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FIRAT UNIVERSITY
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STATISTICAL CONVERGENCE OF DIFFERENCE SEQUENCES IN CONNECTION WITH MODULUS FUNCTIONS AND SOME GENERALIZATIONS

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Master's Thesis

DEPARTMENT OF MATHEMATICS

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DECLARATION

I hereby declare that I wrote this Master's Thesis titled “Statistical Convergence of Difference Sequences in Connection with Modulus Functions and some Generalizations ” in consistent with the thesis writing guide of the Graduate School of Natural and Applied Sciences, Firat University. I also declare that all information in it is correct, that I acted according to scientific ethics in producing and presenting the findings, cited all the references I used, express all institutions or organizations or persons who supported the thesis financially. I have never used the data and information I provide here in order to get a degree in any way.

30 January 2020

Sarkawt Asaad ABDULSAMAD

PREFACE

There were many investigations concerning of difference sequences, and also some researchers studied on statistical convergence difference sequences by using modulus functions. In literature there has not been any study on the relations between the sets of statistically convergent difference sequences which are defined by using different modulus functions. In this study we determine the relations between the sets of statistically convergent difference sequences which are defined by using different modulus functions.

It has been my proud of privilege to have accomplished my thesis under the able guidance of Prof. Dr. Rifat Çolak. I have great pleasure in taking this opportunity of acknowledging my deep sense of gratitude and high indebtedness to Prof. Dr. Rifat Çolak. He certainly is a great advisor for me and he has been very generous in participation his rich and worthy knowledge in the field. I owe more than I can possibly express, for his inspiring supervision, constant help and encouragement throughout to complete this thesis.

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ELAZIG, 2020

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In this thesis by using difference sequences and modulus functions, we give some definitions and theorems about $\Delta_f -$statistical convergence, $\Delta_f -$statistical boundedness, $\Delta_f -$strong Cesàro summability and $\Delta_f -$statistical convergence of order α and strong $\Delta_f -$Cesàro summability of order α with respect to modulus functions. In the beginning we give the relations between the sets of $\Delta -$statistically convergent sequences and $\Delta -$statistically bounded sequences and we give some relations between them. Then we provide the relations between the sets of $\Delta_f -$statistically convergent sequences and $\Delta_f -$statistically bounded sequences. After that we discuss the relations between $S_f(\Delta)$ and $S_g(\Delta)$, $S_f(\Delta)$ and $S(\Delta)$, $BS_f(\Delta)$ and $BS_g(\Delta)$, $BS_f(\Delta)$ and $BS(\Delta)$, $S_f(\Delta)$ and $BS_g(\Delta)$ for different modulus functions f and g under certain conditions. Finally we give some relations between the sets of $\Delta_f -$statistically convergent sequences of order α and $\Delta_f -$strongly Cesàro summable sequences of order α.

Kizmaz defined the difference sequence spaces $l_\infty(\Delta)=\{q=(q_m) : \Delta q \in l_\infty\}$, $c(\Delta)=\{q=(q_m) : \Delta q \in c\}$ and $c_o(\Delta)=\{q=(q_m) : \Delta q \in c_o\}$ which are Banach spaces, where $\Delta q=(\Delta q_m)=(q_m - q_{m+1})$.

Keywords: Statistical convergence, statistical boundedness, strong Cesàro summability, $\Delta_f -$statistical convergence, $\Delta_f -$statistical boundedness, $\Delta_f -$strong Cesàro summability, $\Delta_f -$statistical convergence of order α, $\Delta_f -$strong Cesàro summability of order α.

ÖZET

Modulus Fonksiyonları yardımıyla tanımlanmış Fark Dizilerinin İstatistiksel Yakınsaklığı ve Bazı Genelleştirmeleri

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Bu tezde, ilk önce reel say dizileri için f-istatistiksel yakınsaklık, f-istatistiksel sınırlılık ve ardından f-kuvvetli Cesàro toplanabilirlik kavramları verilmekte ve ilişkili bazı kavramlar incelenmektedir. Sonra fark dizileri için, $\Delta_f -$istatistiksel yakınsaklık olarak adlandırılacak olan $f -$ istatistiksel yakınsaklık, $\Delta_f -$ istatistiksel sınırlılık olarak adlandırılacak olan $f -$ istatistiksel sınırlılık ve ardından $\Delta_f -$ kuvvetli Cesàro toplanabilirlik kavramları verilmekte, kavramlar arasındaki ilişkiler ortaya konulmaktadır. Bundan sonra, bazı şartlara sahip farklı f ve g modülüs fonksiyonları için $S_f(\Delta)$ ve $S_g(\Delta)$, $BS_f(\Delta)$ ve $BS_g(\Delta)$, $w^f(\Delta)$ ve $w^g(\Delta)$, $S^f(\Delta)$ ve $w^f(\Delta)$ kümeleri arasındaki kapsama bağıntıları elde edilmektedir. Ayrıca bazı özel modülüs fonksiyonları için $w^f(\Delta)$ ve $w(\Delta)$, $S^f(\Delta)$ ve $S(\Delta)$ sınıfları arasındaki ilişkiler elde edilmektedir. Daha sonar $0<\alpha\leq 1$ şartına sahip herhangi bir α için α. dereceden Δ_f -istatistiksel yakınsaklık ve α . dereceden Δ_f -kuvvetli Cesàro toplanabilirlik üzerinde çalışılıp, bu iki kavram arasındaki ilişkiler de verilmektedir.

Anahtar Kelimeler: İstatistiksel yakınsaklık, istatistiksel sınırlılık, kuvvetli Cesàro toplanabilirlik, Δ_f -istatistiksel yakınsaklık, $\Delta_f -$istatistiksel sınırlılık, α. dereceden $\Delta_f -$ istatistiksel yakınsaklık, α. dereceden $\Delta_f -$kuvvetli Cesàro toplanabilirlik.

LIST OF SYMBOLS

$\mathbb{N}$	: The set of natural numbers
$\mathbb{R}$	: The set of real numbers
$\mathbb{Z}$	: The set of Integer numbers
$\mathbb{C}$	: The set of complex numbers
s	: The set of all number sequences
c	: The set of convergent sequences
c_0	: The set of null sequences
l_∞	: The set of bounded sequences
$d(H)$	: Natural density of the set H
$d_\alpha(H)$	: $\alpha-$density of the set H
$d_f(H)$	: $f-$density of the set H
$d_\alpha^f(H)$	: $f_\alpha-$density of the set H
S	: Statistically convergent sequences
BS	: Statistically bounded sequences
f	: Modulus function
S_f	: $f-$statistically convergent sequences
BS_f	: $f-$statistically bounded sequences
Δ	: The difference operator
$S(\Delta)$	: $\Delta-$statistically convergent sequences
$S_f(\Delta)$	: Δ_f-statistically convergent sequences
$BS(\Delta)$	: $\Delta-$statistically bounded sequences
$BS_f(\Delta)$	: Δ_f-statistically bounded sequences
w	: Strongly Cesàro summable sequences
$w(\Delta)$	: $\Delta-$strongly Cesàro summable sequences
$w^f(\Delta)$	: Δ_f-strongly Cesàro summable sequences
$w_0^f(\Delta)$	: Δ_f-strongly Cesàro summable null sequences
$w_\infty^f(\Delta)$	: Δ_f-strongly Cesàro summable bounded sequences
$w_\alpha^f(\Delta)$	: Δ_f-strongly Cesàro summable sequences of order α

$w_{\alpha,0}^{f}(\Delta)$: Δ_f – strongly Cesàro summable null sequences of order α

$w_{\alpha,\infty}^{f}(\Delta)$: Δ_f – strongly Cesàro summable bounded sequences of order α

a.a.m : For almost all m

1. INTRODUCTION

The concept of statistical convergence reverts to the monograph of Zygmund [1]. The conception of statistical convergence was explicitly presented by Steinhaus in [2] and Fast in [3] and reintroduced later by Schoenberg [4]. Statistical convergence also appears as an example of density convergence introduced by Buck in [5].

To solve series summation problems, statistical convergence was introduced. Many researchers provided many statistical convergence results and theories in many spaces and statistical convergence has been considered in different setups, and its different speculations, expansions, and variations have been concentrated by different creators up until now. For example, statistical convergence of order α [6], λ-statistical convergence with order α [7], statistical λ-summability [8], lacunary statistical convergence [9], generalized weighted statistical convergence [10] have been given.

Kizmaz introduced the notion of spaces of difference sequences [11], who examined the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, and then by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$, Et and Çolak further generalized this concept [12].

In 1953, Nakano [13] was introduced the concept of a modulus function and subsequently Ruckle [14] and Maddox [15] studied on the concept. Later, using a modulus function, many mathematicians constructed a lot of sequence spaces.

Aizpuru et al. in [16] introduced a different but in the same time much more general and new concept of density in 2014 by including an unbounded modulus function which is called f-density. After then using this concept, they got another non-matrix convergence concept, in other words, $f-$statistical convergence, that is close to ordinary convergence and statistical convergence and it coincide with statistical convergence when the modulus function taken as identity mapping.

2. FUNDAMENTAL DEFINITIONS AND RESULTS

We will provide this chapter with the basic definitions and results related to the subject. Throughout *s*, l_∞, c and c_o symbolize the spaces of all, bounded, convergent and null sequences of real numbers.

2.1. Basic Concepts on Sequences

Definition 2.1.1 A sequence (q_m) is called convergent to $l \in \mathbb{R}$ if for each $\varepsilon > 0$, there exists an integer N such that $|q_m - l| < \varepsilon$ whenever $n \geq N$.

Definition 2.1.2 A sequence (q_m) is called bounded if there is an $M > 0$ such that $|q_m| \leq M$ for all m.

Definition 2.1.3 A sequence (q_m) is called Cauchy if for each $\varepsilon > 0$, there exists an integer N such that $|q_m - q_n| < \varepsilon$ whenever $n, m \geq N$.

Theorem 2.1.4 $c \subset l_\infty$.

Proof Suppose $(q_m) \in c$ and $q_m \to l$ as $m \to \infty$. We choose $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$ such that $|q_m - l| < 1$ for every $m > N$. Then

$$|q_m| = |q_m - l + l| \leq |q_m - l| + |l| < 1 + |l|.$$

for every $n > N$. If we choose $M = \max\{|q_1|, |q_2|, ..., |q_N|, 1 + |l|\}$ then we have $|q_m| \leq M$ for every m So $(q_m) \in l_\infty$.

Remark 2.1.5 The reverse of the above theorem does not hold in general, that is, a bounded sequence may not be convergent. Indeed the sequence $\left\{\left(1 + (-1)^n\right)\right\}$ is not convergent but it is bounded.

2.2. Basic Concepts on Difference Sequences

Kizmaz defined the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ as

$$l_\infty(\Delta) = \{q = (q_m) \ : \ \Delta q \in l_\infty\},$$

$$c(\Delta) = \{q = (q_m) \ : \ \Delta q \in c\},$$

$$c_o(\Delta) = \{ q = (q_m) \ : \ \Delta q \in c_o \}$$

and showed that these are Banach spaces with norm

$$\|q\| = |q_1| + \|\Delta q\|_\infty$$

where $\Delta q = (\Delta q_m) = (q_m - q_{m+1})$ and $\|q\|_\infty = \sup_m |q_m|$ (see [11]).

Definition 2.2.1 A sequence (q_m) is called $\Delta-$convergent to l if for each $\varepsilon > 0,$ there is an integer N such that $|\Delta q_m - l| < \varepsilon$ whenever $m \geq N$.

Definition 2.2.2 A sequence (q_m) is called $\Delta-$bounded if there is an $M > 0$ such that $|\Delta q_m| \leq M$ for all m .

Definition 2.2.3 A sequence (q_m) is called $\Delta-$Cauchy if for each $\varepsilon > 0,$ there is an integer N such that $|\Delta q_m - \Delta q_n| < \varepsilon$ whenever $n, m \geq N$.

Theorem 2.2.4 $c(\Delta) \subset l_\infty(\Delta)$.

We omit the proof, since it is similar to the proof of Theorem 2.1.5.

Remark 2.2.5 The reverse of the above theorem does not hold in general, that is, a $\Delta-$bounded sequence may not be $\Delta-$convergent. Indeed the sequence $(q_m) = (1,0,1,0,...)$ is $\Delta-$bounded but it is not $\Delta-$convergent.

It is clear that we have the following inclusions:

$$c \subset c_o(\Delta) \subset c(\Delta) \subset l_\infty(\Delta) .$$

3. $\Delta-$STATISTICAL CONVERGENCE AND $\Delta-$STATISTICAL BOUNDEDNESS

In this chapter we study on the concepts of $\Delta-$statistical convergence and $\Delta-$statistical boundedness and we give some relations between these concepts and give the inclusions relations between the sets $S(\Delta)$ and $BS(\Delta)$.

3.1. $\Delta-$Statistical Convergence and $\Delta-$Statistical Boundedness

Definition 3.1.1 [17] Consider a set $H \subseteq \mathbb{N}$. Then we define $d(H)$ by

$$d(H) = \lim_{n\to\infty} \frac{1}{n} \left|\{m \le n \ : \ m \in H\}\right|,$$

as a natural density of set $H,$ where $\left|\{m \le n \ : \ m \in H\}\right|$ stands for the number of elements of H which is less than or equal to *n*, so it is clear that $\left|\{m \le n \ : \ m \in H\}\right| \le n.$

Statistical convergence of sequences relies on the density of subsets of $\mathbb{N}$. The natural density of any finite subset of $\mathbb{N}$ obviously is zero and $d(H^c) = 1 - d(H)$. If $d(H) = 1,$ then the set *H* is said to be statistically dense.

Definition 3.1.2 [18] A sequence (q_m) is called $\Delta-$statistically convergent if there is a complex number $l,$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{m \le n \ : \ |\Delta q_m - l| \ge \varepsilon\}\right| = 0,$$

for every $\varepsilon > 0$. We write $q_m \to l(S(\Delta))$ in this case. $S(\Delta)$ denotes the class of $\Delta-$statistically convergent sequences.

Definition 3.1.3 [18] A sequence (q_m) is called $\Delta-$statistically bounded if there is a number $M > 0,$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{m \le n \ : \ |\Delta q_m| > M\}\right| = 0 .$$

$BS(\Delta)$ denotes the class of $\Delta-$statistically bounded sequences.

Theorem 3.1.4 $c(\Delta)\subset S(\Delta)$.

Proof Let $(q_m)\in c(\Delta)$ and $(\Delta q_m)\to l$ as $m\to\infty$. Then the set $\{m\in\mathbb{N} : |\Delta q_m - l|\geq\varepsilon\}$ is finite for each $\varepsilon>0$. Suppose $|\{m\in\mathbb{N} : |\Delta q_m - l|\geq\varepsilon\}| = M$. Then

$$\lim_{n\to\infty}\frac{|\{m\leq n : |\Delta q_m - l|\geq\varepsilon\}|}{n}$$
$$\leq\lim_{n\to\infty}\frac{M}{n}=0.$$

This means that $(q_m)\in S(\Delta)$.

Remark 3.1.5 The reverse of the above theorem does not hold in general. Indeed, define $q=(q_m)$ as

$$q_m=\begin{cases}0, & m\neq r^2\\ \\ 1, & m=r^2\end{cases}\qquad r\in\mathbb{N},$$

that is, the sequence (q_m) is $\Delta-$statistically convergent, since

$$\Delta q_m=\begin{cases}1, & m=r^2\\ -1, & m-1=r^2\\ 0, & \text{otherwise}\end{cases}\quad r=1,2,3,\ldots$$

is statistically convergent, but it is not $\Delta-$convergent, that is $(q_m)\in S(\Delta)-c(\Delta)$.

Theorem 3.1.6 $l_\infty(\Delta)\subset BS(\Delta)$.

Proof Let $(q_m)\in l_\infty(\Delta)$. Then for some $M>0,$ we have $|\Delta q_m|\leq M,$ for all $m\in\mathbb{N}$ and this means that

$$\{m\in\mathbb{N} : |\Delta q_m|>M\}=\varnothing.$$

Thus, we get

$$\lim_{n\to\infty}\frac{1}{n}|\{m\leq n : |\Delta q_m|>M\}|=0.$$

Hence, $(q_m)\in BS(\Delta)$.

Remark 3.1.7 The reverse of the above theorem does not hold in general. Let us define the sequence (q_m) by

$$q_m = v_{r-1} - (r-1)^2, \text{ if } (r-1)^2 < m \le r^2, \; r \in \mathbb{N}, \tag{3.1}$$

where the sequence (v_r) defined via

$$v_r = v_{r-1} - (r-1)^2, \; r \in \mathbb{N},$$

and $v_0 = 0$. Now we have

$$\Delta q_m = \begin{cases} m, & m = r^2 \\ & \qquad r = 1, 2, 3, ... \\ 0, & m \ne r^2 \end{cases}$$

And so that $(q_m) = (0, -1, -1, -1, -5, -5, -5, -5, -5, -14, -14, -14, -14, -14, -14, -14, ...)$, and $(\Delta q_m) = (1, 0, 0, 4, 0, 0, 0, 0, 9, ...)$. Clearly (q_m) is $\Delta -$statistically bounded but it is not a $\Delta -$ bounded sequence.

Theorem 3.1.8 [19] $BS \subset BS(\Delta)$.

The proof is easy that is why we omit it.

Remark 3.1.9 The reverse of the above theorem does not hold in general. Indeed the sequence $(q_m) = (1,2,3,...) \in BS(\Delta)$, since $(\Delta q_m) = (q_m - q_{m+1}) = (-1,-1,-1,...)$ is bounded, but $(q_m) \notin BS$, that is $(q_m) \in BS(\Delta) - BS$.

Theorem 3.1.10 [20] $S(\Delta) \subset BS(\Delta)$.

Proof Assume that $(q_m) \in S(\Delta)$ and $q_m \to l(S(\Delta))$. Then $d(\{m \in \mathbb{N} : |\Delta q_m - l| \ge \varepsilon\}) = 0$ for every $\varepsilon > 0$. If we choose $M > 0$ such that $M > \varepsilon$. Then we have $\{m \in \mathbb{N} : |\Delta q_m - l| > M\} \subset \{m \in \mathbb{N} : |\Delta q_m - l| \ge \varepsilon\}$, so that

$$\frac{1}{n}\left|\{m \in \mathbb{N} : |\Delta q_m - l| > M\}\right| \le \frac{1}{n}\left|\{m \in \mathbb{N} : |\Delta q_m - l| > \varepsilon\}\right|.$$

Since the sequence $(q_m) \in S(\Delta)$, then in the above inequality the right side, and so that the left side tend to 0 as $n \to \infty$ and hence we have that $(q_m) \in BS(\Delta)$. (Note that for some $M_1 > M$ $\lim_{n\to\infty} \frac{1}{n} |\{m \in \mathbb{N} : |\Delta q_m - l| > M\}| = 0$ implies that $\lim_{n\to\infty} \frac{1}{n}\left|\{m \in \mathbb{N} : |\Delta q_m| > M_1\}\right| = 0$.

Remark 3.1.11 The reverse of the above theorem does not hold in general. Indeed the sequence $(q_m) = (0,1,0,1,...)$ is in $BS(\Delta)$, since $(\Delta q_m) = (q_m - q_{m+1}) = (-1,1,-1,1,...) \in BS$, but $(q_m) \notin S(\Delta)$.

Definition 3.1.12 If $\lim_{n\to\infty}\frac{1}{n}\left(\left|\{m\le n\ :\ q_m \text{ does not satisfy } P\}\right|\right)=0,$ then it is said that q_m satisfy property P for almost all m.

3.2. $\Delta-$Statistically Cauchy Sequences

Definition 3.2.1 [21] A sequence (q_m) is named $\Delta-$statistically Cauchy if for any $\varepsilon>0$, there is $N=N(\varepsilon)\in\mathbb{N}$ such that

$$\lim_{n\to\infty}\frac{1}{n}\left|\{m\le n\ :\ \left|\Delta q_m-\Delta q_N\right|\ge\varepsilon\}\right|=0.$$

Theorem 3.2.2 [22] A sequence is $\Delta-$statistically convergent if and only if it is $\Delta-$statistically Cauchy.

Proof Assume that $q_m\to l(S(\Delta))$. Then for every $\varepsilon>0,$ $\left|\Delta q_m-l\right|<\frac{\varepsilon}{2}$ for almost all m. If we choose N such that $\left|\Delta q_N-l\right|<\frac{\varepsilon}{2}$, we may write

$$\left|\Delta q_m-\Delta q_N\right|<\left|\Delta q_m-l\right|+\left|\Delta q_N-l\right|<\varepsilon$$

for almost all m. Hence q is a $\Delta-$statistically Cauchy sequence.

Now let (q_m) be $\Delta-$statistically Cauchy sequence. Then for any $\varepsilon>0$, there is $N\in\mathbb{N}$ such that

$$d\left(\{m\le n\ :\ \left|\Delta q_m-\Delta q_N\right|<\varepsilon\}\right)=1.$$

Hence, we obtain

$$d\left(\{m\le n\ :\ \Delta q_m<\Delta q_N+\varepsilon\}\right)=1,$$

and

$$d\left(\{m\le n\ :\ \Delta q_N-\varepsilon<\Delta q_m\}\right)=1.$$

We define the following sets:

$$A=\left\{a\in\mathbb{R}\ :\ d\left(\{m\le n\ :\ \Delta q_m<a\}\right)=1\right\},$$

and

$$B=\left\{b\in\mathbb{R}\ :\ d\left(\{m\le n\ :\ \Delta q_m>b\}\right)=1\right\}.$$

Now $(\Delta q_N+\varepsilon)\in A$, $(\Delta q_N-\varepsilon)\in B$ and we have $d(\{m\le n : \Delta q_m<a\})=1$ and $d(\{m\le n : \Delta q_m>b\})=1$ for $a\in A$ and $b\in B$.

Therefore, we get

$$d(\{m\le n : b<\Delta q_m<a\})=1.$$

This implies $b<a$. Clearly we have

$$\Delta q_N-\varepsilon\le \sup B\le \inf A\le \Delta q_N+\varepsilon.$$

Since ε was an arbitrary positive number, we get $\sup B=\inf A$. Now given $\varepsilon>0$ there exists $a\in A$ and $b\in B$ such that $l-\varepsilon<b<a<l+\varepsilon,$ if we choose $\sup B=\inf A=l$. From the definition of A and B we may write

$$d(\{m\le n : l-\varepsilon<\Delta q_m<l+\varepsilon\})=1,$$

and hence we obtain

$$d(\{m\le n : |\Delta q_m-l|<\varepsilon\})=1 \text{ or } d(\{m\le n : |\Delta q_m-l|\ge\varepsilon\})=0.$$

Therefore (q_m) is $\Delta-$statistically convergent.

Theorem 3.2.3 [20] Every $\Delta-$statistically Cauchy sequence is $\Delta-$statistically bounded.

Proof Assuming that (q_m) is a $\Delta-$statistically Cauchy sequence. Then for any $\varepsilon>0$, there is $N=N(\varepsilon)\in\mathbb{N}$ such that $|\Delta q_m-\Delta q_N|<\varepsilon$ a.a. m. This means that $|\Delta q_m|\le L$ a.a. m, where $L=\varepsilon+|\Delta q_N|$.

Remark 3.2.4 The reverse of the above theorem does not hold in general. Indeed the sequence $(q_m)=(1,0,1,0,...)$ is $\Delta-$statistically bounded, since $(\Delta q_m)=(q_m-q_{m+1})=$ $(1,-1,1,-1,...)$ is statistically bounded. However, it is not $\Delta-$statistically Cauchy.

Theorem 3.2.5 [23] Let $(q_m)\in s$. Then $(q_m)\in S(\Delta)$ if and only if the following condition is satisfied

$$\lim_{n\to\infty}\frac{1}{n}\left|\{m,r\le n : |\Delta q_m-\Delta v_r|\ge\varepsilon\}\right|=0$$

where (v_r) is a subsequence of (q_m) such that

$$\lim_{r\to\infty}\Delta v_r=l$$

for some l.

Proof Assuming that $(q_m) \in S(\Delta)$. We will prove that

$$\lim_{n\to\infty} \frac{1}{n} \left| \{m, r \le n \ : \ |\Delta q_m - \Delta v_r| \ge \varepsilon\} \right| = 0 .$$

If the sequence $(q_m) \in S(\Delta)$, then by the definition we have,

$$\lim_{n\to\infty} \frac{1}{n} \left| \{m \le n \ : \ |\Delta q_m - l| \ge \varepsilon\} \right| = 0 .$$

Hence, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \left| \{m, r \le n \ : \ |\Delta q_m - \Delta v_r| \ge \varepsilon\} \right|$$

$$\le \lim_{n\to\infty} \frac{1}{n} \left| \{m \le n \ : \ |\Delta q_m - l| \ge \varepsilon\} \right| + \lim_{n\to\infty} \frac{1}{n} \left| \{r \le n \ : \ |\Delta v_r - l| \ge \varepsilon\} \right|$$

$$\le 0 + \lim_{n\to\infty} \frac{1}{n} \left| \{r \le n \ : \ |\Delta v_r - l| \ge \varepsilon\} \right| .$$

Since $(v_r) \in c(\Delta)$, then $(v_r) \in S(\Delta)$. So, we may write

$$\lim_{n\to\infty} \frac{1}{n} \left| \{r \le n \ : \ |\Delta v_r - l| \ge \varepsilon\} \right| = 0 .$$

Hence, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \left| \{m, r \le n \ : \ |\Delta q_m - \Delta v_r| \ge \varepsilon\} \right| = 0 .$$

Conversely, let $\lim_{n\to\infty} \frac{1}{n} \left| \{m, r \le n \ : \ |\Delta q_m - \Delta v_r| \ge \varepsilon\} \right| = 0$. In order to prove that $(q_m) \in S(\Delta)$, we begin with the inequality that comes following

$$\lim_{n\to\infty} \frac{1}{n} \left| \{m \le n \ : \ |\Delta q_m - l| \ge \varepsilon\} \right| = \lim_{n\to\infty} \frac{1}{n} \left| \{m, r \le n \ : \ |\Delta q_m - \Delta v_r + \Delta v_r - l| \ge \varepsilon\} \right|$$

$$\le \lim_{n\to\infty} \frac{1}{n} \left| \{m, r \le n \ : \ |\Delta q_m - \Delta v_r| \ge \varepsilon\} \right| + \lim_{n\to\infty} \frac{1}{n} \left| \{r \le n \ : \ |\Delta v_r - l| \ge \varepsilon\} \right|$$

$$\le \lim_{n\to\infty} \frac{1}{n} \left| \{m, r \le n \ : \ |\Delta q_m - \Delta v_r| \ge \varepsilon\} \right| + 0$$

since it is given that $\lim_{r\to\infty}\Delta v_r = l$ then $\lim_{n\to\infty}\frac{1}{n}\left|\{r \le n \ : \ \left|\Delta v_r - l\right| \ge \varepsilon\}\right| = 0$. Consequently, we get

$$\lim_{n\to\infty}\frac{1}{n}\left|\{m \le n \ : \ \left|\Delta q_m - l\right| \ge \varepsilon\}\right| = 0$$

This means that $(q_m) \in S(\Delta)$.

4. Δ_f –STATISTICAL CONVERGENCE AND Δ_f –STATISTICAL BOUNDEDNESS

In this chapter we study on the concepts of Δ_f – statistical convergence and Δ_f – statistical boundedness and we give some relations between these concepts and give the inclusions relations among the sets $S_f(\Delta)$, $S(\Delta)$, $BS_f(\Delta)$ and $BS(\Delta)$.

4.1. Δ_f –Statistical Convergence and Δ_f –Statistical Boundedness

Definition 4.1.1 A function f from $[0,\infty)$ to $[0,\infty)$ is called a modulus if

i) $f(u)=0$ if and only if $u=0$,

ii) $f(u_1+u_2)\le f(u_1)+f(u_2)$ for every $u_1,u_2\ge 0$,

iii) f is increasing,

iv) From the right, f is continuous at 0.

An f modulus is continuous everywhere on $[0,\infty)$. The modulus functions $f(t)=t^p$ $(0<p\le 1)$ and $g(t)=\frac{t}{t+1}$ are unbounded and bounded functions, respectively. That is why for a modulus it is possible to be bounded or unbounded.

Furthermore given any modulus *f*, the inequality $f(nx)\le nf(x)$, so that $f(n)\le nf(1)$ is satisfied for every positive integer n and real number *x* by (ii).

Lemma 4.1.2 [24] The limit $\lim_{t\to\infty}\frac{f(t)}{t}=\inf_{t\in(0,\infty)}\frac{f(t)}{t}$ exists for any modulus f.

Definition 4.1.3 [16] Let $H\subseteq\mathbb{N}$. The f – density of a set H is defined by

$$d_f(H)=\lim_{n\to\infty}\frac{f\left(\left|\{m\le n\ :\ m\in H\}\right|\right)}{f(n)},$$

whenever the limit exists, where modulus f be unbounded.

When $f(t)=t$, the f – density and natural density are same. For the natural density, we have $d(H)+d(\mathbb{N}-H)=1$ for every $H\subset\mathbb{N}$. But this is not true in case of f – density in general, that is $d_f(H)+d_f(\mathbb{N}-H)=1$ generally, does not hold. For example, if we take $f(t)=\log(t+1)$ and $H=\{2n\ :\ n\in\mathbb{N}\}$, then $d_f(H)=d_f(\mathbb{N}-H)=1$. However, we can say that in the case of f - density, if $d_f(H)=0$ then $d_f(\mathbb{N}-H)=1$. Finite sets have zero f – density, as in the case of natural density and so that $d_f(H)+d_f(\mathbb{N}-H)=1$ for any finite set H.

For any unbounded modulus f and $H \subset \mathbb{N}$, $d_f(H)=0$ implies that $d(H)=0$. But the reverse does not have to be true, in general. For example, taking $f(t)=\log(t+1)$ and $H=\{1,4,9,...\}$, then $d(H)=0$ but $d_f(H)=\frac{1}{2}$. However, for any finite set $H \subset \mathbb{N}$, $d(H)=0$ implies $d_f(H)=0$ is always true, irrespective of selection of unbounded modulus f.

Lemma 4.1.4 [16] If $H \subset \mathbb{N}$ is infinite then at least for an unbounded modulus f we have $d_f(H)=1$.

Definition 4.1.5 [25] Let f be an unbounded modulus function. A sequence (q_m) is called Δ_f- statistically convergent to l, if for each $\varepsilon>0$

$$d_f\left(\left\{m\in\mathbb{N} : |\Delta q_m - l| \geq \varepsilon\right\}\right)=0,$$

i.e.,

$$\lim_{n\to\infty}\frac{1}{f(n)}f\left(\left|\left\{m\leq n : |\Delta q_m - l|\geq\varepsilon\right\}\right|\right)=0.$$

We write this notation ally as $S_f(\Delta)-\lim q_m = l$ or $q_m \to l\left(S_f(\Delta)\right)$. $S_f(\Delta)$ denotes the class of Δ_f-statistically convergent sequences.

Theorem 4.1.6 $S_f(\Delta)\subset S(\Delta)$ for any unbounded modulus f.

Proof Assume that $(q_m)\in S_f(\Delta)$ and $S_f(\Delta)-\lim q_m = l$. Then $d_f(H)=0$ if we choose $H=\left\{m\in\mathbb{N} : |\Delta q_m - l|\geq\varepsilon\right\}$. Now the proof follows from the fact "for any $H\subseteq\mathbb{N}$ and any modulus f, $d_f(H)=0$ implies that $d(H)=0$".

Theorem 4.1.7 Let f be an unbounded modulus. If $\lim_{t\to\infty}\frac{f(t)}{t}>0$, then $S(\Delta)\subset S_f(\Delta)$.

Proof Assume that $q=(q_m)\in S(\Delta)$ and $S(\Delta)-\lim q_m = l$. Then we have

$$\lim_{n\to\infty}\frac{\left(\left|\left\{m\leq n : |\Delta q_m - l|\geq\varepsilon\right\}\right|\right)}{n}=0$$

for every $\varepsilon>0$. Now we may write

$$\frac{f\left(\left|\left\{m\leq n : |\Delta q_m - l|\geq\varepsilon\right\}\right|\right)}{f(n)}\leq\frac{\left(\left|\left\{m\leq n : |\Delta q_m - l|\geq\varepsilon\right\}\right|\right)f(1)}{n}\cdot\frac{n}{f(n)}.$$

Since $\lim\limits_{t\to\infty}\frac{f(t)}{t}>0$ and $S(\Delta)-\lim q_m = l,$ the right hand side of the above inequality tends to 0 and this implies that the left hand side tends to 0 as $n\to\infty$. Therefore $(q_m)\in S_f(\Delta)$.

Theorem 4.1.8 Let f be an unbounded modulus. If $\lim\limits_{t\to\infty}\frac{f(t)}{t}>0,$ then $c(\Delta)\subset S_f(\Delta)$**.**

The proof is derived from Theorem 3.1.4 and Theorem 4.1.7.

Theorem 4.1.9 [25] Let f be an unbounded modulus and $q=(q_m)$, $v=(v_m)$ be any two sequences. Then

(i) If $S_f(\Delta)-\lim q_m = l$ and $c\in\mathbb{C}$, then $S_f(\Delta)-\lim cq_m = cl,$

(ii) If $S_f(\Delta)-\lim q_m = l_1$ and $S_f(\Delta)-\lim v_m = l_2$, then $S_f(\Delta)-\lim(q_m+v_m)=l_1+l_2$.

Remark 4.1.10 It is clear that if $(q_m)\in c$, then every subsequence of (q_m) belongs to *c*, but this situation is no longer true in case of Δ_f-statistical convergence, that is a Δ_f-statistically convergent sequence may have a subsequence which is not Δ_f-statistically convergent. If we consider the modulus function $f(t)=t^p,\ 0<p\le 1$ and the sequence (q_m) defined by (3.1) in Remark 3.1.7，that is

$$(q_m)=(0,-1,-1,-1,-5,-5,-5,-5,-5,-14,-14,-14,-14,-14,-14,-14,...),$$

then we have that $q_m\in S_f(\Delta)$, since $(\Delta q_m)=(1,0,0,4,0,0,0,0,9,...)\in S_f$. But whereas $(1,4,9,...)$ is a subsequence of (Δq_m) which is not $f-$statistically convergent.

4.2. Δ_f-Statistically Cauchy Sequences

Definition 4.2.1. Let f be an unbounded modulus. A sequence (q_m) is called Δ_f-statistically Cauchy or Cauchy sequence if there exists a positive integer $N=N(\varepsilon)$ such that

$$d_f\left(\left\{m\in\mathbb{N}\ :\ |\Delta q_m-\Delta q_N|\ge\varepsilon\right\}\right)=0,$$

i.e.,

$$\lim_{n\to\infty}\frac{1}{f(n)}f\left(\left|\left\{m\le n\ :\ |\Delta q_m-\Delta q_N|\ge\varepsilon\right\}\right|\right)=0\,.$$

for every $\varepsilon>0$.

Theorem 4.2.2 Given any sequence is Δ_f – statistically convergent if and only if it is Δ_f – statistically Cauchy for any unbounded modulus f.

Proof It is clear to demonstrate that any Δ_f – statistically convergent sequence is Δ_f – statistically Cauchy.

To prove that a Δ_f – statistically Cauchy sequence is Δ_f – statistically convergent sequence we may use the technique given in the proof of Theorem 3.3 in [16].

Definition 4.2.3 Let f be an unbounded modulus. A sequence (q_m) of numbers is named Δ_f – statistically bounded if there exists $M > 0$ such that

$$d_f\left(\{m \in \mathbb{N} : |\Delta q_m| > M\}\right) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{f(n)} f\left(|\{m \le n : |\Delta q_m| > M\}|\right) = 0 .$$

$BS_f(\Delta)$ denotes the class of Δ_f – statistically bounded sequences.

Theorem 4.2.4 $BS_f(\Delta) \subset BS(\Delta)$ for any unbounded modulus f.

Proof Let $(q_m) \in BS_f(\Delta)$. Then $d_f(H) = 0$ if we choose $H = \{m \in \mathbb{N} : |\Delta q_m| > M\}$ for an M large enough. Now the proof is based on the fact " for any $H \subset \mathbb{N}$ and any modulus f, $d_f(H) = 0$ implies that $d(H) = 0$".

Remark 4.2.5 The reverse of the above theorem does not hold in general. If we consider the modulus function $f(t) = \log(t+1)$ and the sequence (q_m) defined by (3.1) in Remark 3.1.7, that is $(q_m) = (0,-1,-1,-1,-5,-5,-5,-5,-5,-14,-14,-14,-14,-14,-14,-14,...)$, then we have that $(q_m) \in BS(\Delta)$, since $(\Delta q_m) = (1,0,0,4,0,0,0,0,9,...)$ is statistically bounded, but $(q_m) \notin BS_f(\Delta)$.

Theorem 4.2.6 [19] For every unbounded modulus f we have $S_f(\Delta) \subset BS_f(\Delta)$.

Proof Using the inclusion $\{m \in \mathbb{N} : |\Delta q_m| > |l| + \varepsilon\} \subset \{m \in \mathbb{N} : |\Delta q_m - l| > \varepsilon\}$ the proof is straightforward.

Remark 4.2.7 The reverse of the above theorem does not hold in general. Indeed if we consider the modulus function $f(t) = t$, and define (q_m) as

$$q_m = \begin{cases} 1, & m = 2r \\ & \qquad r \in \mathbb{N}, \\ 0, & m \neq 2r \end{cases}$$

then we have that $(q_m) \in BS_f(\Delta)$, since $(\Delta q_m) = (q_m - q_{m+1}) = (-1,1,-1,1,...)$ is $f-$statistically bounded, but $(q_m) \notin S_f(\Delta)$, that is $(q_m) \in BS_f(\Delta) - S_f(\Delta)$.

Theorem 4.2.8 [19] Every $\Delta_f -$statistically Cauchy sequence is $\Delta_f -$statistically bounded for any modulus f.

Aizpuru et al. in [16] proved that $f - st\lim q_m = l$ if and only if there exists $H \subset \mathbb{N}$ with $d_f(H) = 0$ and $\lim_{m\in\mathbb{N}-H} q_m = l$.

Now we provide an analogous theorem in the same structure for $\Delta_f -$statistically bounded sequences which includes the f-density of related sets.

Theorem 4.2.9 A sequence (q_m) is $\Delta_f -$statistically bounded if and only if there exists $H \subset \mathbb{N}$ such that $d_f(H) = 0$ and $(q_m)_{m\in\mathbb{N}-H} \in l_\infty(\Delta)$.

Proof Assume that (q_m) is $\Delta_f -$statistically bounded. Then we have an integer $M > 0$ which has $d_f(\{m \in \mathbb{N} : |\Delta q_m| > M\}) = 0$. Take $H = \{m \in \mathbb{N} : |\Delta q_m| > M\}$. Then $d_f(H) = 0$ and for $m \in \mathbb{N} - H$, we have $|\Delta q_m| \leq M$, that is, $(q_m)_{m\in\mathbb{N}-H} \in l_\infty(\Delta)$.

Conversely, since $(q_m)_{m\in\mathbb{N}-H} \in l_\infty(\Delta)$ there exists $M > 0$ such that for any $m \in \mathbb{N} - H$ we have $|\Delta q_m| \leq M$. This implies that $\{m \in \mathbb{N} : |\Delta q_m| > M\} \subset H$ and so $d_f(\{m \in \mathbb{N} : |\Delta q_m| > M\}) = 0$. Hence $(q_m)_{m\in\mathbb{N}}$ is $\Delta_f -$statistically bounded.

Theorem 4.2.10 Let f be an unbounded modulus and $(q_m) \in s$. If $(q_m) \in BS_f(\Delta)$, then $(q_m) \in S_f(\Delta)$ in case (Δq_m) is monotone.

Proof Let the sequence (Δq_m) be monotone and $(q_m) \in BS_f(\Delta)$. Now the sequence $(q_m) \in BS_f(\Delta)$ if and only if there exists $H \subset \mathbb{N}$ such that $d_f(H) = 0$ and $(q_m)_{m\in\mathbb{N}-H} \in l_\infty(\Delta)$ by Theorem 30 in [26]. So there exists $l \in \mathbb{C}$ such that $\lim_{m\in\mathbb{N}-H} \Delta q_m = l$. By using Theorem 3.1 of [16], we have $(q_m) \in S_f(\Delta)$.

Theorem 4.2.11 If $(q_m) \in BS_f(\Delta)$ for every unbounded modulus f, then $(q_m) \in l_\infty(\Delta)$.

Proof Suppose $(q_m) \in BS_f(\Delta)$ and if possible $(q_m) \notin l_\infty(\Delta)$. Then the set $H = \{m \in \mathbb{N} : |\Delta q_m| > M\}$ is infinite for every $M > 0$ and hence by Lemma 4.1.4 there exists an unbounded modulus f such that $d_f(H)=1$, which contradicts the assumption that $(q_m)\in BS_f(\Delta)$ for every modulus f.

Remark 4.2.12 We have $l_\infty(\Delta)\subseteq BS_f(\Delta)$ for every unbounded modulus f. By using this fact with the above theorem, we can say that the sequences in $l_\infty(\Delta)$ are those sequences which are Δ_f-statistically bounded for every unbounded modulus f.

Theorem 4.2.13 [26] Let f be an unbounded modulus and $(q_m)\in s$. Then $(q_m)\in BS_f(\Delta)$ if and only if there exists $\Delta v=(\Delta v_m)\in l_\infty$ such that $\Delta q_m=\Delta v_m$ for a.a. m, w.r.t. f.

5. Δ_f – STRONG CESÀRO SUMMABILITY

In this chapter, we study on some concepts related Δ_f – strong Cesàro summability and give some relations between the sets which were created by using a modulus function.

5.1. Δ – Strong Cesàro Summability

Definition 5.1.1 [27] A sequence (q_m) is called Δ – Cesàro summable to l if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\Delta q_m = l .$$

Theorem 5.1.2 Every Δ – convergent sequence is also Δ – Cesàro summable.

Remark 5.1.3 The reverse of the above theorem does not hold in general. Indeed the sequence $(q_m) = (8,8,7,7,6,6,5,5...,-1,-1,-2,-2,...)$ is Δ – Cesàro summable, since $(\Delta q_m) = (0,1,0,1,0,1,...)$ is Cesàro summable to $\frac{1}{2}$. However, (q_m) is not Δ – convergent.

Definition 5.1.4 A number sequence (q_m) is called Δ – strongly Cesàro summable to l if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left|\Delta q_m - l\right| = 0 .$$

Strongly summable sequence spaces were mentioned and studied by Kuttner [28], Maddox ([24], [29]) and some others. The renowned spaces w_o, w and w_∞ of strongly Cesàro summable sequences are identified by

$$w_o = \left\{(q_m)\in s \ : \ \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left|q_m\right| = 0\right\},$$

$$w = \left\{(q_m)\in s \ : \ \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left|q_m - l\right| = 0, \text{ for some number } l\right\},$$

$$w_\infty = \left\{(q_m)\in s \ : \ \sup_{n}\frac{1}{n}\sum_{m=1}^{n} f\left(\left|q_m\right|\right) < \infty\right\}.$$

Maddox [30] using a modulus function *f*, he has expanded this definition to introduce much more general spaces such as w_o^f, w^f and w_∞^f. These spaces of sequences are defined, respectively as

$$w_o^f = \left\{ (q_m) \in s \ : \ \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f\left(|q_m|\right) = 0 \right\},$$

$$w^f = \left\{ (q_m) \in s \ : \ \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f\left(|q_m - l|\right) = 0, \text{ for some number } l \right\},$$

$$w_\infty^f = \left\{ (q_m) \in s \ : \ \sup_n \frac{1}{n} \sum_{m=1}^{n} f\left(|q_m|\right) < \infty \right\}.$$

Definition 5.1.5 Let (q_m) be any number sequence and $p > 0$. Then the sequence (q_m) is called $\Delta-$strongly $p-$Cesàro summable to l if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} |\Delta q_m - l|^p = 0 .$$

Theorem 5.1.6 Let $p \in \mathbb{R}^+$. Then if a sequence is $\Delta-$strongly $p-$Cesàro summable to l, then it is $\Delta-$statistically convergent to l.

Proof Assume that the sequence (q_m) is $\Delta-$strongly $p-$Cesàro summable to l. Then for each $\varepsilon > 0$, we have

$$\sum_{m=1}^{n} |\Delta q_m - l|^p \geq \sum_{m=1, |\Delta q_m - l| \geq \varepsilon}^{n} |\Delta q_m - l|^p \geq \left| \{ m \leq n \ : \ |\Delta q_m - l|^p \geq \varepsilon \} \right| .\varepsilon^p .$$

and so that

$$\frac{1}{n} \sum_{m=1}^{n} |\Delta q_m - l|^p \geq \frac{1}{n} \left| \{ m \leq n \ : \ |\Delta q_m - l|^p \geq \varepsilon \} \right| .\varepsilon^p .$$

By taking limit at both sides as $n \to \infty$ we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} |\Delta q_m - l|^p \geq \lim_{n\to\infty} \frac{1}{n} \left| \{ m < n \ : \ |\Delta q_m - l|^p \geq \varepsilon \} \right| .\varepsilon^p .$$

Since the sequence (q_m) is $\Delta-$strongly $p-$Cesàro summable to l, then from above inequality we get that

$$\lim_{n\to\infty} \frac{1}{n} \left| \{ m < n \ : \ |\Delta q_m - l|^p \geq \varepsilon \} \right| = 0 .$$

Hence, (q_m) is $\Delta-$statistically convergent to l.

5.2. Δ_f –Strong Cesàro Summability

Definition 5.2.1 [31] Let (q_m) be any number sequence and let f be any modulus function. Then we say that (q_m) is Δ_f –strongly Cesàro summable to l if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} f\left(\left|\Delta q_m - l\right|\right)=0 .$$

We use the following notations

$$w_0^f(\Delta)=\left\{(q_m)\in s \ : \ \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} f\left(\left|\Delta q_m\right|\right)=0\right\},$$

$$w^f(\Delta)=\left\{(q_m)\in s \ : \ \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} f\left(\left|\Delta q_m - l\right|\right)=0, \text{ for some number } l\right\},$$

$$w_\infty^f(\Delta)=\left\{(q_m)\in s \ : \ \sup_{n}\frac{1}{n}\sum_{m=1}^{n} f\left(\left|\Delta q_m\right|\right)<\infty\right\}.$$

Theorem 5.2.2 For any modulus f

(i) $w_0^f(\Delta)\subset w_\infty^f(\Delta)$.

(ii) $w^f(\Delta)\subset w_\infty^f(\Delta)$.

Proof The first being obvious, we establish the second inclusion. Let $q\in w^f(\Delta)$. By using (ii) and (iii) in the definition of modulus function, we have

$$\frac{1}{n}\sum_{m=1}^{n} f\left(\left|\Delta q_m\right|\right)\le \frac{1}{n}\sum_{m=1}^{n} f\left(\left|\Delta q_m - l\right|\right)+f\left(\left|l\right|\right)\frac{1}{n}\sum_{m=1}^{n}1 .$$

Since $q\in w^f(\Delta)$, we have $(q_m)\in w_\infty^f(\Delta)$.

Theorem 5.2.3 For any modulus f

(i) $w(\Delta)\subset w^f(\Delta)$.

(ii) $w_0(\Delta)\subset w_0^f(\Delta)$.

(iii) $w_\infty(\Delta)\subset w_\infty^f(\Delta)$.

Proof Now we prove the last inclusion, the first two inclusions are omitted. Let $(q_m)\in w_\infty(\Delta)$, then at least for a number $M>0$

$$\frac{1}{n}\sum_{m=1}^{n}|\Delta q_m| < M,$$

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(h) < \varepsilon$ for $0 < h \le \delta$. Now the equality

$$\frac{1}{n}\sum_{m=1}^{n} f(|\Delta q_m|) = \frac{1}{n}\sum_{\substack{m=1 \\ |\Delta q_m| \le \delta}}^{n} f(|\Delta q_m|) + \frac{1}{n}\sum_{\substack{m=1 \\ |\Delta q_m| > \delta}}^{n} f(|\Delta q_m|)$$

can be written. Then, since $f(|\Delta q_m|) \le \varepsilon$ for $|\Delta q_m| \le \delta$

$$\frac{1}{n}\sum_{\substack{m=1 \\ |\Delta q_m| \le \delta}}^{n} f(|\Delta q_m|) \le \frac{1}{n}\sum \varepsilon \le \frac{n\varepsilon}{n} = \varepsilon .$$

And also for $|\Delta q_m| > \delta$ we have

$$|\Delta q_m| < \frac{|\Delta q_m|}{\delta} < 1 + \left[\frac{|\Delta q_m|}{\delta}\right],$$

where [h] denotes the integral part of real number h. Now since f is a modulus, by (ii) and (iii) in the definition of modulus function, we can write

$$f(|\Delta q_m|) \le \left(1 + \left[\frac{|\Delta q_m|}{\delta}\right]\right) f(1) \le 2f(1)\left[\frac{|\Delta q_m|}{\delta}\right]$$

and so that

$$\frac{1}{n}\sum_{\substack{m=1 \\ |\Delta u_m| > \delta}}^{n} f(|\Delta q_m|) \le \frac{2f(1)}{\delta}\frac{1}{n}\sum_{\substack{m=1 \\ |\Delta u_m| > \delta}}^{n} (|\Delta q_m|) .$$

Now the inequality

$$\frac{1}{n}\sum_{m=1}^{n} f(|\Delta q_m|) \le \varepsilon + \frac{2f(1)}{\delta}\frac{1}{n}\sum_{m=1}^{n}|\Delta q_m| .$$

can be written. Since $(q_m) \in w_\infty(\Delta)$ we have $(q_m) \in w_\infty^f(\Delta)$.

Theorem 5.2.4 Let f be a modulus such that $\lim_{t\to\infty}\frac{f(t)}{t} > 0$. Then $w^f(\Delta) \subset w(\Delta)$.

Proof For any modulus *f*, $\upsilon = \lim_{t\to\infty}\frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t} : \quad t > 0\right\}$ exists by Proposition 1 of Maddox [15].

By definition of υ, we have $f(t) \ge \upsilon t$ for all $t \ge 0$. Since $\upsilon > 0$, we have $t \le \upsilon^{-1} f(t)$ for all $t \ge 0$ and so

$$\frac{1}{n}\sum_{m=1}^{n}|\Delta q_m - l| \leq \upsilon^{-1}\frac{1}{n}\sum_{m=1}^{n}\left(f\,|\Delta q_m - l|\right)$$

from where it follows that $(q_m) \in w(\Delta)$ whenever $(q_m) \in w^f(\Delta)$.

From Theorem 5.2.3 and Theorem 5.2.4 we have the next result.

Corollary 5.2.5 Given a modulus function f if $\lim_{t\to\infty}\frac{f(t)}{t} > 0,$ then $w^f(\Delta) = w(\Delta)$.

6. RELATIONS BETWEEN THE SETS OF Δ_f –STATISTICALLY CONVERGENT SEQUENCES

In this chapter we discuss the relations between $S_f(\Delta)$ and $S_g(\Delta)$, $S_f(\Delta)$ and $S(\Delta)$, $BS_f(\Delta)$ and $BS_g(\Delta)$, $BS_f(\Delta)$ and $BS(\Delta)$, $S_f(\Delta)$ and $BS_g(\Delta)$ for different modulus functions f and g under certain conditions.

6.1. Relations between the sets $S_f(\Delta)$ and $BS_f(\Delta)$

The relations between $f-$densities of a set of positive integers for different modulus functions is given in the following Theorem given by Çolak [32]. This helps us to establish the relations between $\Delta-$statistically convergent and $\Delta-$statistically bounded sequence sets defined by modulus functions.

Theorem 6.1.1 [32] Let f and g be two unbounded modulus functions**.** Then for a set $H \subset \mathbb{N}$

(i) if

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} > 0 \qquad (6.1)$$

then $d_g(H)=0$ implies $d_f(H)=0$ whenever the limit exists,

(ii) if

$$0 < \lim_{t\to\infty} \frac{f(t)}{g(t)} = \alpha < \infty \qquad (6.2)$$

then $d_g(H)=0 \Leftrightarrow d_f(H)=0$ whenever the limit exists.

Corollary 6.1.2 [33] For any $H \subset \mathbb{N}$ and any unbounded modulus f providing

$$\lim_{t\to\infty} \frac{f(t)}{t} > 0 \qquad (6.3)$$

we have $d_f(H)=0 \Leftrightarrow d(H)=0$.

Theorem 6.1.3 Let f and g be two unbounded modulus functions. Then

(i) If (6.1) holds, then $S_g(\Delta) \subset S_f(\Delta)$.

(ii) If (6.2) holds, then $S_g(\Delta) = S_f(\Delta)$.

Proof (i) Suppose (q_m) is Δ_g-statistically convergent to l, that is $S_g(\Delta)-\lim q_m=l$. Define $H=\{m\in\mathbb{N}\ :\ |\Delta q_m-l|\geq\varepsilon\}$. Then

$$d_g(H)=\lim_{n\to\infty}\frac{g\left(\left|\{m\leq n\ :\ |\Delta q_m-l|\geq\varepsilon\}\right|\right)}{g(n)}=0$$

and this implies

$$d_f(H)=\lim_{n\to\infty}\frac{f\left(\left|\{m\leq n\ :\ |\Delta q_m-l|\geq\varepsilon\}\right|\right)}{f(n)}=0$$

if (6.1) holds by Theorem 6.1.1 (i).

The Proof of (ii) is based on the Theorem 6.1.1 (ii).

Remark 6.1.4 The inclusion in (i) of the above Theorem may be strict. It can easily be seen that for the modulus functions $g(t)=\log(t+1)$, $f(t)=t^{\frac{1}{2}}$ and the sequence (q_m) defined by (3.1) in Remark 3.1.7 we get $(q_m)\in S_f(\Delta)-S_g(\Delta)$ and so that the inclusion $S_g(\Delta)\subset S_f(\Delta)$ is strict.

Corollary 6.1.5 $S_f(\Delta)=S(\Delta)$ if (6.3) holds.

Proof Let (q_m) be Δ_f-statistically convergent to l. Then $d_f(H)=0$ if we choose $H=\{m\in\mathbb{N}\ :\ |\Delta q_m-l|\geq\varepsilon\}$. Now the proof is based on the fact for any $H\subset\mathbb{N}$ and any modulus f, $d_f(H)=0$ implies that $d(H)=0$. Thus $S_f(\Delta)\subset S(\Delta)$.

To show that $S(\Delta)\subset S_f(\Delta)$, assume that $\lim_t\frac{f(t)}{t}>0$ and let $(q_m)\in S(\Delta)$ and $S(\Delta)-\lim q_m=l$. Then for every $\varepsilon>0$, we have

$$\lim_{n\to\infty}\frac{\left(\left|\{m\leq n\ :\ |\Delta q_m-l|\geq\varepsilon\}\right|\right)}{n}=0.$$

Now, since $\left|\{m\leq n\ :\ |\Delta q_m-l|\geq\varepsilon\}\right|$ is a positive integer and f is a modulus we may write

$$\frac{f\left(\left|\{m\leq n\ :\ |\Delta q_m-l|\geq\varepsilon\}\right|\right)}{f(n)}\leq\frac{\left(\left|\{m\leq n\ :\ |\Delta q_m-l|\geq\varepsilon\}\right|\right)f(1)}{n}\cdot\frac{n}{f(n)}.$$

Since $\lim_{t\to\infty}\frac{f(t)}{t}>0$ and $S(\Delta)-\lim q_m=l$, the right side and so the left side tend to 0 as $n\to\infty$ in above-mentioned inequality. This means that (q_m) is Δ_f-statistically convergent.

Theorem 6.1.6 Let f and g be two unbounded modulus functions. Then

(i) If the limit exists and (6.1) holds then a $\Delta_g -$ statistically Cauchy sequence is $\Delta_f -$ statistically Cauchy sequence,

(ii) If the limit exists and (6.2) holds then a sequence (q_m) is $\Delta_g -$ statistically Cauchy sequence if it is $\Delta_f -$ statistically Cauchy sequence.

Theorem 6.1.1 (i) and 6.1.1(ii) give the proof if we take $H = \{m \in \mathbb{N} : |\Delta q_m - \Delta q_N| \geq \varepsilon\}$.

We may give the following result by using Theorem 40 in [26].

Theorem 6.1.7 If for every unbounded modulus f, $(q_m) \in BS_f(\Delta)$, then $(q_m) \in l_\infty(\Delta)$.

Proof Let $(q_m) \in BS_f(\Delta)$. Suppose, if possible, $(q_m) \notin \ell_\infty(\Delta)$. Then for any $M > 0$, we have that the set $H = \{m \in \mathbb{N} : |\Delta q_m| > M\}$ is infinite. Now we have an unbounded modulus f with $d_f(H) = 1$ by Lemma 4.1.4, which contradicts the assumption that $(q_m) \in BS_f(\Delta)$ for every modulus f.

Theorem 6.1.8 Let f and g be two unbounded modulus functions.

(i) If (6.1) holds, then $BS_g(\Delta) \subset BS_f(\Delta)$,

(ii) If (6.2) holds, then $BS_g(\Delta) = BS_f(\Delta)$.

Proof Assuming (q_m) is $\Delta_g -$ statistically bounded. Then we have $M > 0$ with $d_g(\{m \in \mathbb{N} : |\Delta q_m| > M\}) = 0$. Theorem 6.1.1 (i) and 6.1.1(ii) give the proof if we take $H = \{m \in \mathbb{N} : |\Delta q_m| > M\}$.

Remark 6.1.9 The inclusion in (i) of the above Theorem may be strict. It can easily be seen that for the modulus functions $g(t) = \log(t+1)$, $f(t) = t^{\frac{1}{2}}$ and the sequence (q_m) defined by (3.1) in Remark 3.1.7 we get $(q_m) \in BS_f(\Delta) - BS_g(\Delta)$ and so that the inclusion $BS_g(\Delta) \subset BS_f(\Delta)$ is strict.

Corollary 6.1.10 For any unbounded modulus f we have $BS_f(\Delta) = BS(\Delta)$ if (6.3) holds.

Proof The proof is immediate by Corollary 6.1.2.

Theorem 6.1.11 If (6.1) holds then $S_g(\Delta) \subseteq BS_f(\Delta)$.

Proof Assume that $(q_m) \in S_g(\Delta)$ and $S_g(\Delta) - \lim q_m = l$. Let $\varepsilon > 0$ be given and define $H(n) = \{m \leq n : |\Delta q_m - l| \geq \varepsilon\}$ and $Q(n) = \{m \leq n : |\Delta q_m - l| > M\}$ for a number $M > \varepsilon$ large enough. Now since clearly $|H(n)| \geq |Q(n)|$ for every $n \in \mathbb{N}$ we have that $d_g(H) \geq d_g(Q)$ and so that $d_g(H) = 0$ implies $d_g(Q) = 0$. If (1) holds then $d_g(Q) = 0$ implies $d_f(Q) = 0$ by Theorem 6.1.1 (i). This means that (q_m) is $\Delta_f -$statistically bounded.

7. Δ_f –STATISTICAL CONVERGENCE OF ORDER α AND Δ_f – STRONG CESÀRO SUMMABILITY OF ORDER α

In this last chapter of the thesis, we study on the relationships between the sets $S_\alpha^f(\Delta)$ and $w_\alpha^f(\Delta)$ for various $\alpha,\beta\in(0,1]$ and modulus functions *f* and *g*.

7.1. Relationship between the Sets $S_\alpha^f(\Delta)$

Çolak was introduced the class S_α, for $0<\alpha\leq 1$ in [6].

Definition 7.1.1 [6] Let $0<\alpha\leq 1$. Define the α –density of a set $H\subset\mathbb{N}$ via

$$d_\alpha(H)=\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\{m\leq n\ :\ m\in H\}\right|.$$

Definition 7.1.2 [33] Let f be an unbounded modulus and $0<\alpha\leq 1$. Define the f_α –density of a set $H\subset\mathbb{N}$ by

$$d_\alpha^f(H)=\lim_{n\to\infty}\frac{1}{f(n^\alpha)}f\left(\left|\{m\leq n\ :\ m\in H\}\right|\right),$$

in case this limit exists.

Definition 7.1.3 Let f be an unbounded modulus and $0<\alpha\leq 1$. A sequence (q_m) is called Δ_f – statistically convergent of order α to l if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{f(n^\alpha)}f\left(\left|\{m\leq n\ :\ |\Delta q_m-l|\geq\varepsilon\ \}\right|\right)=0.$$

We write in this case $S_a^f(\Delta)-\lim q_m=l$ or $q_m\to l\left(S_\alpha^f(\Delta)\right)$. $S_a^f(\Delta)$ denotes the class of Δ_f – statistically convergent sequences of order α .

Remark 7.1.4 The Δ_f –statistical convergence of order α is not well defined in case $\alpha>1$. Indeed assume that f is an unbounded modulus function such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$. Define (q_m) by

$$q_m=9-r,\ \text{if}\ 2r-1\leq m\leq 2r,\ r=1,2,3,...$$

so that

$$\Delta q_m = \begin{cases} 1, & m = 2r \\ & \qquad r = 1,2,3,\ldots \\ 0, & m \neq 2r \end{cases} .$$

Then we have

$$\frac{1}{f(n^\alpha)} f\left(\left|\{m \le n \,:\, |\Delta q_m - 0| \ge \varepsilon\}\right|\right) \le \frac{1}{f(n^\alpha)} f(\frac{n}{2})$$

and

$$\frac{1}{f(n^\alpha)} f\left(\left|\{m \le n \,:\, |\Delta q_m - 1| \ge \varepsilon\}\right|\right) \le \frac{1}{f(n^\alpha)} f(\frac{n}{2}) .$$

Since $\lim_{t\to\infty} \frac{f(t)}{t} > 0,$ we have

$$\lim_{n\to\infty} \frac{1}{f(n^\alpha)} f\left(\left|\{m \le n \,:\, |\Delta q_m - 0| \ge \varepsilon\}\right|\right) = 0$$

and

$$\lim_{n\to\infty} \frac{1}{f(n^\alpha)} f\left(\left|\{m \le n \,:\, |\Delta q_m - 1| \ge \varepsilon\}\right|\right) = 0$$

for each $\varepsilon > 0$ if $\alpha > 1$. Because, under the condition $\lim_{t\to\infty} \frac{f(t)}{t} > 0,$

$$\lim_{n\to\infty} \frac{f\left(\frac{n}{2}\right)}{f(n^\alpha)} = \frac{1}{2} \lim_{n\to\infty} \frac{n}{n^\alpha} \lim_{n\to\infty} \frac{f\left(\frac{n}{2}\right)}{\frac{n}{2}} \lim_{n\to\infty} \frac{n^\alpha}{f(n^\alpha)} = 0$$

if $\alpha > 1$. Hence (q_m) is $S_\alpha^f(\Delta)-$convergent to both 1 and 0, which is impossible.

It is clear that $c(\Delta) \subset S_\alpha^f(\Delta)$ for any an unbounded modulus f and $\alpha \in (0,1]$. But the reverse is not generally true. It is easy to check that $(q_m) \in S_\alpha^f(\Delta)$ for $\alpha \in (\frac{1}{2},1]$ and modulus $f(t) = \sqrt{t}$, where

$$q_m = \begin{cases} 1, & m = n^2 \\ & \qquad n = 1,2,3,\ldots . \\ 0, & m \neq n^2 \end{cases}$$

Indeed

$$\Delta q_m = \begin{cases} 1, & m = n^2 \\ -1, & m-1 = n^2 \quad n = 1,2,3,... \\ 0, & \text{otherwise} \end{cases}$$

and, then

$$\lim_{n\to\infty} \frac{1}{f(n^\alpha)} f\left(\left|\left\{m \le n \ : \ |\Delta q_m - 0| \ge \varepsilon \ \right\}\right|\right) \le \lim_{n\to\infty} \frac{f(2\sqrt{n})}{f(n^\alpha)} = \lim_{n\to\infty} \frac{\sqrt{2\sqrt{n}}}{\sqrt{n^\alpha}} = \sqrt{2} \lim_{n\to\infty} \frac{n^{1/4}}{n^{\alpha/2}} = 0$$

that is $(\Delta q_m) \in S_\alpha^f$ for $\alpha \in (\frac{1}{2}, 1]$ but the sequence is not $\Delta -$convergent.

Theorem 7.1.5 Let (q_m) and (v_m) be any two sequences, f be an unbounded modulus and $0 < \alpha \le 1$. Then

(i) if $S_a^f(\Delta) - \lim_m q_m = l$ and $c \in \mathbb{C}$, then $S_a^f(\Delta) - \lim_m cq_m = cl$,

(ii) if $S_a^f(\Delta) - \lim_m q_m = l_1$ and $S_a^f(\Delta) - \lim_m v_m = l_2$, then $S_a^f(\Delta) - \lim_m (q_m + v_m) = l_1 + l_2$.

Theorem 7.1.6 Let f be an unbounded modulus and $0 < \alpha \le \beta \le 1$. Then $S_a^f(\Delta) \subset S_\beta^f(\Delta)$ and there may be strict inclusion.

Proof Since f is a modulus and $0 < \alpha \le \beta \le 1,$ the inclusion follows easily. For the strict inclusion, define the sequence $q = (q_m)$ by

$$q_m = 10 - r, \text{ if } (r-1)^2 < m \le r^2, \ r = 1,2,3,...$$

so that

$$(q_m) = (9,8,8,8,7,7,7,7,7,6,6,6,6,6,6,6,...) .$$

Now

$$\Delta q_m = \begin{cases} 1, & m = r^2 \\ & \qquad r = 1,2,3,.... \\ 0, & m \ne r^2 \end{cases}$$

If we consider the modulus function $f(t) = t^p, \ 0 < p \le 1$ then $q \in S_\beta^f(\Delta)$ for $\beta \in (\frac{1}{2}, 1],$ but $q \notin S_a^f(\Delta)$ for $\alpha \in (0, \frac{1}{2}]$. Thus $S_a^f(\Delta) \subset S_\beta^f(\Delta)$ is strict.

Corollary 7.1.7 Let $0 < \alpha \le 1$. Then $S_a^f(\Delta) \subset S^f(\Delta)$ for any unbounded modulus f, and there may be strict inclusion.

Theorem 7.1.8 Let $0<\alpha\leq 1$. Then

(i) $S_a^f(\Delta)\subset S_a(\Delta)$,

(ii) $S_a^f(\Delta)\subset S(\Delta)$

for any unbounded modulus *f*. The inclusions may be strict.

Proof To demonstrate that the inclusions are strict, consider the modulus function $f(t)=\log(t+1)$ and the sequence $q=(q_m)$ defined by (3.1) in Remark 3.1.7, that is

$$(q_m)=(0,-1,-1,-1,-5,-5,-5,-5,-5,-14,-14,-14,-14,-14,-14,-14,...),$$

then $q\in S_a(\Delta)$ and $q\in S(\Delta)$ for $0<\alpha\leq 1$. But $q\notin S_a^f(\Delta)$, since as

$$d_\alpha^f\left(\left\{k\in\mathbb{N}\ :\ |\Delta q_m-0|\geq\varepsilon\right\}\right)\geq d^f\left(\left\{m\in\mathbb{N}\ :\ |\Delta q_m-0|\geq\varepsilon\right\}\right)=\frac{1}{2}\neq 0\,.$$

Now we provide the generalizations of the spaces of Δ_f $-$strong Cesàro summability of order α.

7.2. Relationship between the Sets $w_\alpha^f(\Delta)$

Definition 7.2.1 Let f be a modulus and $0<\alpha\leq 1$. We define

$$w_{\alpha,0}^f(\Delta)=\left\{(q_m)\in s\ :\ \lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{m=1}^{n}f\left(|\Delta q_m|\right)=0\right\},$$

$$w_\alpha^f(\Delta)=\left\{(q_m)\in s\ :\ \lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{m=1}^{n}f\left(|\Delta q_m-l|\right)=0,\ \text{for some number } l\right\},$$

$$w_{\alpha,\infty}^f(\Delta)=\left\{(q_m)\in s\ :\ \sup_n\frac{1}{n^\alpha}\sum_{m=1}^{n}f\left(|\Delta q_m|\right)<\infty\right\}.$$

By specializing f and α, some well-known spaces are obtained. These spaces become $w_0^f(\Delta)$, $w^f(\Delta)$, $w_\infty^f(\Delta)$, respectively if $\alpha=1$. If we take $f(t)=t$ and $\alpha=1$, we obtain the familiar spaces $w_0(\Delta)$, $w(\Delta)$, $w_\infty(\Delta)$, respectively.

Now by using the spaces $w_{\alpha,0}^f(\Delta)$, $w_\alpha^f(\Delta)$, $w_{\alpha,\infty}^f(\Delta)$, we establish some inclusion relations.

Theorem 7.2.2 Let f be a modulus and $0<\alpha\leq 1$. Then $w_{\alpha,0}^f(\Delta)\subset w_{\alpha,\infty}^f(\Delta)$.

Theorem 7.2.3 Let f be a modulus and $0<\alpha\leq 1$. If $\lim_{t\to\infty}\frac{f(t)}{t}>0$, then $w_\alpha^f(\Delta)\subset w_\alpha(\Delta)$.

Proof By Proposition 1 of Maddox [25] we have $\upsilon = \lim_{t\to\infty} \frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t}, \quad t>0\right\}$. By definition of υ, we have $f(t) \geq \upsilon t$ for all $t \geq 0$. Since $\upsilon > 0$, we have $t \leq \upsilon^{-1} f(t)$ for all $t \geq 0$ and so

$$\frac{1}{n^{\alpha}} \sum_{m=1}^{n} \left|\Delta q_m - l\right| \leq \upsilon^{-1} \frac{1}{n^{\alpha}} \sum_{m=1}^{n} \left(f\left|\Delta q_m - l\right|\right)$$

from where it follows that $q \in w_{\alpha}(\Delta)$ whenever $q \in w_{\alpha}^{f}(\Delta)$.

Theorem 7.2.4 Let f be any modulus and $0 < \alpha \leq \beta \leq 1$. Then $w_{\alpha}^{f}(\Delta) \subset w_{\beta}^{f}(\Delta)$ and there may be strict inclusion.

Proof To prove $w_{\alpha}^{f}(\Delta) \subset w_{\beta}^{f}(\Delta)$ is straightforward since $\alpha \leq \beta$. In order to show that the inclusion is strict, let us take account of sequence $q = (q_m)$ defined by

$$q_m = 10 - r, \text{ if } (r-1)^2 < m \leq r^2, \ r = 1, 2, 3, \ldots$$

so that

$$\Delta q_m = \begin{cases} 1, & m = r^2 \\ 0, & m \neq r^2 \end{cases} \qquad r = 1, 2, 3, \ldots \ .$$

Since $f(0) = 0$,

$$\frac{1}{n^{\beta}} \sum_{m=1}^{n} f\left(\left|\Delta q_m - 0\right|\right) \leq \frac{\sqrt{n}}{n^{\beta}} f(1)$$

for every $n \in \mathbb{N}$. $\frac{\sqrt{n}}{n^{\beta}} f(1) \to 0$ as $n \to \infty$ for $\beta > \frac{1}{2}$, so $q \in w_{\beta}^{f}(\Delta)$. Also

$$\frac{1}{n^{\alpha}} \sum_{m=1}^{n} f\left(\left|\Delta q_m - 0\right|\right) \geq \frac{\sqrt{n}-1}{n^{\alpha}} f(1)$$

for every $n \in \mathbb{N}$. Since $\frac{\sqrt{n-1}}{n^{\alpha}} f(1) \to \infty$ as $n \to \infty$ for $0 < \alpha < \frac{1}{2}$, we get $q \notin w_{\alpha}^{f}(\Delta)$.

7.3. Relationship between the Sets $S_{\alpha}^{f}(\Delta)$ and $w_{\alpha}^{f}(\Delta)$

Çolak in [6] was shown that the strong Cesàro summability of order α implies statistical convergence of order α with preserving limit, that is $S_{\alpha} - \lim q_m = w_{\alpha} - \lim q_m$ for a number sequence (q_m) in [6]. In this section some relationship between $S_{\alpha}^{f}(\Delta)$ and $w_{\alpha}^{f}(\Delta)$, are obtained and established.

Theorem 7.3.1 Let f be an unbounded modulus such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$ and $0<\alpha\le\beta\le 1$. Then if a sequence is Δ_f – strongly Cesàro summable of order α to l, then it is Δ_f – statistically convergent of order β to l.

Proof Suppose that $\lim_{t\to\infty}\frac{f(t)}{t}>0$. Then by Corollary 3.1.1, we have $\upsilon=\inf_{t\in(0,\infty)}\frac{f(t)}{t}>0$ and so that $\upsilon t\le f(t)$ for every $t\in(0,\infty)$. Now if $q=(q_m)\in w_{\alpha}^{f}(\Delta)$, then since $\left|\{m\le n : |\Delta q_m-l|\ge\varepsilon\}\right|$ is a positive integer we may write

$$\frac{1}{n^{\alpha}}\sum_{m=1}^{n}f(|\Delta q_m-l|)\ge\upsilon\frac{1}{n^{\alpha}}\sum_{m=1}^{n}|\Delta q_m-l|\ge\upsilon\frac{1}{n^{\alpha}}\sum_{m=1,|\Delta q_m-l|\ge\varepsilon}^{n}|\Delta q_m-l|$$

$$\ge\upsilon\frac{1}{n^{\alpha}}\left|\{m\le n : |\Delta q_m-l|\ge\varepsilon\}\right|\varepsilon$$

$$\ge\upsilon\frac{1}{n^{\alpha}}f\left(\left|\{m\le n : |\Delta q_m-l|\ge\varepsilon\}\right|\right)\frac{\varepsilon}{f(1)}$$

$$\ge\upsilon\frac{1}{n^{\beta}}f\left(\left|\{m\le n : |\Delta q_m-l|\ge\varepsilon\}\right|\right)\frac{\varepsilon}{f(1)}$$

$$=\frac{f\left(\left|\{m\le n : |\Delta q_m-l|\ge\varepsilon\}\right|\right)f(\varepsilon)}{f(n^{\beta})}\frac{f(n^{\beta})}{n^{\beta}}\frac{\varepsilon}{f(1)}\upsilon .$$

Since $q\in w_{\alpha}^{f}(\Delta)$ and $\lim_{t\to\infty}\frac{f(t)}{t}>0$, then taking limit on both sides it follows that $q\in S_{\beta}^{f}(\Delta)$.

If we take $\beta=\alpha$ in above Theorem, then we get the next outcome.

Corollary 7.3.2 Let f be an unbounded modulus such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$. Then if a sequence is Δ_f – strongly Cesàro summable of order α to l, then it is Δ_f – statistically convergent of order α to l for any $\alpha\in(0,1]$.

Taking $\alpha=1$, we get the next outcome from above Corollary .

Corollary 7.3.3 Let f be an unbounded modulus such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$. Then if a sequence is Δ_f – strongly Cesàro summable to l, then it is Δ_f – statistically convergent to l.

Theorem 7.3.4 Let f be a modulus function such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$ and $\alpha\in(0,1]$. If a sequence is $\Delta-$strongly Cesàro summable of order α with respect f to l, then it is $\Delta-$statistically convergent of order α to l.

Taking $\alpha=1$ in above Theorem we get the next outcome.

Corollary 7.3.5 Let f be an unbounded modulus function such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$. If a sequence is $\Delta-$strongly Cesàro summable with respect f to l, then it is $\Delta-$statistically convergent to l.

If we take $f(t)=t$ and $\alpha=1$ in Theorem 7.3.4, then we get the next outcome.

Corollary 7.3.6 If a sequence is $\Delta-$strongly Cesàro summable to l, then it is $\Delta-$statistically convergent to l

8. Conclusions

We observed the notion $\Delta -$Statistical convergence, $\Delta -$Statistical boundedness, $\Delta_f -$ Statistical convergence, $\Delta_f -$Statistical boundedness, $\Delta_f -$strong Cesàro summability and $\Delta_f -$ statistical convergence of order α and strong $\Delta -$Cesàro summability of order α with respect to modulus functions. Also we provided the relationships between these conceptions.
Moreover, we also established some inclusion relation between $S_f(\Delta)$ and $S_g(\Delta)$, $S_f(\Delta)$ and $S(\Delta)$, $BS_f(\Delta)$ and $BS_g(\Delta)$, $BS_f(\Delta)$ and $BS(\Delta)$, $S_f(\Delta)$ and $BS_g(\Delta)$ for different modulus functions f and g under certain conditions, that is an original part in this thesis. Also we have given some relations between the sets of $\Delta_f -$statistically convergent sequences of order α and $\Delta_f -$strongly Cesàro summable sequences of order α .

REFERENCES

[1] Zygmund, A. (2002). *Trigonometric series* Vol. 1 Cambridge University Press. Cambridge

[2] Steinhaus, H. (1951). Sur la convergence ordinaire et la convergence asymptotique. *Colloq. Math* 2(1), pp. 73-74.

[3] Fast, H. (1951). Sur la convergence statistique. C*olloq. Math.* 2(3-4), pp. 241-244.

[4] Schoenberg, I. J. (1959). The integrability of certain functions and related summability methods. *The American Mathematical Monthly*, *66*(5), pp. 361-775.

[5] Buck, R. C. (1953). Generalized asymptotic density. *American Journal of Mathematics*, *75*(2), pp. 335-346.

[6] Çolak, R. (2010). Statistical convergence of order α. *Modern Methods in Analysis and Its Applications*, Anamaya Pub New Delhi, India, pp. 121-129.

[7] Çolak, R. and Bektaş, Ç. A. (2011). λ-statistical convergence of order α. *Acta Mathematica Scientia*, *31*(3), pp. 953-959.

[8] Mursaleen, M. and Alotaibi, A. (2011). Statistical summability and approximation by de la Vallée-Poussin mean. *Applied Mathematics Letters*, *24*(3), pp: 320-324.

[9] Fridy, J. A. and Orhan, C. (1993). Lacunary statistical convergence. *Pacific Journal of Mathematics*, *160*(1), pp. 43-51.

[10] Belen, C, and Mohiuddine, S. A. (2013). Generalized weighted statistical convergence and application. *Applied Mathematics and Computation*, 219(18), pp. 9821-9826.

[11] Kizmaz, H. (1981). On certain sequence spaces. *Canadian Mathematical Bulletin*, *24*(2), pp. 169-176.

[12] Çolak, R., & Et, M. (1995). On some generalized difference sequence spaces. *Soochow J. Math*, *21*(4), pp. 377-386.

[13] Nakano, H. (1953). Concave modulars. *Journal of the Mathematical society of Japan*, *5*(1), 29-49.

[14] Ruckle, W. H. (1973). FK spaces in which the sequence of coordinate vectors is bounded. *Canadian Journal of Mathematics*, *25*(5), pp. 973-978.

[15] Maddox, I. J. (1987, May). Inclusions between FK spaces and Kuttner's theorem. *Mathematical Proceedings of the Cambridge Philosophical Society*, 101(3), pp. 523-527). Cambridge University Press.

[16] Aizpuru, A., Listàn-Garcìa, M. C., and Rambla-Barreno, F. (2014). Density by moduli and statistical convergence. *Quaestiones Mathematicae*, *37*(4), pp. 525-530.

[17] Niven, I., Zuckerman, H. S., and Montgomery, H. L. (1991). *An introduction to the theory of numbers*. John Wiley and Sons.

[18] Altin, Y, Altinok, H., and Çolak, R. (2015). Statistical convergence of order α for difference sequences. *Quaestiones Mathematicae*, *38*(4), pp. 505-514.

[19] Altinok, H., & Mursaleen, M. (2011). δ - Statistical boundedness for sequences of fuzzy numbers. *Taiwanese Journal of Mathematics*, *15* (5), pp. 2081-2093.

[20] Bhardwaj, V. K., & Gupta, S. (2014). On some generalizations of statistical boundedness. *Journal of Inequalities and Applications*, *2014*(1), 12.

[21] Isik, Mahmut (2004). On statistical convergence of generalized difference sequences. *Soochow Journal of Mathematics*, *30*(2), pp. 197-206.

[22] A. Karakas¸ Y. Altn, M. Et, (2018). $\Delta_p^m -$Statistical Convergence of Order α. *Faculty of sciences and mathematics, university of Niš, Serbia*. 32 (16), pp. 5565-5572.

[23] Fridy, J. A. (1985). On statistical convergence. *Analysis*, *5*(4), pp. 301-314.

[24] Maddox, I. J. (1967). Spaces of strongly summable sequences. *The Quarterly Journal of Mathematics*,*18* (1), pp. 345-355.

[25] Gidemen, H. and Et, M. (2018). On f–lacunary statistical convergence of generalized difference sequences. *AIP Conference Proceedings*. 2037 (1). pp. 020011.

[26] Bhardwaj, V. K., Dhawan, S. and Gupta, S. (2016). Density by moduli and statistical boundedness. *Abstract and Applied Analysis* Vol. 2016 ID 2143018, 6 pages http://dx.doi.org/10.1155/2016/2143018.

[27] Connor, J. (1988). The statistical and strong p-Cesaro convergence of sequences. *Analysis*, *8*(1-2), pp. 47-64.

[28] Kuttner, B, (1946). Note on strong summability. *J. Lond. Math. Soc*. 80(2), pp. 118-122.

[29] Maddox, I. J. (1968). On Kuttner's theorem. *Journal of the London Mathematical Society*, 1(1), pp. 285-290.

[30] Maddox, I. J. (1986), Sequence spaces defined by a modulus. *Math. Proc. Cambridge Philos. Soc.* 100(1), pp. 161-166.

[31] Connor, J. (1989). On strong matrix summability with respect to a modulus and statistical convergence. *Canadian Mathematical Bulletin*, *32*(2), pp. 194-198.

[32] Çolak R. (2020) Some Relations Between the Sets of *f*-Statistically Convergent Sequences., H. Dutta et al. (Eds.): CMES 2019, AISC 1111, pp. 307–314, Springer, Cham https://doi.org/10.1007/978-3-030-39112-6_25.

[33] Bhardwaj, V. K., Dhawan, S., (2015). $f-$statistical convergence of order α and strong Cesàro summability of order α with respect to a modulus, *Journal of Inequalities and Applications* Vol.332, 14 pages.

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