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THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

WEAK λ –STATISTICAL CONVERGENCE

Kosrat Othman MOHAMMED

Master's Thesis

Program: Analysis and Functions Theory

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This thesis was prepared according to the rules of Firat University, Graduate School of Natural and Applied Sciences. It was evaluated by the jury members who have signed the following signatures and accepted unanimously because of the defense made available to the academic audiences.

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DECLARATION

I declare that the Master Thesis entitled (Weak $\lambda -$Statistical Convergence) is my own research and prepared by myself, and hereby certify that unless stated, all work contained within this thesis is my own independent research and It is being submitted for the Degree of Master of Science (Analysis and Functions Theory) at the Firat University. It has not been submitted for the award of any other degree at any institution.

Kosrat Othman MOHAMMED

ELAZIĞ, 2020

PREFACE

In this thesis Weak λ −Statistical Convergence and weakly $[V, \lambda]$ −summable of order α and weakly $\lambda-$ statistically convergence sequence of order α were defined and the relations between these concepts was investigated. The results presented in this thesis are promising and may contribute to the literature.

First of all, I gratefully acknowledge the support I would like to express my sincere gratitude to my advisor Prof. Dr. Çiğdem BEKTAŞ for his assistance through the college process. Without him, it would be impossible for me to complete this work.

I am indebted to my mother, wife, brothers, sisters and all my friends, who encouraged me to complete my master thesis with their continuous support during the study. Finally, I want to say thanks to everyone that helps me to prepare master thesis.

Kosrat Othman MOHAMMED
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In the first part of this thesis is consisting of three chapters, some basic concepts related to the subject are given.

In the second part, statistical convergence, weak convergence, weak statistical convergence and λ −statistically convergence and weak λ −statistically convergence are investigated and the relations between each of them are given.

In the last part weakly $[V, \lambda]$ −summable of order α and weakly λ − statistically convergence sequence of order α were defined and the relations between these concepts are given.

Keywords: Statistical Convergence, λ −Statistically Convergence, Weak Convergence, Weak Statistical Convergence, Weak λ −Statistically Convergence of order α.

ÖZET

Zayıf λ −İstatistiksel Yakınsaklik

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Üç bölümden oluşan bu tezin ilk bölümünde konuya ilişkin bazı temel kavramlar verilmiştir.

İkinci bölümde, istatistiksel yakınsaklık, zayıf yakınsaklık, zayıf istatistiksel yakınsaklık, λ −istatistiksel olarak yakınsaklık ve zayıf λ −istatistiksel yakınsaklık araştırılmış ve bunlar arasındaki ilişkiler ele alınmıştır.

Son bölümde α. dereceden zayıf $\lambda-$ istatistiksel yakınsaklık ve α. dereceden zayıf $[V,\lambda]$ −toplanabilirlik ve bu kavramlar arasındaki ilişki verilmiştir.

Anahtar kelimeler: İstatistiksel yakınsaklık, λ −istatistiksel yakınsaklık, Zayıf yakınsaklık, Zayıf İstatistiksel yakınsaklık, α. dereceden zayıf $\lambda-$ istatistiksel yakınsaklık.

LIST OF ABBREVIATIONS

Symbols

$\mathbb{N}$	: Set of the natural numbers
$\mathbb{R}$	: Set of the real numbers
$\mathbb{C}$	: Set of the complex numbers
ω	: Space of all sequences
c	: Space of all convergent sequences
c_0	: Space of all null sequences
ℓ_∞	: Space of all bounded sequences
X^*	: The continuous dual of X
S	: Statistically convergence sequences
S_λ	: λ −statistically convergence sequences
WS_λ	: Weakly λ −statistically convergence sequence
WS_λ^α	: Weakly λ −statistically convergence sequence of order α

1. INTRODUCTION

Statistical convergence is the generalization of the ordinary convergence. In 1935 Zygmund [13] provided the idea of statistical convergence in the first edition published in Warsaw. After that Fast [6] reintroduced this idea formally, statistical convergence has explored in number theory, trigonometric series, Fourier analysis, measure theory, Banach spaces. Later It was investigated further from the viewpoint of linked with summability theory and sequence space by Fridy [7], Salat[11], Bhardwaj and Bala [2], Mursaleen [10] etc. In fact, statistical convergence is closely linked to the theory of convergence in probability.The statistical convergence depends on the density of the subsets of N. Let $K \subseteq \mathbb{N}$. Then $\delta(K)$ denotes the natural density of K which defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n: k \in K\}|.$$

whenever the limit exists. $|\{k \leq n: k \in K\}|$denotes the number of elements of K not exceeding n
A sequence (a_k) is statistically convergent if

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: |a_k - \ell| \geq \varepsilon\}| = 0$$

for any $\varepsilon > 0$ and for some ℓ. Sometimes denotes $stat - \lim_{k\to\infty} a_k = \ell$ (see [7]). After that Mursaleen [10] introduced the $\lambda - density$ of $M \subseteq \mathbb{N}$ is defined by

$$\delta_\lambda(M) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k: n - \lambda_n + 1 \leq k \leq n \; and \; k \in M\}|$$

and λ −statistically convergent as follows:
The sequence $a = (a_k)$ of real number is called λ −statistically convergent to the number ℓ if $\forall \, \varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n: |a_k - \ell| \geq \varepsilon \}| = 0.$$

In this condition we write $S_\lambda - \lim_{n\to\infty} a_k = \ell$

Also Çolak and Bektaş introduced λ - statistical convergence of order α in [4]

Definition 1.1 [3] Let X be a set such that the two operation (vector addition and scalar multiplication) are defined and K be a field on X. If the following axioms are satisfied for every x, y and z in X and every scalars $\alpha, \beta \in K$, then X is said to be a vector space (linear space).

$(x, y) \longrightarrow x + y$ from $X \times X$ into X is called vector addition

(1) $x, y \in X$ (Closure under addition)

(2) $x + y = y + x$ (Commutative property)

(3) $x + (y + z) = (x + y) + z$ (Associative property)

(4) X has zero vector θ such that $x + \theta = x$ for any $x \in X$ (Additive identity)

(5) For every $x \in X$, there is a vector $(-x) \in X$ such that $x + (-x) = \theta$ (Additive inverse)

$(\alpha, x) \longrightarrow \alpha.x$ from $K \times X$ into X is called scalar multiplication

(6) $\alpha x \in X$ (Closure under scalar multiplication)

(7) $\alpha(x + y) = \alpha x + \alpha y$ (Distributive property)

(8) $(\alpha + \beta)x = \alpha x + \beta x$ (Distributive property)

(9) $\alpha(\beta x) = (\alpha\beta)x$ (Associative property)

(10) $1.x = x$ (Scalar identity)

Elements of X are said to be vectors. If $K = \mathbb{R}$, then X is said to be a real linearspace, and if $K = \mathbb{C}$, X is said to be a complex linear space.

Definition 1.2 [12] An infinite sequence is a function whose domain is the set of natural numbers. The function values

$$a_1, a_2, a_3, \ldots, a_k, \ldots$$

are the terms of the sequence. If the domain of the function consists of the first k positive integers, the sequence is finite sequence. If the range of the sequence is real, we say that it is real sequence. In general we use this notation

$$a = (a_k)_{k=1}^{\infty}.$$

Example 1.3 [12] The sequence $(a_k)_{k=1}^{\infty}$ such that $a_k = \frac{1}{k}$ is an infinite sequence, this sequence is a function whose domain $\mathbb{N}$, the value at any k is $\frac{1}{k}$. The set of values is $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

Example 1.4 [12] The sequence $(a_k)_{k=1}^{\infty}$ such that $a_k = (-2)^k$ is also an infinite sequence, this sequence is a function whose domain $\mathbb{N}$. The set of values is $\{2, -2\}$.

Definition 1.5 [12] Let $(a_k)_{k=1}^{\infty}$ be a sequence of real number and $\gamma \in \mathbb{R}$ be a constant real number. If $a_k = \gamma$ for every $n \in \mathbb{N}$ then $(a_k)_{k=1}^{\infty}$ is called constant sequence.

Example 1.6 [12] The sequence $(a_k)_{n=1}^{\infty} = (4)_{k=1}^{\infty}$ is the constant sequence $\{4,4,4, \ldots\}$ whose set of values is singleton $\{4\}$.

Definition 1.7 [12] Given $a = (a_n)$ and let $k_1 < k_2 < \cdots < k_n < \cdots$ where $k_n \in \mathbb{N}$. Then the sequence $(a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots)$ is called a subsequence of a.

Example 1.8 Given $a = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, then obviously the sequence$a = (a_{3n}) = (\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \ldots, \frac{1}{3n})$, is a subsequence of $a = (a_n)$.

Definition 1.9 [12] We call the sequence $(a_k)_{k=1}^{\infty}$ is bounded from above if there exist a real number m such that it is satisfies the inequality

$$a_k \leq m \text{ for all } k \in \mathbb{N}.$$

Definition 1.10 [12] We call the sequence $(a_k)_{n=1}^{\infty}$ is bounded from below if there exist a real number m such that it is satisfies the inequality

$$m \leq a_k \text{ for all } k \in \mathbb{N}.$$

Definition 1.11 [12] We call the sequence $(a_k)_{k=1}^{\infty}$ is bounded sequence if there exist a real number M such that it is satisfies the inequality

$$|a_k| \leq M \text{ for all } k \in \mathbb{N}.$$

Lemma 1.12 [12] The sequence $(a_k)_{k=1}^{\infty}$ is said to be bounded sequence if it is bounded from above and below.

Definition 1.13 [12] Let $(a_k)_{k=1}^{\infty}$ be a sequence of real number, then we say that the sequence $(a_k)_{n=1}^{\infty}$ converge to areal number ℓ if and only if $\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N}$ such that

$$|a_k - \ell| < \varepsilon \;\; \forall\, k \geq N$$

in this case we write

$$\lim_{k\to\infty} a_k = \ell \text{ or } a_k \longrightarrow \ell \text{ as } k \longrightarrow \infty.$$

Definition 1.14 [12] A sequence $(a_k)_{k=1}^{\infty}$ of real number converges to zero is said to be null sequence [3].

Theorem 1.15 [12] Every convergent sequence is bounded.

Remark 1.16 The converse above theorem in general may not be true for example the sequence $(a_k)_{k=1}^{\infty} = (-1)^{k+1}$ is bounded sequence but it is not convergent.

Theorem 1.17 [12] Let $a = (a_n)$ be a convergent sequence, then limit of a is unique.

Definition 1.18 [8] Let X be a real or complex vector space. A $\|\,.\,\|$ function from X into the set $\mathbb{R}^+$ of non-negative numbers, the pair $(X, \|\,.\,\|)$ is called normed linear space and $\|\,.\,\|$ is a norm on X, if the following conditions are satisfied for every elements $x, y \in X$ and scalar γ

(N1) $\|x\| \geq 0$,

(N2) $\|x\| = 0$ if and only if $x = \theta$,

(N3) $\|\gamma x\| = |\gamma|.\|x\|$,

(N4) $\|x + y\| \leq \|x\| + \|y\|$.

The condition (N3) is called absolute homogeneity and the condition (N4) is called triangle inequality.

Remark 1.19 If X is a real linear space, then the scalar γ must be real number.

Definition 1.20 [12] Let (a_k) be a sequence in a normed linear space $(X, \|\,.\,\|)$, then it is called convergent to a in X if

$$\|a_k - a\| \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

Lemma 1.21 [12] Let x and y be two elements in a normed linear space X, then

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

Definition 1.22 [8] Let X and Y be two normed linear spaces over the same field of scalars. Then a transformation (or map, operator) $T : X \longrightarrow Y$ is said to be linear if

$$T(x + y) = T(x) + T(y) \text{ and } T(\alpha x) = \alpha T(x)$$

for all $x, y \in X$ and for all scalars α.

Definition 1.23 [8] f is a linear functional on X if $f : X \longrightarrow \mathbb{R}$ or $\mathbb{C}$ is a linear operator. In other words a linear functional is a real or complex-valued linear operator.

Definition 1.24 [8] Let X and Y be two normed spaces. The set $B(X, Y)$ consisting of all bounded linear operators from X into Y is a linear space with respect to the addition and scalar multiplication of operators. If $Y = \mathbb{R}$ or $\mathbb{C}$, then $B(X, Y)$ is called the continuous dual of X and denoted by X^*. X^* forms a normed space with the norm

$$\|f\| = \sup_{\substack{x\in X \\ x\neq\theta}} \left\{\frac{|f(x)|}{\|x\|}\right\} = \sup_{\substack{x\in X \\ \|x\|=1}} \{|f(x)|\}.$$

Theorem 1.25 [8] (Hahn Banach Theorem) Let f be a bounded linear functional on a subspace Z of a normed space X. Then there exist a bounded linear functional $\tilde{f}$ on X which is an extension of f to X and has the same norm. i.e. $\|\tilde{f}\|_x = \|f\|_Z$ where

$$\|\tilde{f}\|_x = \sup_{\substack{x\in X \\ \|x\|=1}} \{|\tilde{f}(x)|\} \text{ and } \|f\|_z = \sup_{\substack{x\in Z \\ \|x\|=1}} \{|f(x)|\}.$$

Theorem 1.26 [8] Let X be a normed space and let $x_0 \neq \theta$ be any element of X. Then there exist a normed linear functional $\tilde{f}$ on X such that

$$\|\tilde{f}\| = 1 \text{ and } \tilde{f}(x) = \|x_0\|.$$

Definition 1.27 [1] (Space ω) The space of all complex sequences defined by

$$\omega = \{z = (z_k) : z_k \in \mathbb{C}, (k \in \mathbb{N}) \}.$$

An element of ω has the form $z = (a_k + b_k)$, where a_k and b_k are both real sequences. It is easy that ω is a linear space according to the usual coordinative addition and scalar multiplication of sequences which are defined by

$$x + y = (a_k) + (b_k) = (a_k + b_k) \text{ and } ca = (ca_k) = c(a_k)$$

where $a = (a_k)$, $b = (b_k) \in \omega$ and $c \in \mathbb{C}$.

Definition 1.28 [8] (Space ℓ_∞) The space of all bounded sequences is indicated by ℓ_∞ and defined by

$$\ell_\infty = \{a = (a_k) \in \omega: \sup_{k\in\mathbb{N}} |a_k| < \infty\}.$$

Definition 1.29 [3] (Space c) The space of all convergent sequences is indicated by c and defined by

$$c = \{a = (a_k) \in \omega: \lim_{k\to\infty} |a_k - l| = 0, \textit{ for some } l \in \mathbb{C} \}$$

Definition 1.30 [3] (Space c_0) The vector space of all null sequences is indicated by c_0 and defined by

$$c = \{a = (a_k) \in \omega: \lim_{k\to\infty} a_k = 0\}.$$

Theorem 1.31 [1] The following statements are strictly true.

(i) $c_0 \subset c$,

(ii) $c \subset \ell_\infty$.

2. WEAK AND STATISTICAL CONVERGENCE

Definition 2.1 [6] Let $K \subseteq \mathbb{N}$ and define

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n: k \in K\}|.$$

The number $\delta(K)$ is said to be the natural density of the set K, if the limit exist. We can examined that every finite subset of the set $\mathbb{N}$ has $\delta(K) = 0$ and $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} - K$ for any $K \subseteq \mathbb{N}$. If $\delta(K) = 1$, then is called statistically dense.

Example 2.2 For the set $K = \{ak + b: k \in \mathbb{N}\}$ we have $\delta(K) = \frac{1}{a}$.

Example 2.3 The set $K = \{k^2: k \in \mathbb{N}\}$ has natural density zero in fact $|K(n)| \leq \sqrt{n}$ so we conclude that $\lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0$.

Definition 2.4 [7] Let $a = (a_k) \in w$. We say that the sequence (a_k) is statistically convergent if

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: |a_k - \ell| \geq \varepsilon\}| = 0$$

for any $\varepsilon > 0$ and for some ℓ. If the condition satisfied we say that a is statistically convergent to the number ℓ. Sometimes denotes

$$stat - \lim_{k\to\infty} a_k = \ell$$

and S delineate the collection all stat-convergent sequence.

Lemma 2.5 [7] If the sequence (a_k) is stat-convergent to a and the sequence b_k is statistically convergent to b and $c \in \mathbb{R}$, then

(i) The sequence (ca_k) is stat-convergent to ca,

(ii) The sequence $(a_k + b_k)$ is stat-convergent to $a + b$.

Theorem 2.6 [11] Every convergence sequence is statistically convergence sequence. In other word if $\lim_{k\to\infty} a_k = \ell$ then $stat - \lim_{k\to\infty} a_k = \ell$.

Proof: Suppose that $\lim_{k\to\infty} a_k = \ell$. Then $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ such that

$$|a_k - \ell| < \varepsilon \quad \forall\, k \geq N.$$

Since $\delta\big(H(\varepsilon)\big) = 0$ where $H(\varepsilon) = \{k \in \mathbb{N} : |a_k - \ell| \geq \varepsilon\}$, we get $stat - \lim_{k\to\infty} a_k = \ell$.

Note the converse above theorem is not true in general.

Example 2.7 [11] Let (a_k) be defined as:

$$a_k = \begin{cases} 1, & if\ k\ is\ a\ square \\ 0, & otherwise. \end{cases}$$

It is clear that (a_k) is not convergent. Because the limit points of (a_k) are 0 and 1. But the sequence (a_k) is statistically convergent and statistical limit is 0 . Indeed suppose that $\varepsilon > 0$, we have

$$\delta(\{k \in \mathbb{N} : |a_k - 0| \geq \varepsilon\}) = \lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : |a_k - 0| \geq \varepsilon\}|.$$

Fix $n \in \mathbb{N}$ and set $|A| = \{k^2 : k^2 \leq n\}$, then $|A| \leq \sqrt{n}$ where $|A|$ denotes cardinality of the set A, so

$$\delta(\{k \in \mathbb{N} : |a_k - 0| \geq \varepsilon\}) = \lim_{n\to\infty} \frac{1}{n} |\{k \in \mathbb{N} : |a_k - 0| \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{\sqrt{n}}{n}.$$

Therefore

$$\delta(\{k \in \mathbb{N} : |a_k - 0| \geq \varepsilon\}) = 0.$$

Hence

$$stat - \lim_{k\to\infty} a_k = 0.$$

Another difference between ordinary and statistical convergence is the boundedness property of ordinary convergence because in the sense of ordinary convergence sequences are all bounded but we may have unbounded and statistical convergence and hence we have some example to illustrate it.

Example 2.8 Let (a_k) be defined as

$$a_k = \begin{cases} \sqrt{k}\ , & if\ k = m^2 \\ 0\ \ , & ifk \neq m^2 \end{cases}$$

Then the sequence a_k is statistically convergence to 0, but it is not bounded.

Example 2.9 Let (a_k) be defined as

$$a_k = \begin{cases} k^2\ , & if\ k = m^2 \\ 1\ \ , & ifk \neq m^2 \end{cases}$$

Then the sequence a_k is statistically convergence to 1, but it is not bounded.

Definition 2.10 [11] A sequence $a = (a_k)$ is said to be statistically bounded if there exist $M > 0$ such that $\lim_{n\to\infty}\frac{1}{n}|\{k \le n: |a_k| > \varepsilon\}| = 0$ i.e $|a_k| \le M\ a.a.k.$

Theorem 2.11 [11] Every bounded number sequences is statistically bounded.

Remark 2.12 The converse of theorem is generally true.

Example 2.13 Let $a = (a_k)$ be defined as

$$a_k = \begin{cases} k^2 & , \quad if\ k = m^2 \\ (-1)^k, & \quad if k \ne m^2 \end{cases}$$

then the sequence $a = \{1,1,-1,16,-1,1,\ldots\}$ is not bounded. Now we must show that $a = (a_k)$ is statistically bounded, suppose that $M > 1$ be given. Then

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: |a_k| > M\}| \le \lim_{n\to\infty}\frac{\sqrt{n}}{n} = 0.$$

Hence a is statistically bounded.

Remark 2.14 Every subsequence of convergence sequences is also convergent. But every subsequence of statistically convergence may not be convergent and may not be statistically bounded.

For example, suppose that $a = (a_k)$ be defined as

$$a_k = \begin{cases} k & , \quad \text{if } k \text{ is prime number} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Since the natural density of the collection of prime numbers is zero, therefore

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: k \text{ is prime number}\}| = 0$$

and hence (a_k) is statistically convergent to zero, but it is obvious that the subsequence $(a_{k'})$ is not convergence and not statistically convergence such that $(a_{k'}) = \{1,2,3,5,7,11,\ldots\}$.

Theorem 2.15 [7] Let (a_k) and (b_k) be two sequences such that (a_k) is convergent to ℓ and (b_k) is stat-convergent to zero, then the sequence $(a_k + b_k)$ is stat-convergent to ℓ.

Proof: Let (a_k) be a convergent sequence. So by definition of convergent sequence $\lim_{k\to\infty} a_k = \ell$, i.e, $\forall \varepsilon > 0, \exists\, N \in \mathbb{N}$ such that $|a_k - \ell| < \varepsilon,\ \ \forall\, k \ge N$.

Also since (b_k) is stat-convergent to zero so Stat-$\lim_{k\to\infty} b_k = 0$, i.e,

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: |b_k - 0| \ge \varepsilon\}| = 0 \tag{1}$$

Now, let stat - $\lim_{k\to\infty}(a_k + b_k) = \ell'$. So

$$stat - \lim_{k\to\infty}(a_k + b_k) = \lim_{n\to\infty}\frac{1}{n}|\{k \le n: |(a_k + b_k) - \ell'| \ge \varepsilon\}| = 0.$$

Note that

$$\left|\lim_{k\to\infty}|a_k - \ell'\,| + \lim_{n\to\infty}\frac{1}{n}|\{k \le n: |b_k - 0| \ge \varepsilon\}|\right| = 0.$$

By using (1) we get

$$\lim_{k\to\infty}|a_k - \ell'\,| = 0 \text{ i.e. } \lim_{k\to\infty} a_k = \ell'.$$

So by our assumption we know that $\lim_{k\to\infty} a_k = \ell$. So we get $\ell' = \ell$. Hence we get

$$stat - \lim_{k\to\infty}(a_k + b_k) = \ell.$$

Theorem 2.16 [7] Suppose that the sequence (a_k) is stat-convergent to ℓ. Then there are sequences (a_k) and (b_k) such that

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: a_k \ne b_k\}| = 0$$

and (z_k) is stat-null sequence, where

$$\lim_{k\to\infty} b_k = \ell,\ a_k = (b_k + z_k)$$

Proof: Consider the sequence (a_k) is stat-convergent to ℓ. So by definition of stat-convergent we have

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: |a_k - \ell| \ge \varepsilon\}| = 0$$

and by assumption we have $|b_k - \ell| \longrightarrow 0$ as $k \longrightarrow \infty$. We must show that

$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: a_k \ne b_k\}| = 0$ and $\lim_{n\to\infty}\frac{1}{n}|\{k \le n: |z_k - \ell| \ge \varepsilon\}| = 0.$

Since (a_k) is stat-convergent to ℓ, so we have $\lim_{n\to\infty}\frac{1}{n}|\{k \le n: |a_k - \ell| \ge \varepsilon\}| = 0.$

Hence

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: |a_k - \ell - b_k + b_k| \ge \varepsilon\}| = 0.$$

In another word

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n: |(a_k-b_k)+(b_k-\ell)|\ge\varepsilon\}|=0.$$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n: |(a_k-b_k)|\ge\varepsilon\}|+\lim_{n\to\infty}\frac{1}{n}|\{k\le n: |(b_k-\ell)|\ge\varepsilon\}|=0.$$

Since $\lim_{k\to\infty} b_k = \ell$, we have

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n: |(a_k-b_k)|\ge\varepsilon\}|+0=0.$$

Hence

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n: |(a_k-b_k)|\ge\varepsilon\}|=0$$

implies that

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n: a_k\ne b_k\}|=0.$$

Since

$$\lim_{k\to\infty} b_k = \ell \text{ and } a_k=(b_k+z_k)$$

and

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n: a_k\ne b_k\}|=0,$$

we get

$$stat-\lim_{k\to\infty}(b_k+z_k)=\lim_{k\to\infty} b_k = \ell$$

and hence

$$stat-\lim_{k\to\infty}(z_k)=0.$$

Therefore we get our results that (z_k) is statistically null sequence.

Let $\lambda=(\lambda_n)$ be a non- decreasing sequence of positive real numbers such that

$$\lim_{n\to\infty}\lambda_n=\infty,\ \lambda_{n+1}\le\lambda_n+1,\ \lambda_1=1$$

and $I_n = [n - \lambda_n + 1, n]$ be the closed interval . The set of all sequences satisfying above conditions will be denoted by Λ.

Example 2.17 The sequences $(\lambda_n) = (n)$ and $(\lambda_n) = (\ln(ne))$ are in Λ.

Lemma 2.18 Let $(\lambda_n) = (n) \in \Lambda$. Then for every n, $\lambda_n \leq n$.

Proof We will prove it by mathematical induction. If $n = 1$, it is clearly that

$$\lambda_1 \leq 1.$$

Suppose that the inequality be true for $n = k$, it means that

$$\lambda_k \leq k.$$

Then we must show that

$$\lambda_{k+1} \leq k + 1. \tag{2}$$

Since $\lambda = (\lambda_n) \in \Lambda$, so we have

$$\lambda_{k+1} \leq \lambda_k + 1, \tag{3}$$

using (2) and (3) we can write

$$\lambda_{k+1} \leq \lambda_k + 1 \leq k + 1.$$

Hence we get our results.

Definition 2.19 [10] Let $\lambda = (\lambda_n) \in \Lambda$. Also suppose that $I_n = [n - \lambda_n + 1, n]$ and $M \subseteq \mathbb{N}$ then the $\lambda - density$ of M is defined by

$$\delta_\lambda(M) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k: n - \lambda_n + 1 \leq k \leq n \; and \; k \in M\}|.$$

Remark 2.20 Obviously in the special case if $(\lambda_n) = (n)$ and then $\lambda -$ density reduce to the definition of natural density.

Definition 2.21 [4] The sequence $a = (a_k)$ of real number is called $\lambda -$statistically convergent to the number ℓ if $\forall \, \varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n: |a_k - \ell| \geq \varepsilon \}| = 0.$$

In this condition we write $S_\lambda - \lim_{n\to\infty} a_k = \ell$ and we denote the set of all $\lambda -$ statistically convergent by S_λ.

Remark 2.22 If $(\lambda_n) = (n)$ then S_λ is same as S.

Theorem 2.23 [4] Let (a_k) and (b_k) be two sequences of real numbers such that $S_\lambda - \lim_{k\to\infty} a_k = a$ and $S_\lambda - \lim_{k\to\infty} b_k = b$ and c be any real number, then

(i) $S_\lambda - \lim_{k\to\infty} ca_k = ca,$

(ii) $S_\lambda - \lim_{k\to\infty} a_k + b_k = a + b.$

Proof (i) Since

$$|\{k \le n: |ca_k - ca| \ge \varepsilon\}| = \left|\left\{k \le n: |a_k - a| \ge \frac{\varepsilon}{|c|}\right\}\right|,$$

we have

$$\frac{1}{\lambda_n}|\{k \le n: |ca_k - ca| \ge \varepsilon\}| = \frac{1}{\lambda_n}\left|\left\{k \le n: |a_k - a| \ge \frac{\varepsilon}{|c|}\right\}\right|.$$

Therefore

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k \le n: |ca_k - ca| \ge \varepsilon\}| = \lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k \le n: |a_k - a| \ge \frac{\varepsilon}{|c|}\right\}\right| = 0,$$

that is $S_\lambda - \lim_{k\to\infty} ca_k = ca.$

(ii) Observe that

$$|\{k \le n: |(a_k + b_k) - (a + b)| \ge \varepsilon\}| = |\{k \le n: |(a_k - \mathrm{a}) + (b_k - b)| \ge \varepsilon\}|$$

$$\le \left|\left\{k \le n: |a_k - a| \ge \frac{\varepsilon}{2}\right\}\right| + \left|\left\{k \le n: |b_k - b| \ge \frac{\varepsilon}{2}\right\}\right|.$$

Hence

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k \le n: |(a_k + b_k) - (a + b)| \ge \varepsilon\}|$$

$$\le \lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k \le n: |a_k - a| \ge \frac{\varepsilon}{2}\right\}\right| + \lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k \le n: |b_k - b| \ge \frac{\varepsilon}{2}\right\}\right|.$$

Therefore

$$\lim_{n\to\infty}\frac{1}{\lambda_n}|\{k \le n: |(a_k + b_k) - (a + b)| \ge \varepsilon\}| = 0.$$

It means that $S_\lambda - \lim_{k\to\infty} a_k + b_k = a + b.$

Definition 2.24 [2] A sequence (a_k) in a normed space X is said to be weakly convergent to the number $a \in X$, if $\lim_{k\to\infty} \varphi(\, a_k - a) = 0$ for any $\varphi \in X^*$, the continuous dual of X. In this condition we write $W - \lim_{k\to\infty} a_k = a$ or simply $a_k \overset{W}{\to} a$.

Theorem 2.25 [2] If a sequence $(a_k) \in X$ such that $W - \lim_{k\to\infty} a_k = a$, then the weak limit $a \in X$ is unique.

Proof: Suppose that there exist $a, b \in X$ such that $a_k \overset{W}{\to} a$ and $a_k \overset{W}{\to} b$ where $a \neq b$. Hence $a - b \neq 0$, so by application of Hahn Banach theorem there exist $\varphi \in X^*$ such that $\varphi(a - b) = \|a - b\|$ with $\|\varphi\| = 1$.

Since $a_k \overset{W}{\to} a$ and $a_k \overset{W}{\to} b$ so for every $\varphi \in X^*$, we have

$$\varphi(a_k) \overset{W}{\to} \varphi(a) \text{ and } \varphi(a_k) \overset{W}{\to} \varphi(b) \text{ as } k \longrightarrow \infty$$

so

$$\varphi(a) = \varphi(b)$$

implies

$$\varphi(a - b) = 0.$$

Therefore

$$\|a - b\| = 0.$$

So $a = b$ this is a contradiction with our assumption. Hence we get the weak limit is unique.

Definition 2.26 [2] A sequence (a_k) in a normed space X is said to be weakly statistically convergent to the number $a \in X$, if $\forall \varepsilon > 0$, $\delta(\{k \leq n: |\varphi(a_k - a)| \geq \varepsilon\}) = 0$ for any $\varphi \in X^*$, the continuous dual of X. In this condition we write $WS - \lim_{k\to\infty} a_k = a$ or simply $a_k \overset{WS}{\longrightarrow} a$. The set of all weak statistically convergent sequence is denoted by WS.

Theorem 2.27 [2] Suppose that the sequence (a_k) is weakly convergent in a normed space X and $W - \lim_{k\to\infty} a_k = a$. Then (a_k) is weakly statistically convergent to a.

Proof: Suppose that $W - \lim_{k\to\infty} a_k = a$. Then by definition of weak convergence sequence we have, for any $\varepsilon > 0$ and each $\varphi \in X^*$, there exists a positive integer N such that

$$|\varphi(a_k - a)| < \varepsilon$$

for all $k \geq N$. Thus the set $H(\varepsilon) = \{k \in \mathbb{N} : |\varphi(a_k - a)| \geq \varepsilon\}$ is finite, therefore $\delta(H(\varepsilon)) = 0$. It means $\mathrm{WS} - \lim_{k\to\infty} a_k = a$.

Remark 2.28 The converse above theorem is not generally true.

Definition 2.29 [9] A sequence (a_k) in a normed space X is said to be weakly λ −statistically convergent to the number $a \in X$, if $\forall \varepsilon > 0$ and for any $\varphi \in X^*$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |\varphi(a_k - a)| \geq \varepsilon\}| = 0.$$

In this condition we write $WS_\lambda - \lim_{k\to\infty} a_k = a$ or simply $a_k \xrightarrow{WS_\lambda} a$.

Theorem 2.30 [9] Suppose that (a_k) be a sequence in a normed space X. If $a_k \xrightarrow{WS_\lambda} a$ then a must be unique.

Proof: Consider there exists $a, b \in X$ such that $a_k \xrightarrow{WS_\lambda} a$ and $a_k \xrightarrow{WS_\lambda} b$. Then for every $\varphi \in X^*$ we have

$$\varphi(a_k) \xrightarrow{S_\lambda} \varphi(a) \text{ and } \varphi(a_k) \xrightarrow{S_\lambda} \varphi(b).$$

Now by the uniqueness of S_λ −limit of a sequence of scalars immediately implies $\varphi(a) = \varphi(b)$; and so by linearity of φ we have $\varphi(a - b) = 0$. Suppose that if possible $a \neq b$. Then $a - b \neq 0$ and so by application of Hahn Banach theorem there exist $\varphi \in X^*$ such that $\varphi(a - b) = \|a - b\|$ with $\|\varphi\| = 1$. Since $\|a - b\| \neq 0$, therefore $\varphi(a - b) \neq 0$ and therefore we obtain a contradiction as $\varphi(a - b) = 0$. Hence $a = b$.

Theorem 2.31 [9] Let (a_k) and (b_k) be two sequences in X such that $WS_\lambda - \lim_{k\to\infty} a_k = a$ and $S_\lambda - \lim_{k\to\infty} b_k = b$ and $a, b \in X$, γ be any scalar. Then

(i) $WS_\lambda - \lim_{k\to\infty} \gamma a_k = \gamma a$,

(ii) $WS_\lambda - \lim_{k\to\infty} a_k + b_k = a + b$.

Theorem 2.32 [9] For every sequence (a_k) in X , if $W - \lim_{k\to\infty} a_k = a$, then $WS_\lambda - \lim_{k\to\infty} a_k = a$. But the converse must not be true in general .

Proof: Suppose that $W - \lim_{k\to\infty} a_k = a$. Then by definition of weak convergence sequence we have, for any $\varepsilon > 0$ and each $\varphi \in X^*$, there exists a positive integer N such that

$$|\varphi(a_k - a)| < \varepsilon,$$

for all $k \geq N$. Thus the set $H(\varepsilon) = \{k \in \mathbb{N} : |\varphi(a_k - a)| \geq \varepsilon\}$ is finite, therefore $\delta_\lambda\big(H(\varepsilon)\big) = 0$. It means that $WS_\lambda - \lim_{k\to\infty} a_k = a$.

Theorem 2.33 [9] S_λ −convergence implies WS_λ −convergence with the same limit in X.

Proof: Suppose that (a_k) be a sequences in X such that $S_\lambda - \lim_{k\to\infty} a_k = a$. Then for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \|a_k - a\| \geq \varepsilon\}| = 0.$$

Now for any $\varepsilon > 0$ and each $\varphi \in X^*$, the expression

$$\frac{1}{\lambda_n} |\{k \in I_n : |\varphi(a_k - a)| \geq \varepsilon\}| \leq \frac{1}{\lambda_n} |\{k \in I_n : \|\varphi\|\|a_k - a\| \geq \varepsilon\}|$$

$$= \frac{1}{\lambda_n} |\{k \in I_n : \|a_k - a\| \geq \frac{\varepsilon}{\|\varphi\|}\}|$$

gives immediately $\frac{1}{\lambda_n} |\{k \in I_n : |\varphi(a_k - a)| \geq \varepsilon\}| = 0$.

Definition 2.34 [9] A sequence (a_k) in X is said to be weakly $[V, \lambda]$ −summable to $\ell \in X$, provided that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |\varphi(a_k - \ell)| = 0,$$

for every $\varphi \in X^*$. In this case we can write $W[V, \lambda] - \lim_{k\to\infty} a_k = \ell$ or $a_k \xrightarrow{W[V,\lambda]} a$.

Theorem 2.35 [9] For any sequence (a_k) in X, $a_k \xrightarrow{W[V,\lambda]} \ell$ if and only if $a_k \xrightarrow{WS_\lambda} \ell$.

Proof: Suppose that $a_k \xrightarrow{W[V,\lambda]} \ell$. Then for every $\varphi \in X^*$ and $\varepsilon > 0$, we have

$$\frac{1}{\lambda_n} \sum_{k\in I_n} |\varphi(a_k - \ell)| \geq \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ |\varphi(a_k-\ell)|\geq\varepsilon}} |\varphi(a_k - \ell)|$$

$$\geq \frac{\varepsilon}{\lambda_n} |\{k \in I_n : |\varphi(a_k - \ell)| \geq \varepsilon\}|$$

so $a_k \xrightarrow{W[V,\lambda]} \ell$ implies $a_k \xrightarrow{WS_\lambda} \ell$.

Conversely suppose that $a_k \xrightarrow{WS_\lambda} \ell$. Since $\varphi \in X^*$, φ is bounded. So there exists some $M > 0$ such that $|\varphi(a_k - \ell)| \leq M$ for all k. For $\varepsilon > 0$,

$$\frac{1}{\lambda_n}\sum_{k\in I_n}|\varphi(a_k-\ell)| = \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\|\varphi(a_k-\ell)|\geq\varepsilon}}|\varphi(a_k-\ell)| + \sum_{\substack{k\in I_n\\|\varphi(a_k-\ell)|<\varepsilon}}|\varphi(a_k-\ell)|$$

$$\leq \frac{M}{\lambda_n}|\{k\in I_n: |\varphi(a_k-\ell)|\geq\varepsilon\}| + \varepsilon$$

which implies that $a_k \xrightarrow{W[V,\lambda]} \ell$.

3. WEAK λ-STATISTICAL CONVERGENCE OF ORDER α

Definition 3.1 [5] Let $\lambda = (\lambda_n) \in \Lambda$ and $0 < \alpha \leq 1$ be given. The sequence $a = (a_k) \in w$ is callled weakly $\lambda -$ statistically convergence of order α if for every $\varepsilon > 0$ there is $\ell \in X$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\varphi(a_k - \ell)| \geq \varepsilon\}| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ and for every $\varphi \in X^*$. In this case we write

$$WS_\lambda^\alpha - \lim_{k\to\infty} a_k = \ell.$$

The set of all weakly $\lambda -$ statistically convergence sequence of order α will be represented by WS_λ^α. The weakly $\lambda -$ statistically convergence of order α is the same with weakly $\lambda -$ statistically convergence, it means $WS_\lambda^\alpha = WS_\lambda$ if $\alpha = 1$.

Definition 3.2 [5] A sequence $a = (a_k)$ in X is said to be strongly $\lambda -$ statistically convergence of order α to $\ell \in X$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \|a_k - \ell\| \geq \varepsilon\}| = 0,$$

where $0 < \alpha \leq 1$. In this case, we write $S_\lambda^\alpha - \lim_{k\to\infty} a_k = \ell$ or $a_k \overset{S_\lambda^\alpha}{\to} \ell$. The set of all strongly $\lambda -$ statistically convergence sequence of order α will be represented by S_λ^α .

Theorem 3.3 [5] Let $0 < \alpha \leq 1$ and $a = (a_k)$, $b = (b_k)$ be sequences of complex numbers, then

(i) For every sequence (a_k) in X , if $WS_\lambda^\alpha - \lim_{k\to\infty} a_k = \ell$, then ℓ must be unique,

(ii) If $WS_\lambda^\alpha - \lim_{k\to\infty} a_k = \ell_0$ and c being a scalar, then $WS_\lambda^\alpha - \lim_{k\to\infty} ca_k = c\ell_0$,

(iii) If $WS_\lambda^\alpha - \lim_{k\to\infty} a_k = \ell_1$ and $WS_\lambda^\alpha - \lim_{k\to\infty} b_k = \ell_2$, then $WS_\lambda^\alpha - \lim_{k\to\infty}(a_k + b_k) = \ell_1 + \ell_2$.

Proof The proof of (i) follows similar as in theorem 2.25 so omitted here.

(ii) Since

$$|\{k \in I_n : |\varphi(ca_k - ca)| \geq \varepsilon\}| = \left|\left\{k \in I_n : |\varphi(a_k - a)| \geq \frac{\varepsilon}{|c|}\right\}\right|,$$

we have

$$\frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\varphi(ca_k - ca)| \geq \varepsilon\}| = \frac{1}{\lambda_n^\alpha} \left|\left\{k \in I_n : |\varphi(a_k - a)| \geq \frac{\varepsilon}{|c|}\right\}\right|.$$

Therefore

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|\{k\in I_n: |\varphi(ca_k-ca)|\geq\varepsilon\}| = \lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n: |\varphi(a_k-a)|\geq\frac{\varepsilon}{|c|}\right\}\right| = 0$$

that is $WS_\lambda^\alpha - \lim_{k\to\infty} ca_k = c\ell_0$.

(iii) Observe that

$$|\{k\in I_n: |\varphi((a_k+b_k)-(\ell_1+\ell_2))|\geq\varepsilon\}|$$

$$= |\{k\in I_n: |\varphi((a_k-\ell_1)+(b_k-\ell_2))|\geq\varepsilon\}|$$

$$\leq \left|\left\{k\in I_n: |\varphi(a_k-\ell_1)|\geq\frac{\varepsilon}{2}\right\}\right| + \left|\left\{k\in I_n: |\varphi(b_k-\ell_2)|\geq\frac{\varepsilon}{2}\right\}\right|.$$

Hence

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|\{k\in I_n: |\varphi((a_k+b_k)-(\ell_1+\ell_2))|\geq\varepsilon\}|$$

$$\leq \lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n: |\varphi(a_k-\ell_1)|\geq\frac{\varepsilon}{2}\right\}\right| + \lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n: |\varphi(b_k-\ell_2)|\geq\frac{\varepsilon}{2}\right\}\right|.$$

Therefore

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}|\{k\in I_n: |(a_k+b_k)-(\ell_1+\ell_2)|\geq\varepsilon\}| = 0.$$

It means that $WS_\lambda^\alpha - \lim_{k\to\infty}(a_k+b_k) = \ell_1+\ell_2$.

Theorem 3.4 [5] Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ belong to Λ such that $\lambda_n\leq\mu_n$ for all $n\in\mathbb{N}_{n_0}$ and let α and β be given such that $0<\alpha\leq\beta\leq 1$.

(i) If

$$\lim_{n\to\infty}\inf\frac{\lambda_n^\alpha}{\mu_n^\beta} > 0 \quad (4)$$

then $WS_\mu^\beta \subseteq WS_\lambda^\alpha$.

(ii) If

$$\lim_{n\to\infty}\frac{\lambda_n^\alpha}{\mu_n^\beta} = 1 \text{ and } \lim_{n\to\infty}\frac{\mu_n}{\mu_n^\beta} = 1 \quad (5)$$

then $WS_\mu^\beta = WS_\lambda^\alpha$.

Proof: (i) Let $\lambda_n \leq \mu_n, n \in \mathbb{N}_{n_0}$ and let (4) be satisfied. Since $I_n \subset J_n$, for given $\varepsilon > 0$ we have

$$|\{k \in J_n: |\varphi(a_k - \ell)| \geq \varepsilon\}| \supset |\{k \in I_n: |\varphi(a_k - \ell)| \geq \varepsilon\}|$$

and so

$$\frac{1}{\mu_n^\beta}|\{k \in J_n: |\varphi(a_k - \ell)| \geq \varepsilon\}| \geq \frac{\lambda_n^\alpha}{\mu_n^\beta} \cdot \frac{1}{\lambda_n^\alpha}|\{k \in I_n: |\varphi(a_k - \ell)| \geq \varepsilon\}|$$

for all $n \in \mathbb{N}_{n_0}$, where $J_n = [n - \mu_n + 1, n]$. By taking limit as $n \longrightarrow \infty$ in the last inequality and using (4) we get $WS_\mu^\beta \subseteq WS_\lambda^\alpha$.

(ii) Suppose that $(a_k) \in WS_\lambda^\alpha$ and (5) be satisfied. Since $I_n \subset J_n$, for given $\varepsilon > 0$ we may write

$$\begin{aligned}
\frac{1}{\mu_n^\beta}|\{k \in J_n: |\varphi(a_k - \ell)| \geq \varepsilon\}| &= \frac{1}{\mu_n^\beta}|\{n - \mu_n + 1 \leq k \leq n - \lambda_n: |\varphi(a_k - \ell)| \geq \varepsilon\}| \\
&\quad + \frac{1}{\mu_n^\beta}|\{k \in I_n: |\varphi(a_k - \ell)| \geq \varepsilon\}| \\
&\leq \frac{\mu_n - \lambda_n}{\mu_n^\beta} + \frac{1}{\mu_n^\beta}|\{k \in I_n: |\varphi(a_k - \ell)| \geq \varepsilon\}| \\
&\leq \frac{\mu_n - \lambda_n^\alpha}{\mu_n^\beta} + \frac{1}{\mu_n^\beta}|\{k \in I_n: |\varphi(a_k - \ell)| \geq \varepsilon\}| \\
&\leq \frac{\mu_n}{\mu_n^\beta} - \frac{\lambda_n^\alpha}{\mu_n^\beta} + \frac{1}{\lambda_n^\alpha}|\{k \in I_n: |\varphi(a_k - \ell)| \geq \varepsilon\}|
\end{aligned}$$

$n \in \mathbb{N}_{n_0}$. Since $\lim_{n\to\infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} = 1$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_n^\beta} = 1$ by (5), therefore the right hand side of latest inequality tends to 0 as $n \longrightarrow \infty$ ($\frac{\mu_n}{\mu_n^\beta} - \frac{\lambda_n^\alpha}{\mu_n^\beta} \geq 0$ for $n \in \mathbb{N}_{n_0}$). Hence $WS_\lambda^\alpha \subset WS_\mu^\beta$. Since (5) implies (4) so we get our results $WS_\mu^\beta = WS_\lambda^\alpha$.

Corollary 3.5 [5] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_0}$ and let (4) be satisfied. Then the following statements hold:

(i) If $\alpha = \beta$, then $WS_\mu^\alpha \subseteq WS_\lambda^\alpha$ for every $\alpha \in (0,1]$,

(ii) If $\beta = 1$, then $WS_\mu \subseteq WS_\lambda^\alpha$ for every $\alpha \in (0,1]$

Corollary 3.6 [5] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \leq \mu_n$ $\forall n \in \mathbb{N}_{n_0}$ and let (5) be satisfied. Then the following statements hold:

(i) If $\alpha = \beta$, then $WS_\lambda^\alpha \subseteq WS_\mu^\alpha$ for every $\alpha \in (0,1]$,

(ii) If $\beta = 1$, then $WS_\lambda^\alpha \subseteq WS_\mu$ for every $\alpha \in (0,1]$.

Theorem 3.7 [5] S_λ^α $-$convergence implies WS_λ^α $-$convergence with the same limit in X, but the converse is not true true in general.

Proof: Suppose that (a_k) in X, be a sequence such that $a_k \xrightarrow{S_\lambda^\alpha} \ell$. Then for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \|a_k - \ell\| \geq \varepsilon\}| = 0.$$

Now, for any $\varepsilon > 0$ and every $\varphi \in X^*$, the expression

$$\frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\varphi(a_k - \ell)| \geq \varepsilon\}| \leq \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \|\varphi\| \|a_k - \ell\| \geq \varepsilon\}|$$

$$= \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n : \|a_k - \ell\| \geq \frac{\varepsilon}{\|\varphi\|} \right\} \right|$$

It means S_λ^α $-$convergence implies WS_λ^α-convergence with the same limit.

Theorem 3.8 [5] Suppose that $0 < \alpha \leq \beta \leq 1$. Then the inclusion $WS_\lambda^\alpha \subseteq WS_\lambda^\beta$ for some α and β is strict such that $\alpha < \beta$.

Proof: Suppose that $0 < \alpha \leq \beta \leq 1$. Then we may write

$$\frac{1}{\lambda_n^\beta} \leq \frac{1}{\lambda_n^\alpha}$$

and so that

$$\frac{1}{\lambda_n^\beta} |\{k \in I_n : |\varphi(a_k - \ell)| \geq \varepsilon\}| \leq \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\varphi(a_k - \ell)| \geq \varepsilon\}|.$$

By taking limit in both sides this equation as n goes to infinity we get results

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k \in I_n : |\varphi(a_k - \ell)| \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |\varphi(a_k - \ell)| \geq \varepsilon\}|$$

Therefore $WS_\lambda^\alpha \subseteq WS_\lambda^\beta$.

Corollary 3.9 [5] Suppose that $0 < \alpha \leq 1$ be given. Then $WS_\lambda^\alpha \subseteq WS_\lambda$

Corollary 3.10

(i) $WS_\lambda^\alpha = WS_\lambda^\beta \Leftrightarrow \alpha = \beta$,

(ii) $WS_\lambda^\alpha = WS_\lambda \Leftrightarrow \alpha = 1$.

Definition 3.11 [5] A sequence (a_k) in X is called weakly $[V,\lambda]-$summable of order α to $\ell \in X$ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} |\varphi(a_k - \ell)| = 0,$$

for every $\varphi \in X^*$ where $0 < \alpha \leq 1$. In this case we can write $W^\alpha[V,\lambda] - \lim_{k\to\infty} a_k = \ell$ and $W^\alpha[V,\lambda]$ represents the collection of all weakly $[V,\lambda]-$summable of order α sequences.

Theorem 3.12 [5] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_0}$and $0 < \alpha \leq \beta \leq 1$. Then the following statements holds:

(i) If $\lim_{n\to\infty} \inf \frac{\lambda_n^\alpha}{\mu_n^\beta} > 0$, then $W^\beta[V,\mu] \subseteq W^\alpha[V,\lambda]$,

(ii) If $\lim_{n\to\infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} = 1$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_n^\beta} = 1$, then $W^\beta[V,\mu] = W^\alpha[V,\lambda]$.

Proof: (i) Let $\lambda_n \leq \mu_n$, $\forall n \in \mathbb{N}$. Since $I_n \subset J_n$ so that we may write

$$\frac{1}{\mu_n^\beta} \sum_{k\in J_n} |\varphi(a_k - \ell)| \geq \frac{1}{\mu_n^\beta} \sum_{k\in I_n} |\varphi(a_k - \ell)|$$

and so

$$\frac{1}{\mu_n^\beta} \sum_{k\in J_n} |\varphi(a_k - \ell)| \geq \frac{\lambda_n^\alpha}{\mu_n^\beta} \; \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} |\varphi(a_k - \ell)|$$

for $n \in \mathbb{N}_{n_0}$. By taking limit in both sides this equation as n goes to infinity we get results and using (4) we get $W^\beta[V,\mu] \subseteq W^\alpha[V,\lambda]$.

(ii) Suppose that $(a_k) \in W^\alpha[V,\lambda]$ and $\lim_{n\to\infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} = 1$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_n^\beta} = 1$. Since $\varphi \in X^*$, φ is bounded. So there exists some $M > 0$ such that $|\varphi(a_k - \ell)| \leq M$ for all k. Since $\lambda_n \leq \mu_n$ and so that $\frac{1}{\mu_n^\beta} \leq \frac{1}{\lambda_n^\alpha}$, and $I_n \subset J_n$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{1}{\mu_n^\beta} \sum_{k\in J_n} |\varphi(a_k - \ell)| &= \frac{1}{\mu_n^\beta} \sum_{k\in J_{n-I_n}} |\varphi(a_k - \ell)| + \frac{1}{\mu_n^\beta} \sum_{k\in I_n} |\varphi(a_k - \ell)| \\
&\leq \left(\frac{\mu_n - \lambda_n}{\mu_n^\beta}\right) M + \frac{1}{\mu_n^\beta} \sum_{k\in I_n} |\varphi(a_k - \ell)| \\
&\leq \left(\frac{\mu_n - \lambda_n^\alpha}{\mu_n^\beta}\right) M + \frac{1}{\mu_n^\beta} \sum_{k\in I_n} |\varphi(a_k - \ell)|
\end{aligned}$$

$$\leq \left(\frac{\mu_n}{\mu_n^\beta} - \frac{\lambda_n^\alpha}{\mu_n^\beta}\right) M + \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |\varphi(a_k - \ell)|$$

for al $n \in \mathbb{N}_{n_0}$. Hence $W^\alpha[V,\mu] \subseteq W^\beta[V,\lambda]$. Since (5) implies (4) therefore we get the equality $W^\beta[V,\mu] = W^\alpha[V,\lambda]$.

Corollary 3.13 [5] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let (4) be satisfied. Then the following statements hold:

(i) If $\alpha = \beta$, then $W^\alpha[V,\mu] \subseteq W^\alpha[V,\lambda]$ for every $\alpha \in (0,1]$,

(ii) If $\beta = 1$, then $W[V,\mu] \subseteq W^\alpha[V,\lambda]$ for every $\alpha \in (0,1]$.

Corollary 3.14 [5] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_0}$and let (4) be satisfied. Then the following statements hold:

(i) If $\alpha = \beta$, then $W^\alpha[V,\lambda] \subseteq W^\alpha[V,\mu]$ for every $\alpha \in (0,1]$,

(ii) If $\beta = 1$, then $W^\alpha[V,\lambda] \subseteq W[V,\mu]$ for every $\alpha \in (0,1]$.

Theorem 3.15 [5] Suppose that $\alpha, \beta \in (0,1]$ be real numbers such that $\alpha \leq \beta$, and $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}_{n_0}$.

(i) Suppose that (4) holds, then if a sequence is $W^\beta[V,\lambda]$ $-$summable of order β to ℓ, then it is $WS_\lambda^\alpha -$ statistically convergent of order α to ℓ,

(ii) Suppose that (5) holds, if a sequence $WS_\lambda^\alpha -$ statistically convergent of order α to ℓ, then it is $W^\beta[V,\lambda]$ $-$summable of order β to ℓ.

Proof For every sequence $a = (a_k)$ and $\varepsilon > 0$, we have

$$\sum_{k \in J_n} |\varphi(a_k - \ell)| = \sum_{\substack{k \in J_n \\ |\varphi(a_k - \ell)| \geq \varepsilon}} |\varphi(a_k - \ell)| + \sum_{\substack{k \in J_n \\ |\varphi(a_k - \ell)| < \varepsilon}} |\varphi(a_k - \ell)|$$

$$\geq \sum_{\substack{k \in I_n \\ |\varphi(a_k - \ell)| \geq \varepsilon}} |\varphi(a_k - \ell)| + \sum_{\substack{k \in I_n \\ |\varphi(a_k - \ell)| < \varepsilon}} |\varphi(a_k - \ell)|$$

$$\geq \sum_{\substack{k \in I_n \\ |\varphi(a_k - \ell)| \geq \varepsilon}} |\varphi(a_k - \ell)|$$

$$\geq \ |\{k \in I_n : |\varphi(a_k - \ell)| \geq \varepsilon\}|.\,\varepsilon$$

and so that

$$\frac{1}{\mu_n^\beta}\sum_{k\in J_n}|\varphi(a_k-\ell)| \ \geq \ \frac{1}{\mu_n^\beta}|\{k\in I_n : |\varphi(a_k-\ell)|\geq\varepsilon\}|.\,\varepsilon$$

$$\geq \frac{\lambda_n^\alpha}{\mu_n^\beta}\ \ \frac{1}{\lambda_n^\alpha}|\{k\in I_n : |\varphi(a_k-\ell)|\geq\varepsilon\}|.\,\varepsilon.$$

Since (4) holds so if $a=(a_k)$ is $W^\beta[V,\lambda]$ −summable of order β to ℓ, then it is $WS_\lambda^\alpha-$ statistically convergent of order α to ℓ.

(ii) Let $WS_\lambda^\alpha-\lim_{k\to\infty}a_k=\ell$. Since φ is bounded. So there exists some $M>0$ such that $|\varphi(a_k-\ell)|\leq M$ for all k, for every $\varepsilon>0$ we have

$$\frac{1}{\mu_n^\beta}\sum_{k\in J_n}|\varphi(a_k-\ell)| \quad = \frac{1}{\mu_n^\beta}\sum_{k\in J_{n-I_n}}|\varphi(a_k-\ell)|+\frac{1}{\mu_n^\beta}\sum_{k\in I_n}|\varphi(a_k-\ell)|$$

$$\leq\left(\frac{\mu_n-\lambda_n}{\mu_n^\beta}\right)M+\frac{1}{\mu_n^\beta}\sum_{k\in I_n}|\varphi(a_k-\ell)|$$

$$\leq\left(\frac{\mu_n-\lambda_n^\alpha}{\mu_n^\beta}\right)M+\frac{1}{\mu_n^\beta}\sum_{k\in I_n}|\varphi(a_k-\ell)|$$

and hence

$$\frac{1}{\mu_n^\beta}\sum_{k\in J_n}|\varphi(a_k-\ell)|$$

$$\leq\left(\frac{\mu_n}{\mu_n^\beta}-\frac{\lambda_n^\alpha}{\mu_n^\beta}\right)M+\frac{1}{\mu_n^\beta}\sum_{\substack{k\in I_n\\ |\varphi(a_k-\ell)|\geq\varepsilon}}|\varphi(a_k-\ell)|+\frac{1}{\mu_n^\beta}\sum_{\substack{k\in I_n\\ |\varphi(a_k-\ell)|<\varepsilon}}|\varphi(a_k-\ell)|$$

$$\leq\left(\frac{\mu_n}{\mu_n^\beta}-\frac{\lambda_n^\alpha}{\mu_n^\beta}\right)M+\frac{M}{\lambda_n^\alpha}|\{k\in I_n : |\varphi(a_k-\ell)|\geq\varepsilon\}|\frac{\lambda_n}{\mu_n^\beta}\varepsilon$$

for all $n\in\mathbb{N}$. Now by using (5) we obtain that $W^\beta[V,\lambda]-\lim_{k\to\infty}a_k=\ell$, whenever $WS_\lambda^\alpha-\lim_{k\to\infty}a_k=\ell$.

Corollary 3.16 [5] Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ belong to Λ such that $\lambda_n\leq\mu_n$ for all $n\in\mathbb{N}$ and let (4) be satisfied. Then the following statements hold:

(i) If $\alpha=\beta$, then $W^\alpha[V,\mu]\subset WS_\lambda^\alpha$ for every $\alpha\in(0,1]$,

(ii) If $\beta=1$, then $W[V,\mu]\subset WS_\lambda^\alpha$ for every $\alpha\in(0,1]$.

Corollary 3.17 [5] Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \leq \mu_n$ $\forall n \in \mathbb{N}$ and let (5) be satisfied. Then the following statements hold:

(i) If $\alpha = \beta$, then $WS_\lambda^\alpha \subset W^\alpha[V,\mu]$ for every $\alpha \in (0,1]$,

(ii) If $\beta = 1$, then $WS_\lambda^\alpha \subset W[V,\mu]$ for every $\alpha \in (0,1]$.

4. CONCLUSION

Weak λ −statistical convergence is studied in this thesis. First, the definitions of natural density, weak convergent and sequence are studied in order to discuss the concept of statistical convergence. Then, a brief summary of λ-statistical convergence, weak statistical and weakly λ-statistical convergences is given. In the last chapter the concept of weak $\lambda -$ statistical convergence of order α where $\alpha \in (0,1]$, which is the main interest of this thesis has been considered. These concepts are much more general than the ideas of statistical convergence and weak statistical convergence which include these ideas in the special case $\alpha = 1$.

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