(s) &= \frac{p}{p^2+q^2}\left[\left(k_2^{''}-2k_1k_2^2\right)q+\left(k_1^{''}-2k_1^2k_2\right)p\right]N_1\\
&\quad +\frac{q}{p^2+q^2}\left[\left(k_2^{''}-2k_1k_2^2\right)q+\left(k_1^{''}-2k_1^2k_2\right)p\right]N_2,
\end{aligned} \tag{4.17}
$$

but from (4.5) we know that

$$
M_3(s)=\left(k_2^{''}-2k_1k_2^2\right)N_1+\left(k_1^{''}-2k_1^2k_2\right)N_2,
$$

so by using the relation (4.5) and (4.17)

$$
\begin{aligned}
\left(k_2^{''}-2k_1k_2^2\right) &= \frac{p}{p^2+q^2}\left[\left(k_2^{''}-2k_1k_2^2\right)q+\left(k_1^{''}-2k_1^2k_2\right)p\right]\\
\left(k_1^{''}-2k_1^2k_2\right) &= \frac{q}{p^2+q^2}\left[\left(k_2^{''}-2k_1k_2^2\right)q+\left(k_1^{''}-2k_1^2k_2\right)p\right],
\end{aligned}
$$

which completes the proof of the theorem.

Theorem 4.2.8. Suppose that γ is a Frenet curve of osculating order 3 in E_1^3, so γ is of weak $AW(3)-$type if and only if

$$k_2'' - 2k_1k_2^2 = \frac{k_2}{k_2^2+k_1^2}\left[k_1k_2'' + k_1''k_2 - 4k_1^2k_2^2\right], \tag{4.18}$$

$$k_1'' - 2k_1^2k_2 = \frac{k_1}{k_2^2+k_1^2}\left[k_1k_2'' + k_1''k_2 - 4k_1^2k_2^2\right]. \tag{4.19}$$

Proof Since γ is of weak $AW(3)-$type, by using (4.2), (4.5), (3.10), (3.11) and (4.15)

$$M_3(s) = \langle M_3(s), M_1^\star(s)\rangle M_1^\star(s)$$

$$M_1^\star(s) = \frac{k_2N_1 + k_1N_2}{\sqrt{k_2^2+k_1^2}}$$

$$\begin{aligned} M_3(s) &= \left\langle \begin{array}{c} \left(k_2'' - 2k_1k_2^2\right)N_1 + \left(k_1'' - 2k_1^2k_2\right)N_2, \\ \frac{k_2N_1+k_1N_2}{\sqrt{k_2^2+k_1^2}} \end{array}\right\rangle \left(\frac{k_2N_1+k_1N_2}{\sqrt{k_2^2+k_1^2}}\right) \\ &= \left\langle \begin{array}{c} \left(k_2'' - 2k_1k_2^2\right)N_1 + \left(k_1'' - 2k_1^2k_2\right)N_2, \\ k_2N_1+k_1N_2 \end{array}\right\rangle \frac{k_2N_1+k_1N_2}{k_2^2+k_1^2} \\ &= \left[\left(k_2'' - 2k_1k_2^2\right)k_1 + \left(k_1'' - 2k_1^2k_2\right)k_2\right]\frac{k_2N_1+k_1N_2}{k_2^2+k_1^2} \\ &= \left[k_1k_2'' - 2k_1^2k_2^2 + k_1''k_2 - 2k_1^2k_2^2\right]\frac{k_2N_1+k_1N_2}{k_2^2+k_1^2} \\ &= \left[k_1k_2'' + k_1''k_2 - 4k_1^2k_2^2\right]\frac{k_2N_1+k_1N_2}{k_2^2+k_1^2} \end{aligned}$$

$$\begin{aligned} M_3(s) = {} & \frac{k_2}{k_2^2+k_1^2}\left[k_1k_2'' + k_1''k_2 - 4k_1^2k_2^2\right]N_1 + \\ & \frac{k_1}{k_2^2+k_1^2}\left[k_1k_2'' + k_1''k_2 - 4k_1^2k_2^2\right]N_2 \end{aligned} \tag{4.20}$$

but

$$M_3(s) = \left(k_2'' - 2k_1k_2^2\right)N_1 + \left(k_1'' - 2k_1^2k_2\right)N_2,$$

so by using the relation (4.5) and (5.20) we get

$$k_2'' - 2k_1k_2^2 = \frac{k_2}{k_2^2+k_1^2}\left[k_1k_2'' + k_1''k_2 - 4k_1^2k_2^2\right]$$

$$k_1'' - 2k_1^2k_2 = \frac{k_1}{k_2^2+k_1^2}\left[k_1k_2'' + k_1''k_2 - 4k_1^2k_2^2\right],$$

the theorem is proved.

4.3. Null cartan curve according to Bishop frame in E_1^3

Theorem 4.3.1. Let γ be a null Cartan curve in E_1^3 with the curvature $k(s) = 1$ and the torsion $\tau(s)$ (s is arc-length parameter) So, the Bishop frame $\{T_1, N_1, N_2\}$ and the Frenet frame $\{T, N, B\}$ of γ are associated with

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -k_2 & 1 & 0 \\ \frac{k_2^2}{2} & -k_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{4.21}$$

and the Cartan equations of according to the Bishop frame get

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} k_2 & k_1 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{4.22}$$

where the first Bishop curvature $k_1(s) = 1$ and the second Bishop curvature satisfies *Riccati differential equations*

$$k_2'(s) = -\frac{1}{2}k_2^2(s) - \tau(s).$$

which satisfies the conditions

$$\begin{aligned} \langle N_1, N_1 \rangle &= 1, \langle T_1, T_1 \rangle = \langle N_2, N_2 \rangle = 0, \\ \langle T_1, N_2 \rangle &= -1, \langle T_1, N_1 \rangle = \langle N_1, N_2 \rangle = 0 \quad [3]. \end{aligned}$$

4.4 Null Cartan Curves of AW(k)- type in E_1^3

Theorem 4.4.1. Let γ be a Frenet curve of osculating order 3 in E_1^3, by using the cartan equations of γ according to the Bishop frame (2.7), then we have

$$\begin{aligned} \gamma'(s) &= T_1(s) \\ \gamma''(s) &= T_1'(s) = k_2T_1 + k_1N_1 \\ \gamma'''(s) &= (k_2T_1 + k_1N_1)' = k_2'T_1 + k_2T_1' + k_1'N_1 + k_1N_1' \\ &= k_2'T_1 + k_2(k_2T_1 + k_1N_1) + k_1'N_1 + k_1(k_1N_2) \\ &= \left(k_2' + k_2^2\right)T_1 + \left(k_1' + k_1k_2\right)N_1 + k_1^2N_2, \end{aligned}$$

$$
\begin{aligned}
\gamma^{(iv)}(s) &= \left(k_2^{'}T_1 + k_2^2 T_1 + k_1^{'}N_1 + k_1k_2N_1 + k_1^2N_2\right) \\
&= k_2^{''}T_1 + k_2^{'}T_1^{'} + 2k_2k_2^{'}T_1 + k_2^2T_1^{'} + k_1^{''}N_1 \\
&\quad + k_1^{'}N_1^{'} + k_1^{'}k_2N_1 + k_1k_2^{'}N_1 + k_1k_2N_1^{'} + 2k_1k_1^{'}N_2 + k_1^2N_2^{'} \\
&= k_2^{''}T_1 + k_2^{'}\left(k_2T_1 + k_1N_1\right) + 2k_2k_2^{'}T_1 + k_2^2\left(k_2T_1 + k_1N_1\right) \\
&\quad + k_1^{''}N_1 + k_1^{'}\left(k_1N_2\right) + k_1^{'}k_2N_1 + k_1k_2^{'}N_1 \\
&\quad + k_1k_2\left(k_1N_2\right) + 2k_1k_1^{'}N_2 + k_1^2\left(-k_2N_2\right) \\
&= \left(k_2^{''} + 3k_2k_2^{'} + k_2^3\right)T_1 \\
&\quad + \left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right)N_1 + 3k_1k_1^{'}N_2
\end{aligned}
$$

Notation 4.4.2 Let us write

$$
\begin{aligned}
M_1(s) &= k_1N_1, && (4.23) \\
M_2(s) &= \left(k_1^{'} + k_1k_2\right)N_1 + k_1^2N_2, && (4.24) \\
M_3(s) &= \left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right)N_1 + 3k_1k_1^{'}N_2. && (4.25)
\end{aligned}
$$

Corollary 4.4.3. $\gamma^{'}(s)$, $\gamma^{''}(s)$, $\gamma^{'''}(s)$ and $\gamma^{(iv)}(s)$ are linearly dependent if and only if $M_1(s)$, $M_2(s)$ and $M_3(s)$ are linearly dependent.

Theorem 4.4.4. Suppose that γ is a Frenet curve of osculating order 3 in E_1^3, so γ is $AW(1)-$type if and only if

i) k_1 is a constant function, and

ii)

$$k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2} = 0. \qquad (4.26)$$

Proof Since γ is a curve of type $AW(1)$, then γ must satisfy (3.13)

$$
\begin{aligned}
M_3(s) &= \left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right)N_1 + 3k_1k_1^{'}N_2 \\
0 &= \left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right)N_1 + 3k_1k_1^{'}N_2
\end{aligned}
$$

Since N_1 and N_2 are linearly independent , the coefficients of the vectors should be zero. Therefore

$$3k_1k_1^{'} = 0,$$

k_1 is a constant function

$$k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2} = 0.$$

Thus the equation (4.26) is obtained.

Theorem 4.4.5. Suppose that γ is a Frenet curve of osculating order 3 in E_1^3, so γ is $AW(2)-$type if and only if

$$3\left(k_1^{'}\right)^2 + 3k_1^{'}k_1k_2 = k_1^{''}k_1 + 2k_1^2k_2^{'} + k_1^{'}k_1k_2 + k_1^2k_2^{2}. \tag{4.27}$$

Proof Assume that γ is a Frenet curve of order 3. From (4.24) and (4.25) we can write

$$\begin{aligned} M_2(s) &= \beta(s)N_1 + \alpha(s)N_2 \\ M_3(s) &= \delta(s)N_1 + \eta(s)N_2, \end{aligned}$$

where $\beta(s)$, $\alpha(s)$, $\delta(s)$ and $\eta(s)$ are differential functions. Since $M_2(s)$ and $M_3(s)$ are linearly dependent , determinant of the coefficients is equal to zero, so we can get

$$\begin{vmatrix} \beta(s) & \alpha(s) \\ \delta(s) & \eta(s) \end{vmatrix} = 0 \tag{4.28}$$

where

$$\begin{aligned} \beta(s) &= k_1^{'} + k_1k_2\ , \qquad \alpha(s) = k_1^2 \\ \delta(s) &= k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}, \\ \eta(s) &= 3k_1k_1^{'}. \end{aligned} \tag{4.29}$$

$$\begin{aligned} 0 &= \beta(s)\eta(s) - \alpha(s)\delta(s)\,, \\ 0 &= 3k_1k_1^{'}\left(k_1^{'} + k_1k_2\right) - k_1^2\left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right) \\ 3k_1k_1^{'}\left(k_1^{'} + k_1k_2\right) &= k_1^2\left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right) \\ 3k_1^{'}\left(k_1^{'} + k_1k_2\right) &= k_1\left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right) \\ 3\left(k_1^{'}\right)^2 + 3k_1^{'}k_1k_2 &= k_1^{''}k_1 + 2k_1^2k_2^{'} + k_1^{'}k_1k_2 + k_1^2k_2^{2}. \end{aligned}$$

Thus the equation (4.26) is obtained.

Theorem 4.4.6. Suppose that γ is a Frenet curve of osculating order 3 in E_1^3, so γ is $AW(3)-$type if and only if $k_1(s)$ is a constant function

Proof Suppose that γ is a Frenet curve of order 3. From (4.23) and (4.25) we can write

$$\begin{aligned} M_1(s) &= \beta(s)N_1 + \alpha(s)N_2 \\ M_3(s) &= \delta(s)N_1 + \eta(s)N_2, \end{aligned}$$

where $\beta(s)$, $\alpha(s)$, $\delta(s)$ and $\eta(s)$ are differential functions. Since $M_1(s)$ and $M_3(s)$ are linearly dependent , hence one can write

$$\begin{vmatrix} \beta(s) & \alpha(s) \\ \delta(s) & \eta(s) \end{vmatrix} = 0 \tag{4.30}$$

where

$$\begin{aligned} \beta(s) &= k_1 , \quad \alpha(s) = 0 \\ \delta(s) &= k_1^{''} + 2k_1 k_2^{'} + k_1^{'} k_2 + k_1 k_2^{2}, \\ \eta(s) &= 3k_1 k_1^{'}. \end{aligned} \tag{4.31}$$

$$\begin{aligned} \beta(s)\eta(s) - \alpha(s)\delta(s) &= 0, \\ k_1\left(3k_1k_1^{'}\right) - 0\left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right) &= 0 \\ k_1^2 k_1^{'} &= 0. \end{aligned}$$

For $k_1^2 k_1^{'}$ to be zero, $k_1(s)$ has to be a constant function.

Theorem 4.4.7. Suppose that γ is a Frenet curve of order 3 in E_1^3, so γ is of weak $AW(2)-$type if and only if

i) $k_1(s)$ is a constant function,

ii)

$$k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2} = 0. \tag{4.32}$$

Proof Since γ is of weak $AW(2)-$type , it must satisfy (3.9), by using (3.11), (3.12) and (4.23)

$$\begin{aligned} M_1^\star(s) &= \frac{M_1(s)}{\|M_1(s)\|} = \frac{k_1 N_1}{\sqrt{k_1^2}} = N_1 \qquad (4.33) \\ M_2^\star(s) &= \frac{M_2(s) - \langle M_2(s), M_1^\star(s)\rangle M_1^\star(s)}{\|M_2(s) - \langle M_2(s), M_1^\star(s)\rangle M_1^\star(s)\|} \end{aligned}$$

$$
\begin{aligned}
M_2^{\star}(s) &= \frac{\begin{pmatrix} \left(k_1' + k_1 k_2\right) N_1 + k_1^2 N_2 \\ -\left\langle \left(k_1' + k_1 k_2\right) N_1 + k_1^2 N_2, N_1 \right\rangle N_1 \end{pmatrix}}{\left\| \begin{matrix} \left(k_1' + k_1 k_2\right) N_1 + k_1^2 N_2 \\ -\left\langle \left(k_1' + k_1 k_2\right) N_1 + k_1^2 N_2, N_1 \right\rangle N_1 \end{matrix} \right\|} \\
M_2^{\star}(s) &= \frac{\left(k_1' + k_1 k_2\right) N_1 + k_1^2 N_2 - \left(k_1' + k_1 k_2\right) N_1}{\left\| \left(k_1' + k_1 k_2\right) N_1 + k_1^2 N_2 - \left(k_1' + k_1 k_2\right) N_1 \right\|} \\
M_2^{\star}(s) &= \frac{k_1^2 N_2}{\left\| k_1^2 N_2 \right\|} = \frac{k_1^2 N_2}{\sqrt{\left(k_1^2\right)^2}} \\
M_2^{\star}(s) &= N_2 \qquad (4.34)
\end{aligned}
$$

Since γ is of weak $AW(2)-$type, then it must satisfy

$$
\begin{aligned}
M_3(s) &= \left\langle M_3(s), M_2^{\star}(s) \right\rangle M_2^{\star}(s) \\
&= \left\langle \left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2 \right) N_1 + 3k_1 k_1' N_2, N_2 \right\rangle N_2 \\
&= 0.
\end{aligned}
$$

Therefore

$$
\left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2 \right) N_1 + 3k_1 k_1' N_2 = 0,
$$

then

$$
\begin{aligned}
k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2 &= 0, \\
3k_1 k_1' &= 0.
\end{aligned}
$$

For $k_1^2 k_1'$ to be zero, $k_1(s)$ has to be a constant function. Hence the theorem is proved.

Theorem 4.4.8. Let γ be a Frenet curve of order 3 in E_1^3, then γ is of weak $AW(3)-$type if and only if $k_1(s)$ is a constant function.

Proof Since γ is of weak $AW(3)-$type , by using (4.25) and (4.33)

$$
\begin{aligned}
M_3(s) &= \left\langle M_3(s), M_1^{\star}(s) \right\rangle M_1^{\star}(s) \\
&= \left\langle \left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2 \right) N_1 + 3k_1 k_1' N_2, N_1 \right\rangle N_1 \\
&= \left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2 \right) N_1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2 \right) N_1 + 3k_1 k_1' N_2 &= \left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2 \right) N_1 \\
k_1 k_1' &= 0.
\end{aligned}
$$

For $k_1^2 k_1'$ to be zero, $k_1(s)$ has to be a constant function. Hence the theorem is proved.

5. HELIX AND SLANT HELICES ACCORDING TO BISHOP FRAME IN MINKOWSKI 3-SPACE E_1^3

5.1 Slant helices of pseudo null curve in Minkowski 3-space E_1^3

Definition 5.1.1. A unit speed curve γ is called a slant helix if there exists a non-zero constant vector field $U \epsilon E_1^3$ such that the function $\langle N(s), U\rangle$ is constant [26].

It is important to point out, in contrast to what happens in E^3 , we cannot define the angle between two vectors (except that both vectors are of time-like). For this reason we avoid to say about the angle between the normal vector field $N(s)$ and U [26].

Theorem 5.1.2. Let γ be a pseudo null curve in E_1^3 , then γ is a general helix if and only if $\frac{k_1}{k_2}$ is constant.

Proof Let γ be a general helix. The slope axis of the curve γ is shown as $sp\{U\}$. Note that

$$\langle T, U\rangle = constant. \tag{5.1}$$

If we differentiate both sides of the equation (5.1), then we have

$$\begin{aligned} \left\langle T', U\right\rangle + \left\langle T, U'\right\rangle &= 0. \\ \left\langle T', U\right\rangle &= 0. \end{aligned} \tag{5.2}$$

By using (5.2) and (4.2)

$$\begin{aligned} \langle k_2 N_1 + k_1 N_2, U\rangle &= 0 \\ k_2 \langle N_1, U\rangle + k_1 \langle N_2, U\rangle &= 0 \\ k_2 \cos\theta + k_1 \sin\theta &= 0 \end{aligned}$$

$$\frac{k_1}{k_2} = -\cot\theta = \text{constant} , \tag{5.3}$$

as desired.

Theorem 5.1.3. Let γ be a pseudo null curve in E_1^3 , then γ is a slant helix if and only if $\frac{k_1}{k_2}$ is constant.

Proof Let γ be a slant helix in E_1^3 and $\langle N(s), U\rangle$ is constant. Then γ a slant helix, from the definition we have

$$\langle N(s), U\rangle = constant, \tag{5.4}$$

where U is a constant vector in E_1^3. By differentiating (5.4), and use (4.2)

$$\begin{aligned} \langle N(s), U\rangle &= constant, \\ \left\langle N_1'(s), U\right\rangle + \left\langle N(s), U'\right\rangle &= 0, \\ \left\langle N_1'(s), U\right\rangle &= 0, \\ \langle -k_1 T_1, U\rangle &= 0 \\ -k_1 \langle T_1, U\rangle &= 0, \quad k_1 \neq 0 \end{aligned}$$

Hence

$$\langle T_1, U\rangle = 0 \tag{5.5}$$

$u \epsilon sp\{N_1, N_2\}$, therefore u$=\cos\theta N_1 + \sin\theta N_2$. U is a linear combination of N_1 and N_2. By differentiating (5.5), and use (4.2)

$$\begin{aligned} \left\langle T', U\right\rangle &= 0 \\ \langle k_2 N_1 + k_1 N_2, U\rangle &= 0 \\ k_2 \langle N_1, U\rangle + k_1 \langle N_2, U\rangle &= 0 \\ k_2 \cos\theta + k_1 \sin\theta &= 0 \\ \frac{k_1}{k_2} = -\cot\theta = constant, \end{aligned}$$

as desired.

Theorem 5.1.4. Let γ be a pseudo null curve in E_1^3 , then γ is a slant helix if and only if

$$det(N_1', N_1'', N_1''') = 0. \tag{5.6}$$

Proof ($\Longrightarrow$) Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$\begin{aligned} N_1' &= -k_1 T \\ N_1'' &= -k_1' T - k_1 T' = -k_1' T - k_1 (k_2 N_1 + k_1 N_2) \\ &= -k_1' T - k_1 k_2 N_1 - k_1^2 N_2 \\ N_1''' &= \left(-k_1' T - k_1 k_2 N_1 - k_1^2 N_2\right)' = -k_1'' T - k_1' T' - k_1' k_2 N_1 \\ &\quad -k_1 k_2' N_1 - k_1 k_2 N_1' - 2k_1 k_1' N_2 - k_1^2 N_2' \\ &= -k_1'' T - k_1' (k_2 N_1 + k_1 N_2) - k_1' k_2 N_1 - k_1 k_2' N_1 \\ &\quad -k_1 k_2 (-k_1 T_1) - 2k_1 k_1' N_2 - k_1^2 (-k_2 T_1) \\ &= \left(2k_1^2 k_2 - k_1''\right) T + \left(-2k_1' k_2 - k_1 k_2'\right) N_1 - 3k_1 k_1' N_2. \end{aligned}$$

So we get

$$
\begin{aligned}
det(N_1', N_1'', N_1''') &= \begin{vmatrix} -k_1 & 0 & 0 \\ -k_1' & -k_1k_2 & -k_1^2 \\ \left(2k_1^2k_2 - k_1''\right) & -\left(2k_1'k_2 + k_1k_2'\right) & -3k_1k_1' \end{vmatrix} \\
&= -k_1\left[-3k_1k_1'\left(-k_1k_2\right) - k_1^2\left(2k_1'k_2 + k_1k_2'\right)\right] \\
&= -k_1\left[3k_1^2k_1'k_2 - 2k_1^2k_1'k_2 - k_1^3k_2'\right] \\
&= -k_1^3\left[k_1'k_2 - k_1k_2'\right] \\
&= -k_1^3\left[\frac{k_1'k_2 - k_1k_2'}{k_2^2}\right]k_2^2 \\
&= -k_1^3k_2^2\left(\frac{k_1}{k_2}\right)'.
\end{aligned}
$$

Since γ is a slant helix, and $\frac{k_1}{k_2}$ is constant. Hence, we have

$$det(N_1', N_1'', N_1''') = 0,\ but\ k_2 \neq 0.$$

($\Longleftarrow$) Suppose that $det(N_1', N_1'', N_1''') = 0$, then it is clear that the $\frac{k_1}{k_2}$ is constant, since $\left(\frac{k_1}{k_2}\right)'$ is zero. Hence the theorem is proved.

Theorem 5.1.5. Let γ be a pseudo null curve in E_1^3 , then γ is a slant helix if and only if

$$det(N_2', N_2'', N_2''') = 0. \tag{5.7}$$

Proof ($\Longrightarrow$) Suppose that $\frac{k_1}{k_2}$ be constant. From (4.2) we have

$$N_2' = -k_2T,$$

therefore

$$
\begin{aligned}
N_2'' &= -k_2'T - k_2T' = -k_2'T - k_2\left(k_2N_1 + k_1N_2\right) \\
&= -k_2'T - k_2^2N_1 - k_1k_2N_2 \\
N_2''' &= \left(-k_2'T - k_2^2N_1 - k_1k_2N_2\right)' \\
&= -k_2''T - k_2'T' - 2k_2k_2'N_1 - k_2^2N_1' - k_1'k_2N_2 - k_1k_2'N_2 - k_1k_2N_2' \\
&= -k_2''T - k_2'\left(k_2N_1 + k_1N_2\right) - 2k_2k_2'N_1 - k_2^2\left(-k_1T_1\right) \\
&\quad -k_1'k_2N_2 - k_1k_2'N_2 - k_1k_2\left(-k_2T_1\right) \\
&= \left(2k_1k_2^2 - k_2''\right)T + \left(-k_2^2 - 2k_2k_2'\right)N_1 + \left(-k_1k_2 - k_1'k_2 - k_1k_2'\right)N_2
\end{aligned}
$$

So we get

$$det(N_2', N_2'', N_2''') = \begin{vmatrix} -k_2 & 0 & 0 \\ -k_2' & -k_2^2 & -k_1k_2 \\ \begin{pmatrix} 2k_1k_2^2 \\ -k_2'' \end{pmatrix} & -\begin{pmatrix} k_2^2 \\ +2k_2k_2' \end{pmatrix} & -\begin{pmatrix} k_1k_2 \\ +k_1'k_2 \\ +k_1k_2' \end{pmatrix} \end{vmatrix}$$

$$\begin{aligned} det(N_2', N_2'', N_2''') &= -k_2\left[k_2^2\left(k_1k_2 + k_1'k_2 + k_1k_2'\right) - k_1k_2\left(k_2^2 + 2k_2k_2'\right)\right] \\ &= -k_2^2\left[k_1k_2^2 + k_1'k_2^2 + k_1k_2k_2' - k_1k_2^2 - 2k_1k_2k_2'\right] \\ &= -k_2^2\left[k_1'k_2^2 - k_1k_2k_2'\right] \\ &= -k_2^3\left[k_1'k_2 - k_1k_2'\right] \\ &= -k_2^3\left[\frac{k_1'k_2 - k_1k_2'}{k_2^2}\right]k_2^2 \\ &= -k_2^5\left(\frac{k_1}{k_2}\right)'. \end{aligned}$$

Since γ is a slant helix, and $\frac{k_1}{k_2}$ is constant. Hence, we have

$$det(N_2', N_2'', N_2''') = 0, \ but \ k_2 \neq 0.$$

($\Longleftarrow$) Suppose that $det(N_2', N_2'', N_2''') = 0$, then it is clear that the $\frac{k_1}{k_2}$ is constant, since $\left(\frac{k_1}{k_2}\right)'$ is zero. Hence the theorem is proved.

Theorem 5.1.6. Let $\gamma : I \longrightarrow E_1^3$ be a unit speed pseudo null curve on M_1 is a general slant helix if and only if

$$N_1''' + 3k_1'T' = \left(k_1'' - 2k_1^2k_2\right)\frac{1}{k_1}N_1'. \tag{5.8}$$

Proof ($\Longrightarrow$) Assume that γ is a general slant helix. Then, from (4.2), we have

$$\begin{aligned} N_1' &= -k_1T_1 \\ N_1'' &= (-k_1T_1)' = -k_1'T_1 - k_1T_1' \\ &= -k_1'T_1 - k_1\left(k_2N_1 + k_1N_2\right) \\ N_1'' &= -k_1'T_1 - k_1k_2N_1 - k_1^2N_2 \end{aligned} \tag{5.9}$$

$$
\begin{aligned}
N_1^{'''} &= \left(-k_1^{'}T_1 - k_1k_2N_1 - k_1^2N_2\right)^{'} \\
&= -k_1^{''}T_1 - k_1^{'}T_1^{'} - k_1^{'}k_2N_1 - k_1k_2^{'}N_1 \\
&\quad -k_1k_2N_1^{'} - 2k_1k_1^{'}N_2 - k_1^2N_2^{'} \\
&= -k_1^{''}T_1 - k_1^{'}T_1^{'} - k_1^{'}k_2N_1 - k_1k_2^{'}N_1 \\
&\quad -k_1k_2N_1^{'} - 2k_1k_1^{'}N_2 - k_1^2\left(-k_2T_1\right) \\
N_1^{'''} &= \left(k_1^2k_2 - k_1^{''}\right)T_1 - k_1^{'}T_1^{'} - \left(k_1^{'}k_2 + k_1k_2^{'}\right)N_1 \\
&\quad -k_1k_2N_1^{'} - 2k_1k_1^{'}N_2. \qquad (5.10)
\end{aligned}
$$

Since γ is a general helix

$$\frac{k_1}{k_2} = c \quad \text{c is constant.} \qquad (5.11)$$

By differentiating (5.11)

$$
\begin{aligned}
\frac{k_1^{'}k_2 - k_1k_2^{'}}{k_2^2} &= 0 \\
k_1^{'}k_2 - k_1k_2^{'} &= 0 \\
k_1^{'}k_2 &= k_1k_2^{'} \\
k_1^{'}k_2 + k_1^{'}k_2 &= k_1k_2^{'} + k_1^{'}k_2 \\
\left(k_1k_2\right)^{'} &= 2k_1^{'}k_2, \qquad (5.12)
\end{aligned}
$$

but

$$
\begin{aligned}
N_1^{'} &= -k_1T, \\
T &= -\frac{1}{k_1}N_1^{'}. \qquad (5.13)
\end{aligned}
$$

By substituting (5.12) and (5.13) in (5.10)

$$
\begin{aligned}
N_1^{'''} &= \left(k_1^2k_2 - k_1^{''}\right)T_1 - k_1^{'}T_1^{'} - \left(k_1^{'}k_2 + k_1k_2^{'}\right)N_1 - k_1k_2N_1^{'} - 2k_1k_1^{'}N_2. \\
N_1^{'''} &= \left(k_1^2k_2 - k_1^{''}\right)\left(-\frac{1}{k_1}N_1^{'}\right) - k_1^{'}T_1^{'} - 2k_1^{'}k_2N_1 - k_1k_2N_1^{'} - 2k_1k_1^{'}N_2 \\
&= \left(k_1^{''} - 2k_1^2k_2\right)\left(\frac{1}{k_1}N_1^{'}\right) - k_1^{'}T_1^{'} - 2k_1^{'}\left(k_2N_1 - k_1N_2\right) \\
&= \left(k_1^{''} - 2k_1^2k_2\right)\left(\frac{1}{k_1}N_1^{'}\right) - k_1^{'}T_1^{'} - 2k_1^{'}T_1^{'}
\end{aligned}
$$

$$
\begin{aligned}
N_1^{'''} &= \left(k_1^{''} - 2k_1^2k_2\right)\left(\frac{1}{k_1}N_1^{'}\right) - 3k_1^{'}T_1^{'} \qquad (5.14) \\
N_1^{'''} + 3k_1^{'}T_1^{'} &= \left(k_1^{''} - 2k_1^2k_2\right)\left(\frac{1}{k_1}N_1^{'}\right).
\end{aligned}
$$

($\Longleftarrow$) We will show that pseudo null curve γ is a slant helix. By differentianting (5.13) covariantly

$$\begin{aligned} T &= -\frac{1}{k_1}N_1^{'} \\ T_1^{'} &= \left(-\frac{1}{k_1}N_1^{'}\right)^{'} \end{aligned}$$

$$T_1^{'} = \frac{k_1^{'}}{k_1^2}N_1^{'} - \frac{1}{k_1}N_1^{''} \tag{5.15}$$

$$\begin{aligned} T_1^{''} &= \left(\frac{k_1^{'}}{k_1^2}N_1^{'} - \frac{1}{k_1}N_1^{''}\right)^{'} \\ T_1^{''} &= \left(\frac{k_1^{'}}{k_1^2}\right)^{'}N_1^{'} + \frac{k_1^{'}}{k_1^2}N_1^{''} + \frac{k_1^{'}}{k_1^2}N_1^{''} - \frac{1}{k_1}N_1^{'''} \\ T_1^{''} &= \left(\frac{k_1^{'}}{k_1^2}\right)^{'}N_1^{'} + \frac{2k_1^{'}}{k_1^2}N_1^{''} - \frac{1}{k_1}N_1^{'''}. \end{aligned} \tag{5.16}$$

By substituting (5.14) in (5.16)

$$\begin{aligned} T_1^{''} &= \left(\frac{k_1^{'}}{k_1^2}\right)^{'}N_1^{'} + \frac{2k_1^{'}}{k_1^2}N_1^{''} - \frac{1}{k_1}\left[\left(k_1^{''} - 2k_1^2k_2\right)\left(\frac{1}{k_1}N_1^{'}\right) - 3k_1^{'}T_1^{'}\right] \\ T_1^{''} &= \left(\frac{k_1^{'}}{k_1^2}\right)^{'}N_1^{'} + \frac{2k_1^{'}}{k_1^2}N_1^{''} - \left(k_1^{''} - 2k_1^2k_2\right)\frac{1}{k_1^2}N_1^{'} + \frac{3k_1^{'}}{k_1}T_1^{'} \end{aligned}$$

$$T_1^{''} = \left[\left(\frac{k_1^{'}}{k_1^2}\right)^{'} - \frac{1}{k_1^2}\left(k_1^{''} - 2k_1^2k_2\right)\right]N_1^{'} + \frac{2k_1^{'}}{k_1^2}D_TD_TN_1 + \frac{3k_1^{'}}{k_1}T_1^{'} \tag{5.17}$$

by substituting (4.2) and (5.9) in (5.17)

$$\begin{aligned} T_1^{''} &= \left[\left(\frac{k_1^{'}}{k_1^2}\right)^{'} - \frac{1}{k_1^2}\left(k_1^{''} - 2k_1^2k_2\right)\right]N_1^{'} \\ &\quad + \frac{2k_1^{'}}{k_1^2}\left(-k_1^{'}T_1 - k_1k_2N_1 - k_1^2N_2\right) + \frac{3k_1^{'}}{k_1}\left(k_2N_1 + k_1N_2\right) \\ T_1^{''} &= \left[\left(\frac{k_1^{'}}{k_1^2}\right)^{'} - \frac{1}{k_1^2}\left(k_1^{''} - 2k_1^2k_2\right)\right]N_1^{'} - 2\left(\frac{k_1^{'}}{k_1}\right)^2T + \frac{k_1^{'}k_2}{k_1}N_1 + k_1^{'}N_2. \end{aligned} \tag{5.18}$$

From (5.8)

$$\begin{aligned} T_1^{'} &= k_2N_1 + k_1N_2 \\ T_1^{''} &= k_2^{'}N_1 + k_2N_1^{'} + k_1^{'}N_2 + k_1N_2^{'} \\ T_1^{''} &= k_2^{'}N_1 + k_2N_1^{'} + k_1^{'}N_2 - k_1k_2T_1. \end{aligned} \tag{5.19}$$

By comparing (5.18) and (5.19)

$$\begin{aligned} \frac{k_1' k_2}{k_1} &= k_2' \\ \frac{k_1'}{k_1} &= \frac{k_2'}{k_2}, \end{aligned} \tag{5.20}$$

by integrating (5.20)

$$\begin{aligned} \int \frac{k_1'}{k_1} &= \int \frac{k_2'}{k_2} \\ \int \frac{dk_1}{ds} . \frac{1}{k_1} ds &= \int \frac{dk_2}{ds} . \frac{1}{k_2} ds \\ \int \frac{1}{k_1} ds &= \int \frac{1}{k_2} ds \\ \ln k_1 &= \ln k_2 + c \\ \ln k_1 - \ln k_2 &= c \\ \ln\left(\frac{k_1}{k_2}\right) &= c \\ e^{\ln\left(\frac{k_1}{k_2}\right)} &= e^c \end{aligned}$$

$$\frac{k_1}{k_2} = e^c \quad \text{constant.}$$

Hence γ is a general slant helix.

Theorem 5.1.7. Let $\gamma : I \longrightarrow E_1^3$ be a unit speed pseudo null curve on M_1 is a general slant helix if and only if

$$N_2''' + 3k_2' T_1' = \left(\frac{k_2''}{k_2} - 2k_1 k_2\right) N_2'. \tag{5.21}$$

Proof ($\Longrightarrow$) Suppose that γ is a general slant helix. Then, from (4.2), we have

$$\begin{aligned} N_2' &= -k_2 T_1 \\ N_2'' &= (-k_2 T_1)' \\ &= -k_2' T_1 - k_2 T_1' \\ &= -k_2' T_1 - k_2 (k_2 N_1 + k_1 N_2) \\ N_2'' &= -k_2' T_1 - k_2^2 N_1 - k_1 k_2 N_2 \end{aligned} \tag{5.22}$$

$$
\begin{aligned}
N_2^{'''} &= \left(-k_2^{'}T_1 - k_2^2 N_1 - k_1 k_2 N_2\right)^{'} \\
&= -k_2^{''}T_1 - k_2^{'}T_1^{\prime} - 2k_2 k_2^{'} N_1 - k_2^2 N_1^{'} - k_1^{'} k_2 N_2 - k_1 k_2^{'} N_2 - k_1 k_2 N_2^{'} \\
&= -k_2^{''}T_1 - k_2^{'}T_1^{\prime} - 2k_2 k_2^{'} N_1 - k_2^2 \left(-k_1 T_1\right) \\
&\quad -k_1^{'} k_2 N_2 - k_1 k_2^{'} N_2 - k_1 k_2 N_2^{'} \\
N_2^{'''} &= \left(k_2^2 k_1 - k_2^{''}\right) T_1 - k_2^{'}T_1^{\prime} - 2k_2 k_2^{'} N_1 - \left(k_1^{'} k_2 + k_1 k_2^{'}\right) N_2 - k_1 k_2 N_2^{'}. \quad (5.23)
\end{aligned}
$$

Since γ is a general helix

$$
\frac{k_1}{k_2} = c \quad \text{c is constant,}
$$

we can differentiate the above equation

$$
\begin{aligned}
\frac{k_1^{'} k_2 - k_1 k_2^{'}}{k_2^2} &= 0 \\
k_1^{'} k_2 - k_1 k_2^{'} &= 0 \\
k_1^{'} k_2 &= k_1 k_2^{'} \\
k_1^{'} k_2 + k_1 k_2^{'} &= k_1 k_2^{'} + k_1 k_2^{'} \\
\left(k_1 k_2\right)^{'} &= 2k_1 k_2^{'}, \qquad (5.24)
\end{aligned}
$$

but

$$
\begin{aligned}
N_2^{'} &= -k_2 T, \\
T &= -\frac{1}{k_2} N_2^{'}. \qquad (5.25)
\end{aligned}
$$

By substituting (5.24) and (5.25) in (5.23)

$$
\begin{aligned}
N_2^{'''} &= \left(k_2^2 k_1 - k_2^{''}\right) T_1 - k_2^{'}T_1^{\prime} - 2k_2 k_2^{'} N_1 - \left(k_1^{'} k_2 + k_1 k_2^{'}\right) N_2 - k_1 k_2 N_2^{'} \\
N_2^{'''} &= \left(k_2^2 k_1 - k_2^{''}\right) \left(-\frac{1}{k_2} N_2^{'}\right) - k_2^{'}T_1^{\prime} - 2k_2 k_2^{'} N_1 - 2k_1 k_2^{'} N_2 - k_1 k_2 N_2^{'} \\
N_2^{'''} &= \left(\frac{k_2^{''}}{k_2} - 2k_1 k_2\right) N_2^{'} - k_2^{'}T_1^{\prime} - 2k_2^{'} \left(k_2 N_1 + k_1 N_2\right) \\
N_2^{'''} &= \left(\frac{k_2^{''}}{k_2} - 2k_1 k_2\right) N_2^{'} - k_2^{'}T_1^{\prime} - 2k_2^{'}T_1^{\prime}
\end{aligned}
$$

$$
\begin{aligned}
N_2^{'''} &= \left(\frac{k_2^{''}}{k_2} - 2k_1 k_2\right) N_2^{'} - 3k_2^{'}T_1^{\prime} \qquad (5.26) \\
N_2^{'''} + 3k_2^{'}T_1^{\prime} &= \left(\frac{k_2^{''}}{k_2} - 2k_1 k_2\right) N_2^{'}.
\end{aligned}
$$

($\Longleftarrow$) We will show that pseudo null curve γ is a slant helix. By (5.25) differentianting covariantly

$$\begin{aligned} T &= -\frac{1}{k_2}N_2' \\ T_1' &= \left(-\frac{1}{k_2}N_2'\right)' \\ T_1' &= \frac{k_2'}{k_2^2}N_2' - \frac{1}{k_2}N_2'' \end{aligned} \tag{5.27}$$

$$\begin{aligned} T_1'' &= \left(\frac{k_2'}{k_2^2}N_2' - \frac{1}{k_2}N_2''\right)' \\ T_1'' &= \left(\frac{k_2'}{k_2^2}\right)' N_2' + \frac{k_2'}{k_2^2}N_2'' + \frac{k_2'}{k_2^2}N_2'' - \frac{1}{k_2}N_2''' \\ T_1'' &= \left(\frac{k_2'}{k_2^2}\right)' N_2' + \frac{2k_2'}{k_2^2}N_2'' - \frac{1}{k_2}N_2''' . \end{aligned} \tag{5.28}$$

By substituting (5.26) in(5.28)

$$\begin{aligned} T_1'' &= \left(\frac{k_2'}{k_2^2}\right)' N_2' + \frac{2k_2'}{k_2^2}N_2'' - \frac{1}{k_2}\left[\left(\frac{k_2''}{k_2} - 2k_1k_2\right)N_2' - 3k_2'T_1'\right] \\ T_1'' &= \left(\frac{k_2'}{k_2^2}\right)' N_2' + \frac{2k_2'}{k_2^2}N_2'' - \left(\frac{k_2''}{k_2} - 2k_1k_2\right)\frac{1}{k_2}N_2' + \frac{3k_2'}{k_2}T_1' \\ T_1'' &= \left[\left(\frac{k_2'}{k_2^2}\right)' - \left(\frac{k_2''}{k_2} - 2k_1k_2\right)\frac{1}{k_2}\right]N_2' + \frac{2k_2'}{k_2^2}N_2'' + \frac{3k_2'}{k_2}T_1', \end{aligned} \tag{5.29}$$

by substituting (4.2) and (5.22) in (5.29)

$$\begin{aligned} T_1'' &= \left[\left(\frac{k_2'}{k_2^2}\right)' - \left(\frac{k_2''}{k_2} - 2k_1k_2\right)\frac{1}{k_2}\right]N_2' \\ &\quad + \frac{2k_2'}{k_2^2}\left(-k_2'T_1 - k_2^2N_1 - k_1k_2N_2\right) + \frac{3k_2'}{k_2}\left(k_2N_1 + k_1N_2\right) \\ T_1'' &= \left[\left(\frac{k_2'}{k_2^2}\right)' - \left(\frac{k_2''}{k_2} - 2k_1k_2\right)\frac{1}{k_2}\right]N_2' \\ &\quad -2\left(\frac{k_2'}{k_2}\right)^2 T_1 + k_2'N_1 + \frac{k_1k_2'}{k_2}N_2 \end{aligned} \tag{5.30}$$

From (4.2)

$$\begin{aligned} T_1' &= k_2N_1 + k_1N_2 \\ T_1'' &= k_2'N_1 + k_2N_1' + k_1'N_2 + k_1N_2' \\ T_1'' &= k_2'N_1 + k_2N_1' + k_1'N_2 - k_1k_2T_1. \end{aligned} \tag{5.31}$$

By comparing (5.30) and (5.31)

$$\begin{aligned}\frac{k_1 k_2^{'}}{k_2} &= k_1^{'} \\ \frac{k_1^{'}}{k_1} &= \frac{k_2^{'}}{k_2},\end{aligned} \tag{5.32}$$

by integrating (5.32)

$$\begin{aligned}\int \frac{k_1^{'}}{k_1} &= \int \frac{k_2^{'}}{k_2} \\ \int \frac{dk_1}{ds}.\frac{1}{k_1}ds &= \int \frac{dk_2}{ds}.\frac{1}{k_2}ds \\ \int \frac{1}{k_1}ds &= \int \frac{1}{k_2}ds \\ \ln k_1 &= \ln k_2 + c \\ \ln k_1 - \ln k_2 &= c \\ \ln\left(\frac{k_1}{k_2}\right) &= c \\ e^{\ln\left(\frac{k_1}{k_2}\right)} &= e^c\end{aligned}$$

$$\frac{k_1}{k_2} = e^c \quad \text{constant.}$$

Hence γ is a general slant helix. Hence the theorem is proved.

5.2. The helices of null Cartan curve in Minkowski 3-space E_1^3

Theorem 5.2.1 Let γ be a null Cartan curve in E_1^3 , then γ is a general helix if and only if $\frac{k_1}{k_2}$ is constant.

Proof Let γ be a general helix in E_1^3 and $\langle T, U\rangle = constant$, then γ is a general helix, from the definition we have

$$\langle T, U\rangle = constant, \tag{5.33}$$

by differentiating the above equation

$$\begin{aligned}\left\langle T^{'}, U\right\rangle + \left\langle T, U^{'}\right\rangle &= 0 \\ \left\langle T^{'}, U\right\rangle &= 0 \\ \langle k_2 T + k_1 N_1, U\rangle &= 0 \\ k_2\langle T, U\rangle + k_1\langle N_1, U\rangle &= 0 \\ k_2\cos\theta + k_1\sin\theta &= 0\end{aligned}$$

$$\frac{k_1}{k_2} = -\cot\theta = \text{constant} \ , \tag{5.34}$$

as desired.

Theorem 5.2.2. Let γ be a null Cartan curve in E_1^3, so γ is a general helix if and only if

$$det(T_1', T_1'', T_1''') = k_1^2 \left(k_1 k_2'' - k_2 k_1'' \right) . \tag{5.35}$$

Proof ($\Longrightarrow$) Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$\begin{aligned}
T_1'(s) &= k_2 T_1 + k_1 N_1 \\
T_1''(s) &= (k_2 T_1 + k_1 N_1)' = k_2' T_1 + k_2 T_1' + k_1' N_1 + k_1 N_1' \\
&= k_2' T_1 + k_2 (k_2 T_1 + k_1 N_1) + k_1' N_1 + k_1 (k_1 N_2) \\
T_1''(s) &= \left(k_2' + k_2^2 \right) T_1 + \left(k_1' + k_1 k_2 \right) N_1 + k_1^2 N_2, \\
T_1'''(s) &= k_2'' T_1 + k_2' T_1' + 2k_2 k_2' T_1 + k_2^2 T_1' + k_1'' N_1 \\
&\quad + k_1' N_1' + k_1' k_2 N_1 + k_1 k_2' N_1 + k_1 k_2 N_1' + 2k_1 k_1' N_2 + k_1^2 N_2' \\
&= k_2'' T_1 + k_2' (k_2 T_1 + k_1 N_1) + 2k_2 k_2' T_1 + k_2^2 (k_2 T_1 + k_1 N_1) \\
&\quad + k_1'' N_1 + k_1' (k_1 N_2) + k_1' k_2 N_1 + k_1 k_2' N_1 \\
&\quad + k_1 k_2 (k_1 N_2) + 2k_1 k_1' N_2 + k_1^2 (-k_2 N_2) \\
T_1'''(s) &= \left(k_2'' + 3k_2 k_2' + k_2^3 \right) T_1 \\
&\quad + \left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2 \right) N_1 + 3k_1 k_1' N_2
\end{aligned}$$

So we get

$$det(T_1', T_1'', T_1''') = \begin{vmatrix} k_2 & k_1 & 0 \\ \left(k_2' + k_2^2 \right) & \left(k_1' + k_1 k_2 \right) & k_1^2 \\ \left(\begin{array}{c} k_2'' + 3k_2 k_2' \\ +k_2^3 \end{array} \right) & \left(\begin{array}{c} k_1'' + 2k_1 k_2' \\ +k_1' k_2 + k_1 k_2^2 \end{array} \right) & 3k_1 k_1' \end{vmatrix}$$

$$
\begin{aligned}
det(T_1', T_1'', T_1''') &= k_2 \left[3k_1 k_1' \left(k_1' + k_1 k_2 \right) - k_1^2 \left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^{2} \right) \right] \\
& \quad -k_1 \left[3k_1 k_1' \left(k_2' + k_2^2 \right) - k_1^2 \left(k_2'' + 3k_2 k_2' + k_2^3 \right) \right] \\
&= -k_1^2 k_2^2 k_1' + k_1^3 k_2 k_2' + 3k_1 k_2 \left(k_1' \right)^2 \\
& \quad -k_1^2 k_2 k_1'' - 3k_1^2 k_1' k_2' + k_1^3 k_2'' \\
&= -k_1^2 k_2 \left(k_2 k_1' - k_1 k_2' \right) + 3k_1 k_1' \left(k_2 k_1' - k_1 k_2' \right) \\
& \quad +k_1^2 \left(k_1 k_2'' - k_2 k_1'' \right) \\
&= -k_1^2 k_2 \left(\frac{k_2 k_1' - k_1 k_2'}{k_2^2} \right) k_2^2 + 3k_1 k_1' \left(\frac{k_2 k_1' - k_1 k_2'}{k_2^2} \right) k_2^2 \\
& \quad +k_1^2 \left(k_1 k_2'' - k_2 k_1'' \right) \\
det(T_1', T_1'', T_1''') &= -k_1^2 k_2^3 \left(\frac{k_1}{k_2} \right)' + 3k_1 k_1' k_2^2 \left(\frac{k_1}{k_2} \right)' + k_1^2 \left(k_1 k_2'' - k_2 k_1'' \right)
\end{aligned}
$$

Since γ is a general helix, and $\frac{k_1}{k_2}$ is constant. Hence, we have

$$
det(T_1', T_1'', T_1''') = k_1^2 \left(k_1 k_2'' - k_2 k_1'' \right), \text{ but } k_2 \neq 0.
$$

($\Longleftarrow$) Suppose that $det(T_1', T_1'', T_1''') = k_1^2 \left(k_1 k_2'' - k_2 k_1'' \right)$, then it is clear that the $\frac{k_1}{k_2}$ is constant, since $\left(\frac{k_1}{k_2} \right)'$ is zero. Hence the theorem is proved.

Theorem 5.2.3. Let γ be a null cartan curve in E_1^3 , then γ is a general helix if and only if

$$
det(N_1', N_1'', N_1''') = 0. \tag{5.36}
$$

Proof ($\Longrightarrow$) Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$
\begin{aligned}
N_1' &= k_1 N_2 \\
N_1'' &= (k_1 N_2)' = k_1' N_2 + k_1 N_2' \\
&= \left(k_1' - k_1 k_2 \right) N_2 \\
N_1''' &= \left[\left(k_1' - k_1 k_2 \right) N_2 \right]' \\
&= \left(k_1'' - k_1' k_2 - k_1 k_2' \right) N_2 + \left(k_1' - k_1 k_2 \right) N_2' \\
&= \left(k_1'' - k_1' k_2 - k_1 k_2' \right) N_2 + \left(k_1' - k_1 k_2 \right) (-k_2 N_2) \\
&= \left(k_1'' - 2k_1' k_2 - k_1 k_2' + k_1 k_2^2 \right) N_2.
\end{aligned}
$$

So we get

$$
\begin{aligned}
det(N_1', N_1'', N_1''') &= \begin{vmatrix} 0 & 0 & k_1 \\ 0 & 0 & -k_1^2 \\ 0 & 0 & \left(k_1'' - 2k_1'k_2 - k_1k_2' + k_1k_2^2\right) \end{vmatrix} \\
det(N_1', N_1'', N_1''') &= 0
\end{aligned}
$$

($\Longleftarrow$) Suppose that $det(N_1', N_1'', N_1''') = 0$, then it is clear that the $\frac{k_1}{k_2}$ is constant, since $\left(\frac{k_1}{k_2}\right)'$ is zero. Hence the equation (5.37) is obtained.

Theorem 5.2.4. Let γ be a null Cartan curve in E_1^3 , so γ is a general helix if and only if

$$det(N_2', N_2'', N_2''') = 0. \tag{5.37}$$

Proof ($\Longrightarrow$) Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$
\begin{aligned}
N_2' &= -k_2N_2 \\
N_2'' &= (-k_2N_2)' = -k_2'N_2 - k_2N_2' \\
&= \left(-k_2' + k_2^2\right) N_2 \\
N_2''' &= \left[\left(-k_2' + k_2^2\right) N_2\right]' \\
&= \left(-k_2'' + 2k_2k_2'\right) N_2 + \left(-k_2' + k_2^2\right) N_2' \\
&= \left(-k_2'' + 2k_2k_2'\right) N_2 + \left(-k_2' + k_2^2\right) (-k_2N_2) \\
&= \left(-k_2'' + 3k_2k_2' - k_2^3\right) N_2.
\end{aligned}
$$

So we get

$$
\begin{aligned}
det(N_2', N_2'', N_2''') &= \begin{vmatrix} 0 & 0 & -k_2 \\ 0 & 0 & (-k_2' + k_2^2) \\ 0 & 0 & (-k_2'' + 3k_2k_2' - k_2^3) \end{vmatrix} \\
det(N_2', N_2'', N_2''') &= 0
\end{aligned}
$$

($\Longleftarrow$) Suppose that $det(N_2', N_2'', N_2''') = 0$, then it is clear that the $\frac{k_1}{k_2}$ is constant, since $\left(\frac{k_1}{k_2}\right)'$ is zero. Hence the theorem is proved.

Theorem 5.2.5. Let $\gamma : I \longrightarrow E_1^3$ be a unit speed null cartan curve on M_1 is a general helix if and only if

$$N_1''' + k_1k_2N_2' = \left(k_1'' - 3k_1'k_2\right) \frac{1}{k_1} N_1'. \tag{5.38}$$

Proof ($\Longrightarrow$) Suppose that γ is a general helix. Then, from (4.22), we have

$$\begin{aligned} N_1^{'} &= k_1 N_2 \\ N_1^{''} &= (k_1 N_2)^{'} = k_1^{'} N_2 + k_1 N_2^{'} \\ &= k_1^{'} N_2 + k_1 (-k_2 N_2) \\ N_1^{''} &= k_1^{'} N_2 - k_1 k_2 N_2 \end{aligned} \tag{5.39}$$

$$\begin{aligned} N_1^{'''} &= \left(k_1^{'} N_2 - k_1 k_2 N_2\right)^{'}, \\ &= k_1^{''} N_2 + k_1^{'} N_2^{'} - k_1^{'} k_2 N_2 - k_1 k_2^{'} N_2 - k_1 k_2 N_2^{'}, \\ N_1^{'''} &= k_1^{''} N_2 - \left(k_1^{'} k_2 + k_1 k_2^{'}\right) N_2 - k_1^{'} k_2 N_2 - k_1 k_2 N_2^{'}. \end{aligned} \tag{5.40}$$

Since γ is a general helix

$$\frac{k_1}{k_2} = c \quad \text{c is constant.} \tag{5.41}$$

we can differentiate (5.41)

$$\begin{aligned} \frac{k_1^{'} k_2 - k_1 k_2^{'}}{k_2^2} &= 0 \\ k_1^{'} k_2 - k_1 k_2^{'} &= 0 \\ k_1^{'} k_2 &= k_1 k_2^{'} \\ k_1^{'} k_2 + k_1^{'} k_2 &= k_1 k_2^{'} + k_1^{'} k_2 \\ (k_1 k_2)^{'} &= 2 k_1^{'} k_2, \end{aligned} \tag{5.42}$$

but

$$\begin{aligned} N_1^{'} &= k_1 N_2, \\ N_2 &= \frac{1}{k_1} N_1^{'}. \end{aligned} \tag{5.43}$$

By substituting (5.42) and (5.43) in (5.40)

$$\begin{aligned} N_1^{'''} &= k_1^{''} N_2 - \left(k_1^{'} k_2 + k_1 k_2^{'}\right) N_2 - k_1^{'} k_2 N_2 - k_1 k_2 N_2^{'}, \\ &= k_1^{''} \left(\frac{1}{k_1} N_1^{'}\right) - 2 k_1^{'} k_2 N_2 - k_1^{'} k_2 N_2 - k_1 k_2 N_2^{'}, \\ &= \frac{k_1^{''}}{k_1} N_1^{'} - 3 k_1^{'} k_2 \left(\frac{1}{k_1} N_1^{'}\right) - k_1 k_2 N_2^{'}, \\ &= \left(k_1^{''} - 3 k_1^{'} k_2\right) \left(\frac{1}{k_1} N_1^{'}\right) - k_1 k_2 N_2^{'}, \end{aligned}$$

$$
\begin{aligned}
N_1^{'''} &= \left(k_1^{''} - 3k_1^{'}k_2\right)\left(\frac{1}{k_1}N_1^{'}\right) - k_1k_2N_2^{'}, && (5.44)\\
N_1^{'''} + k_1k_2N_2^{'} &= \left(k_1^{''} - 3k_1^{'}k_2\right)\left(\frac{1}{k_1}N_1^{'}\right).
\end{aligned}
$$

($\Longleftarrow$) We will show that null cartan curve γ is a general helix. By differentiating (5.43) covariantly

$$
\begin{aligned}
N_2 &= \frac{1}{k_1}N_1^{'}\\
N_2^{'} &= \left(\frac{1}{k_1}N_1^{'}\right)^{'}
\end{aligned}
$$

$$
N_2^{'} = -\frac{k_1^{'}}{k_1^2}N_1^{'} + \frac{1}{k_1}N_1^{''} \tag{5.45}
$$

$$
\begin{aligned}
N_2^{''} &= \left(-\frac{k_1^{'}}{k_1^2}N_1^{'} + \frac{1}{k_1}N_1^{''}\right)^{'}\\
N_2^{''} &= \left(-\frac{k_1^{'}}{k_1^2}\right)^{'}N_1^{'} - \frac{k_1^{'}}{k_1^2}N_1^{''} - \frac{k_1^{'}}{k_1^2}N_1^{''} + \frac{1}{k_1}N_1^{'''}
\end{aligned}
$$

$$
N_2^{''} = \left(-\frac{k_1^{'}}{k_1^2}\right)^{'}N_1^{'} - \frac{2k_1^{'}}{k_1^2}N_1^{''} + \frac{1}{k_1}N_1^{'''}. \tag{5.46}
$$

By substituting (5.39) and (5.44) in (5.46)

$$
\begin{aligned}
N_2^{''} &= \left(-\frac{k_1^{'}}{k_1^2}\right)^{'}N_1^{'} - \frac{2k_1^{'}}{k_1^2}N_1^{''} + \frac{1}{k_1}\left[\left(k_1^{''} - 3k_1^{'}k_2\right)\frac{1}{k_1}N_1^{'} - k_1k_2N_2^{'}\right]\\
N_2^{''} &= \left(-\frac{k_1^{'}}{k_1^2}\right)^{'}N_1^{'} - \frac{2k_1^{'}}{k_1^2}N_1^{''} + \left(k_1^{''} - 3k_1^{'}k_2\right)\frac{1}{k_1^2}N_1^{'} - k_2N_2^{'}
\end{aligned}
$$

$$
\begin{aligned}
N_2^{''} &= \left[\left(-\frac{k_1^{'}}{k_1^2}\right)^{'} + \left(k_1^{''} - 3k_1^{'}k_2\right)\frac{1}{k_1^2}\right]N_1^{'} - \frac{2k_1^{'}}{k_1^2}N_1^{''} - k_2N_2^{'}\\
N_2^{''} &= \left[\left(-\frac{k_1^{'}}{k_1^2}\right)^{'} + \left(k_1^{''} - 3k_1^{'}k_2\right)\frac{1}{k_1^2}\right]N_1^{'} - \frac{2k_1^{'}}{k_1^2}\left(k_1^{'}N_2 + k_1N_2^{'}\right) - k_2N_2^{'}\\
N_2^{''} &= \left[\left(-\frac{k_1^{'}}{k_1^2}\right)^{'} + \left(k_1^{''} - 3k_1^{'}k_2\right)\frac{1}{k_1^2}\right]N_1^{'} - \frac{2\left(k_1^{'}\right)^2}{k_1^2}N_2 - \left(\frac{2k_1^{'}}{k_1} + k_2\right)N_2^{'} && (5.47)
\end{aligned}
$$

From (4.22)

$$
\begin{aligned}
N_2^{'} &= -k_2N_2\\
N_2^{''} &= \left(-k_2N_2\right)^{'} && (5.48)\\
&= -k_2^{'}N_2 - k_2N_2^{'}
\end{aligned}
$$

By comparing (5.47) and (5.48)

$$\begin{aligned}
-\left(\frac{2k_1^{'}}{k_1}+k_2\right) &= -k_2 \\
\frac{2k_1^{'}}{k_1} &= 0 \\
\int \frac{1}{k_1}\frac{dk_1}{ds} &= \int 0 \\
\int \frac{1}{k_1}\frac{dk_1}{ds}ds &= \int 0ds \\
\ln k_1 &= 0 \\
e^{\ln k_1} &= e^0 \\
k_1 &= 1
\end{aligned}$$

$$-\frac{2\left(k_1^{'}\right)^2}{k_1^2} = -k_2^{'}.$$

But $k_1 = 1$, therefore

$$k_2^{'} = 0,$$

which means k_2 is a constant function.

$$\frac{k_1}{k_2} \quad \text{is constant..}$$

Hence γ is a general slant helix.

Theorem 5.2.6. Let $\gamma : I \longrightarrow E_1^3$ be a unit speed null Cartan curve on M_1 is a general helix if and only if

$$N_2^{'''} = \left(k_2^{''} - 3k_2k_2^{'} + k_2^3\right)\left(\frac{1}{k_2}N_2^{'}\right) \tag{5.49}$$

Proof Suppose that γ is a general helix. Then, from (3.9), we have

$$\begin{aligned}
N_2^{'} &= -k_2N_2 \\
N_2^{''} &= \left(-k_2N_2\right)^{'} \\
&= -k_2^{'}N_2 - k_2N_2^{'} \\
&= -k_2^{'}N_2 - k_2\left(-k_2N_2\right) \\
N_2^{''} &= -k_2^{'}N_2 + k_2^2N_2
\end{aligned} \tag{5.50}$$

$$\begin{aligned}
N_2^{'''} &= \left[\left(-k_2^{'}+k_2^2\right)N_2\right]^{'}, \\
&= \left(-k_2^{''}+2k_2k_2^{'}\right)N_2 + \left(-k_2^{'}+k_2^2\right)N_2^{'}, \\
N_2^{'''} &= \left(-k_2^{''}+3k_2k_2^{'}\right)N_2 + k_2^2N_2^{'}.
\end{aligned} \tag{5.51}$$

but

$$N_2' = -k_2 N_2,$$

$$N_2 = -\frac{1}{k_2} N_2'. \tag{5.52}$$

By substituting (5.52) in (5.51)

$$\begin{aligned} N_2''' &= \left(-k_2'' + 3k_2 k_2'\right) N_2 + k_2^2 N_2', \\ N_2''' &= \left(-k_2'' + 3k_2 k_2'\right)\left(-\frac{1}{k_2} N_2'\right) + k_2^2 N_2', \\ N_2''' &= \left(k_2'' - 3k_2 k_2' + k_2^3\right)\left(\frac{1}{k_2} N_2'\right). \end{aligned} \tag{5.53}$$

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Autobiography

I was born and brought up in a city called Zaria in 1992, I completed my primary and junior high school there, then I moved to kaduna state where I completed my high school there In 2009. I got admisson from Dokuz Eylul University Izmir in thesame year, but I had studied Tömer (Turkish language preparatory class) at Ege University. Then I moved to Dokuz Eylul University where I studied mathematics education. I went back to Nigeria as soon as I had graduated from the university where worked as a mathematics teacher from 2014 to 2017. In 2017, I started masters degree at Firat University, Institute of science and technology, department of Mathematics.