REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

A BRIEF REVIEW OF TRANSFERRING OF TRANSIENT SIGNALS ALONG HOLLOW WAVEGUIDE

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MSc. THESIS
DEPARTMENT OF MATHEMATICAL ENGINEERING
PROGRAM OF MATHEMATICAL ENGINEERING

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LIST OF SYMBOLS

$\mathbf{R}$	3-component position vector
∇	3 −component Nabla operator
$\mathbf{r}$	2-component position vector
$\nabla_{\perp}$	Transverse part of ∇
∂_t	Partial derivative operator with respect to time
$\mathbf{z}$	Unit vector directed along the Oz-axis
$\mathbf{n}$	Unit vector outer normal to domain S
$\mathbf{l}$	Unit vector tangential to the contour L
$\mathbf{E}$	Electric field vector
$\mathbf{H}$	Magnetic field vector
ε_0	Permitivity of free space
μ_0	Permeability of free space
c	Speed of light in space
υ_m	Eigenvalues of Neumann problem
ψ_m	Eigenvectors of Neumann problem
κ_m	Eigenvalues of Dirichlet problem
ϕ_m	Eigenvectors of Dirichlet problem
$\mathbf{E}_m(\mathbf{R}, t)$	Transverse part of Electric field vector
$\mathbf{H}_m(\mathbf{R}, t)$	Transverse part of Magnetic field vector
$h_m(z, t)$	Time-dependent modal amplitudes of magnetic field vector
$e_m(z, t)$	Time-dependent modal amplitudes of electric field vector
A_m^h	Normalization constant of magnetic field
A_m^e	Normalization constant of electric field

LIST OF ABBREVIATIONS

EAE	Evolutionary Approach to Electromagnetics
FDTD	Finite Difference Time Domain
KGE	Klein-Gordon Equation
TE	Transverse Electric
TM	Transver Magnetic

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ABSTRACT

A BRIEF REVIEW OF TRANSFERRING OF TRANSIENT SIGNALS ALONG HOLLOW WAVEGUIDE

Sedef ŞEMŞİT

Department of Mathematical Engineering

MSc. Thesis

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In this thesis, the purpose is to bring together the studies done before. Evolutionary Approach to Electromagnetics Theory (EAE), which is an anlytical method and gives the solutions in directly time-domain, is aimed to use. Two main problems are introduced to analyze transferring of transient signals along hollow waveguide. First one, Modal Basis Problem corresponds to Neumann and Dirichlet boundary-eigenvalue problems. These problems gives us eigenfuctions corresponds to eigenvalues. Second one is Modal Amplitude Problem which derived from Klein-Gordon equations. As a result of this, transverse electric (TE) and transverse magnetic (TM) modes are obtained in time-domain. Each component of these modal fields is expressed as a product of modal base element which is a vector function of transverse coordinates and a scalar modal amplitude that depends on time t and axial coordinate z. Also, in this study, Bessel and Airy functions are analyzed in detail and they are used to expressed Modal amplitudes.

Key words: Maxwell’s equation, Neumann and Dirichlet boundary-eigenvalue problems, Klein-Gordon equation, Airy functions, Bessel functions

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ÖZET

BOŞ DALGA KLAVUZU BOYUNCA GEÇİCİ SİNYALLERİN İLETİLMESİNİN KISA BİR İNCELEMESİ

Sedef ŞEMŞİT

Matematik Mühendisliği Anabilim Dalı

Yüksek Lisans Tezi

Tez Danışmanı: Doç. Dr. Kevser KÖKLÜ

Eş Danışman: Yrd. Doç. Dr. Emre EROĞLU

Bu tezde amaç daha önce yapılmış çalışmaları bir araya getirmektir. Analitik bir metod olan ve çözümleri direk olarak zaman-domeininde veren Elektromanyetik Teoriye Evrimsel Yaklaşım (ETEY) metodunu kullanmak amaçlanmıştır. Dalga kılavuzu boyunca geçici sinyallerin iletilmesini incelemek için iki temel problem tanıtılmıştır. Birincisi, Neumann ve Dirichlet sınır-özdeğer problemlerine karşılık gelen Modal Baz Problemidir. Bu problemlerden özdeğerlere bağlı özfonksiyonlar elde edilmiştir. İkincisi ise Klein-Gordon denkleminden elde edilen Modal Genlik Problemidir. Bunların sonucunda enlemsel elektrik (TE) ve enlemsel manyetik modları zaman-domeininde elde edilmiştir. Bu modal alanların her bir elemanı, boylamsal koordinatların vektör fonksiyonu olan modal baz ve t zaman ve z eksenel koordinatına bağlı olan modal genlik çarpımıyla ifade edilmişlerdir. Ayrıca bu çalışmada Bessel ve Airy fonksiyonları deletaylı olarak incelenmiş ve modal genlikler bu fonksiyonlar yardımıyla bulunmuştur.

Anahtar kelimer: Maxwell denklemleri, Neumann ve Dirichlet sınır-özdeğer problemleri, Klein-Gordon denklemi, Airy fonksiyonları, Bessel fonksiyonları.

YILDIZ TEKNİK ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ

CHAPTER 1

INTRODUCTION

1.1 Literature Review

In Electromagnetics, there are three canonical boundary value problem with given initial conditions for the electromagnetic field sought, namely: *Cavity Problem*, *Waveguide Problem*, and *External Problem*. In this thesis, the waveguide problem which is conanical problem is studied in a subdomain of Euclidean space.

The time-domain studies in electromagnetic field theory can be investigated in two groups, in general. The first one is numerical methods; Finite Difference Time Domain (FDTD) is the most popular one among these methods. Efficiency of the method is abundance of data which has obtained during oscillation. Compariable process is permitted to reach exact results via rich data . The second one is analytical methods that divided into two subgroups. The first one consists of frequency domain usually performed concerned with Fourier integral transform or Laplace transform. The solutions obtained from the Fourier integral transform is needed to find time-domain solutions in the final step. A new alternative (to the time-harmonic field concept) approach in studying the time-domain modes was developed within the framework of four-dimensional relativistic formalism in electrodynamics.

Eventually, one more alternative approach called *Evolutionary Approach to Electromagnetics* (EAE) was suggested in 80s. It is destined for the time-domain theory of both the cavity and waveguide modes. The theory is based on evolution equations was kept ∂t in time-domain of Maxwell's equations and Neumann-Dirichlet boundary-value problems, analytically. In the method, electromagnetic fields are solved by extracting Maxwell's equation system in time-domain. The solution is attained by observing along waveguide

The EAE method is based on the solution of sequential two autonomous problems, basically. The first one is a “modal basis problem” corresponds to well-studied Dirichlet and Neumann boundary eigenvalue problems. This involves two complete sets as the Transverse-Electric (TE) and Transverse-Magnetic (TM) modes. Because of the fact that the generating scalar potentials for the TE and TM modes are actually the same as the time-harmonic modes, one can freely use the methods, which have been developed in the frequency domain, and use as well even ready results obtained for the complex waveguide configurations. The second one is a “time-dependent modal amplitude problem” corresponds to Klein-Gordon Equation (KGE) with the axial coordinate and time. Main effort of theory is addressed to obtain the analytical solution to the KGE leading to the timedependent modal amplitudes unlike to those in the time-harmonic waves .

1.2 Objective of the Thesis

In this thesis, the purpose is to solve Maxwell’s equation system analytically via Laplacian and evolution equation that preserve ∂_t in time-domain. Electromagnetic fields will be separated from Maxwell’s equation system and solved in time-domain. The solution is obtained by investigating along the waveguide. This study is composed as follows.

In Section II, the above mentioned executive summary is presented, definition about the waveguide is given and time-domain problems are solved

In Section III, the general properties of the waveguide time-domain modes are considered, namely. Completeness of the modal sets is discussed.. Conservation of energy law is given for every time-domain mode. The initial conditions for the modal amplitudes are mentioned. The causality principle is imposed on the time-domain modes.

In Section IV, a particular case of Bessel and Airy functions are considered in detail and time-dependent modal amplitudes, which are expressible via Bessel and Airy functions with time in their arguments, are obtained.

In Section V, graphical results are presented by using Mapple programme.

1.3 Hypothesis

There are lots of study about EAE method. But all these studies fouced on specific titles. Therefore, it is necessary to gather them in one study. So this is the major contribution of this thesis.

As understood from the title of the thesis, we are going to reconsider all the works have been done up to this time. The works are stated below:

- Serkan Aksoy (Aksoy [1]) gave general information about the problem like basis formed in wave boundary operator, projection of vector fields and Maxwell's equation on the basis and numerical applications for rectangular waveguide
- Özlem Akgün (Akgün [2]) found the solution of Klein-Gordon Equation (modal amplitudes) by using Miller's fifth case as a product of Airy function and she investigated relativistic effects on the wave propagation in a hollow waveguide
- Emre Eroğlu (Eroğlu [3]) solved KGE by using Miller's second case as a product of Bessel function, gave detailed solution of Bessel differential equation, studied spherical and cylindrical Bessel function and he also focused on surplus of energy and energy.
- Özlem Işık (Işık [4]) analyzed wave boundary operator and surplus of enery and expressed the modal amplitudes in terms of Airy functions.
- Nevra Eren (Eren [5]) investigated surplus of enery and airy function in detail.

CHAPTER 2

THE PROBLEM of TIME-DOMAIN MODES

2.1 The Waveguide

The waveguide is a device that transfer electromagnetic energy efficiently which means with minimum extinction and degradation. Propagation of electromagnetic fields can be classfied as guided and unguided. The waves in guided propagation are transmitted from one point to another by following a particular path. The waves in unguided propagation spread or beam on an open space. There are three types of waveguide namely; circular, elliptic and rectangular.

We consider a hollow rectangular waveguide (as in Figure 2.1) directed along Oz −axis and bounded by conductive metal walls that means the waveguide is perfect electric conductor.

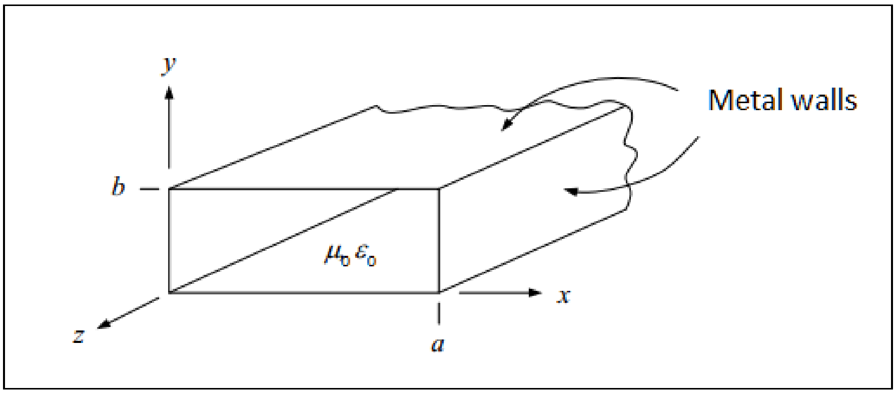

Figure 2.1 The rectangular waveguide along Oz-axis

where ε_0 is dielectric constant for free space (i.e., permitivity) and μ_0 is magnetic constant for free space (i.e., permeability).

2.2 Formulation of the Problem

We consider a hollow waveguide having domain S directed along Oz −axis. The domain S is bounded by closed singly-connected contour L. We also assume that the contour has an arbitrary but smooth enough shape which means any possible inner angles (measured within S) can not exceed π. In the rest of the study, mutually orthogonal unit vectors $(\mathbf{z}, \mathbf{l}, \mathbf{n})$ were used where $\mathbf{z} \times \mathbf{l} = \mathbf{n}$, $\mathbf{l} \times \mathbf{n} = \mathbf{z}$ and $\mathbf{n} \times \mathbf{z} = \mathbf{l}$. The unit vector $\mathbf{n}$ is outward normal to domain S, the unit vectors $\mathbf{z}$ and $\mathbf{l}$ are tangential to the Oz −axis and contour L, respectively.

In this study, 3 −component position vector $\mathbf{R}$ and the operator ∇ are decomposed onto their transverse and longitudinal parts as

$$\mathbf{R} = \mathbf{r} + \mathbf{z}z, \qquad \nabla = \nabla_\perp + \mathbf{z}\partial_z \tag{2.1}$$

where $\mathbf{z}$ is a unit vector directed along Oz −axis, $\mathbf{r}$ is 2 −component position vector in the waveguide cross-section S (and also projection of $\mathbf{R}$ on domain S) and $\nabla_\perp$ is the transverse part of ∇. The differential operator $\nabla_\perp$ transacts only at the coordinates $(\mathbf{r})$ of transver waveguide.

3 −component electromagnetic field strength vectors $\mathbf{E}_m$ and $\mathbf{H}_m$ are introduced

$$\begin{aligned} \mathbf{E}_m(\mathbf{R}, t) &= \mathbf{E}(\mathbf{r}, z, t) + \mathbf{z}E_z(\mathbf{r}, z, t) \\ \mathbf{H}_m(\mathbf{R}, t) &= \mathbf{H}(\mathbf{r}, z, t) + \mathbf{z}H_z(\mathbf{r}, z, t) \end{aligned} \tag{2.2}$$

and the boundary conditions hold over the waveguide surface are

$$(\mathbf{n} \cdot \mathbf{H}_m)|_L = 0, \quad (\mathbf{l} \cdot \mathbf{E}_m)|_L = 0, \quad (\mathbf{z} \cdot \mathbf{E}_m)|_L = 0. \tag{2.3}$$

where $\mathbf{n}$ is unit vector outer normal to domain S, $\mathbf{z}$ and $\mathbf{l}$ are the unit vectors tangential to the Oz −axis and contour L, respectively.

2.3 TE Time-Domain Modes

To obtain components of TE time-domain modes let's solve Neumann boundary eigenvalue problem for $\nabla_\perp$

$$(\nabla_\perp^2 + \upsilon_m^2)\psi_m(\mathbf{r}) = 0$$

$$\left.\frac{\partial \psi_m(\mathbf{r})}{\partial n}\right|_L = 0 \tag{2.4}$$

$$\frac{\upsilon_m^2}{S}\int_S |\psi_m(\mathbf{r})|^2 dS = 1N$$

where $\partial_n = \mathbf{n} \cdot \nabla_\perp$ is normal derivative on L −contour, $\upsilon_m^2 > 0$ are eigenvalues for $m = 1,2, \ldots$, the index m is numerical values sorted by increasing order in the real axis and $\psi_m(\mathbf{r})$'s are eigenvectors corresponding to the eigenvalues. In equation (2.4) the first equation is Helmholtz equation, the second one is boundary condition and the third one is normalization condition. The force dimension in equation (2.4) has to be N (Newton) so that physical dimensions, V_m^{-1}(Volt per meter) and A_m^{-1}(Ampere per meter), are provided for the field vector extensions $\mathbf{E}_m$ and $\mathbf{H}_m$, respectively.

Example 2.1 Let us consider L-contour that is a rectangular waveguide with boundaries $0 \le x \le a$ and $0 \le y \le b$.

Since

$$\nabla_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

we can write Helmholtz equation as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_m + \upsilon_m^2\psi_m = 0 \qquad \Rightarrow \qquad \frac{\partial^2\psi_m}{\partial x^2} + \frac{\partial^2\psi_m}{\partial y^2} + \upsilon_m^2\psi_m = 0$$

By using the separation of variables method, choose $\psi_m(\mathbf{r}) = X(x)Y(y)$ and substitute it into above equation yields

$$\frac{\partial^2[X(x)Y(y)]}{\partial x^2} + \frac{\partial^2[X(x)Y(y)]}{\partial y^2} + \upsilon_m^2 X(x)Y(y) = 0$$

or equavalently

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + \upsilon_m^2 XY = 0.$$

Multiply both sides with $1/XY$ gives

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-\upsilon_x^2}+\underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\upsilon_y^2}+\upsilon_m^2=0 \qquad \Rightarrow \qquad \upsilon_m^2=\upsilon_x^2+\upsilon_y^2$$

The eigenvalues υ_x^2 and υ_y^2 are choosen as negative because if we choose them as positive the solution of Neumann problem becomes $\psi_m(\mathbf{r}) = \exp(\pm i\upsilon x)\exp(\pm i\upsilon y)$ which is a divergent function. So, it is not suitable for the EAE method.

Therefore, the solution of above equation can be written as

$$X(x) = A\cos(\upsilon_x x) + B\sin(\upsilon_x x)$$

$$Y(y) = C\cos(\upsilon_y y) + D\sin(\upsilon_y y)$$

or

$$\psi_m(\mathbf{r}) = X(x)Y(y) = [A\cos(\upsilon_x x) + B\sin(\upsilon_x x)][C\cos(\upsilon_y y) + D\sin(\upsilon_y y)].$$

By using boundary condition, the constants of $\psi_m(\mathbf{r})$ can be found as stated below.

(i) $0 \le x \le a,\ \left.\frac{\partial\psi_m(\mathbf{r})}{\partial y}\right|_{y=0} = 0$

$$\left.\frac{\partial\psi_m}{\partial y}\right|_{y=0} = [A\cos(\upsilon_x x) + B\sin(\upsilon_x x)]\left[-C\upsilon_y\sin(\upsilon_y y) + D\upsilon_y\cos(\upsilon_y y)\right]\Big|_{y=0} = 0$$

which is equivalent to

$$[A\cos(\upsilon_x x) + B\sin(\upsilon_x x)]\left[-C\upsilon_y\sin(0) + D\upsilon_y\cos(0)\right] = 0$$

or

$$[A\cos(\upsilon_x x) + B\sin(\upsilon_x x)]\left[0 + D\upsilon_y 1\right] = 0$$

In this equation $A\cos(\upsilon_x x) + B\sin(\upsilon_x x) \neq 0$ so

$$0 + D\upsilon_y 1 = 0 \qquad \Rightarrow \qquad D = 0$$

(ii) $0 \le x \le a,\ \left.\frac{\partial\psi_m(\mathbf{r})}{\partial y}\right|_{y=b} = 0$

$$\left.\frac{\partial \psi_m}{\partial y}\right|_{y=b} = [A\cos(\upsilon_x x) + B\sin(\upsilon_x x)]\left[-C\upsilon_y \sin(\upsilon_y y) + D\upsilon_y \cos(\upsilon_y y)\right]\Big|_{y=b} = 0$$

which is equivalent to

$$[A\cos(\upsilon_x x) + B\sin(\upsilon_x x)]\left[-C\upsilon_y \sin(\upsilon_y b)\right] = 0$$

Again in this equation $A\cos(\upsilon_x x) + B\sin(\upsilon_x x) \neq 0$ so

$$\left[-C\upsilon_y \sin(\upsilon_y b)\right] = 0 = \sin(0 + p\pi)\,, \qquad p = 0,1,2,\ldots$$

$$\Rightarrow \upsilon_y = \frac{p\pi}{b}$$

(iii) $0 \leq y \leq b,\ \left.\frac{\partial \psi_m(\mathbf{r})}{\partial x}\right|_{x=0} = 0$

$$\left.\frac{\partial \psi_m}{\partial x}\right|_{x=0} = [-A\upsilon_x \sin(\upsilon_x x) + B\upsilon_x \cos(\upsilon_x x)]\left[C\cos(\upsilon_y y) + 0\right]\Big|_{x=0} = 0$$

which is equivalent to

$$[-A\upsilon_x \sin(0) + B\upsilon_x \cos(0)]\left[C\cos(\upsilon_y y)\right] = 0$$

or

$$[0 + B\upsilon_x 1]\left[C\cos(\upsilon_y y)\right] = 0$$

In this equation $C\cos(\upsilon_y y) \neq 0$ so

$$0 + B\upsilon_x 1 = 0 \quad \Rightarrow \quad B = 0$$

(iv) $0 \leq y \leq b,\ \left.\frac{\partial \psi_m(\mathbf{r})}{\partial x}\right|_{x=a} = 0$

$$\left.\frac{\partial \psi_m}{\partial x}\right|_{x=a} = [-A\upsilon_x \sin(\upsilon_x x) + 0]\left[C\cos(\upsilon_y y) + 0\right]\Big|_{x=a} = 0$$

which is equivalent to

$$[-A\upsilon_x \sin(\upsilon_x a)]\left[C\cos(\upsilon_y y)\right] = 0$$

Again in this equation $C\cos(\upsilon_y y) \neq 0$ so

$$-A\upsilon_x \sin(\upsilon_x a) = 0 = \sin(0 + q\pi)\,, \qquad q = 0,1,2, \ldots$$

$$\Rightarrow \upsilon_x = \frac{q\pi}{a}$$

Above results yields

$$\psi_m(\mathbf{r}) = X(x)Y(y) = [A\cos(\upsilon_x x)][C\cos(\upsilon_y y)] = A_m^h \cos(\upsilon_x x)\cos(\upsilon_y y)$$

where A_m^h is a normalization constant.

The eigenvalue is

$$\upsilon_m^2 = \pi^2\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right) \equiv \upsilon_{p,q}^2$$

where the parameters p and q are integers that $p + q \neq 0$ for $p, q = 0,1,2, \ldots$.

To find normalization constant A_m^h let's use normalization condition

$$\frac{\upsilon_m^2}{S}\int_S |\psi_m(\mathbf{r})|^2 dS = 1N \quad \text{where} \quad S = a.b\ .$$

$$\frac{\upsilon_m^2}{S}\int_S |\psi_m(\mathbf{r})|^2 dS = \frac{\upsilon_m^2}{a.b}\int_S (A_m^h)^2 \cos^2(\upsilon_x x)\cos^2(\upsilon_y y)\,dxdy = 1$$

$$\Rightarrow \frac{(A_m^h)^2.\upsilon_m^2}{a.b}\int_{x=0}^{x=a}\int_{y=0}^{y=b} \cos^2(\upsilon_x x)\cos^2(\upsilon_y y)\,dx\,dy = 1$$

$$\Rightarrow \frac{(A_m^h)^2.\upsilon_m^2}{a.b}\int_{x=0}^{x=a} \cos^2(\upsilon_x x)\,dx \int_{y=0}^{y=b} \cos^2(\upsilon_y y)\,dy = 1$$

$$\Rightarrow \frac{(A_m^h)^2.\upsilon_m^2}{a.b}\left[\left(\frac{x}{2} + \frac{1}{2}\frac{\sin(2\upsilon_x x)}{2\upsilon_x x}\right)_{x=0}^{x=a}\right]\left[\left(\frac{y}{2} + \frac{1}{2}\frac{\sin(2\upsilon_y y)}{2\upsilon_y y}\right)_{y=0}^{y=b}\right] = 1$$

$$\Rightarrow A_m^h = \frac{2}{\upsilon_m} \equiv A_{p,q}^h \qquad (2.5)$$

where the parameters p and q are integers that $p + q \neq 0$ for $p, q = 0,1,2, \dots$.

Take potential $\Psi_m(\mathbf{r})$ as $\Psi_m(\mathbf{r}) = A_m^h \psi_m(\mathbf{r})$ where A_m^h is a normalization constant. Then every components of TE time-domain modes can be written as

$$\mathbf{E}_{zm}^h = 0$$

$$v_m^{-1}\mathbf{E}_m^h = \langle -\partial_{(v_m ct)} h_m(z,t) \rangle \left[\sqrt[-2]{\varepsilon_0} \nabla_\perp \Psi_m(\mathbf{r}) \times \mathbf{z} \right]$$

$$v_m^{-1}\mathbf{H}_m^h = \langle \partial_{(v_m z)} h_m(z,t) \rangle \left[\sqrt[-2]{\mu_0} \nabla_\perp \Psi_m(\mathbf{r}) \right] \tag{2.6}$$

$$v_m^{-1}\mathbf{H}_{zm}^h = \langle h_m(z,t) \rangle \left[v_m \sqrt[-2]{\mu_0} \Psi_m(\mathbf{r}) \right]$$

where $\partial_{(v_m ct)} = (1/c\, v_m)\partial_t$, $\partial_{(v_m z)} = (1/v_m)\partial_z$ and $c = \sqrt[-2]{\varepsilon_0 \mu_0}$ is the speed of light in space. The potential $h_m(z,t)$ is obtained from Klein-Gordon equation

$$\left(\partial_{v_m ct}^2 - \partial_{v_m z}^2 + 1\right) h_m(z,t) = 0. \tag{2.7}$$

The problem (2.4) has a zero trivial solution. This solution corresponds to the eigenvalue $v_0^2 = 0$ and generates the problem, $\nabla_\perp^2 \Psi_0(\mathbf{r}) = 0$ and $\partial_n \Psi_0(\mathbf{r})|_L = 0$, for a harmonic function $\Psi_0(\mathbf{r})$. he harmonic functions are attained as $\Psi_0(\mathbf{r}) = c_1$ where $\mathbf{r} \in L + S$ and c_1 is a constant. The potential $\Psi_0(\mathbf{r})$ produce a *TE* mode.

$$\mathbf{E}_0^h(\mathbf{r}, z, t) = 0, \qquad \mathbf{H}_0^h(\mathbf{r}, z, t) = \mathbf{z} c_1 \tag{2.8}$$

where the modal amplitude of the field $\mathbf{H}_0^h$ is constant.

Weyl Theorem (Appendix C) means that *TE* time-domain modes are complete in Hilbert space L_2 (Weyl [6]).

2.4 TM Time-Domain Modes

Dirichlet boundary eigenvalue problem for Laplacian is expressed as

$$(\nabla_\perp^2 + \kappa_m^2)\phi_m(\mathbf{r}) = 0$$

$$\phi_m(\mathbf{r})|_L = 0 \tag{2.9}$$

$$\frac{\kappa_m^2}{S}\int_S |\phi_m(\mathbf{r})|^2 dS = 1N$$

where $\kappa_m^2 > 0$'s are eigenvalues for $m = 1,2, \ldots,$ the index m is numerical values sorted by increasing order in the real axis and $\phi_m(\mathbf{r})$'s are eigenvectors corresponding to the eigenvalues. The physical dimensions of field vector extentions $\mathbf{E}_m$ and $\mathbf{H}_m$ are V_m^{-1}(Volt per meter) and A_m^{-1}(Ampere per meter), respectively. The solution $\phi_0(\mathbf{r})$ corresponding to eigenvalues $\kappa_0^2 = 0$ is zero.

Example 2.2 Consider previous example for the rectangular waveguide.

The solution to Dirichlet problem is same as Neumann's solution.

Since

$$\nabla_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

we can write Helmholtz equation as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi_m + \kappa_m^2\phi_m = 0 \quad \Rightarrow \quad \frac{\partial^2\phi_m}{\partial x^2} + \frac{\partial^2\phi_m}{\partial y^2} + \kappa_m^2\phi_m = 0$$

By using the separation of variables method, choose $\phi_m(\mathbf{r}) = X(x)Y(y)$ and substitute it into above equation yields

$$\frac{\partial^2[X(x)Y(y)]}{\partial x^2} + \frac{\partial^2[X(x)Y(y)]}{\partial y^2} + \kappa_m^2 X(x)Y(y) = 0$$

or equavalently

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + \kappa_m^2 XY = 0.$$

Multiply both sides with $1/XY$ gives

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-\kappa_x^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\kappa_y^2} + \kappa_m^2 = 0 \quad \Rightarrow \quad \kappa_m^2 = \kappa_x^2 + \kappa_y^2$$

The eigenvalues κ_x^2 and κ_y^2 are choosen as negative because if we choose them as positive the solution of Neumann problem becomes $\phi_m(\mathbf{r}) = \exp(\pm i\kappa x)\exp(\pm i\kappa y)$ which is a divergent function. So, it is not suitable for the EAE method.

Therefore, the solution of above equation can be written as

$$X(x) = A\cos(\kappa_x x) + B\sin(\kappa_x x)$$

$$Y(y) = C\cos(\kappa_y y) + D\sin(\kappa_y y)$$

or

$$\phi_m(\mathbf{r}) = X(x)Y(y) = [A\cos(\kappa_x x) + B\sin(\kappa_x x)][C\cos(\kappa_y y) + D\sin(\kappa_y y)].$$

By using boundary condition, the constants of $\phi_m(\mathbf{r})$ can be found as stated below.

(i) $0 \le x \le a,\ \phi_m(\mathbf{r})|_{y=0} = 0$

$$\phi_m(\mathbf{r})|_{y=0} = [A\cos(\kappa_x x) + B\sin(\kappa_x x)][C\cos(\kappa_y y) + D\sin(\kappa_y y)]\big|_{y=0} = 0$$

which is equivalent to

$$[A\cos(\kappa_x x) + B\sin(\kappa_x x)][C\cos(0) + D\sin(0)] = 0$$

or

$$[A\cos(\kappa_x x) + B\sin(\kappa_x x)][C.1 + D.0] = 0$$

In this equation $A\cos(\kappa_x x) + B\sin(\kappa_x x) \neq 0$ so

$$C.1 + D.0 = 0 \quad \Rightarrow \quad C = 0$$

(ii) $0 \le x \le a,\ \phi_m(\mathbf{r})|_{y=b} = 0$

$$\phi_m(\mathbf{r})|_{y=b} = [A\cos(\kappa_x x) + B\sin(\kappa_x x)][0.\cos(\kappa_y y) + D\sin(\kappa_y y)]\big|_{y=b} = 0$$

which is equivalent to

$$[A\cos(\kappa_x x) + B\sin(\kappa_x x)]\big[D\sin(\kappa_y b)\big] = 0$$

Again in this equation $A\cos(\kappa_x x) + B\sin(\kappa_x x) \neq 0$ so

$$D\sin(\kappa_y b) = 0 = \sin(0 + q\pi)\,, \qquad q = 0,1,2,\ldots$$

$$\Rightarrow \kappa_y = \frac{q\pi}{a}$$

(iii) $0 \leq y \leq b,\ \phi_m(\mathbf{r})|_{x=0} = 0$

$$\phi_m(\mathbf{r})|_{x=0} = [A\cos(\kappa_x x) + B\sin(\kappa_x x)]\big[0.\cos(\kappa_y y) + D\sin(\kappa_y y)\big]\big|_{x=0} = 0$$

which is equivalent to

$$[A\cos(0) + B\sin(0)]\big[D\sin(\kappa_y y)\big] = 0$$

or

$$[A.1 + B.0]\big[D\sin(\kappa_y y)\big] = 0$$

In this equation $D\sin(\kappa_y y) \neq 0$ so

$$A.1 + B.0 = 0 \quad \Rightarrow \quad A = 0$$

(iv) $0 \leq y \leq b,\ \phi_m(\mathbf{r})|_{x=a} = 0$

$$\phi_m(\mathbf{r})|_{x=a} = [0.\cos(\kappa_x x) + B\sin(\kappa_x x)]\big[0.\cos(\kappa_y y) + D\sin(\kappa_y y)\big]\big|_{x=a} = 0$$

which is equivalent to

$$[B\sin(\kappa_x a)]\big[D\sin(\kappa_y y)\big]$$

Again in this equation $D\sin(\kappa_y y) \neq 0$ so

$$B\sin(\kappa_x a) = 0 = \sin(0 + p\pi)\,, \qquad p = 0,1,2,\ldots$$

$$\Rightarrow \kappa_x = \frac{p\pi}{a}$$

Above results yields

$$\phi_m(\mathbf{r}) = X(x)Y(y) = [B\sin(\kappa_x x)]\big[D\sin(\kappa_y y)\big] = A_m^e \sin(\kappa_x x)\sin(\kappa_y y)$$

where A_m^e is a normalization constant.

Hence, the eigenvalue is found as

$$\kappa_m^2 = \pi^2\left(\frac{p^2}{a^2}+\frac{q^2}{b^2}\right) \equiv \kappa_{p,q}^2$$

where the parameters p and q are integers that $p+q \neq 0$ for $p, q = 0,1,2, \ldots$.

To find normalization constant A_m^e let's use normalization condition

$$\frac{\kappa_m^2}{S}\int_S |\phi_m(\mathbf{r})|^2 dS = 1N \quad \text{where} \quad S = a.b \ .$$

$$\frac{\kappa_m^2}{S}\int_S |\phi_m(\mathbf{r})|^2 ds = \frac{\kappa_m^2}{a.b}\int_S (A_m^e)^2 \sin^2(\kappa_x x)\sin^2(\kappa_y y)\, dxdy = 1$$

$$\Rightarrow \frac{(A_m^e)^2.\kappa_m^2}{a.b}\int_{x=0}^{x=a}\int_{y=0}^{y=b} \sin^2(\kappa_x x)\sin^2(\kappa_y y)\, dx\, dy = 1$$

$$\Rightarrow \frac{(A_m^e)^2.\kappa_m^2}{a.b}\int_{x=0}^{x=a} \sin^2(\kappa_x x)\, dx \int_{y=0}^{y=b} \sin^2(\kappa_y y)\, dy = 1$$

$$\Rightarrow \frac{(A_m^e)^2.\kappa_m^2}{a.b}\left[\left(\frac{x}{2}-\frac{1}{2}\frac{\sin(2\kappa_x x)}{2\kappa_x x}\right)_{x=0}^{x=a}\right]\left[\left(\frac{y}{2}-\frac{1}{2}\frac{\sin(2\kappa_y y)}{2\kappa_y y}\right)_{y=0}^{y=b}\right] = 1$$

$$\Rightarrow A_m^e = \frac{2}{\kappa_m} \equiv A_{p,q}^e \qquad (2.10)$$

where the parameters p and q are integers that $p+q \neq 0$ for $p, q = 0,1,2, \ldots$. Then take the potential $\Phi_m(\mathbf{r})$ as $\Phi_m(\mathbf{r}) = A_m^e \phi_m(\mathbf{r})$ where A_m^e is normalization constant. Thus, every component of TM time-domain modes can be stated as

$$\mathbf{H}_{zm}^{e} = 0$$

$$\kappa_m^{-1}\mathbf{H}_m^{e} = \langle -\partial_{(\kappa_m ct)} e_m(z,t)\rangle\left[\mathbf{z}\times \sqrt[-2]{\mu_0}\nabla_{\perp}\Phi_m(\mathbf{r})\right]$$

$$\kappa_m^{-1}\mathbf{E}_m^{e} = \langle \partial_{(\kappa_m z)} e_m(z,t)\rangle\left[\sqrt[-2]{\varepsilon_0}\nabla_{\perp}\Phi_m(\mathbf{r})\right] \tag{2.11}$$

$$\kappa_m^{-1}\mathbf{E}_m^{e} = \langle e_m(z,t)\rangle\left[\kappa_m \sqrt[-2]{\varepsilon_0}\Phi_m(\mathbf{r})\right]$$

where $\partial_{(\kappa_m ct)} = (1/c\,\kappa_m)\partial_t$, $\partial_{\kappa_m z} = (1/\kappa_m)\partial_z$ and $c = \sqrt[-2]{\varepsilon_0\mu_0}$ is the speed of light in space. The potential $e_m(z,t)$ is obtained from Klein-Gordon equation

$$\left(\partial^2_{\upsilon_m ct} - \partial^2_{\upsilon_m z} + 1\right)e_m(z,t) = 0. \tag{2.12}$$

Weyl Theorem (Appendix C) states that *TM* time-domain modes are complete in Hilbert space L_2 (Weyl [6]).

CHAPTER 3

GENERAL PROPERTIES of TIME-DOMAIN MODES

3.1 Completeness of Time-Domain Modes

The set of TE and TM modes is complete due to the completeness of their generating potentials in the same energetic space. The completeness comes from Weyl theorem (Appendix C) in functional analysis about orthogonal detachments of Hilbert space L_2 Proof of completeness is stated in the references (Tretyakov [7]), (Aksoy and Tretyakov [8]), (Tretyakov[9]).

This energetic space can be specified by an inner product as

$$(X_1, X_2) = \frac{1}{S}\int_S (\varepsilon_0 \mathbf{E}_1 \cdot \mathbf{E}_2^* + \mu_0 \mathbf{H}_1 \cdot \mathbf{H}_2^*)\, ds < \infty \qquad (3.1)$$

where $X_1 = col(\mathbf{E}_1, \mathbf{H}_1)$, $X_2 = col(\mathbf{E}_2, \mathbf{H}_2)$ and the meaning of "col" is column and "·" is scalar product of 3 $-$component vectors.

Since TE and Tm time-domain modes are complete, we can talk about their orthogonality. Let's choose the set of *TE* and *TM* modes (2.6) and (2.11) as

$$\mathbf{G} = \{X_m^h\}_{m=0}^{\infty}\,, \qquad \mathfrak{J} = \{X_m^e\}_{m=0}^{\infty}\,. \qquad (3.2)$$

respectively, where $X_m^h = col(\mathbf{E}_m^h, \mathbf{H}_m^h)$ and $X_m^e = col(\mathbf{E}_m^e, \mathbf{H}_m^e)$.

If we set $m = m'$ and substitute the pair X_m^h, $X_{m'}^h$ into the pair X_1, X_2 in equation (3.1), then it yields $\left(X_m^h, X_{m'}^h\right) = 0$. It shows that all elements of the set **G** are mutually orthogonal. The same is true for the elements of the set $\mathfrak{J}$**.** Let's take an element X_m^h from the set **G** and take another element X_m^e from the set $\mathfrak{J}$. Now, if we plug them into equation (3.1), again we have the same result $\left(X_m^h, X_{m'}^e\right) = 0$ where $m = m'$ is arbitrary.

Therefore, any pair of the TE and TM time-domain modes is orthogonal in the sense of inner product as in Figure 3.1.

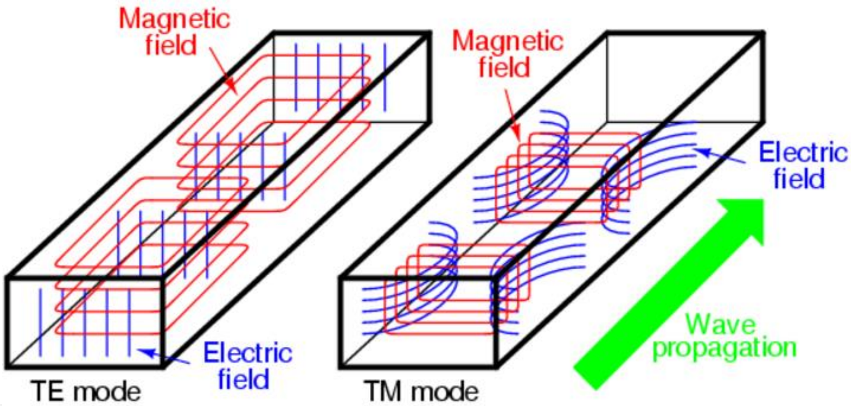

Figure 3.1 Orthogonality of TE and TM modes

The proofs of the completeness of **G** and $\mathfrak{I}$ are in the studies of (Tretyakov [10]) and (Tretyakov [7]).

3.2 Energy Conservation Law for Time-Domain Modes

Let's select a volume V located between two waveguide cross-sections along the coordinates z and $z + \delta_z$ where z and $\delta_z > 0$ are arbitrary. After applying Poynting theorem (Appendix D) to Maxwell's equations for this volume with the limiting case of $\delta_z \to 0$, it yields law of conservation for any modal field energy as

$$\frac{\partial}{\partial_\xi} P_z(\xi, \tau) + \frac{\partial}{\partial_\tau} W(\xi, \tau) = 0 \tag{3.3}$$

where $P_z(\xi, \tau)$ and $W(\xi, \tau)$ is the z −component of Poynting vector averaged over the waveguide cross-section and averaged modal field energy stored in the same cross-section, respectively, and they defined as

$$W(\xi, \tau) = \left[(\partial_\tau f)^2 + \left(\partial_\xi f\right)^2 + f^2\right]/2$$
$$P_z(\xi, \tau) = c(-\partial_\tau f)\left(\partial_\xi f\right). \tag{3.4}$$

Equation (3.3) known as "continuity equation" holds for every modal field. Detailed solution is in (Akgun [2]) and (Eroglu [3]).

3.3 Initial Conditions for Klein-Gordon Equations

Klein-Gordon equation should has a pair of initial conditions like any other second order partial differential equations. Physically, they are involved in the excitation of a suitable signal source.Let's assume that this source is at rest before time $t = 0$ but it starts excitation at time $t = 0$. So, the initial condition can be written as

$$f(\xi,\tau)|_{\xi=0} = \begin{cases} \varphi(\tau), & \tau \geq 0 \quad \text{for} \quad t \geq 0 \\ 0, & \tau < 0 \quad \text{for} \quad t < 0 \end{cases}$$

$$\frac{\partial}{\partial\tau} f(\xi,\tau)|_{\xi=0} = \begin{cases} \hat{\varphi}(\tau), & \tau \geq 0 \quad \text{for} \quad t \geq 0 \\ 0, & \tau < 0 \quad \text{for} \quad t < 0 \end{cases} \tag{3.5}$$

where $\varphi(\tau)$ and $\hat{\varphi}(\tau)$ should be given and $\xi = 0 \;\; \Rightarrow \;\; z = 0$.

3.4 The Causality Principle

The solutions of Klein-Gordon equation is based on the causality principle. There are two kinds of causality. If the sources are zero at the beginning then all the fields are zero at weak causality. In our problem this correponds to $\tau < 0$. The strong causality condition is the Einstein's postulate that claims any magnetic field propagates a signal in space with the speed of light c. In our problem, the source is in the waveguide cross-section at $\xi = 0$. It states that the solution of Klein-Gordon must be zero at beyond the source point $\xi = 0$, after $\xi = \tau$ (i.e., $z = ct$). Hence, if the signal propagates along Oz −axis then Klein-Gordon equation should be read as

$$f(\xi,\tau) = \begin{cases} f(\xi,\tau) = 0 & \text{if} \quad \tau < 0 \\ f(\xi,\tau) \neq 0 & \text{if} \quad 0 \leq \xi \leq \tau \\ f(\xi,\tau) = 0 & \text{if} \quad \xi > \tau \end{cases} \tag{3.6}$$

CHAPTER 4

MODAL AMPLITUDE PROBLEM

The wave equation which moves along the $z-$axis at a particular time t is Klein-Gordon equation and it is defined by

$$(\partial_{ct}^2 - \partial_z^2 + \kappa^2)F(z,t) = 0$$

where c is the speed of light and κ^2 is an eigenvalue of Wave-Boundary Operators (which are inside the parenthesis of the equation).

Let us consider

$$(\partial_{ct}^2 - \partial_z^2 + 1)F(z,t) = 0, \tag{4.1}$$

where $\kappa^2 \equiv 1$. The function F sought, in equation (4.1), is depend on two variables which are z and t: $F \equiv F(z,t)$. To apply the method of separation of variables, we need to ask a question like "is it possible to write (z,t) as a combination of $u(z,t)$ and $v(z,t)$". To make it possible (i) we should take into new independent variables provided that (ii) these new variables can be separated in Klein-Gordon equation. In other words, we consider separation of variables in regard to the function F as follows:

$$F \equiv F(z,t) = F[u(z,ct), v(z,ct)] \equiv F(u,v) = U(u)V(v) \tag{4.2}$$

To answer above question, we should look at which form acquires equation (4.1) in terms of new variables, $u \equiv u(z,t)$ and $v \equiv v(z,t)$, as some functions of the original ones. For reforming equation (4.1) in terms of u and v, we have to do the initial work as stated below:

$$\frac{1}{c}\frac{\partial}{\partial t}F(u,v) = \frac{1}{c}\left\{\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial t}\right\}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}F(u,v) = \frac{1}{c^2}\left\{\frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 F}{\partial v^2}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{\partial F}{\partial v}\frac{\partial^2 v}{\partial t^2}\right\}$$

$$+ \frac{1}{c^2}\left\{\frac{\partial^2 F}{\partial u \partial v}\frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \frac{\partial^2 F}{\partial v \partial u}\frac{\partial v}{\partial t}\frac{\partial u}{\partial t}\right\}$$

Since u and v are independent variables, the last equation can be rearrange as

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}F(u,v) = \frac{1}{c^2}\left\{\frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{\partial^2 F}{\partial v^2}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial t^2} + \frac{\partial F}{\partial v}\frac{\partial^2 v}{\partial t^2}\right.$$

$$\left. + 2\frac{\partial^2 F}{\partial u \partial v}\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\right\}. \tag{4.3}$$

Equally,

$$\frac{\partial^2}{\partial t^2}F(u,v) = \left\{\frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{\partial^2 F}{\partial v^2}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial t^2} + \frac{\partial F}{\partial v}\frac{\partial^2 v}{\partial t^2}\right.$$

$$\left. + 2\frac{\partial^2 F}{\partial u \partial v}\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\right\}. \tag{4.4}$$

Substituting equations (4.3) and (4.4) in (4.1) lead us to

$$\left[\left(\frac{1}{c}\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial z}\right)^2\right]\frac{\partial^2 F}{\partial u^2} + \left[\left(\frac{1}{c}\frac{\partial v}{\partial t}\right)^2 - \left(\frac{\partial v}{\partial z}\right)^2\right]\frac{\partial^2 F}{\partial v^2} + \left[\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2}\right]\frac{\partial F}{\partial u}$$

$$+ \left[\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial z^2}\right]\frac{\partial F}{\partial v} + 2\left[\frac{1}{c}\frac{\partial u}{\partial t}\frac{1}{c}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right]\frac{\partial^2 F}{\partial u \partial v} + F = 0 \tag{4.5}$$

At this point, the following comments may be needed:

- Equation (4.5) is reformulated version of equation (4.1) with the transformation of the original variables to $u(z,t)$ and $v(z,t)$
- The terms in the square brackets of equation (4.5) has variable coefficients;
- The dependences of $u(z,t)$ and $v(z,t)$ should be defined to separate new variables u and v and find a solution of equation (4.5) in the form of

$F(u,v) = U(u)V(v)$. The definition can not be understood with a simple look at equation (4.5).

Klein-Gordon equation is well-studied about separation of variables in the mathematical book (Miller [11]). In this book, 11 orthogonal coordinate systems (Appendix A) were found which makes it possible to separate new variables, by using group theory approach. In addition, another set of nonorthogonal coordinate systems, in which separation of the variables is possible, can be attained with the aid of the same approach. The list of the orhogonal coordinate systems are cited below.

Group theory gives positive answer on this question. In the list, there is 11 versions of ct and z in terms of the functions of u and v. Inverse functional depences, $u \equiv u(z, ct)$ and $v \equiv v(z, ct)$ can be found easily.

4.1 An Overview of Bessel Functions

4.1.1 Definition

Consider the homogenous linear n −th order ordinary differential equation of is

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_0(x)y = 0. \tag{4.6}$$

In this case;

1. If the coefficients $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ are not analytic at $x = x_0$ then x_0 is called a singular point of given differential equation.
2. If all the coefficients $p_k(x)$ are not analytic but all points $(x - x_0)^{n-k}$ for $k = 0,1, \ldots, (n-2)$ are analytic, then the point x_0 is regular singular.
3. If the point x_0 is neither an ordinary point nor a regular singular point of equation (4.6) then it is called as irregular singular point.

4.1.2 The Method of Frobenius

Let us consider a homogenous linear 2 −nd order ordinary differential equation. If $x = 0$ is a regular singular point then both $xP(x)$ and $x^2Q(x)$ can be written as power series

$$xP(x) = \sum_{n=0}^{\infty} P_n x^n, \qquad |x| < r$$

$$x^2Q(x) = \sum_{n=0}^{\infty} Q_n x^n, \qquad |x| < r$$

or equivalenly,

$$P(x) = \sum_{n=0}^{\infty} P_n x^{n-1}, \qquad |x| < r, \;\; x \neq 0$$

$$Q(x) = \sum_{n=0}^{\infty} Q_n x^{n-2}, \qquad |x| < r, \qquad x \neq 0.$$

Frobenius series solution of given differential equation is suggested as

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} a_n x^{n+\alpha} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \quad 0 < x < r$$

Differentiating y with respect to x, in turn, yields

$$y'(x) = \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1},$$

$$y''(x) = \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2}$$

and substituting these results into the differential equation, we get

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2} + \sum_{n=0}^{\infty} P_n x^{n-1} \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1}$$
$$+ \sum_{n=0}^{\infty} Q_n x^{n-2} \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0 \qquad (4.7)$$

In this equation, the second and third power series can be stated as

$$\sum_{n=0}^{\infty} P_n x^{n-1} \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_{n-m} x^{n-m-1} (m+\alpha) a_m x^{m+\alpha-1}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} (m+\alpha) P_{n-m} \, a_m \right] x^{n+\alpha-2} \qquad (4.8)$$

and

$$\sum_{n=0}^{\infty} Q_n x^{n-2} \sum_{n=0}^{\infty} a_n x^{n+\alpha} = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} Q_{n-m} \, a_m \right] x^{n+\alpha-2} \ , \qquad (4.9)$$

respectively. Plugging equations (4.8) and (4.9) into (4.7) gives

$$\sum_{n=0}^{\infty} \left\{ (n+\alpha)(n+\alpha-1) a_n + \sum_{m=0}^{n} [(m+\alpha) P_{n-m} + Q_{n-m}] \, a_m \right\} x^{n+\alpha-2} = 0. \qquad (4.10)$$

However, for equation (4.10) to hold the coefficient $x^{n+\alpha-2}$, $n = 0,1, \ldots$ should be zero. Our task is to find a recurrence relation involving a_n, for $n = 0,1, \ldots$. So, setting $n = 0$ equation (4.10) becomes

$[\alpha(\alpha-1) + \alpha P_0 + Q_0] a_0 = 0$.

In this case, $a_0 = 0$ or $\alpha(\alpha-1) + \alpha P_0 + Q_0 = 0$.

Setting $n = 1$ in (4.10)

$$(n+\alpha)(n+\alpha-1) a_n + \sum_{m=0}^{n} [(m+\alpha) P_{n-m} + Q_{n-m}] \, a_m = 0$$

or,

$$a_n = -\frac{1}{(n+\alpha)(n+\alpha-1) + (n+\alpha) P_0 + Q_0} \sum_{m=0}^{n-1} [(m+\alpha) P_{n-m} + Q_{n-m}] \, a_m.$$

1. If $a_0 = 0$ then $a_1 = a_2 = \cdots = 0$ and so $y(x) = 0$.
2. If $a_0 \neq 0$ then the indicial equation

$$\alpha(\alpha-1) + \alpha P_0 + Q_0 = 0 \qquad (4.11)$$

is obtained.

We assume that $a_0 \neq 0$, so equation (4.11) must be hold. Thus, α_1 and $\alpha_2$are the roots of the indicial equation associated with $x = 0$.

4.1.3 Fuchs Theorem (Series Solution Near a Regular Singular Point)

We already know that if $x = 0$ is a singular point for 2nd order homogenous linear ordinary differential equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \tag{4.12}$$

then

$$xP(x) = \sum_{n=0}^{\infty} P_n x^n, \quad x^2 Q(x) = \sum_{n=0}^{\infty} Q_n x^n, \qquad |x| < r.$$

Let the indicial equation (4.11) has two real roots α_1 and α_2 where $\alpha_1 \geq \alpha_2$.

1. If $\alpha_1 - \alpha_2$is not an integer, then there exist two linearly independent solutions of the form

$$y_1(x) = x^{\alpha_1} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0, \ 0 < x < r \tag{4.13}$$

$$y_2(x) = x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n, \quad a_0 \neq 0, \ 0 < x < r. \tag{4.14}$$

The coefficients b_n is found by substituting $y_1(x)$ into the differential equation.

2. If $\alpha_1 = \alpha_2 = \alpha$ then the first and second linearly independent solutions are

$$y_1(x) = x^{\alpha_1} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0, \ 0 < x < r$$

$$y_2(x) = y_1(x) \ln x + x^{\alpha} \sum_{n=0}^{\infty} b_n x^n, \qquad 0 < x < r. \tag{4.15}$$

In this case, second solution $y_2(x)$ is not a Frobenius series solution.

3. If $\alpha_1 - \alpha_2$ is an integer then

$$y_1(x) = x^{\alpha_1} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0, \ \ 0 < x < r$$

$$y_2(x) = a y_1(x) \ln x + x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n, \qquad 0 < x < r. \tag{4.16}$$

where a is a constant that could be zero. The solution $y_2(x)$ can be a second solution of Frobenius series solution.

Consequently, the general solution of differential equation is

$$y(x) = C_1(x) y_1(x) + C_2(x) y_2(x). \tag{4.17}$$

4.1.4 Bessel Differential Equation and its Solution

The Bessel differential equation is the linear second-order ordinary differential equation given by

$$x^2 y''(x) + x y'(x) + (x^2 - p^2) y(x) = 0, \quad x > 0 \tag{4.18}$$

where $p \geq 0$ is a constant number. The method of series solution near a regular singular point is used for solving the Bessel differential equation.

Clearly,

$$y''(x) + P(x) y'(x) + Q(x) y(x) = 0 \tag{4.19}$$

has a regular singular point at $x = 0$ where $P(x) = \frac{1}{x}$ and $Q(x) = \frac{x^2 - p^2}{x^2}$. Since,

$$xP(x) = 1 = 1 + 0.x + 0.x^2 + \cdots \qquad \Rightarrow \qquad P_0 = 1 \tag{4.20}$$

$$x^2 Q(x) = x^2 - p^2 = -p^2 + 0.x + x^2 + 0.x^3 + \cdots \qquad \Rightarrow \qquad Q_0 = -p^2 \tag{4.21}$$

$xP(x)$ and $x^2 Q(x)$ are both analytic at the point $x = 0$, can be written as a series expansion and these series are convergent for $|x| < \infty$, respectively.

By the indicial equation (4.11)

$$\alpha(\alpha-1)+\alpha.1-p^2=0 \quad \Rightarrow \quad \alpha-p^2=0 \quad \Rightarrow \quad \alpha_1=p,\;\; \alpha_2=-p. \tag{4.22}$$

The Bessel differential equation (4.18) has a series solution in the form of

$$y_1(x)=x^p\sum_{n=0}^{\infty}a_n\,x^n=\sum_{n=0}^{\infty}a_n\,x^{n+p}, \quad a_0\neq 0, \quad 0<x<\infty. \tag{4.23}$$

We begin by differentiating $y_1(x)$ in (4.23) with respect to x to obtain

$$y_1'(x)=\sum_{n=0}^{\infty}(n+p)a_n\,x^{n+p-1} \tag{4.24}$$

$$y_1''(x)=\sum_{n=0}^{\infty}(n+p)(n+p-1)a_n\,x^{n+p-2}. \tag{4.25}$$

Substituting $y_1(x)$, $y_1'(x)$ and $y_1''(x)$ into equation (4.18) yields

$$x^2\sum_{n=0}^{\infty}(n+p)(n+p-1)a_n\,x^{n+p-2}+x\sum_{n=0}^{\infty}(n+p)a_n\,x^{n+p-1}$$

$$+(x^2-p^2)\sum_{n=0}^{\infty}a_n\,x^{n+p}=0$$

or more precisely,

$$\sum_{n=0}^{\infty}[(n+p)(n+p-1)+(n+p)-p^2]a_n\,x^{n+p}+\sum_{n=0}^{\infty}a_n\,x^{n+p+2}=0 \tag{4.26}$$

To simplify the addition of two summations in (4.26), let's shift the index of second

$$\sum_{n=0}^{\infty}a_n\,x^{n+p+2} \;\xrightarrow{n+2=m}\; \sum_{m=2}^{\infty}a_{n-2}\,x^{n+p}.$$

With these changes equation (4.26) becomes

$$x^p\left\{\sum_{n=0}^{\infty}[(n+p)(n+p-1)+(n+p)-p^2]a_n\,x^n+\sum_{n=2}^{\infty}a_{n-2}\,x^n\right\}=0. \tag{4.27}$$

Since $x^p\neq 0$, then

$$\sum_{n=0}^{\infty}n(n+2p)a_n\,x^n+\sum_{n=2}^{\infty}a_{n-2}\,x^n=0 \tag{4.28}$$

holds for $x^n\neq 0$, $n=0,1,\dots$.

Separating the terms corresponding to $n=0$ and $n=1$ and combining the rest under one summation, we have

$$0.(0+2p)a_0+1.(1+2p)a_1x+\sum_{n=2}^{\infty}(n(n+2p)a_n+a_{n-2})\,x^n=0\,.$$

Setting the coefficients equal to zero gives

$0.(0+2p)a_0=0$

where $a_0\neq 0$is an arbitrary constant,

$1.(1+2p)a_1=0 \quad\Rightarrow\quad a_1=0$

and the recurrence relation for $n\geq 2$ is

$n(n+2p)a_n\ +a_{n-2}=0$

$$a_n=-\frac{a_{n-2}}{n(n+2p)} \tag{4.29}$$

Solving the recurrence relation (4.29) for a_{2n+1}, $n=0,1,\dots$,we obtain

$a_{2n+1}=0.$

For $n=2$ and 4. this gives

$$a_2=-\frac{a_0}{2(2+2p)}=-\frac{a_0}{2^2.1(1+p)}$$

$$a_4=-\frac{a_2}{4(4+2p)}=-\frac{a_2}{2^2.2(2+p)}=(-1)^2\frac{a_0}{2^4.2!\,(1+p)(2+p)}$$

it follows by induction

$$a_{2n} = (-1)^n \frac{a_0}{2^{2n}.\, n!\,(1+p)(2+p)\dots(n+p)}.$$

So, equation (4.23) is

$$y_1(x) = a_0 x^p \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(1+p)(2+p)\dots(n+p)} \left(\frac{x}{2}\right)^{2n}, \quad 0 < x < \infty. \tag{4.30}$$

In order to write simplified solution, the Gamma function can be used which is defined as

$$\Gamma(p+1) = \int_0^{\infty} t^p e^{-t} dt, \quad p > 0$$

(Xie [12]). Using integration by part

$$\Gamma(p+1) = -t^p e^{-t}\Big|_{t=0}^{\infty} + \int_0^{\infty} p t^{p-1} e^{-t} dt = p \int_0^{\infty} t^{p-1} e^{-t} dt = p\Gamma(p), \tag{4.31}$$

is obtained. Thus

$$\Gamma(n+p+1) = (n+p)\Gamma(n+p) = (n+p)(n+p-1)\Gamma(n+p-1)$$

$$= \dots = (n+p)(n+p-1)\dots(1+p)\Gamma(1+p).$$

In equation (4.31), for $p = 0$

$$\Gamma(1) = +\int_0^{\infty} e^{-t} dt = e^{-t}\Big|_{t=0}^{\infty} = 1,$$

for $p = 1$

$$\Gamma(2) = 2.\Gamma(1) = 1,$$

for $p = 2$

$$\Gamma(3) = 2.\Gamma(2) = 2.1 = 2!,$$

$$\vdots$$

for the case $p = k$ is an integer

$\Gamma(k+1) = k.\Gamma(k) = k!\,.$

Therefore, if we choose

$a_0 = [2^p\Gamma(1+p)]^{-1}$

and, first Frobenius series solution is

$$y_1(x) = J_p(x) \qquad (4.32)$$

where $J_p(x)$ is called as the Bessel function of the first kind and defined by

$$J_p(x) = \frac{1}{2^p\Gamma(1+p)}x^p\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!\,(1+p)(2+p)\dots(n+p)}\left(\frac{x}{2}\right)^{2n}$$

or equivalently,

$$J_p(x) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!\,\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p}$$

In Fuch's theory, linearly independent second solution changes according to the difference between the roots of the indicial equation (4.11).The difference $\alpha_1 - \alpha_2 = 2p$ is not an integer, equal to zero or a positive integer

4.1.4.1 2p Nonbeing an Integer

Second Frobenius series solution (4.14) of the Bessel equation (4.18) is

$$y_2(x) = x^{-p}\sum_{n=0}^{\infty}b_nx^n\,, \qquad 0 < x < \infty$$

Similarly, in the previous section,

$b_{2n-1} = 0,$

and th recurrence relation is

$$b_{2n} = (-1)^n\frac{b_0}{2^{2n}.\,n!\,(1-p)(2-p)\dots(n-p)}.$$

Hence, the second Frobenius series solution is

$$y_2(x) = b_0 x^{-p} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(1-p)(2-p)\dots(n-p)} \left(\frac{x}{2}\right)^{2n}, \qquad 0 < x < \infty$$

or equivalently,

$$y_2(x) = x^{-p} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n-p} = J_{-p}(x), \quad 0 < x < \infty \tag{4.33}$$

where $b_0 = [2^{-p}\Gamma(1-p)]^{-1}$.

As a result, the general solution of the Bessel equation is

$$y(x) = C_1 J_p(x) + C_2 J_{-p}(x) \tag{4.34}$$

or

$$y(x) = D_1 J_p(x) + D_2 Y_p(x) y(x) = C_1 J_p(x) + C_2 J_{-p}(x) \tag{4.35}$$

where $Y_p(x)$ is the Bessel function of the second kind of order two and represented by

$$Y_p(x) = \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)} \tag{4.36}$$

4.1.4.2 $p = 0, \alpha_1 = \alpha_2 = 0$

The solution of first Frobenius series solution simplified from equation (4.32) as

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \qquad 0 < x < \infty \tag{4.37}$$

and linearly independent second solution is

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^n. \qquad 0 < x < \infty \tag{4.38}$$

We differentiate termwise to obtain

$$y_2'(x) = y_1'(x)\ln x + \frac{y_1}{x} + \sum_{n=0}^{\infty} nb_n x^{n-1} \qquad 0 < x < \infty \tag{4.39}$$

$$y_2''(x) = y_1''(x)\ln x + \frac{2y_1'}{x} - \frac{y_1}{x^2} + \sum_{n=2}^{\infty} n(n-1)b_n x^{n-2}. \tag{4.40}$$

and substituting them into the Bessel equation (4.18) for $p = 0$ yields

$$(x^2 y_1'' + xy_1' + x^2 y_1)\ln x + 2xy_1' + \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} nb_n x^n + \sum_{n=0}^{\infty} b_n x^{n+2} = 0 \tag{4.41}$$

(Xie [12]).

Since $y_1(x)$ is a solution of the Bessel equation, we can write $x^2 y_1'' + xy_1' + x^2 y_1 = 0$ and

$$2xy_1' = 2x\sum_{n=0}^{\infty}(-1)^n \frac{1}{(n!)^2}\frac{2n\, x^{2n-1}}{2^{2n}} = \sum_{n=1}^{\infty}(-1)^n \frac{4n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}. \tag{4.42}$$

Plugging above equations in equation (4.41) gives

$$\sum_{n=1}^{\infty}(-1)^n \frac{4n}{(n!)^2}\left(\frac{x}{2}\right)^{2n} + \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} nb_n x^n + \sum_{n=2}^{\infty} b_{n-2} x^n = 0 \tag{4.43}$$

where $x^n$is need be equal zero to provide this equality for $n = 0,1, \dots$.

Next, if we separate the x^1 terms from others and set the coefficients of the power series equal to zeroe get

$b_1 = 0,$

$b_{2n+1} = 0$

and the recurrence relation

$$(-1)^n \frac{4n}{(n!)^2}\left(\frac{1}{2}\right)^{2n} + [2n(2n-1)+2n]b_{2n} + b_{2n-2} = 0$$

or

$$b_{2n} = (-1)^{n+1}\frac{1}{n(n!)^2}\left(\frac{1}{2}\right)^{2n} - \frac{b_{2n-2}}{(2n)^2} \quad .$$

Assuming $b_0 = 0$ yields

$$b_{2n} = (-1)^{n+1}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{(n!)^2}\left(\frac{1}{2}\right)^{2n} .$$

Accordingly, the linearly independent second solution becomes

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{(n!)^2}\left(\frac{1}{2}\right)^{2n} . \qquad 0 < x < \infty \qquad (4.44)$$

is a term of the Bessel funtion of the second kind of order zero. Or it can be expressed by

$$y_2(x) = \frac{\pi}{2}Y_0(x) + (\ln 2 - \gamma)J_0(x) \qquad 0 < x < \infty \qquad (4.45)$$

where $Y_0(x)$ is the Bessel funtion of the second kind of order zero. It is defined by

$$Y_0 = \frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)J_0(x) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{(n!)^2}\left(\frac{1}{2}\right)^{2n}\right]$$

where $\gamma = 0{,}57721566490153 \ldots = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$ is Euler constant.

The general solution is

$$y(x) = C_0 J_p(x) + C_2 Y_0\,(x). \qquad (4.46)$$

4.1.4.3 *p* Being a Positive Integer

The solution of first Frobenius series solution was

$$y_1(x) = J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)(n+p)!} \left(\frac{x}{2}\right)^{2n+p}. \qquad 0 < x < \infty \tag{4.47}$$

Linearly independent second solution is

$$y_2(x) = a y_1(x) \ln x + x^{-p} \sum_{n=0}^{\infty} b_n x^n. \qquad 0 < x < \infty \tag{4.48}$$

Our goal is to determine the coefficients b_n by substituting $y_2(x)$ directly into equation (4.18). Let's take the derivatives of equation (4.48) with respect to x

$$y_2'(x) = a\left(y_1'(x) \ln x + \frac{y_1}{x}\right) + \sum_{n=0}^{\infty} (n-p) b_n x^{n-p-1} \tag{4.49}$$

$$y_2''(x) = a\left(y_1''(x) \ln x + \frac{2y_1'}{x} - \frac{y_1}{x^2}\right) + \sum_{n=0}^{\infty} (n-p)(n-p-1) b_n x^{n-p-2} \tag{4.50}$$

and substitute them into the Bessel equation (4.18)

$$a(x^2 y_1'' + x y_1' + (x^2 - p^2) y_1) \ln x + 2ax y_1' + \sum_{n=0}^{\infty} (n-p)(n-p-1) b_n x^{n-p}$$

$$+ \sum_{\substack{n= \\ n=0}}^{\infty} (n-p) b_n x^{n-p} + \sum_{n=0}^{\infty} b_n x^{n-p-2} - \sum_{n=0}^{\infty} p^2 b_n x^{n-p} = 0. \tag{4.51}$$

Since $y_1(x)$ is a solution of Bessel equation, we can write

$$x^2 y_1'' + x y_1' + (x^2 - p^2) y_1 = 0 \tag{4.52}$$

and

$$2ax y_1' = 2ax \sum_{n=0}^{\infty} (-1)^n \frac{(2n+p).\, x^{2n+p-1}}{n!\,(n+p)!\, 2^{2n+p}} = \sum_{n=1}^{\infty} (-1)^n \frac{2a(2n+p)}{n!\,(n+p)!} \left(\frac{x}{2}\right)^{2n+p}.$$

Multiply the equation by x^p and substituting it and equation (4.52) into equation (4.51) gives

$$\sum_{n=0}^{\infty}(-1)^n \frac{2^{p+2}a(2n+p)}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2(n+p)} + \sum_{n=0}^{\infty} n(2n-p)b_n x^n + \sum_{n=0}^{\infty} b_{n-2}x^n = 0 \quad (4.53)$$

where $x^n$is need to be equal zero to provide this equality for $n = 0,1,\dots$ (Xie [12]). In equation (4.53), there is no contribution of x^n: $0 \le n < 2p$.

Separating the terms correponding to $n = 0$ and $n = 1$ and setting the coefficients equal to zero yields

$$0.(0-2p)b_0 = 0$$

where b_0 is an arbitrary and

$$1.(1-2p)b_1 = 0 \quad \Rightarrow \quad b_1 = 0$$

Let's assume that $b_0 = 1$. The recurrence relation for $2 \le n < 2p$ is

$$n(n-2p)b_n + b_{n-2} = 0 \quad \Rightarrow \quad b_n = \frac{b_{n-2}}{n(2p-n)}. \quad (4.54)$$

Thus,

$$b_2 = \frac{b_0}{2(2p-2)} = \frac{1}{2^2.1(p-1)} \quad (n=2),$$

$$b_4 = \frac{b_2}{4(2p-4)} = \frac{1}{2^4.2!\,(p-1)(p-2)} \quad (n=4).$$

The pattern for the coefficients is now apparent. We find

$$b_{2k} = \frac{b_{2k-2}}{2k(2p-2k)} = \frac{1}{2^{2k}.k!\,(p-1)(p-2)\dots(p-k)} = \frac{(p-k-1)!}{2^{2k}.k!\,(p-1)!}$$

where $b_{2n+1} = 0$ is obvious for $n = 0,1,2,\dots$.

From the coefficient x^{2p} in equation (4.53)

$$\frac{2^{p+1}ap}{p!}\left(\frac{x}{2}\right)^{2p} + b_{2p-2} = 0$$

then

$$a = -2^{p-1}(p-1)!\, b_{2(p-1)} = -\frac{1}{2^{p-1}(p-1)}. \tag{4.55}$$

Let's take the arbitrary contant $b_{2p} = 0$.

By the cofficient $x^{2(n+p)}$, $n \geq 1$

$$(-1)^n \frac{2^{p+1}a(2n+p)}{n!\,(n+p)!}\left(\frac{1}{2}\right)^{2(n+p)} + (2n+2p)(2n)b_{2(n+p)} + b_{2(n-1+p)} = 0$$

Leave the term $b_{2(n+p)}$ alone

$$b_{2(n+p)} = (-1)^{n+1}\frac{2^{p-1}a(2n+p)}{n(n+p)n!\,(n+p)!}\left(\frac{1}{2}\right)^{2(n+p)} - \frac{b_{2(n-1+p)}}{2^2 n(n+p)}.$$

or equivalently,

$$b_{2(n+p)} = (-1)^{n+1}\frac{2^{p-1}aA_n}{n!\,(n+p)!}\left(\frac{1}{2}\right)^{2(n+p)}$$

where $A_n = \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right)\left(\frac{1}{1+p} + \frac{1}{2+p} + \cdots + \frac{1}{n+p}\right)$. So, linealy independent second solution is

$y_2(x) = aJ_p(x)\ln x$

$$+x^{-p}\left(\sum_{n=0}^{p-1}\frac{(p-n-1)!}{n!\,(p-1)!}\left(\frac{x}{2}\right)^{2n} + a\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{p-1}aA_n}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2(n+p)}\right). \tag{4.56}$$

or it can be expressed by

$$y_2(x) = a\left(J_p(x)\ln x + \frac{x^{-p}}{a}\sum_{n=0}^{p-1}\frac{(p-n-1)!}{n!\,(p-1)!}\left(\frac{x}{2}\right)^{2n}\right.$$

$$\left. + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\psi(n+1)+\psi(n+p+1)-\psi(p+1)+\gamma}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2n+p}\right)$$

$$= a\left(J_p(x)\ln\frac{x}{2} - \frac{1}{2}\sum_{n=0}^{p-1}\frac{(p-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n-p}\right.$$

$$-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n}\frac{\psi(n+1)+\psi(n+p+1)}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2n+p}$$

$$\left. -\frac{1}{2}(\gamma - \psi(p+1) - 2\ln 2)J_p(x)\right)$$

$$= a\left(\frac{\pi}{2}Y_p(x) - \frac{1}{2}(\gamma - \psi(p+1) - 2\ln 2)J_p(x)\right), \quad 0 < x < \infty \qquad (4.57)$$

where the function ψ is a term of the Bessel function which is defined by

$$\gamma + \psi(n+1) = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$\psi(1) = -\gamma$$

$$\psi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$$

and the Bessel function of the second kind is formulated as

$$Y_p(x) = \frac{2}{\pi}J_p(x)\ln\frac{x}{2} - \frac{1}{\pi}\sum_{n=0}^{p-1}\frac{(p-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n-p}$$

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n}\frac{\psi(n+1)+\psi(n+p+1)}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2n+p}.$$

The general solution can be expressed in terms of the Bessel function of the first and second kind as follows:

$$y(x) = C_1 J_p(x) + C_2 Y_p\,(x). \tag{4.58}$$

This solution is the case of $\alpha_1 - \alpha_2$ being an integer. The second solution contains the logaritmic function $\ln x$ (Xie [12]).

4.1.4.4 $\boldsymbol{p = k + \frac{1}{2}, k = 0, 1, \ldots}$ and $\boldsymbol{\alpha_1 - \alpha_2 = 2k + 1}$ Being an Integer

The solution of first Frobenius series solution is

$$y_1(x) = J_{k+\frac{1}{2}}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)\Gamma\left(n + k + \frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2n+k+\frac{1}{2}} \qquad 0 < x < \infty \tag{4.59}$$

and first derivative of the solution is

$$y_1'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{2n + k + \frac{1}{2}}{2n!\,\Gamma\left(n + k + \frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2n+k-\frac{1}{2}} \tag{4.60}$$

By using equation (4.60), linearly independent second solution can be derived as

$$y_2(x) = a y_1(x) \ln x + x^{-\left(k+\frac{1}{2}\right)} \sum_{n=0}^{\infty} b_n x^n . \qquad 0 < x < \infty \tag{4.61}$$

Let’s take the derivatives of this equation with respect to x

$$y_2'(x) = a\left(y_1'(x) \ln x + \frac{y_1}{x}\right) + \sum_{n=0}^{\infty} \left(n - k - \frac{1}{2}\right) b_n x^{n-k-\frac{3}{2}}$$

$$y_2''(x) = a\left(y_1''(x) \ln x + \frac{2y_1'}{x} - \frac{y_1}{x^2}\right) + \sum_{n=0}^{\infty} \left(n - k - \frac{1}{2}\right)\left(n - k - \frac{3}{2}\right) b_n x^{n-k-\frac{5}{2}}$$

and substitute them into the Bessel equation

$$a(x^2y_1'' + xy_1' + (x^2 - p^2)y_1)\ln x + 2axy_1' + \sum_{n=0}^{\infty}(n - 2k - 1)b_n x^{n-k-\frac{1}{2}}$$

$$+ \sum_{n=0}^{\infty} b_n x^{n-k+\frac{3}{2}} = 0\,. \tag{4.62}$$

Since $y_1(x)$ is a solution of Bessel equation, we can write

$$x^2y_1'' + xy_1' + (x^2 - p^2)y_1 = 0$$

By multipling equation (4.62) by $x^{k+\frac{1}{2}}$, we obtain

$$a\sum_{n=0}^{\infty}(-1)^n \frac{2^{k+\frac{3}{2}}\left(2n + k + \frac{1}{2}\right)}{n!\,\Gamma\left(n + k + \frac{3}{2}\right)!}\left(\frac{x}{2}\right)^{2n+2k+1} + \sum_{n=0}^{\infty} n(n - 2k - 1)b_n x^n$$

$$+ \sum_{n=0}^{\infty} b_n x^{n+2} = 0 \tag{4.63}$$

where $x^{2n} = 0 =$ to hold this equality for $n = 0,1, \dots$. In equation (4.63), there is no contribution of x^n: $0 \le n < 2k + 1$.

For $n = 0$,

$$x^0: 0.(0 - 2k - 1)b_0 = 0$$

where b_0 is an arbitrary.

For $n = 1$,

$$x^1: 1.(1 - 2k - 1)b_1 = 0 \qquad \Rightarrow \qquad b_1 = 0$$

For n: $2 \le n < 2p$ we can easily see that

$$b_{2m+1} = -\frac{b_{2m-1}}{(2m + 1)(2m - 2k)} = 0\,. \quad 1 < m < k$$

From the coefficient x^{2k+1}

$$a\frac{k+\frac{1}{2}}{2^{k-\frac{1}{2}}\Gamma\left(k+\frac{3}{2}\right)}+(2k+1).0.b_{2k+1}+b_{2k-1}=0$$

then

$$a=0 \tag{4.64}$$

Plugging equation (4.64) into equation (4.63)and by the coefficient x^n

$$n(n-2p)b_n+b_{n-2}=0 \quad \Rightarrow \quad b_{2m-1}=0$$

and

$$b_{2m}=-\frac{b_{2m-2}}{2^2m(m-p)} \qquad \text{for } m=1,2,\ldots$$

or equivalently,

$$b_{2m}=(-1)^m\frac{b_0}{2^{2m}m!\,(1-p)(2-p)\ldots(m-p)}.$$

If we choose $b_0=\left(2^{-p}\Gamma(1-p)\right)^{-1}$, then

$$b_{2m}=\frac{(-1)^m}{m!\,\Gamma(m-p+1)}\left(\frac{1}{2}\right)^{2m-p}.$$

Thus, linealy independent second solution is

$$y_2(x)=x^{-p}\sum_{n=0}^{\infty}b_{2n}\,x^{-2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n-p+1)}\left(\frac{x}{2}\right)^{2n+p}=J_{-p}(x). \qquad 0<x<\infty$$

The general solution can be expressed by

$$y(x)=C_1J_p(x)+C_2J_{-p}(x).$$

This solution is same as the case of $\alpha_1-\alpha_2$ being a positive integer. The second solution is a Frobenius seriesand does not contain the logaritmic function $\ln x$ (Xie [12]).

In application, the Bessel function is rarely in standart form.

Let’s consider the second order ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{1-2\alpha}{x}\frac{dy}{dx} + \left(\beta\rho x^{\rho-1} + \frac{\alpha^2 - p^2\rho^2}{x^2}\right)y = 0, \qquad x > 0 \tag{4.65}$$

where α, β, p and ρ are constant numbers.

Now, we will interchange the Bessel equation

$$\xi^2\frac{d^2\eta}{d\xi^2} + \xi\frac{d\eta}{d\xi} + (\xi^2 - p^2)\eta = 0, \quad \xi > 0 \tag{4.66}$$

to equation (4.65) as stated below.

By using change of variables method $\xi = \beta x^\rho$, we can write

$$\xi\frac{dy}{d\xi} = \xi\frac{dy/dx}{d\xi/dx} = \beta x^\rho \frac{1}{\beta\rho x^{\rho-1}}\frac{dy}{dx} = \frac{x}{\rho}\frac{dy}{dx} \quad \Rightarrow \quad \xi\frac{d}{d\xi} = \frac{x}{\rho}\frac{d}{dx} \tag{4.67}$$

which simplifies equation (4.66) as

$$\xi\frac{d}{d\xi}\left(\xi\frac{d\eta}{d\xi}\right) + (\xi^2 - p^2)\eta = 0 \tag{4.68}$$

and then substitute equation (4.67) in eq. (4.68)

$$\frac{x}{\rho}\frac{d}{dx}\left(\frac{x}{\rho}\frac{d\eta}{dx}\right) + (\beta^2x^{2\rho} - p^2)\eta = 0.$$

For $\eta = x^{-\alpha}y$

$$x\frac{d\eta}{dx} = x^{1-\alpha}\frac{dy}{dx} - \alpha x^{-\alpha}y$$

and

$$x\frac{d}{dx}\left(x\frac{d\eta}{dx}\right) = x^{2-\alpha}\frac{d^2y}{dx^2} + (1-2\alpha)x^{1-\alpha}\frac{dy}{dx} + \alpha^2x^{-\alpha}y.$$

Finally, the equation (4.66) becomes

$$x^{2-\alpha}\frac{d^2y}{dx^2} + (1-2\alpha)x^{1-\alpha}\frac{dy}{dx} + \alpha^2x^{-\alpha}y + (\beta^2x^{2\rho} - p^2)\rho^2x^{-\alpha}y = 0.$$

which is equal to equation (4.65).

4.2 Expression of Modal Amplitudes via Bessel Functions

Let's investigate Miller's second case.

Case 2: $ct = u\cosh v,\ z = u\sinh v;\ 0 \le u < \infty,\ -\infty < v < \infty$. Initially, we need to find relations for $u(z,t)$ and $v(z,t)$. In order to do that, let us take square of two equations mentioned

$$c^2t^2 = u^2\cosh^2 v, \quad z^2 = u^2\sinh^2 v$$

and then

$$c^2t^2 - z^2 = u^2(\cosh^2 v - \sinh^2 v) = u^2.$$

Hence,

$$u = \sqrt{c^2t^2 - z^2} \tag{4.69}$$

since $0 \le u < \infty$, by definition. Similarly,

$$\frac{z}{ct} = \frac{u\sinh v}{u\cosh v} = \tanh v$$

and thus,

$$v = \operatorname{arctanh}\frac{z}{ct} \equiv \frac{1}{2}\ln\frac{ct+z}{ct-z}. \tag{4.70}$$

The next step is to calculate the partial derivatives of $u(z,t)$ and $v(z,t)$ in equation (4.5).

$$\frac{1}{c}\frac{\partial u}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\sqrt{c^2t^2 - z^2} = \frac{ct}{\sqrt{c^2t^2 - z^2}} = \frac{u\cosh v}{u} = \cosh v\,; \tag{4.71}$$

$$\frac{1}{c}\frac{\partial v}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\ln\frac{ct+z}{ct-z}\right) = -\frac{z}{c^2t^2 - z^2} = -\frac{u\sinh v}{u^2} = -\frac{\sinh v}{u};$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial u}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}\cosh v = \frac{1}{c}\frac{\partial v}{\partial t}\sinh v = -\frac{\sinh^2 v}{u}; \tag{4.72}$$

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial v}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{\sinh v}{u}\right) = \frac{\sinh 2v}{u^2}; \tag{4.73}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\sqrt{c^2t^2 - z^2} = -\frac{z}{\sqrt{c^2t^2 - z^2}} = -\frac{u\sinh v}{u} = -\sinh v\,; \tag{4.74}$$

$$\frac{\partial v}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{2}\ln\frac{ct + z}{ct - z}\right) = \frac{ct}{c^2t^2 - z^2} = \frac{u\cosh v}{u^2} = \frac{\cosh v}{u}; \tag{4.75}$$

$$\frac{\partial^2 u}{\partial z^2} = -\frac{\cosh^2 v}{u}; \tag{4.76}$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{\sinh 2v}{u^2}; \tag{4.77}$$

Substituting these results in Klein-Gordon equation (4.5), a new representation in terms of the variables u and v arises as

$$[\cosh^2 v - \sinh^2 v]\frac{\partial^2 F}{\partial u^2} + \left[\frac{\sinh^2 v}{u^2} - \frac{\cosh^2 v}{u^2}\right]\frac{\partial^2 F}{\partial v^2} + \left[-\frac{\sinh^2 v}{u} + \frac{\cosh^2 v}{u}\right]\frac{\partial F}{\partial u}$$
$$+ \left[\frac{\sinh 2v}{u^2} - \frac{\sinh 2v}{u^2}\right]\frac{\partial F}{\partial v} + 2\left[-\frac{\sinh v\cosh v}{u} + \frac{\sinh v\cosh v}{u}\right]\frac{\partial^2 F}{\partial u \partial v}$$
$$+ F = 0$$

The last equation can be simplified as

$$\frac{\partial^2 F}{\partial u^2} + \frac{1}{u}\frac{\partial F}{\partial u} + F - \frac{1}{u^2}\frac{\partial^2 F}{\partial v^2} = 0 \tag{4.78}$$

Now, the solution of equation (4.78) can be obtained by separation of the variables (u, v). So, if we choose

$$F(u, v) = U(u)V(v)\,, \tag{4.79}$$

then equation (4.78) can be rewritten as

$$\left(\frac{\partial^2}{\partial u^2}+\frac{1}{u}\frac{\partial}{\partial u}+1-\frac{1}{u^2}\frac{V''(v)}{V(v)}\right)U(u)=0. \tag{4.80}$$

In equation (4.80), $V''(v)=\alpha^2 V(v)$ where α is a free parameter that is a constant of separation of the variables u and v. Solution of this equation is

$$V(v)=e^{\pm\alpha v}. \tag{4.81}$$

Thus, the function $U(u)$ satisfies Bessel equation

$$\left(\frac{\partial^2}{\partial u^2}+\frac{1}{u}\frac{\partial}{\partial u}+1-\frac{\alpha^2}{u^2}\right)U(u)=0. \tag{4.82}$$

which has two linearly independent solutions, in particularly,

$$U(u)=A_\alpha J_\alpha(u)+B_\alpha Y_\alpha(u), \tag{4.83}$$

where A_α and B_α are arbitrary constants, J_α and Y_α are Bessel functions.

Example 4.1 Let us consider

$V(v)=e^{-\alpha v}, \qquad U(u)=J_\alpha(\kappa u).$

So

$$F(z,t)=U(u)V(v)=e^{-\alpha v}J_\alpha(\kappa u). \tag{4.84}$$

Since $v=\frac{1}{2}\ln\frac{ct+z}{ct-z}$, $u=\sqrt{c^2t^2-z^2}$ and

$$e^{-\alpha v}=e^{-\frac{\alpha}{2}\ln\frac{ct+z}{ct-z}}=e^{\ln\left(\frac{ct-z}{ct+z}\right)^{\alpha/2}}=\left(\frac{ct-z}{ct+z}\right)^{\alpha/2}$$

then right-hand side of equation (4.84) become

$$F(z,t)=\left(\frac{ct-z}{ct+z}\right)^{\alpha/2}J_\alpha\left(\kappa\sqrt{c^2t^2-z^2}\right)\equiv F_\alpha(z,t) \tag{4.85}$$

where α is free numerical parameter. The equation (4.85) transforms into

$$F_\alpha(z,t) = J_\alpha(\kappa ct) \quad at\ z = 0. \tag{4.86}$$

When we choose the free parameter α as integer: $\alpha = 0,1,2,\ldots$, then a set of function

$$\{J_n(\kappa ct)\}_{n=0}^{\infty}, \tag{4.87}$$

is complete. It means that arbitrary function of time $\varphi(t)$ can be expressed in the form of Neumann series, that is,

$$\varphi(t) = \sum_{n=0}^{\infty} C_n J_n(\kappa ct)$$

where C_n's are appropriate constants.

4.3 Differential Equation and Functions of Airy

Consider the Airy's differential equation which is named after the applied mathematician and astronomer G. B. Airy, who investigated the second order differential equation,

$$y'' - zy = 0. \tag{4.88}$$

Since the term $-z$ is analytic at $z = 0$ then $z = 0$ is an ordinary point for equation (4.88). Hence, we can express its general solution as a power series in the form

$$y(z) = \sum_{n=0}^{\infty} a_n z^n \tag{4.89}$$

The first and second derivatives of equation (4.89) with respect to z are

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \tag{4.90}$$

and

$$y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}, \tag{4.91}$$

respectively.

Substituting these power series into equation (4.88) we find

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} - z\sum_{n=0}^{\infty} a_n z^n = 0$$

or, equivalently,

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} - \sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

The next goal is simplfy this expression by combining everything into a single summation. Let's shift the indeces so that the general term in each is a constant times z^n, that is,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n - \sum_{n=1}^{\infty} a_{n-1}z^n = 0\,. \tag{4.92}$$

For the first summation, we split of the 0-th term as

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n = 2.1.\,a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

With these changes of indeces, equation (4.92) becomes

$$2a_2 + \sum_{n=1}^{\infty}\big((n+2)(n+1)a_{n+2}z^n - a_{n-1}z^n\big) = 0$$

or

$$2a_2 + \sum_{n=1}^{\infty}\big((n+2)(n+1)a_{n+2} - a_{n-1}\big)z^n = 0\,.$$

Since the power series is equal to zero, all coefficients are supposed to be equal to zero, too. Therefore,

$a_2 = 0$

and the recurrence relation is

$$(n+2)(n+1)a_{n+2} = a_{n-1}$$

or

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \qquad \text{for all } n = 1,2,3, \dots \, . \tag{4.93}$$

To find the series solution to equation (4.88), the values of a_n's are needed to determine.

For $n = 1$, equation (4.93) is

$$a_3 = \frac{a_0}{3.2}$$

Continuing,

$$a_4 = \frac{a_1}{4.3}$$

$$a_5 = \frac{a_2}{5.4} = 0$$

$$a_6 = \frac{a_3}{6.5} = \frac{a_0}{(6.5)(3.2)}$$

$$a_7 = \frac{a_4}{7.6} = \frac{a_1}{(7.6)(4.3)}$$

$$a_8 = \frac{a_5}{8.7} = 0$$

$$a_9 = \frac{a_6}{9.8} = \frac{a_0}{(9.8)(6.5)(3.2)}$$

The pattern for the coefficients is now apparent. So, let's write down all the cases:

(i) The terms $a_2, a_5, a_8, \dots$ are equal to zero, that is,

$$a_{3n-1} = 0 \quad \text{for all } n = 1,2, \dots$$

(ii) The terms $a_3, a_6, a_9, \dots$ are multiples of a_0, so

$$a_{3n} = \frac{a_0}{[(3n)(3n-1)][(3n-3)(3n-4)]\ldots[6.5][3.2]}$$

for all $n = 1,2,\ldots$.

(iii) All the terms $a_4, a_7, a_{10}, \ldots$ are multiples of a_1 and so

$$a_{3n+1} = \frac{a_1}{[(3n+1)(3n)][(3n-2)(3n-3)]\ldots[7.6][4.3]}$$

for all $n = 1,2,\ldots$.

Finally, the general solution of Airy's equation is given by

$$y(z) = a_0\left[\sum_{n=1}^{\infty}\frac{z^{3n}}{(3n)(3n-1)(3n-3)(3n-4)\ldots 3.2}\right]$$

$$+a_1\left[\sum_{n=1}^{\infty}\frac{z^{3n+1}}{(3n+1)(3n)(3n-2)(3n-3)\ldots 4.3}\right]. \qquad (4.94)$$

and this is also called as the Airy functions.

After G. B. Airy, the Airy functions were well-studied by H.Jeffreys and J.C.P Miller. It was them who introduced the Airy functions Ai and Bi as special solutions of Airy's equation. So, the solution of the Airy's equation was expressed by

$$y(z) = c_1 Ai + c_2 Bi, \qquad z \in \mathbb{R}$$

where Ai is the first solution of the Airy's differential equation which is called as the Airy function of first kind and Bi is the second solution of the differential equation which is named as the Airy function of second kind.

The Airy functions Ai and Bi have rather simple integral representations through sine, cosine and power functions as follows:

$$Ai(z) = \frac{1}{\pi}\int_0^{\infty}\cos\left(\frac{t^3}{3} + zt\right)dt, \qquad z \in \mathbb{R} \qquad (4.95)$$

and

$$Bi(z) = \frac{1}{\pi}\int_0^\infty \left[\sin\left(\frac{t^3}{3} + zt\right) + \exp\left(-\frac{t^3}{3} + zt\right)\right] dt\,, \qquad z \in \mathbb{R}. \tag{4.96}$$

4.4 Expression of Modal Amplitudes via Airy Functions

Let's investigate Miller's fifth case.

Case 5: $ct = \frac{1}{2}(u-v)^2 + u + v,\ \ z = -\frac{1}{2}(u-v)^2 + u + v,\ \ -\infty < u, v < \infty$. First, the relation for $u(z,t)$ and $v(z,t)$ should be found. So,

$$ct + z = 2(u+v), \qquad ct - z = (u-v)^2$$

Adding above equations side by side gives

$$u = \frac{ct+z}{4} - \frac{\sqrt{ct-z}}{2} \tag{4.97}$$

and

$$v = \frac{ct+z}{4} + \frac{\sqrt{ct-z}}{2} \tag{4.98}$$

Taking partial derivatives of equation (4.97) and (4.98) to substitute them into equation (4.5)

$$\frac{1}{c}\frac{\partial u}{\partial t} = \frac{1}{4} - \frac{1}{4\sqrt{ct-z}} = \frac{1}{4} - \frac{1}{4(u-v)}$$

$$\frac{1}{c}\frac{\partial v}{\partial t} = \frac{1}{4} + \frac{1}{4\sqrt{ct-z}} = \frac{1}{4} + \frac{1}{4(u-v)}$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial u}{\partial t}\right) = \frac{1}{32(u-v)^{3/2}}$$

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial v}{\partial t}\right) = -\frac{1}{32(u-v)^{3/2}}$$

$$\frac{\partial u}{\partial z} = \frac{1}{4} + \frac{1}{4(u-v)}$$

$$\frac{\partial v}{\partial z} = \frac{1}{4} - \frac{1}{4(u-v)}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{32(u-v)^{3/2}}$$

$$\frac{\partial^2 v}{\partial z^2} = -\frac{1}{32(u-v)^{3/2}}$$

Substituting above results to equation (4.5) yields

$$\frac{1}{4(u-v)}\frac{\partial^2 F}{\partial u^2} - \frac{1}{4(u-v)}\frac{\partial^2 F}{\partial v^2} + F = 0 \tag{4.99}$$

(Aksoy and Tretyakov [8]), (Akgun [2]).

Now, the solution of equation (4.99) can be obtained by separation of the variables (u, v). So, if we choose

$F(u, v) = U(u)V(v)$

then equation (4.99) can be rewritten as

$$\frac{1}{U(u)}\frac{d^2 U(u)}{du^2} + 4u = \frac{1}{V(v)}\frac{d^2 V(v)}{dv^2} + 4v = 4\alpha. \tag{4.100}$$

where α is a constant of separation of the variables.

At this point, it is reasonable to change notations for the variables u and v as

$$\bar{u} = \sqrt[3]{4}(\alpha - u) \quad \text{and} \quad \bar{v} = \sqrt[3]{4}(\alpha - v),$$

then equation (4.100) becomes

$$\frac{d^2 U(\bar{u})}{d\bar{u}^2} - \bar{u}U(\bar{u}) = 0 \quad \text{and} \quad \frac{d^2 V(\bar{v})}{d\bar{v}^2} - \bar{v}V(\bar{v}) = 0. \tag{4.101}$$

Since equation (4.101) is standart Airy's differential equation, it has two linearly independet solution in the form of

$$U(\bar{u}) = C_1 Ai(\bar{u}) + D_1 Bi(\bar{u}) \tag{4.102}$$

and

$$V(\bar{v}) = C_2 Ai(\bar{v}) + D_2 Bi(\bar{v}) \tag{4.103}$$

where C_1, C_2, D_1 and D_2 are arbitrary constants and Ai and Bi are Airy Functions. Consequently, we shown that Miller's fifth case can be written in terms of Airy Functions.

CHAPTER 5

GRAPHICS

The energy W_α and surplus of energy sW_α are exhibited in the following figures for the changing values of α. The Mapple programme (Appendix B) is used for drawings.

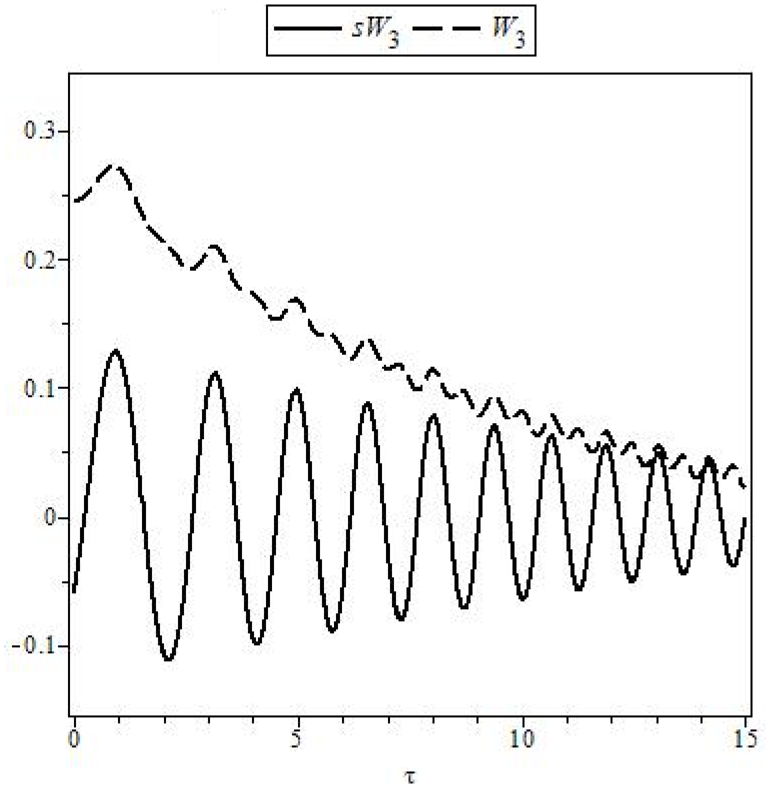

Figure 5.1 Time dependence of W_3 and sW_3 for $0 \leq \tau \leq 15$.

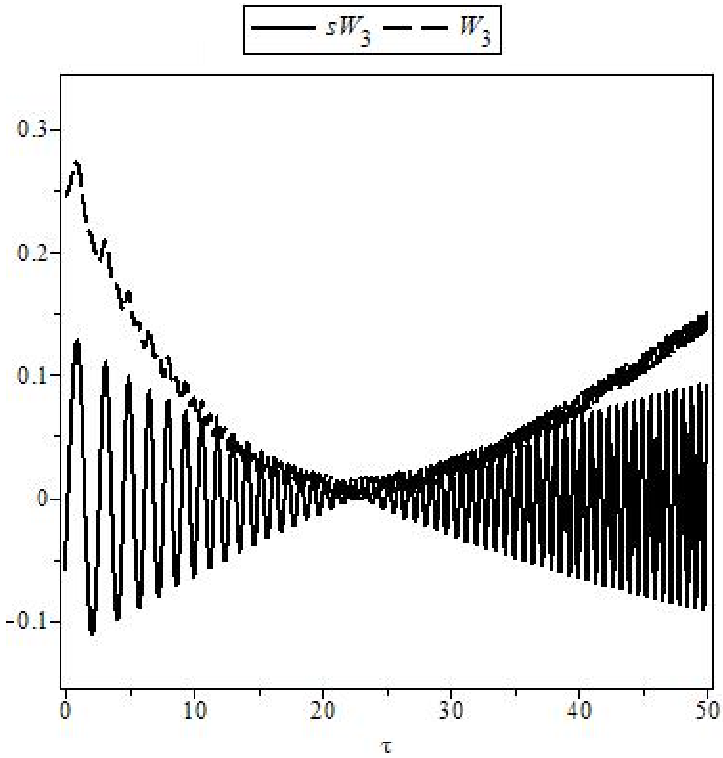

Figure 5.2 Time dependence of W_3 and sW_3 for $0 \leq \tau \leq 50$.

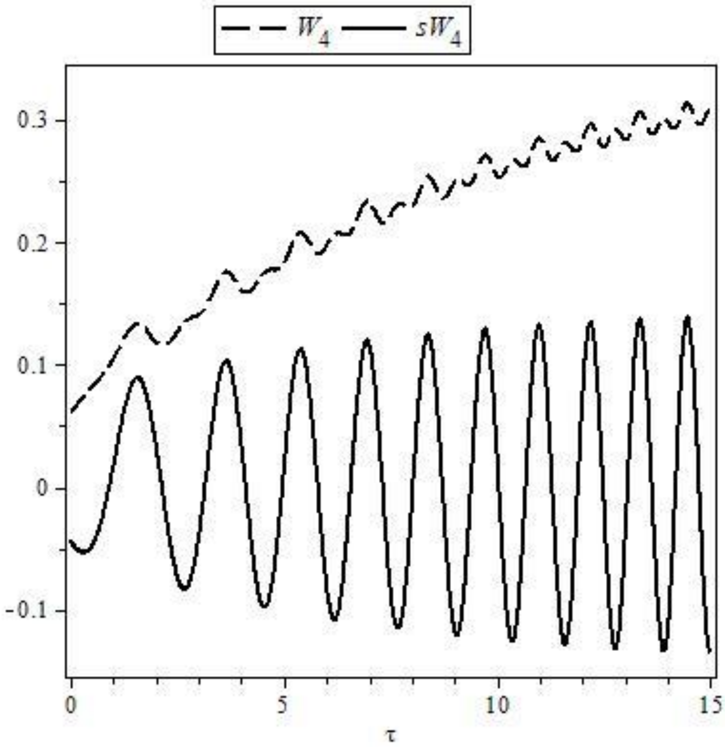

Figure 5.3 Time dependence of W_4 and sW_4 for $0 \leq \tau \leq 15$.

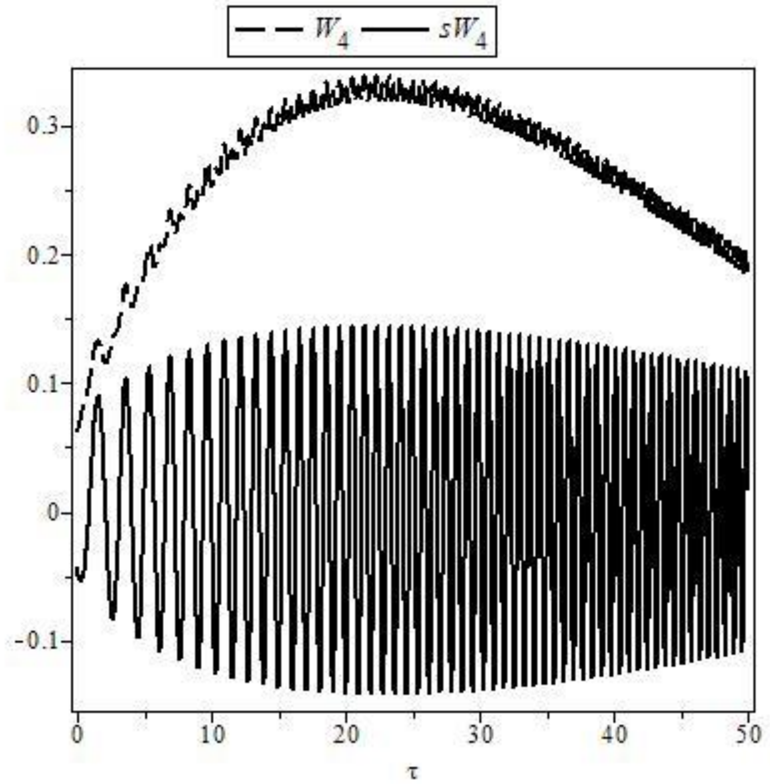

Figure 5.4 Time dependence of W_4 and sW_4 for $0 \leq \tau \leq 50$.

The following figures are exhibited the changes of modal ampitudes A and B for $\alpha = 0,1,2,\ldots$ which are depend on the solution F of KGE functions, in the cylindrical and spherical form. These oscillations are in time t (i.e., in τ) and axial coordinate z (i.e., in ξ).

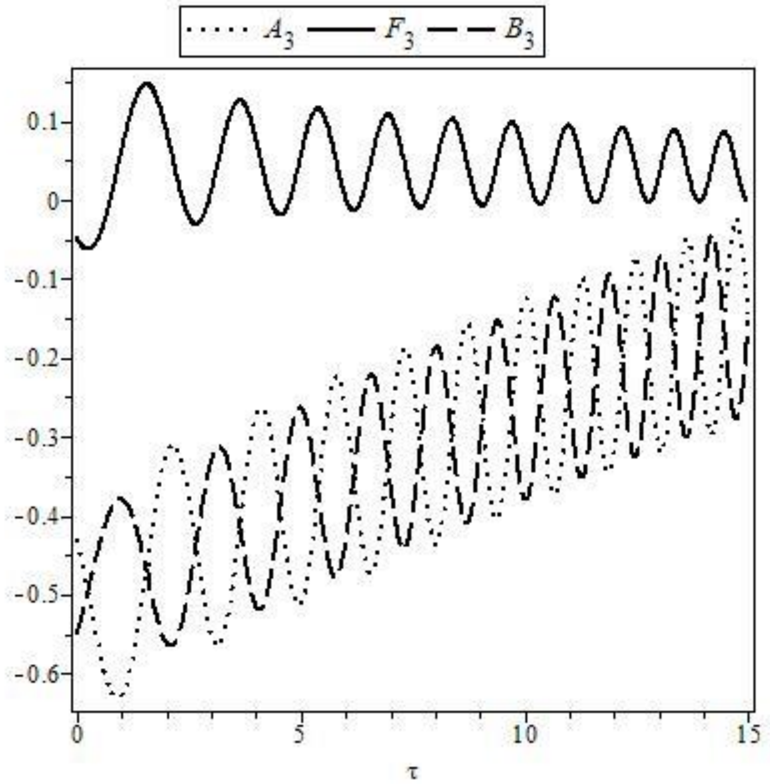

Figure 5.5 The changes of modal amplitudes in interval for $0 \leq \tau \leq 15$ where $\alpha = 3$.

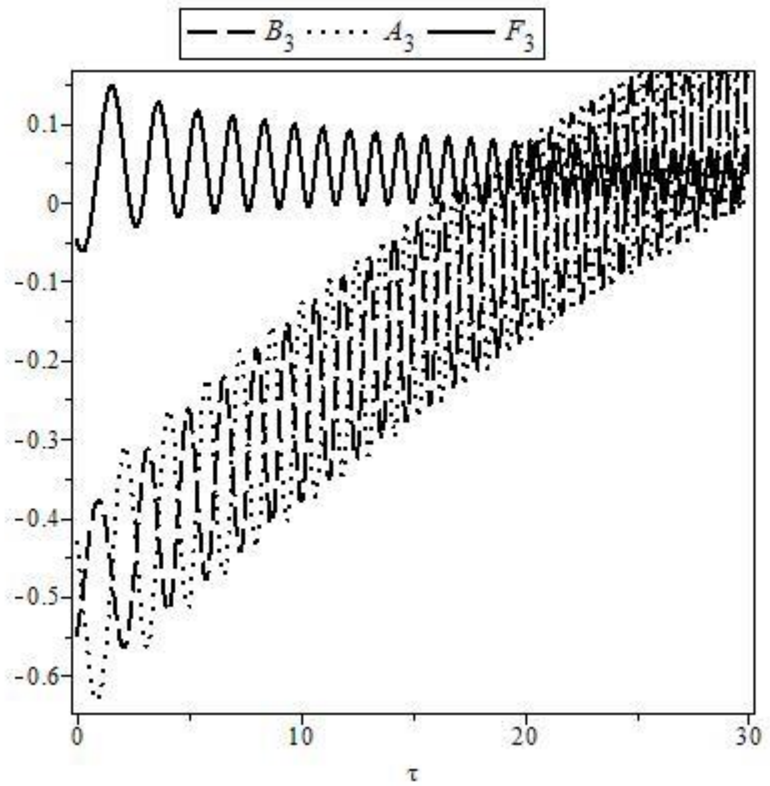

Figure 5.6 The changes of modal amplitudes in interval for $0 \leq \tau \leq 30$ where $\alpha = 3$.

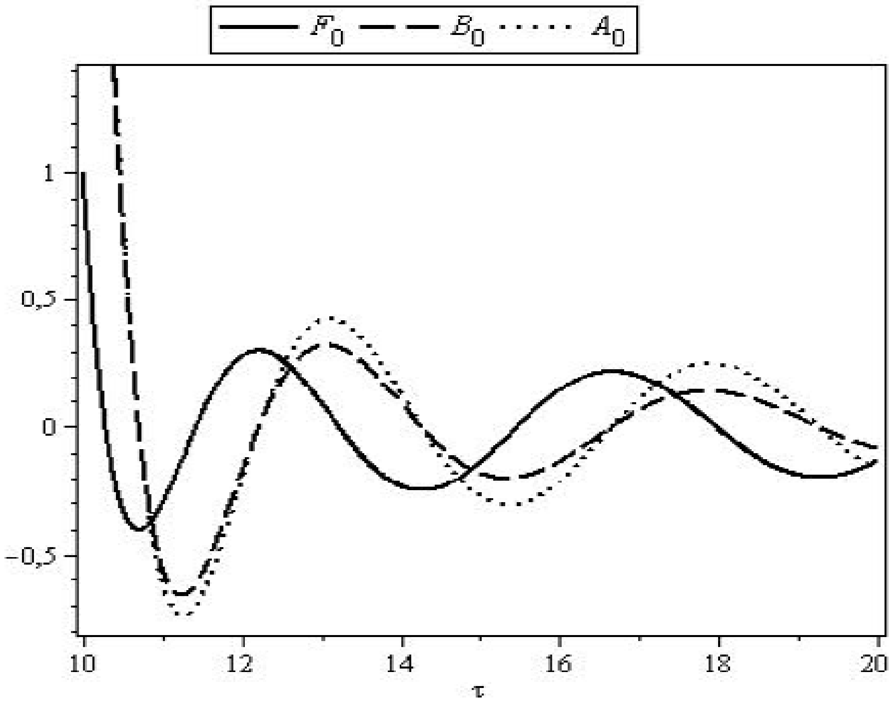

Figure 5.7 The changes of modal amplitudes in interval for $10 \leq \tau \leq 20$ where $\alpha = 0$, $\xi = 10$.

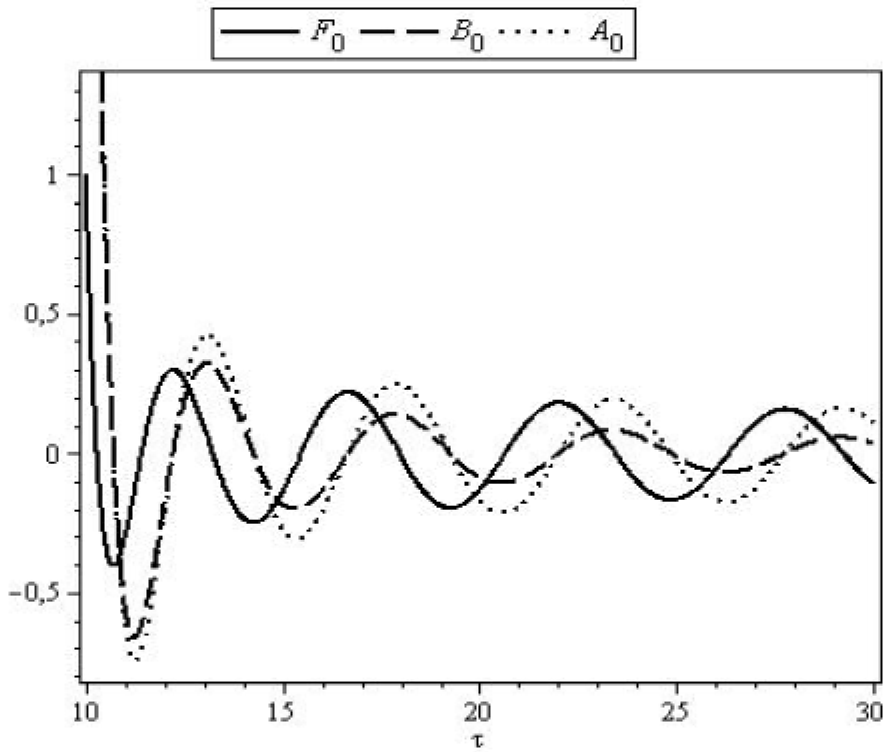

Figure 5.8 The changes of modal amplitudes in interval for $10 \leq \tau \leq 30$ where $\alpha = 0$, $\xi = 10$.

In the following graphs, the speed v_α is calculated with transverse $w_\alpha^t$and longitudinal w_α^l parts of energy, separately.

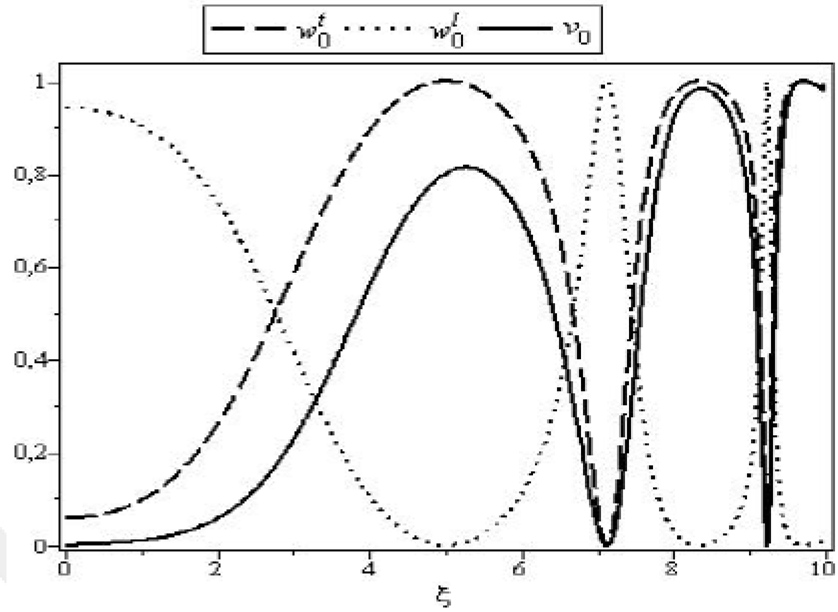

Figure 5.9 The change of v_0, w_0^t and w_0^l in $0 \leq \xi \leq \tau$ where $\tau = 10$ and $\alpha = 0$

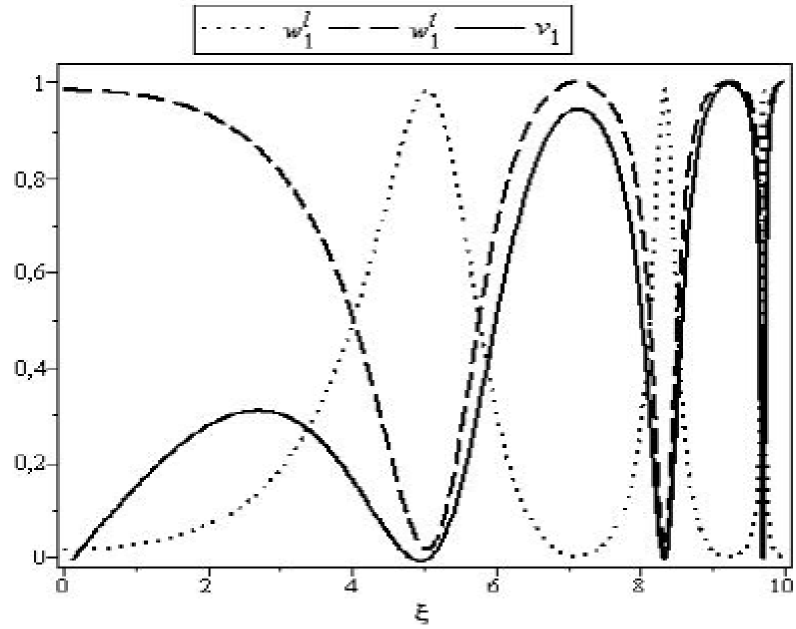

Figure 5.10 The change of v_1, w_1^t and w_1^l in $0 \leq \xi \leq \tau$ where $\tau = 10$ and $\alpha = 1$

CHAPTER 6

RESULTS AND DISCUSSION

In this study, the time-domain waveguide modes are expressed analytically by a method of Evolutionary Approach to Electromagnetics (EAE). A hollow waveguide is considered with the perfect electric conductor surfaces.

Initialy, relevant definition of the waveguide were given. To obtain the components of transverse electric (TE) and transverse magnetic (TM) time-domain modes, Maxwell's equation and fields are decomposed into their transverse and longitudinal parts. Two main problems are introduced to analyze transferring of transient signals along hollow waveguide. Modal bases are determined by obtaining the solution of Dirichlet and Neumann boundary eigenvalue problems. Solving Klein-Gordon Equation yields modal amplitudes as a function of time and axial coordinate . As a result, every component of time-domain modes are expressed as a product of modal bases and modal amplitudes.

According to this new method, we expressed the solutions of Klein-Gordon equations (modal amplitudes) in terms of some sellected proper functions. By using Miller's list, the modal amplitudes were obtained in terms of Airy and Bessel functions with especially sellected parameters. The informations such as type (or mode) of wave spanning over the waveguide or form and dimension of waveguide cross-section can be attained by using modal amplitudes.

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APPENDIX-A

MILLER'S LIST

Coordinate Systems for Separation of Variables in Klein-Gordon Equation

1. $ct = u, z = v$: $U(u)V(v)$ is a product of the exponentials
2. $ct = u\cosh v, \quad z = u\sinh v; \quad 0 \le u < \infty, \quad -\infty < v < \infty$: $U(u)V(v)$ is a product of Bessel function and an exponential.
3. $ct = \frac{1}{2}(u^2 + v^2), z = uv; -\infty < u < \infty, 0 \le v < \infty$: $U(u)V(v)$ is a product of parabolic cylinder functions.
4. $ct = uv, z = \frac{1}{2}(u^2 + v^2); -\infty < u < \infty, 0 \le v < \infty$: $U(u)V(v)$ is a product of parabolic cylinder functions.
5. $ct = \frac{1}{2}(u - v)^2 + u + v, \quad z = -\frac{1}{2}(u - v)^2 + u + v; \quad -\infty < u, v < \infty$: $U(u)V(v)$ is a product of Airy functions.
6. $2ct = \cosh\frac{u-v}{2} + \sinh\frac{u+v}{2}, \quad 2z = \cosh\frac{u-v}{2}\sinh\frac{u+v}{2}$: $\quad -\infty < u, v < \infty$: $U(u)V(v)$ is a product of Mathieu functions.
7. $ct = \sinh(u - v) + \frac{1}{2}e^{u+v}, \quad z = \sinh(u - v) - \frac{1}{2}e^{u+v}$: $\quad -\infty < u, v < \infty$: $U(u)V(v)$ is a product of Bessel functions.
8. $ct = \cosh(u - v) + \frac{1}{2}e^{u+v}, \quad z = \cosh(u - v) - \frac{1}{2}e^{u+v}$: $\quad -\infty < u, v < \infty$: $U(u)V(v)$ is a product of Bessel functions
9. $ct = \sinh u \cosh v, z = \cosh u \sinh v; -\infty < u, v < \infty$: $U(u)V(v)$ is a product of Mathieu functions.
10. $ct = \cosh u \cosh v, z = \sinh u \sinh v; -\infty < u < \infty, 0 \le v < \infty$: $U(u)V(v)$ is a product of Mathieu functions.
11. $ct = \cos u \cos v, \quad z = \sin u \sin v; \quad 0 < u < 2\pi, \quad 0 \le v < \pi$: $U(u)V(v)$ is a product of Mathieu functions.

APPENDIX-B

MAPPLE CODES

Mapple Code for Figure 5.1 and Figure 5.2

```
> ##f22:Bi(u)*Bi(v)
> restart;
  alpha:=-1;
  y1:=-0.15;
  y2:=0.34;
  tau1:=0;
  tau2:=15;
  u:=(4^(1/3))*(alpha-(tau+xi)/4+(((tau-xi)^(1/2))/2));
  v:=(4^(1/3))*(alpha-(tau+xi)/4-(((tau-xi)^(1/2))/2));

  f:=AiryBi((4^(1/3))*(alpha-(tau+xi)/4+(((tau-
xi)^(1/2))/2)))*AiryBi((4^(1/3))*(alpha-(tau+xi)/4-(((tau-
xi)^(1/2))/2)));
  A:=-diff(f,tau);
  B:=diff(f,xi);

  sW4:=(1/2)*((A)^2-(B)^2);
  WW:=(1/2)*(f^2+A^2+B^2);

  N:=1500;          #N=numpoints
  th:=2;            #th is thickness
  h4:=piecewise(0>tau,0,f);

  xi:=tau-0.05;
```

```
 with(plots):
#a4:=plot(h4,tau=tau1..tau2,numpoints=N,thickness=th,view=[tau1..tau2,y1..y2],axes=boxed,legend=typeset(F4),color=black,symbol=box,linestyle=solid);
#t7:=plot(ttau,tau=tau1..tau2,numpoints=N,thickness=th,view=[tau1..tau2,y1..y2],axes=boxed,legend=typeset(A4),color=black,symbol=box,linestyle=dot);
#t8:=plot(txi,tau=tau1..tau2,numpoints=N,thickness=th,view=[tau1..tau2,y1..y2],axes=boxed,legend=typeset(B4),color=black,symbol=box,linestyle=dash);

w:=plot(WW,tau=tau1..tau2,numpoints=N,thickness=th,view=[tau1..tau2,y1..y2],axes=boxed,legend=typeset(W[4]),color=black,symbol=box,linestyle=dash);
SW:=plot(sW4,tau=tau1..tau2,numpoints=N,thickness=th,view=[tau1..tau2,y1..y2],axes=boxed,legend=typeset(sW[4]),color=black,symbol=box,linestyle=solid);

display({SW,w});
```

Mapple Code for Figure 5.3 and Figure 5.4

```
> ##f21:Bi(u)*Ai(v)
> restart;
alpha:=-1;

y1:=-0.64;
y2:=0.16;
tau1:=0;
tau2:=15;
u:=(4^(1/3))*(alpha-(tau+xi)/4+(((tau-xi)^(1/2))/2));
v:=(4^(1/3))*(alpha-(tau+xi)/4-(((tau-xi)^(1/2))/2));

f3:=AiryBi((4^(1/3))*(alpha-(tau+xi)/4+(((tau-xi)^(1/2))/2)))*AiryAi((4^(1/3))*(alpha-(tau+xi)/4-(((tau-xi)^(1/2))/2)));
ttau:=-diff(f3,tau);
txi:=diff(f3,xi);
```

```
N:=1500;          #N=numpoints
th:=2;            #th is thickness
h3:=piecewise(0>tau,0,f3);

xi:=tau-0.05;

with(plots):
a3:=plot(h3,tau=tau1..tau2,numpoints=N,thickness=th,view=[t
au1..tau2,y1..y2],axes=boxed,legend=typeset(F3),color=black
,symbol=box,linestyle=solid);
t5:=plot(ttau,tau=tau1..tau2,numpoints=N,thickness=th,view=
[tau1..tau2,y1..y2],axes=boxed,legend=typeset(A3),color=bla
ck,symbol=box,linestyle=dot);
t6:=plot(txi,tau=tau1..tau2,numpoints=N,thickness=th,view=[
tau1..tau2,y1..y2],axes=boxed,legend=typeset(B3),color=blac
k,symbol=box,linestyle=dash);

display({a3,t5,t6});
```

APPENDIX-C

WEYL THEOREM

Let G be a Lie group and let ρ be a group representation of $\boldsymbol{G}$ on $\mathbb{C}^n$ (for some natural number n) which is continuous in the sense that the function $G \times \mathbb{C}^n \to \mathbb{C}^n$ defined by $(g, v) \to \rho(g)(v)$ is continuous. Then for each $v \in \mathbb{C}^n$ and each $\alpha \in (\mathbb{C}^n)^*$, the function $G \to \mathbb{C}^n$ defined by $g \to \alpha\big(\rho(g)(v)\big)$ is continuous. The vector space span of all such functions is called the space of representative functions.

The Weyl theorem says that, if G is compact, then

1) The representative functions are dense in the space of all continuous functions, with respect to the supremum norm;
2) The representative functions are dense in the space $L^2(G)$ of all square-integrable functions, with respect to aHaar measure on G;
3) The vector space span of the characters of the irreducible continuous representations of G are dense in the space of all continuous functions from G into $\mathbb{C}^n$ which are constant on each conjugacy class of G, with respect to the supremum norm.

This theorem is easy to deduce from the Stone-Weierstrass theorem if it is assumed that G is a matrix group. On the other hand, it is a corollary of the Weyl theorem that every compact Lie group is isomorphic to some matrix group

APPENDIX-D

POYNTING'S THEOREM (ENERGY CONSERVATION THEOREM)

Poynting's theorem is a powerful statement of energy conservation. It can be used to relate power absorption in an object to incident field. The theorem allows us to calculate the rate at which energy changes within a given volume due to the action of electromagnetic fields.. The theorem says if S is any closed mathematical surface and V is the volume inside S, then

$$\frac{\partial}{\partial t}\int_V (W_c + \varepsilon_0 \mathbf{E}.\mathbf{E} + \mu_0 \mathbf{H}.\mathbf{H})dV + \oint_S (\mathbf{E} \times \mathbf{H})dS = 0$$

where W_c is the energy possessed by charged particles at a given point in V, $\varepsilon_0 \mathbf{E}.\mathbf{E}$ is the enery stored in the $\mathbf{E}$ electric field at a given point in V and $\mu_0 \mathbf{H}.\mathbf{H}$ is the enery stored in the $\mathbf{H}$ emagnetic field at a given point in V. The first integral gives the rate at which energy stored within the fields changes within the volume V and the second integral gives the rate at which energy flows across the surface S enclosing V.

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PUBLISHMENTS

Conference Papers

1. **S. Semsit**, E. Eroglu, K. Koklu, O. Isık, (2014) 3rd International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2014 Book of Abstracts), "Wave Boundary Operators for Time-Domain Electromagnetic Fields", 259 pp., Vienna, Austria, 25-28 August,.

2. **S. Semsit**, E. Eroglu, K. Koklu, O. Isık, (2014) 3rd International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2014 Book of Abstracts), “Decomposition of Components of Energy During Signal Transferring”, 260 pp., Vienna, Austria, 25-28 August.

3. E. Eroglu, **S. Semsit**, E. Sener, U.S. Sener, (2016) World Academy of Science, Engineering and Technology – International Journal of Mathematical and Computational Sciences, “Transferring of Transient Signals Along Hollow Waveguide”, 3(5)