LIMIT SETS OF BEST-REPLY PROCESSES

A Master's Thesis

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by

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

LIMIT SETS OF BEST-REPLY PROCESSES

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I analyze limiting behavior of best-reply processes. I find that without inertia Nash Equilibria are not limit sets. Moreover, even for processes with inertia, Nash Equilibria are not stable.

I argue that minimal CURB sets are reasonable candidates for limit sets if best-reply processes are indeterminate or Nash Equilibria satisfy evolutionary stability (Oechssler 1997). In such cases, limit sets necessarily contain a Nash Equilibrium. Otherwise limit sets may not be close to any Nash Equilibria unless they satisfy some support consistency condition.

Keywords: Best-Reply Processes, Limit Sets, Nash Equilibria, Minimal CURB sets.

ÖZET

EN-İYİ-TEPKİ SÜREÇLERİNİN LİMİT KÜMELERİ

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En-iyi-tepki süreçlerinin uzun vadedeki davranışlarını inceliyorum. Bu çalışmada atalet olmadan Nash Dengelerinin limit kümesi oluşturamayacağını buldum. Ayrıca, ataletli süreçler için bile Nash Dengeleri istikrarlı değiller.

İndirgenemez CURB kümeleri en-iyi-tepki süreçleri belirsiz olduğunda ya da Nash Dengeleri saf stratejilerde evrimsel istikrar koşulunu sağladığında limit kümeleri için makul adaylar oluyorlar.Bu durumlarda limit kümelerinde mutlaka bir Nash Dengesi bulunuyor. Diğer durumlarda ise, Nash Dengesi belli bir saf strateji tutarlılığı koşulunu sağlamadığında limit kümeleri hiçbir Nash Dengesine yakın olmayabiliyorlar.

Anahtar Kelimeler: En-iyi-tepki süreçleri, Limit kümeleri, Nash Dengeleri, İndirgenemez CURB kümeleri

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CHAPTER 1

Introduction

Nash Equilibrium is an intellectually compelling model for human behavior in game theory. Nonetheless it requires both rationality and coordination of beliefs. We will concentrate on whether rational players learn to coordinate their beliefs. One way to do this is to define a dynamic best-reply process among rational players and investigate whether coordination of beliefs is an outcome of a limit point this best-reply process. In a standard repeated game a natural model for analysis is rational learning, as was done in Kalai and Lehrer (1993). In a large matching game where a player will probably not interact with the same person again the intellectual appeal of this approach is weaker. In this environment a reasonable alternative is to assume some type of simple social learning rule and analyze the limiting behavior of the population.

The common approach in this literature is to analyze the mean of the limiting distribution. For example Hopkins (1999) considers mean dynamics for best-response processes and compares a perturbed version of best-response processes with mean dynamics in evolutionary game theory. Although this approach makes it possible to employ methods of differential equations or inclusions, it can be misleading. The population distribution may not converge when mean converges and so the mean does not summarize the limiting behavior of the population. A limiting distribution in mean dynamics may correspond to a non-singleton absorbing set in distribution dynamics. Moreover, under any payoff monotone mean dynamics Nash Equilibrium is a fixed point (Friedman 1991, Ritzberger and Weibull 1995), but the mixed equilibria may either not be in the limit distribution (called *limit sets* in this paper) or form a singleton limit set. Therefore in this paper we partially characterize the limit sets, or the possible limiting distributions of various distributional best-reply processes.

Oechssler (1997) is one of the rare studies on learning mixed equilibria using distributions in best-reply processes. He finds conditions under which the mixed equilibrium is accessible by a best-reply process in finite time in games with a unique mixed equilibrium. Under the assumption that no one changes strategies when they are best replying, Oechssler finds sufficient conditions for the mixed equilibrium to be absorbing in two person symmetric games. We study two basic best-reply processes—the no-worse and all-best—in general symmetric games. The all-best process is similar to that analyzed by Oechssler, under this dynamic all Nash equilibria will be singleton limit sets. Other limit sets may not include a Nash equilibrium at all. With the no-worse dynamic the only singleton limit sets will be pure strategy strict equilibria of the stage game.

We also analyze the relationship between Nash Equilibria and best-reply learning dynamics. As Oechssler (1997) observes for some cases there are limit sets that are one mutation away from a mixed equilibrium irrespective of population size. In such cases, it would be artificial to argue that no one in the population might make a "mistake" or use some alternative best-reply process. Thus we allow for finite mutations, a finite number that is fixed irrespective of population size. If each of these "mutations" occurs with positive and independent probability then as the population size goes to infinity this finite number of mutations occurs with a probability approaching one, thus a reasonable analysis should include this possibility. This allows us to analyze the proper closure of our limit sets.

We find that the closure of limit sets provide a deeper understanding than minimal CURB sets (Basu and Weibull (1991). Not all equilibria are in minimal CURB sets and the support of minimal CURB sets may not be the support of any equilibrium. However, some of the Nash Equilibria outside minimal CURB sets are contained in closure of limit sets. We prove that consistent Nash Equilibria belong to closure of limit set, where consistency is defined for best-replies of its support. Moreover if a Nash Equilibrium satisfies evolutionary stability for pure strategies, ESPS (Oechssler 1997), then there is a limit set contained in the support of the Nash Equilibrium. However, we do not have a clear characterization for when an equilibrium will be in closure of a limit set.

The existence of limit sets outside minimal CURB sets means that for some initial conditions no minimal CURB set is accessible. This might seem contradictory to affirmative results for accessibility of minimal CURB sets in the literature (Kosfeld, Drost, and Voorneveld 2002, Young 2001, Sanchirico 1996). However Kosfeld, Drost, and Voorneveld (2002) define CURB sets in an unconventional way so that any pure strategy best-reply cycle forms a CURB set. It is easy toobserve that any learning rule that puts positive weights to best-replies will access to a minimal CURB set and what they prove is consistent with this observations. When the CURB set is defined as in Young (2001) , we might not have same results. One needs some form of indeterminateness in best-reply processes, that is some positive share of the population should be allowed to play something else from best-reply to the current distribution to prove accessibility of minimal CURB sets.

Young (Young2001) and Sanchirico (Sanchirico1996) get positive results concerning accessibility of minimal CURB sets by introducing indeterminateness to the best-reply process. When we translate the belief update procedures in both models from sampling in time series to cross section over other players in the population, we see that both models require a form of noisy sampling, where large noise is possible with small probabilities. Evidently noisy sampling is not the only way to induce indeterminateness. For example, each period a positive share of the population might adopt a different learning rule or discover that all individuals are learning too and decide to make use of it like in the clever agents model developed by Saez-Marti and Weibull (1999). Clearly all these deviations will look like indeterminateness to an analyst. Under such dynamics we prove that from any initial point some minimal CURB set is accessible.

In the next chapter we review the relevant literature in stochastic evolution and learning. Chapter 3 sets up the basic framework of our model and in chapter 4 we define and analyze best-reply processes. Succeeding two chapters capture indeterminate and determinate processes. We investigate stability of limit sets in chapter 7. In chapter 8 we conclude.

CHAPTER 2

Literature Review

One can broadly break the learning literature into two classes of learning dynamics, reduced-form models and models that do not have a reduced-form formulation. In reduced-form models there is a recursive formulation of behaviors and belief update procedures. In contrast, in the other type of models beliefs are represented as probability measure over all future periods. Correspondingly, behavior rules are responses to such probability measures. One leading model of this class is Bayesian learning , first formulated by Kalai and Lehrer (1993). In this model players update their beliefs using Bayes Rule and are forward-looking. Kalai and Lehrer (1993) consider "absolute continuity" as a restriction on prior beliefs. This condition requires that if an event is possible given the realized history, then all players should believe that this event is possible. Given this condition they show that players learn to predict the future in the long-run. Their main result that if players optimize the play converges to a Nash Equilibrium of the infinitely repeated game is stated as a corollary to this assertion.

Although convergence to Nash Equilibrium is a strong result, this model has received many critiques due to the necessity of absolute continuity. Fudenberg and Levine (1998) point out that since the set of possible future paths is continuum it is impossible to impose full support property on beliefs.

Therefore prior beliefs will be consistent with the path that will be realized but inconsistent with almost all of other possible paths. In this respect, absolute continuity is a weak version of equilibrium beliefs. Kalai and Lehrer (1993) also show that absolute continuity provides a relation between prediction of play and optimization against learning opponents. Nachbar (1997) argues that in general this relationship is non-trivial. He proves that when we require prior beliefs to satisfy some intuitive *computability* properties, simultaneous achievement of prediction and optimization is impossible. Foster and Young (2001) focus on prediction only and prove that if the stage game is a perturbed version of matching pennies then no player who employs rational learning can predict the play at all.

Sanchirico's (1996) model also does not have a reduced-form formulation, since there is not an explicitly defined learning procedure. Instead, he assumes that players' beliefs satisfy some assumptions which can be justified by various learning procedures. This paper only has three assumptions. The first is that there is common knowledge of rationality. The second assumption guarantees that if the play has stayed in a set long enough then players believe that the play is in the set. These two assumptions make CURB sets absorbing. The third assumption requires that it is possible for players to play best-reply to strategy profiles that are in their memory. Under the second and third assumptions if the memory is large enough the play converges to a minimal CURB set.

In reduced-form learning models we do not have to put restrictions on initial beliefs over infinite paths. Moreover reduced-form models can be a solution to the computability problem that Nachbar (1997) poses. However, since a behavioral rule is assumed a-priori, there is no justification of the behavior rule in these models. A reduced-form model is basically a triple: initial beliefs over the strategy space of the stage game, a belief update process and a behavioral rule. The literature on this type of model can be classified under two branches, mean dynamics and distributional dynamics.

One leading model of mean dynamics has been borrowed from evolutionary biology. Under Replicator Dynamics fitter animals produce more offspring. In game theory this corresponds to a dynamic where players switch to strategies with higher payoffs, or payoff monotonicity. Friedman (1991) and Ritzberger and Weibull (1995) are two important examples of economic interpretations of Replicator Dynamics. While Friedman (1991) proves that Nash Equilibria are absorbing under payoff monotone dynamics, Ritzberger and Weibull (1995) proves same result for upper semi-continuous behavioral rules. Borgers and Sarin (1997, 2000) provide a psychological justification for Replicator Dynamics.

Another leading model of mean dynamices is continuous time best-response. Hopkins (1999) compares a perturbed best-response with evolutionary models. Berger (2002) adds a role game prior to stage game at each period. By this way, players first decide their role in the stage game, then play the game. Thus, it becomes possible to analyze asymmetric interactions in one population. Benaim and Hirsh (1996) prove that continuous time best-response models serve as an approximation to fictitious play.

The asymptotical similarity between fictitious play and continuous best-reply processes is not coincidental, since fictitious play behaves like mean dynamics asymptotically. Although in fictitious play players play in discrete time, as beliefs are statistical distributions of strategies over the realized history the play converges to the mean-dynamics.

Fictitious play is one the most studied models in the theory of learning. Fudenberg and Levine (1998) presents various versions of fictitious play in detail. Conditional fictitious play developed by Fudenberg and Levine (1999) is one of the important versions fictitious play. In this model players are able to detect patterns in the play.

The other subclass of reduced-form learning models is distribution dynamics. In this type of model the state of the world is the distribution of strategies over the players in population in the current period and possibly the past periods. Since it is possible to use Markov Chains to analyze discrete time reduced-form models, one can find the limiting distributions. However, for games with multiple equilibria limiting distributions are not unique. Introducing continual random mutations or experimentation will make the Markov Chain irreducible and thus imply a unique limiting distribution. This type of models is classified as Stochastic Evolution models. Kandori, Mailath and Rob (1993) is one of the seminal papers in the Stochastic Evolution literature. . They consider uniform matching between two populations. The learning dynamic is a best-reply process which is perturbed by persistent mutations. They prove that risk dominant equilibrium survives in the long-run in 2x2 coordination games. Ellison (2000) generalizes the framework of Kandori, Mailath and Rob (1993) and prove that 1/2-dominant equilibria survive in the long-run. Ellison (Ellison1993) addresses the same problem for local interactions and find same result but that convergence occurs much faster. Fudenberg and Kreps (1995) focus on extensive form games. Under the condition that players experiment enough (there is a lower bound for probability of experimentation), non-Nash Equilibrium profiles are unstable and Nash Equilibria are weakly stable. As Jordan (1993) points out, mixed equilibria are hard to justify as a limiting distribution of a learning process. Thus there are limited studies which consider mixed equilibria. Oechssler (1997) is one of the rare ones, which characterizes learning mixed-equilibria in 2x2 and 3x3 games and gives a sufficient condition for general symmetric two-person games.

Fudenberg and Kreps (1993) adopts an approach similar to Harsanyi (1973). By assuming uncertain payoffs they purify mixed equilibria and obtain a smooth version of fictitious play. Gorodeisky (2006) proves that unique mixed equilibrium in many 2x2 games is stable. Stability follows from the fact that population distribution is not allowed to make big jumps. Since we allow big jumps in population distribution, no mixed-equilibria is stable in our model.

Young (1993) develops a learning model related to fictitious play. He assumes that players play best-reply to a statistical distribution derived from past plays. However players have a finite memory length and each period they draw a sample from the past plays remains in the memory. He proves that the learning process converges to a convention, which is a repetition of pure strategy equilibrium for games that there is no pure strategy best-response cycle. He then switches to a perturbed dynamics, which leads to the concept of stochastically stable limit sets. Stochastic stability is risk dominance in 2x2 coordination games but this result does not generalize.

CHAPTER 3

The Model

Let $G = \langle I, A, u \rangle$ be a finite symmetric game, where I is the finite set of players, A is the finite set of strategies, and $u : A^{|I|} \to \mathbb{Q}$ is the payoff function.

Suppose there is a single population with cardinality N. At each period $t \in \{0, 1, ...\}$ players in the population uniformly match in $|I|$-groups to play the game G. If N is finite, one can assume $N \equiv 0 \pmod{|I|}$ to guarantee that everyone plays the game every period.

Let $\Delta = \Delta(A)$ be the set of population distributions over strategies; that is, $\forall \mu \in \Delta \quad \mu = (\mu(a))_{a \in A}$, such that $\forall a \in A \quad \mu(a)$ is the share of the population that plays a. For any subset C of A the set of population distributions over C is $\Delta(C) = \{\mu \in \Delta : \forall a \in A \backslash C \; \mu(a) = 0\}$. The support of a distribution μ is $Supp(\mu) = \{a \in A : \mu(a) > 0\}$ and $\forall X \subseteq \Delta$ $Supp(X) = \cup_{\mu \in X} Supp(\mu)$. $\forall a \in A$ the population distribution that all players play a is denoted by δ_a. For any subset C of A let $\delta_C = \cup_{a \in C} \delta_a$. Let $\mu_1, \mu_2 \in \Delta$, then the line between μ_1 and μ_2 is

$$[\mu_1, \mu_2] = \{\mu' \in \Delta : \exists \alpha \in (0,1) \text{ such that } \mu' = \alpha\mu_1 + (1-\alpha)\mu_2\}.$$

Each player plays a strategy in A against the population distribution, so each player's strategy is independent. We assume that each player ignores her contribution to the population distribution, essentially N is large enough that a given player's impact is negligible. Let $\bar{u} : A \times \Delta(A) \to \mathbb{R}$ be the expected payoff function for mixed strategies. For any player, expected payoff of playing $a \in A$ against $\mu \in \Delta$ is

$$\bar{u}(a, \mu) = \sum_{\hat{a} \in A^{|I|-1}} \left(\Pi_{i=1}^{|I|-1} \mu(\hat{a}_i) \right) u(a, \hat{a}).$$

Then the best reply correspondence is given by

$$\forall \mu \in \Delta \;\; BR(\mu) = \arg\max_{a \in A} \bar{u}(a, \mu),$$

and correspondingly the best response region is given by $\forall\, a \in A \;\; BR^{-1}(a) = \{\mu \in \Delta : a \in BR(\mu)\}$. We have two non-degeneracy assumptions regarding best-reply regions:

Assumption 1 *1.* $\forall a \in A \; |BR(a)| = 1$

2. $\forall a, \forall C \subseteq A$, $BR^{-1}(a) \cap \Delta(C) = \emptyset$ *or* $(\overset{\circ}{BR^{-1}(a)} \cap \Delta(C)) \neq \emptyset$ *where the interior is taken with respect to relative topology defined over* $\Delta(C)$*.}*

We will be considering various best-reply learning dynamics of how players choose their strategies. Each process will imply that given the population distribution $\mu = \mu_t$ there is a certain set of distributions that can occur in period $t+1$. We will call this set as the *successor set* of μ.

Definition 1 *Let* $\{X_t\}_{t \geq 0}$ *be a Markov process with state space* Δ *and for any period* t*, the transition function of* X_t *is* $F|_{X_{t-1}}$*. The successor set of* $\mu \in \Delta$ *is* $S(\mu)$*, where* $S : \Delta \rightrightarrows \Delta$ *maps* μ *to the set of distributions that can occur in* $t+1$ *given* $X_t = \mu$*; that is,*

$$S(\mu) = S(X_t) = Supp\left(F|_{X_t = \mu}\right) = \{\mu' \in \Delta : F|_{X_t=\mu}(\mu') > 0\}.$$

For any best-reply process if $[\forall \mu \in \Delta \; \mu \in S(\mu)]$ does not hold, then we cannot talk about singleton successor sets. However, we expect that successors pure strategy strict equilibria are themselves. Thus we assume this here. Define $\forall X \subseteq \Delta \quad S(X) = \cup_{\mu \in X} S(\mu)$, $S^0(\mu) = \mu$, $\forall k \geq 1$ $S^k(\mu) = S\left(S^{k-1}(\mu)\right)$, and $S^\infty(\mu) = \left\{\bar{\mu} \in \Delta : \exists k < \infty \ \bar{\mu} \in S^k(\mu)\right\}$.

We will analyze ergodic sets of the Markov Processes derived by the best-reply processes, which we call limit sets after proving that they are same.

Definition 2 *Let $X \subseteq \Delta$ be non-empty. X is a limit set if $\forall \mu \in X \ \ S^\infty(\mu) = X$.*

Note that this definition is different than usual definition of ergodic sets. The defining properties of ergodic sets are that they are *absorbing*, one can easily show that limit sets are *minimal absorbing sets*.

Definition 3 *Let $X \subseteq \Delta$ be non-empty. X is absorbing if $S^\infty(X) \subseteq X$. X is called a minimal absorbing set if there 's no $X' \subset X$ such that X' is an absorbing set.*

Notice that:

Remark 1 *Let $X \subseteq \Delta(A)$ be non-empty. X is absorbing if and only if $S(X) \subseteq X$.*

Proof Suppose X is absorbing, then $S(X) \subseteq S^\infty(X) \subseteq X$. Suppose $S(X) \subseteq X$, then by induction over the degree of successor, $S^\infty(X) \subseteq X$. □

Lemma 1 *Let $X \subseteq \Delta(A)$ be non-empty. X is a limit set if and only if X is a minimal absorbing set.*

Proof Suppose X is a limit set, then $S(X) \subseteq S^\infty(X) = X$, which implies that X is absorbing. $\forall X' \subseteq X \ S^\infty(X') = X$, which follows directly from the definition of limit sets, implies that X is minimal.

Suppose X is a minimal absorbing set. Then $\forall \mu \in X \quad S^\infty(\mu) \subseteq X$. Suppose $\exists \mu \in X$ such that $S^\infty(\mu) \neq X$. However, this is a contradiction with that X is minimal since $S^\infty(\mu)$ is absorbing. □

Before passing to best-reply processes, we need following two important concepts from Game Theory; minimal CURB sets and Nash Equilibria.

Definition 4 *A set* $\breve{A} \subseteq A$ *is a CURB set iff* $\forall \mu \in \Delta\left(\breve{A}\right)$ $BR(\mu) \in \breve{A}$. $\breve{A}$ *is a minimal CURB set if it contains no other CURB set.*

Definition 5 $\mu^* \in \Delta$ *is a Nash Equilibrium iff* $Supp(\mu^*) \subseteq BR(\mu^*)$. μ^* *is called a strict Nash Equilibrium if* μ^* *is a Nash Equilibrium and* $|BR(\mu^*)| = |Supp(\mu^*)| = 1$.

CHAPTER 4

Best-Reply Processes

We have two basic best-reply processes, which are no-worse and all-best processes. For the no-worse process, successor set is defined as follows:

$$\forall \mu \in \Delta \quad S_{nw}(\mu) = \{\mu' \in \Delta : \forall a \notin BR(\mu)\,,\ \mu'(a) \leq \mu(a)\}\,.$$

For the all-best process we put the additional restriction that best-replies cannot decrease; that is,

$$\forall \mu \in \Delta \quad S_{ab}(\mu) = \{\bar{\mu} \in \Delta : \forall a \in BR(\mu)\,,\ \bar{\mu}(a) \geq \mu(a)\} \cap S_{nw}(\mu)\,.$$

Note that the all-best process can be regarded as no-worse process with inertia. In the all-best process, the share of any strategy which is a best-reply should not decrease. Thus when players do not switch strategies when they are playing a best-reply we will realize all-best process.

These processes are determinate processes, that is, the population moves to the direction of best replies. We will have an extension to indeterminate processes, in which a (small) portion of the population might move in some other direction.The following definition characterizes indeterminate best-reply processes.

Definition 6 *A best-reply process is called as $\varepsilon-$indeterminate with respect to $C(\cdot)$ if and only if it is defined by the following successor relation*

$$\forall \mu \in \Delta \;\; S(\mu, \varepsilon) = S\left(B_{\varepsilon}^{\Delta(C(\mu))}(\mu)\right)$$

where $B_{\varepsilon}^{\Delta(C(\mu))}(\mu) = B_{\varepsilon}(\mu) \cap \Delta(C(\mu))$, and $Supp(\mu) \subseteq C(\mu)$.

Here $B_{\varepsilon}(\mu)$ is the usual open ball around μ when N is continuum. If N is finite, $B_{\varepsilon}(\cdot)$ can be defined as

$$\forall \mu \in \Delta \;\; B_{\varepsilon}(\mu) = \left\{ \mu' \in \Delta : \sum_{a \in A} |\mu'(a) - \mu(a)| < \varepsilon \right\}.$$

The first assertion in the following proposition indicates the importance of the inertia. The second assertion can be regarded as an existence result for limit sets as there always exists a CURB set.

Proposition 1 *The following hold:*

1. *Let μ^* be a Nash Equilibrium. If $\mu^*$is a strict Nash Equilibrium, μ^* is absorbing both in all-best and no-worse processes. Otherwise, μ^* is absorbing in all-best, but not in no-worse.*

2. *Let $C \subseteq A$. C is a CURB set then $\Delta(C)$ is absorbing and contains a limit set.*

Proof *μ^*being a strict Nash Equilibrium implies that $BR(\mu^*) = \mu^*$, this makes $\{\mu^*\}$ absorbing for no-worse processes so for all-best processes. Otherwise $Supp(\mu^*) \subseteq BR(\mu^*)$ implies that $S_{ab}(\mu^*) = \{\mu^*\}$ but $S_{nw}(\mu^*) = \Delta(BR(\mu^*))$. For the second assertion; clearly, $\Delta(C)$ is absorbing since $BR(\Delta(C)) \subseteq \Delta(C)$ so $S(\Delta(C)) \subseteq \Delta(C)$. This implies by well-ordering principle that there exists a minimal absorbing set X in $\Delta(C)$. By the lemma above, X is a limit set.* □

From now on, unless otherwise is stated, we will mean no-worse processes whenever we refer to a best-reply processes.

The next proposition gives a useful characterization of equivalence between limit sets and population distributions over a minimal CURB set. However to prove the proposition, following lemma will be needed.

Lemma 2 *Let $L \subseteq \Delta(A)$ be a limit set for the no-worse process. Then $\delta_{Supp(L)} \subset L$ and there exists unique limit set in $\Delta(Supp(L))$.*

Proof Since $\forall \mu \in L\ \delta_{BR(\mu)} \subset L$. Assuming $\exists a \in Supp(L)$ such that $\delta_a \notin L$ implies that $\nexists \mu \in L$ such that $a \in BR(\mu)$. But this requires that

$$\forall \mu' \in L \cap (\Delta(Supp(L) \setminus \{a\}))\, S(\mu') \subseteq L \cap (\Delta(Supp(L) \setminus \{a\}))$$

contradicting with $a \in Supp(L)$. The second assertion directly follows from the first. □

Proposition 2 *Let $L \subseteq \Delta(A)$ be a limit set for no-worse process. Then L is convex if and only if there exists a minimal CURB set $C \subseteq A$ such that $L = \Delta(C)$.*

Proof Assume L is convex. Then by the lemma 2 $L = \Delta(Supp(L))$. But this implies $\forall \mu \in \Delta(Supp(L))\ BR(\mu) \subseteq Supp(L) \Rightarrow Supp(L)$ is a CURB set. If $Supp(L)$ is not minimal, $\exists$ a limit set $L' \underset{\neq}{\subset} L$, which is impossible. Thus $Supp(L)$ is a minimal CURB set. The converse is true by definition. □

In Proposition 1, we assert that if C is a minimal CURB set, then $\Delta(C)$ is a good place to look for a limit set. If all limit sets are convex, then population distributions of minimal CURB sets are the only places to look for a limit set. In the next chapter, we prove that all limit sets of indeterminate processes are convex.

CHAPTER 5

Indeterminate Processes

Note that both Young (2001) and Sanchirico (1996) proved convergence to minimal CURB sets. They assume some form of indeterminacy in their models, which makes limit sets convex. Then by the proposition 2, the limit sets correspond to minimal CURB sets.

Note that $\forall\mu\ \forall\varepsilon > 0\ \ S(\mu) \subseteq S(\mu,\varepsilon)$, thus proposition 2 applies for indeterminate processes.

We will now consider some examples of indeterminate processes to see the scope of this definition.

Example 1 *Suppose* $C(\mu) = Supp(\mu)$. *This process might be implied by following individual behavior rule. In each period, a given portion of the population imitates instead of best-replying. Some people in the population choose a strategy by imitating some individual, and others best-reply to the current distribution of the population.*

It is easy to formulate social learning models by successor correspondences. This method allows to generalize various learning processes by small variations on the defining successor correspondence.

Noisy beliefs is common way to assume indeterminacy. Young's dynamics (1993) is a best-reply process with noisy beliefs. However, he defined this noise in beliefs by a sampling process, which allows high noise in beliefs.

Similarly, Sanchirico's (1996) assumption of best-reply entropy is also a form of indeterminacy. Following example illustrates that learning processes in both models are forms of $\varepsilon-$indeterminate processes.

For all succeeding examples,

$$\forall C \subseteq A \;\; \beta^0(C) = C, \;\; \beta^{i+1}(C) = BR(C) \cup \beta^i(C),$$

and

$$\beta^\infty(C) = \left\{a \in A : \exists k \;\; s.t \;\; a \in \beta^k(C)\right\}.$$

Example 2 *Suppose $C(\mu) = \beta^1(Supp(\mu))$. The corresponding process might be implied by a noisy belief process. For noise large enough, a portion of the population have wrong beliefs about the current distribution, thus it is possible that they play a best-reply to any strategy in the support of the population. When we put a restriction for each period on number of players who can switch, this process will be Young's dynamics (1993). For ε large, this process will be equivalent to the one in Sanchirico (1996).*

Example 3 *A learning process with clever individuals is also an element of indeterminate processes. Saez-Marti and Weibull (1999) studies such a behavior in bargaining games. Suppose $C(\mu) = \beta^2(Supp(\mu))$. In this case, a portion of the population understands that people in the population play best-reply to current distribution. Thus they anticipate the distribution after everyone played their best reply with a large noise, like in Young (2001), and play best-reply to that.*

Example 4 *Suppose $C(\mu) = A$. The process correspond to this case is a best-reply process with continual mutations. In this case, a portion of the population choose their strategy randomly from the whole strategy set. This modification switches a learning process to a stochastic evolution process. We will not analyze this case, but it is possible to characterize stochastically stable* *limit sets, which might be defined as an extension of stochastically stable states in Kandori Mailath and Rob (1993) and Young (1993).*

As the last example illustrates the class of processes that is defined above contains stochastic evolutionary processes. However, we will restrict our attention to learning processes so we assume $\forall \mu \in \Delta(A)\ C(\mu) \subseteq \beta^{\infty}(Supp(\mu))$ throughout the analysis. Following lemma states that there is an interior point in the limit set.

Lemma 3 *$\exists \mu \in L$ such that $Supp(\mu) = Supp(L)$*

Proof If $|Supp(L)| = 1$, L itself a Nash Equilibrium, and we are done. Now suppose $|Supp(L)| > 1$. We will use induction for this proof. Let $a \in Supp(L)$ be arbitrary. Then $\exists\, a_1 \in Supp(L)$ such that $\{a_1\} = BR(\delta_a)$, then by the first non-degeneracy assumption, $\exists \mu_1 \in S(\delta_a)$ such that $BR(\mu_1) = BR(\delta_{a_1})$, $\mu_1(a) > 0$ and $\mu_1(a_1) > 0$. Assume the inductive hypothesis: $\exists \mu_i \in L$ such that $Supp(\mu_i) = \{a, a_1, \ldots, a_i\}$. If $\{a, a_1, \ldots, a_i\} = Supp(L)$, we are done. So suppose not. Then since $\{a, a_1, \ldots, a_i\}$ is not a $CURB$ set,

$$\exists a_{i+1} \in Supp(L) \setminus \{a, a_1, \ldots, a_i\} \text{ s.t.} \left(\overset{\circ}{BR^{-1}}(a_{i+1})\right) \cap \Delta(\{a, a_1, \ldots, a_i\}) \neq \emptyset$$

and $\exists l < \infty$ such that

$$S^l(\mu_i) \cap \left(BR^{-1}(a_{i+1})\right) \cap \Delta(\{a, a_1, \ldots, a_i\}) \neq \emptyset.$$

By repeated use of the non-degeneracy assumption, $\exists \{\mu_{i1}, \ldots, \mu_{il-1}\}$ such that

$$\forall j \in \{1, \ldots, l\}\ Supp(\mu_{ij}) = \{a, a_1, \ldots, a_i\},$$

$$\forall j \in \{1, \ldots, l-1\}\ \mu_{j+1} \in S(\mu_j), \mu_{il} \in \left(\overset{\circ}{BR^{-1}}(\delta_{a_{i+1}})\right).$$

Then $\exists \mu_{i+1}$ such that $Supp(\mu_{i+1}) = \{a, a_1, \ldots, a_i, a_{i+1}\}$ and $BR(\mu_{i+1}) = BR\left(\delta_{a_{i+1}}\right) \Rightarrow \exists \mu_{n-1} \in \Delta(\overset{\circ}{Supp}(L))$. $\square$

From an interior point an indeterminate process can reach any point with same support as the limit set. Thus the limit set should be convex.

Lemma 4 *Let L_ε be any limit set of an ε- indeterminate process. Then L_ε is convex.*

Proof Let $\mu, \mu' \in \Delta(Supp(\mu))$ be any pair of population distributions such that $Supp(\mu) = Supp(L_\varepsilon)$ and $\mu \in L_\varepsilon$. Take a finite sequence $\{\mu_0, \ldots, \mu_T\}$ for some T such that $\forall i \in \{0, \ldots, T\}$ $\mu_i \in [\mu, \mu']$, $\mu_0 = \mu$, $\mu_T = \mu'$ and $\forall i \in \{1, \ldots, T\}$ $d(\mu_{i-1}, \mu_i) = \frac{\varepsilon}{2}$. Note that $\mu_i \in S^i(\mu_0, \varepsilon)$ then $\mu_T \in S^T(\mu_0, \varepsilon) \subseteq S^\infty(\mu, \varepsilon) = L_\varepsilon$. Then as μ' is arbitrary $\Delta(Supp(L)) = L_\varepsilon$. □

Following theorem characterizes limit sets of indeterminate processes.

Theorem 1 *Let $X \subseteq \Delta(A)$ be a limit set for $\varepsilon-$indeterminate process where $\varepsilon > 0$. Then there exists a minimal CURB set $C \subseteq A$ s.t $X = \Delta(C)$.*

Proof This trivially follows from lemma 4 and proposition 2. □

In next chapter, we consider the case where the population distribution evolves in the direction of best-replies.

CHAPTER 6

Determinate Processes

We have argued that minimal CURB sets are natural sets to start the analysis with. However, minimal CURB sets are not perfect candidates for limit sets of determinate bets-reply processes. Kalai and Samet (1984) consider Nash Retracts, which could be regarded as a generalization of limit sets except that they require Nash Retracts to be convex, effectively they considered only minimal CURB sets by Proposition 2. However, as the following example illustrates, there may be limit sets outside minimal CURB sets.

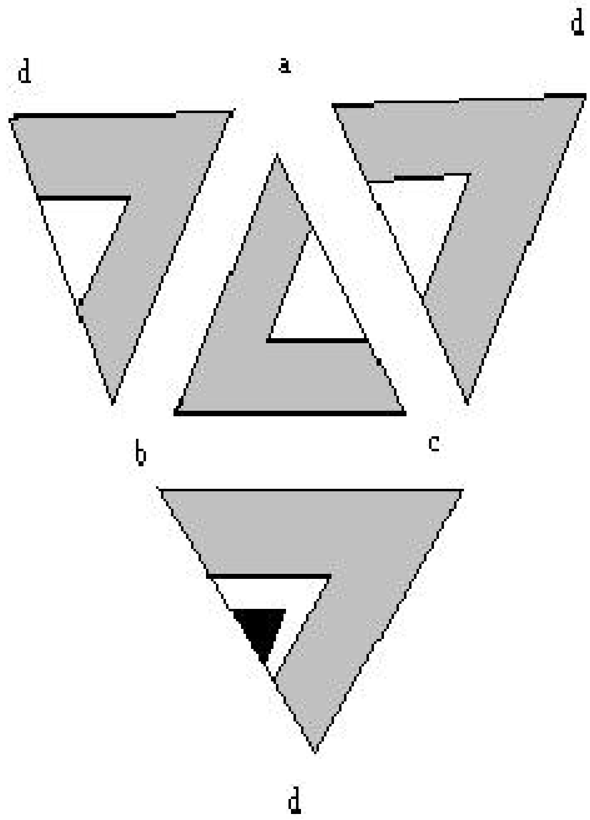
Example 5 *Consider the following game:*

1\2	*a*	*b*	*c*	*d*	*e*
a	*8*	*1*	*2*	*10*	*0*
b	*10*	*8*	*1*	*2*	*0*
c	*2*	*10*	*8*	*1*	*0*
d	*1*	*2*	*10*	*8*	*0*
e	*1*	*6*	*1*	*6*	*1*

.

This game has two limit sets. First one is a minimal CURB set and consists of $\{e\}$*. The second one is in* $\Delta(\{a,b,c,d\})$*. The set of points in* $\Delta(\{a,b,c,d\})$ *to which e is best response is not contained in the limit set, so is not accessible from any pure strategy. The triangles are faces of the simplex*

$\Delta(\{a,b,c,d\})$. *The grey region represents the limit set in* $\Delta(\{a,b,c,d\})$. *The small black triangle is the region that e is a best-response.*



As the example and proposition 1 suggests there may be more limit sets than minimal CURB sets in a given game. Since minimal CURB sets are not places to look for a limit set, we have to find another set-valued solution concept. Kosfeld, Drost, and Voorneveld (2002)'s definition of CURB sets gives an insight about which sets we should consider. We will define pre-CURB sets which is basically a slightly stronger pure-strategy best-reply cycle.

Definition 7 *Let* C *be a subset of* A. C *is called a pre-CURB set if and only if for any strategy* a *in* C $BR\left(\left[\delta_a, \delta_{BR(\delta_a)}\right]\right) \subseteq C$. C *is called minimal pre-CURB set iff there is not proper subset* C' *of* C *such that* C' *is also a pre-CURB set.*

As the following remark indicates, there is a relationship between pre-CURB set and limit sets.

Remark 2 *Let $L \subseteq \Delta$ be a limit set. Then $Supp(L)$ is a pre-CURB set.*

The proof follows from the absorbing property of limit sets. We can now find an upper and lower bound for number of limit sets.

Proposition 3 *Let $\mathbb{PC}$ be the set of minimal pre-CURB sets, $\mathbb{C}$ be the set of minimal CURB sets, and $\mathbb{L}$ be the set of limit sets. Then $|\mathbb{PC}| \geq |\mathbb{L}| \geq |\mathbb{C}|$.*

The proof of the proposition follows directly from remark 2 and proposition 1.

To apply some of the results below one has to be able to find the limit sets of a given game. Limit sets are minimal absorbing sets, but that does not help much. By using the lemma above one can eliminate pure strategies that are not elements of any minimal pre-CURB sets. Then one calculates $S^{\infty}(\delta_a)$ for any pure strategy a in any minimal pre-CURB set C. The following algorithm will be helpful in finding limit sets.

Algorithm 1 *Start with $\mu^0 = \delta_a$ for any $a \in C$ where C is a minimal pre-CURB set. We will calculate a branching path strating from μ^0, so $\forall i \in \{1, 2, \ldots\}$ μ^i is a set of points in $R^{|A|}$. Now $\forall b \in BR(\mu^0)$, $\forall c \in A \backslash \{b\}$ such that $BR^{-1}(c) \cap S(\mu^0) \neq \emptyset$. Let $\mu^0_{bc} \in \mathbb{R}^{|A|}$ be defined as*

$$\mu^0_{bc}(c) := \min_{\mu \in BR^{-1}(c) \cap S(\mu^0)} \mu(c) \quad \textit{and} \quad \forall d \in A \backslash \{c\} \;\; \mu^0_{bc}(d) := \max_{\mu \in BR^{-1}(c) \cap S(\mu^0)} \mu(d) \quad .$$

Notice that μ^0_{bc} may not belong to Δ. Define

$$S\left(\mu^0_{bc}\right) := \left\{\mu \in \Delta : \forall d \in A \backslash \{b\} \;\; \mu(d) \leq \mu^0_{bc}(d)\right\}.$$

Then

$$\left(BR^{-1}(c) \cap S\left(\mu^0\right)\right) \subset S\left(BR^{-1}(c) \cap S\left(\mu^0\right)\right) \subseteq S\left(\mu^0_{bc}\right).$$

And define

$$\mu^1 := \left\{ \begin{array}{c} \mu^0_{bc} \in \mathbb{R}^{|A|} : \forall b \in BR\left(\mu^0\right) \;\; \textit{and } \forall c \in A \backslash \{b\} \\ \textit{such that } BR^{-1}\left(c\right) \cap S\left(\mu^0\right) \;\; \neq \emptyset \end{array} \right\}.$$

$\forall i \in \{2, 3, \ldots\}$ *calculate* μ^i_{bc} *as in* μ^1_{bc}*, however define* μ^i *as follows*

$$\mu^1 := \left\{ \begin{array}{c} \mu^0_{bc} \in R^{|A|} : \forall b \in A \;\; \textit{such that } BR^{-1}\left(a\right) \cap S\left(\mu^{i-1}\right) \cap S\left(\mu^{i-2}\right) \neq \emptyset \\ \textit{and } \forall c \in A \backslash \{b\} \;\; \textit{such that } BR^{-1}\left(c\right) \cap S\left(\mu^0\right) \neq \emptyset \end{array} \right\}.$$

Note that this algorithm is sufficient but not necessary as $\forall i \; S\left(\mu^i_{bc}\right)$ might be larger than $S\left(BR^{-1}\left(c\right) \cap S\left(\mu^{i-1}\right)\right)$. To make algorithm necessary one has to calculate $\left(S\left(\mu^i_{bc}\right) \cap S\left(\mu^{i-1}\right)\right) \setminus \left(BR^{-1}\left(c\right) \cap S\left(\mu^{i-1}\right)\right)$. If this set is empty there is no problem; if not, excluding successor of this set will do the job.

Both pre-CURB sets and CURB sets are set valued solution concepts. We have found relationships betweenlimit sets with these solution concepts. One natural question at this point that what is the relation between limit sets and singleton solution concepts. The first natural candidate is Nash Equilibria. We will prove that some Nash Equilibria are "close" to a limit set. Before stating and proving this result we need to formalize the concept of closeness in best-reply processes.

By the closure of a limit set we mean the set of distributions that can be reached from the limit set with "zero cost"; that is, the set of points such that the minimum prabability that is needed for reaching to any point in this set with mutations is zero. We will call this set limit set with finite mutations. If N is infinite, this set corresponds to the usual closure of the limit set.

If N is finite, we will define a discrete probability distribution on the possible number of individuals who may mutate. We will keep the maximum number limited such that even in the extreme case that all possible mutations occur, only a small part of the population will mutate. The probability distribution is

$\eta : \{0,1,\dots,m\} \longrightarrow [0,1]$ s.t $\sum_{i=0}^{m} \eta(i) = 1$ and $\eta(i) > 0$ $\forall i \in \{0,1,\dots,m\}$ where $m > 0$ is fixed for all population sizes N. Then the successors with mutations of any μ is:

$$\tilde{S}(\mu, N) = S\left(B_{\frac{m}{N}}(\mu)\right).$$ [1]

Then given any limit set L a *m-transitional limit set* is defined as:

$$\tilde{L}^N = \bigcup_{\mu \in L} \tilde{S}(\mu, N)$$

and a *finite transitional limit set* is defined as:

$$L_\infty^+ = \lim_{N \to \infty} \tilde{L}^N$$

any point in $L_\infty^+ \cap \Delta_N$ can always be reached with a finite number of mutations as $N \to \infty$, and this set is unique. Thus our *transitional limit set* is

$$\bar{L} = L_\infty^+ \cap \Delta_N.$$

Note that analyzing the closure $\bar{L}$ of a limit set is equivalent to analyzing transitional limit set, since each point in the boundary of the limit set is one mutation away from the limit set. Let μ^* be Nash Equilibrium whose support is contained in the support of limit set. We will show that if best-reply of any element in the support of μ^* is an element of the set of best-replies of μ^*, then μ^* belongs to the closure of the limit set.

Theorem 2 *Let μ^* be a Nash Equilibrium such that $Supp(\mu^*) \subseteq Supp(L)$. Then if $\forall a \in Supp(\mu^*)$ $BR(\delta_a) \in BR(\mu^*)$ then $\mu^* \in \bar{L}$.*

Before proving the theorem, we will need following two lemmas.

[1] this is not the exact definition, but for large N the difference will not affect any result.

Lemma 5 $\forall a \in Supp\,(L)$, $\forall \mu_1, \mu_2 \in [\delta_a, \mu^*]$

$$d(\mu_2, \mu^*) < d(\mu_1, \mu^*) \Rightarrow \inf_{\mu \in S(\mu_2) \cap [\delta_b, \mu^*]} d(\mu, \mu^*) < \inf_{\mu \in S(\mu_1) \cap [\delta_b, \mu^*]} d(\mu, \mu^*)$$

where $d(\cdot, \cdot)$ *is the usual metric defined over* $R^{|A|}$.

Proof $\forall a \in Supp\,(L)$, $\forall \mu_1, \mu_2 \in [\delta_a, \mu^*]$ we will first show that

$$d(\mu_2, \mu^*) < d(\mu_1, \mu^*) \Leftrightarrow \mu_2(a) < \mu_1(a).$$

For each $i = 1, 2$ $\exists \lambda_i \in [0, 1]$ $\mu_i = \lambda_i \delta_a + (1 - \lambda_i)\mu^*$. Thus $d(\mu_2, \mu^*) < d(\mu_1, \mu^*) \Rightarrow$

$$\lambda_2 < \lambda_1 \Rightarrow \mu_2(a) = \lambda_2 + (1 - \lambda_2)\mu^*(a) < \lambda_1 + (1 - \lambda_1)\mu^*(a) = \mu_1(a).$$

Following the steps in reverse order proves the converse.

Let $\bar{\mu}_1, \bar{\mu}_2 \in [\delta_b, \mu^*]$ be such that $d(\bar{\mu}_i, \mu^*) = \inf_{\mu \in S(\mu_i) \cap [\delta_b, \mu^*]} d(\mu, \mu^*)$. Note that $\forall i \in \{1, 2\}$ $\bar{\mu}_i$ exists and unique since both and $S(\mu_i)$ and $[\delta_b, \mu^*]$ are compact and $[\delta_b, \mu^*]$ is a line. Then

$$\bar{\mu}_i(b) = \inf_{\mu \in S(\mu_i) \cap [\delta_b, \mu^*]} \mu(b).$$

This requires that we will choose $\bar{\mu}_i$ such that $\bar{\mu}_i(b)$ is minimum in $S(\mu_i) \cap [\delta_b, \mu^*]$. Since $BR(\mu_1) = BR(\mu_2) = b$, share of b must increase while passing from μ_i to $\bar{\mu}_i$. So we will keep this increase at minimum.

Let $\mu_i' \in [\delta_b, \mu^*]$ be such that

$$\forall c \in A \backslash \{a, b\} \;\; \mu_i'(c) = \mu_i(c), \mu_i'(a) = \mu_i(b)\, and \mu_i'(b) = \mu_i(a).$$

Note that μ_i' is uniquely defined for each $i \in \{1, 2\}$. Since $\forall \mu \in [\delta_b, \mu^*]$ $\forall c, c'$ $\in Supp\,(L) \setminus \{b\}$ $\mu(c) = \mu(c')$, $\forall \mu \in [\delta_b, \mu^*]$ such that $d(\mu, \mu^*) < d(\mu_i', \mu^*)$

$\Rightarrow \mu(b) < \mu_i'(b) \Rightarrow$

$$\forall c \in Supp(L) \setminus \{b\} \;\; \mu(c) > \mu_i'(c).$$

Then $\mu_i' = \bar{\mu}_i$. But then $\mu_2(a) < \mu_1(a) \Rightarrow \bar{\mu}_2(b) < \mu_1(b) \Rightarrow d(\bar{\mu}_2, \mu^*) < d(\bar{\mu}_1, \mu^*)$. $\square$

Before proceeding to the next lemma, we have to define the following; given $Supp(\mu^*) \subseteq Supp(L)$ $\forall a \in Supp(L)$, let $\bar{\mu}_a \in [\delta_a, \mu^*]$ be such that $d(\bar{\mu}_a, \mu^*) = \inf_{\mu \in L \cap [\delta_a, \mu^*]} d(\mu, \mu^*)$. The next lemma states that there is no such a maximal point in the limit set that is not equal to μ^*.

Lemma 6 *Either for any $a \in Supp(L)$ $BR(\delta_a) \in BR(\bar{\mu}_a)$ or for any $a \in Supp(L)$ $\bar{\mu}_a \neq \mu^*$.*

Proof Assume

$$[[\forall a \in Supp(L) \;\; BR(\delta_a) \in BR(\bar{\mu}_a)] \wedge [\forall a \in Supp(L) \;\; \bar{\mu}_a \neq \mu^*]],$$

and let $a \in Supp(L)$ be arbitrary. For $\bar{\mu}_a$ $\exists \lambda \in [0,1]$ such that $\forall b \in Supp(L) \setminus \{a\}$ $\bar{\mu}_a(b) = \lambda \mu^*(b)$. Then let $\mu_b^\lambda \in [\delta_b, \mu^*]$ be such that $\forall c \in Supp(L) \setminus \{b\}$ $\mu_b^\lambda(c) = \lambda \mu^*(c)$. Now we will consider

$$Co\left\{\mu_b^\lambda : b \in Supp(L)\right\}$$

as our new simplex. Note that $\mu^* \in Co\left\{\mu_b^\lambda : b \in Supp(L)\right\}$ and

$$\forall \mu \in \partial(\Delta(Supp(L)))\, \exists \mu^\lambda \in \partial\left(Co\left\{\mu_b^\lambda : b \in Supp(L)\right\}\right)$$

such that $\mu^\lambda = \lambda\mu^* + (1-\lambda)\mu$. If $BR\left(\mu^\lambda\right) \cap Supp(L) = \emptyset$ for some $\mu^\lambda \in S^\infty(\bar{\mu}_a)$ $\exists \hat{\mu} \in L$ close enough to $\bar{\mu}_a$ such that $\exists \hat{\mu}^\lambda \in S^\infty(\hat{\mu})$ such that $BR\left(\hat{\mu}^\lambda\right) \cap Supp(L) = \emptyset$ by lemma 5, which is a contradiction. Moreover let $\mu \in \partial(\Delta(Supp(L)))$ such that $|BR(\mu)| = 1$, then $|BR(\mu^\lambda)| = 1$. Since otherwise

$$\exists \{\mu_i^\lambda\}_{i\geq 1} \subset Co\{\mu_b^\lambda : b \in Supp(L)\}$$

such that $BR(\mu_i^\lambda) \neq BR(\mu^\lambda)$ and $\mu_i^\lambda \to \mu^\lambda$ as $i \to \infty$. But by convexity of best-reply regions and by hypothesis μ^*, $\exists \{\mu_i\}_{i\geq 1} \subset \partial(\Delta(Supp(L)))$ such that $BR(\mu_i) \neq BR(\mu)$ and $\mu_i \to \mu$ as $i \to \infty$, which is impossible by 2^{nd} non-degeneracy assumption. Then we can extend the proof in lemma 3 to $Co\{\mu_b^\lambda : b \in Supp(L)\}$. Then $\exists \mu \in Co\{\mu_b^\lambda : b \in Supp(L)\}$ such that $\forall b \in Supp(L)\ \mu(\mu_b^\lambda) > 0$. But by again repeated use of the second non-degeneracy assumption $\exists \varepsilon > 0$ such that

$$(B_\varepsilon(\bar{\mu}_a) \cap \Delta(Supp(L))) \subseteq S(\mu')$$

for some $\mu' \in Co\{\mu_b^\lambda : b \in Supp(L)\} \Rightarrow$

$$\exists \bar{\mu}'_a \in [\delta_a, \mu^*] \cap L \text{ such that } d(\bar{\mu}'_a, \mu^*) < d(\bar{\mu}_a, \mu^*),$$

which is a contradiction with the definition of $\bar{\mu}_a$. □

With these two lemmas, the proof of the Theorem is immediate.

Proof [Proof of Theorem] $Supp(\mu^*) = Supp(L) \Rightarrow u(\delta_a, \mu^*) = u(\delta_b, \mu^*)\ \forall a, b \in Supp(L)$. Now since

$$[[\forall a \in Supp(L)\quad BR(\delta_a) \subseteq BR(\bar{\mu}_a)] \wedge [\forall a \in Supp(L)\quad \bar{\mu}_a \neq \mu^*]]$$

leads to a contradiction, either $\exists a \in Supp(L)$ such that $\bar{\mu}_a = \mu^*$, in which case we are done, or $\exists a \in Supp(L)$ such that $BR(\bar{\mu}_a) \cap Supp(L) = \emptyset$, which is a contradiction since in that case $\exists \mu \in L$ close enough to $\bar{\mu}_a$ such that $BR(\mu) \cap Supp(L) = \emptyset$ by lemma 5. Thus if μ^* is not a Nash Equilibrium, then we will have a contradiction by second lemma above. Moreover, μ^* is a

Nash Equilibrium $\Rightarrow [\forall a \in Supp(L)\ \bar{\mu}_a \neq \mu^*]$ is wrong so $\mu^* \in \bar{L}$. $\square$

The condition in the theorem is sufficient; however, as the following example shows this condition is not necessary:

Example 6 *Consider the following game:*

1\2	a	b	c
a	5	10	10
b	10	5	5
c	11	3	3

The only Nash Equilibrium in this game $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ *lies in the interior of the limit set.*

Although the requirement in the theorem is not necessery, it is critical. When we relax the condition we can find limit sets such that there is no Nash Equilibrium near the limit set. Following is an example to this observation.

Example 7 *Consider the following game:*

1\2	a	b	c	d
a	9	10	3	1
b	3	9	10	9
c	8	5	9	10
d	10	1	1	9

.

In this game the unique limit set lies in the set of distributions over the unique minimal CURB set $\{a, b, c, d\}$. *Here there are three Nash Equilibria, which are* $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$, $(0.858, 0, 0.142, 0)$, $\left(\frac{64}{174}, \frac{8}{25}, 0, \frac{11}{35}\right)$, *and none of them belongs to the closure of the limit set. At this stage one can ask whether there exists a Nash Equilibrium in* $\Delta(Supp(L))$ *necessarily for any limit set* L. *Answer to that question will be negative. A modification of the example above will* *provide a counter-example to this assertion:*

1\2	a	b	c	d	e_1	e_2	e_3
a	9	10	3	1	0	0	0
b	3	9	10	9	0	0	0
c	8	5	9	10	0	0	0
d	10	1	1	9	0	0	0
e_1	7	7	7	0	0	0	0
e_2	7	7	0	7	0	0	0
e_3	$7+\varepsilon$	0	$7+\varepsilon$	0	1	1	1

where $\varepsilon > 0$. *In this game there are two limit sets one in* $\Delta(\{a,b,c,d\})$ *and the other is* $\{e_3\}$. *However, there is a unique Nash Equilibrium* e_3. *Thus there is no Nash Equilibrium in* $\Delta(\{a,b,c,d\})$.

The theorem 2 provides a sufficient condition for a Nash Equilibrium to be included in closure of a limit set. Oechssler (1997) gives a sufficient condition which he calls evolutionary stability for pure strategies (ESPS) for a Nash Equilibrium to be in the limit set. However he defines ESPS for the unique, full support Nash Equilibrium. Thus ESPS is not a natural restriction for the general case we analyze. We will modify that condition in the next definition. Before that we have to define the most overrepresented strategies. let C be a nonempty subset of A. For any $\mu, \mu^* \in \Delta(C)$ $O(\mu,\mu^*)$ represents the set of strategies that are the most overrepresented ones with respect to μ^*at μ; that is,

$$\forall \mu \in \Delta \;\; O(\mu,\mu^*) = \{a \in A : \mu(a) - \mu^*(a) \geq \mu(b) - \mu^*(b) \quad \forall b \in A\}.$$

Definition 8 μ^* *satisfies evolutionary stability for pure strategies (ESPS) if for any* $\mu \in \Delta(Supp(\mu^*)) \setminus \{\mu^*\}$ *there exists* $a \in O(\mu,\mu^*)$ *and* $b \in Supp(\mu^*) \setminus O(\mu,\mu^*)$ *such that* $a \notin BR(\mu)$ *and* $b \in BR(\mu)$.

Note that this means that μ^* does not have any Nash Equilibrium that have support on subset of μ^*.

Let μ^* be a Nash Equilibrium that satisfies ESPS, then it does not necessarily follow that there is a limit set in $\Delta(Supp(\mu^*))$. The following proposition gives a characterization regarding existence of a limit set in $\Delta(Supp(\mu^*))$. The proof follows from a modification of the proof in Oechssler (1997).

Proposition 4 *Let μ^* be a Nash Equilibrium that satisfies ESPS. Then the set of population distributions over $Supp(\mu^*)$ is a limit set if and only if $Supp(\mu^*)$ is a minimal CURB set.*

Proof Suppose $\exists$ a limit set $L = \Delta(Supp(\mu^*))$. If there exists $\mu \in \Delta(Supp(\mu^*))$ such that $BR(\mu) \nsubseteq Supp(\mu^*)$ then $L = \Delta(Supp(\mu^*))$ would be wrong.

Suppose $Supp(\mu^*)$ is a minimal CURB set. Then there is a limit set such that

$$L \subseteq \Delta(Supp(\mu^*)).$$

Let $a_1 \in Supp(L)$ be arbitrary and $\mu^0 := \delta_{a_1}$. If $\delta_{a_1} = L$, we are done.Otherwise $O(\mu^0, \mu^*) = \{a_1\}$. Then $\exists a_2 \in Supp(L)$ such that $a_2 = BR(\mu^0)$. If $\{a_1, a_2\} = Supp(L)$, we are done. Otherwise let μ^1 be such that

$$\mu^1(a_1) = \mu^*(a_1)\, and \mu^1(a_2) = 1 - \mu^*(a_2) \Rightarrow O\left(\mu^1, \mu^*\right) = \{a_2\}.$$

So we have two cases. In the first case, $a_1 \in BR(\mu^1) \Rightarrow \exists\bar{\mu}^1 \in [\delta_{a_1}, \mu^1]$ such that $O(\bar{\mu}^1, \mu^*) = \{a_1, a_2\}$ or $\{a_1, a_2\} \subseteq BR(\bar{\mu}^1) \Rightarrow \bar{\mu}^1 = \mu^*$ in which case we are done. Thus suppose $O(\bar{\mu}^1, \mu^*) = \{a_1, a_2\} \Rightarrow \exists a_3 \in Supp(L) \setminus \{a_1, a_2\}$ such that $a_3 \in BR(\bar{\mu}^1)$, then proceed. In the second case $a_1 \notin BR(\mu^1)$ then proceed.

Let $\mu^2 = (\mu^*(a_1),\ \mu^*(a_2),\ 1 - \mu^*(a_1) - \mu^*(a_2),\ 0, \ldots, 0)$. If $\{a_1, a_2, a_3\} = Supp(\mu^*)$ then we are done. Otherwise

$$O\left(\mu^2, \mu^*\right) = \{a_3\} \Rightarrow a_3 \notin BR\left(\mu^2\right).$$

If $\{a_1, a_2\} \subset BR(\mu^2) \Rightarrow \exists \bar{\mu}^2$ such that

$$\bar{\mu}^2(a_i) - \mu^*(a_i) = \bar{\mu}^2(a_j) - \mu^*(a_j) \quad \forall i, j \in \{1, 2, 3\}$$

$\Rightarrow \exists\, a_4 \in Supp(L) \setminus \{a_1, a_2, a_3\}$ such that $a_4 \in BR(\bar{\mu}^2)$ then proceed.

If $\{a_1, a_2\} \nsubseteq BR(\mu^2)$, and if a_1,$a_2 \notin BR(\mu^2)$ then proceed. Otherwise WLOG assume that $a_1 \in BR(\mu^2)$ but $a_2 \notin BR(\mu^2)$. Then

$$\exists \hat{\mu}^2 \in S\left(\mu^2\right) \text{ such that } O\left(\hat{\mu}^2, \mu^*\right) = \{a_1, a_2\}.$$

If a_1,$a_3 \notin BR(\hat{\mu}^2)$ then proceed. Otherwise WLOG assume that $a_1 \in BR(\hat{\mu}^2)$ but $a_3 \notin BR(\hat{\mu}^2)$. We have two cases here. In the first case $a_2 \in BR(\hat{\mu}^2) \Rightarrow \bar{\mu}^2 \in S(\hat{\mu}^2)$ then proceed. In the second case

$$a_2 \notin BR\left(\hat{\mu}^2\right) \Rightarrow \exists a_4 \in Supp(L) \setminus \{a_1, a_2, a_3\}$$

such that $a_4 \in BR(\bar{\mu}^2)$ then proceed. Continuing in this fashion we get $\mu^* \in L$. But this implies that $L = \Delta(Supp(\mu^*))$. □

CHAPTER 7

Stability

We have proved that in remark that limit sets are minimal absorbing sets. This result can be interpreted as that limit sets are stable in the sense that once the population enters a limit set then it does not leave the set. However, as in the notion of "perfectness" in game theory, a common understanding of stability in economic theory is such that stability is a stronger condition than being absorbing. We will call a limit set stable if and only if "small" deviations from the limit set does not lead to large deviations in the long-run. But before giving the rigorious definition of stability, we have to formalize what "small" is.

If N is continuum, small deviations from limit set L is just for small $\varepsilon > 0$ $B_{\varepsilon}(L) := \cup_{\mu \in L} B_{\varepsilon}(\mu)$. If N is finite, we will be playing with population size N, since for small populations no deviation is small. Then $\forall (\varepsilon, N) \in \mathbb{R} \times \mathbb{N}$ such that $\varepsilon > \frac{1}{N}$

$$B_{\varepsilon}(L) := \left\{ \mu \in \Delta : \exists \mu' \in L \text{ such that } \sum_{a \in A} |\mu(a) - \mu'(a)| < \varepsilon \right\}.$$

Now we can define stability as follows:

Definition 9 *Let L be a limit set. Then L is stable if and only if* $\lim_{\varepsilon \to 0} S^{\infty}(B_{\varepsilon}(L)) = S^{\infty}(\bar{L})$ *where $\bar{L}$ is closure of the limit set L.*

In the proposition 1, we claimed that inertia matters in terms of limit sets. However, for stable limit sets inertia does not induce any difference; stable all-best limit sets are also stable no-worse limit sets.

Lemma 7 *No non-strict Nash Equilibrium constitutes a stable all-best limit set.*

Proof of this lemma follows directly from definition of a non-strict Nash Equilibrium. This lemma proves following proposition

Proposition 5 *Assume that* $L_{ab} \subseteq L_{nw}$ *then* L_{ab} *is stable if and only if* L_{nw} *is stable.*

This proposition is a necessary condition for limit set for being stable. Following proposition gives a useful sufficient condition for a no-worse limit set to be stable.

Proposition 6 *If* L *is a non-singleton convex limit set, then* L *is stable.*

Proof By proposition 2 $\exists$ a minimal CURB set C such that $L = \Delta(C)$. But then $\exists \varepsilon > 0$ such that $\forall \varepsilon' \in (0, \varepsilon)$ $BR(B_{\varepsilon'}(\Delta(C))) \subseteq C$. Then $S(B_{\varepsilon'}(\Delta(C))) \subseteq B_{\varepsilon'}(\Delta(C))$, which proves the result after applying induction over degrees of the successor correspondence. $\square$

CHAPTER 8

Conclusion

We have analyzed limit sets of no-worse and all-best processes. We have found that limit sets of these two processes differ. Then for no-worse processes, we showed that minimal CURB sets are not perfect candidates of limit sets for determinate processes. However, indeterminateness and ESPS may justify minimal CURB sets as limit sets under some circumstances. Moreover, we proves that full support Nash Equilibria are close to limit sets, but a characterization of this relation remains to be an interesting open problem.

As we note in the chapter 5, one can define and study stochastically stable limit sets within this framework. Such a study will be a generalization of Kandori Mailath and Rob (1993) and Young (1993). However, we conjecture that this generalization will be costly in the sense that many results in those studies will not hold. Such a study will be a considerable contribution to the literature of learning in games.

As an application of this paper, one can study dynamic general equilibrium models in the theory of economic growth. This framework can be regarded as an alternative approach to representative-agent framework for such models. One can ask under which conditions limit sets coincide competitive equilibria. Moreoever, it would be interesting to know that when they do not concide, whether cylic behavior in non-singleton limit sets could explain business cycles as deterministic movements.

We have considered best-reply processes in this paper. As Josephson et al. showed that different behavior types lead to different minimal closed sets. However, limit sets of processes that are generated from a behavior type might not coincide with the corresponding minimal closed sets. Understanding when these two sets coincide for different learning processes is crucial for forming a general theory of distributional learning dynamics. However, there is so much to do compared to things that have been done in this literature, as many fields in game theory.

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