ON STABILITY AND EFFICIENCY IN DIFFERENT ECONOMIC ENVIRONMENTS

A Ph.D. Dissertation

by

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Ankara

July, 2011

To my parents,
RAZİYE and FEYZİ,
with my love

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The Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

MEHMET KARAKAYA

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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ABSTRACT

ON STABILITY AND EFFICIENCY IN DIFFERENT ECONOMIC ENVIRONMENTS

Mehmet Karakaya
Ph.D. in Economics
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July, 2011
This thesis consists of four main chapters. In the first main part, hedonic coalition formation games where each player's preferences rely only upon the members of her coalition are studied. A new stability notion under free exit-free entry membership rights, referred to as strong Nash stability, is introduced which is stronger than both core and Nash stabilities studied earlier in the literature. The weak top-choice property is introduced and shown to be sufficient for the existence of a strongly Nash stable partition. It is also shown that descending separable preferences guarantee the existence of a strongly Nash stable partition. Strong Nash stability under different membership rights is also studied. In the first main part, hedonic coalition formation games are also extended to cover formation games, where a player can be a member of several different coalitions, and these games are studied. In the second main part, Nash implementability of a social choice rule (via a mechanism) which is implementable via a Rechtsstaat is studied. A new condition on a Rechtsstaat, referred to as *equal treatment of equivalent alternatives* ($ETEA$), is introduced, and it is shown that if a social choice rule is implementable via some Rechtsstaat satisfying $ETEA$ then it is Nash implementable via a mechanism provided that there are at least three agents in the society. In the third main part, a characterization of the Borda rule on the domain of weak preferences is studied. A new property, which is referred to as the *degree equality*, is introduced, and it is shown that the Borda rule is characterized by weak neutrality, reinforcement, faithfulness and degree equality. In the fourth main part, the graduate admissions problem with quota and budget constraints is studied as a two sided many to one matching market. The students proposing algorithm, which is an extension of the Gale-Shapley algorithm, is constructed, and it is shown that the students proposing algorithm ends up with a core stable matching if the algorithm stops. However, there exist graduate admissions problems for which there exist core stable matchings, while neither the departments proposing nor the students proposing algorithm stops. It is proved that the students proposing algorithm stops if and only if no cycle occurs in

the algorithm. It is also shown that no random path to core stability for the graduate admissions problem exists.

Keywords: Hedonic coalition formation games, core stability, Nash stability, strong Nash stability, membership rights, cover formation games, implementation via a Rechtsstaat, Nash implementation via a mechanism, equal treatment of equivalent alternatives, the Borda rule, degree equality, graduate admissions problem, the Gale-Shapley algorithm, quota and budget constraints, random paths to core stable matching.

ÖZET

ÇEŞİTLİ İKTİSADİ ORTAMLARDA KARARLILIK VE VERİMLİLİK ÜZERİNE

Mehmet Karakaya
Ekonomi, Doktora
Tez Yöneticsi: Prof. Dr. Semih Koray
Temmuz, 2011

Bu tez çalışması dört ana kısımdan oluşmaktadır. Birinci ana kısımda her oyuncunun tercihinin sadece kendisinin içinde bulunduğu koalisyonun üyelerine bağlı olduğu hazcı koalisyon oluşum oyunları çalışılmıştır. Kuvvetli Nash kararlılığı adıyla yeni bir kararlılık kavramı herhangi bir koalisyona giriş ve çıkışın izne bağlı olmadığı üyelik hakları çerçevesinde tanımlanmıştır, bu yeni tanımlanan kararlılık kavramı daha önceleri çalışılmış olan çekirdek ve Nash kararlılık kavramlarının her ikisinden daha kuvvetlidir. En iyi zayıf seçim özelliği tanımlanmış ve bu özelliğin kuvvetli Nash kararlı koalisyon yapılarının varlığı için gerek şart olduğu gösterilmiştir. Azalan ayrılabilir tercihlerin de kuvvetli Nash kararlı koalisyon yapılarının varlığını garantilediği gösterilmiştir. Ayrıca kuvvetli Nash kararlılığı farklı üyelik hakları altında da çalışılmıştır. Yine, birinci ana kısımda hazcı koalisyon oluşum oyunları oyuncuların aynı anda birden fazla koalisyonun üyesi olabildiği örtüşük koalisyonların oluşum oyunlarına genişletilmiş ve bu oyunlar incelenmiştir. İkinci ana kısımda haklar yapısı aracılığı ile uygulanabilir olan bir sosyal seçim kuralının bir mekanizma vasıtasıyla Nash uygulanabilirliği çalışılmıştır. Haklar yapısı üzerinde *eşdeğer seçeneklere eşit muamele* adıyla yeni bir şart tanımlanmış ve bu şartı sağlayan bir haklar yapısı ile uygulanabilen bir sosyal seçim kuralının, en az üç kişinin olduğu bir toplumda, bir mekanizma vasıtasıyla Nash uygulanabilir olduğu gösterilmiştir. Üçüncü ana kısımda Borda kuralının bir karakterizasyonu tanım bölgesi zayıf tercihler demeti olmak suretiyle çalışılmıştır. *Derece eşitliği* diye adlandırılan yeni bir özellik tanımlanmış ve Borda kuralının karakterizasyonu zayıf nötrlük, pekiştirme, sadakatlilik ve derece eşitliği özellikleri ile yapılmıştır. Dördüncü ana kısımda kota ve bütçe kısıtları altında doktora kabul problemi iki taraflı eşleşme olarak incelenmiştir. Gale-Shapley algoritmasının bir uzantısı olan ve öğrencilerin teklif götürdüğü bir algoritma yazılmış ve bu algoritma durursa oluşan eşleşmenin çekirdek kararlı olduğu gösterilmiştir. Bununla beraber, ne bölümlerin teklif götürdüğü ne de öğrencilerin teklif götürdüğü algoritmaların durduğu ve çekirdek kararlı bir eşleşmenin bulunduğu durumlar mevcuttur. “Eğer ve sadece eğer algoritma içerisinde bir döngü oluşmazsa öğrencilerin teklif götürdüğü algoritma durur” önermesi ispat edilmiştir. Ayrıca, doktora kabul problemi için rastgele patika aracılığı ile çekirdek kararlı bir eşleşmeye ulaşılamayacağı da gösterilmiştir.

Anahtar sözcükler: Hazcı koalisyon oluşum oyunları, çekirdek kararlılığı, Nash kararlılığı, kuvvetli Nash kararlılığı, üyelik hakları, örtüşük koalisyonların oluşum oyunları, haklar yapısı aracılığı ile uygulanabilirlik, mekanizma aracılığı ile Nash uygulanabilirlik, eşdeğer seçeneklere eşit muamele, Borda kuralı, derece eşitliği, doktora kabul problemi, Gale-Shapley algoritması, kota ve bütçe kısıtları, çekirdek karalı eşleşmeye rastgele patika aracılığı ile ulaşılabilirlik.

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CHAPTER 1

INTRODUCTION

In every field of economic theory, the common main question is what outcomes are stable and what outcomes are efficient. The next natural question concerns the relationship between stability and efficiency. When does stability imply efficiency, and under what circumstances can efficient outcomes be reached as equilibrium outcomes?

The first theorem of welfare economics states that a competitive equilibrium allocation is Pareto efficient. An allocation is Pareto efficient if there does not exist any feasible allocation that makes some agents better off without hurting some others. Pareto optimality is the most natural efficiency notion for agents with non-transferable utilities who are to act individually in a decentralized way. It is worth to note that the first welfare theorem holds under two important conditions. One is that all goods are private goods. The theorem does not hold in the presence of public goods. The other hypothesis is that every agent's preferences depend only on her own consumption. Hence, an agent is not allowed either to be concerned or to be jealous about what happens to her neighbor or to the rest of the world. In game theory, on the other hand, there is no counterpart of the first theorem. That is, it is not the case that every Nash equilibrium of a game is Pareto optimal. To the contrary, the main problem that game theory seems to deal with is the tension between stability and efficiency. In contrast to the first theorem of welfare economics, players in a game may be equipped with preferences that reflect altruism as well as envy towards their opponents.

The second theorem of welfare economics starts with an outcome which is Pareto efficient and specifies sufficient conditions under which the efficient allocation can be obtained as an equilibrium outcome by redistributing initial endowments in an economy. Not every efficient outcome may be socially desirable, however, as is typically exemplified by dictatoriality, where all the goods in the economy go to the dictator. The second theorem deals with the problem of designing a social configuration under which efficient and socially desirable outcomes arise as equilibrium outcomes. The counterpart of the second theorem in game theory can be thought of as implementation via a mechanism. Roughly said, a mechanism -conjoined with a game-theoretic solution concept- redistributes the power among the players so as to achieve “socially desirable” outcomes, if possible, paralleling the redistribution of initial endowments in the economy.

An alternative way of dealing with design problems is introduced by Sertel (2002). He proposes to explicitly introduce a rights structure (or a code of rights), specifying what coalition is entitled to approve what changes in the states of affairs. The notion of a rights structure can easily be seen to reduce to the notion of core-stability in the very special case, where every coalition is entitled to approve any change in the state of affairs. The notion of core-stability -by allowing every coalition to get formed and to take joint binding decisions- combines efficiency and stability.

In this thesis, we deal with different environments as hedonic coalition formation games or cover formation games, implementation via codes of rights or graduate admissions problem under quota and budget constraints. Although the environments considered exhibit a wide variety, what combines them is the efficiency-stability or the invisible hand-design axes along which they are dealt with.

The first chapter studies hedonic coalition formation games. A hedonic coalition formation game consists of a finite non-empty set of players and a list of players’ preferences where every player’s preferences depend only on the members of her coalition. Hedonic coalition formation games are used to model certain economic and political circumstances such as the provision of public goods in local communities or forming clubs and organizations. An outcome of such a game is a partition of the player set (coalition structure) -that is, a collection of pairwise disjoint coalitions whose union is equal to the set of players. Given a hedonic coalition formation game, the main concern is the existence of partitions that are stable in some sense. A partition is core stable if there is no coalition each of whose members strictly prefers it to the coalition to which she belongs under the given partition. We introduce the framework of "membership rights" of Sertel (1992) into the context of hedonic games. Given a hedonic game and a partition, the membership rights employed specify the set of agents whose approval is needed for each particular deviation of a subset of players. We define a new stability notion under free exit-free entry membership rights, referred to as strong Nash stability, which is stronger than the core stability studied earlier in the literature. Strong Nash stability has an analogue in non-cooperative games and it is the strongest stability notion fitting the context of hedonic coalition formation games. We introduce the weak top-choice property, and show that it guarantees the existence of a strongly Nash stable partition. We prove that descending separable preferences suffice for a hedonic game to have a strongly Nash stable partition. We also study varying versions of strong Nash stability under different membership rights.

In the first chapter, we also extend hedonic coalition formation games to cover formation games, where a player can be a member of several different coalitions. For example, a researcher can be a member of several research teams at the same time. A collection of coalitions is referred to as a cover if its union is equal to the set of players. We define stability concepts based on individual movements as well as movements by subsets of players under different membership rights, and provide existence results for covers which are stable in the corresponding senses.

In the second chapter, we consider an environment with a finite non-empty set of alternatives and a finite non-empty set of agents, where each agent has complete, reflexive and transitive preferences over the set of alternatives. A list of agents' preferences is called a preference profile. A social choice rule (SCR) is a rule which chooses a nonempty subset of alternatives at each preference profile. However, agents' preferences are not known to a designer (or planner) and an agent may benefit by not revealing her true preferences. The "implementation" problem arises from this situation as it gives rise to the question of whether it is possible to design a mechanism (game form) which provides no incentives for misrepresentation of preferences. So, we are back at design problem with which the second welfare theorem deals. A mechanism (game form) consists of a nonempty strategy set for each agent (messages) and an outcome function which maps from joint messages into alternatives. A mechanism with a preference profile on the set of alternatives induces a game in strategic form. A mechanism is said to implement an SCR according to a game theoretic solution concept σ if the σ-equilibrium outcomes of the induced game coincide with the set of alternatives assigned by the SCR at each preference profile of the society.

Sertel (2002) introduced the notion of a "Rechtsstaat" through which he explicitly specifies a rights structure based on two functions, namely, the *benefit* function and the *code of rights* function. Given a pair of alternatives and a preference profile, a benefit gives us the set of all coalitions that strictly prefer the second alternative in the pair to the first one at the given preference profile. A code of rights specifies, for every pair of alternatives, a family of coalitions in which each coalition is given the right to approve the alteration of first alternative to the second one. So, a code of rights is independent of agents' preferences. An alternative is said to be an *equilibrium* of a Rechtsstaat at a given preference profile if there is no coalition which is given the right to approve the alteration of this alternative to some other one such that every agent in the coalition benefits from this alteration, i.e., all agents in the coalition strictly prefer the latter alternative to the former one. It is clear that in a Rechtsstaat, the rights structure in the society are explicitly given by its code of rights. An SCR is said to be *implementable via a Rechtsstaat* if, at every preference profile, alternatives which are chosen by the SCR coincide with the equilibria of the Rechtsstaat (Koray and Yıldız (2008)).

In the second chapter, we study Nash implementability of an SCR (via a mechanism) which is implementable via a Rechtsstaat, i.e., what properties of a Rechtsstaat implementing an SCR ensure that the SCR is also Nash implementable via a mechanism. We introduce a condition on a Rechtsstaat which is referred to as the *equal treatment of equivalent alternatives* ($ETEA$). We say that a Rechtsstaat satisfies $ETEA$, if all agents are indifferent between two alternatives under any preference profile, then one of these alternatives being an equilibrium of our Rechtsstaat implies that the other alternative is also an equilibrium. We show that if an SCR is implementable via some Rechtsstaat satisfying $ETEA$ then it is Nash implementable via a mechanism when there are at least three agents in the society. However, an SCR which is implementable via a Rechtsstaat that violates $ETEA$ may not be Nash implementable. We also show that a Rechtsstaat satisfies $ETEA$ if and only if its code of rights is as follows: for any alternative x and any alternatives y and z (different from x), those and only those coalitions bearing the right to approve the alteration of y to x are also the coalitions which have the right to approve the alteration of z to x. We define oligarchic Rechtsstaats and show that if an SCR is implementable via an oligarchic Rechtsstaat then it is Nash implementable provided that there are at least three agents in the society.

In chapter three, we study a characterization of the Borda rule on the domain of weak preferences, where the Borda rule is defined for each finite set of voters having preferences over a fixed set of alternatives. In the case of a collective decision problem where each agent in a society has preferences over a finite set of alternatives, either a social welfare function is employed to aggregate a list of agents' preferences into a social ordering of alternatives (social preference), or a social choice rule (SCR) is employed to specify a set of selected alternatives at the given preference profile (social choice). Since, in either approach, for all individuals in the society the outcome is the same, the situation falls into the realm of the second theorem of welfare economics. Our concern is to employ an SCR which is used to make a choice over alternatives for each preference profile of a society. Many different SCRs have been established to determine which alternative(s) should be selected when a preference profile of a society is considered. An SCR should satisfy some desirable properties such as being Pareto optimal, non-dictatorial and independent of the names of alternatives and voters. However, there are many SCRs which are Pareto optimal and non-dictatorial, which necessitates us to look for further specifications that fully distinguish a desirable SCR from others. We say that a set of specific properties characterize an SCR if the SCR is the only one that satisfies these properties. When players have strict preference relations over alternatives, the Borda rule is characterized by neutrality, reinforcement, faithfulness and Young's cancellation property (Young (1974), Hansson and Sahlquist (1976)). Neutrality means that the names of the alternatives do not affect the selected alternatives. An SCR satisfies reinforcement if there exist common selected alternatives for any two disjoint voter sets and these common choices are considered the exact selected alternatives for the combined society. Faithfulness is satisfied by an SCR if there is only one agent in the society and the SCR chooses her top-ranked alternative. An SCR satisfies Young's cancellation property if, for every pair of alternatives, the number of agents who strictly prefer the first alternative to the second one is equal to the number of agents who strictly prefer the second alternative to the first one implies the selection of all alternatives.

We introduce a new property which is referred to as the *degree equality*; an SCR satisfies degree equality if, for any two profiles of a finite set of voters, equality between the sums of the degrees of every alternative under the two profiles implies that the same alternatives get chosen by the SCR at these two profiles. We show that the Borda rule is characterized by the conjunction of weak neutrality, reinforcement, faithfulness and degree equality on the domain of weak preferences. As it is not often easy to show the independence of neutrality from other axioms when it is used in a characterization, we could not show that weak neutrality is independent of the other three axioms. We also show that the Borda rule is the unique scoring rule which satisfies the degree equality. In addition, we introduce a new cancellation property and show that it characterizes the Borda rule among all scoring rules.

In the fourth chapter, we study the graduate admissions problem with quota and budget constraints as a two sided many to one matching market as a continuation of Karakaya and Koray (2003). One side of the market consists of the departments of a university, while there is a set of students (applicants) on the other side. Each department faces both quota and budget constraints set by the central university administration. Karakaya and Koray (2003) constructed the departments proposing algorithm, and showed that if the algorithm stops then the resulting matching is core stable, and it is possible that the algorithm does not stop while there is a core stable matching. They also showed that the departments proposing algorithm stops if and only if no cycle occurs in the algorithm, i.e., a finite sequence of matchings does not repeat itself infinitely many times in the algorithm. The existence of either a departments-optimal or a students-optimal matching is not guaranteed in the graduate admissions problem with both quota and budget constraints.

We construct the students proposing algorithm, and show that the students proposing algorithm ends up with a core stable matching if the algorithm stops. However, there exist graduate admissions problems for which there exist core stable matchings, while neither the algorithm proposing side being the departments nor that proposing side being the students stops. We show that the students proposing algorithm stops if and only if no cycle occurs in the algorithm. Moreover, we show that there is no random path to core stability for the graduate admissions problem, i.e., a core stable matching can not be reached starting with an arbitrary matching and satisfying a randomly chosen blocking coalition at each step. We also consider the model with the assumption that the students care only about their reservation prices and do not derive any further utility from money transfers over and above their reservation prices. Under this model we get results similar to those obtained in the general model.

The thesis is organized as follows: Hedonic coalition formation games and cover formation games are studied in chapter 2. Chapter 3 studies Nash implementation of social choice rules which are implementable via a Rechtsstaat. Chapter 4 studies the characterization of the Borda rule on the domain of weak preferences. Graduate admissions problem with quota and budget constraints is studied in chapter 5. Chapter 6 constitutes the conclusion. Omitted proofs and examples are provided in the Appendix.

CHAPTER 2

HEDONIC COALITION FORMATION GAMES AND COVER FORMATION GAMES

2.1 Hedonic coalition formation games

2.1.1 Introduction

Individuals act by forming coalitions under certain economic and political circumstances such as the provision of public goods in local communities or forming clubs and organizations. One way to describe such an environment is to model it as a (pure) hedonic coalition formation game.

A hedonic coalition formation game consists of a finite non-empty set of players and a list of players' preferences where every player's preferences depend only on the members of her coalition.[1] An outcome of such a game is a partition of the player set (coalition structure) -that is, a collection of coalitions whose union is equal to the set of players, and which are pairwise disjoint. Marriage problems and roommate problems (Gale and Shapley (1962), Roth and Sotomayor (1990b)) can be seen as special cases of hedonic coalition formation games, where each agent only considers who will be his/her mate. In fact, hedonic games are reduced forms of general coalition

[1]The dependence of a player's utility on the identity of members of her coalition is referred to as the "hedonic aspect" in Drèze and Greenberg (1980), and the formal model of (pure) hedonic coalition formation games was introduced by Banerjee et al. (2001) and Bogomolnaia and Jackson (2002).

formation games where, for each coalition, how its total payoff is to be divided among its members is fixed in advance and made known to all agents.

Given a hedonic coalition formation game, the main concern is the existence of partitions that are stable in some sense. The stability concepts that have been mostly studied so far are core stability and Nash stability of coalition structures.[2] A partition is core stable if there is no coalition each of whose members strictly prefers it to the coalition to which she belongs under the given partition. A partition is said to be Nash stable if there is no player who benefits from leaving her present coalition to join another coalition of the partition which might be the "empty coalition" in this context. Note that a Nash stable partition need not be core stable, and a core stable partition need not be Nash stable.

One needs to focus attention on two key points when considering or comparing stability concepts, namely: (i) who can deviate from the given partition (e.g., a coalition of players as in core stability, a singleton as in Nash stability), and (ii) what the deviators are entitled to do (e.g., form a new, self standing coalition as in core stability, join an already existing coalition -irrespective of how the incumbent members are effected- as in Nash stability). For hedonic coalition formation games, the second point can be examined by introducing membership rights. Sertel (1992) introduced four possible membership rights in an abstract setting. Given a hedonic game and a partition, the membership rights employed specify the set of agents whose approval is needed for each particular deviation of a subset of players.

Under *free exit-free entry* (FX-FE) membership rights, every agent is entitled to make any movements among the coalitions of a given partition without taking any permission of members of the coalitions that she leaves or joins. An example in the context of the roommate problem would be that whenever an agent finds a place in a room she has the right to move into that room. So, two agents in different rooms may benefit by exchanging their rooms without asking anyone else. Another example is that a citizen of a country which is a member of the EU can move to another country in the EU without the permission of either country.

[2]See the taxonomy introduced in Sung and Dimitrov (2007) for all stability concepts which were studied in the literature.

Under *free exit-approved entry* (FX-AE) membership rights, an agent can leave her current coalition without the permissions of her current partners, but she can join another coalition only if all members of that coalition welcome her, that is her joining does not hurt any member of the coalition she joins. A typical example is provided by club membership, where a member of a club can leave her current club without taking into account whether her leaving hurts some members of that club. However, she needs the approval of the members of a club that she wants to join. Another example is that of a researcher, who is a member of a research team and can leave the team without the permissions of other team members, while her joining another team is usually subject to the approval of that team's present members.

Under *approved exit-free entry* (AX-FE), every agent is endowed with rights, under which she can leave her current coalition only if that coalition's members approve her leaving, while her joining requires no one else's permission. An example would be that of an army recruiting volunteers. Every healthy citizen in a certain age interval may enter the army if he volunteers to do so, but is not allowed to freely exit once he is in.

Under *approved exit-approved entry* (AX-AE) membership rights each player needs to get the unanimous permission of the coalition that she leaves or joins. A typical example is that of a criminal organization. An agent who is a member of a criminal organization cannot leave it without permission as she may have information about some secrets of the organization. Similarly, one cannot join a criminal organization without permission by a similar token.

Note that under the definition of Nash stability, a player can deviate by leaving her current coalition to join another coalition of the partition without any permission of the players of the coalitions that she leaves or joins, although she might thereby be hurting some of these. In other words, Nash stability is defined under FX-FE membership rights. Other stability concepts that consider individual deviations under different membership rights have already been studied in the literature. That is, *individual stability* is defined under FX-AE membership rights (Bogomolnaia and Jackson (2002)), *contractual Nash stability* is defined under AX-FE membership rights (Sung and Dimitrov (2007)), and *contractual individual stability* is defined under AX-AE membership rights (Bogomolnaia and Jackson (2002) and Ballester (2004)).

The aim of this section is to study coalitional extension of Nash stability under FX-FE membership rights, referred to as strong Nash stability, which has not been studied yet. Note that strong Nash stability is not defined in Sung and Dimitrov (2007) but they identified some weaker versions of strong Nash stability.

Two approaches will be employed while defining a strongly Nash stable partition. The first approach is posed in terms of an induced non-cooperative game. A hedonic coalition formation game induces a non-cooperative game in which each player chooses a “label”; players who choose the same label are placed in a common coalition. Strong Nash (respectively, Nash) stability in this induced game then corresponds to strong Nash (respectively, Nash) of the corresponding partition in the coalitional form of the game. The second approach is posed in terms of movements and reachability. A partition is said to be strongly Nash stable if there is no subset of players who reach a new partition via certain admissible movements such that these players strictly prefer the new partition to the initial one.

Banerjee et al. (2001) introduced the *top-coalition* and the *weak top-coalition* properties and proved that each property suffices for a hedonic game to have a core stable partition. They also showed that if a game is *anonymous* and *separable*, then it has a core stable partition. Bogomolnaia and Jackson (2002) introduced two conditions, called *ordinal balancedness* and *weak consecutiveness*. They showed that if a hedonic game is ordinally balanced or weakly consecutive, then there exists a core stable partition. Iehlé (2007) introduced *pivotal balancedness* and showed that it is both a necessary and sufficient condition for the existence of a core stable partition. Alcalde and Romero-Medina (2006) introduced four different restrictions on the domain of each player's preferences called as the *union responsiveness condition*, the *intersection responsiveness condition*, *singularity* and *essentiality*. They showed that each of these conditions is sufficient for the existence of a core stable partition under the assumption that players have strict preferences. Alcalde and Revilla (2004) proposed a condition in each player's preferences called as *top responsiveness* and showed that if each player's preferences satisfy top responsiveness then there exists a core stable partition. Dimitrov et al. (2006) studied core stability in a hedonic game if players' preferences derived from *appreciation of friends* or *aversion to enemies*. They showed that if players' preferences are derived from either appreciation of friends or aversion to enemies then a core stable partition exists. Pápai (2004) studied unique core stability of hedonic games and introduced *single-lapping property*. She showed that single-lapping property is both a necessary and sufficient condition for a hedonic game to have a unique core stable partition. We note that none of the above conditions which suffices for the existence of a core stable partition guarantees the existence of a strongly Nash stable partition.

Bogomolnaia and Jackson (2002) showed that a hedonic game which is *additively separable* and satisfies *symmetry* has a Nash stable partition. However, Banerjee et al. (2001) provided an example of a hedonic game which is additively separable and satisfies symmetry, but has no core stable partition. Burani and Zwicker (2003) considered *descending separable preferences* posed in the form of several ordinal axioms, and showed that it is sufficient for the simultaneous existence of Nash and core stable partition.

The weak top-choice property is introduced by borrowing the definition of weak top-coalition from Banerjee et al. (2001), and shown that it guarantees the existence of a strongly Nash stable partition (Proposition 1). It is also shown that descending separable preferences suffice for a hedonic game to have a strongly Nash stable partition (Proposition 2).

How the concept of strong Nash stability changes under different membership rights is also examined. It is shown that under FX-AE membership rights, a partition is FX-AE strictly strongly Nash stable if and only if it is strictly core stable (Proposition 3), showing that core stability entails an FX-AE rights structure. Sung and Dimitrov (2007) defined *contractual strict core* stability and showed that for any hedonic game such a partition always exists. It is proved that under AX-AE membership rights, a partition is AX-AE strictly strongly Nash stable if and only if it is contractual strictly core stable (Proposition 4).

This section is organized as follows: Section 2.1.2 presents the basic notions. Section 2.1.3 introduces the weak top-choice property and provides an existence result. Descending separable preferences are studied in section 2.1.4 and it is shown that there always exists a strongly Nash stable partition if players have descending separable preferences. In section 2.1.5, strong Nash stability under different membership rights is studied. Section 2.1.6 concludes.

2.1.2 Basic notions

Let $N = \{1, 2, \ldots, n\}$ be a nonempty finite set of players. A nonempty subset H of N is called a coalition. Let $i \in N$ be a player, and $\sigma_i = \{H \subseteq N \mid i \in H\}$ denote the set of coalitions each of which contains player i. Each player i has a reflexive, complete and transitive preference relation $\succeq_i$ over σ_i. So, a player's preferences depend only on the members of her coalition. The strict and indifference preference relations associated with $\succeq_i$ will be denoted by $\succ_i$ and $\sim_i$, respectively. Let $\succeq = (\succeq_1, \ldots, \succeq_n)$ denote a preference profile for the set of players.

Definition 1 A pair $G = (N, \succeq)$ denote a ***hedonic coalition formation game***, or simply a ***hedonic game***.

Given a hedonic game, it is required that the set of coalitions which might form to be a partition of N.

Definition 2 ***A partition (coalition structure)*** of a finite set of players $N = \{1, \ldots, n\}$ is a set $\pi = \{H_1, H_2, \ldots, H_K\}$ ($K \leq n$ is a positive integer) such that

(i) for any $k \in \{1, \ldots, K\}$, $H_k \neq \emptyset$,

(ii) $\bigcup_{k=1}^{K} H_k = N$, and

(iii) for any $k, l \in \{1, \ldots, K\}$ with $k \neq l$, $H_k \cap H_l = \emptyset$.

Let $\Pi(N)$ denote the set of all partitions of N. Given any $\pi \in \Pi(N)$ and any $i \in N$, let $\pi(i) \in \pi$ denote the unique coalition which contains the player i. Since we are working with hedonic games, for any player $i \in N$, the preference relation $\succeq_i$ over σ_i can be extended over the set of all partitions $\Pi(N)$ in a usual way as follows: For any $\pi, \acute{\pi} \in \Pi(N)$, $[\pi \succeq_i \acute{\pi}]$ if and only if $[\pi(i) \succeq_i \acute{\pi}(i)]$.

Definition 3 Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***individually rational for player*** i if $\pi(i) \succeq_i \{i\}$ and is ***individually rational*** if it is individually rational for every player $i \in N$.

A partition is individually rational if each player prefers the coalition that she is a member of to being single, i.e., each agent i prefers $\pi(i)$ to $\{i\}$.

Definition 4 Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***core stable*** if there does not exist a coalition $T \subseteq N$ such that for all $i \in T$, $T \succ_i \pi(i)$. If such a coalition T exists, then it is said that T ***blocks*** π.[3]

Definition 5 Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ a partition. We say that a player $i \in N$ ***Nash blocks*** π if there exists a coalition $H \in (\pi \cup \{\emptyset\})$ such that $H \cup \{i\} \succ_i \pi(i)$. A partition is ***Nash stable*** if there does not exist a player who Nash blocks it.

Two approaches will be employed while defining the strongly Nash stable partition. In the first one, the non-cooperative game induced by a hedonic game is used.

Every hedonic game induces a non-cooperative game as defined below.

Let $G = (N, \succeq)$ be a hedonic game with $\mid N \mid = n$ players. Consider the following induced non-cooperative game $\Gamma^G = \big(N, (S_i)_{i \in N}, (R_i)_{i \in N}\big)$ which is defined as follows:

- The *set of players* in Γ^G is the player set N of G.

[3] A partition $\pi \in \Pi(N)$ is ***strictly core stable*** if there does not exist a coalition $T \subseteq N$ such that for all $i \in T$, $T \succeq_i \pi(i)$, and for some $i \in T$, $T \succ_i \pi(i)$. If such a coalition T exists, then it is said that T ***weakly blocks*** π.

- Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be a finite set of labels such that $m = n + 1$. Take $\mathcal{L}$ to be the *set of strategies* available to each player, so $S_i = \mathcal{L}$ for each $i \in N$. Let $S = \prod_{i \in N} S_i$ denote the strategy space. A strategy profile $s = (s_1, \ldots, s_n) \in S$ induces a partition π_s of N as follows: two players i, j of N are in the same piece of π_s if and only if $s_i = s_j$ (i and j choose the same strategy according to s).

- *Preferences* for Γ^G is defined as follows: a player i prefers the strategy profile s to the strategy profile $\acute{s}$, $sR_i\acute{s}$, if and only if $\pi_s(i) \succeq_i \pi_{\acute{s}}(i)$, i.e., player i prefers the coalition of those who choose the same strategy as she does according to s, to the coalition of those who choose the same strategy as she does according to $\acute{s}$.

Now, the main stability concept of this section will be defined by using the induced non-cooperative game approach.

Definition 6 Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***strongly Nash stable*** if it is induced by a strategy profile which is a strong Nash equilibrium of the induced non-cooperative game Γ^G.

Thus, the Nash equilibria of Γ^G correspond to the Nash stable partitions of G, and the strong Nash equilibria of Γ^G correspond to the strongly Nash stable partitions of G. Hence, strong Nash stability has an analogue in non-cooperative games, and it is the strongest natural stability notion appropriate to the context of hedonic games.

If the strategy profile s which induces the partition π_s is not a strong Nash equilibrium of Γ^G, then there is a subset of players $H \subseteq N$ which deviates from s (according to s) and this deviation is beneficial to all agents in H. In such a case, it is said that H ***strongly Nash blocks*** the partition π_s.

The second approach is posed in terms of movements and reachability which is derived from the first one.

Let π_s be a partition which is induced by the strategy profile s, and $H \subseteq N$ be a deviating subset of players. The deviation of these players from s can be explained as movements among the coalitions of the partition π_s, where the allowable movements of these players are as follows:[4]

(i) All players in $H \notin \pi_s$ choose a label which is not chosen by any player under s.[5] Let $\acute{s}$ denote the strategy profile that is obtained by this deviation. Now, $H \in \pi_{\acute{s}}$. This deviation means in terms of movements that all players in H leave their current coalitions and form the coalition $H \in \pi_{\acute{s}}$ (which is the movement used in the definition of blocking in the core stability).

(ii) All players in H [6] choose the label which is chosen by members of a coalition $T \in \pi_s$. Let $\widetilde{s}$ denote the strategy profile that is obtained by this deviation. Now, $(H \cup T) \in \pi_{\widetilde{s}}$. This deviation means all players in H leave their current coalitions and join another coalition T of π_s, so for each $i \in H$, $\pi_{\widetilde{s}}(i) = T \cup H$.

(iii) Players in $H \notin \pi_s$ partition among themselves as $\{H_1, \ldots, H_t\}$, and for any $k \in \{1, \ldots, t\}$, agents in H_k choose the label which is chosen under s by an agent $j \in H_{k+1}$, where it is taken $t + 1 = 1$. Let $\widehat{s}$ denote the strategy profile that is obtained by this deviation. Now, for any $i \in H_k$, $\pi_{\widehat{s}}(i) = (\pi_s(j) \setminus H) \cup H_k$. This deviation means individual players in H (or subsets of H) exchange their current coalitions in the partition π_s. For instance, let $H = \{i, j\} \notin \pi_s$ and player i leaves $\pi_s(i)$ and joins $\pi_s(j) \setminus \{j\}$, and player j leaves $\pi_s(j)$ and joins $\pi_s(i) \setminus \{i\}$. So, $\pi_{\widehat{s}}(i) = (\pi_s(j) \setminus \{j\}) \cup \{i\}$ and $\pi_{\widehat{s}}(j) = (\pi_s(i) \setminus \{i\}) \cup \{j\}$. Note that more complicated movements are possible when the size of H increases.[7]

Given a partition π and a subset of players $H \subseteq N$, by any movements of H among the coalitions of the partition π, players of H obtain a new partition $\acute{\pi}$, and it is said that $\acute{\pi}$ is reachable from the partition π via H.

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[4]Movements of H are coordinated and simultaneous.

[5]Such a label always exists, since $m = n + 1$.

[6]It is possible in here that $H \in \pi_s$.

[7]Movements of H among the coalitions of the partition π_s can also be explained as follows: Each player in H leaves the coalition that she belongs under partition π_s. Let $\pi_s^{-H} = \{T \setminus H \mid T \in \pi_s$ and $T \setminus H \neq \emptyset\}$ denote the set of coalitions after each player in H leaves her current coalition. Now, individual players or subsets of H can join any coalition (or an empty set) of $(\pi_s^{-H} \cup \{\emptyset\})$. This approach is similar to the one given by Conley and Konishi (2002). In their approach, a set of agents is only allowed to form coalitions among themselves, i.e., individual players or subsets of H are only permitted to join the empty set. However, in our approach individual players or subsets of H are allowed to join not only the empty set but also any coalition of π_s^{-H}.

Definition 7 Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ be a partition. Another partition $\acute{\pi} \in (\Pi(N) \setminus \{\pi\})$ is said to be ***reachable from*** π ***by movements of a subset of players*** $H \subseteq N$, denoted by $\pi \xrightarrow{H} \acute{\pi}$, if, for all $i, j \in (N \setminus H)$ with $i \neq j$, $\pi(i) = \pi(j) \Leftrightarrow \acute{\pi}(i) = \acute{\pi}(j)$.

Reachability by movements of a subset of agents simply says that agents who are not deviators are passive, and a non-deviator remains with all former mates who are not deviators. Notice that a subset of players $\widehat{H} \supseteq H$ can do all movements that H can. Note that for any $\pi \in \Pi(N)$ and $\acute{\pi} \in (\Pi(N) \setminus \{\pi\})$, $\pi \xrightarrow{N} \acute{\pi}$, i.e., given any partition π all other partitions can be reached by movements of the grand coalition N.

Now, the strong Nash stability of a partition can also be defined in terms of movements and reachability.

Definition 8 Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***strongly Nash stable*** if there does not exist a pair $(\acute{\pi}, H)$ (where $\acute{\pi} \in (\Pi(N) \setminus \{\pi\})$ and $\emptyset \neq H \subseteq N$) such that

(i) $\pi \xrightarrow{H} \acute{\pi}$ ($\acute{\pi}$ is reachable from π by movements of H), and

(ii) for all $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$.

If such a pair $(\acute{\pi}, H)$ exists, then it is said that H ***strongly Nash blocks*** π (by inducing $\acute{\pi}$).

Note that the two definitions of strongly Nash stable partitions are equivalent (definitions 6 and 8).

It is clear that a strongly Nash stable partition is both core and Nash stable. However, a hedonic game which has a partition that is both core and Nash stable may not have a strongly Nash stable partition.

Example 1 Let $G = (N, \succeq)$, where $N = \{1, 2, 3, 4\}$ and the preferences of players are as follows:

$\{1,4\} \succ_1 \{1,2\} \succ_1 \{1,3,4\} \succ_1 \{1,3\} \succ_1 \{1\} \succ_1 \ldots,$[8]

$\{2,4\} \succ_2 \{1,2\} \succ_2 \{2,3,4\} \succ_2 \{2\} \succ_2 \ldots,$

$\{1,3\} \succ_3 \{3,4\} \succ_3 \{1,2,3\} \succ_3 \{3\} \succ_3 \ldots,$

$\{3,4\} \succ_4 \{1,2,4\} \succ_4 \{2,4\} \succ_4 \{4\} \succ_4 \ldots.$

The partitions $\widetilde{\pi} = \{\{1,2\},\{3,4\}\}$ and $\widehat{\pi} = \{\{1,3\},\{2,4\}\}$ are the only partitions which are both core stable and Nash stable, and there is no partition $\pi \in (\Pi(N) \setminus \{\widetilde{\pi}, \widehat{\pi}\})$ which is either core stable or Nash stable. However, neither $\widetilde{\pi}$ nor $\widehat{\pi}$ is strongly Nash stable.

Let $\widetilde{s}$ denote the strategy profile in Γ^G which induces the partition $\widetilde{\pi}$. So, players 1 and 2 choose the same label under $\widetilde{s}$, say $\acute{L}$, and players 3 and 4 choose the same label under $\widetilde{s}$, say $\bar{L}$. Thus $\widetilde{s} = (\acute{L}, \acute{L}, \bar{L}, \bar{L})$. The strategy profile $\widetilde{s}$ is not a strong Nash equilibrium of Γ^G, since players 2 and 3 deviate from $\widetilde{s}$ as follows:[9] Player 2 chooses label $\bar{L}$ and player 3 chooses label $\acute{L}$. Let $\widehat{s} = (\acute{L}, \bar{L}, \acute{L}, \bar{L})$ denote the strategy profile that is obtained by the deviation of players 2 and 3. Now, the strategy profile $\widehat{s}$ induces the partition $\widehat{\pi} = \{\{1,3\},\{2,4\}\}$.[10] This deviation is beneficial to both players 2 and 3, since $\widehat{\pi}(2) \succ_2 \widetilde{\pi}(2)$ and $\widehat{\pi}(3) \succ_3 \widetilde{\pi}(3)$. Therefore, $\widetilde{\pi}$ is not strongly Nash stable.

Now consider the partition $\widehat{\pi} = \{\{1,3\},\{2,4\}\}$. $\widehat{\pi}$ is not strongly Nash stable, since players 1 and 4 strongly Nash block the partition $\widehat{\pi}$ by exchanging their current coalitions, i.e., $\widehat{\pi} \xrightarrow{\{1,4\}} \widetilde{\pi}$, and $\widetilde{\pi}(1) \succ_1 \widehat{\pi}(1)$ and $\widetilde{\pi}(4) \succ_4 \widehat{\pi}(4)$.

Hence the partitions $\widetilde{\pi}$ and $\widehat{\pi}$ are not strongly Nash stable, whereas they are both core and Nash stable. Therefore there is no strongly Nash stable partition for this game.

[8]Note that only individually rational coalitions are listed in a player's preference list, since remaining coalitions for the player can be listed in any way.

[9]Note that players 2 and 3 dislike each other, that is $\{2\} \succ_2 \{2,3\}$ and $\{3\} \succ_3 \{2,3\}$.

[10]This deviation means in terms of movements that players 2 and 3 exchange the coalitions that they are in under $\widetilde{\pi}$, and the partition $\widehat{\pi}$ is reached by this movement.

Iehlé (2007) introduced *pivotal balancedness* and showed that it is both necessary and sufficient for the existence of a core stable partition. As strong Nash stability implies core stability, and the hedonic game in Example 1 has a core stable partition but lacks any strongly Nash stable partitions, it follows that pivotal balancedness is a necessary but not sufficient condition for strong Nash stability.

2.1.3 The weak top-choice property

Banerjee et al. (2001) introduced two top-coalition properties and showed that each property is sufficient for a hedonic game to have a core stable partition.

Given a nonempty set of players $\widetilde{N} \subseteq N$, a nonempty subset $H \subseteq \widetilde{N}$ is a ***top-coalition*** of $\widetilde{N}$ if for any $i \in H$ and any $T \subseteq \widetilde{N}$ with $i \in T$, we have $H \succeq_i T$.
A game $G = (N, \succeq)$ satisfies the ***top-coalition property*** if for any nonempty set of players $\widetilde{N} \subseteq N$, there exists a top-coalition of $\widetilde{N}$.

Given a nonempty set of players $\widetilde{N} \subseteq N$, a nonempty subset $H \subseteq \widetilde{N}$ is a ***weak top-coalition*** of $\widetilde{N}$ if H has an ordered partition $\{H^1, \ldots, H^l\}$ such that

(i) for any $i \in H^1$ and any $T \subseteq \widetilde{N}$ with $i \in T$, we have $H \succeq_i T$, and

(ii) for any $k > 1$, any $i \in H^k$ and any $T \subseteq \widetilde{N}$ with $i \in T$, we have

$T \succ_i H \Rightarrow T \cap (\bigcup_{m<k} H^m) \neq \emptyset$.
A game $G = (N, \succeq)$ satisfies the ***weak top-coalition property*** if for any nonempty set of players $\widetilde{N} \subseteq N$, there exists a weak top-coalition of $\widetilde{N}$.

For any nonempty set of players $H \subseteq N$, let $W(H)$ denote the weak top-coalitions of H. Thus, $W(N)$ denote the weak top-coalitions of the grand coalition N.

Definition 9 A hedonic game $G = (N, \succeq)$ satisfies the ***weak top-choice property*** if $W(N)$ partitions N.

Proposition 1 *If a hedonic game satisfies the weak top-choice property, then it has a strongly Nash stable partition.*

Proof Let $G = (N, \succeq)$ be a hedonic game which satisfies the weak top-choice property. Let $W(N) = \{H_1, \ldots, H_K\}$ with corresponding partitions $\{H_1^1, \ldots, H_1^{l(1)}\}, \ldots, \{H_K^1, \ldots, H_K^{l(K)}\}$. Clearly, $W(N)$ is a partition for N since the game satisfies the weak top-choice property. Let $W(N) = \pi^\star$. It will be shown that $\pi^\star$ is strongly Nash stable. Suppose that $\pi^\star$ is not strongly Nash stable. Then, there exists a nonempty subset of players $H \subseteq N$ which strongly Nash blocks the partition $\pi^\star$.

Note that $H \cap (\bigcup_{j=1}^{K} H_j^1) = \emptyset$, since for any $j \in \{1, \ldots, K\}$, for any $i \in H_j^1$ and any $T \in \sigma_i$, $H_j \succeq_i T$. Now it will be shown that $H \cap (\bigcup_{j=1}^{K} H_j^2) = \emptyset$. For any $j \in \{1, \ldots, K\}$, any agent $i \in H_j^2$ needs the cooperation of at least one agent in H_j^1 in order to form a better coalition than H_j. That is, for any $i \in H_j^2$ and any $T \in \sigma_i$, $T \succ_i H_j$ implies $T \cap H_j^1 \neq \emptyset$. However, it is known that $H \cap H_j^1 = \emptyset$ for all $j \in \{1, \ldots, K\}$, so $H \cap (\bigcup_{j=1}^{K} H_j^2) = \emptyset$.

Continuing with similar arguments it is shown that $H \cap (\bigcup_{j=1}^{K} H_j^k) = \emptyset$ for all $k \in \{1, \ldots, \bar{l}\}$, where $\bar{l} = \max \{l(1), \ldots, l(K)\}$. However, this implies that there does not exist a nonempty subset of players $H \subseteq N$ which strongly Nash blocks the partition $\pi^\star$, a contradiction. Hence $\pi^\star$ is strongly Nash stable. □

We have constructed examples showing that the weak top-choice property and the weak top-coalition property are independent of each other.[11] If a game satisfies the weak top-choice property and players have strict preferences, then the game may have more than one strongly Nash stable partition.

A stronger version of the weak top-choice property can be defined as follows (by using the definition of top-coalition): A hedonic game $G = (N, \succeq)$ satisfies the ***top-choice property*** if the top-coalitions of the grand coalition N form a partition of N. Now, if a hedonic game satisfies the top-choice property then it has a strongly Nash stable partition. Moreover, if every player's best coalition is unique then there exists a unique strongly Nash stable partition which consists of the top-coalitions of N. We

[11] These examples are provided in the Appendix.

have constructed examples showing that the top-choice property and the top-coalition property (respectively, the weak top-coalition property) are independent of each other. It is clear that if a hedonic game satisfies the top-choice property then it also satisfies the weak top-choice property. However, a hedonic game satisfying the weak top-choice property may fail to satisfy the top-choice property.

An application of the weak top-choice property is *Benassy (1982)'s uniform reallocation rule*.[12] Banerjee et al. (2001) showed that a hedonic game which is induced by the uniform reallocation rule satisfies the weak top-coalition property, by proving that any subset $\widetilde{N} \subseteq N$ is a weak top-coalition of itself. Hence, the weak top-choice property is satisfied, and the partition $\{N\}$ is strongly Nash stable. Note that a hedonic game which is induced by the uniform reallocation rule may violate the top-choice property.[13]

2.1.4 Descending separable preferences

In a well established paper, Burani and Zwicker (2003) study hedonic games when players have *descending separable preferences*, and show that such a hedonic game always has a partition, which is called the *top segment partition*, that is both core and Nash stable. Burani and Zwicker (2003) will be followed to define descending separable preferences and the top segment partition.[14]

Let $p : N \to N$ be a permutation of the set of players and assume that p yields a strict reference ranking of players

$$p_1 > p_2 > \ldots > p_n. \tag{2.1}$$

The following conditions are defined for an individual player's preferences.

Condition 1. (Common ranking of individuals, CRI) For any three distinct players p_i, p_j and p_k, if $p_j > p_k$ then $\{p_i, p_j\} \succeq_{p_i} \{p_i, p_k\}$.

[12]See Banerjee et al. (2001) for details of the hedonic game derived from the uniform reallocation rule.

[13]See example 3 (page 152) of Banerjee et al. (2001) for such an example.

[14]The reader is referred to Burani and Zwicker (2003) for more details of descending separable preferences and the construction of the top segment partition.

Condition 2. (Descending desire, DD) For any pair p_i, p_j of distinct players with $p_i > p_j$ and for any coalition C containing neither player p_i nor p_j, if $\{p_j\} \cup C \succeq_{p_j} \{p_j\}$ then $\{p_i\} \cup C \succeq_{p_i} \{p_i\}$ and if $\{p_j\} \cup C \succ_{p_j} \{p_j\}$ then $\{p_i\} \cup C \succ_{p_i} \{p_i\}$.

Condition 3. (Separable preferences, SP) A profile of players' preferences is *separable* if, for every $i, j \in N$ and every coalition C such that $C \in \sigma_i$ and $j \notin C$, $\{i, j\} \succeq_i \{i\} \Leftrightarrow C \cup \{j\} \succeq_i C$ and $\{i, j\} \succ_i \{i\} \Leftrightarrow C \cup \{j\} \succ_i C$.

Condition SP implies the property of iterated separable preferences.

Definition 10 (Iterated separable preferences) For any player p_i and for any two disjoint coalitions C and D with $C \ni p_i$, if $\{p_i, d\} \succeq_{p_i} \{p_i\}$ for every $d \in D$ then $C \cup D \succeq_{p_i} C$, and if $\{p_i, d\} \succ_{p_i} \{p_i\}$ for every $d \in D$ then $C \cup D \succ_{p_i} C$.

Condition 4. (Group separable preferences, GSP) For any player p_i and for any two disjoint coalitions C and D with $C \ni p_i$, if $\{p_i\} \cup D \succeq_{p_i} \{p_i\}$ then $C \cup D \succeq_{p_i} C$ and if $\{p_i\} \cup D \succ_{p_i} \{p_i\}$ then $C \cup D \succ_{p_i} C$.

Condition 5. (Responsive preferences, RESP) For any triple of players p_i, p_j, p_k and for any coalition C such that $p_j, p_k \notin C$ and $p_i \in C$, $\{p_i, p_j\} \succeq_{p_i} \{p_i, p_k\}$ if and only if $\{p_j\} \cup C \succeq_{p_i} \{p_k\} \cup C$ and $\{p_i, p_j\} \succ_{p_i} \{p_i, p_k\}$ if and only if $\{p_j\} \cup C \succ_{p_i} \{p_k\} \cup C$.

Condition 6. (Replaceable preferences, REP) For any pair p_i, p_j of distinct players with $p_i > p_j$ and for any coalition C containing neither player p_i nor p_j, if $\{p_i, p_j\} \cup C \succeq_{p_j} \{p_j\}$ then $\{p_i, p_j\} \cup C \succeq_{p_i} \{p_i\}$ and if $\{p_i, p_j\} \cup C \succ_{p_j} \{p_j\}$ then $\{p_i, p_j\} \cup C \succ_{p_i} \{p_i\}$.

Condition REP implies descending mutual preferences.

Definition 11 (Descending mutual preferences) For any pair p_i, p_j of distinct players with $p_i > p_j$, if $\{p_i, p_j\} \succeq_{p_j} \{p_j\}$ then $\{p_i, p_j\} \succeq_{p_i} \{p_i\}$ and if $\{p_i, p_j\} \succ_{p_j} \{p_j\}$ then $\{p_i, p_j\} \succ_{p_i} \{p_i\}$.

Definition 12 A profile of agents' preferences is ***descending separable*** if there exists a reference ordering (2.1) under which Conditions 1 (CRI), 2 (DD), 3 (SP), 4 (GSP), 5 (RESP), and 6 (REP) all hold.

Let $G = (N, \succeq)$ be a hedonic game where players have descending separable preferences. A partition $\pi^\star = \{T^\star, \{p_{l+1}\}, \ldots, \{p_n\}\}$ is called a *top-segment partition* which is obtained in terms of the reference ordering (2.1) as follows: First, the *top-segment coalition* $T^\star$ is formed. Player p_1, the first agent in the ordering, belongs to the top-segment coalition. If the next agent, player p_2, strictly prefers being alone to joining p_1, then $T^\star$ is completed and $T^\star = \{p_1\}$. If, however, $\{p_1, p_2\} \succeq_{p_2} \{p_2\}$, then player p_2 is added to $T^\star$. Continue to add players from left to right until a player, denoted as p_{l+1}, is reached who strictly prefers staying alone to joining the growing coalition (or until everyone joins, if such an agent p_{l+1} is never reached). The top-segment coalition is denoted by $T^\star = \{p_1, \ldots, p_l\}$. Second, let players from p_{l+1} to p_n each form a one member coalition.

Following results are taken from Burani and Zwicker (2003) which will be helpful while proving that a hedonic game with descending separable preferences always has a strongly Nash stable partition.

Lemma 1 *(Burani and Zwicker (2003), Lemma 1, page 37) Every individually rational coalition contains at most l members.*

It is shown in Burani and Zwicker (2003) that there exists a coalition $\emptyset \neq T^{\star\star} = \{p_1, \ldots, p_f\}$ contained in $T^\star$ such that $\{p_i, p_l\} \succeq_{p_i} \{p_i\}$ holds for each agent $p_i \in T^{\star\star}$, where such an agent with the highest index is denoted by p_f.

Lemma 2 *(Burani and Zwicker (2003), Lemma 3, page 38) For each of the players in $T^{\star\star} = \{p_1, \ldots, p_f\} \subset T^\star$, coalition $T^\star$ is top-ranked among individually rational coalitions (or tied for top). Therefore, no deviating coalition can contain any of the players in $T^{\star\star}$.*

We will also need the following lemma.

Lemma 3 *For each player* $p_k \in \{p_{l+1}, \ldots, p_n\}$*,* $\{p_k\} \succ_{p_k} \{p_j, p_k\}$ *holds for any* $p_j \in \{p_{f+1}, \ldots, p_l\} = T^\star \setminus T^{\star\star}$.

Proof First, it is shown that the lemma holds for agent p_{l+1}. Consider agent p_{f+1}. Since $p_{f+1} \notin T^{\star\star}$, $\{p_{f+1}\} \succ_{p_{f+1}} \{p_{f+1}, p_l\}$. Then, condition CRI and transitivity of preferences imply, $\{p_{f+1}\} \succ_{p_{f+1}} \{p_{f+1}, p_{l+1}\}$. This fact, together with descending mutual preferences, yields that $\{p_{l+1}\} \succ_{p_{l+1}} \{p_{f+1}, p_{l+1}\}$. Now, by condition CRI, $\{p_{l+1}\} \succ_{p_{l+1}} \{p_j, p_{l+1}\}$ holds for any $p_j \in \{p_{f+1}, \ldots, p_l\}$. It is also needed to show independently that $\{p_{l+1}\} \succ_{p_{l+1}} \{p_l, p_{l+1}\}$ holds, in case $T^{\star\star} = \{p_1, \ldots, p_{l-1}\}$. Suppose not. Condition CRI then implies that $\{p_j, p_{l+1}\} \succeq_{p_{l+1}} \{p_{l+1}\}$ for all $p_j \in T^\star$. Now, iterated separable preferences imply that $(T^\star \cup \{p_{l+1}\}) \succeq_{p_{l+1}} \{p_{l+1}\}$ which is in contradiction with $p_{l+1} \notin T^\star$. So, $\{p_{l+1}\} \succ_{p_{l+1}} \{p_l, p_{l+1}\}$ also holds. Hence, $\{p_{l+1}\} \succ_{p_{l+1}} \{p_j, p_{l+1}\}$ for any $p_j \in \{p_{f+1}, \ldots, p_l\}$.

Second, by condition DD, it holds for any $p_k < p_{l+1}$ that $\{p_k\} \succ_{p_k} \{p_j, p_k\}$ for every $p_j \in \{p_{f+1}, \ldots, p_l\}$, completing the proof. □

Our main result with descending separable preferences is now stated and proved.

Proposition 2 *Let* $G = (N, \succeq)$ *be a hedonic game. If players have descending separable preferences, then there always exists a strongly Nash stable partition.*

Proof Let $G = (N, \succeq)$ be a hedonic game where players have descending separable preferences. Let $\pi^\star$ be a top-segment partition. It is known by Burani and Zwicker (2003) that $\pi^\star$ is both core and Nash stable. It will be shown that $\pi^\star$ is strongly Nash stable. Suppose that $\pi^\star$ is not strongly Nash stable. Then, there exists a pair (π, H) where $\pi \in (\Pi(N) \setminus \{\pi^\star\})$ and $\emptyset \neq H \subseteq N$ such that $\pi^\star \xrightarrow{H} \pi$ and for all $i \in H$, $\pi(i) \succ_i \pi^\star(i)$. Note that $\mid H \mid > 1$ since $\pi^\star$ is Nash stable.

Since $\pi^\star$ is both core and Nash stable, and it is supposed that H strongly Nash blocks the partition $\pi^\star$, another remaining four possible cases will be checked.

Case 1. $H \subseteq \{p_{l+1}, \ldots, p_n\}$ and H strongly Nash blocks the top-segment partition $\pi^\star$ by joining $T^\star$.[15]

Since H strongly Nash blocks the partition $\pi^\star$ by joining $T^\star$, $(T^\star \cup H) \succ_{p_j} \{p_j\}$ for all $p_j \in H$. For any $p_i \in T^\star$ and any $p_j \in H$, $p_i > p_j$. So, by condition REP, it holds for each $p_i \in T^\star$ that $(T^\star \cup H) \succ_{p_i} \{p_i\}$. Hence, $(T^\star \cup H)$ would be an individually rational coalition which contradicts with Lemma 1, since $|\ (T^\star \cup H)\ | > l$. So, there is no subset H of $\{p_{l+1}, \ldots, p_n\}$ which strongly Nash blocks the top-segment partition $\pi^\star$ by joining $T^\star$.

Case 2. $H \subsetneq \{p_{l+1}, \ldots, p_n\}$, $p_i \in [N \setminus (T^\star \cup H)]$, and H strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$.[16]

Since H strongly Nash blocks the partition $\pi^\star$ by joining $\{p_i\}$, $(H \cup \{p_i\}) \succ_{p_j} \{p_j\}$ for all $p_j \in H$. Note that since $\pi^\star$ is Nash stable, it is true for every $p_j \in H$ that $\{p_j\} \succeq_{p_j} \{p_j, p_k\}$ for all $p_k \in [(H \setminus \{p_j\}) \cup \{p_i\}]$. Then, iterated separable preferences imply that $\{p_j\} \succeq_{p_j} (H \cup \{p_i\})$ for every $p_j \in H$. This is in contradiction with the fact that H strongly Nash blocks the partition $\pi^\star$ by joining $\{p_i\}$. Hence, there does not exist a proper subset H of $\{p_{l+1}, \ldots, p_n\}$ which strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$, where $p_i \in [N \setminus (T^\star \cup H)]$.

Case 3. $H \subseteq T^\star$, $p_i \in \{p_{l+1}, \ldots, p_n\}$, and H strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$.[17]

Since $H \subseteq T^\star$ strongly Nash blocks the partition $\pi^\star$ by joining $\{p_i\}$, $(H \cup \{p_i\}) \succ_{p_j} T^\star$ for all $p_j \in H$. This fact, together with Lemma 2, implies that $H \cap T^{\star\star} = \emptyset$. Let $p_h \in H$ be a player such that $p_h > p_j$ for all $p_j \in (H \setminus \{p_h\})$. Note that $p_h \neq p_l$, because $|\ H\ | > 1$. Since $p_h \notin T^{\star\star}$, agent p_h has preferences such that $\{p_h\} \succ_{p_h} \{p_h, p_l\}$. Condition CRI yields that $\{p_h, p_l\} \succeq_{p_h} \{p_h, p_i\}$ because $p_l > p_i$, and transitivity of preferences implies, $\{p_h\} \succ_{p_h} \{p_h, p_i\}$. Then, descending mutuality implies, $\{p_j\} \succ_{p_j} \{p_j, p_i\}$ holds for each $p_j \in H$. This result combined with

[15] So, $\pi = \{T^\star \cup H\} = \{\{N\}\}$ if $H = \{p_{l+1}, \ldots, p_n\}$, and $\pi = \{T^\star \cup H, \{\{p_j\} \mid p_j \in N \setminus (T^\star \cup H)\}\}$ if $H \subsetneq \{p_{l+1}, \ldots, p_n\}$.

[16] So, $\pi = \{T^\star, H \cup \{p_i\}, \{\{p_j\} \mid p_j \in [N \setminus (T^\star \cup H \cup \{p_i\})]\}\}$ if $H \neq N \setminus (T^\star \cup \{p_i\})$, and $\pi = \{T^\star, H \cup \{p_i\}\}$ if $H = N \setminus (T^\star \cup \{p_i\})$.

[17] Now, $\pi = \{T^\star \setminus H, H \cup \{p_i\}, \{\{p_j\} \mid p_j \in N \setminus (T^\star \cup \{p_i\})\}\}$ if $H \subsetneq T^\star$, and $\pi = \{H \cup \{p_i\}, \{\{p_j\} \mid p_j \in N \setminus (T^\star \cup \{p_i\})\}\}$ if $H = T^\star$.

condition SP implies that $H \succ_{p_j} (H \cup \{p_i\})$ for every $p_j \in H$. Now, transitivity of preferences yields for each $p_j \in H$ that $H \succ_{p_j} T^\star$. However, this is in contradiction with $\pi^\star$ being core stable, i.e., H would block the partition $\pi^\star$. Hence, there is no subset H of $T^\star$ which strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$, where $p_i \in \{p_{l+1}, \ldots, p_n\}$.

Case 4. $H = H_1 \cup H_2$, where $H_1 \subseteq T^\star$ and $H_2 \subsetneqq \{p_{l+1}, \ldots, p_n\}$, $p_i \in N \setminus (T^\star \cup H_2)$, and H strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$.[18]

So, $(H \cup \{p_i\}) \succ_{p_j} T^\star$ for all $p_j \in H_1$, and $(H \cup \{p_i\}) \succ_{p_k} \{p_k\}$ for all $p_k \in H_2$. Since $\pi^\star$ is Nash stable, it holds for each $p_k \in H_2$ that, $\{p_k\} \succeq_{p_k} \{p_k, p_h\}$ for any $p_h \in [(H_2 \setminus \{p_k\}) \cup \{p_i\}]$. Now, Lemma 2 implies that $H_1 \cap T^{\star\star} = \emptyset$, i.e., $H_1 \subseteq \{p_{f+1}, \ldots, p_l\}$. This fact, together with Lemma 3, implies that, for each $p_k \in H_2$, $\{p_k\} \succ_{p_k} \{p_k, p_j\}$ for any $p_j \in H_1$. Hence, for each $p_k \in H_2$ it holds that $\{p_k\} \succeq_{p_k} \{p_k, p_x\}$ for all $p_x \in [(H \setminus \{p_k\}) \cup \{p_i\}]$. Then, iterated separable preferences imply that $\{p_k\} \succeq_{p_k} (H \cup \{p_i\})$ for all $p_k \in H_2$, which is the desired contradiction. Hence, there does not exist $H = H_1 \cup H_2$, where $H_1 \subseteq T^\star$ and $H_2 \subsetneqq \{p_{l+1}, \ldots, p_n\}$, which strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$, where $p_i \in N \setminus (T^\star \cup H_2)$.

Since the four cases cover all possibilities it is concluded that there does not exist a subset of players $\emptyset \neq H \subseteq N$ which strongly Nash blocks the top-segment partition $\pi^\star$. Hence $\pi^\star$ is strongly Nash stable. □

Based on Proposition 2, one can argue that Burani and Zwicker (2003) were studying the wrong solution concept; they really should have been applying their methods to strong Nash stability. We have constructed examples showing that preferences are descending separable and the weak top-choice properties are independent of each other.

Burani and Zwicker (2003) also studied hedonic games on additively separable and symmetric domain of preferences where players' preferences are *purely cardinal*.

[18] So, $\pi = \{T^\star \setminus H_1, H \cup \{p_i\}, \{\{p_j\} \mid p_j \in [N \setminus (T^\star \cup H_2 \cup \{p_i\})]\}\}$ if $H_1 \subsetneqq T^\star$, and $\pi = \{H \cup \{p_i\}, \{\{p_j\} \mid p_j \in [N \setminus (T^\star \cup H_2 \cup \{p_i\})]\}\}$ if $H_1 = T^\star$. Note that $H_1 \neq \emptyset$ by case 2 and $H_2 \neq \emptyset$ by case 3.

Definition 13 A hedonic game $G = (N, \succeq)$ is ***additively separable*** if for any $i \in N$, there exists a function $v_i : N \to \mathbb{R}$ such that for any $H, T \in \Sigma_i$, $H \succeq_i T \Leftrightarrow \sum_{j \in H} v_i(j) \geq \sum_{j \in T} v_i(j)$, where $v_i(j) = 0$ for $i = j$.

Definition 14 An additively separable hedonic game satisfies ***symmetry*** if for any $i, j \in N$, $v_i(j) = v_j(i)$.

Definition 15 A profile of additively separable and symmetric preferences is ***purely cardinal*** if there exists an assignment of individual weights $w(i)$ to the players for which the following vector v represents the profile: for all $i, j \in N$,

$$v(i,j) = \begin{cases} w(i) + w(j) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

For any player i, her individual weight $w(i)$ represents the fixed individual contribution that she brings to any member of the coalition that she belongs. Purely cardinal preferences are descending separable, where the reference ranking (2.1) of agents is the permutation that ranks them in non-increasing order of their weights. Hence, a hedonic game with purely cardinal preferences always has a strongly Nash stable partition. However, we have provided an example showing that purely cardinal preferences is not a necessary condition for a game to have a strongly Nash stable partition.

We have constructed examples showing that preferences being purely cardinal and the weak top-choice property are independent of each other. Note that players' preferences need not be purely cardinal for a separable[19] and *anonymous* game.

A hedonic game $G = (N, \succeq)$ satisfies ***anonymity*** if for any $i \in N$, for any $H, T \in \Sigma_i$ with $\mid H \mid = \mid T \mid$, $H \sim_i T$.

[19]A hedonic game is *separable* if players' preferences satisfy Condition 3 (SP).

Lemma 4 *If a hedonic game is anonymous, additively separable and symmetric, then players' preferences are purely cardinal (hence has a strongly Nash stable partition).*

Proof Let $G = (N, \succeq)$ be an anonymous, additively separable and symmetric hedonic game. Anonimity and additive separability imply that for any $i \in N$ and any $j, k \in (N \setminus \{i\})$ with $j \neq k$ we have $v_i(j) = v_i(k)$. This fact, together with symmetry, implies that for any pair $i, j \in N$ and any $k \in (N \setminus \{i, j\})$ we have $v_i(k) = v_j(k)$. So, players' preferences for this game is represented by the functions $v = (v_i)_{i \in N}$ illustrated below:

	1	2	3	$\ldots$	$n-2$	$n-1$	n
v_1	0	x	x	$\ldots$	x	x	x
v_2	x	0	x	$\ldots$	x	x	x
v_3	x	x	0	$\ldots$	x	x	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
v_{n-1}	x	x	x	$\ldots$	x	0	x
v_n	x	x	x	$\ldots$	x	x	0

where $x \in \mathbb{R}$.

Players' preferences are purely cardinal, i.e., there exists an assignment of individual weights $w(i)$ to the players, where for any player $i \in N$, $w(i) = \frac{x}{2}$. So, for any $i, j \in N$, $v_i(j) = v_j(i) = w(i) + w(j) = x$ if $i \neq j$, and $v_i(j) = 0$ if $i = j$.

It is clear that $x = 0$ if and only if $w(i) = 0$ for all $i \in N$, $x > 0$ if and only if $w(i) > 0$ for all $i \in N$, $x < 0$ if and only if $w(i) < 0$ for all $i \in N$. So, any partition $\pi \in \Pi(N)$ is strongly Nash stable if $w(i) = 0$ for all $i \in N$. The partition $\{N\}$ is strongly Nash stable if $w(i) > 0$ for all $i \in N$ and the partition $\{\{i\} \mid i \in N\}$, which contains all singletons, is strongly Nash stable if $w(i) < 0$ for all $i \in N$. □

It is clear by the proof of Lemma 4 that if a hedonic game is anonymous, additively separable and symmetric then it satisfies the top-choice property.[20] However, we have given an example showing that a hedonic game which satisfies the top-choice property may not be additively separable and symmetric.

[20]If $x = 0$, then every partition is a top-coalition of N. If $x < 0$, then $\{\{i\} \mid i \in N\}$, which contains all singletons, is the top-coalition of N. If $x > 0$, then the grand coalition is a top-coalition of itself.

The strong Nash stability for hedonic games is not the unique stability notion which has not been studied earlier. In fact, two other stability notions for hedonic games can be defined.

Definition 16 Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ a partition. We say that a subset of players $T \subseteq N$ ***coalitionally Nash blocks*** π if there exists a coalition $H \in (\pi \cup \{\emptyset\})$ such that for each player $i \in T$, $(H \cup T) \succ_i \pi(i)$. A partition is ***coalitionally Nash stable*** if there does not exist a subset of players which coalitionally Nash blocks it.

Definition 17 Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***core-exchange stable*** if it is core stable and there does not exist a subset of players $T \subseteq N$ such that individual players in T or subsets of T (strongly Nash) block π by exchanging their current coalitions under π.

It is clear that these two concepts are independent of each other, and each of these concepts is weaker than strong Nash stability. Moreover, a partition is both coalitionally Nash stable and core-exchange stable if and only if it is strongly Nash stable.

We have constructed an example showing that neither the weak top-choice property nor the preferences being descending separable is necessary for a hedonic game to have a strongly Nash stable partition. Hence, it is an open question to find a condition which is both necessary and sufficient for the existence of a strongly Nash stable partition.

2.1.5 Strong Nash stability under different membership rights

Different societies may have different membership rights, and a designer employs a certain rights structure to achieve some aims. This section studies how the concept of strong Nash stability changes under different membership rights. We will see that strong Nash stability under different membership rights fits with the earlier concepts.

FX-FE strong Nash stability is what has been called strong Nash stability in previous sections. Now, its strict version is defined.

Definition 18 Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***free exit-free entry strictly strongly Nash stable*** (FX-FE strictly strongly Nash stable) if there does not exist a pair $(\acute{\pi}, H)$ (where $\acute{\pi} \in (\Pi(N) \setminus \{\pi\})$ and $\emptyset \neq H \subseteq N$) such that

(i) $\pi \xrightarrow{H} \acute{\pi}$ ($\acute{\pi}$ is reachable from π by movements of H),

(ii) for all $i \in H$, $\acute{\pi}(i) \succeq_i \pi(i)$, and for some $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$.

Definition 19 Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***free exit-approved entry strongly Nash stable*** (FX-AE strongly Nash stable) if there does not exist a pair $(\acute{\pi}, H)$ such that

(i) $\pi \xrightarrow{H} \acute{\pi}$,

(ii) for all $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$, and

(iii) for all $i \in H$, for all $k \in (\acute{\pi}(i) \setminus \{i\})$, $\acute{\pi}(k) \succeq_k \pi(k)$.

Definition 20 Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***approved exit-approved entry strongly Nash stable*** (AX-AE strongly Nash stable) if there does not exist a pair $(\acute{\pi}, H)$ such that

(i) $\pi \xrightarrow{H} \acute{\pi}$,

(ii) for all $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$, and

(iii) for all $k \in (N \setminus H)$, $\acute{\pi}(k) \succeq_k \pi(k)$.

Definition 21 Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***approved exit-free entry strongly Nash stable*** (AX-FE strongly Nash stable) if there does not exist a pair $(\acute{\pi}, H)$ such that

(i) $\pi \xrightarrow{H} \acute{\pi}$,

(ii) for all $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$, and

(iii) for all $i \in H$, for all $j \in (\pi(i) \setminus \{i\})$, $\acute{\pi}(j) \succeq_j \pi(j)$.

Strict versions of concepts given in definitions 19-21 are defined by replacing item (ii) with [for all $i \in H$, $\acute{\pi}(i) \succeq_i \pi(i)$, and for some $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$].

Lemma 5 *Let $G = (N, \succeq)$ be a hedonic game. If a partition $\pi \in \Pi(N)$ is FX-AE strongly Nash stable, then it is core stable.*

Proof Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ be an FX-AE strongly Nash stable partition. Suppose that π is not core stable. Then, there is a coalition $T \subseteq N$ such that for all $i \in T$, $T \succ_i \pi(i)$. Let $\acute{\pi} = \{T, \{\{H \setminus T\} \mid H \in \pi$ and $H \setminus T \neq \emptyset\}\}$ denote the partition that is obtained from coalition T's blocking of π. Now, it is shown that the pair $(\acute{\pi}, T)$ satisfies the three conditions of FX-AE strong Nash stability. First, it is clear that $\acute{\pi}$ is reachable from π by T, i.e., $\pi \xrightarrow{T} \acute{\pi}$. Second, since it is supposed that T blocks π, i.e., for any $i \in T$, $T = \acute{\pi}(i) \succ_i \pi(i)$. Third, for any $i \in T$, $\acute{\pi}(i) \setminus \{i\} = T \setminus \{i\}$. So, for all $i \in T$, for all $k \in (\acute{\pi}(i) \setminus \{i\})$, we have $\acute{\pi}(k) \succ_k \pi(k)$. Hence, the pair $(\acute{\pi}, T)$ satisfies the three conditions of FX-AE strong Nash stability, in contradiction with π being FX-AE strongly Nash stable, i.e., coalition T would block the partition π under FX-AE membership rights. Hence, π is core stable. □

Note that this lemma implies that if a partition is FX-AE strictly strongly Nash stable, then it is strictly core stable. Now, it is shown that the converse of this lemma is true under the assumption that players have strict preferences.

Lemma 6 *Let $G = (N, \succ)$ be a hedonic game where players have strict preferences. If a partition $\pi \in \Pi(N)$ is core stable, then it is FX-AE strongly Nash stable.*

Proof Let $G = (N, \succ)$ be a hedonic game where players have strict preferences. Let $\pi \in \Pi(N)$ be a core stable partition. Suppose that π is not FX-AE strongly Nash stable. Then, there exists a pair $(\acute{\pi}, H)$ such that players in H strongly Nash block the partition π by inducing $\acute{\pi}$ under FX-AE membership rights.

Since π is core stable, H cannot block π. So, H strongly Nash blocks the partition π by either players in H (or subsets of H) exchange their current coalitions that they belong under π or all players in H leave their current coalitions and join another coalition of the partition π. In either case, there exists a coalition $T \in \acute{\pi}$ such that $T \cap H \neq \emptyset$. Since the membership rights is FX-AE and players have strict preferences, we have $\acute{\pi}(j) \succ_j \pi(j)$ for all players $j \in (T \setminus H)$. This result, together with the fact that H strongly Nash blocks the partition π, implies that $\acute{\pi}(i) \succ_i \pi(i)$ for all $i \in T$. However, this is in contradiction with π being core stable, i.e., coalition T would block the partition π. Hence, π is FX-AE strongly Nash stable. □

This lemma is not true without the assumption of strict preferences.

Example 2 Let $G = (N, \succeq)$, where $N = \{1, 2\}$ and players' preferences are as follows: $\{1, 2\} \sim_1 \{1\}$, and $\{1, 2\} \succ_2 \{2\}$.

The partition $\pi = \{\{1\}, \{2\}\}$ is core stable. However π is not FX-AE strongly Nash stable, since player 2 strongly Nash blocks the partition π by joining $\{1\}$ under FX-AE membership rights, i.e., $\pi \xrightarrow{\{2\}} \acute{\pi} = \{\{1, 2\}\}$, and $\acute{\pi}(2) \succ_2 \pi(2)$ and $\acute{\pi}(1) \sim_1 \pi(1)$.

Lemma 6 implies, if a partition is strictly core stable, then it is FX-AE strictly strongly Nash stable. The following proposition is an implication of lemmata 5 and 6.

Proposition 3 *Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is FX-AE strictly strongly Nash stable if and only if it is strictly core stable.*

Note that if players have strict preferences then a partition is FX-AE strictly strongly Nash stable if and only if it is FX-AE strongly Nash stable, and a partition is strictly core stable if and only if it is core stable. Proposition 3 shows that core stability entails an FX-AE rights structure.

Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is *contractual core stable* (defined in Sung and Dimitrov (2007)) if there does not exist a coalition $T \subseteq N$ such that

(i) for all $i \in T$, $T \succ_i \pi(i)$ and

(ii) for all $j \in (N \setminus T)$, $\pi(j) \setminus T \succeq_j \pi(j)$.

Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is *contractual strictly core stable* (defined in Sung and Dimitrov (2007)) if there does not exist a coalition $T \subseteq N$ such that
(i) for all $i \in T$, $T \succeq_i \pi(i)$,
(ii) for some $i \in T$, $T \succ_i \pi(i)$, and
(iii) for all $j \in (N \setminus T)$, $\pi(j) \setminus T \succeq_j \pi(j)$.

Lemma 7 *Let $G = (N, \succeq)$ be a hedonic game. If a partition $\pi \in \Pi(N)$ is AX-AE strongly Nash stable, then it is contractual core stable.*

Proof Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ be an AX-AE strongly Nash stable partition. Suppose that π is not contractual core stable. Then, there is a coalition $T \subseteq N$ such that for all $i \in T$, $T \succ_i \pi(i)$ and for all $j \in (N \setminus T)$, $\pi(j) \setminus T \succeq_j \pi(j)$.

Let $\acute{\pi} = \{T, \{\{\pi(j) \setminus T\} \mid j \in (N \setminus T) \text{ and } \pi(j) \setminus T \neq \emptyset\}\}$ denote the partition that is obtained from coalition T's blocking of π. Now, it is shown that the pair $(\acute{\pi}, T)$ satisfies the three conditions of AX-AE strong Nash stability. The first two conditions are trivially satisfied, i.e., $\pi \xrightarrow{T} \acute{\pi}$, and for all $i \in T$, $T = \acute{\pi}(i) \succ_i \pi(i)$.

Let $H = \{j \in N \mid j \notin T \text{ and } j \in \pi(i) \text{ for some } i \in T\}$ and $\bar{H} = \{\bar{j} \in N \mid \bar{j} \notin \pi(i) \text{ for any } i \in T\}$. Note that $H, \bar{H}$ and T are pairwise disjoint, and $N = H \cup \bar{H} \cup T$. Since it is supposed that T blocks π and this blocking does not hurt any player, for any $j \in H$ we have $\acute{\pi}(j) \succeq_j \pi(j)$. Note that $\bar{H} = \{\bar{j} \in N \mid \acute{\pi}(\bar{j}) = \pi(\bar{j})\}$, so for any $\bar{j} \in \bar{H}$ we have $\acute{\pi}(\bar{j}) \sim_{\bar{j}} \pi(\bar{j})$. Hence, for any $k \in (N \setminus T)$ we have $\acute{\pi}(k) \succeq_k \pi(k)$, i.e., the third condition of AX-AE strong Nash stability is also satisfied by the pair $(\acute{\pi}, T)$. Hence, the pair $(\acute{\pi}, T)$ satisfies the three conditions of AX-AE strong Nash stability, this contradicts with π being AX-AE strongly Nash stable. That is, T would strongly Nash block the partition π under AX-AE membership rights. Hence, π is contractual core stable. □

By this lemma, it can be said that if a partition is AX-AE strictly strongly Nash stable, then it is contractual strictly core stable. Now, it is shown that the converse of lemma 7 is true under the assumption that players have strict preferences.

Lemma 8 *Let $G = (N, \succ)$ be a hedonic game where players have strict preferences. If a partition $\pi \in \Pi(N)$ is contractual core stable, then it is AX-AE strongly Nash stable.*

Proof Let $G = (N, \succ)$ be a hedonic game where players have strict preferences. Let $\pi \in \Pi(N)$ be a contractual core stable partition. Suppose that π is not AX-AE strongly Nash stable. Then, there exists a pair $(\acute{\pi}, H)$ such that H strongly Nash blocks the partition π by inducing $\acute{\pi}$ under AX-AE membership rights.

Let $T = \{i \in N \mid \acute{\pi}(i) \neq \pi(i)\}$ denote the set of agents whose coalitions changed from π to $\acute{\pi}$. Note that $T \neq \emptyset$. Now, for any $i \in T$ we have $\acute{\pi}(i) \succ_i \pi(i)$, since players have strict preferences and it is supposed that π is not AX-AE strongly Nash stable.

However, each player in T leaves her current coalition under π, and forms the coalitions $T_1, \ldots, T_K$ which are pairwise disjoint and their union is equal to T such that for any $k \in \{1, \ldots, K\}$, $T_k \in \acute{\pi}$. Now, for any $k \in \{1, \ldots, K\}$, we have, for all $i \in T_k$, $T_k \succ_i \pi(i)$ and for all $j \in (N \setminus T_k)$, $\pi(j) \setminus T_k \succeq_j \pi(j)$. This is in contradiction with π being contractual core stable, i.e., for any $k \in \{1, \ldots, K\}$, a coalition T_k would block the partition π without hurting other players. Hence, π is AX-AE strongly Nash stable. □

Lemma 8 may fail to be true if the assumption that players have strict preferences is relaxed.[21] By lemma 8, it can be said that, if a partition is contractual strictly core stable, then it is AX-AE strictly strongly Nash stable. The next proposition follows from lemmata 7 and 8.

Proposition 4 *Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is AX-AE strictly strongly Nash stable if and only if it is contractual strictly core stable.*

Sung and Dimitrov (2007) showed that for any hedonic game a contractual strictly core stable partition always exists. This result together with Proposition 4 implies that an AX-AE strictly strongly Nash stable partition always exists for any hedonic game.

[21]Consider example 2. The partition $\pi = \{\{1\}, \{2\}\}$ is contractual core stable. However, it is not AX-AE strongly Nash stable, since $\pi \xrightarrow{\{2\}} \acute{\pi} = \{\{1, 2\}\}$, and $\acute{\pi}(2) \succ_2 \pi(2)$ and $\acute{\pi}(1) \sim_1 \pi(1)$.

Note that, if a partition is AX-FE strongly Nash stable, then it is AX-AE strongly Nash stable. This fact and lemma 7 imply, if a partition $\pi \in \Pi(N)$ is AX-FE strongly Nash stable then it is contractual core stable. However, the converse is not true.

Example 3 Let $G = (N, \succeq)$, where $N = \{1, 2\}$ and players' preferences are as follows: $\{1\} \succ_1 \{1, 2\}$, and $\{1, 2\} \succ_2 \{2\}$.

The partition $\pi = \{\{1\}, \{2\}\}$ is contractual core stable. However π is not AX-FE strongly Nash stable, since player 2 strongly Nash blocks the partition π by joining $\{1\}$ under AX-FE membership rights, i.e., $\pi \xrightarrow{\{2\}} \acute{\pi} = \{\{1, 2\}\}$, and $\acute{\pi}(2) \succ_2 \pi(2)$ and $\pi(2) \setminus \{2\} = \emptyset$, i.e., there is no player that player 2 needs to get a permission to leave from the coalition $\pi(2)$.[22]

2.1.6 Conclusion

In this section, we studied hedonic coalition formation games where each player's preferences rely only upon the members of her coalition. A new stability notion under free exit-free entry membership rights, referred to as strong Nash stability, is introduced which is stronger than both core and Nash stabilities studied earlier in the literature. Strong Nash stability has an analogue in non-cooperative games and it is the strongest stability notion appropriate to the context of hedonic coalition formation games. The weak top-choice property is introduced and shown to be sufficient for the existence of a strongly Nash stable partition. It is also shown that descending separable preferences guarantee the existence of a strongly Nash stable partition. Strong Nash stability under different membership rights is also studied.

[22]The partition $\acute{\pi} = \{\{1, 2\}\}$ is AX-FE strongly Nash stable since player 2 does not permit player 1 to leave from $\{1, 2\}$. However, $\acute{\pi}$ is not individually rational, since $\{1\} \succ_1 \acute{\pi}(1)$. So, there is no individually rational and AX-FE strongly Nash stable partition for this game.

2.2 Cover formation games

2.2.1 Introduction

In this section, we define cover formation games as an extension of hedonic coalition formation games. A collection of coalitions is referred to as a cover if its union is equal to the set of players. Thus, a player can be a member of several different coalitions in a cover formation game, whereas a player can be a member of only one coalition under a hedonic coalition formation game. For instances, a researcher can be a member of several research teams at the same time, and an individual may be a member of several clubs or she may have more than one nationalities. A country possibly has memberships of some free trade agreements.

A cover formation game consists of a finite set of players, each of whom is endowed with preferences over nonempty collections of coalitions each of which contains herself. We define stability concepts based on individual movements as well as movements by a subset of players under different membership rights.

Although purely cardinal preferences guarantee the existence of an FX-FE strongly Nash stable partition for hedonic coalition formation games (by Proposition 2), purely cardinal preferences do not even guarantee the existence of an FX-FE Nash stable cover for cover formation games (Lemma 9). We introduce the notion of a top-choice property for cover formation games and show that it suffices for the existence of an FX-FE strongly Nash stable cover (Proposition 5).

We show that if players have additively separable preferences, then there exists an FX-AE Nash stable cover (Proposition 6). We also show that there always exists an AX-AE strictly strongly Nash stable cover for any cover formation game (Proposition 7).

This section is organized as follows: Section 2.2.2 presents the basic notions. Results are given in Section 2.2.3, and Section 2.2.4 concludes.

2.2.2 Basic notions

Let $N = \{1, 2, \ldots, n\}$ be a nonempty finite set of players. A nonempty subset H of N is called a coalition. Let $i \in N$ be a player, and $\sigma_i = \{H \subseteq N \mid i \in H\}$ denote the set of all coalitions each of which contains player i. For any player i, let $\Sigma_i = (2^{\sigma_i} \setminus \{\emptyset\})$ denote the non-empty power set of σ_i. Each player i has a reflexive, complete and transitive preference relation $\succeq_i$ over Σ_i.[23] So, a player's preferences depend only on the members of her coalitions, i.e., each player considers only who will be her partners in the coalitions that she belongs, and she is not interested in what other coalitions her partners belong.

The strict and indifference preference relations associated with $\succeq_i$ will be denoted by $\succ_i$ and $\sim_i$, respectively, and defined as follows: For all $\mathcal{X}(i), \mathcal{Y}(i) \in \Sigma_i$,
$[\mathcal{X}(i) \succ_i \mathcal{Y}(i)]$ if and only if $[\mathcal{X}(i) \succeq_i \mathcal{Y}(i)$ and not $\mathcal{Y}(i) \succeq_i \mathcal{X}(i)]$, and
$[\mathcal{X}(i) \sim_i \mathcal{Y}(i)]$ if and only if $[\mathcal{X}(i) \succeq_i \mathcal{Y}(i)$ and $\mathcal{Y}(i) \succeq_i \mathcal{X}(i)]$.

Let $\succeq = (\succeq_1, \ldots, \succeq_n)$ denote a preference profile for the set of players.

Definition 22 A pair $E = (N, \succeq)$ denote ***a cover formation game***.

Given a cover formation game we are interested in the coalitions that might form. We require that the union of the coalitions be equal to the set of agents, i.e., we require that the set of coalitions be a cover of N.

Definition 23 We say that a set of coalitions $\Theta = \{H \mid H \in (2^N \setminus \{\emptyset\})\}$ is a ***cover*** of N if $\bigcup_{H \in \Theta} H = N$.

Let $\Psi(N)$ denote the set of all covers of N.[24] Given any $\Theta \in \Psi(N)$ and any $i \in N$, we let $\Theta(i) = \{H \in \Theta \mid i \in H\}$ denote the set of all coalitions each of which contains the player i under cover Θ. The preference relation $\succeq_i$ of player i over Σ_i can be extended over the set of all covers $\Psi(N)$ in a usual way: For any covers $\Theta, \acute{\Theta} \in \Psi(N)$, $[\Theta \succeq_i \acute{\Theta}]$ if and only if $[\Theta(i) \succeq_i \acute{\Theta}(i)]$.

[23] We abuse notation that we also use $\succeq_i$ to denote preferences of agent i over Σ_i.

[24] We note that a partition is also a cover since the union of the coalitions in a partition is equal to the set of agents. So, $\Pi(N) \subsetneqq \Psi(N)$.

Given a cover formation game, our concern is the existence of covers which are stable in some sense. We will define some stability concepts based on individual as well as coalitional deviations under different membership rights.

Definition 24 Let $E = (N, \succeq)$ be a cover formation game. A cover $\Theta \in \Psi(N)$ is ***individually rational for player*** i if $\Theta(i) \succeq_i \{i\}$ and is ***individually rational*** if it is individually rational for every player $i \in N$.

A cover is individually rational if each player prefers the set of coalitions that she belongs to being single, i.e., each agent i prefers $\Theta(i)$ to $\{i\}$.

We will employ two approaches while defining FX-FE strongly Nash stable cover. In the first one, we will use the non-cooperative game induced by a cover formation game.

Every cover formation game induces a non-cooperative game as defined below.

Let $E = (N, \succeq)$ be a cover formation game with $\mid N \mid= n$ players. Consider the following induced non-cooperative game $\Gamma^E = \big(N, (S_i)_{i\in N}, (R_i)_{i\in N}\big)$ which is defined as follows:

- The *set of players* in Γ^E is the player set N of E.
- Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be a finite set of labels such that $m = 2^n$. We take $\mathcal{L}$ to be the *set of strategies* available to each player, so $S_i = \mathcal{L}$ for each $i \in N$. Let $S = \prod_{i\in N} S_i$ denote the strategy space. A strategy profile $s = (s_1, \ldots, s_n) \in S$ induces a cover Θ_s of N as follows: a coalition H belongs to Θ_s if and only if all agents in H choose the same label as part of their strategies under s, i.e., there exists $L \in \mathcal{L}$ such that $L \in s_i$ for all $i \in H$.
- *Preferences* for Γ^E is defined as follows: a player i prefers the strategy profile s to the strategy profile $\acute{s}$, $sR_i\acute{s}$, if and only if $\Theta_s(i) \succeq_i \Theta_{\acute{s}}(i)$, i.e., player i prefers the coalitions of $\Theta_s(i)$ to which she belongs to the coalitions of $\Theta_{\acute{s}}(i)$ that she is a member of.

We now define FX-FE Nash and strong Nash stable covers by using the induced non-cooperative game approach.

Definition 25 Let $E = (N, \succeq)$ be a cover formation game. A cover $\Theta \in \Psi(N)$ is ***FX-FE Nash stable*** (respectively, ***FX-FE strongly Nash stable***) if it is induced by a strategy profile which is a Nash equilibrium (respectively, strong Nash equilibrium) of the induced non-cooperative game Γ^E.

Thus, the Nash equilibria of Γ^E correspond to the Nash stable covers of E, and the strong Nash equilibria of Γ^E correspond to the strongly Nash stable covers of E.

If the strategy profile s which induces the cover Θ_s is not a Nash equilibrium (respectively, strong Nash equilibrium) of Γ^E, then there is a player $i \in N$ (respectively, a subset of players $H \subseteq N$) which deviates from s (according to s) and this deviation is beneficial to agent i (respectively, all agents in H). In such a case, it is said that i ***Nash blocks*** (respectively, H ***strongly Nash blocks***) the cover Θ_s under FX-FE membership rights.

The second approach is posed in terms of movements and reachability.

We first explain what we mean by individual movements. Let Θ_s be a cover which is induced by the strategy profile s, and $i \in N$ be a deviating agent. The deviation of player i from s can be explained as movements among the coalitions of the cover Θ_s as follows: Player i leaves some of her current coalitions and joins some coalitions that she does not belong under Θ_s. We let $\mathcal{B}_{\Theta_s}(i) \subseteq (\{B \in \Theta_s \mid i \in B\} \cup \{\emptyset\})$ [25] denote the set of coalitions that player i leaves, and $\mathcal{T}_{\Theta_s}(i) \subseteq (\{T \in \Theta_s \mid i \notin T\} \cup \{\emptyset\})$ [26] denote the set of coalitions that player i joins by her individual movements among the coalitions of Θ_s. Note that following cases are possible:

(i) $\mathcal{B}_{\Theta_s}(i) \neq \emptyset$ and $\mathcal{H}_{\Theta_s}(i) \neq \emptyset$, that is player i leaves some coalitions that she is a member of under Θ_s and joins some coalitions that she is not in under Θ_s by her individual movements.

[25]That is, $\mathcal{B}_{\Theta_s}(i) \subseteq (\Theta_s(i) \cup \{\emptyset\})$.
[26]That is, $\mathcal{T}_{\Theta_s}(i) \subseteq [(\Theta_s \setminus \Theta_s(i)) \cup \{\emptyset\}]$.

(ii) $\mathcal{B}_{\Theta_s}(i) = \emptyset$ and $\mathcal{H}_{\Theta_s}(i) \neq \emptyset$, that is player i does not leave any coalition that she is in under Θ_s and joins some coalitions that she is not a member of under Θ_s by her individual movements.

(iii) $\mathcal{B}_{\Theta_s}(i) \subsetneqq \Theta(i)$ and $\mathcal{H}_{\Theta_s}(i) = \emptyset$, that is player i leaves some coalitions that she is in under Θ_s and does not join any coalition that she is not in under Θ_s by her individual movements.

So, it is not possible to have both $\mathcal{B}_{\Theta_s}(i) = \emptyset$ and $\mathcal{H}_{\Theta_s}(i) = \emptyset$ by individual movements of player $i \in N$.

Secondly, we will define what we mean by movements of a subset of players $\emptyset \neq H \subseteq N$ among the coalitions of a cover Θ_s. Given a cover Θ_s and a subset of players $H \subseteq N$, agents of H coordinate among themselves, and some players of H leave some of their current coalitions and some players of H join some coalitions that they are not in under Θ_s. Note that movements of H are simultaneous. Again we use the notation that for any $i \in H$, $\mathcal{B}_{\Theta_s}(i) \subseteq (\{B \in \Theta_s \mid i \in B\} \cup \{\emptyset\})$ denote the set of coalitions that player $i \in H$ leaves, and $\mathcal{T}_{\Theta_s}(i) \subseteq (\{T \in \Theta_s \mid i \notin T\} \cup \{\emptyset\})$ denote the set of coalitions that player $i \in H$ joins by movements of H among the coalitions of Θ_s.

We note that it is possible that for all $i \in H$, $\mathcal{B}_{\Theta_s}(i) = \emptyset$ and $\mathcal{T}_{\Theta_s}(i) = \emptyset$ by movements of a subset of players $H \subseteq N$. That is, each agent in H keeps her existing coalitions $\Theta_s(i)$ and does not join any coalition that she is not in under Θ_s, but agents in H form new coalitions $H_1, \ldots, H_K$ among themselves such that $\bigcup_{k=1}^{k=K} H_k = H$ and for any $k \in \{1, \ldots, K\}$, $H_k \notin \Theta_s$.

Given a cover Θ and a subset of players $H \subseteq N$, by any movements of H among the coalitions of Θ, players of H obtain a new cover $\acute{\Theta} \in (\Psi(N) \setminus \{\Theta\})$, and it is said that $\acute{\Theta}$ is reachable from the cover Θ via H.

Definition 26 Let $E = (N, \succeq)$ be a cover formation game and $\Theta \in \Psi(N)$ be a cover. Another cover $\acute{\Theta} \in (\Psi(N) \setminus \{\Theta\})$ is said to be ***reachable from*** Θ ***by movements of a subset of players*** $H \subseteq N$, denoted by $\Theta \xrightarrow{H} \acute{\Theta}$, if, for any $j, k \in (N \setminus H)$ and any coalition $T \in \Theta$ with $j, k \in T$ there exists a coalition $\acute{T} \in \acute{\Theta}$ such that $j, k \in \acute{T}$.

Reachability by movements of a subset of agents says that a non-deviating agent remains with all former mates who are not deviators. Notice that a subset of players $\widehat{H} \supseteq H$ can do all movements that H can. Note that for any covers $\Theta, \acute{\Theta} \in \Psi(N)$ we have $\Theta \xrightarrow{N} \acute{\Theta}$.

In the sequel we use the notation that $\mathcal{B}_{\Theta}(i) \subseteq (\{B \in \Theta \mid i \in B\} \cup \{\emptyset\})$ denote the set of coalitions that player i leaves, with a generic coalition $B \in \mathcal{B}_{\Theta}(i)$ and a generic player $b \in B$, and $\mathcal{T}_{\Theta}(i) \subseteq (\{T \in \Theta \mid i \notin T\} \cup \{\emptyset\})$ denote the set of coalitions to which player i joins, with a generic coalition $T \in \mathcal{H}_{\Theta}(i)$ and a generic player $t \in T$.

We now define FX-FE strong Nash stability of a cover in terms of movements and reachability.

Definition 27 Let $E = (N, \succeq)$ be a cover formation game. A cover $\Theta \in \Psi(N)$ is ***FX-FE strongly Nash stable*** if there does not exist a pair $(\acute{\Theta}, H)$ (where $\acute{\Theta} \in (\Psi(N) \backslash \{\Theta\})$ and $\emptyset \neq H \subseteq N$) such that

(i) $\Theta \xrightarrow{H} \acute{\Theta}$ ($\acute{\Theta}$ is reachable from Θ by movements of H), and

(ii) for all $i \in H$, $\acute{\Theta}(i) \succ_i \Theta(i)$.

If such a pair $(\acute{\Theta}, H)$ exists, then it is said that H ***strongly Nash blocks*** Θ ***under FX-FE membership rights*** (by inducing $\acute{\Theta}$).

Given a cover formation game, note that a cover is FX-FE Nash stable if there is no singleton coalition that blocks it in the above sense, i.e., a cover is FX-FE Nash stable if there is no player such that it is beneficial for her to make individual movements among the coalitions of the given cover without taking into account that her movements may hurt some of other players.[27]

[27]Note that the two definitions of FX-FE (strongly) Nash stable covers are equivalent (definitions 25 and 27).

We now define strong Nash stability under other membership rights.

Definition 28 Let $E = (N, \succeq)$ be a cover formation game. A cover $\Theta \in \Psi(N)$ is ***FX-AE strongly Nash stable*** if there does not exist a pair $(\acute{\Theta}, H)$ such that

(i) $\Theta \xrightarrow{H} \acute{\Theta}$,

(ii) for all $i \in H$, $\acute{\Theta}(i) \succ_i \Theta(i)$, and

(iii) for all $i \in H$, all $T \in \mathcal{T}_\Theta(i)$ and all $t \in T$, $\acute{\Theta}(t) \succeq_t \Theta(t)$.

If such a pair $(\acute{\Theta}, H)$ exists, then it is said that H ***strongly Nash blocks*** Θ ***under FX-AE membership rights*** (by inducing $\acute{\Theta}$).

Definition 29 Let $E = (N, \succeq)$ be a cover formation game. A cover $\Theta \in \Psi(N)$ is ***AX-AE strongly Nash stable*** if there does not exist a pair $(\acute{\Theta}, H)$ such that

(i) $\Theta \xrightarrow{H} \acute{\Theta}$,

(ii) for all $i \in H$, $\acute{\Theta}(i) \succ_i \Theta(i)$, and

(iii) for all $j \in (N \setminus T)$, $\acute{\Theta}(j) \succeq_j \Theta(j)$.

If such a pair $(\acute{\Theta}, H)$ exists, then it is said that H ***strongly Nash blocks*** Θ ***under AX-AE membership rights*** (by inducing $\acute{\Theta}$).

Definition 30 Let $E = (N, \succeq)$ be a cover formation game. A cover $\Theta \in \Psi(N)$ is ***AX-FE strongly Nash stable*** if there does not exist a pair $(\acute{\Theta}, H)$ such that

(i) $\Theta \xrightarrow{H} \acute{\Theta}$,

(ii) for all $i \in H$, $\acute{\Theta}(i) \succ_i \Theta(i)$, and

(iii) for all $i \in H$, all $B \in \mathcal{B}_\Theta(i)$ and all $b \in B$, $\acute{\Theta}(b) \succeq_b \Theta(b)$.

If such a pair $(\acute{\Theta}, H)$ exists, then it is said that H ***strongly Nash blocks*** Θ ***under AX-FE membership rights*** (by inducing $\acute{\Theta}$).

Since movements of a deviating subset of players H are simultaneous, we assume for above definitions that complete movements of H are first announced, then knowing these movements, players of the coalitions that members of H leave and/or join approve or disapprove these movements if they are entitled with such a right. There may exist a player $j \in (N \setminus H)$ such that for some $i, h \in H$ (possibly $i = h$), $j \in B$ for some $B \in \mathcal{B}_{\Theta}(i)$ and $j \in T$ for some $T \in \mathcal{T}_{\Theta}(h)$, i.e., movements of H may affect player j via both coalition B that i leaves and coalition T to which h joins.

If we restrict our attention to singleton coalitions for the concepts given in definitions 28-30, we obtain corresponding Nash stable notions.

We can define strict versions of concepts given in definitions 27-30 by replacing item (ii) with [for all $i \in H$, $\acute{\Theta}(i) \succeq_i \Theta(i)$, and for some $i \in H$, $\acute{\Theta}(i) \succ_i \Theta(i)$].

2.2.3 Results

We will study the existence of (strongly) Nash stable covers under different membership rights.

Bogomolnaia and Jackson (2002) showed that for hedonic coalition formation games if players' preferences are additively separable and symmetric[28] then there exists an FX-FE Nash stable partition. We know by Proposition 2 that for hedonic coalition formation games if players have purely cardinal preferences[29] then there always exists an FX-FE strongly Nash stable partition. A question arises that whether these results hold for cover formation games.

[28]These properties are given in definitions 13 and 14, respectively. Since we are working with cover formation games, if players' preferences are additively separable, then a player i compares two collections of coalitions $\mathcal{X}(i), \mathcal{Y}(i) \in \Sigma_i$ as follows: $[\mathcal{X}(i) \succeq_i \mathcal{Y}(i)] \iff [\sum_{j \in X : X \in \mathcal{X}(i)} v_i(j) \geq \sum_{j \in Y : Y \in \mathcal{Y}(i)} v_i(j)]$, where $v_i : N \to \mathbb{R}$ is a function.

[29]See definition 15 and recall that purely cardinal preferences are descending separable.

Lemma 9 *Purely cardinal preferences do not guarantee the existence of an FX-FE Nash stable cover for cover formation games.*

Proof We will provide a cover formation game where players have purely cardinal preferences, but has no FX-FE Nash stable cover.

Let $E = (N, \succeq)$, where $N = \{1, 2, 3\}$ and individual weights are $w(1) = -15$, $w(2) = -65$ and $w(3) = 115$. So, additively separable and symmetric preferences of players are represented by the following functions $v = (v_i)_{i \in N}$:

	1	2	3
v_1	0	-80	100
v_2	-80	0	50
v_3	100	50	0

We will show that there does not exist an FX-FE Nash stable cover.[30] Suppose that there exists a cover Θ which is FX-FE Nash stable.

Note that any cover including $\{1, 2, 3\}$ (respectively, $\{1, 2\}$) is not FX-FE Nash stable, since player 2 benefits by leaving $\{1, 2, 3\}$ (respectively, $\{1, 2\}$). So, $\{1, 2, 3\} \notin \Theta$ and $\{1, 2\} \notin \Theta$.

If $\{2, 3\} \in \Theta$, then we must have that $\{1, 2, 3\} \in \Theta$, otherwise player 1 benefits by joining $\{2, 3\}$. However we know that $\{1, 2, 3\} \notin \Theta$, so $\{2, 3\} \notin \Theta$.

If $\{2\} \in \Theta$, then we must have that $\{2, 3\} \in \Theta$, otherwise player 3 benefits by joining $\{2\}$. However we know that $\{2, 3\} \notin \Theta$, so $\{2\} \notin \Theta$.

So, we have that $\{1, 2, 3\} \notin \Theta$, $\{1, 2\} \notin \Theta$, $\{2, 3\} \notin \Theta$ and $\{2\} \notin \Theta$. This means that there is no coalition in Θ which contains player 2, in contradiction with that Θ is a cover of N. Hence there does not exist a cover which is FX-FE Nash stable. $\square$

[30]Note that the partition $\pi = \{\{1, 3\}, \{2\}\}$ is the unique FX-FE Nash stable partition.

Although purely cardinal preferences property suffices for the existence of an FX-FE strongly Nash stable partition for hedonic coalition formation games, it does not even guarantee the existence of an FX-FE Nash stable cover for cover formation games.

We introduce following notation.
Given a nonempty set of players $T \subseteq N$ with $i \in T$, let $Ch_i(T) = \{\mathcal{Y} \in (2^T \cap \Sigma_i) \mid \mathcal{Y} \succeq_i \mathcal{X}$ for all $\mathcal{X} \in (2^T \cap \Sigma_i)\}$ denote the set of best alternatives of player i on $(2^T \cap \Sigma_i)$ under $\succeq_i$. Thus, for any player i, $Ch_i(N)$ denote the set of best alternatives of player i over Σ_i under $\succeq_i$, i.e., $Ch_i(N) = \{\mathcal{Y} \in \Sigma_i \mid \mathcal{Y} \succeq_i \mathcal{X}$ for all $\mathcal{X} \in \Sigma_i\}$.

Definition 31 A cover formation game $E = (N, \succeq)$ satisfies the ***top-choice property*** if the following two conditions are satisfied:

(i) for all $i \in N$, $\mid Ch_i(N) \mid= 1$, and

(ii) for all $i \in N$, all $T \in Ch_i(N)$ and all $j \in T$, $T \in Ch_j(N)$.

Proposition 5 *If a cover formation game $E = (N, \succeq)$ satisfies the top-choice property, then it has a unique FX-FE strongly Nash stable cover, $\Theta^\star$, where for each player $i \in N$ we have $\Theta^\star(i) = Ch_i(N)$.*

Proof Let $E = (N, \succeq)$ be a cover formation game which satisfies the top-choice property. Let $\Theta^\star = \bigcup_{i \in N} Ch_i(N) = \{H_1, \ldots, H_K\}$. It is clear that $\Theta^\star$ is a cover of N. Note that $\Theta^\star(i) = Ch_i(N)$ for every player $i \in N$. This fact, together with the top-choice property, implies that for all $i \in N$ we have $\Theta^\star \succ_i \Theta$ for any $\Theta \in (\Psi(N) \setminus \{\Theta^\star\})$. Hence $\Theta^\star$ is an FX-FE strongly Nash stable cover.

Since for any covers $\Theta, \acute{\Theta} \in \Psi(N)$ we have $\Theta \xrightarrow{N} \acute{\Theta}$ and the game satisfies the top-choice property, any FX-FE strongly Nash stable cover must include H_1,. . ., H_{K-1} and H_K. Hence, $\Theta^\star$ is the unique FX-FE strongly Nash stable cover. □

We note that the top-choice property is not necessary for a cover formation game to have an FX-FE strongly Nash stable cover.

Proposition 6 *Let $E = (N, \succeq)$ be a cover formation game. If players have additively separable preferences, then there always exists an FX-AE Nash stable cover.*

Proof Let $E = (N, \succeq)$ be a cover formation game where players have additively separable preferences. We will construct a cover, $\Theta^\star$, and show that it is FX-AE Nash stable.

For any player i, let $\mathcal{H}^\star(i) = \{H \in \Sigma_i \mid \sum_{j\in H} v_i(j) \geq 0\}$ denote the set of all coalitions each of which gives a non-negative payoff to player i. We define $\Theta^\star$ as follows: $\Theta^\star = \{H \subseteq N \mid \text{for all } i \in H,\ H \in \mathcal{H}^\star(i)\}$. That is, a coalition H is in $\Theta^\star$ if each player in H gets a non-negative payoff from being a member of H, and $\Theta^\star$ contains all such coalitions.

We note that $\Theta^\star$ is a cover of N.[31] Suppose that $\Theta^\star$ is not FX-AE Nash stable. Then there exists a player $i \in N$ who Nash blocks $\Theta^\star$ under FX-AE membership rights.

Claim 1: $\mathcal{T}_{\Theta^\star}(i) = \emptyset$, i.e., for any coalition $T \in \Theta^\star$ with $i \notin T$, there exists a player $t \in T$ such that she disapproves player i's joining to coalition T.

Proof of Claim 1. Let $T \in \Theta^\star$ with $i \notin T$. Since $(T \cup \{i\}) \notin \Theta^\star$, there exists a player $t \in T$ such that $(T \cup \{i\}) \notin \mathcal{H}^\star(t)$, i.e., $\sum_{j\in(T\cup\{i\})} v_t(j) < 0$. This fact, together with FX-AE membership rights, implies that player t disapproves player i's joining to T. Hence, $\mathcal{T}_{\Theta^\star}(i) = \emptyset$, i.e., player i cannot join any coalition $T \in \Theta^\star$ such that $i \notin T$.

Claim 2: $\mathcal{B}_{\Theta^\star}(i) = \emptyset$, i.e., player i does not leave any coalition that she is a member of under $\Theta^\star$.

Proof of Claim 2. Note that for all $H \in \Theta^\star(i)$, we have $\sum_{j\in H} v_i(j) \geq 0$. So, if player i leaves a coalition that she belongs under $\Theta^\star$, then she gains at most nothing. So, it is not a profitable movement for player i to leave a coalition that she is a member of under $\Theta^\star$. Hence, player i does not leave any coalition that she is in under $\Theta^\star$, i.e., $\mathcal{B}_{\Theta^\star}(i) = \emptyset$.

[31] For all $i \in N$ we have $\{i\} \in \mathcal{H}^\star(i)$ as $v_i(i) = 0$. So, $\Theta^\star$ contains all singleton coalitions, and hence $\bigcup_{H\in\Theta^\star} H = N$.

By claims 1 and 2, $\mathcal{T}_{\Theta^\star}(i) = \emptyset$ and $\mathcal{B}_{\Theta^\star}(i) = \emptyset$, which is the desired contradiction. Hence $\Theta^\star$ is FX-AE Nash stable.[32] □

Ballester (2004) showed that there always exists an AX-AE Nash stable partition for hedonic coalition formation games. Sung and Dimitrov (2007) defined *contractual strict core stability* and showed that every hedonic coalition formation game has such a partition. We know by Proposition 4 that a partition is AX-AE strictly strongly Nash stable if and only if it is contractual strictly core stable. Hence, an AX-AE strictly strongly Nash stable partition always exists for any hedonic game.

Following result shows that for cover formation games there always exists an AX-AE strictly strongly Nash stable cover.[33]

Proposition 7 *There always exists an AX-AE strictly strongly Nash stable cover for any cover formation game.*

Proof Let $E = (N, \succeq)$ be a cover formation game and $\Theta_1 \in \Psi(N)$ be an individually rational cover.[34] Now we will define an algorithm.

Period 1. Start with cover Θ_1 and search for a pair (Θ_2, H_1) (where $\Theta_2 \in (\Psi(N) \setminus \{\Theta_1\})$ and $\emptyset \neq H_1 \subseteq N$) such that following conditions are satisfied:

(i) $\Theta_1 \xrightarrow{H_1} \Theta_2$,

(ii) for all $i \in H_1$, $\Theta_2(i) \succeq_i \Theta_1(i)$, and for some $j \in H_1$, $\Theta_2(j) \succ_j \Theta_1(j)$, and

(iii) for all $l \in (N \setminus H_1)$, $\Theta_2(l) \succeq_l \Theta_1(l)$.

If there exists such a pair (Θ_2, H_1), then we go to next period, and next period starts with Θ_2. If there does not exist such a pair, then we stop and we say that the result of the algorithm is Θ_1.

[32] We note that $\Theta^\star$ need not to be FX-AE strongly Nash stable.

[33] We note that our proof is similar to the one given in Sung and Dimitrov (2007) to show the existence of a contractual strictly core stable partition.

[34] An individually rational cover always exists for any cover formation game, e.g., a cover consisting of only all singleton coalitions is individually rational.

In general, we have following in Period k:

Period k. Start with Θ_k and search for a pair (Θ_{k+1}, H_k) such that following conditions are satisfied:

(i) $\Theta_k \xrightarrow{H_k} \Theta_{k+1}$,

(ii) for all $i \in H_k$, $\Theta_{k+1}(i) \succeq_i \Theta_k(i)$, and for some $j \in H_k$, $\Theta_{k+1}(j) \succ_j \Theta_k(j)$, and

(iv) for all $l \in (N \setminus H_k)$, $\Theta_{k+1}(l) \succeq_l \Theta_k(l)$.

If there exists such a pair (Θ_{k+1}, H_k), then we go to next period, and next period starts with Θ_{k+1}. If there does not exist such a pair, then we stop and we say that the result of the algorithm is Θ_k.

Claim. The algorithm stops in a finite period.

Proof of Claim. Note that while passing from one period to another period at least one player is made strictly better off and no player is made worse off. This fact, together with AX-AE membership rights, implies that a cycle never occurs in the algorithm. Note that $\mid \sigma_i \mid= 2^{n-1}$ and let $2^{n-1} - 1 = c$. A player $i \in N$ can be better off at most $\mid \Sigma_i \mid -1 = 2^c - 1$ times without being worse off. Hence the algorithm stops at most at period $n2^c - n$ which is finite since n is finite. Hence the algorithm stops in a finite period.

Let the algorithm stops at period k and Θ_k is the result of the algorithm. Clearly Θ_k is an AX-AE strictly strongly Nash stable cover, otherwise the algorithm would not stop at period k. □

2.2.4 Conclusion

We introduced cover formation games, and for each membership rights we defined corresponding Nash and strongly Nash stable covers. We showed that purely cardinal preferences are not sufficient for the existence of an FX-FE Nash stable cover. We introduced the top-choice property and showed that it suffices for the existence of an FX-FE strongly Nash stable cover. We also proved that additively separable preferences guarantee the existence of an FX-AE Nash stable cover. We also showed that for any cover formation game there always exists an AX-AE strictly strongly Nash stable cover.

CHAPTER 3

NASH IMPLEMENTATION OF SOCIAL CHOICE RULES WHICH ARE IMPLEMENTABLE VIA RECHTSSTAAT

3.1 Introduction

We consider an environment with a finite non-empty set of agents and a finite non-empty set of alternatives. Each agent has preferences over the set of alternatives where indifferences are allowed and a list of agents' preferences is called a preference profile. A social choice rule (SCR) is a rule which chooses a nonempty subset of the set of alternatives at each preference profile. An SCR can be seen as a reflection of some social planner's values at each preference profile, it gives the list of alternatives which are considered as desirable for the society, i.e., it makes a choice in the name of the society. Therefore there may be, and generally will be information problem in the society and the question of how a planner can learn individuals' information in order to implement an SCR, is an interesting one (Hurwicz (1972)).

If the planner has all relevant information, then her problem would just to choose the alternatives indicated by the SCR. The implementation problem arises when the preferences of individuals are private information; if the social planner does not know the preferences of the individuals, so the preference profile of the society, how can she decide on the alternatives which should be chosen?

The planner's problem is to design a mechanism in order to get the necessary information in the process of choosing an alternative. The simple approach to mechanism design is to ask agents to reveal their preferences. However this approach will generally be unsatisfactory, because agents will be able to benefit by misreporting their preferences. As their preferences are private information, this misreporting will not be detectable. So we can take a more general approach to mechanism design and ask each individual to send an abstract "message" to the planner and the planner's problem is to determine an outcome function, i.e., an appropriate relation between the messages sent by the agents and the set of alternatives. Hence, a mechanism (game form) consists of a nonempty strategy set (messages) for each agent and an outcome function which maps from joint messages into alternatives. A mechanism with a preference profile on the set of alternatives induces a game in strategic form. A mechanism is said to implement an SCR according to a game theoretic solution concept σ if the σ-equilibrium outcomes of the induced game coincide with the set of alternatives assigned by the SCR at each preference profile of the society. At this point implementation theory is on a different footing from game theory. Whereas game theory is concerned with how a *given* game will be played, implementation theory deals with the *design* of games.

Maskin (1999) gave a partial characterization of Nash implementability in terms of two conditions called *Maskin monotonicity* and *no veto power*. Maskin monotonicity turns out to be a necessary condition (which is not sufficient) for Nash implementability. In a society with at least three agents, Maskin monotonicity combined with no veto power is a sufficient condition for Nash implementability, although no veto power condition is not necessary. Danilov (1992) introduced *essential monotonicity* and showed that with at least three agents, essential monotonicity is both necessary and sufficient for an SCR to be Nash implementable when agents have strict preference relations over the set of alternatives, and Yamato (1992) extended this result to weak domain of preferences, i.e., agents are allowed to have indifferences over alternatives. Moore and Repullo (1990) introduced a condition, referred to as *Condition* μ, and showed that when there are at least three agents in the society, an SCR is Nash implementable if and only if it satisfies Condition μ.

Sertel (2002) studied designing rights by introducing the notion of a *Rechtsstaat*. A Rechtsstaat is a pair $\omega = (\beta, \gamma)$, where β and γ are functions, called a *benefit* and a *code of rights*, respectively. Given a pair (x, y) of alternatives and a preference profile, a benefit gives us the set of all coalitions that strictly prefer y to x at the given preference profile. Given a pair (x, y) of alternatives, a code of rights specifies a family $\gamma(x, y)$ of coalitions in which each coalition is given the right to approve the alteration of x to y. Given a preference profile, a coalition in $\gamma(x, y)$ approves the alteration of x to y if all members of the coalition strictly prefer y to x at the given preference profile. An alternative x is said to be an *equilibrium* of a Rechtsstaat at a given preference profile if, there is no coalition which is given the right to approve the alteration of x to some other alternative y such that every agent in the coalition strictly prefers y to x. Sertel (2002) investigated Rechtsstaats which posses invisible hand property and the preservation of the best public interest which are parallel to the first and the second theorems of welfare economics.

Koray and Yıldız (2008) studied implementation via a Rechtsstaat. An SCR is said to be *implementable via a Rechtsstaat* if, at every preference profile, alternatives which are chosen by the SCR coincide with the equilibria of the Rechtsstaat. They characterized the SCRs which are implementable via Rechtsstaats. Moreover, they identified some properties of a Rechtsstaat which guarantee Nash implementability of an SCR which is implementable via Rechtsstaat. We note that in the model of Koray and Yıldız (2008), every agent has strict preference relations over alternatives, and an SCR may be empty-valued at some preference profiles. However, we allow agents to have weak preference relations over alternatives, and in our context an SCR is non-empty valued. Hence, our results are not comparable with the results of Koray and Yıldız (2008).

We will study Nash implementation of social choice rules which are implementable via a Rechtsstaat in this chapter. We will introduce a property, called as the *intersection property*, and show that every Rechtsstaat satisfying this property has non-empty equilibria at each preference profile (Proposition 8). We will then show that if an SCR is implementable via a Rechtsstaat then it is is weakly Pareto optimal and Maskin monotonic (Proposition 9).

We say that a Rechtsstaat ω satisfies *equal treatment of equivalent alternatives* ($ETEA$), if for any alternatives x and y, and any preference profile such that all agents are indifferent between x and y, then x being an equilibrium of ω implies that y is also an equilibrium of ω. However, there exists an SCR which is implementable via a Rechtsstaat that violates $ETEA$, but it is not Nash implementable (Example 5). We will show that, when there are at least three agents, if an SCR is implementable via some Rechtsstaat satisfying $ETEA$ then it is essentially monotonic, hence Nash implementable by Yamato (1992) (Theorem 1). We will also show that a Rechtsstaat $\omega = (\beta, \gamma)$ satisfies $ETEA$ if and only if for any alternative x and any alternatives y and z (different from x) we have $\gamma(y, x) = \gamma(z, x)$ (Proposition 11).

We will define oligarchic Rechtsstaats and show that if an SCR is implementable via an oligarchic Rechtsstaat then it is Nash implementable when there are three or more agents, and we will prove that such an SCR also satisfies neutrality.

This chapter is organized as follows. Section 3.2 presents the basic notions. The notion of Rechtsstaat is introduced in section 3.3. Main results are given in section 3.4. Section 3.5 introduces oligarchic Rechtsstaats. Section 3.6 concludes.

3.2 Basic notions

For any non-empty finite set T, we let $\mathcal{P}(T) = 2^T$, $\mathcal{P}_0(T) = \mathcal{P}(T) \setminus \{\emptyset\}$, and $\#T$ denote the cardinality of T.

Alternatives
Let A denote a non-empty finite set of alternatives, and we assume that $\#A \geq 2$.

Agents
Let $N = \{1, \ldots, n\}$ denote a non-empty finite set of agents, and we assume that $n \geq 2$.

Preferences
A *preference* on $B \subseteq A$ is a complete, reflexive and transitive binary relation on B. Let $W(B)$ denote the set of all preferences on B. An n-tuple $R_N \in W(A)^N$ is called a *preference profile*. For a coalition $H \in \mathcal{P}_0(N)$, $R_H \in W(A)^H$ denote a preference profile for coalition H. For any $i \in N$, let P_i denote the strict preference relation

associated with R_i and I_i denote the indifference relation associated with R_i. For any $R \in W(A)$, let $top(R) = \{x \in A \mid xRy$ for all $y \in A\}$ denote the set of best alternatives at R.

Social Choice Rules

A ***social choice rule*** (SCR) is a map $F : W(A)^N \to \mathcal{P}_0(A)$, i.e., for every preference profile $R_N \in W(A)^N$, an SCR F assigns a non-empty subset $F(R_N)$ of A.

Mechanism and Implementation

A ***mechanism*** (or game form) is an $(n+1)$-tuple $G = (S_1, \ldots, S_n; g)$ where

(i) for each $i \in N$, S_i is a nonempty set of *strategies* for player i,

(ii) $g : S^N = S_1 \times \cdots \times S_n \to A$ is a map called the *outcome function*.

For $H \in \mathcal{P}_0(N)$, $S^H = \prod_{i \in H} S_i$, and an element s_H of S^H is called a strategy profile for coalition H. A mechanism $G = (S_1, \ldots, S_n; g)$ at each preference profile $R_N \in W(A)^N$ induces a game (in strategic form) (G, R_N).

A strategy profile $s_N \in S^N$ is called a ***Nash equilibrium*** of (G, R_N) if for every $i \in N$, $g(s_N) R_i g(\acute{s}_i, s_{N \setminus \{i\}})$ for any $\acute{s}_i \in S_i$. Let $NE(G, R_N)$ denote the set of all Nash equilibria of the game (G, R_N).

Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR and $G = (S_1, \ldots, S_n; g)$ a mechanism. We say that G ***implements*** F ***in Nash equilibrium*** if $g(NE(G, R_N)) = F(R_N)$ for all $R_N \in W(A)^N$. We say that F is ***Nash implementable*** if there is a mechanism G which implements it in Nash equilibrium.

3.3 Rechtsstaat

Sertel (2002) studied designing rights by introducing *Rechtsstaat*. We will follow Sertel (2002) to define a Rechtsstaat.

Given N and A, a ***Rechtsstaat*** is a pair $\omega = (\beta, \gamma)$, where β and γ are functions, called a *benefit* and a *code of rights*, respectively, which are defined as follows:[1]

- A function $\beta : A \times A \times W(A)^N \to \mathcal{P}(\mathcal{P}_0(N))$ is called a ***benefit*** and defined as follows: For any $(x, y) \in A \times A$ with $x \neq y$, $H \in \mathcal{P}_0(N)$ and $R_N \in W(A)^N$, $H \in \beta(x, y; R_N)$ if and only if yP_ix for all $i \in H$. Let $\beta(x, y; R_N)$ gives us the set of all willing coalitions for an alteration of x to y at R_N, i.e., $\beta(x, y; R_N) = \{H \in \mathcal{P}_0(N) \mid yP_ix \text{ for all } i \in H\}$.
- A ***code of rights*** is a function $\gamma : A \times A \to \mathcal{P}(\mathcal{P}_0(N))$ which specifies for every $x, y \in A$ with $x \neq y$, $\gamma(x, y)$ of family of coalitions in which each coalition is given the right to approve the alteration of alternative x to alternative y, that is, for any $H \in \mathcal{P}_0(N)$, $H \in \gamma(x, y)$ if and only if the coalition H is given the right to approve the alteration of x to y. Given any $R_N \in W(A)^N$, we say that coalition $H \in \gamma(x, y)$ *approves* the alteration of x to y at R_N if and only if yP_ix for all $i \in H$. For any $x, y \in A$, we let $\underline{\gamma}(x, y)$ denote the set of all minimal coalitions each of which has the right to approve the alteration of x to y, i.e., for any $H \in \mathcal{P}_0(N)$, $H \in \underline{\gamma}(x, y)$ if and only if $H \in \gamma(x, y)$ and there does not exist $\acute{H} \subsetneqq H$ such that $\acute{H} \in \gamma(x, y)$.

In the sequel, we assume that a code γ satisfies following axioms:

(A1) For all $x, y \in A$ and all $H \in \mathcal{P}_0(N)$, if $H \in \gamma(x, y)$, then for all $\acute{H} \supseteq H$ we have $\acute{H} \in \gamma(x, y)$.[2]

(A2) For all $x, y \in A$, there exists $H \in \mathcal{P}_0(N)$ such that $H \in \gamma(x, y)$.

[1] The original definition of Rechtsstaat given in Sertel (2002) also contains a function $\alpha : A \times A \to \mathcal{P}(\mathcal{P}_0(N))$ which is called *ability*. For any $x, y \in A$ with $x \neq y$, the ability α gives a family $\alpha(x, y)$ of coalitions which are able to bring the alteration of alternative x to alternative y. We assume that for any $x, y \in A$ with $x \neq y$, $\alpha(x, y) = \mathcal{P}_0(N)$. So, we do not include ability in the definition of Rechtsstaat.

[2] By axiom A1, γ is monotonic with respect to the players.

Axioms A1 and A2 imply that for all $x, y \in A$, $N \in \gamma(x, y)$, i.e., for all $x, y \in A$, the grand coalition N has the right to approve the alteration of x to y.

Given N and A, we let Ω denote the set of all Rechtsstaats (on N and A) each of which satisfies the axioms A1 and A2.

An alternative $x \in A$ is an ***equilibrium*** of Rechtsstaat ω at R_N (defined in Sertel (2002)) if and only if for all $y \in (A \setminus \{x\})$, $\beta(x, y; R_N) \cap \gamma(x, y) = \emptyset$.

Let $\varepsilon(\omega, R_N)$ denote the set of equilibria of Rechtsstaat ω at $R_N \in W(A)^N$.

Following example shows that $\varepsilon(\omega, R_N) = \emptyset$ for some $\omega \in \Omega$ at some $R_N \in W(A)^N$.

Example 4 Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Let $\widetilde{\omega} = (\beta, \widetilde{\gamma})$, where $\widetilde{\gamma}$ is defined as follows:
$\widetilde{\gamma}(a, b) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{N\}\}$, $\widetilde{\gamma}(a, c) = \{\{2\}, \{1, 2\}, \{2, 3\}, \{N\}\}$,
$\widetilde{\gamma}(b, a) = \{\{3\}, \{1, 3\}, \{2, 3\}, \{N\}\}$, $\widetilde{\gamma}(b, c) = \{\{2\}, \{1, 2\}, \{2, 3\}, \{N\}\}$,
$\widetilde{\gamma}(c, a) = \{\{3\}, \{1, 3\}, \{2, 3\}, \{N\}\}$, $\widetilde{\gamma}(c, b) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{N\}\}$.

We consider following profile $R_N \in W(A)^N$:

R_1	R_2	R_3
a	b	c
b	c	a
c	a	b

Now, $a \notin \varepsilon(\omega, R_N)$ since $\beta(a, c; R_N) \cap \widetilde{\gamma}(a, c) = \{2, 3\}$, $b \notin \varepsilon(\omega, R_N)$ since $\beta(b, a; R_N) \cap \widetilde{\gamma}(b, a) = \{1, 3\}$, and $c \notin \varepsilon(\omega, R_N)$ since $\beta(c, b; R_N) \cap \widetilde{\gamma}(c, b) = \{1, 2\}$. Hence, $\varepsilon(\widetilde{\omega}, R_N) = \emptyset$.

We now define another Rechtsstaat $\widehat{\omega} = (\beta, \widehat{\gamma})$, where for any pair $x, y \in A$ and a coalition $H \in \mathcal{P}_0(N)$, $H \in \widehat{\gamma}(x, y)$ if and only if $\#H \geq 2$. Hence, for every pair $x, y \in A$ we have $\widehat{\gamma}(x, y) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{N\}\}$. We again consider the profile $R_N \in W(A)^N$. Now, $a \notin \varepsilon(\widehat{\omega}, R_N)$ since $\beta(a, c; R_N) \cap \widehat{\gamma}(a, c) = \{2, 3\}$, $b \notin \varepsilon(\widehat{\omega}, R_N)$ since $\beta(b, a; R_N) \cap \widehat{\gamma}(b, a) = \{1, 3\}$, and $c \notin \varepsilon(\widehat{\omega}, R_N)$ since $\beta(c, b; R_N) \cap \widehat{\gamma}(c, b) = \{1, 2\}$. Hence, $\varepsilon(\widehat{\omega}, R_N) = \emptyset$.

Let $\Omega^* \subset \Omega$ denote the set of Rechtsstaats such that for every $\omega \in \Omega^*$, $\varepsilon(\omega, R_N) \neq \emptyset$ for all $R_N \in W(A)^N$.

We will introduce a property, called intersection property, and show that any Rechtsstaat satisfying this property has non-empty equilibria at every preference profile.

Definition 32 Let $\omega = (\beta, \gamma)$ be a Rechtsstaat. We say that ω satisfies the ***intersection property*** if there exists $T \in \mathcal{P}_0(N)$ such that for all $x, y \in A$, and all $H \in \gamma(x, y)$, $T \subseteq H$, i.e., $\bigcap_{(x,y)\in(A\times A)} \gamma(x, y) \neq \emptyset$.

Proposition 8 *Let $\omega = (\beta, \gamma)$ be a Rechtsstaat. If ω satisfies the intersection property, then $\omega \in \Omega^*$.*

Proof Let $\omega = (\beta, \gamma)$ be a Rechtsstaat which satisfies the intersection property. We will show that $\omega \in \Omega^*$. Since ω satisfies the intersection property, there exists $T \in \mathcal{P}_0(N)$ such that $T = \bigcap_{(x,y)\in(A\times A)} \gamma(x, y)$. We will show that for all $R_N \in W(A)^N$, all $j \in T$ and all $x \in A$, if $x \in top(R_j)$, then $x \in \varepsilon(\omega, R_N)$.

Let $R_N \in W(A)^N$, $j \in T$ and $x \in A$ be such that $x \in top(R_j)$. Suppose that $x \notin \varepsilon(\omega, R_N)$. Then, there exist $y \in (A \setminus \{x\})$ and $H \in \mathcal{P}_0(N)$ such that $H \in [\beta(x, y; R_N) \cap \gamma(x, y)]$. Since $T = \bigcap_{(x,y)\in(A\times A)} \gamma(x, y)$, we have $T \subseteq H$. However, since $j \in T$ and $x \in top(R_j)$, for any $y \in (A \setminus \{x\})$ we have xR_jy. So, $j \notin H$ for all $H \in \beta(x, y; R_N)$. This contradicts with that $H \in \beta(x, y; R_N)$. That is, agent j does not approve the alteration of x to y since $x \in top(R_j)$. So, $x \in \varepsilon(\omega, R_N)$. Hence, $\omega \in \Omega^*$. □

Definition 33 An SCR $F : W(A)^N \to \mathcal{P}_0(A)$ is said to be ***implementable via Rechtsstaat*** ω (defined in Koray and Yıldız (2008)) if for all $R_N \in W(A)^N$, $F(R_N) = \varepsilon(\omega, R_N)$.

We note that since $F(R_N) \neq \emptyset$ for every $R_N \in W(A)^N$, if F is implementable via a Rechtsstaat ω, then $\omega \in \Omega^*$.

For example, a dictatorial SCR F^d is implementable via dictatorial Rechtsstaat ω^d.[3]

3.4 Results

In this section, we will study what properties of a Rechtsstaat implementing an SCR ensure that the SCR is also Nash implementable.

Definition 34 We say that alternative x ***strongly Pareto dominates*** alternative y at $R_N \in W(A)^N$, if xP_iy for all $i \in N$. For any $R_N \in W(A)^N$, let $SPO(R_N)$ denote the set of all strongly Pareto undominated alternatives at R_N. We say that an SCR F is ***weakly Pareto optimal*** if for all $R_N \in W(A)^N$, $F(R_N) \subseteq SPO(R_N)$.

Let $L(x, R_i) = \{y \in A \mid xR_iy\}$ denote the **lower counter set of** R_i **at** $x \in A$. A preference profile $\widetilde{R}_N \in W(A)^N$ is obtained by **a monotonic transformation of** $R_N \in W(A)^N$ **at** $x \in A$, if $L(x, R_i) \subseteq L(x, \widetilde{R}_i)$ for all $i \in N$. We let $MT(R_N, x)$ denote the set of preference profiles which are obtained by a monotonic transformation of R_N at $x \in A$.

Definition 35 We say that an SCR $F : W(A)^N \to \mathcal{P}_0(A)$ is ***Maskin monotonic*** if for all $R_N, \widetilde{R}_N \in W(A)^N$ and all $x \in F(R_N)$, if $\widetilde{R}_N \in MT(R_N, x)$ then $x \in F(\widetilde{R}_N)$.

Maskin (1999) showed that if an SCR is Nash implementable then it is Maskin monotonic. He also showed that when $\#N \geq 3$, if an SCR satisfies Maskin monotonicity and no veto power condition,[4] then it is Nash implementable.

The next proposition shows that an SCR which is implementable via a Rechtsstaat is both weakly Pareto optimal and Maskin monotonic.

[3] An SCR $F^d : W(A)^N \to \mathcal{P}_0(A)$ is *dictatorial* if there exists $d \in N$ such that for all $R_N \in W(A)^N$, $F^d(R_N) = top(R_d)$, and a Rechtsstaat $\omega^d = (\beta, \gamma^d)$ is *dictatorial* if there exists $d \in N$ such that for all pairs $x, y \in A$, $\underline{\gamma}(x, y) = \{\{d\}\}$, where $d \in N$ is called the dictator.

[4] We say that an SCR $F : W(A)^N \to \mathcal{P}_0(A)$ satisfies ***no veto power condition*** if for all $R_N \in W(A)^N$ and all $x \in A$ such that there is $i \in N$, for all $j \in (N \setminus \{i\})$, $L(x, R_j) = A$, then $x \in F(R_N)$.

Proposition 9 *Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR. If F is implementable via Rechtsstaat $\omega = (\beta, \gamma)$, then F is weakly Pareto optimal and Maskin monotonic.*

Proof Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR which is implementable via Rechtsstaat $\omega = (\beta, \gamma)$. So, for all $R_N \in W(A)^N$, $F(R_N) = \varepsilon(\omega, R_N)$.

Let $x, y \in A$, $R_N \in W(A)^N$ be such that xP_iy for all $i \in N$. By axiom A2, we have $\gamma(y, x) \neq \emptyset$. So, $\beta(y, x; R_N) \cap \gamma(y, x) \neq \emptyset$, i.e., $y \notin \varepsilon(\omega, R_N)$. As F is implementable via ω, we have $y \notin F(R_N)$. Hence, F is weakly Pareto optimal.

We will now show that F is Maskin monotonic. Let $R_N, \widetilde{R}_N \in W(A)^N$, $x \in F(R_N)$ and $\widetilde{R}_N \in MT(R_N, x)$. We will show that $x \in F(\widetilde{R}_N)$. Since F is implementable via ω and $x \in F(R_N)$, we have $x \in \varepsilon(\omega, R_N)$. Suppose that $x \notin \varepsilon(\omega, \widetilde{R}_N)$. So, there exist $z \in A$ and $H \in \mathcal{P}_0(N)$ such that $H \in [\beta(x, z; \widetilde{R}_N) \cap \gamma(x, z)]$. Since $\widetilde{R}_N \in MT(R_N, x)$, i.e., for all $i \in N$, $L(x, R_i) \subseteq L(x, \widetilde{R}_i)$, we have that the set of agents who prefer x to z under R_N continues to prefer x to z under $\widetilde{R}_N$. Now, this fact together with $H \in \beta(x, z; \widetilde{R}_N)$ implies that for all $i \in H$ we have zP_ix, i.e., $H \in \beta(x, z; R_N)$. So, $H \in [\beta(x, z; R_N) \cap \gamma(x, z)]$, which is in contradiction with $x \in \varepsilon(\omega, R_N)$. Hence, $x \in \varepsilon(\omega, \widetilde{R}_N)$. As F is implementable via ω, we have $x \in F(\widetilde{R}_N)$. Hence, F is Maskin monotonic. $\square$

Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR, and $B \in \mathcal{P}_0(A)$. We say that an alternative $x \in A$ **is essential for** $i \in N$ **in** B **for** F if there exists $R_N \in W(A)^N$ such that $L(x, R_i) \subseteq B$ and $x \in F(R_N)$. Let $Ess(F, i, B)$ denote the set of essential alternatives for $i \in N$ in B for F.

Definition 36 Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR. We say that F is ***essentially monotonic*** if for all $R_N, \widetilde{R}_N \in W(A)^N$ and for all $x \in F(R_N)$, if $Ess(F, i, L(x, R_i)) \subseteq L(x, \widetilde{R}_i)$ for all $i \in N$, then $x \in F(\widetilde{R}_N)$.

Danilov (1992) showed that when $\#N \geq 3$ and players have strict preference relations over alternatives, an SCR F is Nash implementable if and only if it is essentially monotonic. Yamato (1992) extended this result and showed that when $\#N \geq 3$ and players have weak preferences over alternatives, essential monotonicity is both necessary and sufficient for an SCR to be Nash implementable.

We now define a new condition on a Rechtsstaat, referred to as the equal treatment of equivalent alternatives.

Definition 37 Let $\omega = (\beta, \gamma)$ be a Rechtsstaat. We say that ω satisfies the ***equal treatment of equivalent alternatives*** $(ETEA)$ if for all $x, y \in A$, and all $R_N \in W(A)^N$ with xI_iy for all $i \in N$, then [$x \in \varepsilon(\omega, R_N)$ implies $y \in \varepsilon(\omega, R_N)$].

We note that a dictatorial Rechtsstaat satisfies $ETEA$, but it violates no veto power condition.[5] Another example of Rechtsstaat which satisfies $ETEA$ but violates no veto power condition is the weakly Pareto optimal and individually rational Rechtsstaat $\omega_{IR-WPO} = (\beta, \gamma_{IR-WPO})$ with respect to some $a_0 \in A$, where for any $b \in (A \setminus \{a_0\})$, $\gamma_{IR-WPO}(b, a_0) = \mathcal{P}_0(N)$, $\gamma_{IR-WPO}(a_0, b) = \{\{N\}\}$, and for any $b, c \in (A \setminus \{a_0\})$ with $b \neq c$, $\gamma_{IR-WPO}(b, c) = \{\{N\}\}$. Thus, for any $R_N \in W(A)^N$ we have $\emptyset \neq \varepsilon(\omega_{IR-WPO}, R_N) = \{b \in A \mid bR_ia_0 \text{ for all } i \in N\} \cap SPO(R_N)$, and ω_{IR-WPO} satisfies $ETEA$[6] but it violates no veto power condition.

We now state a lemma which will be useful while determining essential sets of a subset of alternatives for an SCR that is implementable via a Rechtsstaat, and we will provide its proof in the Appendix.

Lemma 10 *Let F be an SCR which is implementable via Rechtsstaat $\omega = (\beta, \gamma)$. For any $i \in N$, any $B \in \mathcal{P}_0(A)$ and any $x \in B$,*
(i) if $B \neq A$ and $\{i\} \notin \gamma(x, y)$ for all $y \in (A \setminus B)$, then $x \in Ess(F, i, B)$,
(ii) if $B \neq A$ and $\{i\} \in \gamma(x, y)$ for some $y \in (A \setminus B)$, then $x \notin Ess(F, i, B)$,
(iii) if $B = A$, then $Ess(F, i, B) = A$.

An SCR which is implementable via a Rechtsstaat violating $ETEA$ may not be Nash implementable as shown by the following example.

Example 5 Let $A = \{x, y, z\}$, $N = \{1, 2, 3\}$ and $\omega = (\beta, \gamma)$ where γ is as follows: $\underline{\gamma}(y, z) = \underline{\gamma}(z, x) = \{\{3\}\}$, and $\underline{\gamma}(x, y) = \underline{\gamma}(x, z) = \underline{\gamma}(y, x) = \underline{\gamma}(z, y) = \{\{N\}\}$.

[5] We say that a Rechtsstaat ω satisfies no veto power condition if for all $R_N \in W(A)^N$ and all $x \in A$ such that there is $i \in N$, for all $j \in (N \setminus \{i\})$, $L(x, R_j) = A$, then $x \in \varepsilon(\omega, R_N)$.

[6] So, when $\#N \geq 3$, an SCR which is implementable via ω_{IR-WPO} is Nash implementable by Theorem 1.

Note that ω satisfies the intersection property since for every pair $a, b \in A$ and any coalition $H \in \gamma(a, b)$ we have $3 \in H$. Hence, $\omega \in \Omega^*$ by Proposition 8. However, ω violates $ETEA$.[7] Consider the profile R_N:

R_1	R_2	R_3
x, y, z	x, y, z	z
		x, y

Now, $x \in \varepsilon(\omega, R_N)$ and xI_iy for all $i \in N$. However $y \notin \varepsilon(\omega, R_N)$, since $\beta(y, z; R_N) \cap \gamma(y, z) = \{3\}$.

We define an SCR F^ω as follows: $F^\omega(R_N) = \varepsilon(\omega, R_N)$ for all $R_N \in W(A)^N$. We will now show that F^ω is not essentially monotonic, hence it is not Nash implementable. Consider following profiles $\acute{R}_N$ and $\widehat{R}_N$:

$\acute{R}_1$	$\acute{R}_2$	$\acute{R}_3$	$\widehat{R}_1$	$\widehat{R}_2$	$\widehat{R}_3$
y	y	x	y	y	x
x, z	x, z	y	x, z	x, z	z
		z			y

We have $y \in F^\omega(\acute{R}_N)$. Now, $Ess(F^\omega, 1, L(y, \acute{R}_1)) = Ess(F^\omega, 2, L(y, \acute{R}_2)) = A$ by Lemma 10-(iii) since $L(y, \acute{R}_1) = L(y, \acute{R}_2) = A$. By Lemma 10-(ii), $z \notin Ess(F^\omega, 3, L(y, \acute{R}_3))$ since $x \notin L(y, \acute{R}_3)$ and $\underline{\gamma}(z, x) = \{\{3\}\}$. So, $Ess(F^\omega, 3, L(y, \acute{R}_3)) = \{y\}$. Note that for all $i \in N$ we have $Ess(F^\omega, i, L(y, \acute{R}_i)) = L(y, \widehat{R}_i)$. However, $y \notin F^\omega(\widehat{R}_N)$ since $\beta(y, z; \widehat{R}_N) \cap \gamma(y, z) = \{3\}$. Hence, F^ω is not essentially monotonic.

Proposition 10 *Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR. If F is implementable via some Rechtsstaat $\omega = (\beta, \gamma)$ satisfying $ETEA$, then for any $i \in N$, any $R_N \in W(A)^N$ and any $x \in A$ with $x \in F(R_N)$, we have $Ess(F, i, L(x, R_i)) = L(x, R_i)$.*

Proof Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR that is implementable via some Rechtsstaat $\omega = (\beta, \gamma)$ satisfying $ETEA$. Let $R_N \in W(A)^N$ and $x \in A$ be such that $x \in F(R_N)$. We will show that $Ess(F, i, L(x, R_i)) = L(x, R_i)$ for all $i \in N$.

[7] Hence, a Rechtsstaat ω satisfying the intersection property may violate $ETEA$.

It is clear that for all $i \in N$, $Ess(F, i, L(x, R_i)) \subseteq L(x, R_i)$. So, suppose that there exist $j \in N$ and $y \in L(x, R_j)$ such that $y \notin Ess(F, j, L(x, R_j))$, i.e., alternative y is not essential for agent j in $L(x, R_j)$ for F. So, there is no $\widehat{R}_N \in W(A)^N$ such that $L(y, \widehat{R}_j) \subseteq L(x, R_j)$ and $y \in F(\widehat{R}_N)$, that is

$$\text{for all} \quad \widehat{R}_N \in W(A)^N, \quad \text{if} \quad L(y, \widehat{R}_j) \subseteq L(x, R_j) \quad \text{then} \quad y \notin F(\widehat{R}_N). \tag{3.1}$$

We now consider the preference profile $\overline{R}_N \in W(A)^N$, where for any $i \in N$, $\overline{R}_i$ is obtained from R_i by putting y indifferent to x without changing the ordering of other alternatives, i.e., for all $i \in N$, $x\overline{I}_i y$, and for any $z, \acute{z} \in (A \setminus \{y\})$, $z\overline{R}_i\acute{z}$ if and only if $zR_i\acute{z}$.

Now, for all $i \in N$ we have $L(x, R_i) \cup \{y\} = L(x, \overline{R}_i)$. So, for all $i \in N$ we have $L(x, R_i) \subset L(x, \overline{R}_i)$, i.e., $\overline{R}_N \in MT(R_N, x)$. Therefore, $x \in F(\overline{R}_N)$ since $x \in F(R_N)$ and F is Maskin monotonic.

Since $y \in L(x, R_j)$, we have $L(x, R_j) = L(x, \overline{R}_j)$. Moreover, since $x\overline{I}_j y$, we have $L(x, \overline{R}_j) = L(y, \overline{R}_j)$. So, $L(y, \overline{R}_j) = L(x, \overline{R}_j) = L(x, R_j)$, i.e., $L(y, \overline{R}_j) = L(x, R_j)$. This fact, together with the statement in (3.1), implies that $y \notin F(\overline{R}_N)$. However, $x\overline{I}_i y$ for all $i \in N$ and $x \in F(\overline{R}_N)$ imply that $y \in F(\overline{R}_N)$, since ω satisfies $ETEA$, which is the desired contradiction. So, there do not exist $j \in N$ and $y \in L(x, R_j)$ such that $y \notin Ess(F, j, L(x, R_j))$. Therefore, for all $i \in N$ we also have $L(x, R_i) \subseteq Ess(F, i, L(x, R_i))$, and hence $Ess(F, i, L(x, R_i)) = L(x, R_i)$. □

We now show that an SCR which is implementable via some Rechtsstaat satisfying $ETEA$ is essentially monotonic.[8]

Theorem 1 *Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR. If F is implementable via some Rechtsstaat $\omega = (\beta, \gamma)$ satisfying $ETEA$, then F is essentially monotonic (hence Nash implementable, when $\#N \geq 3$).*

Proof Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR that is implementable via some Rechtsstaat $\omega = (\beta, \gamma)$ satisfying $ETEA$. We will show that F is essentially monotonic.

[8]For similar results in matching problems, see Kara and Sönmez (1996), Kara and Sönmez (1997) and Sönmez (1996).

Let $R_N, \widetilde{R}_N \in W(A)^N$ and $x \in A$ be such that $x \in F(R_N)$ and $Ess(F, i, L(x, R_i)) \subseteq L(x, \widetilde{R}_i)$ for all $i \in N$. We will show that $x \in F(\widetilde{R}_N)$. By Proposition 10, for all $i \in N$ we have $Ess(F, i, L(x, R_i)) = L(x, R_i)$. Hence, $L(x, R_i) \subseteq L(x, \widetilde{R}_i)$ for all $i \in N$. This fact combined with Maskin monotonicity (by Proposition 9) implies that $x \in F(\widetilde{R}_N)$. So, F is essentially monotonic. Hence, when $\#N \geq 3$, F is Nash implementable by Yamato (1992)'s result. □

Moore and Repullo (1990) showed that when $\#N \geq 3$, an SCR F is Nash implementable if and only if it satisfies Condition μ.

Definition 38 We say that an SCR $F : W(A)^N \to \mathcal{P}_0(A)$ satisfies ***Condition*** μ if there is a set $B \subseteq A$ and for all triplets $(i, R_N, a) \in N \times W(A)^N \times A$ such that $a \in F(R_N)$, there is a set $C_i = C_i(a, R_N)$ such that $a \in C_i \subseteq L(a, R_i) \cap B$, and conditions (i), (ii) and (iii) are satisfied:

(i). Let $\widehat{R}_N \in W(A)^N$, if $C_i \subseteq L(a, \widehat{R}_i)$ for all $i \in N$, then $a \in F(\widehat{R}_N)$.

(ii). Let $\widehat{R}_N \in W(A)^N$, if there exist $i \in N$ and $c \in A$ such that $c \in C_i \subseteq L(c, \widehat{R}_i)$, and $B \subseteq L(c, \widehat{R}_j)$ for all $j \in (N \setminus \{i\})$, then $c \in F(\widehat{R}_N)$.

(iii). Let $\widehat{R}_N \in W(A)^N$, if there exists $c \in A$ such that $c \in B \subseteq L(c, \widehat{R}_i)$ for all $i \in N$, then $c \in F(\widehat{R}_N)$.

Next result follows from Theorem 1 and its proof is given in the Appendix.

Corollary 1 *Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR. If F is implementable via some Rechtsstaat $\omega = (\beta, \gamma)$ satisfying $ETEA$, then F satisfies Condition μ (hence, when $\#N \geq 3$, F is Nash implementable).*

The question of how much $ETEA$ does impose a restriction on a Rechtsstaat is an interesting one.

Proposition 11 *A Rechtsstaat $\omega = (\beta, \gamma)$ satisfies $ETEA$ if and only if γ has following property:*

$$\textit{For any} \quad x \in A, \quad \textit{any} \quad y, z \in (A \setminus \{x\}), \quad \gamma(y, x) = \gamma(z, x). \tag{3.2}$$

Proof "⇒" Let $\omega = (\beta, \gamma)$ be a Rechtsstaat which satisfies $ETEA$. Suppose that γ violates property (3.2). Then there exist $x \in A$ and $y, z \in (A \setminus \{x\})$ such that $\gamma(y, x) \neq \gamma(z, x)$. So, we have that $\underline{\gamma}(y, x) \neq \underline{\gamma}(z, x)$. Then, at least one of the following cases occur:

Case 1. There exists $\widehat{H} \in \underline{\gamma}(y, x)$ such that for all $H \in \mathcal{P}_0(\widehat{H})$, $H \notin \underline{\gamma}(z, x)$.

Case 2. There exists $\widetilde{H} \in \underline{\gamma}(z, x)$ such that for all $H \in \mathcal{P}_0(\widetilde{H})$, $H \notin \underline{\gamma}(y, x)$.

If case 1 occurs, then we consider following profile $\widehat{R}_N$:

$\widehat{R}_{\widehat{H}}$	$\widehat{R}_{N \setminus \widehat{H}}$
x	x, y, z
$A \setminus \{x\}$	$A \setminus \{x, y, z\}$

Now, $z \in \varepsilon(\omega, \widehat{R}_N)$ and $y\widehat{I}_i z$ for all $i \in N$. However $y \notin \varepsilon(\omega, \widehat{R}_N)$ since $\widehat{H} \in [\beta(y, x; \widehat{R}_N) \cap \gamma(y, x)]$, which is in contradiction with that ω satisfies $ETEA$.

If case 2 occurs, then we consider following profile $\widetilde{R}_N$:

$\widetilde{R}_{\widetilde{H}}$	$\widetilde{R}_{N \setminus \widetilde{H}}$
x	x, y, z
$A \setminus \{x\}$	$A \setminus \{x, y, z\}$

Now, $y \in \varepsilon(\omega, \widetilde{R}_N)$ and $y\widetilde{I}_i z$ for all $i \in N$. However $z \notin \varepsilon(\omega, \widetilde{R}_N)$ since $\widetilde{H} \in [\beta(z, x; \widetilde{R}_N) \cap \gamma(z, x)]$, which contradicts with that ω satisfies $ETEA$.

"⇐" Let $\omega = (\beta, \gamma)$ be a Rechtsstaat such that γ satisfies property (3.2). Suppose that ω violates $ETEA$. Then, there exist $x, y \in A$ and $R_N \in W(A)^N$ with $xI_i y$ for all $i \in N$ such that $x \in \varepsilon(\omega, R_N)$ but $y \notin \varepsilon(\omega, R_N)$. Since $y \notin \varepsilon(\omega, R_N)$, there exist $z \in (A \setminus \{x, y\})$ and $H \in \mathcal{P}_0(N)$ such that $H \in [\beta(y, z; R_N) \cap \gamma(y, z)]$. The fact that $H \in \beta(y, z; R_N)$, together with $xI_i y$ for all $i \in N$, implies that $H \in \beta(x, z; R_N)$.

Since $H \in \gamma(y,z)$ and γ satisfies property (3.2), we have that $H \in \gamma(x,z)$. Hence, $H \in [\beta(x,z;R_N) \cap \gamma(x,z)]$ which is in contradiction with that $x \in \varepsilon(\omega, R_N)$. $\square$

We note in here that Rechtsstaats $\widetilde{\omega}$ and $\widehat{\omega}$ given in Example 4 satisfy property (3.2), so $\widetilde{\omega}$ and $\widehat{\omega}$ satisfy $ETEA$. However, as shown in Example 4 that $\widetilde{\omega}, \widehat{\omega} \notin \Omega^*$. Hence, for a Rechtsstaat ω satisfying $ETEA$ we may have $\omega \notin \Omega^*$.

3.5 Oligarchic Rechtsstaats

In this section, we will define oligarchic Rechtsstaats and show that if an SCR is implementable via an oligarchic Rechtsstaat then it is Nash implementable when there are at least three agents in the society.

Definition 39 We say that a Rechtsstaat $\omega^K = (\beta, \gamma^K)$ is ***oligarchic*** if there exists $K \in \mathcal{P}_0(N)$ such that for all pairs $x, y \in A$, $\underline{\gamma}^K(x,y) = \{\{K\}\}$, i.e., $\gamma^K(x,y)$ consists of the coalition K and all of its supersets.

Such a coalition $K \in \mathcal{P}_0(N)$ is called an oligarchy.[9] For example, for a dictatorial Rechtsstaat ω^d we have $K = \{d\}$. We note that $K = \{N\}$ is also possible.

It is clear that an oligarchic Rechtsstaat satisfies the intersection property. So, for any oligarchic Rechtsstaat ω^K we have $\omega^K \in \Omega^*$ by Proposition 8. It is also clear that for an oligarchic Rechtsstaat $\omega^K = (\beta, \gamma^K)$, γ^K satisfies property (3.2). So, an oligarchic Rechtsstaat satisfies $ETEA$.[10] Hence, when $\#N \geq 3$, if an SCR F is implementable via an oligarchic Rechtsstaat ω^K then F is Nash implementable by Theorem 1.

Definition 40 We say that an SCR $F : W(A)^N \to \mathcal{P}_0(A)$ satisfies ***neutrality*** if for any permutation $\tau : A \to A$ and any $R_N \in W(A)^N$, we have $F\big((R_N)^\tau\big) = \tau\big(F(R_N)\big)$, where for each $i \in N$ and $x, y \in A$, $\tau(x) R_i^\tau \tau(y) \Leftrightarrow x R_i y$.

[9]Note that Rechtsstaats $\widetilde{\omega}$ and $\widehat{\omega}$ given in Example 4 are not oligarchic.

[10]Note that an oligarchic Rechtsstaat $\omega^K = (\beta, \gamma^K)$ satisfies following stronger version of $ETEA$: for all $x, y \in A$, all $R_N \in W(A)^N$ with $x I_k y$ for all $k \in K$, if $x \in \varepsilon(\omega^K, R_N)$ then $y \in \varepsilon(\omega^K, R_N)$.

Moulin (1983) showed that when $\#N \geq 3$, if an SCR F is Maskin monotonic and satisfies neutrality then it is Nash implementable. We now show that an SCR which is implementable via an oligarchic Rechtsstaat satisfies neutrality.

Proposition 12 *Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR. If F is implementable via an oligarchic Rechtsstaat $\omega^K = (\beta, \gamma^K)$, then F satisfies neutrality.*

Proof Let F be implementable via an oligarchic Rechtsstaat $\omega^K = (\beta, \gamma^K)$, $R_N \in W(A)^N$, and $x \in F(R_N)$. Let $\tau : A \to A$ be a permutation such that $\tau(x) = y$. We will show that $y \in F\big((R_N)^\tau\big)$. Suppose that $y \notin F\big((R_N)^\tau\big)$. Then, there exist $z \in (A \setminus \{y\})$ and $H \in \mathcal{P}_0(N)$ such that $H \in [\beta\big(y, z; (R_N)^\tau\big) \cap \gamma(y, z)]$. Since ω^K is oligarchic with the oligarchy K, we have $K \subseteq H$. So, for all $i \in K$ we have $zP_i^\tau y$. Note that $\{i \in N \mid zP_i^\tau y\} = \{i \in N \mid \tau^{-1}(z)P_i x\}$. So, for all $i \in K$ we have $\tau^{-1}(z)P_i x$, i.e., $K \in \beta\big(x, \tau^{-1}(z); R_N\big)$. So, $K \in [\beta\big(x, \tau^{-1}(z); R_N\big) \cap \gamma\big(x, \tau^{-1}(z)\big)]$, which contradicts with that $x \in F(R_N)$. Hence, $\tau(x) = y \in F\big((R_N)^\tau\big)$, i.e., F satisfies neutrality. □

Propositions 9 and 12, together with Moulin (1983)'s result, imply that when $\#N \geq 3$, if an SCR F is implementable via an oligarchic Rechtsstaat ω^K then F is Nash implementable.

3.6 Conclusion

We studied Nash implementation of an SCR which is implementable via a Rechtsstaat. We introduced a condition on a Rechtsstaat which is referred to as the equal treatment of equivalent alternatives ($ETEA$), and showed that if an SCR is implementable via some Rechtsstaat satisfying $ETEA$ then it is essentially monotonic, hence Nash implementable via a mechanism when there are at least three agents in the society. We defined oligarchic Rechtsstaats and showed that if an SCR is implementable via an oligarchic Rechtsstaat then it is Nash implementable, and we also showed that such an SCR also satisfies neutrality.

CHAPTER 4

A CHARACTERIZATION OF THE BORDA RULE ON THE DOMAIN OF WEAK PREFERENCES

4.1 Introduction

We consider an environment with a non-empty finite set of alternatives and variable number of finite sets of agents (voters). Each voter is endowed with weak preferences over the set of alternatives, i.e., indifferences are allowed. The set of all such preference profiles of a finite society of voters is referred to as the weak domain of preferences. A social choice rule (SCR) chooses a nonempty subset of the set of alternatives at each preference profile for a finite set of voters.

Many different social choice rules have been established to determine which alternative(s) should be selected when a preference profile of a society is considered. One rule that has received a great deal of attention in the literature is attributed to Borda, Borda (1781), which is also our line of interest in this chapter.

Young (1974) provided an axiomatic characterization of the Borda rule. He showed that when players' have strict preferences over alternatives, the Borda rule is characterized by neutrality, reinforcement, faithfulness and Young's cancellation property. Hansson and Sahlquist (1976) provided another proof for Young's characterization of the Borda rule, again assuming that voters have strict preferences.

The main purpose of this chapter is to give a characterization of the Borda rule on the domain of weak preferences. We introduce a new property, referred to as the *degree equality*, that an social choice rule (SCR) satisfies degree equality if, for any two profiles of a finite set of voters, equality between the sums of the degrees of every alternative under the two profiles implies that the same alternatives get chosen by the SCR at these two profiles.

We show that the Borda rule is characterized by the conjunction of weak neutrality, reinforcement, faithfulness and degree equality on the domain of weak preferences (Theorem 2). Moreover, the Borda rule is the unique scoring rule which satisfies the degree equality (Proposition 13). We also introduce a new cancellation property and show that it characterizes the Borda rule among all scoring rules (Proposition 17).

This chapter is organized as follows: Section 4.2 introduces basic notions. Our characterization result and its proof are provided in section 4.3. We introduce a new cancellation property in section 4.4. Section 4.5 concludes.

4.2 Basic notions

Let A be a non-empty finite set of alternatives with $\#A = m \geq 3$. The universal set of voters is denoted by positive integers $\mathbb{N}$, and we let $\mathcal{N}$ denote all finite subsets of $\mathbb{N}$ and $N = \{1, 2, \ldots, n\} \in \mathcal{N}$ a finite set of voters.

Each voter $i \in \mathbb{N}$ has a complete, reflexive and transitive preference relation R_i over A. Let $W(A)$ denote the set of all preference relations over A. An n-tuple, $R^N = (R_1, \ldots, R_n) \in W(A)^N$ denote a preference profile for a finite set of voters N, where $\#N = n$.

For any $i \in N$, let P_i denote the strict preference relation associated with R_i and I_i denote the indifference relation associated with R_i. We let $L(A)$ denote the set of all strict preference relations over A. An n-tuple, $P^N = (P_1, \ldots, P_n) \in L(A)^N$ denote a strict preference profile for a finite set of voters N.

Given any $x \in A$ and any $R \in W(A)$, we let

- $U(x, R) = \{y \in A \mid yRx\}$ denote the upper contour set of x at R, and
- $SU(x, R) = \{y \in A \mid yPx\}$ denote the strict upper contour set of x at R.

For any $R \in W(A)$, let $top(R) = \{x \in A \mid xRy$ for all $y \in A\}$ denote the best alternatives at R.

Definition 41 A ***social choice rule*** (SCR) is a map $F : \bigcup_{N \in \mathcal{N}} W(A)^N \to 2^A \setminus \{\emptyset\}$, i.e., for every preference profile $R^N \in W(A)^N$ of a finite set of voters N, an SCR F assigns a nonempty subset $F(R^N)$ of A.

Thus, our social choice rule operates on a fixed set of alternatives and every finite set of voters.

For any alternative x and any $R \in W(A)$, we let $d(x, R) \in \mathbb{R}_{++}$ denote the ***degree of*** x ***at*** R and is defined as follows:[1]

$$d(x, R) = \frac{\#SU(x, R) + \#U(x, R) + 1}{2}.$$

Let $s = (s_1, s_2, \ldots, s_m) \in \mathbb{R}^m$ denote a score vector, where $s_1 \geq s_2 \geq \ldots \geq s_m$ and $s_1 > s_m$.

Given a score vector $s = (s_1, s_2, \ldots, s_m)$, $x \in A$ and $R \in W(A)$, we determine the score of x at R, $s(x, R) \in \mathbb{R}$, as follows:

$$s(x, R) = \begin{cases} s_{d(x,R)} & \text{if } d(x, R) \in \mathbb{Z}_{++} \\ (s_{\lfloor d(x,R) \rfloor} + s_{\lfloor d(x,R) \rfloor + 1})/2 & \text{otherwise} \end{cases}$$

where for any $\delta \in \mathbb{R}$, $\lfloor \delta \rfloor$ denote the maximal integer which is smaller than or equal to δ.

Given any $N \in \mathcal{N}$ and any profile $R^N \in W(A)^N$, the total score of $x \in A$ at R^N, $S(x, R^N)$, is defined by $S(x, R^N) = \sum_{i \in N} s(x, R_i)$.

[1] I am grateful to my friend Serhat Doğan for suggesting this definition.

A *scoring rule* selects the alternatives with the maximal total score. *Plurality rule* is a scoring rule defined by the scoring vector $(1, 0, \ldots, 0)$. *Inverse plurality rule* is a scoring rule defined by the scoring vector $(1, 1, \ldots, 1, 0)$. *Borda rule* is a scoring rule defined by the scoring vector $s = (s_1, s_2, \ldots, s_m)$ such that $s_1 - s_2 = s_2 - s_3 = \ldots = s_{m-1} - s_m$, i.e., $s_k - s_{k+1} = s_{k+1} - s_{k+2}$ for all $1 \leq k \leq m - 2$.

We will now define some axioms.

Definition 42 We say that an SCR F satisfies ***neutrality*** (N) if for any finite set of voters N, any $R^N \in W(A)^N$ and any permutation $\tau : A \to A$, we have $F\big((R_N)^\tau\big) = \tau\big(F(R_N)\big)$, where for each $i \in N$ and $x, y \in A$, $\tau(x) R_i^\tau \tau(y) \Leftrightarrow x R_i y$.

Definition 43 We say that an SCR F satisfies ***weak neutrality*** (WN) if for any finite set of voters N, any $R^N \in W(A)^N$, any $x \in F(R^N)$ and any permutation $\tau : A \to A$ such that $\tau(x) = x$, we have $x \in F\big((R_N)^\tau\big)$.

Definition 44 We say that an SCR F satisfies ***anonymity*** (A) if for any finite set of voters N, any $R^N \in W(A)^N$ and any permutation $\sigma : N \to N$, we have $F\big((R_{\sigma(i)})_{i \in N}\big) = F\big((R_i)_{i \in N}\big)$.

Definition 45 We say that an anonymous SCR F satisfies ***continuity*** (CO) if for any finite sets of voters N and H, any $R^N \in W(A)^N$ with $\#F(R^N) = 1$ and any $R^H \in W(A)^H$, there exists an integer k (k is sufficiently large) such that $F(R^{kN} + R^H) = F(R^N)$, where R^{kN} denote the k copies of R^N.

Definition 46 We say that an SCR F satisfies ***reinforcement*** (RE) if for any finite sets of voters N and H with $N \cap H = \emptyset$, any $R^N \in W(A)^N$ and any $R^H \in W(A)^H$, $F(R^N) \cap F(R^H) \neq \emptyset$ implies that $F(R^N + R^H) = F(R^N) \cap F(R^H)$, where $(R^N + R^H) \in W(A)^{N \cup H}$.

Definition 47 We say that an SCR F satisfies ***Young's cancellation*** (Y-Ca) property if for any finite set of voters N, any $R^N \in W(A)^N$ such that for all pairs x and y, $\#\{i \in N \mid x P_i y\} = \#\{i \in N \mid y P_i x\}$, then we have $F(R^N) = A$.

Definition 48 We say that an SCR F satisfies ***faithfulness*** (F) if whenever $N = \{i\}$, then for any $R_i \in W(A)$, $F(R_i)$ chooses agent i's most preferred alternative(s), i.e., $F(R_i) = top(R_i)$.

We now give Young's characterizations of scoring rules and the Borda rule.

- **Theorem** (Young (1975)). An SCR $F : \bigcup_{N \in \mathcal{N}} L(A)^N \to 2^A \setminus \{\emptyset\}$ is a scoring rule if and only if it satisfies anonymity (A), neutrality (N), reinforcement (RE) and continuity (CO), where agents have strict preference relations.

- **Theorem** (Young (1974), Hansson and Sahlquist (1976)). An SCR $F : \bigcup_{N \in \mathcal{N}} L(A)^N \to 2^A \setminus \{\emptyset\}$ is the Borda rule if and only if it satisfies neutrality (N), reinforcement (RE), faithfulness (F) and Young's cancellation (Y-Ca) property, where agents have strict preference relations.

For any finite set of voter N, any $R^N \in W(A)^N$ and any $x \in A$, we let $D(x, R^N) = \sum_{R_i \in R^N} d(x, R_i)$ denote the total degree of x at R^N.

We now define our degree equality axiom.

Definition 49 We say that an SCR F satisfies ***degree equality*** (DE) if, for any finite sets of voters N and $\acute{N}$ with $\#N = \#\acute{N}$ and any $R^N, \acute{R}^{\acute{N}} \in W(A)^N$, $D(x, R^N) = D(x, \acute{R}^{\acute{N}})$ for all $x \in A$ implies $F(R^N) = F(\acute{R}^{\acute{N}})$.

An social choice rule (SCR) satisfies degree equality if, for any profiles of two finite sets of voters with equal cardinality, equality between the total degree of every alternative under these two profiles implies that the same alternatives get chosen by the SCR at these two profiles.

Lemma 11 *If an SCR satisfies degree equality (DE) then it satisfies anonymity (A).*

Proof Let F be an SCR satisfying degree equality (DE). We will show that F satisfies anonymity (A). Let $R^N \in W(A)^N$ be a profile for a finite set of voters N and $\sigma : N \to N$ be a permutation of N. Let $R^{\sigma(N)} = (R_{\sigma(i)})_{i \in N} \in W(A)^N$ denote the profile that is obtained from R^N by the permutation σ. Now, for any $x \in A$ and any $i \in N$ we have $d(x, R_i) = d(x, R_{\sigma(i)})$ because $R_i = R_{\sigma(i)}$ for any $i \in N$. So, we have $D(x, R^N) = D\big(x, R^{\sigma(N)}\big)$ for all $x \in A$. Since F satisfies degree equality (DE), $F\big(R^{\sigma(N)}\big) = F(R^N)$, i.e., F satisfies anonymity (A). $\square$

We note that an SCR satisfying degree equality (DE) may violate Young's cancellation property (Y-Ca). For instance, let $\widehat{F}$ denote the constant social choice rule which is defined as follows: For any finite set of voters N and any $R^N \in W(A)^N$, $\widehat{F}(R^N) = \{a\}$, where $a \in A$. It is clear that $\widehat{F}$ satisfies degree equality (DE), but it violates Young's cancellation property (Y-Ca).[2] Similarly, an SCR satisfying Young's cancellation property (Y-Ca) may violate degree equality (DE), such an SCR is provided in the Appendix.

We now show that degree equality (DE) characterizes the Borda rule among all scoring rules.

Proposition 13 *A scoring rule satisfies degree equality (DE) if and only if it is the Borda rule.*

Proof It is clear that the Borda rule satisfies degree equality (DE). For the other part of the proof, let F be a scoring rule which satisfies degree equality (DE). Suppose that F is not the Borda rule.

Let $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$. We consider following profiles $\widetilde{R}^N$ and $\acute{R}^N$:

$\widetilde{R}_1$	$\widetilde{R}_2$	$\widetilde{R}_3$	$\acute{R}_1$	$\acute{R}_2$	$\acute{R}_3$
a	b, c	a, b	a, b, c	a, b, c	a, b
b, c	a	c			c

Note that for all $x \in A$ we have $D(x, \widetilde{R}^N) = D(x, \acute{R}^N)$. So, $F(\widetilde{R}^N) = F(\acute{R}^N)$ since F satisfies degree equality (DE).

Let $s = (s_1, s_2, s_3)$, where $s_1 \geq s_2 \geq s_3$ and $s_1 > s_3$. Since it is supposed that F is not the Borda rule, we have, $s_1 - s_2 \neq s_2 - s_3$. Let $t_{12} = s_1 - s_2$ and $t_{23} = s_2 - s_3$. So, $s_1 = s_3 + t_{12} + t_{23}$ and $s_2 = s_3 + t_{23}$.

[2] The constant social choice rule also satisfies reinforcement (RE) and weak neutrality (WN), and violates faithfulness (F) and neutrality (N).

We calculate the total score of every alternative at $\widetilde{R}^N$ and $\acute{R}^N$:
$S(a, \widetilde{R}^N) = s_1 + s_3 + \frac{s_1+s_2}{2} = 3s_3 + \frac{3}{2}t_{12} + 2t_{23}$,
$S(b, \widetilde{R}^N) = \frac{s_2+s_3}{2} + 2(\frac{s_1+s_2}{2}) = 3s_3 + t_{12} + \frac{5}{2}t_{23}$,
$S(c, \widetilde{R}^N) = \frac{s_2+s_3}{2} + \frac{s_1+s_2}{2} + s_3 = 3s_3 + \frac{1}{2}t_{12} + \frac{3}{2}t_{23}$,
$S(a, \acute{R}^N) = 2(\frac{s_1+s_2+s_3}{3}) + \frac{s_1+s_2}{2} = 3s_3 + t_{12} + \frac{7}{3}t_{23}$,
$S(b, \acute{R}^N) = 2(\frac{s_1+s_2+s_3}{3}) + \frac{s_1+s_2}{2} = 3s_3 + t_{12} + \frac{7}{3}t_{23}$,
$S(c, \acute{R}^N) = 2(\frac{s_1+s_2+s_3}{3}) + s_3 = 3s_3 + \frac{2}{3}t_{12} + \frac{4}{3}t_{23}$.

Note that $S(a, \acute{R}^N) = S(b, \acute{R}^N)$. This fact, together with $s_1 \geq s_2 \geq s_3$ and $s_1 > s_3$, implies that $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$. Then, F being a scoring rule yields that $F(\acute{R}^N) = \{a, b\}$. Hence, $F(\widetilde{R}^N) = \{a, b\}$. Since F is a scoring rule and $F(\widetilde{R}^N) = \{a, b\}$, we have $S(a, \widetilde{R}^N) = S(b, \widetilde{R}^N)$ which in turn yields that $t_{12} = t_{23}$, a contradiction. Hence, F is the Borda rule. □

4.3 Main theorem and its proof

We now state our main result which is a characterization of the Borda rule on the domain of weak preferences, and provide its proof.

Theorem 2 *An SCR* $F : \bigcup_{N \in \mathcal{N}} W(A)^N \to 2^A \setminus \{\emptyset\}$ *satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE) if and only if it is the Borda rule.*

It is clear that the Borda rule satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). For the other part, let F be an SCR satisfying the given axioms. We will show that F is the Borda rule. First, we will show that such an SCR completely depends on the Borda scores. Second, we will show that the SCR depends on the Borda scores in the right way, i.e., it chooses the alternatives with the highest Borda score. Hence, our proof is similar to the one given in Hansson and Sahlquist (1976).

We take the Borda score $s^B = (m-1, m-3, \ldots, -(m-3), -(m-1))$ as in Young (1974) and Hansson and Sahlquist (1976),[3] i.e., $s^B_k - s^B_{k+1} = 2$ for all $1 \le k \le m-1$ and $\sum_{k=1}^m s^B_k = 0$.[4]

Notice that when the Borda score vector is s^B, then for any $R \in W(A)$ we have $\sum_{x \in A} s(x, R) = 0$. So, for any finite set of voters N and any $R^N \in W(A)^N$ we have $\sum_{x \in A} S(x, R^N) = 0$. It is straightforward to check that for any finite set of voters N with $\#N = n$, any $R^N \in W(A)^N$ and any $x \in A$, we have,[5]

$$S(x, R^N) = 0 \text{ if and only if } D(x, R^N) = [(m+1)n]/2, \tag{4.1}$$

$$S(x, R^N) > 0 \text{ if and only if } D(x, R^N) < [(m+1)n]/2, \tag{4.2}$$

$$S(x, R^N) < 0 \text{ if and only if } D(x, R^N) > [(m+1)n]/2. \tag{4.3}$$

We will prove that our SCR completely depends on the Borda scores without using weak neutrality (WN).

For any $x \in A$, let Ψ_x denote the set of all permutations $\tau : A \to A$ such that $\tau(x) = x$, i.e., x is kept fixed.

For any finite set of voters N and any $R^N \in W(A)^N$, we let $\widehat{R}^N \in W(A)^N$ denote the preference profile obtained from R^N by reversing each voter's preferences.

[3]So, if m is even then $s^B = (m-1, \ldots, 3, 1, -1, -3, \ldots, -(m-1))$, and if m is odd then $s^B = (m-1, \ldots, 2, 0, -2, \ldots, -(m-1))$.

[4]Note that any positive affine transformation of s^B is also a Borda score vector.

[5]For any $R \in W(A)$ and any $x \in A$,

- if m is even, then we have
  - $s(x, R) > 0$ if and only if $d(x, R) \le \frac{m}{2}$,
  - $s(x, R) = 0$ if and only if $d(x, R) = [\frac{m}{2} + (\frac{m}{2} + 1)]/2 = \frac{m+1}{2}$,
  - $s(x, R) < 0$ if and only if $d(x, R) \ge \frac{m}{2} + 1$,
- if m is odd, then we have
  - $s(x, R) > 0$ if and only if $d(x, R) < \frac{m+1}{2}$,
  - $s(x, R) = 0$ if and only if $d(x, R) = \frac{m+1}{2}$,
  - $s(x, R) < 0$ if and only if $d(x, R) > \frac{m+1}{2}$.

Lemma 12 *Let F be an SCR which satisfies faithfulness (F), reinforcement (RE) and degree equality (DE). For any finite set of voters N, any $R^N \in W(A)^N$ and any $x \in A$,*
(i) $F(R^N + \widehat{R}^N) = A$, and
(ii) $F\big(\sum_{\tau\in\Psi_x}(R^N + \widehat{R}^N)^\tau\big) = A$.

Proof It is given in the Appendix. □

Lemma 13 *Let F be an SCR which satisfies faithfulness (F), reinforcement (RE) and degree equality (DE). For any finite set of voters N and any $R^N \in W(A)^N$, if $D(x, R^N) = \frac{(m+1)n}{2}$ for all $x \in A$, then $F(R^N) = A$.*

Proof Let F be an SCR which satisfies faithfulness (F), reinforcement (RE) and degree equality (DE). Let $R^N \in W(A)^N$ be such that for any $x \in A$, $D(x, R^N) = \frac{(m+1)n}{2}$. We will show that $F(R^N) = A$. Consider $2(m-1)!$ copies of R^N,[6] denoted by $R^{2N(m-1)!}$. Note that by reinforcement (RE) we have $F(R^N) = F(R^{2N(m-1)!})$.

For any $R \in W(A)$ and any $x \in A$, we have $d(x,R) + d(x,\widehat{R}) = m+1$. So, for any $x \in A$ and any $R^N \in W(A)^N$ we have $D(x,R^N) + D(x,\widehat{R}^N) = (m+1)n$. For any $x \in A$, we now consider the profile $\sum_{\tau\in\Psi_x}(R^N + \widehat{R}^N)^\tau = \sum_{\tau\in\Psi_x}(R^N)^\tau + \sum_{\tau\in\Psi_x}(\widehat{R}^N)^\tau = \mathcal{R}_x + \widehat{\mathcal{R}}_x$. For every $a \in A$, we have $D(a, \mathcal{R}_x + \widehat{\mathcal{R}}_x)$ $= D(a, R^{2N(m-1)!})$ $=(m-1)!(m+1)n$. Then degree equality (DE) implies that $F(\mathcal{R}_x + \widehat{\mathcal{R}}_x) = F(R^{2N(m-1)!})$. By Lemma 12-(ii), we have $F(\mathcal{R}_x + \widehat{\mathcal{R}}_x) = A$. So, $F(R^{2N(m-1)!}) = A = F(R^N)$. □

We now show that our SCR completely depends on the Borda scores.

[6] We note that each copy is taken on a different voter set.

Lemma 14 *Let F be an SCR satisfying reinforcement (RE), faithfulness (F) and degree equality (DE). Now, for any finite sets of voters N and H, any $R^N \in W(A)^N$ and any $R^H \in W(A)^H$, if for all $x \in A$, $S(x, R^N) = S(x, R^H)$, then we have $F(R^N) = F(R^H)$.*

Proof Let N and H be finite voter sets with $\#N = n$ and $\#H = h$, and $R^N \in W(A)^N$ and $R^H \in W(A)^H$ be such that for all $x \in A$, $S(x, R^N) = S(x, R^H)$. We will show that $F(R^N) = F(R^H)$.

We know that for any $x \in A$, $S(x, R^N + \widehat{R}^N) = S(x, R^N) + S(x, \widehat{R}^N) = 0$. Since for all $x \in A$, $S(x, R^N) = S(x, R^H)$, we have $S(x, R^H) + S(x, \widehat{R}^N) = 0$ for all $x \in A$. So, $S(x, R^H + \widehat{R}^N) = 0$ for all $x \in A$. Then, by (4.1), we have $D(x, R^H + \widehat{R}^N) = [(m+1)(n+h)]/2$ for all $x \in A$, where there are $n + h$ voters at the profile $R^H + \widehat{R}^N$. This fact together with Lemma 13 implies that $F(\widehat{R}^N + R^H) = A$.

Now we have that

$$\begin{aligned}
F(R^N) &= F(R^N) \cap A, \\
&= F(R^N) \cap \underbrace{F(\widehat{R}^N + R^H)}_{A}, \\
&= F(R^N + \widehat{R}^N + R^H), \\
&= \underbrace{F(R^N + \widehat{R}^N)}_{A} \cap F(R^H), \\
&= A \cap F(R^H), \\
&= F(R^H),
\end{aligned}$$

where $F(R^N + \widehat{R}^N) = A$ by Lemma 12-(i). Hence, $F(R^N) = F(R^H)$. □

Hence, an SCR satisfying reinforcement (RE), faithfulness (F) and degree equality (DE) completely depends on the Borda scores.

It is left to prove that an SCR satisfying weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE) depends on the Borda scores in the right way. We will first prove some lemmata without using the weak neutrality axiom.

Lemma 15 *Let F be an SCR satisfying faithfulness (F), reinforcement (RE) and degree equality (DE). For any finite set of voters N, any $R^N \in W(A)^N$ and any $x \in A$, followings are true:*
(i) If $D(x, R^N) = \frac{(m+1)n}{2}$, then $F\big(\sum_{\tau\in\Psi_x}(R^N)^\tau\big) = A$.
(ii) If $D(x, R^N) < \frac{(m+1)n}{2}$, then $F\big(\sum_{\tau\in\Psi_x}(R^N)^\tau\big) = \{x\}$.
(iii) If $D(x, R^N) > \frac{(m+1)n}{2}$, then $F\big(\sum_{\tau\in\Psi_x}(R^N)^\tau\big) \subseteq A \setminus \{x\}$.

Proof It is given in the Appendix. □

For any $x, y \in A$ with $x \neq y$, let Ψ_{xy} denote the set of all permutations $\tau : A \to A$ such that $\tau(x) = x$ and $\tau(y) = y$, i.e., x and y are kept fixed.

Lemma 16 *Let F be an SCR which satisfies reinforcement (RE), faithfulness (F) and degree equality (DE). Now, for any finite set of voters N, any $R^N \in W(A)^N$ and any $x, y \in A$, if $D(x, R^N) = D(y, R^N) < \frac{(m+1)n}{2}$, then $F\big(\sum_{\tau\in\Psi_{xy}}(R^N)^\tau\big) = \{x, y\}$.*

Proof It is given in the Appendix. □

Lemma 17 *Let F be an SCR which satisfies reinforcement (RE), faithfulness (F) and degree equality (DE). Now, for any finite set of voters N, any $R^N \in W(A)^N$ and any $x, y \in A$, if $D(x, R^N) < D(y, R^N) < \frac{(m+1)n}{2}$, then $y \notin F\big(\sum_{\tau\in\Psi_{xy}}(R^N)^\tau\big)$ and $x \in F\big(\sum_{\tau\in\Psi_{xy}}(R^N)^\tau\big)$.*

Proof It is given in the Appendix. □

We will now have some results by using weak neutrality (WN).

Proposition 14 *Let F be an SCR which satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). Now, for any finite set of voters N, any $R^N \in W(A)^N$ and any $x \in A$, if $D(x, R^N) > \frac{(m+1)n}{2}$ then $x \notin F(R^N)$.*

Proof Let F be an SCR which satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). Let N be a finite set of voters. Let $R^N \in W(A)^N$ and $x \in A$ be such that $D(x, R^N) > \frac{(m+1)n}{2}$. Suppose that $x \in F(R^N)$.

Since $D(x,R^N) > \frac{(m+1)n}{2}$, we have $x \notin F\big(\sum_{\tau\in\Psi_x}(R^N)^\tau\big)$ by Lemma 15-(iii). However, $x \in F\big((R^N)^\tau\big)$ for each $\tau \in \Psi_x$ by weak neutrality (WN). Then, $x \in F\big(\sum_{\tau\in\Psi_x}(R^N)^\tau\big)$ by reinforcement (RE), a contradiction. Hence, $x \notin F(R^N)$. □

Lemma 18 *Let F be an SCR which satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). Now, for any finite set of voters N, any $R^N \in W(A)^N$ and any $x \in A$, if $x \in F(R^N)$ and $F(R^N) \neq A$ then $D(x,R^N) < \frac{(m+1)n}{2}$ and hence $F\big(\sum_{\tau\in\Psi_x}(R^N)^\tau\big) = \{x\}$.*

Proof Let F be an SCR which satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). Let N be a finite set of voters. Let $R^N \in W(A)^N$ and $x \in A$ be such that $x \in F(R^N)$ and $F(R^N) \neq A$. Suppose that $D(x,R^N) \geq \frac{(m+1)n}{2}$. Then Proposition 14 implies that $D(x,R^N) = \frac{(m+1)n}{2}$.

We now consider the profile $\sum_{\tau\in\Psi_x}(R^N)^\tau = \mathcal{R}_x$. Since $D(x,R^N) = \frac{(m+1)n}{2}$, we have $F(\mathcal{R}_x) = A$ by Lemma 15-(i). The profile $\mathcal{R}_x$ can be written as a sum of the profiles $\mathcal{R}_x \setminus R^N$ and R^N,[7] i.e., $\mathcal{R}_x = \big(\mathcal{R}_x \setminus R^N\big) + R^N$. Since $F(\mathcal{R}_x) = A$ and $F(R^N) \neq A$, $F\big(\mathcal{R}_x \setminus R^N\big) \cap F(R^N) = \emptyset$. Then, $x \in F(R^N)$ yields that $x \notin F\big(\mathcal{R}_x \setminus R^N\big)$. However, $x \in F\big((R^N)^\tau\big)$ for any $\tau \in \Psi_x$ by weak neutrality (WN), and then reinforcement (RE) implies that $x \in F\big(\mathcal{R}_x \setminus R^N\big)$, a contradiction. Hence, $D(x,R^N) < \frac{(m+1)n}{2}$. Then, $F\big(\sum_{\tau\in\Psi_x}(R^N)^\tau\big) = \{x\}$ by Lemma 15-(ii). □

Lemma 19 *Let F be an SCR which satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). Now, for any finite set of voters N, any $R^N \in W(A)^N$ and any $x \in A$, if $x \in F(R^N)$ and $D(x,R^N) = \frac{(m+1)n}{2}$, then for all $y \in A$ we have $D(y,R^N) = \frac{(m+1)n}{2}$ and hence $F(R^N) = A$.*

Proof Let N be a finite set of voters. Let $R^N \in W(A)^N$ and $x \in A$ be such that $x \in F(R^N)$ and $D(x,R^N) = \frac{(m+1)n}{2}$. Suppose that $D(y,R^N) \neq \frac{(m+1)n}{2}$ for some $y \in (A \setminus \{x\})$. Then, there exists at least an alternative $z \in (A \setminus \{x\})$ such that $D(z,R^N) > \frac{(m+1)n}{2}$. Proposition 14 implies that $z \notin F(R^N)$, so $F(R^N) \neq A$. Now, since $x \in F(R^N)$ and $F(R^N) \neq A$, we have, $F\big(\sum_{\tau\in\Psi_x}(R^N)^\tau\big) = \{x\}$ by Lemma 18.

[7] That is, $\mathcal{R}_x \setminus R^N = \sum_{\tau\in(\Psi_x\setminus\{\bar\tau\})}(R^N)^\tau$ and $R^N = (R^N)^{\bar\tau}$, where $\bar\tau \in \Psi_x$ is the identity permutation over alternatives.

However, the fact that $D(x, R^N) = \frac{(m+1)n}{2}$, together with Lemma 15-(i), implies that $F\big(\sum_{\tau \in \Psi_x} (R^N)^\tau\big) = A$, a contradiction. So, for all $y \in A$ we have $D(y, R^N) = \frac{(m+1)n}{2}$. Hence, $F(R^N) = A$ by Lemma 13. □

For any finite set of voters N, any $R^N \in W(A)^N$, we let
$M(R^N) = \{x \in A \mid D(x, R^N) \leq D(y, R^N) \text{ for all } y \in A\}$.

Proposition 15 *Let F be an SCR which satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). Now, for any finite set of voters N, any $R^N \in W(A)^N$ and any $x, y \in A$, if $D(y, R^N) > D(x, R^N)$ then $y \notin F(R^N)$.*

Proof Let F be an SCR which satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). Let N be a finite set of voters. Let $R^N \in W(A)^N$ and $x, y \in A$ be such that $D(y, R^N) > D(x, R^N)$. Suppose that $y \in F(R^N)$. Then, we have $D(y, R^N) \leq \frac{(m+1)n}{2}$ by Proposition 14. We will consider two cases.

Case 1. $D(y, R^N) = \frac{(m+1)n}{2}$.

Since $y \in F(R^N)$ and $D(y, R^N) = \frac{(m+1)n}{2}$, for all $z \in A$ we have $D(z, R^N) = \frac{(m+1)n}{2}$ by Lemma 19. However, since $\frac{(m+1)n}{2} = D(y, R^N) > D(x, R^N)$, we have $D(x, R^N) < \frac{(m+1)n}{2}$, a contradiction. Hence, $y \notin F(R^N)$.

Case 2. $D(y, R^N) < \frac{(m+1)n}{2}$.

We consider the profile $\sum_{\tau \in \Psi_{xy}} (R^N)^\tau = \mathcal{R}_{xy}$. Weak neutrality (WN) and reinforcement (RE) imply that $y \in F(\mathcal{R}_{xy})$. However, Lemma 17 implies that $y \notin F(\mathcal{R}_{xy})$, a contradiction. Hence, $y \notin F(R^N)$. □

Proposition 15 implies that for an SCR F satisfying weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE), we have $F(R^N) \subseteq M(R^N)$ for any finite set of voters N and any $R^N \in W(A)^N$.

Proposition 16 *Let F be an SCR which satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). Now, for any finite set of voters N, any $R^N \in W(A)^N$ and any $x, y \in A$, if $x \in M(R^N)$ and $y \in F(R^N)$ then $x \in F(R^N)$.*

Proof Let F be an SCR which satisfies weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE). Let N be a finite set of voters. Let $R^N \in W(A)^N$ and $x, y \in A$ be such that $x \in M(R^N)$ and $y \in F(R^N)$. Since $y \in F(R^N)$, we have $y \in M(R^N)$ by Proposition 15. Hence, $D(y, R^N) = D(x, R^N)$. Note that, by Proposition 14, we have $D(y, R^N) \leq \frac{(m+1)n}{2}$ since $y \in F(R^N)$. We will consider two cases.

Case 1. $D(y, R^N) = \frac{(m+1)n}{2}$.

Since $y \in F(R^N)$ and $D(y, R^N) = \frac{(m+1)n}{2}$, we have $F(R^N) = A$ by Lemma 19. Hence, $x \in F(R^N)$.

Case 2. $D(y, R^N) < \frac{(m+1)n}{2}$.

We consider the profile $\sum_{\tau \in \Psi_{xy}} (R^N)^\tau = \mathcal{R}_{xy}$. Since $D(y, R^N) = D(x, R^N) < \frac{(m+1)n}{2}$, we have $F(\mathcal{R}_{xy}) = \{x, y\}$ by Lemma 16. Since $y \in F(R^N)$, $y \in F\big((R^N)^\tau\big)$ for all $\tau \in \Psi_{xy}$ by weak neutrality (WN). Hence, $\bigcap_{\tau \in \Psi_{xy}} (R^N)^\tau \neq \emptyset$. This fact, together with $F(\mathcal{R}_{xy}) = \{x, y\}$, implies that $x \in F\big((R^N)^\tau\big)$ for each $\tau \in \Psi_{xy}$. Hence, $x \in F(R^N)$. $\square$

Proposition 16 implies that for any finite set of voters N and any $R^N \in W(A)^N$, we have $M(R^N) \subseteq F(R^N)$. Propositions 15 and 16 imply that for an SCR F satisfying weak neutrality (WN), reinforcement (RE), faithfulness (F) and degree equality (DE), we have $F(R^N) = M(R^N)$ for any finite set of voters N and any $R^N \in W(A)^N$.

For any finite voter sets N, any $R^N \in W(A)^N$ and any $x, y \in A$, $S(x, R^N) > S(y, R^N)$ if and only if $D(x, R^N) < D(y, R^N)$, and $S(x, R^N) = S(y, R^N)$ if and only if $D(x, R^N) = D(y, R^N)$. So, $M(R^N) = \{x \in A \mid D(x, R^N) \leq D(y, R^N)$ for all $y \in A\} = \{x \in A \mid S(x, R^N) \geq S(y, R^N)$ for all $y \in A\}$. By Propositions 15 and 16 we have $F(R^N) = M(R^N)$ showing that our SCR depends on the Borda scores in the right way, completing the proof of Theorem 2.

In order to show that the axioms used in Theorem 2 are independent, we need to provide four SCRs at each of which it violates the given axiom but satisfies the other three axioms. However, we could not provide an SCR which violates weak neutrality and satisfies the other axioms.

(1) Degree equality (DE)
The plurality rule satisfies weak neutrality (WN), faithfulness (F) and reinforcement (RE). However, plurality rule does not satisfy degree equality (DE) by Proposition 13.

(2) Faithfulness (F)
For any finite set of voters N, any $R^N \in W(A)^N$, we define $\bar{F}(R^N) = A$, i.e., $\bar{F}$ always chooses the set of all alternatives. It is clear that $\bar{F}$ violates faithfulness (F), and satisfies weak neutrality (WN), reinforcement (RE) and degree equality (DE).

(3) Reinforcement (RE)
We define $\acute{F}$ as follows:
For $\#N = 1$, $\acute{F}(R) = top(R)$ for any $R \in W(A)$,
for $\#N \geq 2$, $\acute{F}(R^N) = A$ for any $R^N \in W(A)^N$.
It is clear that $\acute{F}$ satisfies faithfulness (F), weak neutrality (WN), and degree equality (DE). However, $\acute{F}$ violates reinforcement (RE).

4.4 The cancellation property

In this section, we will introduce a new cancellation property and show that the Borda rule is the unique scoring rule which satisfies this property.

For any positive integer h, $1 \leq h \leq m$, let $r_h(R_i)$ denote the h^{th} level best alternatives at R_i.[8]

[8] For any R_i, any $1 \leq h \leq m$ and any alternative $x \in r_h(R_i)$, an alternative way to determine the score of x at R_i is as follows:

$$s(x, R_i) = \frac{s_{\#SU(x,R_i)+1} + \ldots + s_{\#SU(x,R_i)+\#r_h(R_i)}}{\#r_h(R_i)} = \frac{\sum_{k=\#SU(x,R_i)+1}^{k=\#SU(x,R_i)+\#r_h(R_i)} s_k}{\#r_h(R_i)}.$$

Let $R^N \in W(A)^N$ be a profile such that there exist $i, j \in N$ and $\alpha, \beta \in \{1, \ldots, m-1\}$ such that $r_\alpha(R_i) = r_{\beta+1}(R_j)$ and $r_{\alpha+1}(R_i) = r_\beta(R_j)$, i.e.,

R^N :

	R_1	$\ldots$	R_i	$\ldots$	R_j	$\ldots$	R_n
r_1	$r_1(R_1)$	$\ldots$	$r_1(R_i)$	$\ldots$	$r_1(R_j)$	$\ldots$	$r_1(R_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r_α	$r_\alpha(R_1)$	$\ldots$	$r_\alpha(R_i) = r_{\beta+1}(R_j)$	$\ldots$	$r_\alpha(R_j)$	$\ldots$	$r_\alpha(R_n)$
$r_{\alpha+1}$	$r_{\alpha+1}(R_1)$	$\ldots$	$r_{\alpha+1}(R_i) = r_\beta(R_j)$	$\ldots$	$r_{\alpha+1}(R_j)$	$\ldots$	$r_{\alpha+1}(R_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r_β	$r_\beta(R_1)$	$\ldots$	$r_\beta(R_i)$	$\ldots$	$r_\beta(R_j) = r_{\alpha+1}(R_i)$	$\ldots$	$r_\beta(R_n)$
$r_{\beta+1}$	$r_{\beta+1}(R_1)$	$\ldots$	$r_{\beta+1}(R_i)$	$\ldots$	$r_{\beta+1}(R_j) = r_\alpha(R_i)$	$\ldots$	$r_{\beta+1}(R_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now, we derive $\acute{R}^N$ from R^N as follows:

- for all voters $l \in (N \setminus \{i, j\})$, $\acute{R}_l = R_l$,
- for voter i, $r_\alpha(\acute{R}_i) = r_\alpha(R_i) \cup r_{\alpha+1}(R_i)$,
  for all $h < \alpha$, $r_h(\acute{R}_i) = r_h(R_i)$,
  for all $h > \alpha + 1$, $r_h(\acute{R}_i) = r_{h+1}(R_i)$,
- for voter j, $r_\beta(\acute{R}_j) = r_\beta(R_j) \cup r_{\beta+1}(R_j)$,
  for all $h < \beta$, $r_h(\acute{R}_j) = r_h(R_j)$,
  for all $h > \beta + 1$, $r_h(\acute{R}_j) = r_{h+1}(R_j)$,

i.e., $\acute{R}^N$:

	$\forall l \in (N \setminus \{i,j\}) : \acute{R}_l = R_l$	$\acute{R}_i$	$\acute{R}_j$
r_1	$r_1(\acute{R}_l) = r_1(R_l)$	$r_1(\acute{R}_i) = r_1(R_i)$	$r_1(\acute{R}_j) = r_1(R_j)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_{\alpha-1}$	$r_{\alpha-1}(\acute{R}_l) = r_{\alpha-1}(R_l)$	$r_{\alpha-1}(\acute{R}_i) = r_{\alpha-1}(R_i)$	$r_{\alpha-1}(\acute{R}_j) = r_{\alpha-1}(R_j)$
r_α	$r_\alpha(\acute{R}_l) = r_\alpha(R_l)$	$r_\alpha(\acute{R}_i) = r_\alpha(R_i) \cup r_{\alpha+1}(R_i)$	$r_\alpha(\acute{R}_j) = r_\alpha(R_j)$
$r_{\alpha+1}$	$r_{\alpha+1}(\acute{R}_l) = r_{\alpha+1}(R_l)$	$r_{\alpha+1}(\acute{R}_i) = r_{\alpha+2}(R_i)$	$r_{\alpha+1}(\acute{R}_j) = r_{\alpha+1}(R_j)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_{\beta-1}$	$r_{\beta-1}(\acute{R}_l) = r_{\beta-1}(R_l)$	$r_{\beta-1}(\acute{R}_i) = r_\beta(R_i)$	$r_{\beta-1}(\acute{R}_j) = r_{\beta-1}(R_j)$
r_β	$r_\beta(\acute{R}_l) = r_\beta(R_l)$	$r_\beta(\acute{R}_i) = r_{\beta+1}(R_i)$	$r_\beta(\acute{R}_j) = r_\beta(R_j) \cup r_{\beta+1}(R_j)$
$r_{\beta+1}$	$r_{\beta+1}(\acute{R}_l) = r_{\beta+1}(R_l)$	$r_{\beta+1}(\acute{R}_i) = r_{\beta+2}(R_i)$	$r_{\beta+1}(\acute{R}_j) = r_{\beta+2}(R_j)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Given any profile $R^N \in W(A)^N$, let $\acute{\mathcal{R}}(R^N)$ denote the set of all profiles which are derived from R^N for any $i, j \in N$ and any $\alpha, \beta \in \{1, \ldots, m-1\}$ as defined above.

Definition 50 We say that an SCR F satisfies the ***cancellation property*** (CA) if for any finite set of voters N, any $R^N \in W(A)^N$ and any $\acute{R}^N \in \acute{\mathcal{R}}(R^N)$, we have $F(R^N) = F(\acute{R}^N)$.

Proposition 17 *A scoring rule satisfies the cancellation property (CA) if and only if it is the Borda rule.*

Proof It is clear that the Borda rule satisfies the cancellation property (CA). For the other part of the proof, let F be a scoring rule which satisfies the cancellation property (CA). Suppose that F is not the Borda rule.

Let $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$. We consider following profile R^N:

R_1	R_2	R_3
a	c	a, b
b	b	c
c	a	

We now consider $\acute{R}^N \in \acute{\mathcal{R}}(R^N)$ for voters 1 and 2, and $\alpha = 1$, $\beta = 2$, i.e., $\acute{R}^N$ is as follows:

$\acute{R}_1$	$\acute{R}_2$	$\acute{R}_3 = R_3$
a, b	c	a, b
c	a, b	c

Since F is a scoring rule satisfying cancellation property (CA), we have $F(R^N) = F(\acute{R}^N)$. Let $s = (s_1, s_2, s_3)$, where $s_1 \geq s_2 \geq s_3$ and $s_1 > s_3$. Since we supposed that F is not the Borda rule, we have $s_1 - s_2 \neq s_2 - s_3$. Let $t_{12} = s_1 - s_2$ and $t_{23} = s_2 - s_3$. So, $s_1 = s_3 + t_{12} + t_{23}$ and $s_2 = s_3 + t_{23}$.

We calculate the total score of every alternative at R^N and $\acute{R}^N$:

$$S(a, R^N) = s_1 + s_3 + \tfrac{s_1+s_2}{2} = 3s_3 + \tfrac{3}{2}t_{12} + 2t_{23},$$
$$S(b, R^N) = 2s_2 + \tfrac{s_1+s_2}{2} = 3s_3 + \tfrac{1}{2}t_{12} + 3t_{23},$$
$$S(c, R^N) = s_1 + 2s_3 = 3s_3 + t_{12} + t_{23},$$
$$S(a, \acute{R}^N) = 2(\tfrac{s_1+s_2}{2}) + \tfrac{s_2+s_3}{2} = 3s_3 + t_{12} + \tfrac{5}{2}t_{23},$$
$$S(b, \acute{R}^N) = 2(\tfrac{s_1+s_2}{2}) + \tfrac{s_1+s_3}{2} = 3s_3 + t_{12} + \tfrac{5}{2}t_{23},$$
$$S(c, \acute{R}^N) = 2s_3 + s_1 = 3s_3 + t_{12} + t_{23}.$$

Note that $S(a, \acute{R}^N) = S(b, \acute{R}^N)$. We will now show that $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$. Since F is a scoring rule, we do not have $t_{12} = t_{23} = 0$. If $t_{12} = 0$ and $t_{23} > 0$, then we have $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$. If $t_{12} > 0$ and $t_{23} = 0$, then we have $S(a, \acute{R}^N) = S(b, \acute{R}^N) = S(c, \acute{R}^N)$ which implies that $F(\acute{R}^N) = \{a, b, c\}$. Hence, we have $F(R^N) = \{a, b, c\}$. Therefore, $S(a, R^N) = S(b, R^N) = S(c, R^N)$ implying that $t_{12} = t_{23} = 0$, a contradiction. Hence, $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$.

The fact that $S(a, \acute{R}^N) = S(b, \acute{R}^N) > S(c, \acute{R}^N)$, together with F being a scoring rule, implies that $F(\acute{R}^N) = \{a, b\}$. So, $F(R^N) = \{a, b\}$. Hence, we have $S(a, R^N) = S(b, R^N)$ which implies that $t_{12} = t_{23}$, a contradiction. Hence, F is the Borda rule. □

Propositions 13 and 17 imply that when we restrict ourselves to scoring rules degree equality (DE) is equivalent to cancellation property (CA). However, in general, degree equality (DE) is stronger than cancellation property (CA).

Lemma 20 *(i) If an SCR satisfies degree equality (DE) then it also satisfies the cancellation property (CA).*
(ii) There exists an SCR which satisfies the cancellation property (CA) but violates degree equality (DE).

Proof It is given in the Appendix. □

4.5 Conclusion

We studied a characterization of the Borda rule on the domain of weak preferences. We introduced a new property which is referred to as degree equality, and showed that the Borda rule is characterized by weak neutrality, reinforcement, faithfulness and degree equality on the domain of weak preferences. We also showed that the Borda rule is the unique scoring rule which satisfies degree equality. We introduced a new cancellation property and shown that it characterizes the Borda rule among all scoring rules.

CHAPTER 5

GRADUATE ADMISSIONS PROBLEM WITH QUOTA AND BUDGET CONSTRAINTS

5.1 Introduction

A typical two-sided matching market consists of two disjoint finite sets, for example a set of men and a set of women; colleges and students; firms and workers. A matching is called a one-to-one matching if a member of one set is allowed to match with at most one member of other set, for example a man (woman) can match with only one woman (man). However, a firm hires many workers, but a worker works for one firm only. This type of matching is called a many to one matching.

There is a rich literature on matching theory (see Roth and Sotomayor (1990b) for an excellent survey for a period covering all classical results in the field and Roth (2008) for a recent survey) including both theoretical and empirical studies. Even though there is an extensive literature on matching theory, there were no study considering both quota and budget constraints simultaneously until Karakaya and Koray (2003). There were studies where colleges (or firms) have either quota constraint or budget constraint but not both. Karakaya and Koray (2003) studied the graduate admissions problem under quota and budget constraints. There is a set of departments belonging to one university and a set of students (applicants) who wish to enter these departments. Each department faces both quota and budget constraints which are determined by the university.

Gale and Shapley (1962) described a model for *college admissions problem*.[1] A college admission problem consists of a finite set of students and a finite set of colleges where each college faces a quota constraint. Each student has a linear preference relation over colleges and each college has a linear preference relation over sets of students. A student matches with a college or with herself (i.e., stays unmatched) and a college matches with a group of students whose size does not exceed its quota. A matching is blocked by a student if and only if she prefers to match with herself to getting matched with the college that she is assigned under that matching. A matching is blocked by a college if and only if it prefers a proper subset of the group of students that it matched under the given matching. A matching is blocked by a student-college pair if and only if the student prefers that college to her match and the college prefers the union of a subset of its match with the student to its present match. A matching is stable if and only if it is not blocked by a student, by a college and by a student-college pair. From each given set of students a college selects its most preferred such set of students obeying the quota constraint. This most preferred set of students is referred as the choice of that college from among the group of students it faces. A stable matching is students-optimal if and only if each student likes this matching at least as well as any other stable matching. A stable matching is colleges-optimal if and only if each college likes this matching at least as well as any other stable matching.

The following algorithm is referred as *the Gale-Shapley student optimal algorithm*:
Step 1: Each student proposes to her most preferred college. Each college rejects all but those who comprise its choice among its proposers.
In general, at step k,
Step k: Each student who was rejected in the previous step proposes to her next preferred college. Each college rejects all but those who comprise its choice within the students it has been holding together with its new proposers.
The algorithm stops if there is no student such that her proposal is rejected. Then each student is matched with a college that she proposed at the last step and was not rejected

[1] See Abdulkadiroğlu et al. (2005a), Abdulkadiroğlu et al. (2005b), Abdulkadiroğlu and Sönmez (2002), Balinski and Sönmez (1999), Roth (1985), Roth (1986) and Roth and Sotomayor (1990a).

by that college. The Gale-Shapley college optimal algorithm is similarly defined with colleges proposing to group of students by obeying their quota constraints.

A college has *substitutable* preferences if it regards students as substitutes rather than as complements, i.e., the college prefers to enroll a student who is in its choice set even if some of the other students in its choice set become unavailable. When colleges have substitutable preferences the set of stable matchings is non-empty. That is the Gale-Shapley student optimal algorithm produces a stable students-optimal matching (similarly the Gale-Shapley college optimal algorithm produces a stable colleges-optimal matching).[2]

Kelso and Crawford (1982) considered a model for labor markets as a many to one matching market.[3] There are a finite set of workers and a finite set of firms. Firms do not face quota or budget constraints. It is assumed that all workers are gross substitutes from the viewpoint of each firm. This assumption is referred as the *gross substitutes condition* which states that "all workers be (weak) gross substitutes to each firm, in the sense that increases in other workers' salaries can never cause a firm to withdraw an offer from a worker whose salary has not risen." Thus the production technology is such that workers are not complements. Kelso and Crawford (1982) showed the existence of a core stable matching by an extension of the Gale-Shapley algorithm. That is, there is a matching such that there is no subgroup consisting of firms and workers which blocks that matching. They also showed that there is a firms-optimal core stable matching, i.e., there is a core stable matching that each firm likes at least as well as any other core stable matching.[4]

Mongell and Roth (1986) considered the model of Kelso and Crawford together with budget constraints for firms. They showed by an example that the set of core stable matchings may be empty. They also gave an example to show that if the set of core stable matchings is non-empty, it is possible that there be no firms-optimal core

[2] The Gale-Shapley algorithm has been used since 1951 in the United States to match medical residents to hospitals. See Crawford (2008), Roth (1984a), Roth (2002), Roth and Peranson (1999) for this matching program.

[3] See also Crawford and Knoer (1981).

[4] Roth (1984b) considered the same labor market model as many to many matchings and showed that, under the assumption that both firms and workers preferences satisfy the gross substitutes condition, the set of core stable matchings is non-empty and there exist firms-optimal and workers-optimal core stable matchings.

stable matching.

Karakaya and Koray (2003) considered the *graduate admissions problem* as a two sided many to one matching market. There are a set of students and a set of departments which belong to one university. Each department faces quota and budget constraints which are determined centrally by the university. Students apply to these departments for their graduate studies and each student has a value added to each department. If a student matches with a department she may be paid by the department or she may pay to the department. If a student pays for her graduate study, that payment is not added to the department's budget for graduate admissions. That payment goes to the university which gives some percentage of that payment to the department for its office expenditures. Departments use their budgets for the payments to graduate students, and if a department has some of its budget left after these payments, the remaining part is used for office expenditures by the department. Each department gets a benefit from its accepted students and its office expenditures. The total benefit of a department from its accepted students is the sum of each accepted student's value added to the department. Each department wants to maximize its total benefit which is sum of the benefits from accepted students and from office expenditures. It is assumed that, for any department, the largest benefit from office expenditures is less than any qualified student's benefit to the department no matter how large the office expenditures are. Therefore, each department wants to maximize its total benefit by accepting more qualified students at a minimum cost. Each student wants to make graduate study at her most preferred department.

This model differs from the previous models in the sense that departments face both quota and budget constraints. Karakaya and Koray (2003) constructed the departments proposing algorithm which is an extension of the Gale-Shapley algorithm and showed that, if the algorithm stops then the resulting matching is core stable (and thus Pareto optimal). They showed that the algorithm may not stop while there is a core stable matching. They proved that the departments proposing algorithm stops if and only if no cycle occurs in the algorithm, i.e., a finite sequence of matchings does not repeat itself infinitely many times in the algorithm. The existence of either a departments-optimal or a students-optimal matching is not guaranteed in the graduate admissions problem with both quota and budget constraints.

In this chapter, we will continue to study graduate admissions problem with quota and budget constraints. Here we construct the students proposing algorithm, and show that the students proposing algorithm ends up with a core stable matching if the algorithm stops. However, there exist graduate admissions problems for which there exist core stable matchings, while neither the departments proposing nor the students proposing algorithm stops. We showed that the students proposing algorithm stops if and only if no cycle occurs in the algorithm. Moreover, we show that there is no random path to core stability for the graduate admissions problem, i.e., starting from an arbitrary matching and satisfying a randomly chosen blocking coalition at each step, a core stable matching can not be reached. We will also consider the model with the assumption that the students care only about their reservation prices and do not derive any further utility from money transfers over and above their reservation prices.

This chapter is organized as follows: We present the model and definitions in section 5.2. Section 5.3 defines the algorithms and related results. Section 5.4 shows the nonexistence of random paths to core stability in our model. Section 5.5 studies the model where the students consider only their reservation prices. Section 5.6 concludes the chapter.

5.2 Basic notions

We denote the finite non-empty set of departments of our university by $\mathcal{D} = \{d_1, d_2, \ldots, d_m\}$. A finite nonempty set of students denoted by $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$, is regarded as comprising the applicants to this university for graduate programs offered by its departments.

Each department $d \in \mathcal{D}$ has a quota q_d and a budget b_d for its graduate program; both of which are determined centrally by the university. A student can enroll to at most one department, and each department accepts a group of students obeying its quota and budget constraints.

We assume that each student $s \in \mathcal{S}$ has a qualification level for each department $d \in \mathcal{D}$. The qualification level of student s for department d is an integer and denoted by a_d^s. The qualification levels of student s for the departments are denoted by a vector $a_{\mathcal{D}}^s = (a_{d_1}^s, a_{d_2}^s, \ldots, a_{d_m}^s)$. Also we assume that each department has a minimal qualification level as a threshold for accepting students. The minimal qualification level of department d is a positive integer and denoted by a^d.

Each student yields a benefit (or adds a value) to each department if accepted to that department. These values are independent of who the other accepted students are, i.e., there are no externalities in this regard. The benefit of department d obtained from accepting a group of students $S^d \subseteq \mathcal{S}$ is denoted by $y^d(S^d)$. We assume that department d's benefit $y^d(S^d)$ is additive, i.e., it is the sum of the accepted students' benefits to the department. We assume that the benefit student s provides to department d is equal to her qualification level for department d, i.e., $y^d(\{s\}) = a_d^s$. Therefore the total benefit of department d from accepting a group of students $S^d \subseteq \mathcal{S}$ is $y^d(S^d) = \sum_{s \in S^d} a_d^s$.

If a student gets enrolled to a department for graduate study, she may be paid by the department or she may pay to the department. The amount of payment made by department d to student s is an integer m_{sd}. In other words, student s is paid by department d the amount m_{sd} if $m_{sd} > 0$; there is no payment if $m_{sd} = 0$; student s pays to department d the amount m_{sd} if $m_{sd} < 0$. If an accepted student pays for her graduate study at department d, this payment is not added to department d's budget. That payment is taken by the university and the university gives some fixed percentage of this payment to department d, solely to be used, say, for its office expenditures.

We assume that each student s has a reservation price for each department d (the lowest amount of money that student s will accept from department d) which will be denoted by an integer σ_{sd}. We assume that for all $s \in \mathcal{S}$ and for all $d \in \mathcal{D}$, $\sigma_{sd} \leq b_d$. Student s's reservation prices for departments will be denoted by a vector $\sigma^s = (\sigma_{sd_1}, \ldots, \sigma_{sd_m})$. Note that a reservation price may also be negative, representing the level of willingness on the part of the student to pay to the department in question to get accepted.

If department d has some remaining budget after payments, the remaining money is only used for office expenditures by the department. Let $\overline{B}$ be the total budget of the university, and let student s be the least qualified student for department d among all students who are qualified for department d, i.e., $a^s_d \geq a^d$ and for all $h \in (\mathcal{S} \setminus \{s\})$ with $a^h_d \geq a^d$, we have $a^s_d \leq a^h_d$. Let $\epsilon^d_{\overline{B}}$ be the benefit of department d if it uses the university's entire budget $\overline{B}$ for its office expenditures. We assume that $y^d(\{s\}) > \epsilon^d_{\overline{B}}$. Therefore, the benefit which is gained by spending $\overline{B}$ for the office expenditures is less than any qualified student's benefit to department d. This means that one can take $a^d = 1$ and $0 < \epsilon^d_{\overline{B}} < 1$ for each $d \in \mathcal{D}$.

The total benefit of department d is denoted by Y^d and it is the sum of benefits from accepted students and office expenditures. Therefore when $S^d \subseteq \mathcal{S}$ is the accepted group of students by department d and ϵ^d is the benefit that department d gets from office expenditures, we have that $Y^d(S^d, \epsilon^d) = y^d(S^d) + \epsilon^d$.

Definition 51 A graduate admission problem is a list $(\mathcal{D}, \mathcal{S}, q, b, a^{\mathcal{S}}, \sigma)$ where

1. $\mathcal{D}$ is a finite nonempty set of departments,
2. $\mathcal{S}$ is a finite nonempty set of students,
3. $q = (q_d)_{d \in \mathcal{D}}$ is the departments' quotas with $q_d \in \mathbb{N}$ for each $d \in \mathcal{D}$,
4. $b = (b_d)_{d \in \mathcal{D}}$ is the departments' budgets with $b_d \in \mathbb{N}_0$ for each $d \in \mathcal{D}$,[5]
5. $a^{\mathcal{S}} = (a^s_{\mathcal{D}})_{s \in \mathcal{S}}$ is the students' qualification levels for departments with $a^s_d \in \mathbb{Z}$ for each $s \in \mathcal{S}$, $d \in \mathcal{D}$,
6. $\sigma = (\sigma^s)_{s \in \mathcal{S}}$ is the students' reservation prices for departments with $\sigma_{sd} \in \mathbb{Z}$ and $\sigma_{sd} \leq b_d$ for each $s \in \mathcal{S}$, $d \in \mathcal{D}$.

Now, we will define the preferences of departments and students.

Let $2^{\mathcal{S}} \times \mathbb{R}^{|\mathcal{S}|} = \{S \times \mathbb{R}^{|S|} \mid S \in 2^{\mathcal{S}}\}$ denote the set of all pairs where a pair consists of a group of students $S = \{s_1, \ldots, s_h\} \in 2^{\mathcal{S}}$ and an associated transfer vector $m^S_d = \{m_{s_1 d}, \ldots, m_{s_h d}\} \in \mathbb{R}^{|S|}$ of students S for department $d \in \mathcal{D}$.

The strict preference relation of department d is denoted by P_d. For any $d \in \mathcal{D}$, P_d is a linear order [6] on $2^{\mathcal{S}} \times \mathbb{R}^{|\mathcal{S}|}$.

[5] $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

[6] A linear order on a set X is a complete, transitive and antisymmetric (binary) relation.

Consider two group of students $T = \{s_1, \ldots, s_h\}$ and $\widehat{T} = \{\widehat{s}_1, \ldots, \widehat{s}_k\}$ with associated transfer vectors $m_d^T = \{m_{s_1 d}, \ldots, m_{s_h d}\}$ and $m_d^{\widehat{T}} = \{m_{\widehat{s}_1 d}, \ldots, m_{\widehat{s}_k d}\}$ for department d, respectively, where $\mid T \mid= h$ and $\mid \widehat{T} \mid= k$. Let c_d^T denote the cost of group of students T to department d when the associated transfer vector is m_d^T, i.e., $c_d^T = \sum_{s \in \overline{T}} m_{sd}$ with $\overline{T} = \{s \in T \mid m_{sd} > 0\}$, and $c_d^{\widehat{T}}$ the cost of group of students $\widehat{T}$ to department d when the associated transfer vector is $m_d^{\widehat{T}}$, i.e., $c_d^{\widehat{T}} = \sum_{\widehat{s} \in \overline{\widehat{T}}} m_{\widehat{s}d}$ with $\overline{\widehat{T}} = \{\widehat{s} \in \widehat{T} \mid m_{\widehat{s}d} > 0\}$. Let ϵ_d^T denote the benefit of office expenditures that department d obtains by accepting the group of students T with transfers m_d^T at cost c_d^T, and $\epsilon_d^{\widehat{T}}$ the benefit of office expenditures that department d obtains by accepting the group of students $\widehat{T}$ with transfers $m_d^{\widehat{T}}$ at cost $c_d^{\widehat{T}}$.

Now we can define P_d formally as follows: For any $(T, m_d^T), (\widehat{T}, m_d^{\widehat{T}}) \in 2^{\mathcal{S}} \times \mathbb{R}^{|\mathcal{S}|}$ with $T \neq \widehat{T}$, we have $[(T, m_d^T) P_d (\widehat{T}, m_d^{\widehat{T}})]$ if and only if

- $[Y^d(T, \epsilon_d^T) > Y^d(\widehat{T}, \epsilon_d^{\widehat{T}})]$, or
- $[Y^d(T, \epsilon_d^T) = Y^d(\widehat{T}, \epsilon_d^{\widehat{T}})$ and $c_d^T < c_d^{\widehat{T}}]$, or
- $[Y^d(T, \epsilon_d^T) = Y^d(\widehat{T}, \epsilon_d^{\widehat{T}})$, $c_d^T = c_d^{\widehat{T}}$ and T leximin preferred[7] to $\widehat{T}]$.

That is, department d strictly prefers T (with m_d^T) to $\widehat{T}$ (with $m_d^{\widehat{T}}$) if $Y^d(T, \epsilon_d^T) > Y^d(\widehat{T}, \epsilon_d^{\widehat{T}})$. If $Y^d(T, \epsilon_d^T) = Y^d(\widehat{T}, \epsilon_d^{\widehat{T}})$ then department d considers the associated costs of T and $\widehat{T}$. That is, whenever $Y^d(T, \epsilon_d^T) = Y^d(\widehat{T}, \epsilon_d^{\widehat{T}})$, department d strictly prefers T (with m_d^T) to $\widehat{T}$ (with $m_d^{\widehat{T}}$) if $c_d^T < c_d^{\widehat{T}}$. If $Y^d(T, \epsilon_d^T) = Y^d(\widehat{T}, \epsilon_d^{\widehat{T}})$ and $c_d^T = c_d^{\widehat{T}}$, then department d strictly prefers T (with m_d^T) to $\widehat{T}$ (with $m_d^{\widehat{T}}$) if T leximin preferred to $\widehat{T}$.

Let R_d denote a preference relation of department d induced from P_d, and defined as follows: For any $(T, m_d^T), (\widehat{T}, m_d^{\widehat{T}}) \in 2^{\mathcal{S}} \times \mathbb{R}^{|\mathcal{S}|}$,

- $[(T, m_d^T) R_d (\widehat{T}, m_d^{\widehat{T}})]$ if and only if [not $(\widehat{T}, m_d^{\widehat{T}}) P_d (T, m_d^T)$].

[7] If $Y^d(T, \epsilon_d^T) = Y^d(\widehat{T}, \epsilon_d^{\widehat{T}})$, and $c_d^T = c_d^{\widehat{T}}$, then department d makes a lexicographic comparison among T and $\widehat{T}$ in the following way: Remember that $\mid T \mid= h$ and $\mid \widehat{T} \mid= k$. Let $f : \{1, \ldots, h\} \rightarrow \{i \mid s_i \in T\}$ be a function such that $f(1) < f(2) < \ldots < f(h)$. Let $g : \{1, \ldots, k\} \rightarrow \{j \mid \widehat{s}_j \in \widehat{T}\}$ be a function such that $g(1) < g(2) < \ldots < g(k)$. We say that department d *leximin prefers* T to $\widehat{T}$ if and only if $f(1) < g(1)$ or there exists $k \in \{1, \ldots, \tilde{n}\}$ where $\tilde{n} < \min\{h, k\}$ such that for all $t \in \{1, \ldots, \tilde{n}\}$, $f(t) = g(t)$ but $f(t+1) < g(t+1)$.

We note that for any $(T, m_d^T), (\widehat{T}, m_d^{\widehat{T}}) \in 2^{\mathcal{S}} \times \mathbb{R}^{|\mathcal{S}|}$ with $T \neq \widehat{T}$, we have either $[(T, m_d^T) P_d (\widehat{T}, m_d^{\widehat{T}})]$ or $[(\widehat{T}, m_d^{\widehat{T}}) P_d (T, m_d^T)]$.

The strict preference relation of student s is denoted by P_s. For all $s \in \mathcal{S}$, P_s is a linear order on $(\mathcal{D} \times \mathbb{R}) \bigcup \{(\emptyset, 0)\}$.

We assume that, given any $s \in \mathcal{S}$, $\sigma_{sd} = \sigma_{s\widetilde{d}}$ if and only if $d = \widetilde{d}$. We also assume that $(d, \sigma_{sd}) P_s (\emptyset, 0)$ for all $s \in \mathcal{S}$ and all $d \in \mathcal{D}$, where $(\emptyset, 0)$ stands for the situation that student s is unmatched (or she is matched with herself).[8]

For all $s \in \mathcal{S}$, P_s is defined as follows:
For any $(d, m_{sd}), (\widetilde{d}, m_{s\widetilde{d}}) \in (\mathcal{D} \times \mathbb{R}) \bigcup \{(\emptyset, 0)\}$,
$[(d, m_{sd}) P_s (\widetilde{d}, m_{s\widetilde{d}})]$ if and only if

- $[m_{sd} - \sigma_{sd} > m_{s\widetilde{d}} - \sigma_{s\widetilde{d}}]$, or
- $[m_{sd} - \sigma_{sd} = m_{s\widetilde{d}} - \sigma_{s\widetilde{d}}$ and $\sigma_{sd} < \sigma_{s\widetilde{d}}]$.

Let R_s denote a preference relation of student s induced from P_s and defined as follows: For any $(d, m_{sd}), (\widetilde{d}, m_{s\widetilde{d}}) \in (\mathcal{D} \times \mathbb{R}) \bigcup \{(\emptyset, 0)\}$,

- $[(d, m_{sd}) R_s (\widetilde{d}, m_{s\widetilde{d}})]$ if and only if [not $(\widetilde{d}, m_{s\widetilde{d}}) P_s (d, m_{sd})$].

Note that for any $(d, m_{sd}), (\widetilde{d}, m_{s\widetilde{d}}) \in [(\mathcal{D} \times \mathbb{R}) \bigcup \{(\emptyset, 0)\}]$ with $d \neq \widetilde{d}$, we have either $[(d, m_{sd}) P_s (\widetilde{d}, m_{s\widetilde{d}})]$ or $[(\widetilde{d}, m_{s\widetilde{d}}) P_s (d, m_{sd})]$. And being unmatched is not the worst situation for a student $s \in \mathcal{S}$, because for any student $s \in \mathcal{S}$, any department $d \in \mathcal{D}$ and any transfer $m_{sd} < \sigma_{sd}$ we have $[(\emptyset, 0) P_s (d, m_{sd})]$.

[8] We assume that for all $s \in \mathcal{S}$, $\sigma_{s\emptyset} = 0$.

Now we will define what we mean by a matching.

Definition 52 By a **matching** we mean a function $\mu : \mathcal{S} \longrightarrow (\mathcal{D} \times \mathbb{R}) \bigcup \{(\emptyset, 0)\}$ which matches each student s with a member $\mu_1(s)$ of $\mathcal{D} \bigcup \{\emptyset\}$ and also specifies the amount of transfer $\mu_2(s)$ made from $\mu_1(s)$ to s such that the following are satisfied:

1. (*Quota constraint*) For all $d \in \mathcal{D}$, $\mid S^d_\mu \mid \leq q_d$, where $S^d_\mu = \{s \in \mathcal{S} \mid \mu_1(s) = d\}$,
2. (*Budget constraint*) For all $d \in \mathcal{D}$, $c^d_\mu \leq b_d$, where $c^d_\mu = \sum_{s \in \overline{S}^d_\mu} m^\mu_{sd}$ with $m^\mu_{sd} = \mu_2(s)$ for $\mu_1(s) = d$ and $\overline{S}^d_\mu$=$\{s \in S^d_\mu \mid \ m^\mu_{sd} > 0\}$.

Student s is matched with a department if $\mu_1(s) \in \mathcal{D}$, she is unmatched if $\mu_1(s) = \emptyset$ under μ. Let $\mu(s) = (\mu_1(s), m^\mu_{s\mu_1(s)})$ denote the department that student s is matched and the associated transfer under μ.

Let Y^d_μ denote the total benefit of department d under μ. Let y^d_μ denote the benefit of department d that it obtains by accepting the group of students S^d_μ and ϵ^d_μ the benefit of department d that it gets from office expenditures under μ. When $S^d_\mu = \{s_1, \ldots, s_h\}$, we let $m^d_\mu = (m^\mu_{s_1 d}, \ldots, m^\mu_{s_h d})$ denote the associated transfer vector.

Department d's preference relation R_d induces a preference relation R^μ_d over matchings in a natural fashion as follows: For any matchings $\overline{\mu}$ and $\widetilde{\mu}$,

- $\overline{\mu} R^\mu_d \widetilde{\mu}$ if and only if $(S^d_{\overline{\mu}}, m^d_{\overline{\mu}}) R_d (S^d_{\widetilde{\mu}}, m^d_{\widetilde{\mu}})$.

We abuse notation and we use R_d for R^μ_d.

Students s's preference relation R_s similarly induces a preference relation R^μ_s over matchings as follows: For any matchings $\overline{\mu}$ and $\widetilde{\mu}$,

- $\overline{\mu} R^\mu_s \widetilde{\mu}$ if and only if $(\overline{\mu}_1(s), m^{\overline{\mu}}_{s\overline{\mu}_1(s)}) R_s (\widetilde{\mu}_1(s), m^{\widetilde{\mu}}_{s\widetilde{\mu}_1(s)})$.

We abuse notation and we use R_s for R^μ_s.

To present a matching μ, we will use a matrix consisting of three rows and n columns, where $n = \mid \mathcal{S} \mid$. The first row lists the set of students respecting their original labelling; the second row specifies the departments the students are assigned to and the third row consists of the associated money transfers. That is,

$$\mu = \begin{pmatrix} s_1 & s_2 & \dots & s_n \\ \mu_1(s_1) & \mu_1(s_2) & \dots & \mu_1(s_n) \\ m^{\mu}_{s_1\mu_1(s_1)} & m^{\mu}_{s_2\mu_1(s_2)} & \dots & m^{\mu}_{s_n\mu_1(s_n)} \end{pmatrix}.$$

Now, we define what we mean for a matching to be individually rational.

Definition 53 A matching μ is **individually rational** if and only if it satisfies the following properties

1. For all $s \in \mathcal{S}$, $\big(\mu_1(s), m^{\mu}_{s\mu_1(s)}\big) R_s \big(\emptyset, 0\big)$, that is $m^{\mu}_{s\mu_1(s)} \geq \sigma_{s\mu_1(s)}$, and
2. For all $d \in \mathcal{D}$, for all $S \subsetneqq S^d_\mu$, $(S^d_\mu, m^d_\mu) R_d (S, m^S_d)$, i.e., $Y^d_\mu(S^d_\mu, \epsilon^d_\mu) \geq Y^d(S, \epsilon)$, where m^S_d denote the transfer vector for S with m^{μ}_{sd} for each $s \in S$ and ϵ denote the benefit that department d gets from office expenditures by accepting set of students S with m^S_d.

We say that a matching μ is not individually rational for student s, or student s blocks μ, if $\big(\emptyset, 0\big) P_s \big(\mu_1(s), m^{\mu}_{s\mu_1(s)}\big)$. So, student s blocks μ if the associated transfer $m^{\mu}_{s\mu_1(s)}$ between student s and department $\mu_1(s)$ is smaller than her reservation price for department $\mu_1(s)$, i.e., $m^{\mu}_{s\mu_1(s)} < \sigma_{s\mu_1(s)}$. Hence, we say that μ is individually rational for student s if $m^{\mu}_{s\mu_1(s)} \geq \sigma_{s\mu_1(s)}$.

We say that a matching μ is not individually rational for department d, or department d blocks μ, if $(S, m^S_d) P_d (S^d_\mu, m^d_\mu)$ for any $S \subsetneqq S^d_\mu$. We note that for each student $s \in S$ with whom department d does not break its tie, the amount of transfer m^{μ}_{sd} between student s and department d which is determined by matching μ does not change. So, department d blocks μ if for any set of students $S \subsetneqq S^d_\mu$ we have $Y^d_\mu(S^d_\mu, \epsilon^d_\mu) < Y^d(S, \epsilon)$. Hence, we say that μ is individually rational for department d

if $Y^d_\mu(S^d_\mu, \epsilon^d_\mu) \geq Y^d(S, \epsilon)$ for any $S \subsetneqq S^d_\mu$.[9]

We say that a matching μ is *individually rational* if and only if it is individually rational for all students and all departments.

Definition 54 We say that a matching μ is **blocked by a student - department pair** $(s,d) \in \mathcal{S} \times \mathcal{D}$ with $\mu_1(s) \neq d$ if and only if there exists a transfer $\widetilde{m}_{sd}$ such that

1. $\big(d, \widetilde{m}_{sd}\big) P_s \big(\mu_1(s), m^\mu_{s\mu_1(s)}\big)$, and
2. $[(S^d_\mu \setminus T) \cup \{s\}, \widehat{m}^d] P_d [S^d_\mu, m^d_\mu]$, for some $T \subseteq S^d_\mu$,
   where $\widehat{m}^d$ denote the transfer vector for the set of students $[(S^d_\mu \setminus T) \cup \{s\}]$ with m^μ_{hd} for each student $h \in (S^d_\mu \setminus T)$ and $\widetilde{m}_{sd}$ for student s, such that the group of students $[(S^d_\mu \setminus T) \cup \{s\}]$ with transfers $\widehat{m}^d$ satisfies the quota and budget constraints of department d, i.e., $\mid (S^d_\mu \setminus T) \cup \{s\} \mid \leq q_d$ and $\widehat{c}^d \leq b_d$, where

$$\widehat{c}^d = \begin{cases} \big(\sum_{h \in (\overline{S}^d_\mu \setminus T)} m^\mu_{hd}\big) + \widetilde{m}_{sd} & \text{if } \widetilde{m}_{sd} > 0 \\ \sum_{h \in (\overline{S}^d_\mu \setminus T)} m^\mu_{hd} & \text{otherwise.} \end{cases}$$

A pair (s,d) that satisfies above two conditions is called **a blocking pair** for matching μ.

Thus, a matching μ is blocked by a pair (s,d) where student s and department d is not matched under μ, if there exists a transfer $\widetilde{m}_{sd}$ between s and d such that student s strictly prefers d with transfer $\widetilde{m}_{sd}$ to her match $\mu(s) = \big(\mu_1(s), m^\mu_{s\mu_1(s)}\big)$, and department d strictly prefers the set of students $[(S^d_\mu \setminus T) \cup \{s\}]$ with $\widehat{m}^d$ to S^d_μ with m^d_μ such that the group of students $[(S^d_\mu \setminus T) \cup \{s\}]$ with transfers $\widehat{m}^d$ satisfies its quota and budget constraints. Note that department d may break its ties with a set of students $T \subseteq S^d_\mu$ in order to form a blocking pair with student s. For each student $h \in (S^d_\mu \setminus T)$ with whom department d does not break its tie while forming a blocking pair with student s, the amount of transfer m^μ_{hd} between student h and department d which is determined by matching μ does not change.

[9]So, a matching μ is not individually rational for department d if there exists a student $s \in S^d_\mu$ such that $[a^s_d < 0]$ or $[a^s_d = 0$ and $m^\mu_{sd} > 0]$.

Definition 55 A matching μ is **pairwise stable** if and only if it is individually rational and there does not exist a pair $(s, d) \in \mathcal{S} \times \mathcal{D}$ which blocks it.

Now we will define group blocking of a matching μ.

Definition 56 We say that a matching μ is **blocked by a group** $(\widetilde{D}, \widetilde{S})$ with $\emptyset \neq \widetilde{D} \subseteq \mathcal{D}$ and $\emptyset \neq \widetilde{S} \subseteq \mathcal{S}$ if and only if the following two conditions are satisfied:

1. For all $s \in \widehat{S}^d \subseteq \widetilde{S}$, $\big(d, \widetilde{m}_{sd}\big) P_s \big(\mu_1(s), m^{\mu}_{s\mu_1(s)}\big)$,
   where $d \in \widetilde{D}$, $\widehat{S}^d \subseteq \widetilde{S}$ denote the group of students who matched with department $d \in \widetilde{D}$ by group blocking of μ with for all $s \in \widehat{S}^d$, $\mu_1(s) \neq d$, such that $\bigcup_{d\in\widetilde{D}} \widehat{S}^d = \widetilde{S}$, and $\widetilde{m}_{sd}$ denote the transfer between department $d \in \widetilde{D}$ and a student $s \in \widehat{S}^d \subseteq \widetilde{S}$,

2. For all $d \in \widetilde{D}$, $[(S^d_\mu \setminus T) \cup \widehat{S}^d, \widehat{m}^d] P_d [S^d_\mu, m^d_\mu]$, for some $T \subseteq S^d_\mu$,
   where $\widehat{m}^d$ denote the transfer vector for the set of students $[(S^d_\mu \setminus T) \cup \widehat{S}^d]$ with m^{μ}_{hd} for each student $h \in (S^d_\mu \setminus T)$ and $\widetilde{m}_{sd}$ for each student $s \in \widehat{S}^d$, such that the group of students $[(S^d_\mu \setminus T) \cup \widehat{S}^d]$ with transfers $\widehat{m}^d$ satisfies the quota and budget constraints of department d, i.e., $\mid (S^d_\mu \setminus T) \cup \widehat{S}^d \mid \leq q_d$ and $\widehat{c}^d \leq b_d$, where $\widehat{c}^d = \sum_{h\in(\overline{S}^d_\mu \setminus T)} m^{\mu}_{hd} + \sum_{s\in\overline{\widehat{S}}^d} \widetilde{m}_{sd}$ with $\overline{\widehat{S}}^d = \{s \in \widehat{S}^d \mid \widetilde{m}_{sd} > 0\}$.

Definition 57 We say that a matching μ is **core stable** if and only if μ is individually rational and there does not exist a group $(\widetilde{D}, \widetilde{S})$ which blocks μ.

Since a student can match with at most one department, whenever a matching μ is blocked by a group $(\widetilde{D}, \widetilde{S})$, we can consider the group $(\widetilde{D}, \widetilde{S})$ as a collection of groups, where each group consists of a department $d \in \widetilde{D}$ and a set of students $\widehat{S}^d \subseteq \widetilde{S}$ who matched with department $d \in \widetilde{D}$ by group blocking of μ such that $\bigcup_{d\in\widetilde{D}} \widehat{S}^d = \widetilde{S}$. So, if the group $(\widetilde{D}, \widetilde{S})$ blocks a matching μ then each group $(d, \widehat{S}^d)$ also blocks μ. That is, as shown in Karakaya and Koray (2003), an essential coalition for group blocking of a matching consists of a department and a group of students.

- **Proposition** (Karakaya and Koray (2003)). A matching μ is core stable if and only if μ is individually rational and there does not exist a group (consisting of a department d and a group of students $\widetilde{S} \subseteq \mathcal{S}$) $(d, \widetilde{S})$ which blocks μ.

We say that a core stable matching is ***departments-optimal*** if every department likes it at least as well as any other core stable matching, and a core stable matching is ***students-optimal*** if every student likes it at least as well as any other core stable matching.

Karakaya and Koray (2003) showed that there exists neither a departments-optimal nor a students-optimal matching for the graduate admissions problem with quota and budget constraints.

Definition 58 We say that a matching μ is Pareto dominated by another matching $\widetilde{\mu}$ if and only if

1. for all $i \in (\mathcal{S} \cup \mathcal{D})$, $\widetilde{\mu} R_i \mu$, and
2. for some $i \in (\mathcal{S} \cup \mathcal{D})$, $\widetilde{\mu} P_i \mu$.

Definition 59 A matching μ is **Pareto optimal** if and only if there does not exist another matching which Pareto dominates μ.

Relations between core stability, pairwise stability and Pareto optimality

It is clear that if a matching is core stable then it is both pairwise stable and Pareto optimal. However, a matching which is both pairwise stable and Pareto optimal may not be core stable. We note that a pairwise stable matching need not be Pareto optimal, and a Pareto optimal matching need not be pairwise stable.

5.3 Graduate admission algorithms

Karakaya and Koray (2003) constructed the *departments proposing graduate admission algorithm* $(DPGAA)$, and showed that when the algorithm $DPGAA$ stops then the resulting matching is core stable. They proved that the departments proposing algorithm stops for a given problem if and only if no cycle occurs in the algorithm, i.e., a finite sequence of matchings does not repeat itself infinitely many times in the algorithm. They also showed that the algorithm $DPGAA$ may not stop while there exists a core stable matching.

In this section we will define the students proposing algorithm, to which we will refer to as the *students proposing graduate admission algorithm* ($SPGAA$). We note that the departments proposing and the students proposing algorithms are extensions of the Gale-Shapley algorithm for the graduate admissions problem. Each algorithm is a centralized algorithm, i.e., the departments' and students' preferences are assumed to be known to a planner (or to a computer program) who matches students with departments according to the rule of the algorithms. Hence, there is no agent who behaves strategically to manipulate the algorithm.

We will show that when the algorithm $SPGAA$ stops then the resulting matching is core stable (Proposition 19). However the departments proposing and the students proposing algorithms may not stop for some graduate admissions problems. To clarify this situation, we will give three examples. In Example 6, the algorithms $DPGAA$ and $SPGAA$ do not stop and there is no core stable matching. In Example 7, the algorithm $DPGAA$ does not stop, but the algorithm $SPGAA$ stops (hence there is a core stable matching). In Example 8, the algorithm $SPGAA$ does not stop, but the algorithm $DPGAA$ stops (hence there is a core stable matching).

We will show that the students proposing algorithm stops for a given problem if and only if no cycle occurs in the algorithm (Proposition 21). We will also show that the algorithms $DPGAA$ and $SPGAA$ are not complementary in the sense that for a given graduate admission problem if its core is non-empty then at least one of the algorithms stops, i.e., there exist graduate admissions problems for which there are core stable matchings, while neither of the two algorithms stops (Example 9).

We will also define another algorithm which is a mix of the algorithms $DPGAA$ and $SPGAA$, referred to as the *mix algorithm*. We show that there exists a graduate admission problem with non-empty set of core stable matchings, but by using the mix algorithm we reach a matching which is not core stable (Example 10), and there is a graduate admission problem such that by using the mix algorithm we reach a problem at some period which is equal to the one that we have from previous period, but there is a core stable matching for the given problem. (Example 11).

5.3.1 The departments proposing graduate admission algorithm

We will now define the departments proposing algorithm following Karakaya and Koray (2003).

Time is measured discretely in the algorithm. Let $m_{sd}(t)$ denote the offer that department d makes to student s at time t.

According to the scenario behind the algorithm, given a graduate admission problem and what offers are permitted, at each time t, department d will maximize its total benefit $Y_t^d = y^d(S_t^d) + \epsilon_t^d$ when it makes a permitted offer to a group of students S_t^d such that its quota and budget constraints are satisfied, i.e., $|\ (S_t^d)\ | \leq q_d$ and $\sum_{s \in \overline{S}_t^d} m_{sd}(t) \leq b_d$. Students who have taken offer(s) accept at most one offer and reject the others. Then, at the end of time t, department d is tentatively matched with the group of students who accepted its offers.

Now we can give the details of how the algorithm $DPGAA$ works.

$t = 1$. a) Each department d determines the group of students S_1^d that maximizes its total benefit subject to its quota and budget constraints with $m_{sd}(1) = \sigma_{sd}$ for all $s \in S_1^d$. That is, department d offers to students in S_1^d first their reservation prices.

b) Each students who has taken one or more offers accept at most one offer and reject the others.

c) Each department d tentatively accepts the group of students who accepted its offers. Let T_1^d denote the group of students who accepted department d's offers at time $t = 1$, $T_1^d \subseteq S_1^d$.[10]

Now, at the end of time $t = 1$ we have a matching μ_1 with $S_{\mu_1}^d = T_1^d$ for all $d \in \mathcal{D}$.

$t = 2$. a) Again each department d determines the group of students S_2^d that maximizes its total benefit subject to its constraints where the offers now be of the form:

$$m_{sd}(2) = \begin{cases} \sigma_{sd} + 1 & \text{if } s \in (S_1^d \setminus T_1^d) \\ \sigma_{sd} & \text{otherwise.} \end{cases}$$

[10] $S_1^d \setminus T_1^d$ is now the group of students who took an offer from department d and rejected it at $t = 1$.

b) Students who have taken one or more offers accept at most one offer and reject the others.

c) Department d tentatively accepts the group of students $T_2^d \subseteq S_2^d$ who accepted its offers.

In general, at time k,

$t = k$. a) Consider a student s to whom department d made offers before period k the last of which took place in period $\tilde{t}_s < k$. In case this offer was rejected by s because she accepted department $\widehat{d}$'s offer with which she got again matched at the end of period $k-1$, i.e., $s \in S^{\widehat{d}}_{\mu_{k-1}}$, call such a student a rejector of d prior to k. Let F_k^d denote the group of all rejectors of d prior to k.[11]

Each department d determines the group of students S_k^d solving the same kind of optimization problem as before, where the offers are now of the following form:

$$m_{sd}(k) = \begin{cases} \sigma_{sd} & \text{if } s \notin \bigcup_{t=1}^{t=k-1} S_t^d \\ m_{sd}(\tilde{t}_s) + 1 & \text{if } s \in F_k^d \\ m_{sd}(\tilde{t}_s) & \text{otherwise} \end{cases}$$

b) Students who have taken one or more offers accept at most one offer and reject the others.

c) Department d tentatively accepts the group of students $T_k^d \subseteq S_k^d$ who accepted its offers.

Stopping Rule

$t = t^\star$. The algorithm stops at time $t^\star$ if each department d makes offers to exactly the set of students who accepted its offers in the preceding period, i.e., if we have for all $d \in \mathcal{D}$, $S_{t^\star}^d = T_{t^\star-1}^d$ then the algorithm stops at time $t^\star$.

If the algorithm stops at $t^\star$ the final matching $\mu_{t^\star}$ is regarded as the outcome of the algorithm.

[11]Note that at $t = 1$, we have $F_1^d = \emptyset$ for all $d \in \mathcal{D}$, and at $t = 2$, we have $F_2^d = S_1^d \setminus T_1^d$ for all $d \in \mathcal{D}$.

Proposition 18 *(Karakaya and Koray (2003)). If the algorithm $DPGAA$ stops, then the final matching of the algorithm is core stable (and thus Pareto optimal).*

5.3.2 The students proposing graduate admission algorithm

Time is measured discretely in the algorithm. Let $m_{sd}(t)$ denote the offer that student s makes to department d at time t.

First note that the algorithm $SPGAA$ does not permit a student s to make offers to department d if $[a_d^s < 0]$ or $[a_d^s = 0$ and $\sigma_{sd} > 0]$.

Second, let us make a distinction between a *new offer* and a *holding offer*.

Let student s offered $m_{sd}(t)$ to department d at period t and department d accepted this offer. At the next period $t+1$, if s makes an offer $m_{sd}(t+1) = m_{sd}(t)$ to department d, this offer is called a **new offer**. Let student s offered $m_{sd}(t)$ to department d at period t and department d rejected this offer. At the next period $t+1$, if s makes an offer $m_{s\widetilde{d}}(t+1)$ to a department $\widetilde{d} \in (\mathcal{D} \setminus \{d\})$, this offer is called a **new offer**, and student s's offer $m_{sd}(t)$ made to department d and got rejected remains valid at this period $t+1$ as a **holding offer** $\ddot{m}_{sd}(t+1) = m_{sd}(t)$ to department d.

Let us explain why student s makes a new offer $m_{s\widetilde{d}}(t+1)$ to department $\widetilde{d}$ but not to department d at period $t+1$. If student s would make a new offer $m_{sd}(t+1)$ to department d at period $t+1$, the algorithm $SPGAA$ may require that $m_{sd}(t+1) = m_{sd}(t) - 1$. However, student s may prefer department $\widetilde{d}$ with $m_{s\widetilde{d}}(t+1)$ to department d with $m_{sd}(t+1)$, i.e., $[\widetilde{d}, m_{s\widetilde{d}}(t+1)] P_s [d, m_{sd}(t+1)]$.[12] So, in such a case, student s makes a new offer $m_{s\widetilde{d}}(t+1)$ to department $\widetilde{d}$ but not to department d at period $t+1$. Hence, student s makes following offers at period $t+1$:

- Student s makes a new offer $m_{s\widetilde{d}}(t+1)$ to department $\widetilde{d} \in (\mathcal{D} \setminus \{d\})$, and

- her last new offer $m_{sd}(t)$ made to department d and got rejected remains valid as a holding offer $\ddot{m}_{sd}(t+1) = m_{sd}(t)$ to department d.

[12]Also note that we have $[d, \ddot{m}_{sd}(t+1)] P_s [\widetilde{d}, m_{s\widetilde{d}}(t+1)]$ where $\ddot{m}_{sd}(t+1) = m_{sd}(t)$ is a holding offer to department d at period $t+1$.

According to the scenario behind our algorithm $SPGAA$, given what offers are permitted, at each time t, students s makes at most one new offer to a department which is best for her given the permitted offers, and her last new offer made to a different department and got rejected stays valid as a holding offer. Note that a student s does not make an offer $m_{sd}(t)$ to department d at any time t in the algorithm if $m_{sd}(t) < \sigma_{sd}$. So, the minimal offer if student s makes to department d is equal to σ_{sd}. Note that if student s made offers to all departments her reservation prices and got rejected, then her last new offer remains valid as a holding offer.

Once students made offers to departments, each department d considers the group of students S_t^d who made offers to department d at period t, and accepts the offers of the group of students $T_t^d \subseteq S_t^d$ that maximizes its total benefit subject to its quota and budget constraints. Note that department d does not have any discrimination between new offers and holding offers. Then, each student s who has taken acceptance(s) tentatively accepts at most one of them and rejects the others.

Now we can give the details of how the algorithm $SPGAA$ works.

$t = 1$. a) Each student s makes an offer to her most preferred department d to which she is permitted to make offers, where $m_{sd}(1) = b_d$. That is, student s offers $m_{sd}(1) = b_d$ to department d, where $(d, b_d)P_s(\tilde{d}, b_{\tilde{d}})$ for any $\tilde{d} \neq d$ to which she can make offers. Note that at $t = 1$, there is no holding offer and each student s makes a new offer.

b) Let S_1^d denote the group of students who offered department d at $t = 1$. Each department d accepts the offers of the group of students $T_1^d \subseteq S_1^d$ that maximizes its total benefit subject to its quota and budget constraints.

As we have $m_{sd}(1) = b_d$ if s offers to d at $t = 1$, department d accepts the offer of student $s \in S_1^d$ with $(s, b_d)P_d(\widehat{s}, b_d)$ for any $\widehat{s} \in (S_1^d \setminus \{s\})$.

c) Student s who has taken acceptance(s) tentatively accepts at most one of them and rejects the others.

Now, at the end of time $t = 1$ we have a matching μ_1 with $S_{\mu_1}^d \subseteq T_1^d$. We have $S_{\mu_1}^d = T_1^d$ for all $d \in \mathcal{D}$ at period $t = 1$, since a student s can get at most one acceptance at $t = 1$.

$t = 2$. a) Each student s makes at most one new offer where the new offers be of the form:

$$m_{sd}(2) = \begin{cases} b_d - 1 & \text{if } s \in (S_1^d \setminus T_1^d) \\ b_d & \text{otherwise} \end{cases}$$

Note that if student s offered to department d and was rejected at period $t = 1$, and now (at $t = 2$) if she offers to another department $\tilde{d}$, her offer $m_{sd}(1)$ made to d and got rejected remains valid at period $t = 2$ as a holding offer $\ddot{m}_{sd}(2) = m_{sd}(1)$ to department d.

b) Each department d considers the group of students S_2^d who made new and holding offers to department d at period 2, and accepts the offers of the group of students $T_2^d \subseteq S_2^d$ that maximizes its total benefit subject to its quota and budget constraints.

c) Each student s who has taken acceptance(s) tentatively accepts at most one of them and rejects the others.

Now, at the end of period $t = 2$ we have a matching μ_2 such that for each department d, $S_{\mu_2}^d \subseteq T_2^d$.

In general, at time k,

$t = k$. a) Consider a department d that student s made some offers before period k, and the last new offer was made at period $\tilde{t}_d$ by s to d, $\tilde{t}_d < k$. In case this offer was rejected by department d because of the group of students $T_{\tilde{t}_d}^d$ and department d matched with $T_{\tilde{t}_d}^d$ at the end of period $k-1$, i.e., $S_{\mu_{k-1}}^d = T_{\tilde{t}_d}^d$.

We call such a department d a rejector of student s prior to period k. Let F_k^s denote all rejectors of student s prior to period k.[13]

Each student s makes at most one new offer where the new offers be of the form:

[13]For all $s \in \mathcal{S}$, $F_1^s = \emptyset$, and $F_2^s = \begin{cases} d & \text{if } s \in (S_1^d \setminus T_1^d) \\ \emptyset & \text{otherwise.} \end{cases}$

$$m_{sd}(k) = \begin{cases} b_d & \text{if } s \notin \bigcup_{t=1}^{t=k-1} S_t^d \\ m_{sd}(\tilde{t}_d) - 1 & \text{if } d \in F_k^s \\ m_{sd}(\tilde{t}_d) & \text{otherwise} \end{cases}$$

Note that the last new offer student s made to some department and got rejected remains valid as a holding offer if she makes a new offer to another department at this period k.

b) Each department d accepts the offers of the group of students $T_k^d \subseteq S_k^d$ that maximizes its total benefit subject to its quota and budget constraints.

c) Each student s who has taken acceptance(s) tentatively accepts at most one of them and rejects the others.

Now, at the end of time $t = k$ we have a matching μ_k with $S_{\mu_k}^d \subseteq T_k^d$.

Stopping Rule

$t = t^\star$: The algorithm stops at time $t^\star$ if each student s makes same offer(s) (new and/or holding) to exactly the same department(s) that she offered in the preceding period. That is the algorithm stops at $t^\star$ if for all $d \in \mathcal{D}$ we have $S_{t^\star}^d = S_{t^\star-1}^d$ with for any $s \in S_{t^\star}^d$, $m_{sd}(t^\star) = m_{sd}(t^\star - 1)$ if s made new offers to d at periods $t^\star - 1$ and $t^\star$, and $\ddot{m}_{sd}(t^\star) = \ddot{m}_{sd}(t^\star - 1)$ if s made holding offers to d at periods $t^\star - 1$ and $t^\star$.

If the algorithm stops at $t^\star$ the final matching $\mu_{t^\star}$ is regarded as the outcome of the algorithm.

Proposition 19 *If the algorithm $SPGAA$ stops, then the final matching of the algorithm is core stable (and thus Pareto optimal).*

Proof Assume that the algorithm $SPGAA$ stops. Let the algorithm stop at time $t^\star$ with $\mu_{t^\star}$ denoting the final matching of the algorithm. So we have that, for all $d \in \mathcal{D}$, $S_{t^\star}^d = S_{t^\star-1}^d$ such that for any $d \in \mathcal{D}$ and for any $s \in S_{t^\star}^d$ we have $m_{sd}(t^\star) = m_{sd}(t^\star - 1)$ if s made new offers to d at periods $t^\star - 1$ and $t^\star$, and $\ddot{m}_{sd}(t^\star) = \ddot{m}_{sd}(t^\star - 1)$ if s made holding offers to d at periods $t^\star - 1$ and $t^\star$. We abuse notation that we use $\mu^\star$ for $\mu_{t^\star}$.

It is clear that $\mu^\star$ is individually rational. Now suppose that $\mu^\star$ is not core stable. So, there is a group $(d, \widetilde{S})$ which blocks $\mu^\star$. So we have that

1. for all $s \in \widetilde{S}$, $\mu_1^\star(s) \neq d$,
2. for all $s \in \widetilde{S}$, $(d, \widetilde{m}_{sd}) P_s (\mu_1^\star(s), m^{\mu^\star}_{s\mu_1^\star(s)})$,
3. $[(S^d_{\mu^\star} \setminus T) \cup \widetilde{S}, \widehat{m}^d] P_d [S^d_{\mu^\star}, m^d_{\mu^\star}]$ for some $T \subseteq S^d_{\mu^\star}$.

Note that the algorithm requires that each student $s \in \widetilde{S}$ make the offers $\widetilde{m}_{sd}$ to department d at period $t^\star$. Now, there are three possible cases.

Case 1. If there is a student $s \in \widetilde{S}$ such that $s \notin \bigcup_{t=1}^{t=t^\star-1} S^d_t$, then we have $\widetilde{m}_{sd} = b_d$.

Case 2. If $d \in F^s_{t^\star}$ (that is department d is a rejector of student $s \in \widetilde{S}$ prior to period $t^\star$), and let $\tilde{t}_d$ denote the period that student s made a new offer to department d the last time before period $t^\star$. Now $\widetilde{m}_{sd} = m_{sd}(\tilde{t}_d) - 1$ if s makes a new offer to d at $t^\star$ and $\widetilde{m}_{sd} = m_{sd}(\tilde{t}_d)$ if s makes a holding offer to d at $t^\star$.

Case 3. If $d \notin F^s_{t^\star}$ and let $s \in \widetilde{S}$ made a new offer to d at period $\tilde{t}_d$ the last time before $t^\star$. Now, $\widetilde{m}_{sd} = m_{sd}(\tilde{t}_d)$.

Therefore each student $s \in \widetilde{S}$ would make the offers $\widetilde{m}_{sd}$ to department d (by 2), and department d would accept the offers of the group of students $(S^d_{\mu^\star} \setminus T) \cup \widetilde{S}$ (by 3), i.e., $T^d_{t^\star} = (S^d_{\mu^\star} \setminus T) \cup \widetilde{S}$, and each student in $T^d_{t^\star}$ would accept department d's acceptance. So, the group of students $\widetilde{S}$ and department d would match at the end of period $t^\star$, in contradiction with (1). Hence $\mu^\star$ is core stable. □

Proposition 18 (and respectively, Proposition 19) shows that if the algorithm $DPGAA$ stops (respectively, if the algorithm $SPGAA$ stops) then the resulting matching is core stable. However, Karakaya and Koray (2003) provided a graduate admission problem for which the algorithm $DPGAA$ does not stop. We will now see that for the same problem the algorithm $SPGAA$ does not stop. Hence, there is a graduate admission problem that neither the algorithm $DPGAA$ nor the algorithm $SPGAA$

stops. The following example taken from Karakaya and Koray (2003) demonstrates this situation.[14]

Example 6 Neither $DPGAA$ nor $SPGAA$ stops and there is no core stable matching

Let $\mathcal{D} = \{A, B\}$, $\mathcal{S} = \{1, 2, 3\}$, $q_A = 1$, $q_B = 2$, $b_A = 440$, $b_B = 1075$, and the qualification levels and reservation prices of the students are as given in table 5.1.

$a_A^1 = 7$	$a_B^1 = 11$
$a_A^2 = 0$	$a_B^2 = 15$
$a_A^3 = 8$	$a_B^3 = 12$
$\sigma_{1A} = 400$	$\sigma_{1B} = 300$
$\sigma_{2A} = 440$	$\sigma_{2B} = 1075$
$\sigma_{3A} = 400$	$\sigma_{3B} = 700$

Table 5.1: Qualification levels and reservation prices of students for example 6

If we apply either the algorithm $DPGAA$ or the algorithm $SPGAA$, then following finite sequence of matchings $(\mu_{\bar{t}}, \mu_{\bar{t}+1}, \mu_{\bar{t}+2}, \mu_{\bar{t}+3})$ repeats itself infinitely many times in the algorithms (for different periods $\bar{t}$ for $DPGAA$ and $SPGAA$):[15]

$$\mu_{\bar{t}} = \begin{pmatrix} 1 & 2 & 3 \\ A & \emptyset & B \\ 435 & 0 & 741 \end{pmatrix}, \mu_{\bar{t}+1} = \begin{pmatrix} 1 & 2 & 3 \\ A & B & \emptyset \\ 435 & 1075 & 0 \end{pmatrix},$$

$$\mu_{\bar{t}+2} = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & A \\ 0 & 1075 & 440 \end{pmatrix}, \mu_{\bar{t}+3} = \begin{pmatrix} 1 & 2 & 3 \\ B & \emptyset & B \\ 334 & 0 & 741 \end{pmatrix}.$$

Hence neither the algorithm $DPGAA$ nor the algorithm $SPGAA$ stops in this example. We note that there is no core stable matching in this example, since there

[14]This example is a modification of the example of Mongell and Roth (1986).

[15]How the algorithm $DPGAA$ works for this example can be found in Karakaya and Koray (2003), and how the algorithm $SPGAA$ works for this example is provided in the Appendix.

exists neither a core stable matching such that student 2 is matched with a department, nor a core stable matching under which she is unmatched.

In Example 6, both the algorithms $DPGAA$ and $SPGAA$ do not stop since a finite sequence of matchings repeats itself infinitely many times in the algorithms, that is a cycle occurs both in $DPGAA$ and $SPGAA$.

Definition 60 We say that **a cycle occurs in the algorithm** if there is a finite sequence of matchings $(\mu_{t_0}, \mu_{t_0+1}, \ldots, \mu_{\bar{t}-1})$ ($t_0 < \bar{t}$) such that, for every $t > t_0$, $\mu_t = \mu_{t_0+r}$, where $0 \leq r < \bar{t} - t_0$ and $t \equiv r$ (mod $\bar{t} - t_0$).

Proposition 20 *(Karakaya and Koray (2003)). The algorithm $DPGAA$ stops if and only if no cycle occurs in the algorithm.*

We have seen it is possible that the algorithm $SPGAA$ does not stop. But is it also possible that the algorithm $SPGAA$ does not stop while no cycle occurs in the algorithm?

Proposition 21 *The algorithm $SPGAA$ stops if and only if no cycle occurs in the algorithm.*

Proof It is obvious that if the algorithm $SPGAA$ stops, then no cycle occurs in the algorithm.

For the other part of the proof, assume that the algorithm $SPGAA$ does not stop. Let $\boldsymbol{M}^{SPGAA}$ denote the set of all matchings that occur in the algorithm. Since the set of all individually rational matchings for a given graduate admission problem is finite and we have an individually rational matching in the end of every period in the algorithm $SPGAA$, we have $\boldsymbol{M}^{SPGAA}$ is finite.

Let $\boldsymbol{O}^{SPGAA}$ denote the set of all pairs $(s, d) \in \mathcal{S} \times \mathcal{D}$ such that s makes an offer to d in the algorithm. In the algorithm, there is a period $\underline{t}$ such that for any $(s, d) \in \boldsymbol{O}$, student s proposes its minimal offer to department d at any $t < \underline{t}$ such that s makes an offer to d in period t. We let $\underline{m}_{sd}$ denote the minimal offer that student s makes to department d in the algorithm. So, for any $t > \underline{t}$, we have $m_{sd}(t) = \underline{m}_{sd}$, if s makes an offer to d at period t.

Since M^{SPGAA} is finite and the algorithm does not stop, there is a matching $\overline{\mu}$ such that it occurs infinitely many times in the algorithm. Let t_k be a period such that $t_k > \underline{t}$ and $\mu_{t_k} = \overline{\mu}$.

Claim 1. It is impossible that for all periods $t > t_k$, $\mu_t = \overline{\mu}$.

Proof of claim 1. Suppose not, i.e., suppose that for all times $t > t_k$, $\mu_t = \overline{\mu}$. Since the algorithm does not stop, at each period t, there is at least one department d such that $S_t^d \neq S_{t-1}^d$. Moreover, for all times $t > t_k$, we have, for any $(s, d) \in O^{SPGAA}$, $m_{sd}(t) = \underline{m}_{sd}$ if d gets an offer from s at t. However, this fact together with the finiteness of $\mathcal{D}$ and $\mathcal{S}$ implies that there is some time $t^\star > t_k$ such that for all $d \in \mathcal{D}$, $S_{t^\star}^d = S_{t^\star-1}^d$, in contradiction with that the algorithm does not stop. Hence it is impossible for all times $t > t_k$ to have $\mu_t = \overline{\mu}$. This completes the proof of Claim 1.

Claim 1 implies that there is a matching $\widetilde{\mu}$ which is different than $\overline{\mu}$ such that $\mu_{t_k+1} = \widetilde{\mu}$. A claim for $\widetilde{\mu}$ similar to Claim 1 can be proved, so we can say that it is impossible for all times $t > t_k + 1$ to have $\mu_t = \widetilde{\mu}$. So, there is another matching $\widehat{\mu}$ which is different than $\widetilde{\mu}$ such that $\mu_{t_k+2} = \widehat{\mu}$. As matching $\overline{\mu}$ occurs infinitely many times in the algorithm, at some further time, again we have matching $\overline{\mu}$. That is, there is a time $t_l > t_k$ such that $\mu_{t_l} = \overline{\mu}$. Hence, we get a finite sequence of matchings $(\overline{\mu},\widetilde{\mu},\widehat{\mu},\ldots,\mu_{t_l-1})$. Let C denote this finite sequence of matchings.

Claim 2. $\mu_{t_l+1} = \widetilde{\mu}$.

Proof of claim 2. Note that $\mu_{t_k} = \overline{\mu}$ and $\mu_{t_k+1} = \widetilde{\mu}$ such that $\widetilde{\mu}$ is different than $\overline{\mu}$. So, there exists at least a department d and a student s such that $\overline{\mu}_1(s) \neq d$ but $\widetilde{\mu}_1(s) = d$. That is, student s makes an offer to department d at period $t_k + 1$ and d accepts this offer, i.e., $s \in T^d(t_k+1)$, and student s accepts department d's acceptance, so that we have $\widetilde{\mu}_1(s) = d$ in the end of period $t_k + 1$.

We will show that the algorithm requires that student s makes an offer to department d at period $t_l + 1$. We have two cases to consider that either student s makes a new offer or a holding offer to department d at period $t_k + 1$.

If student s makes a new offer to department d at period $t_k + 1$, then we have $d \notin F^s_{t_k+1}$. Since $\mu_{t_k} = \mu_{t_l} = \overline{\mu}$ and $d \notin F^s_{t_k+1}$, we have $d \notin F^s_{t_l+1}$, i.e., the algorithm requires that student s makes a new offer to department d at period $t_l + 1$. So, student s makes a new offer to department d at period $t_l + 1$ and d accepts this offer, i.e., $s \in T^d(t_l + 1)$, and student s accepts department d's acceptance.

If student s makes a holding offer to department d at period $t_k + 1$, then we have $d \in F^s_{t_k+1}$. Since $\mu_{t_k} = \mu_{t_l} = \overline{\mu}$ and $d \in F^s_{t_k+1}$, we have $d \in F^s_{t_l+1}$, i.e., the algorithm requires that student s makes a holding offer to department d at period $t_l + 1$. So, at period $t_l + 1$, student s makes a holding offer to department d and d accepts this offer, i.e., $s \in T^d(t_l + 1)$, and student s accepts department d's acceptance.

Note that this is true for all pairs (s, d) such that $\overline{\mu}_1(s) \neq d$ but $\widetilde{\mu}_1(s) = d$. So, we have $\mu_{t_l+1} = \mu_{t_k+1} = \widetilde{\mu}$, which completes the proof of Claim 2.

A claim for each matching in C similar to Claim 2 can be proved. Hence, C repeats itself infinitely many times in the algorithm $SPGAA$. This completes the proof of proposition. □

In Example 6, the algorithm $DPGAA$ does not stop and there is no core stable matching. The following example taken from Karakaya and Koray (2003) shows that it is possible the algorithm $DPGAA$ does not stop but the algorithm $SPGAA$ stops and hence there is a core stable matching.

Example 7 The algorithm $DPGAA$ does not stop but the algorithm $SPGAA$ stops

Let $\mathcal{D} = \{A, B, C, D\}$ be the set of departments, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ the set of students, where the quotas and budgets of the departments are as follows: $q_A = 1$, $q_B = 2$, $q_C = 1$, $q_D = 2$; $b_A = 440$, $b_B = 1075$, $b_C = 440$, $b_D = 1075$. The qualification levels and reservation prices of the students are as given in table 5.2.

It is shown in Karakaya and Koray (2003) that if we apply the algorithm $DPGAA$, then a cycle occurs consisting of the following three matchings:

$a^1_A=$	7	$a^1_B=$	11	$a^1_C=$	4	$a^1_D=$	0
$a^2_A=$	0	$a^2_B=$	15	$a^2_C=$	0	$a^2_D=$	2
$a^3_A=$	8	$a^3_B=$	12	$a^3_C=$	0	$a^3_D=$	1
$a^4_A=$	4	$a^4_B=$	0	$a^4_C=$	7	$a^4_D=$	11
$a^5_A=$	0	$a^5_B=$	2	$a^5_C=$	0	$a^5_D=$	15
$a^6_A=$	0	$a^6_B=$	1	$a^6_C=$	8	$a^6_D=$	12
$\sigma_{1A}=$	400	$\sigma_{1B}=$	300	$\sigma_{1C}=$	-500	$\sigma_{1D}=$	440
$\sigma_{2A}=$	440	$\sigma_{2B}=$	1075	$\sigma_{2C}=$	400	$\sigma_{2D}=$	-500
$\sigma_{3A}=$	400	$\sigma_{3B}=$	700	$\sigma_{3C}=$	420	$\sigma_{3D}=$	-500
$\sigma_{4A}=$	-500	$\sigma_{4B}=$	450	$\sigma_{4C}=$	400	$\sigma_{4D}=$	300
$\sigma_{5A}=$	400	$\sigma_{5B}=$	-500	$\sigma_{5C}=$	440	$\sigma_{5D}=$	1075
$\sigma_{6A}=$	420	$\sigma_{6B}=$	-500	$\sigma_{6C}=$	400	$\sigma_{6D}=$	700

Table 5.2: Qualification levels and reservation prices of students for example 7

$$\mu_{\bar{t}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & \emptyset & B & C & \emptyset & D \\ 435 & 0 & 741 & 435 & 0 & 741 \end{pmatrix}, \mu_{\bar{t}+1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & D & \emptyset & C & B & \emptyset \\ 435 & -500 & 0 & 435 & -500 & 0 \end{pmatrix},$$

$$\mu_{\bar{t}+2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \emptyset & D & B & \emptyset & B & D \\ 0 & -500 & 741 & 0 & -500 & 741 \end{pmatrix}.$$

So, the algorithm $DPGAA$ does not stop for this problem. Although the algorithm $DPGAA$ does not stop, the algorithm $SPGAA$ stops and the following matching $\widehat{\mu}$ is the outcome of the algorithm $SPGAA$:

$$\widehat{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ C & D & D & A & B & B \\ 440 & 1075 & 0 & 440 & 1075 & 0 \end{pmatrix}.$$

Note that the matching $\widehat{\mu}$ is core stable since there is no student who wants to form a blocking coalition with any department, i.e., for all $s \in \mathcal{S}$ we have $\big(\widehat{\mu}_1(s), m^{\widehat{\mu}}_{s\widehat{\mu}_1(s)}\big) P_s(d, b_d)$ for any $d \in (\mathcal{D} \setminus \{\widehat{\mu}_1(s)\})$. Hence, it is possible that the algorithm $DPGAA$ does not stop, but the algorithm $SPGAA$ stops and hence there is a core stable matching.

The algorithm $SPGAA$ does not stop and there is no core stable matching in Example 6. The following example shows it is also possible that the algorithm $SPGAA$ does not stop but the algorithm $DPGAA$ stops and hence there is a core stable matching.

Example 8 The algorithm $SPGAA$ does not stop but the algorithm $DPGAA$ stops

Let $\mathcal{D} = \{E, F, G, H\}$ be the set of departments, $\mathcal{S} = \{7, 8, 9, 10, 11, 12\}$ the set of students, where the quotas and budgets of the departments are as follows: $q_E = 1$, $q_F = 2$, $q_G = 1$, $q_H = 2$; $b_E = 440$, $b_F = 1075$, $b_G = 440$, $b_H = 1075$. The qualification levels and reservation prices of the students are as given in table 5.3.

$a_E^7 = 7$	$a_F^7 = 6$	$a_G^7 = 10$	$a_H^7 = 0$
$a_E^8 = 0$	$a_F^8 = 15$	$a_G^8 = 6$	$a_H^8 = 0$
$a_E^9 = 8$	$a_F^9 = 11$	$a_G^9 = 0$	$a_H^9 = 27$
$a_E^{10} = 10$	$a_F^{10} = 0$	$a_G^{10} = 7$	$a_H^{10} = 6$
$a_E^{11} = 6$	$a_F^{11} = 0$	$a_G^{11} = 0$	$a_H^{11} = 15$
$a_E^{12} = 0$	$a_F^{12} = 27$	$a_G^{12} = 8$	$a_H^{12} = 11$
$\sigma_{7E} = 400$	$\sigma_{7F} = 300$	$\sigma_{7G} = 440$	$\sigma_{7H} = 1075$
$\sigma_{8E} = 400$	$\sigma_{8F} = 1000$	$\sigma_{8G} = 440$	$\sigma_{8H} = 1075$
$\sigma_{9E} = 400$	$\sigma_{9F} = 700$	$\sigma_{9G} = 440$	$\sigma_{9H} = 1075$
$\sigma_{10E} = 440$	$\sigma_{10F} = 1075$	$\sigma_{10G} = 400$	$\sigma_{10H} = 300$
$\sigma_{11E} = 440$	$\sigma_{11F} = 1075$	$\sigma_{11G} = 400$	$\sigma_{11H} = 1000$
$\sigma_{12E} = 440$	$\sigma_{12F} = 1075$	$\sigma_{12G} = 400$	$\sigma_{12H} = 700$

Table 5.3: Qualification levels and reservation prices of students for example 8

If we apply the algorithm $SPGAA$, then a cycle occurs consisting of the following four matchings:

$$\mu_{\bar{k}} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ E & \emptyset & F & G & \emptyset & H \\ 435 & 0 & 741 & 435 & 0 & 741 \end{pmatrix}, \mu_{\bar{k}+1} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ E & F & \emptyset & G & H & \emptyset \\ 435 & 1001 & 0 & 435 & 1001 & 0 \end{pmatrix},$$

$$\mu_{\bar{k}+2} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ \emptyset & F & E & \emptyset & H & G \\ 0 & 1001 & 440 & 0 & 1001 & 440 \end{pmatrix}, \mu_{\bar{k}+3} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ F & \emptyset & F & H & \emptyset & H \\ 334 & 0 & 741 & 334 & 0 & 741 \end{pmatrix}.$$

However, if we apply the algorithm $DPGAA$ it stops at the end of period two and following matching $\widetilde{\mu}$ is the outcome of the algorithm $DPGAA$:

$$\widetilde{\mu} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ G & \emptyset & H & E & \emptyset & F \\ 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}.$$

Note that each department is matched with its best group of students among all groups satisfying its constraints under matching $\widetilde{\mu}$, so there is no department which forms a blocking coalition with any set of students. Hence, $\widetilde{\mu}$ is core stable.

Hence, it is possible that the algorithm $SPGAA$ does not stop, but the algorithm $DPGAA$ stops and hence there is a core stable matching. Therefore, we cannot say that if the algorithm $SPGAA$ does not stop, then the set of core stable matchings is empty.

We have seen in Example 7 that the algorithm $DPGAA$ does not stop but the algorithm $SPGAA$ stops (hence there is a core stable matching), and in Example 8 that the algorithm $SPGAA$ does not stop but the algorithm $DPGAA$ stops (hence there is a core stable matching). Because of these examples we ask following question: Whether the departments proposing ($DPGAA$) and the students proposing ($SPGAA$) algorithms are complementary in the sense that for a given graduate admission problem if there is a core stable matching then at least one of the two algorithms stops? The next example answers this question that it is possible neither the algorithm $DPGAA$ nor the algorithm $SPGAA$ stops and there is a core stable matching.

Example 9 Neither $DPGAA$ nor $SPGAA$ stops and there is a core stable matching

We will construct an example of a graduate admission problem where its set of core stable matchings is non-empty and neither of the algorithms stops. Our example will be a union of previously constructed two examples: Examples 7 and 8. That is, we let $\mathcal{D} = \{A, B, C, D, E, F, G, H\}$ be the set of departments and $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ the set of students, where the quotas and budgets of the departments are as follows: $q_A = 1$, $q_B = 2$, $q_C = 1$, $q_D = 2$, $q_E = 1$, $q_F = 2$, $q_G = 1$, $q_H = 2$; $b_A = 440$, $b_B = 1075$, $b_C = 440$, $b_D = 1075$, $b_E = 440$, $b_F = 1075$, $b_G = 440$, $b_H = 1075$. The qualification levels and reservation prices of the students are as given at examples 7 and 8, and we assume that
for any $s \in \{1, 2, 3, 4, 5, 6\}$ and any $d \in \{E, F, G, H\}$, we have $a_d^s < 0$, and
for any $s \in \{7, 8, 9, 10, 11, 12\}$ and any $d \in \{A, B, C, D\}$, we have $a_d^s < 0$.[16]

So, the qualification levels and reservation prices of the students are as given in table 5.4.

Now we apply the algorithm $DPGAA$ to this problem:
Note that a department $d \in \{A, B, C, D\}$ does not make any offer to a student $s \in \{7, 8, 9, 10, 11, 12\}$ since $a_d^s < 0$ for any $s \in \{7, 8, 9, 10, 11, 12\}$ and any $d \in \{A, B, C, D\}$, and a department $d \in \{E, F, G, H\}$ does not make any offer to a student $s \in \{1, 2, 3, 4, 5, 6\}$ since $a_d^s < 0$ for any $s \in \{1, 2, 3, 4, 5, 6\}$ and any $d \in \{E, F, G, H\}$. Therefore, applying the algorithm $DPGAA$ to this problem is equivalent to applying it to examples 7 and 8 separately. We know that the algorithm $DPGAA$ does not stop for Example 7 and it stops for Example 8. So, it does not stop for this problem, i.e., if we apply the algorithm $DPGAA$ to this problem then a cycle occurs consisting of the following three matchings:

[16]Note that for any $s \in \{1, 2, 3, 4, 5, 6\}$ and any $d \in \{E, F, G, H\}$ (and for any $s \in \{7, 8, 9, 10, 11, 12\}$ and any $d \in \{A, B, C, D\}$), we let σ_{sd} be any integer satisfying our model assumptions that for each $s \in \mathcal{S}$ and each $d \in \mathcal{D}$, $\sigma_{sd} \leq b_d$, and for any $s \in \mathcal{S}$, $\sigma_{sd} = \sigma_{s\tilde{d}}$ if and only if $d = \tilde{d}$.

$a^1_A=7$	$a^1_B=11$	$a^1_C=4$	$a^1_D=0$
$a^2_A=0$	$a^2_B=15$	$a^2_C=0$	$a^2_D=2$
$a^3_A=8$	$a^3_B=12$	$a^3_C=0$	$a^3_D=1$
$a^4_A=4$	$a^4_B=0$	$a^4_C=7$	$a^4_D=11$
$a^5_A=0$	$a^5_B=2$	$a^5_C=0$	$a^5_D=15$
$a^6_A=0$	$a^6_B=1$	$a^6_C=8$	$a^6_D=12$
$\sigma_{1A}=400$	$\sigma_{1B}=300$	$\sigma_{1C}=-500$	$\sigma_{1D}=440$
$\sigma_{2A}=440$	$\sigma_{2B}=1075$	$\sigma_{2C}=400$	$\sigma_{2D}=-500$
$\sigma_{3A}=400$	$\sigma_{3B}=700$	$\sigma_{3C}=420$	$\sigma_{3D}=-500$
$\sigma_{4A}=-500$	$\sigma_{4B}=450$	$\sigma_{4C}=400$	$\sigma_{4D}=300$
$\sigma_{5A}=400$	$\sigma_{5B}=-500$	$\sigma_{5C}=440$	$\sigma_{5D}=1075$
$\sigma_{6A}=420$	$\sigma_{6B}=-500$	$\sigma_{6C}=400$	$\sigma_{6D}=700$
$a^7_E=7$	$a^7_F=6$	$a^7_G=10$	$a^7_H=0$
$a^8_E=0$	$a^8_F=15$	$a^8_G=6$	$a^8_H=0$
$a^9_E=8$	$a^9_F=11$	$a^9_G=0$	$a^9_H=27$
$a^{10}_E=10$	$a^{10}_F=0$	$a^{10}_G=7$	$a^{10}_H=6$
$a^{11}_E=6$	$a^{11}_F=0$	$a^{11}_G=0$	$a^{11}_H=15$
$a^{12}_E=0$	$a^{12}_F=27$	$a^{12}_G=8$	$a^{12}_H=11$
$\sigma_{7E}=400$	$\sigma_{7F}=300$	$\sigma_{7G}=440$	$\sigma_{7H}=1075$
$\sigma_{8E}=400$	$\sigma_{8F}=1000$	$\sigma_{8G}=440$	$\sigma_{8H}=1075$
$\sigma_{9E}=400$	$\sigma_{9F}=700$	$\sigma_{9G}=440$	$\sigma_{9H}=1075$
$\sigma_{10E}=440$	$\sigma_{10F}=1075$	$\sigma_{10G}=400$	$\sigma_{10H}=300$
$\sigma_{11E}=440$	$\sigma_{11F}=1075$	$\sigma_{11G}=400$	$\sigma_{11H}=1000$
$\sigma_{12E}=440$	$\sigma_{12F}=1075$	$\sigma_{12G}=400$	$\sigma_{12H}=700$

Table 5.4: Qualification levels and reservation prices of students for example 9

$$\mu_{\bar{t}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ A & \emptyset & B & C & \emptyset & D & G & \emptyset & H & E & \emptyset & F \\ 435 & 0 & 741 & 435 & 0 & 741 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix},$$

$$\mu_{\bar{t}+1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ A & D & \emptyset & C & B & \emptyset & G & \emptyset & H & E & \emptyset & F \\ 435 & -500 & 0 & 435 & -500 & 0 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix},$$

$$\mu_{\bar{t}+2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \emptyset & D & B & \emptyset & B & D & G & \emptyset & H & E & \emptyset & F \\ 0 & -500 & 741 & 0 & -500 & 741 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}.$$

Hence, the algorithm $DPGAA$ does not stop for this problem.

We now apply the algorithm $SPGAA$ to this problem:
A student $s \in \{1, 2, 3, 4, 5, 6\}$ is not permitted to make an offer to a department $d \in \{E, F, G, H\}$ in the algorithm $SPGAA$ since for any $s \in \{1, 2, 3, 4, 5, 6\}$ and any $d \in \{E, F, G, H\}$ we have $a_d^s < 0$, and a student $s \in \{7, 8, 9, 10, 11, 12\}$ is not permitted to make an offer to a department $d \in \{A, B, C, D\}$ since for any $s \in \{7, 8, 9, 10, 11, 12\}$ and any $d \in \{A, B, C, D\}$ we have $a_d^s < 0$. Hence applying the algorithm $SPGAA$ to this problem is equivalent to applying it to examples 7 and 8 separately. We know that the algorithm $SPGAA$ stops for Example 7 and it does not stop for Example 8. So, it does not stop for this problem, i.e., if we apply the algorithm $SPGAA$ to this problem then a cycle occurs consisting of the following four matchings:

$$\mu_{\bar{k}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & E & \emptyset & F & G & \emptyset & H \\ 440 & 1075 & 0 & 440 & 1075 & 0 & 435 & 0 & 741 & 435 & 0 & 741 \end{pmatrix},$$

$$\mu_{\bar{k}+1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & E & F & \emptyset & G & H & \emptyset \\ 440 & 1075 & 0 & 440 & 1075 & 0 & 435 & 1001 & 0 & 435 & 1001 & 0 \end{pmatrix},$$

$$\mu_{\bar{k}+2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & \emptyset & F & E & \emptyset & H & G \\ 440 & 1075 & 0 & 440 & 1075 & 0 & 0 & 1001 & 440 & 0 & 1001 & 440 \end{pmatrix},$$

$$\mu_{\bar{k}+3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & F & \emptyset & F & H & \emptyset & H \\ 440 & 1075 & 0 & 440 & 1075 & 0 & 334 & 0 & 741 & 334 & 0 & 741 \end{pmatrix}.$$

Hence, the algorithm $SPGAA$ does not stop for this problem.

Now, consider the following matching:

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & G & \emptyset & H & E & \emptyset & F \\ 440 & 1075 & 0 & 440 & 1075 & 0 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}.$$

We will show that the matching μ is core stable.[17] Suppose that μ is not core stable. Then there is a group $(\widetilde{d}, \widetilde{S})$ which blocks μ, where $\widetilde{d} \in \mathcal{D}$ and $\widetilde{S} \subseteq \mathcal{S}$. Note that each department $d \in \{E, F, G, H\}$ is matched with its best group of students among all groups satisfying its constraints under μ. So, there is no department $d \in \{E, F, G, H\}$ which forms a blocking coalition with any set of students. Hence, $\widetilde{d} \notin \{E, F, G, H\}$. So, $\widetilde{d} \in \{A, B, C, D\}$. There is no department $d \in \{A, B, C, D\}$ which forms a blocking coalition with any group of students $S \subseteq \{7, 8, 9, 10, 11, 12\}$, since for any $s \in \{7, 8, 9, 10, 11, 12\}$ and any $d \in \{A, B, C, D\}$, we have $a_d^s < 0$. So, we have that $\widetilde{S} \subseteq \{1, 2, 3, 4, 5, 6\}$. However, for each student $s \in \{1, 2, 3, 4, 5, 6\}$ we have $\big(\mu_1(s), m^{\mu}_{s\mu_1(s)}\big) P_s (d, b_d)$ for any $d \in \big(\{A, B, C, D\} \setminus \{\mu_1(s)\}\big)$. So, there does not exist a student $s \in \{1, 2, 3, 4, 5, 6\}$ which forms a blocking coalition with any $d \in \{A, B, C, D\}$. Hence, we also have $\widetilde{d} \notin \{A, B, C, D\}$, contradiction. So, the matching μ is core stable.

Hence, the algorithms $DPGAA$ and $SPGAA$ are not complementary in the sense that given a graduate admission problem if it has a core stable matching then at least one of the two algorithms stops, i.e., it is possible that neither $DPGAA$ nor $SPGAA$ stops for a given graduate admission problem while it has a core stable matching.

[17]Note that $\mu = \widehat{\mu} \cup \widetilde{\mu}$ where $\widehat{\mu}$ is the outcome of the algorithm $SPGAA$ when applied to Example 7 and $\widetilde{\mu}$ is the outcome of the algorithm $DPGAA$ when applied to Example 8.

Notice that that for any student $s \in \{7, 8, 9, 10, 11, 12\}$ we have $\mu_{\bar{t}}(s) = \mu_{\bar{t}+1}(s) = \mu_{\bar{t}+2}(s)$, and for any department $d \in \{E, F, G, H\}$ we have $S^d_{\mu_{\bar{t}}} = S^d_{\mu_{\bar{t}+1}} = S^d_{\mu_{\bar{t}+2}}$, where $(\mu_{\bar{t}}, \mu_{\bar{t}+1}, \mu_{\bar{t}+2})$ is the set of matchings which repeats infinitely many times if we apply the algorithm $DPGAA$ to Example 9. That is, we can say that the positions of the set of students $\{7, 8, 9, 10, 11, 12\}$ and the set of departments $\{E, F, G, H\}$ are constant in the cycle that occurs in the algorithm $DPGAA$. Let $\overline{\mathcal{S}} \subsetneq \mathcal{S}$ denote the set of students such that their positions are constant in the cycle and $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$ the set of students such that their positions are inconstant in the cycle.[18] In Example 9, we have $\overline{\mathcal{S}} = \{7, 8, 9, 10, 11, 12\}$ and $\widetilde{\mathcal{S}} = \{1, 2, 3, 4, 5, 6\}$ when we apply the algorithm $DPGAA$. Similarly, we let $\overline{\mathcal{D}} \subsetneq \mathcal{D}$ denote the set of departments such that their positions are constant in the cycle and $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$ the set of departments such that their positions are inconstant in the cycle.[19] In Example 9, we have $\overline{\mathcal{D}} = \{E, F, G, H\}$ and $\widetilde{\mathcal{D}} = \{A, B, C, D\}$ when we apply the algorithm $DPGAA$.

When the algorithm $SPGAA$ is applied to Example 9, we have $\overline{\mathcal{S}} = \{1, 2, 3, 4, 5, 6\}$, $\widetilde{\mathcal{S}} = \{7, 8, 9, 10, 11, 12\}$, $\overline{\mathcal{D}} = \{A, B, C, D\}$ and $\widetilde{\mathcal{D}} = \{E, F, G, H\}$.

Hence, given a problem if $DPGAA$ or $SPGAA$ does not stop then we can partition the given problem into the constant part $(\overline{\mathcal{S}}, \overline{\mathcal{D}})$ and the inconstant part $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$ by using the cycle that occurs in the algorithm.

We know that there is a core stable matching for Example 9, however both $DPGAA$ and $SPGAA$ do not stop. Now observe that we can find the core stable matching μ by applying an algorithm which is a mix of the algorithms $DPGAA$ and $SPGAA$.

First, we apply the algorithm $DPGAA$ to Example 9 which we know that it does not stop, then we determine the constant part of the cycle that occurs in $DPGAA$.

[18] It is possible that $\overline{\mathcal{S}} = \emptyset$ and $\widetilde{\mathcal{S}} = \mathcal{S}$, e.g., Example 6.

[19] There may exist a department d such that for some non-empty group of students $\bar{S} \subsetneq \mathcal{S}$ we have $\mu_t(\bar{s}) = (d, m_{\bar{s}d}(t))$ for all $\bar{s} \in \bar{S}$ and all periods t in the cycle, and for some periods t in the cycle we also have $\bar{S} \subsetneq S^d_{\mu_t}$. That is, in all periods of the cycle department d is matched with a group students such that their positions are constant, and at some periods of the cycle department d is matched with some students whose positions are inconstant. If such a case occurs, we consider department d as two departments $\bar{d}$ and $\acute{d}$ where the quotas and budgets are as follows: For department $\bar{d}$, $q_{\bar{d}} =\mid \bar{S} \mid$, $b_{\bar{d}} = \sum_{\bar{s} \in \bar{\bar{S}}} m_{\bar{s}d}(t)$ where $\bar{\bar{S}} = \{\bar{s} \in \bar{S} \mid m_{\bar{s}d}(t) > 0\}$, and for department $\acute{d}$, $q_{\acute{d}} = q_d - q_{\bar{d}}$, $b_{\acute{d}} = b_d - b_{\bar{d}}$. Now, $\bar{d} \in \overline{\mathcal{D}}$ and $\acute{d} \in \widetilde{\mathcal{D}}$.

The constant part consists of the set students $\overline{\mathcal{S}} = \{7, 8, 9, 10, 11, 12\}$ and the set of departments $\overline{\mathcal{D}} = \{E, F, G, H\}$, and this constant part has a matching $\widetilde{\mu} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ G & \emptyset & H & E & \emptyset & F \\ 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}$. Note that the matching $\widetilde{\mu}$ is core stable for the constant part $(\bar{\mathcal{S}}, \bar{\mathcal{D}})$, i.e., there is no department $d \in \bar{\mathcal{D}}$ and group of students $S \subseteq \bar{\mathcal{S}}$ such that (d, S) blocks the matching $\widetilde{\mu}$. Secondly, we determine the inconstant part of the cycle which occurs in $DPGAA$. The inconstant part consists of the set students $\widetilde{\mathcal{S}} = \{1, 2, 3, 4, 5, 6\}$ and the set of departments $\widetilde{\mathcal{D}} = \{A, B, C, D\}$. Now, we apply the algorithm $SPGAA$ to this inconstant part $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$. It stops and the resulting matching is $\widehat{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ C & D & D & A & B & B \\ 440 & 1075 & 0 & 440 & 1075 & 0 \end{pmatrix}$ which is core stable for the inconstant part $(\tilde{\mathcal{S}}, \tilde{\mathcal{D}})$. Now, the union of the matchings $\widetilde{\mu}$ and $\widehat{\mu}$ gives us the matching μ which is core stable for Example 9. That is, we found the core stable matching μ by using an algorithm which is a mix of the algorithms $DPGAA$ and $SPGAA$.

Similarly, we can reach the core stable matching μ by first applying $SPGAA$. That is, we first apply the algorithm $SPGAA$ to Example 9 and determine the constant part of the cycle which consists of $\overline{\mathcal{S}} = \{1, 2, 3, 4, 5, 6\}$ and $\overline{\mathcal{D}} = \{A, B, C, D\}$. This constant part has a matching $\widehat{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ C & D & D & A & B & B \\ 440 & 1075 & 0 & 440 & 1075 & 0 \end{pmatrix}$ which is core stable for $(\overline{\mathcal{S}}, \overline{\mathcal{D}})$. Secondly, we determine the inconstant part of the cycle which consists of $\widetilde{\mathcal{S}} = \{7, 8, 9, 10, 11, 12\}$ and $\widetilde{\mathcal{D}} = \{E, F, G, H\}$, and apply the algorithm $DPGAA$ to this inconstant part. It stops, and the outcome is the matching $\widetilde{\mu} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ G & \emptyset & H & E & \emptyset & F \\ 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}$ which is core stable for the inconstant part $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$. Now, we have $\mu = \widehat{\mu} \cup \widetilde{\mu}$.

We formally define the mix algorithm at the next section.

5.3.3 The mix algorithm

We know that for a given graduate admission problem if the algorithm $DPGAA$ does not stop then a cycle occurs in the algorithm, and a cycle consists of two parts, a constant part (possibly empty) and an inconstant part. So, we construct another algorithm which is a mix of the algorithms $DPGAA$ and $SPGAA$, referred to as the *mix algorithm by applying $DPGAA$ first*, that works as follows: In the first period, we apply the algorithm $DPGAA$ to the given problem, if it stops then the mix algorithm stops, and the resulting matching is core stable. If it does not stop, then a cycle occurs in the algorithm $DPGAA$. We determine the constant part (with a matching for this part) and the inconstant part of the cycle that occurs in $DPGAA$, and move to next period. Now, in the second period, we apply the algorithm $SPGAA$ to the inconstant part that comes from the first period. If $SPGAA$ stops for this inconstant part then the mix algorithm stops, and the resulting matching for the mix algorithm is the union of the matchings that we have from first period (the matching for the constant part of the cycle which occurs in $DPGAA$) and second period (the matching we obtain by applying $SPGAA$ to the inconstant part). If $SPGAA$ does not stop for the inconstant part, then we determine the constant part (with a matching for this part) and the inconstant part of the cycle that occurs in $SPGAA$. If this inconstant part is equal to the one that we have from first period, then the mix algorithm stops. Otherwise, we move to next period. In the third period, we apply the algorithm $DPGAA$ to the inconstant part that we have from second period. That is, in general, we apply the algorithms $DPGAA$ and $SPGAA$ recursively to the problems that we have from inconstant parts of the cycles. By this algorithm, we either end up with a matching which is the union of all matchings that we have from constant parts, or a problem (inconstant part) which is equal to the one that we have from previous period. Similarly, by applying the algorithm $SPGAA$ in the first period, we can define the *mix algorithm by applying $SPGAA$ first*.

Now, we formally define the mix algorithm.

Let $\theta = (\mathcal{D}, \mathcal{S})$ simply denote a graduate admission problem. Given a graduate admission problem $\theta = (\mathcal{D}, \mathcal{S})$, the mix algorithm works as follows:

$t = 1$. Apply the algorithm $DPGAA$ to the problem $\theta = (\mathcal{D}, \mathcal{S})$. If it stops, then the mix algorithm stops and the resulting matching is core stable. If it does not stop then we go to next period.

$t = 2$. From the first period, the original problem $\theta = (\mathcal{D}, \mathcal{S})$ is partitioned into two problems which we get from constant and inconstant parts of the cycle that occurs in $DPGAA$. Let $\overline{\theta}_1 = (\overline{\mathcal{S}}, \overline{\mathcal{D}})$ denote the problem that we get from constant part and $\widetilde{\theta}_1 = (\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$ from the inconstant part. Let also that $\bar{\mu}_1$ denote the matching for the constant part $\overline{\theta}_1 = (\overline{\mathcal{S}}, \overline{\mathcal{D}})$. Now we apply the algorithm $SPGAA$ to the inconstant part $\widetilde{\theta}_1 = (\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$ that we obtained from the cycle which occurs in $DPGAA$.

If $SPGAA$ stops for $\widetilde{\theta}_1 = (\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$, then the mix algorithm stops. Let $\widetilde{\mu}_2$ denote the resulting matching of the algorithm $SPGAA$ for the problem $\widetilde{\theta}_1 = (\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$. Now we have a matching $\mu_2^\star = \bar{\mu}_1 \cup \widetilde{\mu}_2$ as an outcome of the mix algorithm.

If $SPGAA$ does not stop for $\widetilde{\theta}_1 = (\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$, then a cycle occurs in the algorithm and we determine the constant and inconstant parts of the cycle. Let $\overline{\theta}_2$ denote the constant part of the cycle with associated matching $\bar{\mu}_2$ and $\widetilde{\theta}_2$ the inconstant part. If $\widetilde{\theta}_2 = \widetilde{\theta}_1$ then the mix algorithm stops. Otherwise, we go to next period.

$t = 3$. We apply the algorithm $DPGAA$ to the problem $\widetilde{\theta}_2$ that we get from previous period. If $DPGAA$ stops for $\widetilde{\theta}_2$, then the mix algorithm stops. Let $\widetilde{\mu}_3$ denote the resulting matching of the algorithm $DPGAA$ for the problem $\widetilde{\theta}_2$. Now we have a matching $\mu_3^\star = \bar{\mu}_1 \cup \bar{\mu}_2 \cup \widetilde{\mu}_3$ as an outcome of the mix algorithm.

If $DPGAA$ does not stop for $\widetilde{\theta}_2$, then a cycle occurs in the algorithm. Let $\overline{\theta}_3$ denote the constant part of the cycle that occurs in $DPGAA$ with associated matching $\bar{\mu}_3$ and $\widetilde{\theta}_3$ denote the inconstant part. If $\widetilde{\theta}_3 = \widetilde{\theta}_2$ then the mix algorithm stops. Otherwise, we go to next period.

In general, at period $k > 1$ we have following:

$t = k$. If k is odd we apply the algorithm $DPGAA$ to the problem $\widetilde{\theta}_{k-1}$ that we get from previous period. If k is even we apply the algorithm $SPGAA$ to the problem $\widetilde{\theta}_{k-1}$. If the applied algorithm stops for $\widetilde{\theta}_{k-1}$, then the mix algorithm stops. Let $\widetilde{\mu}_k$ denote the resulting matching of the applied algorithm for the problem $\widetilde{\theta}_{k-1}$. Now we have a matching $\mu_k^\star = (\bigcup_{t=1}^{t=k-1} \bar{\mu}_t) \cup \widetilde{\mu}_k$ as an outcome of the mix algorithm. If the applied algorithm does not stop for $\widetilde{\theta}_{k-1}$, then we determine the constant part $\overline{\theta}_k$ (with associated matching $\bar{\mu}_k$) and the inconstant part $\widetilde{\theta}_k$ of the cycle. If $\widetilde{\theta}_k = \widetilde{\theta}_{k-1}$ then the mix algorithm stops. Otherwise, we go to next period.

Note that in the above mix algorithm, we apply the algorithm $DPGAA$ in the first period, so we call it the **mix algorithm applying** $DPGAA$ **first**. Similarly, we can define the **mix algorithm applying** $SPGAA$ **first** by applying the algorithm $SPGAA$ in the first period.

By using the mix algorithm, at some period $\bar{t}$, we have, either obtain a matching $\mu^{\star}_{\bar{t}} = (\bigcup_{t=1}^{t=\bar{t}-1} \bar{\mu}_t) \cup \widetilde{\mu}_{\bar{t}}$ as an outcome, or a problem $\widetilde{\theta}_{\bar{t}}$ such that $\widetilde{\theta}_{\bar{t}} = \widetilde{\theta}_{\bar{t}-1}$. For instance, for Example 9 we obtain a core stable matching by using the mix algorithm independent of which algorithm is applied first. For Example 6, if we use the mix algorithm by applying first either $DPGAA$ or $SPGAA$ then at period $t = 2$ we reach a problem $\widetilde{\theta}_2$ which is equivalent to the given problem and note that there is no core stable matching for Example 6. These observations lead us following conjectures:

Conjecture 1 *If we obtain a matching $\mu^{\star}_{\bar{t}} = (\bigcup_{t=1}^{t=\bar{t}-1} \bar{\mu}_t) \cup \widetilde{\mu}_{\bar{t}}$ as an outcome of the mix algorithm at some period $\bar{t}$, then the matching $\mu^{\star}_{\bar{t}}$ is core stable.*

Conjecture 2 *If we reach a problem $\widetilde{\theta}_{\bar{t}}$ such that $\widetilde{\theta}_{\bar{t}} = \widetilde{\theta}_{\bar{t}-1}$ by using the mix algorithm at some period $\bar{t}$, then there is no core stable matching for the given problem.*

Following example shows that Conjecture 1 is not correct.

Example 10 The mix algorithm produces a matching which is not core stable

We know that for Example 9 neither $DPGAA$ nor $SPGAA$ stops and there is a core stable matching. We also know that for the same problem the mix algorithm produces a core stable matching regardless of which algorithm is applied first. Now we construct an example where neither $DPGAA$ nor $SPGAA$ stops, but the mix algorithm produces a matching which is not core stable. The set of departments and their quotas and budgets are as in Example 9. The set of students and their qualification levels and reservation prices are again as in Example 9 except that the qualification level of student 10 for department A is 5, i.e., $a^{10}_A = 5$, and the reservation price of student 10 for department A is 420, i.e., $\sigma_{10A} = 420$.

We now apply the mix algorithm by applying $DPGAA$ first.

$t = 1$. We apply the algorithm $DPGAA$ to this problem. We will show that all departments behave in this example as they behaved in Example 9. Note that this is true for all departments other than department A since for this example everything is same to that of Example 9 except that we now have $a_A^{10} = 5$ and $\sigma_{10A} = 420$. So, we only need to show that department A behaves in $DPGAA$ for this example as it did in Example 9. To do so, we will show that department A never makes an offer to student 10 in $DPGAA$ for this example. Note that in Example 9, department A makes offers only to students 1 and 3 in $DPGAA$, and the maximal offer that department A makes to student 1 is $\overline{m}_{1A} = 435$ and the maximal offer that department A makes to student 3 is $\overline{m}_{3A} = 440$. Now, for department A we have $(\{1\}, \overline{m}_{1A}) P_A (\{10\}, \sigma_{10A})$ since $a_A^1 > a_A^{10}$, and $(\{3\}, \overline{m}_{3A}) P_A (\{10\}, \sigma_{10A})$ since $a_A^3 > a_A^{10}$. So, department A never makes an offer to student 10 for this example in $DPGAA$.

Hence, the algorithm $DPGAA$ for this problem gives us the same result as that of Example 9, i.e., $DPGAA$ does not stop and a cycle occurs consisting of three matchings $\mu_{\bar{t}}$, $\mu_{\bar{t}+1}$ and $\mu_{\bar{t}+2}$ that are given in Example 9.

Since the algorithm $DPGAA$ does not stop, we determine the constant and inconstant parts of the cycle. We know that the constant part consists of the set of departments $\overline{\mathcal{D}} = \{E, F, G, H\}$ and the set of students $\overline{\mathcal{S}} = \{7, 8, 9, 10, 11, 12\}$ and this constant part has a matching $\widetilde{\mu} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ G & \emptyset & H & E & \emptyset & F \\ 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}$. The inconstant part consists of the set students $\widetilde{\mathcal{S}} = \{1, 2, 3, 4, 5, 6\}$ and the set of departments $\widetilde{\mathcal{D}} = \{A, B, C, D\}$.

$t = 2$. We apply the algorithm $SPGAA$ to the inconstant part $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$. It stops and the resulting matching is $\widehat{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ C & D & D & A & B & B \\ 440 & 1075 & 0 & 440 & 1075 & 0 \end{pmatrix}$. Hence the mix algorithm by applying first $DPGAA$ stops in the end of period 2, and its outcome is the union of the matchings $\widetilde{\mu}$ and $\widehat{\mu}$ which is the matching μ given in example 9, i.e.,

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & G & \emptyset & H & E & \emptyset & F \\ 440 & 1075 & 0 & 440 & 1075 & 0 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}.$$

The matching μ is not core stable for this example, i.e., $(A, \{10\})$ blocks the matching μ with $\tilde{m}_{10A} = b_A = 440$ since $(A, 440)P_{10}(E, 440)$[20] and $(\{10\}, 440)P_A(\{4\}, 440)$[21].

Hence, the mix algorithm by applying $DPGAA$ first produces a matching which is not core stable.

Now, let us see what happens if we apply the algorithm $SPGAA$ first.

$t = 1$. We apply the algorithm $SPGAA$ to this problem. We will show that all students behave in this example as they behaved in Example 9. Note that this is true for all students other than student 10. So, we only need to show that student 10 behaves in $SPGAA$ for this example as she behaved in Example 9, i.e., we will show that student 10 never makes an offer to department A in $SPGAA$ for this example. Note that in Example 9, student 10 makes offers only to departments H and G in $SPGAA$, and the minimal offer that student 10 makes to department H is $\underline{m}_{10H} = 334$ and the minimal offer that student 10 makes to department G is $\underline{m}_{10G} = 435$. For student 10 we have $(H, \underline{m}_{10H})P_{10}(A, b_A)$[22] and $(G, \underline{m}_{10G})P_{10}(A, b_A)$[23]. So, student 10 never makes an offer to department A for this example in $SPGAA$.

Hence, the algorithm $SPGAA$ for this problem gives us the same result as that of Example 9, i.e., $SPGAA$ does not stop and a cycle occurs consisting of four matchings $\mu_{\bar{k}}$, $\mu_{\bar{k}+1}$, $\mu_{\bar{k}+2}$ and $\mu_{\bar{k}+3}$ that are given in Example 9. Since the algorithm $SPGAA$ does not stop, we determine the constant and inconstant parts of the cycle. We know that the constant part consists of the set of departments $\overline{\mathcal{D}} = \{A, B, C, D\}$ and the set of students $\overline{\mathcal{S}} = \{1, 2, 3, 4, 5, 6\}$ and this constant part has a matching

[20] Since $b_A - \sigma_{10A} = 440 - 420 = 20 > 0 = 440 - 440 = m^{\mu}_{10E} - \sigma_{10E}$.
[21] Since $a^{10}_A = 5 > 4 = a^4_A$
[22] Since $\underline{m}_{10H} - \sigma_{10H} = 334 - 300 = 34 > 20 = 440 - 420 = b_A - \sigma_{10A}$.
[23] Since $\underline{m}_{10G} - \sigma_{10G} = 435 - 400 = 35 > 20 = 440 - 420 = b_A - \sigma_{10A}$.

$\widehat{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ C & D & D & A & B & B \\ 440 & 1075 & 0 & 440 & 1075 & 0 \end{pmatrix}$. The inconstant part consists of the set students $\widetilde{\mathcal{S}} = \{7, 8, 9, 10, 11, 12\}$ and the set of departments $\widetilde{\mathcal{D}} = \{E, F, G, H\}$.

$t = 2$. We apply the algorithm $DPGAA$ to the inconstant part $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$. It stops and the resulting matching is $\widetilde{\mu} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ G & \emptyset & H & E & \emptyset & F \\ 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}$. Hence the mix algorithm by applying first $SPGAA$ stops and its outcome is the union of the matchings $\widehat{\mu}$ and $\widetilde{\mu}$ which is again the matching μ. We know that the matching μ is not core stable for this example. Hence, the mix algorithm by applying $SPGAA$ first produces a matching which is not core stable. However, the set of core stable matchings is non-empty for this problem, e.g., the following matching $\bar{\mu}$ is core stable:

$$\bar{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ A & D & B & D & B & C & G & \emptyset & H & E & \emptyset & F \\ 440 & 500 & 741 & 575 & 334 & 440 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}.$$

We will now show that Conjecture 2 is not correct, i.e., we will construct an example where it has a core stable matching, but by using the mix algorithm we reach a problem $\widetilde{\theta}_{\bar{t}}$ such that $\widetilde{\theta}_{\bar{t}} = \widetilde{\theta}_{\bar{t}-1}$.

Example 11 The mix algorithm reaches a problem $\widetilde{\theta}_{\bar{t}}$ such that $\widetilde{\theta}_{\bar{t}} = \widetilde{\theta}_{\bar{t}-1}$ and there is a core stable matching

Let $\mathcal{D} = \{A, B, C, D, E, F, G, H\}$ be the set of departments and the set of students be $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ where the quotas and budgets of the departments are as follows: $q_A = 1$, $q_B = 2$, $q_C = 1$, $q_D = 2$, $q_E = 1$, $q_F = 2$, $q_G = 1$, $q_H = 2$; $b_A = 440$, $b_B = 1075$, $b_C = 440$, $b_D = 1000$, $b_E = 440$, $b_F = 1075$, $b_G = 440$, $b_H = 1075$. We assume that for any $s \in \{1, 2, 3, 4, 5, 6\}$ and any $d \in \{E, F, G, H\}$ we have $a_d^s < 0$, and for any $s \in \{7, 8, 9, 10, 11, 12\}$ and any $d \in \{A, B, C, D\}$ we have $a_d^s < 0$. So, the qualification levels and reservation prices of the students are as given in table 5.5.

We apply the mix algorithm by applying $DPGAA$ first.

$a_A^1 = 7$	$a_B^1 = 11$	$a_C^1 = 1$	$a_D^1 = 0$
$a_A^2 = 0$	$a_B^2 = 15$	$a_C^2 = 0$	$a_D^2 = 5$
$a_A^3 = 8$	$a_B^3 = 12$	$a_C^3 = 0$	$a_D^3 = 5$
$a_A^4 = 2$	$a_B^4 = 0$	$a_C^4 = 10$	$a_D^4 = 0$
$a_A^5 = 0$	$a_B^5 = 2$	$a_C^5 = 0$	$a_D^5 = 15$
$a_A^6 = 0$	$a_B^6 = 2$	$a_C^6 = 0$	$a_D^6 = 16$
$\sigma_{1A} = 400$	$\sigma_{1B} = 300$	$\sigma_{1C} = -500$	$\sigma_{1D} = 440$
$\sigma_{2A} = 440$	$\sigma_{2B} = 1075$	$\sigma_{2C} = 400$	$\sigma_{2D} = -500$
$\sigma_{3A} = 400$	$\sigma_{3B} = 700$	$\sigma_{3C} = 420$	$\sigma_{3D} = -500$
$\sigma_{4A} = 100$	$\sigma_{4B} = 450$	$\sigma_{4C} = 440$	$\sigma_{4D} = 300$
$\sigma_{5A} = 400$	$\sigma_{5B} = 1$	$\sigma_{5C} = 440$	$\sigma_{5D} = 500$
$\sigma_{6A} = 420$	$\sigma_{6B} = 1$	$\sigma_{6C} = 400$	$\sigma_{6D} = 500$
$a_E^7 = 7$	$a_F^7 = 11$	$a_G^7 = 10$	$a_H^7 = 0$
$a_E^8 = 0$	$a_F^8 = 15$	$a_G^8 = 0$	$a_H^8 = 0$
$a_E^9 = 8$	$a_F^9 = 12$	$a_G^9 = 0$	$a_H^9 = 15$
$a_E^{10} = 15$	$a_F^{10} = 0$	$a_G^{10} = 7$	$a_H^{10} = 0$
$a_E^{11} = 0$	$a_F^{11} = 0$	$a_G^{11} = 0$	$a_H^{11} = 6$
$a_E^{12} = 0$	$a_F^{12} = 30$	$a_G^{12} = 0$	$a_H^{12} = 6$
$\sigma_{7E} = 400$	$\sigma_{7F} = 300$	$\sigma_{7G} = 440$	$\sigma_{7H} = 1075$
$\sigma_{8E} = 440$	$\sigma_{8F} = 1075$	$\sigma_{8G} = 400$	$\sigma_{8H} = 1075$
$\sigma_{9E} = 400$	$\sigma_{9F} = 700$	$\sigma_{9G} = 440$	$\sigma_{9H} = 1075$
$\sigma_{10E} = 440$	$\sigma_{10F} = 1075$	$\sigma_{10G} = 100$	$\sigma_{10H} = 300$
$\sigma_{11E} = 440$	$\sigma_{11F} = 1075$	$\sigma_{11G} = 400$	$\sigma_{11H} = 100$
$\sigma_{12E} = 440$	$\sigma_{12F} = 1075$	$\sigma_{12G} = 400$	$\sigma_{12H} = 100$

Table 5.5: Qualification levels and reservation prices of students for example 11

$t = 1$. We apply the algorithm $DPGAA$. It does not stop and a cycle occurs consisting of following four matchings:

$$\mu_{\bar{r}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ A & \emptyset & B & C & D & D & G & \emptyset & H & E & \emptyset & F \\ 435 & 0 & 741 & 440 & 500 & 500 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix},$$

$$\mu_{\bar{r}+1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ A & B & \emptyset & C & D & D & G & \emptyset & H & E & \emptyset & F \\ 435 & 1075 & 0 & 440 & 500 & 500 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix},$$

$$\mu_{\bar{r}+2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \emptyset & B & A & C & D & D & G & \emptyset & H & E & \emptyset & F \\ 0 & 1075 & 440 & 440 & 500 & 500 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix},$$

$$\mu_{\bar{r}+3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ B & \emptyset & B & C & D & D & G & \emptyset & H & E & \emptyset & F \\ 334 & 0 & 741 & 440 & 500 & 500 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}.$$

The constant part consists of the set of departments $\overline{\mathcal{D}} = \{C, D, E, F, G, H\}$ and the set of students $\overline{\mathcal{S}} = \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$. The inconstant part consists of the set of departments $\widetilde{\mathcal{D}} = \{A, B\}$ and the set students $\widetilde{\mathcal{S}} = \{1, 2, 3\}$, so $\widetilde{\theta}_1 = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}})$.

$t = 2$. We apply the algorithm $SPGAA$ to the problem $\widetilde{\theta}_1 = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}})$. Note that $\widetilde{\theta}_1$ is equal to the problem given in Example 6 and we know that the algorithm $SPGAA$ does not stop for that problem. So, $SPGAA$ does not stop for $\widetilde{\theta}_1$ and a cycle occurs consisting of the matchings $\mu_{\bar{t}}$, $\mu_{\bar{t}+1}$, $\mu_{\bar{t}+2}$, $\mu_{\bar{t}+3}$ that are given in Example 6. For the cycle consisting of these four matchings we have $\overline{\mathcal{D}} = \emptyset$ and $\overline{\mathcal{S}} = \emptyset$, and $\widetilde{\mathcal{D}} = \{A, B\}$ and $\widetilde{\mathcal{S}} = \{1, 2, 3\}$. So, $\widetilde{\theta}_2 = \widetilde{\theta}_1$ and the mix algorithm stops. Hence by using the mix algorithm by applying $DPGAA$ first, at the end of period 2 we reach a problem $\widetilde{\theta}_2$ such that $\widetilde{\theta}_2 = \widetilde{\theta}_1$.

We now apply the mix algorithm by applying $SPGAA$ first.

$t = 1$. We apply the algorithm $SPGAA$. It does not stop and a cycle occurs consisting of following four matchings:

$$\mu_{\bar{l}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & E & \emptyset & F & G & H & H \\ 440 & 500 & 500 & 440 & 538 & 537 & 435 & 0 & 741 & 440 & 538 & 537 \end{pmatrix},$$

$$\mu_{\bar{t}+1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & E & F & \emptyset & G & H & H \\ 440 & 500 & 500 & 440 & 538 & 537 & 435 & 1075 & 0 & 440 & 538 & 537 \end{pmatrix},$$

$$\mu_{\bar{t}+2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & \emptyset & F & E & G & H & H \\ 440 & 500 & 500 & 440 & 538 & 537 & 0 & 1075 & 440 & 440 & 538 & 537 \end{pmatrix},$$

$$\mu_{\bar{t}+3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & F & \emptyset & F & G & H & H \\ 440 & 500 & 500 & 440 & 538 & 537 & 334 & 0 & 741 & 440 & 538 & 537 \end{pmatrix}.$$

The constant part consists of the set of departments $\overline{\mathcal{D}} = \{A, B, C, D, G, H\}$ and the set of students $\overline{\mathcal{S}} = \{1, 2, 3, 4, 5, 6, 10, 11, 12\}$. The inconstant part consists of the set of departments $\widetilde{\mathcal{D}} = \{E, F\}$ and the set students $\widetilde{\mathcal{S}} = \{7, 8, 9\}$, so $\widetilde{\theta}_1 = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}})$.

$t = 2$. We apply the algorithm $DPGAA$ to the problem $\widetilde{\theta}_1 = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}})$. Note that $\widetilde{\theta}_1$ is equivalent to the problem given in Example 6 and we know that the algorithm $DPGAA$ does not stop for that problem. So, $DPGAA$ does not stop for $\widetilde{\theta}_1$ and a cycle occurs consisting of the matchings $\mu_{\bar{t}}$, $\mu_{\bar{t}+1}$, $\mu_{\bar{t}+2}$, $\mu_{\bar{t}+3}$ that are given in Example 6.[24] We have $\overline{\mathcal{D}} = \emptyset$ and $\overline{\mathcal{S}} = \emptyset$, and $\widetilde{\mathcal{D}} = \{E, F\}$ and $\widetilde{\mathcal{S}} = \{7, 8, 9\}$. So, $\widetilde{\theta}_2 = \widetilde{\theta}_1$ and the mix algorithm stops. Hence by using the mix algorithm by applying $SPGAA$ first, at the end of period 2 we reach a problem $\widetilde{\theta}_2$ such that $\widetilde{\theta}_2 = \widetilde{\theta}_1$.

Now, consider the following matching:

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & G & \emptyset & H & E & \emptyset & F \\ 440 & 500 & 500 & 440 & 538 & 537 & 440 & 0 & 1075 & 440 & 0 & 1075 \end{pmatrix}.$$

[24] We replace the names of A and B with E and F, respectively, and the names of 1,2 and 3 with 7, 8 and 9, respectively.

We will show that the matching μ is core stable. Suppose that μ is not core stable. Then there is a group $(\widetilde{d}, \widetilde{S})$ which blocks μ, where $\widetilde{d} \in \mathcal{D}$ and $\widetilde{S} \subseteq \mathcal{S}$. Note that each department $d \in \{E, F, G, H\}$ is matched with its best group of students among all groups satisfying its constraints under μ. So, there is no department $d \in \{E, F, G, H\}$ which forms a blocking coalition with any set of students. Hence, $\widetilde{d} \notin \{E, F, G, H\}$. So, $\widetilde{d} \in \{A, B, C, D\}$. There is no department $d \in \{A, B, C, D\}$ which forms a blocking coalition with any group of students $S \subseteq \{7, 8, 9, 10, 11, 12\}$, since for any $s \in \{7, 8, 9, 10, 11, 12\}$ and any $d \in \{A, B, C, D\}$ we have $a_d^s < 0$. So, we have $\widetilde{S} \subseteq \{1, 2, 3, 4, 5, 6\}$. However, for each student $s \in \{1, 2, 3, 4, 5, 6\}$ we have $\big(\mu_1(s), m^{\mu}_{s\mu_1(s)}\big) P_s(d, b_d)$ for any $d \in \big(\{A, B, C, D\} \setminus \{\mu_1(s)\}\big)$. So, there is no student $s \in \{1, 2, 3, 4, 5, 6\}$ which forms a blocking coalition with any $d \in \{A, B, C, D\}$. Hence, we also have $\widetilde{d} \notin \{A, B, C, D\}$, a contradiction. So, the matching μ is core stable.

5.4 Nonexistence of random paths to core stability

Gale and Shapley (1962) also described a one to one matching model which is known as the marriage problem,[25] and showed by a centralized algorithm (the Gale-Shapley algorithm) that the set of stable matchings is nonempty for any marriage problem (see Sotomayor (1996) for a nonconstructive proof).[26] Knuth (1976) gave an example that

[25] A **marriage problem** consists of a triplet (M, W, R), where M denote a finite set of men, W denote a finite set of women, and $R = (R_i)_{i \in (M \cup W)}$ denote a preference profile for the set of agents. Every man $m \in M$ has a complete, reflexive and transitive preference relation R_m over $(W \cup \{m\})$, and every woman $w \in W$ has a complete, reflexive and transitive preference relation R_w over $(M \cup \{w\})$.

[26] A **matching** for the marriage problem is the function $\mu : M \cup W \to M \cup W$ such that

- for all $m \in M$, $\mu(m) \in (W \cup \{m\})$,
- for all $w \in W$, $\mu(w) \in (M \cup \{w\})$, and
- for all $m \in M$ and all $w \in W$, $\mu(m) = w$ if and only if $\mu(w) = m$.

A matching μ is *blocked by an agent* $i \in (M \cup W)$ at preference profile R if $iP_i\mu(i)$, where P_i denote the strict preference relation of agent i associated with R_i. A matching μ is *individually rational* at preference profile R if it is not blocked by any agent, i.e., for all $i \in (M \cup W)$ we have $\mu(i)R_i i$. A matching μ is *blocked by a pair* $(m, w) \in M \times W$ at preference profile R if $wP_m\mu(m)$ and $mP_w\mu(w)$. A matching μ is ***stable*** at preference profile R if it is individually rational and there is no pair which blocks it at R. We note that for the marriage problem, the set of stable matchings coincide with the set of core stable matchings (Roth and Sotomayor (1990b)).

the process of allowing blocking pairs to match may have cycles and may not lead to a stable matching, and raised the question that whether there exists a decentralized procedure which converges to a stable matching. Roth and Vande Vate (1990) solved this question by showing that starting from an arbitrary matching and satisfying a randomly chosen blocking pair at each step, we reach a stable matching.[27] That is, they showed that given an arbitrary matching μ for a marriage problem (M, W, R), there exists a finite sequence of matchings $\mu_1, \mu_2, \ldots, \mu_r$ such that $\mu = \mu_1$, μ_r is stable at preference profile R, and for each $j \in \{1, 2, \ldots, r-1\}$, there is a blocking pair (m_j, w_j) for μ_j such that μ_{j+1} is obtained from μ_j by satisfying the blocking pair (m_j, w_j). Such a decentralized procedure is called as ***random paths to stability***.[28]

We will show that there does not exist a random path to core stability for the graduate admissions problem. That is, we will provide an example of graduate admission problem such that its set of core stable matchings is nonempty and an individually rational matching for this problem, and show that starting with this matching and satisfying a blocking coalition at each step, a core stable matching can not be reached. Let us first define what we mean by satisfying a blocking coalition.

Definition 61 We say that matching $\acute{\mu}$ is **obtained from matching** μ by **satisfying a blocking coalition** (d, S), where $d \in \mathcal{D}$ and $\emptyset \neq S \subseteq \mathcal{S}$, with for all $s \in S$, $\mu_1(s) \neq d$, if and only if

- for any $s \in S$, $\acute{\mu}(s) = (d, \widetilde{m}_{sd})$ with $(d, \widetilde{m}_{sd}) P_s \big(\mu_1(s), m^{\mu}_{s\mu_1(s)}\big)$,

[27]Let (M, W, R) be a marriage problem and μ be a matching with a blocking pair $(\acute{m}, \acute{w}) \in M \times W$. We say that *matching* $\widehat{\mu}$ *is obtained from* μ *by satisfying the blocking pair* $(\acute{m}, \acute{w})$ if and only if

- $\widehat{\mu}(\acute{m}) = \acute{w}$,
- for all $m \in [M \setminus \{\acute{m}, \mu(\acute{w})\}]$, $\widehat{\mu}(m) = \mu(m)$, and for all $w \in [W \setminus \{\acute{w}, \mu(\acute{m})\}]$, $\widehat{\mu}(w) = \mu(w)$, and
- if $\mu(\acute{w}) = m$ for some $m \in M$ then $\widehat{\mu}(m) = m$, and if $\mu(\acute{m}) = w$ for some $w \in W$ then $\widehat{\mu}(w) = w$.

[28]See Ma (1996) and Klaus and Klijn (2007a) for a detailed explanation of random paths to stability for the marriage problem. For other studies of random paths to stability, see Chung (2000), Diamantoudi et al. (2004) and Inarra et al. (2008) for the roommate problem, Klaus and Klijn (2007b) for matching markets with couples, and Kojima and Ünver (2008) for many to many matching problems.

- for department d, $S^d_{\acute{\mu}} = [(S^d_\mu \setminus T) \cup S]$ for some $T \subseteq S^d_\mu$, with $(S^d_{\acute{\mu}}, m^d_{\acute{\mu}}) P_d (S^d_\mu, m^d_\mu)$, where the transfer vector $m^d_{\acute{\mu}}$ is as follows: for any student $s \in S^d_{\acute{\mu}}$,

$$m^{\acute{\mu}}_{sd} = \begin{cases} m^\mu_{sd} & \text{if } s \in (S^d_\mu \setminus T) \\ \widetilde{m}_{sd} & \text{if } s \in S, \end{cases}$$

  (of course, the quota and budget constraints of department d under matching $\acute{\mu}$ are satisfied, i.e., $\mid S^d_{\acute{\mu}} \mid \leq q_d$ and $c^d_{\acute{\mu}} \leq b_d$, where $c^d_{\acute{\mu}} = \sum_{h \in (\overline{S}^d_\mu \setminus T)} m^\mu_{hd} + \sum_{s \in \overline{S}} \widetilde{m}_{sd}$ with $\overline{S}^d_\mu = \{k \in S^d_\mu \mid m^\mu_{kd} > 0\}$ and $\overline{S} = \{s \in S \mid \widetilde{m}_{sd} > 0\}$),

- for all students $l \in T \subseteq S^d_\mu$ with whom department d break ties, $\acute{\mu}(l) = (\emptyset, 0)$, i.e., each student in T is unmatched under $\acute{\mu}$,

- for any student $\widehat{s} \in [\mathcal{S} \setminus (S \cup T)]$, $\acute{\mu}(\widehat{s}) = \mu(\widehat{s})$, i.e., each student not in $(S \cup T)$ is matched with the same department under $\acute{\mu}$ as she was matched under μ,

- for any department $\widehat{d} \in (\mathcal{D} \setminus \{d\})$, $S^{\widehat{d}}_{\acute{\mu}} = (S^{\widehat{d}}_\mu \setminus S)$.

Example 12 There does not exist a random path to core stability for the graduate admissions problem

Let $\mathcal{D} = \{A, B, C, D\}$ be the set of departments and $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ the set of students. The quotas and budgets of the departments are as follows: $q_A = 1$, $q_B = 2$, $q_C = 1$, $q_D = 2$; $b_A = 440$, $b_B = 1075$, $b_C = 440$, $b_D = 1000$. The qualification levels and reservation prices of the students are as given in table 5.6.

The set of core stable matchings is non-empty for this problem, e.g., following matching $\mu^\star$ is core stable:

$$\mu^\star = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ D & \emptyset & A & C & D & B \\ 500 & 0 & 440 & 440 & 500 & 1075 \end{pmatrix}.$$

We consider the following individually rational matching μ_1,

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \emptyset & B & A & C & D & D \\ 0 & 1075 & 440 & 440 & 100 & 800 \end{pmatrix}.$$

$a^1_A = 7$	$a^1_B = 11$	$a^1_C = 10$	$a^1_D = 1$
$a^2_A = 0$	$a^2_B = 15$	$a^2_C = 0$	$a^2_D = 0$
$a^3_A = 8$	$a^3_B = 12$	$a^3_C = 0$	$a^3_D = 15$
$a^4_A = 15$	$a^4_B = 0$	$a^4_C = 11$	$a^4_D = 0$
$a^5_A = 0$	$a^5_B = 0$	$a^5_C = 0$	$a^5_D = 9$
$a^6_A = 0$	$a^6_B = 30$	$a^6_C = 0$	$a^6_D = 6$
$\sigma_{1A} = 434$	$\sigma_{1B} = 334$	$\sigma_{1C} = 440$	$\sigma_{1D} = 500$
$\sigma_{2A} = 440$	$\sigma_{2B} = 1075$	$\sigma_{2C} = 400$	$\sigma_{2D} = 1000$
$\sigma_{3A} = 440$	$\sigma_{3B} = 740$	$\sigma_{3C} = 420$	$\sigma_{3D} = 1000$
$\sigma_{4A} = 440$	$\sigma_{4B} = 1075$	$\sigma_{4C} = 100$	$\sigma_{4D} = 300$
$\sigma_{5A} = 440$	$\sigma_{5B} = 1075$	$\sigma_{5C} = 400$	$\sigma_{5D} = 100$
$\sigma_{6A} = 440$	$\sigma_{6B} = 1075$	$\sigma_{6C} = 400$	$\sigma_{6D} = 600$

Table 5.6: Qualification levels and reservation prices of students for example 12

We will show that starting with this matching μ_1 and at each time satisfying a blocking coalition, a core stable matching can not be reached.

The unique coalition which blocks μ_1 is $(B, \{1, 3\})$ where $\widetilde{m}_{1B} = \sigma_{1B} = 334$ and $\widetilde{m}_{3B} = 741$. We reach matching μ_2 by satisfying this blocking coalition, where

$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ B & \emptyset & B & C & D & D \\ 334 & 0 & 741 & 440 & 100 & 800 \end{pmatrix}.$$

Now, the coalition $(A, \{1\})$ blocks μ_2 with $b_A = 440 \geq \widetilde{m}_{1A} \geq 435$, and there is no other coalition which blocks μ_2. By satisfying this blocking coalition we reach six different matchings for each integer value of the transfer $\widetilde{m}_{1A}$. Let μ_3 denote these matchings that we reach by satisfying this blocking coalition, so

$$\mu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & \emptyset & B & C & D & D \\ m^{\mu_3}_{1A} & 0 & 741 & 440 & 100 & 800 \end{pmatrix}, \text{ where } 440 \geq m^{\mu_3}_{1A} \geq 435.$$

The unique coalition which blocks μ_3 is $(B, \{2\})$ where $\widetilde{m}_{2B} = \sigma_{2B} = 1075$. When we satisfy this blocking coalition we reach matching μ_4,

$$\mu_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & B & \emptyset & C & D & D \\ m_{1A}^{\mu_4} & 1075 & 0 & 440 & 100 & 800 \end{pmatrix}, \text{ where } 440 \geq m_{1A}^{\mu_4} \geq 435.$$

We now consider two cases:

Case 1. $440 \geq m_{1A}^{\mu_4} \geq 436$.

Case 2. $m_{1A}^{\mu_4} = 435$.

First, we consider case 1. When $440 \geq m_{1A}^{\mu_4} \geq 436$, we denote this matching by $\overline{\mu}_4$,

$$\overline{\mu}_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & B & \emptyset & C & D & D \\ m_{1A}^{\overline{\mu}_4} & 1075 & 0 & 440 & 100 & 800 \end{pmatrix}, \text{ where } 440 \geq m_{1A}^{\overline{\mu}_4} \geq 436.$$

The unique coalition which blocks $\overline{\mu}_4$ is $(A, \{3\})$ where $\widetilde{m}_{3A} = \sigma_{3A} = 440$.[29] When we satisfy this blocking coalition for $\overline{\mu}_4$ we reach matching μ_1.

Second, we consider case 2 that $m_{1A}^{\mu_4} = 435$. We denote this matching by $\widehat{\mu}_4$,

$$\widehat{\mu}_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & B & \emptyset & C & D & D \\ 435 & 1075 & 0 & 440 & 100 & 800 \end{pmatrix}.$$

There are two blocking coalitions for $\widehat{\mu}_4$. The pair $(A, \{3\})$ blocks $\widehat{\mu}_4$ with $\widetilde{m}_{3A} = \sigma_{3A} = 440$, and when we satisfy this blocking coalition we reach matching μ_1. The coalition $(B, \{1, 3\})$ also blocks $\widehat{\mu}_4$ with $\widetilde{m}_{1B} = 335$ and $\widetilde{m}_{3B} = \sigma_{3B} = 740$. By satisfying this blocking coalition we reach matching μ_5, where

$$\mu_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ B & \emptyset & B & C & D & D \\ 335 & 0 & 740 & 440 & 100 & 800 \end{pmatrix}.$$

[29]Note that the coalition $(B, \{1, 3\})$ cannot block $\overline{\mu}_4$. If we suppose that $m_{1A}^{\overline{\mu}_4} = 436$, then the minimal amount that department B should pay to student 1 is equal to $\widetilde{m}_{1B} = 336$, and department B has to pay student 3 at least her reservation price $\widetilde{m}_{3B} = \sigma_{3B} = 740$. However, $\widetilde{m}_{1B} + \widetilde{m}_{3B} = 336 + 740 = 1076 > 1075 = b_B$.

The pair $(A, \{3\})$ blocks μ_5 with $\widetilde{m}_{3A} = \sigma_{3A} = 440$. (See part I.)

The pair $(A, \{1\})$ blocks μ_5 with $440 \geq \widetilde{m}_{1A} \geq 436$. By satisfying this blocking pair we reach matching

$$\mu_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & \emptyset & B & C & D & D \\ m_{1A}^{\mu_6} & 0 & 740 & 440 & 100 & 800 \end{pmatrix}, \text{ where } 440 \geq m_{1A}^{\mu_6} \geq 436.$$

The pair $(B, \{2\})$ blocks μ_6 with $\widetilde{m}_{2B} = \sigma_{2B} = 1075$, and by satisfying this blocking pair we reach matching $\overline{\mu}_4$.
The pair $(A, \{3\})$ also blocks μ_6 with $\widetilde{m}_{3A} = \sigma_{3A} = 440$, and by satisfying this blocking pair we reach

$$\mu_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \emptyset & \emptyset & A & C & D & D \\ 0 & 0 & 440 & 440 & 100 & 800 \end{pmatrix}.$$

Now we have following blocking coalitions for μ_7:
- The pair $(B, \{2\})$ blocks μ_7 with $\widetilde{m}_{2B} = \sigma_{2B} = 1075$, and by satisfying this blocking pair we reach matching μ_1.
- The coalition $(B, \{1, 3\})$ blocks μ_7 with $\widetilde{m}_{1B} = \sigma_{1B} = 334$ and $\widetilde{m}_{3B} = 741$, and by satisfying this blocking coalition we reach matching μ_2.
- The pair $(B, \{1\})$ blocks μ_7 with $b_B = 1075 \geq \widetilde{m}_{1B} \geq 334 = \sigma_{1B}$. (See part II.)
- The pair $(B, \{3\})$ blocks μ_7 with $b_B = 1075 \geq \widetilde{m}_{3B} \geq 741$. By satisfying this blocking pair we reach matching μ_8,

$$\mu_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \emptyset & \emptyset & B & C & D & D \\ 0 & 0 & m_{3B}^{\mu_8} & 440 & 100 & 800 \end{pmatrix}, \text{ where } 1075 \geq m_{3B}^{\mu_8} \geq 741.$$

Following coalitions block μ_8:
- The pair $(B, \{2\})$ blocks μ_8 with $\widetilde{m}_{2B} = \sigma_{2B} = 1075$. (See part III.)
- If $m_{3B}^{\mu_8} = 741$, then the pair $(B, \{1\})$ blocks μ_8 with $\widetilde{m}_{1B} = \sigma_{1B} = 334$. Note that department B does not break its tie with student 3 while forming this blocking pair. By satisfying this blocking pair we reach matching μ_2.

- The pair $(A, \{1\})$ blocks μ_8 with $440 \geq \widetilde{m}_{1A} \geq 435$. (See part IV.)
- The pair $(A, \{1\})$ blocks μ_8 with $\widetilde{m}_{1A} = \sigma_{1A} = 434$. When we satisfy this blocking pair we reach matching μ_9,

$$\mu_9 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & \emptyset & B & C & D & D \\ 434 & 0 & m_{3B}^{\mu_9} & 440 & 100 & 800 \end{pmatrix}, \text{ where } 1075 \geq m_{3B}^{\mu_9} \geq 741.$$

Blocking coalitions for μ_9:
- If $m_{3B}^{\mu_9} = 741$, then the pair $(B, \{1\})$ blocks μ_9 with $\widetilde{m}_{1B} = \sigma_{1B} = 334$, and department B does not break its tie with student 3 while forming this blocking pair. By satisfying this blocking pair we reach matching μ_2.
- The pair $(B, \{2\})$ blocks μ_9 with $\widetilde{m}_{2B} = \sigma_{2B} = 1075$, and we reach matching μ_{10} by satisfying this blocking pair where

$$\mu_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & B & \emptyset & C & D & D \\ 434 & 1075 & 0 & 440 & 100 & 800 \end{pmatrix}.$$

Blocking coalitions for μ_{10}:
- The pair $(A, \{3\})$ blocks μ_{10} with $\widetilde{m}_{3A} = \sigma_{3A} = 440$, and by satisfying this blocking pair we reach matching μ_1.
- The coalition $(B, \{1, 3\})$ blocks μ_{10} with $\widetilde{m}_{1B} = \sigma_{1B} = 334$ and $\widetilde{m}_{3B} = 741$, and by satisfying this blocking coalition we reach matching μ_2.
- The coalition $(B, \{1, 3\})$ blocks μ_{10} with $\widetilde{m}_{1B} = 335$ and $\widetilde{m}_{3B} = \sigma_{3B} = 740$, and by satisfying this blocking coalition we reach matching μ_5.
- The coalition $(B, \{1, 3\})$ blocks μ_{10} with $\widetilde{m}_{1B} = \sigma_{1B} = 334$ and $\widetilde{m}_{3B} = \sigma_{3B} = 740$, and by satisfying this blocking coalition we reach matching μ_{11}, where

$$\mu_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ B & \emptyset & B & C & D & D \\ 334 & 0 & 740 & 440 & 100 & 800 \end{pmatrix}.$$

Blocking coalitions for μ_{11}:

- The pair $(A, \{1\})$ blocks μ_{11} with $440 \geq \widetilde{m}_{1A} \geq 436$, and by satisfying this blocking pair we reach matching μ_6.
- The pair $(A, \{1\})$ blocks μ_{11} with $\widetilde{m}_{1A} = 435$. (See part V.)
- The pair $(A, \{3\})$ blocks μ_{11} with $\widetilde{m}_{3A} = \sigma_{3A} = 440$, and by satisfying this blocking pair we reach matching μ_{12} where

$$\mu_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ B & \emptyset & A & C & D & D \\ 334 & 0 & 0 & 440 & 100 & 800 \end{pmatrix}.$$

Blocking coalitions for μ_{12}:
- The pair $(B, \{3\})$ blocks μ_{12} with $\widetilde{m}_{3B} = 741$ and department B does not break its tie with student 1. By satisfying this blocking pair we reach matching μ_2.
- The pair $(B, \{3\})$ blocks μ_{12} with $1075 \geq \widetilde{m}_{3B} \geq 741$ and department B breaks its tie with student 1 while forming this blocking pair. By satisfying this blocking pair we reach matching μ_8.
- The pair $(B, \{2\})$ blocks μ_{12} with $\widetilde{m}_{2B} = \sigma_{2B} = 1075$, and we reach matching μ_1 by satisfying this blocking pair.

We now complete parts I, II, III, IV and V.

Part I. The pair $(A, \{3\})$ blocks matching μ_5 with $\widetilde{m}_{3A} = \sigma_{3A} = 440$, and we reach matching $\overline{\mu}_6$ by satisfying this blocking pair where

$$\overline{\mu}_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ B & \emptyset & A & C & D & D \\ 335 & 0 & 440 & 440 & 100 & 800 \end{pmatrix}.$$

The pair $(B, \{3\})$ blocks $\overline{\mu}_6$ with $1075 \geq \widetilde{m}_{3B} \geq 741$ and department B breaks its tie with student 1 while forming this blocking pair. By satisfying this blocking pair we reach matching μ_8.

The pair $(B, \{2\})$ also blocks $\overline{\mu}_6$ with $\widetilde{m}_{2B} = \sigma_{2B} = 1075$, and we reach matching μ_1 by satisfying this blocking coalition. We note that there is no other blocking coalition for $\overline{\mu}_6$.

Part II. The pair $(B, \{1\})$ blocks μ_7 with $b_B = 1075 \geq \widetilde{m}_{1B} \geq 334 = \sigma_{1B}$, and by satisfying this blocking pair we reach matching $\overline{\mu}_8$,

$$\overline{\mu}_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ B & \emptyset & A & C & D & D \\ m_{1B}^{\overline{\mu}_8} & 0 & 440 & 440 & 100 & 800 \end{pmatrix} \text{where } 1075 \geq m_{1B}^{\overline{\mu}_8} \geq 334.$$

Blocking coalitions for $\overline{\mu}_8$:

- The pair $(B, \{2\})$ blocks $\overline{\mu}_8$ with $\widetilde{m}_{2B} = \sigma_{2B} = 1075$, and we reach matching μ_1 by satisfying this blocking pair.
- If $m_{1B}^{\overline{\mu}_8} = 334$, then the pair $(B, \{3\})$ blocks $\overline{\mu}_8$ with $\widetilde{m}_{3B} = 741$, and department B does not break its tie with student 1 while forming this blocking pair. By satisfying this blocking pair we reach matching μ_2.
- The pair $(B, \{3\})$ blocks $\overline{\mu}_8$ with $1075 \geq \widetilde{m}_{3B} \geq 741$ and department B breaks its tie with student 1 while forming this blocking pair. By satisfying this blocking pair we reach matching μ_8.

Part III. The pair $(B, \{2\})$ blocks μ_8 with $\widetilde{m}_{2B} = \sigma_{2B} = 1075$, and by satisfying this blocking pair we reach matching $\overline{\mu}_9$, where

$$\overline{\mu}_9 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \emptyset & B & \emptyset & C & D & D \\ 0 & 1075 & 0 & 440 & 100 & 800 \end{pmatrix}.$$

Blocking coalitions for $\overline{\mu}_9$:

- The pair $(A, \{1\})$ blocks $\overline{\mu}_9$ with $440 \geq \widetilde{m}_{1A} \geq 435$, and by satisfying this blocking pair we reach matching μ_4.
- The pair $(A, \{1\})$ blocks $\overline{\mu}_9$ with $\widetilde{m}_{1A} = \sigma_{1A} = 434$. When we satisfy this blocking pair we reach matching μ_{10}.
- The coalition $(B, \{1, 3\})$ blocks $\overline{\mu}_9$ with $\widetilde{m}_{1B} = \sigma_{1B} = 334$ and $\widetilde{m}_{3B} = 741$, and by satisfying this blocking coalition we reach matching μ_2.
- The coalition $(B, \{1, 3\})$ blocks $\overline{\mu}_9$ with $\widetilde{m}_{1B} = 335$ and $\widetilde{m}_{3B} = \sigma_{3B} = 740$, and by satisfying this blocking coalition we reach matching μ_5.
- The coalition $(B, \{1, 3\})$ blocks $\overline{\mu}_9$ with $\widetilde{m}_{1B} = \sigma_{1B} = 334$ and $\widetilde{m}_{3B} = \sigma_{3B} = 740$, and by satisfying this blocking coalition we reach matching μ_{11}.

Part IV. The pair $(A, \{1\})$ blocks μ_8 with $440 \geq \widetilde{m}_{1A} \geq 435$, and by satisfying this blocking pair we reach matching $\widehat{\mu}_9$,

$$\widehat{\mu}_9 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & \emptyset & B & C & D & D \\ m_{1A}^{\widehat{\mu}_9} & 0 & m_{3B}^{\widehat{\mu}_9} & 440 & 100 & 800 \end{pmatrix},$$

where $440 \geq m_{1A}^{\widehat{\mu}_9} \geq 435$ and $1075 \geq m_{3B}^{\widehat{\mu}_9} \geq 741$.[30]

The unique coalition which blocks $\widehat{\mu}_9$ is $(B, \{2\})$ where $\widetilde{m}_{2B} = \sigma_{2B} = 1075$, and we reach matching μ_4 by satisfying this blocking coalition.

Part V. The pair $(A, \{1\})$ blocks μ_{11} with $\widetilde{m}_{1A} = 435$, and we reach matching $\overline{\mu}_{12}$ by satisfying this blocking pair, where

$$\overline{\mu}_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & \emptyset & B & C & D & D \\ 435 & 0 & 740 & 440 & 100 & 800 \end{pmatrix}.$$

Blocking coalitions for $\overline{\mu}_{12}$:
- The pair $(A, \{3\})$ blocks $\overline{\mu}_{12}$ with $\widetilde{m}_{3A} = \sigma_{3A} = 440$, and we reach matching μ_7 by satisfying this blocking pair.
- The pair $(B, \{2\})$ blocks $\overline{\mu}_{12}$ with $\widetilde{m}_{2B} = \sigma_{2B} = 1075$, and we reach matching $\widehat{\mu}_4$ by satisfying this blocking pair.
- The pair $(B, \{1\})$ blocks $\overline{\mu}_{12}$ with $\widetilde{m}_{1B} = 335$ and department B does not break its tie with student 3 while forming this blocking pair. By satisfying this blocking pair we reach matching μ_5.

Hence, for the graduate admissions problem with quota and budget constraints there does not exist a random path to core stability (see figure 5.1).

[30] Note that if $m_{3B}^{\widehat{\mu}_9} = 741$ then matching $\widehat{\mu}_9$ is equal to matching μ_3.

5.5 Students consider only their reservation prices

Karakaya and Koray (2003) also studied the model under restrictions on students' preferences. The main restriction they imposed to that end is the assumption that the students care only about their reservation prices and do not derive any further utility from money transfers over and above their reservation prices. They constructed another departments proposing graduate admission algorithm ($\widetilde{DPGAA}$) by taking the reservation prices of students equal to the money transfers from the department to which they are accepted. They showed that if the algorithm $\widetilde{DPGAA}$ stops then the resulting matching is core stable. However, like $DPGAA$, $\widetilde{DPGAA}$ does not always stop, and it is possible that there exists a core stable matching although $\widetilde{DPGAA}$ does not stop.

In this section we will construct the students proposing graduate admission algorithm ($\widetilde{SPGAA}$) when students consider only their reservation prices. We will study whether the mix algorithm works for this new model, i.e., we will check whether conjectures 1 and 2 are correct if students consider only their reservation prices. Random paths to core stability will also be studied.

We will first define the departments proposing algorithm ($\widetilde{DPGAA}$) following Karakaya and Koray (2003).

The Departments Proposing Algorithm

The structure of $\widetilde{DPGAA}$ is the same as that of $DPGAA$, the only difference being that a department d which makes an offer to a student s is ready to pay σ_{sd} to s no matter at what stage of the algorithm this offer is made. In other words, $m_{sd}(t) = \sigma_{sd}$ for all $s \in S$, all $d \in D$ and all times t at which d makes an offer to s.

At each time t in the algorithm $\widetilde{DPGAA}$, each department d chooses a group of admissible students S_t^d satisfying its quota and budget constraints so as to maximize its total benefit Y_t^d.

We now explain the details of how the algorithm $\widetilde{DPGAA}$ works.

$t = 1$. a) Each department d determines a group of students $S_1^d \subseteq \mathcal{S}$ as denoted above and offers to each student $s \in S_1^d$.

b) Students who have taken one or more offers accept exactly one offer and reject the others.

c) Department d accepts the group of students who accepted its offers. Let T_1^d denote the group of students who accepted department d's offers at time $t = 1$, where clearly $T_1^d \subseteq S_1^d$.

Now, at the end of period $t = 1$ we have a matching μ_1, and so $S_{\mu_1}^d = T_1^d$.

$t = 2$. a) Each department d determines a group of students $S_2^d \subseteq \mathcal{S} \setminus (S_1^d \setminus T_1^d)$ and makes an offer to each student $s \in S_2^d$.

b) Students who have taken one or more offers accept exactly one offer and reject the others.

c) Department d accepts the group of students who accepted its offers.

In general, at time k, the algorithm works as follows.

$t = k$. a) Now we will define an admissible group of students for department d, i.e, we will define the set $F_k^d \subseteq \mathcal{S}$ for department d at time k.

Assume that $\tilde{t} < k$ was the last time that d made an offer to s before time k where s rejected d's offer because of another department $\widehat{d}$'s offer. Department d cannot make an offer to student s at time k, if $s \in S_{\mu_{k-1}}^{\widehat{d}}$. The set F_k^d denote the group of all such students for department d at time k, i.e., the group of students to whom department d cannot make offers at time k.[31]

Each department d chooses its group of students S_k^d from $\mathcal{S} \setminus F_k^d$ and offers to each student $s \in S_k^d$.

b) Students who have taken one or more offers accept exactly one offer and reject the others.

[31] At time $t = 1$, $F_1^d = \emptyset$ for all $d \in D$, so each department d determines its group of students S_1^d over the set of all students $\mathcal{S}$. At time $t = 2$, for all $d \in D$, $F_2^d = S_1^d \setminus T_1^d$, so the admissible group for department d at time 2 is $\mathcal{S} \setminus (S_1^d \setminus T_1^d)$.

c) Department d accepts the group of students $T_k^d \subseteq S_k^d$ who accepted its offers.

Stopping Rule

$t = t^\star$: The algorithm stops at time $t^\star$ if each department d makes offers exactly to the group of students who accepted its offers at $t^\star - 1$, i.e., if we have $S_{t^\star}^d = T_{t^\star-1}^d$ for all $d \in D$.

If the algorithm stops at time $t^\star$, the matching $\mu_{t^\star}$ is regarded as the outcome of the algorithm.

Proposition 22 *(Karakaya and Koray (2003)) If the algorithm $\widetilde{DPGAA}$ stops, then the final matching of the algorithm is core stable (and thus Pareto optimal).*

We now define the students proposing graduate admission algorithm $(\widetilde{SPGAA})$.

The Students Proposing Algorithm

The structure of $\widetilde{SPGAA}$ is the same as that of $SPGAA$, the only difference is that a student s offers her reservation price σ_{sd} to a department d at any stages of the algorithm if she makes an offer to department d. That is, for all $s \in S$, all $d \in D$ and all periods t we have $m_{sd}(t) = \sigma_{sd}$.

The algorithm $\widetilde{SPGAA}$ does not permit a student s to make offers to department d if she is unqualified for department d. At every period in the algorithm, students s makes at most one new offer to a department which is best for her, and her last new offer made to a different department and got rejected stays valid as a holding offer.

$t = 1$. a) Each student s makes an offer to her most preferred department d to which she is permitted to make offers. That is, student s offers σ_{sd} to department d, where $(d, \sigma_{sd})P_s(\widetilde{d}, \sigma_{s\widetilde{d}})$ for any $\widetilde{d} \neq d$ to which she can make offers. Note that at $t = 1$, there is no holding offer and each student s makes a new offer.

b) Let S_1^d denote the group of students who offered department d at $t = 1$. Each department d accepts the offers of the group of students $T_1^d \subseteq S_1^d$ that maximizes its total benefit subject to its quota and budget constraints.

c) Each student s who has taken acceptance(s) tentatively accepts at most one of them and rejects the others.

Now, at the end of time $t = 1$ we have a matching μ_1 with $S^d_{\mu_1} = T^d_1$ for all $d \in \mathcal{D}$ since a student s can get at most one acceptance at $t = 1$.

$t = 2$. a) Each student s makes at most one new offer. Student s cannot make a new offer to a department d if she offered to d and got rejected at time $t = 1$. In such a case, she makes a new offer to another department at this period, and her offer made to d and got rejected in the preceding period remains valid at period 2 as a holding offer.

b) Each department d considers the group of students S^d_2 who made new and holding offers to department d at period 2, and accepts the offers of the group of students $T^d_2 \subseteq S^d_2$ that maximizes its total benefit subject to its constraints.

c) Each student s who has taken acceptance(s) tentatively accepts at most one of them and rejects the others.

Now, at the end of period $t = 2$ we have a matching μ_2 such that for each department d, $S^d_{\mu_2} \subseteq T^d_2$.

In general, at time k, the algorithm works as follows.

$t = k$. a) We will define the set $F^s_k \subseteq \mathcal{D}$ for student s at period k.

Assume that $\tilde{t}$ was the last period such that student s made a new offer to department d before period k, where this offer was rejected by department d because of the group of students $T^d_{\tilde{t}}$. Student s cannot make a new offer to department d at period k if $S^d_{\mu_{k-1}} = T^d_{\tilde{t}}$. The set F^s_k denote the set of all such departments for student s at period k, i.e., student s cannot make a new offer to a department $d \in F^s_k$.

Each student s makes at most one new offer to a department $d \in (\mathcal{D} \setminus F^s_k)$, and the last new offer student s made to some department and got rejected remains valid as a holding offer if she makes a new offer to another department at this period k.

b) Each department d accepts the offers of the group of students $T^d_k \subseteq S^d_k$ that maximizes its total benefit subject to its constraints.

c) Each student s who has taken acceptance(s) tentatively accepts at most one of them and rejects the others.

Now, at the end of time $t = k$ we have a matching μ_k with $S^d_{\mu_k} \subseteq T^d_k$.

Stopping Rule

$t = t^\star$: The algorithm stops at time $t^\star$ if each student s makes offer(s) (new and/or holding) to exactly the same department(s) that she offered in the preceding period. That is the algorithm stops at $t^\star$ if for all $d \in \mathcal{D}$ we have $S^d_{t^\star} = S^d_{t^\star-1}$ with for any $s \in S^d_{t^\star}$, if s made new offers to d at periods $t^\star - 1$ and $t^\star$, and if s made holding offers to d at periods $t^\star - 1$ and $t^\star$.

If the algorithm stops at $t^\star$ the final matching $\mu_{t^\star}$ is regarded as the outcome of the algorithm.

Proposition 23 *If the algorithm $\widetilde{SPGAA}$ stops, then the final matching of the algorithm is core stable (and thus Pareto optimal).*

Proof Analogous to the proof of Proposition 19. □

Following example taken from Karakaya and Koray (2003) shows that there exists a graduate admission problem when students consider only their reservation prices such that neither $\widetilde{DPGAA}$ nor $\widetilde{SPGAA}$ stops and there is no core stable matching.

Example 13 Neither $\widetilde{DPGAA}$ nor $\widetilde{SPGAA}$ stops and there is no core stable matching

Let $\mathcal{D} = \{A, B\}$, $\mathcal{S} = \{1, 2, 3\}$, $q_A = 1$, $q_B = 2$, $b_A = 50$, $b_B = 70$, and the qualification levels and reservation prices of the students are as given in in table 5.7.

As shown in Karakaya and Koray (2003), the algorithm $\widetilde{DPGAA}$ does not stop and a cycle occurs consisting of following matchings μ_1, μ_2, μ_3 and μ_4:[32]

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ B & \emptyset & B \end{pmatrix}, \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ B & \emptyset & A \end{pmatrix}, \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & A \end{pmatrix}, \mu_4 = \begin{pmatrix} 1 & 2 & 3 \\ A & B & \emptyset \end{pmatrix}.$$

[32]When writing a matching, we will not write the money transfers between matched agents, since all money transfers between matched agents are the reservation prices of the students.

a_A^1=10	a_B^1=10
a_A^2= 1	a_B^2=15
a_A^3= 8	a_B^3= 9
σ_{1A}=40	σ_{1B}=30
σ_{2A}=50	σ_{2B}=45
σ_{3A}=30	σ_{3B}=40

Table 5.7: Qualification levels and reservation prices of students for example 13

If we apply the algorithm $\widetilde{SPGAA}$, it does not stop and a cycle occurs consisting of following matchings $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ and $\widetilde{\mu}_3$:

$$\widetilde{\mu}_1 = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & A \end{pmatrix}, \widetilde{\mu}_2 = \begin{pmatrix} 1 & 2 & 3 \\ A & B & \emptyset \end{pmatrix}, \widetilde{\mu}_3 = \begin{pmatrix} 1 & 2 & 3 \\ B & \emptyset & B \end{pmatrix}.$$

Hence, neither $\widetilde{DPGAA}$ nor $\widetilde{SPGAA}$ stops for this example. We note that there is no core stable matching for this example, since there is neither a core stable matching such that student 2 is matched with a department, nor a core stable matching under which she is unmatched.

Karakaya and Koray (2003) showed that it is impossible that the algorithm $\widetilde{DPGAA}$ does not stop and no cycle occurs in the algorithm.

Proposition 24 *(Karakaya and Koray (2003)) The algorithm $\widetilde{DPGAA}$ stops if and only if no cycle occurs in the algorithm.*

Following result shows that the same is also true for the algorithm $\widetilde{SPGAA}$.

Proposition 25 *The algorithm $\widetilde{SPGAA}$ stops if and only if no cycle occurs in the algorithm.*

Proof The proof is similar to that of Proposition 21. □

The following example taken from Karakaya and Koray (2003) shows that it is possible that the algorithm $\widetilde{DPGAA}$ does not stop but the algorithm $\widetilde{SPGAA}$ stops and hence there is a core stable matching.

Example 14 The algorithm $\widetilde{DPGAA}$ does not stop but the algorithm $\widetilde{SPGAA}$ stops

Let $\mathcal{D} = \{A, B, C, D\}$ be the set of departments, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ the set of students and the quotas and budgets of the departments are given by $q_A = 1$, $q_B = 2$, $q_C = 1$, $q_D = 2$; $b_A = 50$, $b_B = 70$, $b_C = 50$, $b_D = 70$. The qualification levels and reservation prices of the students are as given in table 5.8.

a_A^1=10	a_B^1=10	a_C^1= 5	a_D^1= 0
a_A^2= 1	a_B^2=15	a_C^2= 0	a_D^2= 3
a_A^3= 8	a_B^3= 9	a_C^3= 0	a_D^3= 3
a_A^4= 6	a_B^4= 0	a_C^4=10	a_D^4=10
a_A^5= 0	a_B^5= 3	a_C^5= 1	a_D^5=15
a_A^6= 0	a_B^6= 3	a_C^6= 8	a_D^6= 9
σ_{1A}=50	σ_{1B}=30	σ_{1C}=20	σ_{1D}=40
σ_{2A}=50	σ_{2B}=45	σ_{2C}=40	σ_{2D}=30
σ_{3A}=30	σ_{3B}=40	σ_{3C}=45	σ_{3D}=28
σ_{4A}=20	σ_{4B}=40	σ_{4C}=50	σ_{4D}=30
σ_{5A}=40	σ_{5B}=30	σ_{5C}=50	σ_{5D}=45
σ_{6A}=45	σ_{6B}=28	σ_{6C}=30	σ_{6D}=40

Table 5.8: Qualification levels and reservation prices of students for example 14

Karakaya and Koray (2003) showed that when the algorithm $\widetilde{DPGAA}$ is applied, a cycle occurs in the algorithm consisting of matchings μ_1, μ_2, μ_3 and μ_4:

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ B & \emptyset & B & D & \emptyset & D \end{pmatrix}, \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ B & \emptyset & A & D & \emptyset & C \end{pmatrix},$$

$$\mu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \emptyset & B & A & \emptyset & D & C \end{pmatrix}, \mu_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & B & \emptyset & C & D & \emptyset \end{pmatrix}.$$

Hence, the algorithm $\widetilde{DPGAA}$ does not stop for this example. When we apply the algorithm $\widetilde{SPGAA}$, it stops at the end of period 2, and matching $\widetilde{\mu}$ is the outcome of the algorithm, where

$$\widetilde{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ C & D & D & A & B & B \end{pmatrix}.$$

The matching $\widetilde{\mu}$ is core stable by Proposition 23. Hence, it is possible that the algorithm $\widetilde{DPGAA}$ does not stop, but the algorithm $\widetilde{SPGAA}$ stops.

Example 15 The algorithm $\widetilde{SPGAA}$ does not stop but the algorithm $\widetilde{DPGAA}$ stops

Let $\mathcal{D} = \{E, F, G, H\}$ be the set of departments, $\mathcal{S} = \{7, 8, 9, 10, 11, 12\}$ the set of students and the quotas and budgets of the departments are given by $q_E = 1$, $q_F = 2$, $q_G = 1$, $q_H = 2$; $b_E = 50$, $b_F = 70$, $b_G = 50$, $b_H = 70$. The qualification levels and reservation prices of the students are as given in table 5.9.

$a_E^7=10$	$a_F^7=10$	$a_G^7=13$	$a_H^7=0$
$a_E^8=1$	$a_F^8=15$	$a_G^8=0$	$a_H^8=0$
$a_E^9=8$	$a_F^9=9$	$a_G^9=0$	$a_H^9=28$
$a_E^{10}=15$	$a_F^{10}=0$	$a_G^{10}=10$	$a_H^{10}=10$
$a_E^{11}=0$	$a_F^{11}=0$	$a_G^{11}=1$	$a_H^{11}=15$
$a_E^{12}=0$	$a_F^{12}=27$	$a_G^{12}=8$	$a_H^{12}=9$
$\sigma_{7E}=40$	$\sigma_{7F}=30$	$\sigma_{7G}=50$	$\sigma_{7H}=45$
$\sigma_{8E}=50$	$\sigma_{8F}=45$	$\sigma_{8G}=30$	$\sigma_{8H}=40$
$\sigma_{9E}=30$	$\sigma_{9F}=40$	$\sigma_{9G}=45$	$\sigma_{9H}=50$
$\sigma_{10E}=50$	$\sigma_{10F}=45$	$\sigma_{10G}=40$	$\sigma_{10H}=30$
$\sigma_{11E}=30$	$\sigma_{11F}=40$	$\sigma_{11G}=50$	$\sigma_{11H}=45$
$\sigma_{12E}=45$	$\sigma_{12F}=50$	$\sigma_{12G}=30$	$\sigma_{12H}=40$

Table 5.9: Qualification levels and reservation prices of students for example 15

When we apply the algorithm $\widetilde{SPGAA}$, a cycle occurs in the algorithm consisting of three matchings μ_1, μ_2 and μ_3:

$$\mu_1 = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ \emptyset & F & E & \emptyset & H & G \end{pmatrix}, \mu_2 = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ E & F & \emptyset & G & H & \emptyset \end{pmatrix},$$

$$\mu_3 = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ F & \emptyset & F & H & \emptyset & H \end{pmatrix}.$$

So, the algorithm $\widetilde{SPGAA}$ does not stop for this example. However, the algorithm $\widetilde{DPGAA}$ stops at the end of period 2, and matching $\widehat{\mu}$ is the outcome of the algorithm, where

$$\widehat{\mu} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ G & \emptyset & H & E & \emptyset & F \end{pmatrix}.$$

The matching $\widehat{\mu}$ is core stable by Proposition 22. Hence, it is possible that the algorithm $\widetilde{SPGAA}$ does not stop, but the algorithm $\widetilde{DPGAA}$ stops.

Example 16 Neither $\widetilde{DPGAA}$ nor $\widetilde{SPGAA}$ stops and there exists a core stable matching

Our example is the union of examples 14 and 15. That is, we let $\mathcal{D} = \{A, B, C, D, E, F, G, H\}$ be the set of departments and $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ the set of students, where the quotas and budgets of the departments are as follows: $q_A = 1$, $q_B = 2$, $q_C = 1$, $q_D = 2$, $q_E = 1$, $q_F = 2$, $q_G = 1$, $q_H = 2$; $b_A = 50$, $b_B = 70$, $b_C = 50$, $b_D = 70$, $b_E = 50$, $b_F = 70$, $b_G = 50$, $b_H = 70$. The qualification levels and reservation prices of the students are as given in examples 14 and 15, and we assume that for any $s \in \{1, 2, 3, 4, 5, 6\}$ and any $d \in \{E, F, G, H\}$ we have $a_d^s < 0$, and for any $s \in \{7, 8, 9, 10, 11, 12\}$ and any $d \in \{A, B, C, D\}$ we have $a_d^s < 0$. So, the qualification levels and reservation prices of the students are as given in table 5.10.

a_A^1=10	a_B^1=10	a_C^1= 5	a_D^1= 0	a_E^7=10	a_F^7=10	a_G^7=13	a_H^7= 0
a_A^2= 1	a_B^2=15	a_C^2= 0	a_D^2= 3	a_E^8= 1	a_F^8=15	a_G^8= 0	a_H^8= 0
a_A^3= 8	a_B^3= 9	a_C^3= 0	a_D^3= 3	a_E^9= 8	a_F^9= 9	a_G^9= 0	a_H^9=28
a_A^4= 6	a_B^4= 0	a_C^4=10	a_D^4=10	a_E^{10}=15	a_F^{10}= 0	a_G^{10}=10	a_H^{10}=10
a_A^5= 0	a_B^5= 3	a_C^5= 1	a_D^5=15	a_E^{11}= 0	a_F^{11}= 0	a_G^{11}= 1	a_H^{11}=15
a_A^6= 0	a_B^6= 3	a_C^6= 8	a_D^6= 9	a_E^{12}= 0	a_F^{12}=27	a_G^{12}= 8	a_H^{12}= 9
σ_{1A}=50	σ_{1B}=30	σ_{1C}=20	σ_{1D}=40	σ_{7E}=40	σ_{7F}=30	σ_{7G}=50	σ_{7H}=45
σ_{2A}=50	σ_{2B}=45	σ_{2C}=40	σ_{2D}=30	σ_{8E}=50	σ_{8F}=45	σ_{8G}=30	σ_{8H}=40
σ_{3A}=30	σ_{3B}=40	σ_{3C}=45	σ_{3D}=28	σ_{9E}=30	σ_{9F}=40	σ_{9G}=45	σ_{9H}=50
σ_{4A}=20	σ_{4B}=40	σ_{4C}=50	σ_{4D}=30	σ_{10E}=50	σ_{10F}=45	σ_{10G}=40	σ_{10H}=30
σ_{5A}=40	σ_{5B}=30	σ_{5C}=50	σ_{5D}=45	σ_{11E}=30	σ_{11F}=40	σ_{11G}=50	σ_{11H}=45
σ_{6A}=45	σ_{6B}=28	σ_{6C}=30	σ_{6D}=40	σ_{12E}=45	σ_{12F}=50	σ_{12G}=30	σ_{12H}=40

Table 5.10: Qualification levels and reservation prices of students for example 16

When we apply the departments proposing algorithm $\widetilde{DPGAA}$, a cycle occurs in the algorithm consisting of matchings μ_1, μ_2, μ_3 and μ_4:

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ B & \emptyset & B & D & \emptyset & D & G & \emptyset & H & E & \emptyset & F \end{pmatrix},$$

$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ B & \emptyset & A & D & \emptyset & C & G & \emptyset & H & E & \emptyset & F \end{pmatrix},$$

$$\mu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \emptyset & B & A & \emptyset & D & C & G & \emptyset & H & E & \emptyset & F \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ A & B & \emptyset & C & D & \emptyset & G & \emptyset & H & E & \emptyset & F \end{pmatrix}.$$

Hence, the departments proposing algorithm $\widetilde{DPGAA}$ does not stop for this example.

When we apply the students proposing algorithm $\widetilde{SPGAA}$, a cycle occurs in the algorithm consisting of three matchings $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ and $\widetilde{\mu}_3$:

$$\widetilde{\mu}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & \emptyset & F & E & \emptyset & H & G \end{pmatrix},$$

$$\widetilde{\mu}_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & E & F & \emptyset & G & H & \emptyset \end{pmatrix},$$

$$\widetilde{\mu}_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & F & \emptyset & F & H & \emptyset & H \end{pmatrix}.$$

So, the students proposing algorithm $\widetilde{SPGAA}$ does not stop for this example.

Consider following matching μ:

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & G & \emptyset & H & E & \emptyset & F \end{pmatrix}.$$

Note that $\mu = \widetilde{\mu} \cup \widehat{\mu}$, where $\widetilde{\mu}$ is the resulting matching of the algorithm $\widetilde{SPGAA}$ when applied to Example 14, and $\widehat{\mu}$ is the resulting matching of the algorithm $\widetilde{DPGAA}$ when applied to Example 15. It is easy to check that the matching μ is core stable.

Hence, it is possible that neither $\widetilde{DPGAA}$ nor $\widetilde{SPGAA}$ stops, but there is a core stable matching. So, the algorithms $\widetilde{DPGAA}$ and $\widetilde{SPGAA}$ are not complementary in the sense that for a given graduate admission problem if the set of its core stable matching is non-empty, then either $\widetilde{DPGAA}$ or $\widetilde{SPGAA}$ stops.

We note that for the graduate admission problem given in Example 16, the core stable matching μ can be reached by using the mix algorithm independent of whether $\widetilde{DPGAA}$ or $\widetilde{SPGAA}$ is applied first. Because of this observation, the question of whether Conjecture 1 is correct when students consider only their reservation prices is asked. Following example shows that it is not correct.

Example 17 The mix algorithm produces a matching which is not core stable

The set of departments and their quotas and budgets are as in Example 16. The set of students and their qualification levels and reservation prices are again as in Example 16 except that the qualification level of student 10 for department A is now 7, and the reservation price of student 10 for department A is now 42, i.e., $a_A^{10} = 7$ and $\sigma_{10A} = 42$.

We now apply the mix algorithm by applying first $\widetilde{DPGAA}$.

$t = 1$. All departments behave in this example as they behaved in Example 16.[33] Hence, the algorithm $\widetilde{DPGAA}$ does not stop and a cycle occurs consisting of matchings μ_1, μ_2, μ_3 and μ_4 which are given in Example 16. The constant part consists of the set of departments $\overline{\mathcal{D}} = \{E, F, G, H\}$ and the set of students $\overline{\mathcal{S}} = \{7, 8, 9, 10, 11, 12\}$ with a matching $\widehat{\mu} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ G & \emptyset & H & E & \emptyset & F \end{pmatrix}$. The inconstant part consists of the set students $\widetilde{\mathcal{S}} = \{1, 2, 3, 4, 5, 6\}$ and the set of departments $\widetilde{\mathcal{D}} = \{A, B, C, D\}$.

[33] We note that department A makes offers only to students 1 and 3 in Example 16, and we now have $a_A^{10} = 7$ which is smaller than $a_A^1 = 10$ and $a_A^3 = 8$. So, department A does not make any offer to student 10 in this example, showing that department A behaves in this example as it behaved in Example 16.

$t = 2$. We apply the algorithm $\widetilde{SPGAA}$ to the inconstant part $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$. It stops and the resulting matching is $\widetilde{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ C & D & D & A & B & B \end{pmatrix}$. Hence the mix algorithm by applying first $\widetilde{DPGAA}$ stops at the end of period 2, and its outcome is the union of the matchings $\widehat{\mu}$ and $\widetilde{\mu}$ which is the matching μ given in Example 16, i.e.,

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & G & \emptyset & H & E & \emptyset & F \end{pmatrix}.$$

However, the matching μ is not core stable for this example, since $(A, \{10\})$ blocks the matching μ.[34]

Hence, the mix algorithm by applying first $\widetilde{DPGAA}$ produces a matching which is not core stable.

We now apply the mix algorithm by applying first $\widetilde{SPGAA}$.

$t = 1$. All students behave in this example as they behaved in Example 16.[35] Hence, the algorithm $\widetilde{SPGAA}$ does not stop and a cycle occurs consisting of matchings $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ and $\widetilde{\mu}_3$ which are given in Example 16. The constant part consists of the set of departments $\overline{\mathcal{D}} = \{A, B, C, D\}$ and the set of students $\overline{\mathcal{S}} = \{1, 2, 3, 4, 5, 6\}$ with a matching $\widetilde{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ C & D & D & A & B & B \end{pmatrix}$. The inconstant part consists of the set students $\widetilde{\mathcal{S}} = \{7, 8, 9, 10, 11, 12\}$ and the set of departments $\widetilde{\mathcal{D}} = \{E, F, G, H\}$.

$t = 2$. We apply the algorithm $\widetilde{DPGAA}$ to the inconstant part $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{D}})$. It stops and the resulting matching is $\widehat{\mu} = \begin{pmatrix} 7 & 8 & 9 & 10 & 11 & 12 \\ G & \emptyset & H & E & \emptyset & F \end{pmatrix}$. Hence the mix algorithm by applying first $\widetilde{SPGAA}$ stops, and its outcome is the union of the matchings $\widetilde{\mu}$ and $\widehat{\mu}$ which is the matching μ given in Example 16. We know that the matching μ is not core stable for this example.

[34] Note that $(\{10\}, \sigma_{10A})P_A(\{4\}, \sigma_{4A})$ since $a_A^{10} > a_A^4$, and $(A, \sigma_{10A})P_{10}(E, \sigma_{10E})$ since $\sigma_{10A} < \sigma_{10E}$.

[35] We note that student 10 makes offers only to departments H and G in Example 16, and we now have $\sigma_{10A} = 42$ which is greater than $\sigma_{10H} = 30$ and $\sigma_{10G} = 40$. So, student 10 does not make any offer to department A in this example, showing that student 10 behaves in this example as she behaved in Example 16.

Hence, it is possible that the mix algorithm produces a matching which is not core stable, i.e., Conjecture 1 is not correct when students consider only their reservation prices.

We note that for the graduate admission problem given in Example 13, by using the mix algorithm regardless of which type of algorithm is applied first, we reach the same problem, and we know that there is no core stable matching for Example 13. This observation leads us to the question that whether Conjecture 2 is correct when students consider only their reservation prices. Following example shows that it is not correct.

Example 18 The mix algorithm reaches a problem $\widetilde{\theta}_{\bar{t}}$ such that $\widetilde{\theta}_{\bar{t}} = \widetilde{\theta}_{\bar{t}-1}$ and there is a core stable matching

Let $\mathcal{D} = \{A, B, C, D, E, F, G, H\}$ be the set of departments and $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ the set of students, where the quotas and budgets of the departments are as follows: $q_A = 1$, $q_B = 2$, $q_C = 1$, $q_D = 2$, $q_E = 1$, $q_F = 2$, $q_G = 1$, $q_H = 2$; $b_A = 50$, $b_B = 70$, $b_C = 50$, $b_D = 70$, $b_E = 50$, $b_F = 70$, $b_G = 50$, $b_H = 70$. We assume that for any $s \in \{1, 2, 3, 4, 5, 6\}$ and any $d \in \{E, F, G, H\}$ we have $a_d^s < 0$, and for any $s \in \{7, 8, 9, 10, 11, 12\}$ and any $d \in \{A, B, C, D\}$ we have $a_d^s < 0$. So, the qualification levels and reservation prices of the students are as given in table 5.11.

a_A^1=10	a_B^1=10	a_C^1= 2	a_D^1= 0	a_E^7=10	a_F^7=10	a_G^7= 8	a_H^7= 0
a_A^2= 1	a_B^2=15	a_C^2= 0	a_D^2= 5	a_E^8= 1	a_F^8=15	a_G^8= 0	a_H^8= 0
a_A^3= 8	a_B^3= 9	a_C^3= 0	a_D^3= 5	a_E^9= 8	a_F^9= 9	a_G^9= 0	a_H^9=10
a_A^4= 3	a_B^4= 0	a_C^4= 5	a_D^4= 0	a_E^{10}=12	a_F^{10}= 0	a_G^{10}= 5	a_H^{10}= 0
a_A^5= 0	a_B^5= 3	a_C^5= 0	a_D^5=10	a_E^{11}= 0	a_F^{11}= 0	a_G^{11}= 0	a_H^{11}= 4
a_A^6= 0	a_B^6= 3	a_C^6= 0	a_D^6=10	a_E^{12}= 0	a_F^{12}=20	a_G^{12}= 0	a_H^{12}= 4
σ_{1A}=40	σ_{1B}=30	σ_{1C}=20	σ_{1D}=50	σ_{7E}=40	σ_{7F}=30	σ_{7G}=50	σ_{7H}=70
σ_{2A}=50	σ_{2B}=45	σ_{2C}=30	σ_{2D}=20	σ_{8E}=50	σ_{8F}=45	σ_{8G}=40	σ_{8H}=70
σ_{3A}=30	σ_{3B}=40	σ_{3C}=50	σ_{3D}=20	σ_{9E}=30	σ_{9F}=40	σ_{9G}=50	σ_{9H}=60
σ_{4A}=20	σ_{4B}=30	σ_{4C}=40	σ_{4D}=50	σ_{10E}=50	σ_{10F}=40	σ_{10G}=30	σ_{10H}=45
σ_{5A}=45	σ_{5B}=30	σ_{5C}=40	σ_{5D}=35	σ_{11E}=40	σ_{11F}=50	σ_{11G}=45	σ_{11H}=30
σ_{6A}=40	σ_{6B}=30	σ_{6C}=50	σ_{6D}=35	σ_{12E}=45	σ_{12F}=70	σ_{12G}=50	σ_{12H}=35

Table 5.11: Qualification levels and reservation prices of students for example 18

Now, we apply the mix algorithm by applying $\widetilde{DPGAA}$ first.

$t = 1$. We apply the algorithm $\widetilde{DPGAA}$. It does not stop and a cycle occurs consisting of following four matchings:

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ B & \emptyset & B & C & D & D & G & \emptyset & H & E & \emptyset & F \end{pmatrix},$$

$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ B & \emptyset & A & C & D & D & G & \emptyset & H & E & \emptyset & F \end{pmatrix},$$

$$\mu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \emptyset & B & A & C & D & D & G & \emptyset & H & E & \emptyset & F \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ A & B & \emptyset & C & D & D & G & \emptyset & H & E & \emptyset & F \end{pmatrix}.$$

The constant part consists of the set of departments $\overline{\mathcal{D}} = \{C, D, E, F, G, H\}$ and the set of students $\overline{\mathcal{S}} = \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$. The inconstant part consists of the set of departments $\widetilde{\mathcal{D}} = \{A, B\}$ and the set students $\widetilde{\mathcal{S}} = \{1, 2, 3\}$, so $\widetilde{\theta}_1 = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}})$.

$t = 2$. We apply the algorithm $\widetilde{SPGAA}$ to the problem $\widetilde{\theta}_1 = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}})$. Note that $\widetilde{\theta}_1$ is equal to the problem given in Example 13 and we know that the algorithm $\widetilde{SPGAA}$ does not stop for that problem. So, $\widetilde{SPGAA}$ does not stop for $\widetilde{\theta}_1$ and a cycle occurs consisting of the matchings $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ and $\widetilde{\mu}_3$ that are given in Example 13. For the cycle consisting of these three matchings we have $\overline{\mathcal{D}} = \emptyset$ and $\overline{\mathcal{S}} = \emptyset$, and $\widetilde{\mathcal{D}} = \{A, B\}$ and $\widetilde{\mathcal{S}} = \{1, 2, 3\}$. So, $\widetilde{\theta}_2 = \widetilde{\theta}_1$ and the mix algorithm stops. Hence by using the mix algorithm by applying $\widetilde{DPGAA}$ first, at the end of period 2 we reach a problem $\widetilde{\theta}_2$ such that $\widetilde{\theta}_2 = \widetilde{\theta}_1$.

We now apply the mix algorithm by applying $\widetilde{SPGAA}$ first.

$t = 1$. We apply the algorithm $\widetilde{SPGAA}$. It does not stop and a cycle occurs consisting of following three matchings:

$$\widetilde{\mu}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & \emptyset & F & E & G & H & H \end{pmatrix},$$

$$\widetilde{\mu}_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & E & F & \emptyset & G & H & H \end{pmatrix},$$

$$\widetilde{\mu}_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & F & \emptyset & F & G & H & H \end{pmatrix}.$$

The constant part consists of the set of departments $\overline{\mathcal{D}} = \{A, B, C, D, G, H\}$ and the set of students $\overline{\mathcal{S}} = \{1, 2, 3, 4, 5, 6, 10, 11, 12\}$. The inconstant part consists of the set of departments $\widetilde{\mathcal{D}} = \{E, F\}$ and the set students $\widetilde{\mathcal{S}} = \{7, 8, 9\}$, so $\widetilde{\theta}_1 = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}})$.

$t = 2$. We apply the algorithm $\widetilde{DPGAA}$ to the problem $\widetilde{\theta}_1 = (\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}})$. Note that $\widetilde{\theta}_1$ is equivalent to the problem given in Example 13 and we know that the algorithm $\widetilde{DPGAA}$ does not stop for that problem.[36] So, $\widetilde{DPGAA}$ does not stop for $\widetilde{\theta}_1$ and a cycle occurs consisting of the matchings μ_1, μ_2, μ_3 and μ_4 which are given in Example 13. We have $\overline{\mathcal{D}} = \emptyset$ and $\overline{\mathcal{S}} = \emptyset$, and $\widetilde{\mathcal{D}} = \{E, F\}$ and $\widetilde{\mathcal{S}} = \{7, 8, 9\}$. So, $\widetilde{\theta}_2 = \widetilde{\theta}_1$ and the mix algorithm stops. Hence by using the mix algorithm by applying $\widetilde{SPGAA}$ first, at the end of period 2 we reach a problem $\widetilde{\theta}_2$ such that $\widetilde{\theta}_2 = \widetilde{\theta}_1$.

We now consider matching μ, where

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ C & D & D & A & B & B & G & \emptyset & H & E & \emptyset & F \end{pmatrix}.$$

It is easy to check that the matching μ is core stable. Hence, Conjecture 2 is not correct when students consider only their reservation prices.

Example 19 There does not exist a random path to core stability when students consider only their reservation prices

Let $\mathcal{D} = \{A, B, C, D\}$ be the set of departments, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ the set of students and the quotas and budgets of the departments are given by $q_A = 1$, $q_B = 2$, $q_C = 1$, $q_D = 2$; $b_A = 400$, $b_B = 1000$, $b_C = 400$, $b_D = 1000$. The qualification levels and reservation prices of the students are as given in table 5.12.

[36]We replace the names of A and B with E and F, respectively, and the names of 1, 2 and 3 with 7, 8 and 9, respectively.

$a_A^1= \ 8$	$a_B^1= \ 8$	$a_C^1= \ 0$	$a_D^1= \ 8$
$a_A^2= \ 0$	$a_B^2= \ 15$	$a_C^2= \ 0$	$a_D^2= \ 0$
$a_A^3= \ 10$	$a_B^3= \ 10$	$a_C^3= \ 0$	$a_D^3= \ 0$
$a_A^4= \ 0$	$a_B^4= \ 0$	$a_C^4= \ 5$	$a_D^4= \ 15$
$a_A^5= \ 0$	$a_B^5= \ 0$	$a_C^5= \ 10$	$a_D^5= \ 10$
$a_A^6= \ 0$	$a_B^6= \ 20$	$a_C^6= \ 7$	$a_D^6= \ 12$
$\sigma_{1A}=200$	$\sigma_{1B}= \ 300$	$\sigma_{1C}=150$	$\sigma_{1D}=100$
$\sigma_{2A}=300$	$\sigma_{2B}= \ 900$	$\sigma_{2C}=200$	$\sigma_{2D}=500$
$\sigma_{3A}=400$	$\sigma_{3B}= \ 300$	$\sigma_{3C}=100$	$\sigma_{3D}=600$
$\sigma_{4A}=200$	$\sigma_{4B}= \ 600$	$\sigma_{4C}=400$	$\sigma_{4D}=900$
$\sigma_{5A}=100$	$\sigma_{5B}= \ 800$	$\sigma_{5C}=300$	$\sigma_{5D}=200$
$\sigma_{6A}=200$	$\sigma_{6B}=1000$	$\sigma_{6C}=400$	$\sigma_{6D}=300$

Table 5.12: Qualification levels and reservation prices of students for example 19

The set of core stable matchings is nonempty for this problem, e.g., following matching μ^* is core stable:

$$\mu^* = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ D & \emptyset & A & D & C & B \end{pmatrix}.$$

We consider the following individually rational matching $\widehat{\mu}$:

$$\widehat{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \emptyset & B & A & C & D & D \end{pmatrix}.$$

We will show that starting from this matching $\widehat{\mu}$ and at each time satisfying a blocking coalition, a core stable matching will not be reached.

The unique coalition which blocks $\widehat{\mu}$ is $(B, \{1, 3\})$, and by satisfying this blocking coalition we reach matching $\widetilde{\mu}$,

$$\widetilde{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ B & \emptyset & B & C & D & D \end{pmatrix}.$$

The unique coalition which blocks $\widetilde{\mu}$ is $(A, \{1\})$. We reach matching $\overline{\mu}$ by satisfying this blocking coalition, where

$$\overline{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & \emptyset & B & C & D & D \end{pmatrix}.$$

The unique coalition which blocks $\overline{\mu}$ is $(B, \{2\})$, and by satisfying this blocking coalition we reach matching $\acute{\mu}$,

$$\acute{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ A & B & \emptyset & C & D & D \end{pmatrix}.$$

The unique coalition which blocks $\acute{\mu}$ is $(A, \{3\})$, and by satisfying this blocking coalition we reach the matching $\widehat{\mu}$ that we started with.

Hence for the graduate admission problem with quota and budget constraints when students consider only their reservation prices, there does not exist a random path to core stability.

We close this section by noting that Karakaya and Koray (2003) also showed that there exists neither a departments-optimal matching nor a students-optimal matching if students consider only their reservation prices.

5.6 Concluding remarks

As a continuation of Karakaya and Koray (2003), we studied the graduate admissions problem with quota and budget constraints as a two sided many to one matching market. We constructed the students proposing algorithm which is an extension of the Gale-Shapley algorithm. We showed that the algorithm ends up with a core stable matching if the algorithm stops. However, the algorithm may not stop for some graduate admission problems. Also it is possible that the departments proposing algorithm (constructed in Karakaya and Koray (2003)) or the students proposing algorithm does not stop and there is a core stable matching. We proved that the students proposing algorithm stops for a given problem if and only if no cycle occurs in the algorithm. We showed that the departments proposing and the students proposing algorithms are not complementary in the sense that for a given graduate admission problem if its core is non-empty then at least one of the two algorithms stops, i.e., we showed that there exist graduate admissions problems for which there exist core stable matchings, while neither of the two algorithms stops. Moreover, we showed that there does not exist a random path to core stability for the graduate admissions problem. We continued our study by modifying students' preferences in such a way that the students care only about their reservation prices. Under this model we got results similar to those obtained in the general model.

Hence, we can say that for the model defined in this paper (two sided matching market with quota and budget constraints), straightforward extensions of the Gale-Shapley algorithm do not function as well as it works for college admissions and labor market models without budget constraints. That is, the picture changes entirely when one imposes the two constraints simultaneously.

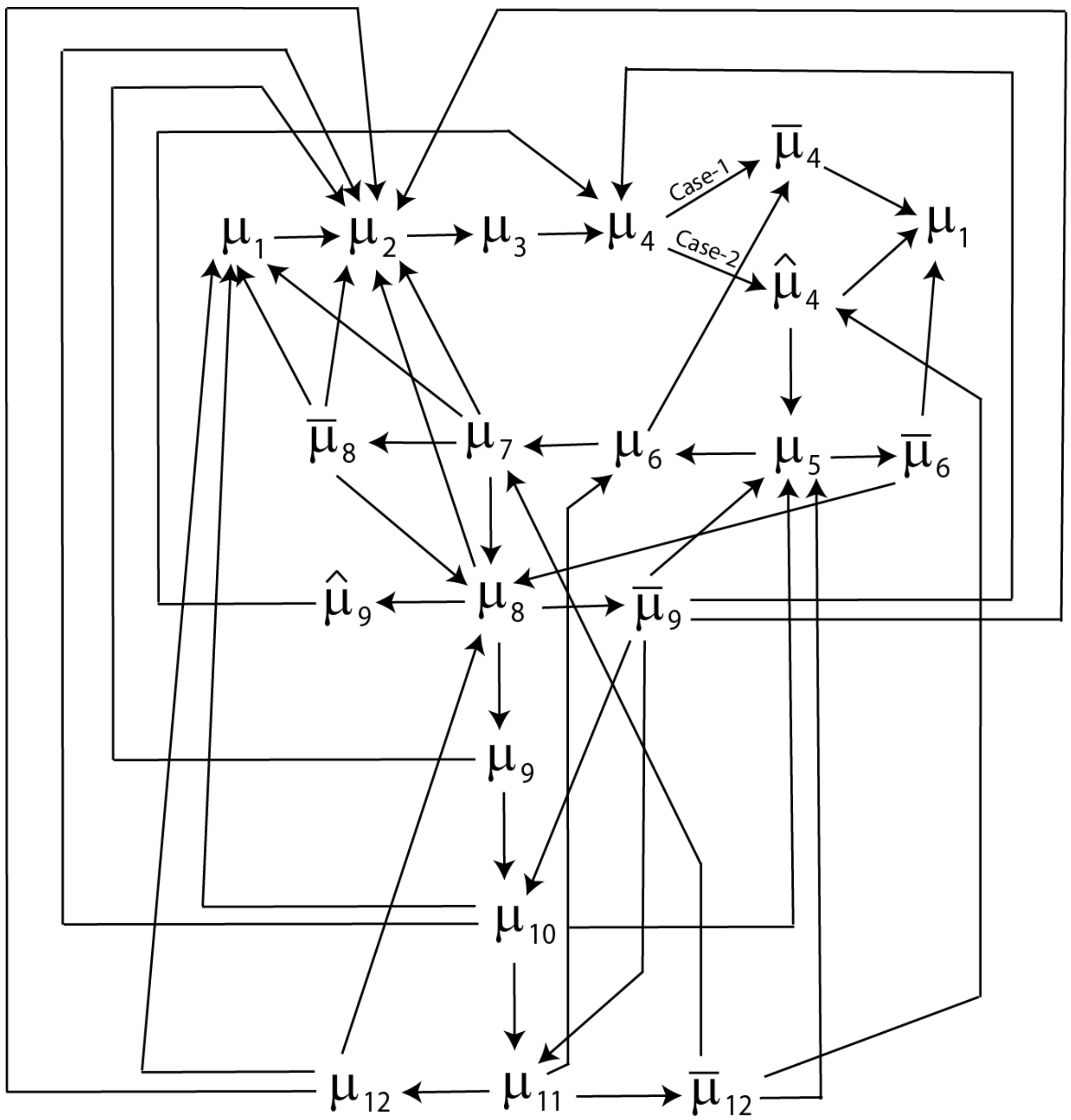

Figure 5.1: Nonexistence of random path to core stability (Example 12)

CHAPTER 6

CONCLUSION

In the previous chapters, we dealt with different economic environments as hedonic coalition formation games or cover formation games, implementation via codes of rights, a characterization of the Borda rule or graduate admissions problem with quota and budget constraints. Although the environments considered exhibit a wide variety, what combines them is the efficiency-stability or the invisible hand-design axes along which they are dealt with.

In the first main part, we introduced the framework of “membership rights” of Sertel (1992) into the context of hedonic games. We proposed a new stability notion under free exit-free entry membership rights, referred to as strong Nash stability, which had not been studied earlier. We provided sufficient conditions for a hedonic game to have a strongly Nash stable partition, and studied varying versions of strong Nash stability under different membership rights. We gave a unitary flavor to our work using the membership rights approach and thus classified all existing and newly proposed stability concepts in terms of approved vs free entry or exit. This study completes the picture on the myopic concepts of stability in hedonic games. We also extended hedonic games to cover formation games where a player can be a member of several different coalitions, and studied these games. In the second main part, we studied Nash implementation of social choice rules which are implementable via a Rechtsstaat. We showed that if a social choice rule is implementable via some Rechtsstaat satisfying equal treatment of equivalent alternatives then it is Nash implementable via a mechanism when there are three or more agents in the society. In the third main part, we studied the question of which set of axioms characterizes the Borda rule when agents have weak preferences over the set of alternatives. We showed that an social choice rule satisfies weak neutrality, reinforcement, faithfulness and degree equality if and only if it is the Borda rule. In the fourth main part, we studied the graduate admissions problem with quota and budget constraints as a two sided many to one matching market as a continuation of Karakaya and Koray (2003). Given a graduate admission problem, the main concern is to determine whether the set of core stable matchings is empty or not. If it is non-empty, then the natural question is how one can obtain a core stable matching. We constructed algorithms which are extensions of the Gale-Shapley algorithm and showed that if the algorithms stops then the resulting matchings are core stable. However, there are problems for which there exist core stable matchings, while none of the algorithms stops. Moreover, no random path to core stability exists, and the existence of either a departments-optimal or a students-optimal matching is not guaranteed in the graduate admissions problem.

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APPENDIX

We will provide omitted proofs and examples.

We now show that the weak top-choice property and the weak top-coalition property are independent of each other.

Lemma 21 *The weak top-choice property neither implies nor is implied by the weak top-coalition property.*

Proof **[the weak top-choice property $\not\Rightarrow$ the weak top-coalition property]**

Let $G = (N, \succeq)$ where $N = \{1, 2, 3, 4\}$ and players' preferences are as follows:

$\{1, 2, 3, 4\} \succ_1 \{1, 2\} \succ_1 \{1, 3\} \succ_1 \{1\} \succ_1 \ldots$,

$\{1, 2, 3, 4\} \succ_2 \{2, 3\} \succ_2 \{2\} \succ_2 \ldots$,

$\{1, 2, 3, 4\} \succ_3 \{1, 3\} \succ_3 \{2, 3\} \succ_3 \{3\} \succ_3 \ldots$,

$\{1, 4\} \succ_4 \{2, 4\} \succ_4 \{3, 4\} \succ_4 \{1, 2, 3, 4\} \succ_4 \{4\} \succ_4 \ldots$.

This hedonic game satisfies the weak top-choice property, i.e., $W(N) = \{\{N\}\}$ with $H^1 = \{1, 2, 3\}$ and $H^2 = \{4\}$. However, it fails to satisfy the weak top-coalition property, since there is no weak top-coalition of $\{1, 2, 3\}$.

[the weak top-coalition property $\not\Rightarrow$ the weak top-choice property]

Let $G = (N, \succeq)$ where $N = \{1, 2, 3, 4\}$ and players' preferences are as follows:

$\{1, 4\} \succ_1 \{1, 3\} \succ_1 \{1, 2, 4\} \succ_1 \{1, 2\} \succ_1 \{1, 2, 3, 4\} \succ_1 \{1, 2, 3\} \succ_1 \{1, 3, 4\} \succ_1 \{1\}$,

$\{2,3,4\} \succ_2 \{2,3\} \succ_2 \{2,4\} \succ_2 \{1,2\} \succ_2 \{1,2,4\} \succ_2 \{1,2,3,4\} \succ_2 \{1,2,3\} \succ_2 \{2\}$,

$\{2,3,4\} \succ_3 \{1,3,4\} \succ_3 \{2,3\} \succ_3 \{3,4\} \succ_3 \{1,2,3\} \succ_3 \{1,3\} \succ_3 \{1,2,3,4\} \succ_3 \{3\}$,

$\{1,4\} \succ_4 \{2,4\} \succ_4 \{3,4\} \succ_4 \{2,3,4\} \succ_4 \{1,2,3,4\} \succ_4 \{1,2,4\} \succ_4 \{1,3,4\} \succ_4 \{4\}$.

This hedonic game satisfies the weak top-coalition property, where
the weak top-coalition of $\{1,2,3,4\}$ is $\{1,4\}$ with $H^1 = \{1,4\}$,
the weak top-coalition of $\{1,2,3\}$ is $\{2,3\}$ with $H^1 = \{2,3\}$,
the weak top-coalition of $\{1,2,4\}$ is $\{1,4\}$ with $H^1 = \{1,4\}$,
the weak top-coalition of $\{1,3,4\}$ is $\{1,4\}$ with $H^1 = \{1,4\}$,
the weak top-coalition of $\{2,3,4\}$ is $\{2,3,4\}$ with $H^1 = \{2,3\}$ and $H^2 = \{4\}$,
and for any remaining coalition $\widetilde{N}$, the weak top-coalition of $\widetilde{N}$ is equal to itself with $H^1 = \widetilde{N}$.[1]

However, this hedonic game fails to satisfy the weak top-choice property, since $W(N) = \{\{1,4\}\}$ which is not a partition for $N = \{1,2,3,4\}$.[2] □

We introduce following notation:
For each $i \in N$ and $H \in \sigma_i$, let $Ch_i(H) = \{T \in (2^H \cap \sigma_i) \mid T \succeq_i \acute{T}$ for each $\acute{T} \in (2^H \cap \sigma_i)\}$ denote the set of maximals of i on H under $\succeq_i$. Hence for any player i, $Ch_i(N)$ denote the set of best coalitions of player i over σ_i under $\succeq_i$, i.e., $Ch_i(N) = \{H \in \sigma_i \mid H \succeq_i T$ for each $T \in \sigma_i\}$.

[1] Note that that there is no top-coalition of $\{2,3,4\}$, since for all $i \in \{2,3\}$ we have $\{2,3,4\} \succ_i T$ for any $T \subsetneq \{2,3,4\}$ with $i \in T$, however $\{2,4\} \succ_4 \{2,3,4\}$. Hence, this game does not satisfy the top-coalition property.

[2] The unique core stable partition for this game is $\pi = \{\{1,4\},\{2,3\}\}$. However π is not strongly Nash stable, since player 3 strongly Nash blocks the partition π by joining $\{1,4\}$, i.e., $\pi \xrightarrow{\{3\}} \acute{\pi} = \{\{1,3,4\},\{2\}\}$, and $\acute{\pi}(3) \succ_3 \pi(3)$. So, there does not exist a strongly Nash stable partition for this game. Hence, the weak top-coalition property does not guarantee the existence of a strongly Nash stable partition.

We now construct a hedonic game such that players have strict preferences and the weak top-choice property is satisfied, and such that the game has more than one strongly Nash stable partition.

Example 20 Let $G = (N, \succeq)$ where $N = \{1, 2, 3, 4, 5, 6\}$ and players' preferences are as follows:

$\{1, 2, 3, 4, 5, 6\} \succ_1 \{1, 6\} \succ_1 \{1\} \succ_1 \ldots$,

$\{1, 2, 3, 4, 5, 6\} \succ_2 \{2, 5\} \succ_2 \{1, 2\} \succ_2 \{2\} \succ_2 \ldots$,

$\{1, 3\} \succ_3 \{2, 3\} \succ_3 \{1, 2, 3, 4, 5, 6\} \succ_3 \{3, 4\} \succ_3 \{3\} \succ_3 \ldots$,

$\{3, 4\} \succ_4 \{2, 4\} \succ_4 \{1, 4\} \succ_4 \{1, 2, 3, 4, 5, 6\} \succ_4 \{4\} \succ_4 \ldots$,

$\{2, 5\} \succ_5 \{4, 5\} \succ_5 \{3, 5\} \succ_5 \{1, 5\} \succ_5 \{1, 2, 3, 4, 5, 6\} \succ_5 \{5\} \succ_5 \ldots$,

$\{1, 6\} \succ_6 \{5, 6\} \succ_6 \{4, 6\} \succ_6 \{3, 6\} \succ_6 \{2, 6\} \succ_6 \{1, 2, 3, 4, 5, 6\} \succ_6 \{6\} \succ_6 \ldots$.

Note that $W(N) = \{\{N\}\}$ with $H^1 = \{1, 2\}$, $H^2 = \{3\}$, $H^3 = \{4\}$, $H^4 = \{5\}$ and $H^5 = \{6\}$, i.e., this game satisfies the weak top-choice property. So, $\pi^\star = W(N) = \{\{N\}\}$ is a strongly Nash stable partition.

Now we will show that the partition $\widehat{\pi} = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$ is also a strongly Nash stable partition. Suppose that $\widehat{\pi} = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$ is not strongly Nash stable. Then, there exist a nonempty set of players $H \subseteq N$ and a partition $\pi \in (\Pi(N) \setminus \{\widehat{\pi}\})$ such that $\widehat{\pi} \xrightarrow{H} \pi$, and $\pi(i) \succ_i \widehat{\pi}(i)$ for all $i \in H$.

Note that $Ch_4(N) = \widehat{\pi}(4) = \{3, 4\}$, $Ch_5(N) = \widehat{\pi}(5) = \{2, 5\}$ and $Ch_6(N) = \widehat{\pi}(6) = \{1, 6\}$, so for any nonempty set of players H which strongly Nash blocks the partition $\widehat{\pi}$ we have $H \cap \{4, 5, 6\} = \emptyset$. So, the candidates for H which strongly Nash blocks the partition $\widehat{\pi}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$.

The only partition $\pi \in (\Pi(N) \setminus \{\widehat{\pi}\})$ for player 1 that satisfies $\pi(1) \succ_1 \widehat{\pi}(1)$ is the partition $\bar{\pi} = \{1, 2, 3, 4, 5, 6\}$, i.e., for all partitions $\tilde{\pi} \in (\Pi(N) \setminus \{\widehat{\pi}, \bar{\pi}\})$ we have $\widehat{\pi}(1) \succ_1 \tilde{\pi}(1)$, and the only partition $\pi \in (\Pi(N) \setminus \{\widehat{\pi}\})$ for player 2 that satisfies

$\pi(2) \succ_2 \widehat{\pi}(2)$ is the partition $\bar{\pi} = \{1, 2, 3, 4, 5, 6\}$.

Now, the partition $\bar{\pi}$ cannot be reachable from $\widehat{\pi}$ by any of the candidates of H that contains player 1 or player 2 or both of them which strongly Nash blocks the partition $\widehat{\pi}$, that is $\bar{\pi}$ cannot be reachable from $\widehat{\pi}$ by $\{1\}$, $\{2\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. Hence, $1 \notin H$ and $2 \notin H$ for any H which strongly Nash blocks the partition $\widehat{\pi}$. So, the only remaining candidate for H is the singleton $\{3\}$. Now, we have that

$\widehat{\pi} \xrightarrow{\{3\}} \{\{1, 6\}, \{2, 5\}, \{3\}, \{4\}\}$, but $\widehat{\pi}(3) = \{3, 4\} \succ_3 \{3\}$;

$\widehat{\pi} \xrightarrow{\{3\}} \{\{1, 3, 6\}, \{2, 5\}, \{4\}\}$, but $\widehat{\pi}(3) = \{3, 4\} \succ_3 \{1, 3, 6\}$;

$\widehat{\pi} \xrightarrow{\{3\}} \{\{1, 6\}, \{2, 3, 5\}, \{4\}\}$, but $\widehat{\pi}(3) = \{3, 4\} \succ_3 \{2, 3, 5\}$.

So, $H \neq \{3\}$.

So, there is no subset of players H which strongly Nash blocks the partition $\widehat{\pi}$, contradiction. Hence, $\widehat{\pi}$ is strongly Nash stable.

We will prove that if a hedonic game satisfies the top-choice property and every player's best coalition is unique then there exists a unique strongly Nash stable partition which consists of the top-coalitions of N.

Proposition 26 *If a hedonic game satisfies the top-choice property and* $\mid Ch_i(N) \mid= 1$ *is satisfied for each player* $i \in N$*, then there exists a unique strongly Nash stable partition* $\pi^\star$ *with* $\pi^\star = R(N)$*, where* $R(N)$ *denote the top-coalitions of the grand coalition* N*.*

Proof Let $G = (N, \succeq)$ be a hedonic game which satisfies the top-choice property and $\mid Ch_i(N) \mid= 1$ for each player $i \in N$. Let $R(N) = \{H_1, \ldots, H_K\}$ denote top-coalitions of N. $R(N)$ is a partition since the game satisfies the top-choice property.

Let $R(N) = \pi^\star$. We will show that $\pi^\star$ is strongly Nash stable. Since the game satisfies the top-choice property and $\mid Ch_i(N) \mid= 1$ is satisfied for any player $i \in N$, each player is in her unique best coalition under $\pi^\star$, that is we have $Ch_i(N) = \pi^\star(i)$ for all $i \in N$. So, for all $i \in N$, we have $\pi^\star(i) \succeq_i \pi(i)$ for any $\pi \in \Pi(N)$. Hence $\pi^\star$

is strongly Nash stable.

For uniqueness, notice that a strongly Nash stable partition must include H_1, otherwise H_1 blocks the partition since for all $i \in H_1$ we have that $H_1 \succ_i T$ for any $T \in (\sigma_i \setminus \{H_1\})$. With the same argument we say that a strongly Nash stable partition must include $H_2, \ldots, H_{K-1}$ and H_K. Hence $\pi^\star$ is the unique strongly Nash stable partition. □

We now show that the top-choice property and the top-coalition property are independent of each other.

Lemma 22 *The top-choice property neither implies nor is implied by the top-coalition property.*

Proof **[the top-choice property $\not\Rightarrow$ the top-coalition property]**

Let $G = (N, \succeq)$ where $N = \{1, 2, 3, 4\}$ and players' preferences are as follows:

$\{1, 2, 3, 4\} \succ_1 \{1, 2\} \succ_1 \ldots$,

$\{1, 2, 3, 4\} \succ_2 \{2, 3\} \succ_2 \ldots$,

$\{1, 2, 3, 4\} \succ_3 \{1, 3\} \succ_3 \ldots$,

$\{1, 2, 3, 4\} \succ_4 \{4\} \succ_4 \ldots$.

This hedonic game satisfies the top-choice property, i.e., $R(N) = \{\{1, 2, 3, 4\}\}$. However, it fails to satisfy the top-coalition property, since there is no top-coalition of $\widetilde{N} = \{1, 2, 3\}$.

[the top-coalition property $\not\Rightarrow$ the top-choice property]

Let $G = (N, \succeq)$ where $N = \{1, 2, 3\}$ and players' preferences are as follows:

$\{1\} \succ_1 \{1, 2\} \succ_1 \{1, 3\} \succ_1 \{1, 2, 3\}$,

$\{2\} \succ_2 \{1, 2\} \succ_2 \{2, 3\} \succ_2 \{1, 2, 3\}$,

$\{1,2,3\} \succ_3 \{1,3\} \succ_3 \{2,3\} \succ_3 \{3\}$.

This hedonic game satisfies the top coalition property, where for any nonempty set of players $\widetilde{N} \subseteq N$ with $1 \in \widetilde{N}$ and $2 \notin \widetilde{N}$, the top coalition of $\widetilde{N}$ is $\{1\}$, and for any nonempty set of players $\widehat{N} \subseteq N$ with $2 \in \widehat{N}$ and $1 \notin \widehat{N}$, the top coalition of $\widehat{N}$ is $\{2\}$. For any nonempty set of players $\acute{N} \subseteq N$ containing players 1 and 2, the top coalition of $\acute{N}$ is both $\{1\}$ and $\{2\}$.

However, this game does not satisfy the top-choice property, since $R(N) = \{\{1\},\{2\}\}$ which is not a partition for $N = \{1,2,3\}$.[3] □

We now show that the top-choice property and the weak top-coalition property are independent of each other.

Lemma 23 *The top-choice property neither implies nor is implied by the weak top-coalition property.*

Proof **[the top-choice property $\nRightarrow$ the weak top-coalition property]**

Let $G = (N, \succeq)$ where $N = \{1,2,3,4\}$ and players' preferences are as follows:[4]

$\{1,2,3,4\} \succ_1 \{1,2\} \succ_1 \{1,3\} \succ_1 \{1\} \succ_1 \{1,2,3\} \succ_1 \ldots$,

$\{1,2,3,4\} \succ_2 \{1,2,3\} \succ_2 \{2,3\} \succ_2 \{1,2\} \succ_2 \{2\} \succ_2 \ldots$,

$\{1,2,3,4\} \succ_3 \{1,2,3\} \succ_3 \{2,3\} \succ_3 \{1,3\} \succ_3 \{3\} \succ_3 \ldots$,

$\{1,2,3,4\} \succ_4 \{4\} \succ_4 \ldots$.

This hedonic game satisfies the top-choice property, i.e., $R(N) = \{\{1,2,3,4\}\}$. However, it fails to satisfy the weak top-coalition property since there does not exist a weak top-coalition of $\widetilde{N} = \{1,2,3\}$.

[3]The unique core stable partition for this game is $\pi = \{\{1\},\{2\},\{3\}\}$. However π is not strongly Nash stable, since player 3 strongly Nash blocks the partition π by joining $\{1\}$, i.e., $\pi \xrightarrow{\{3\}} \acute{\pi} = \{\{1,3\},\{2\}\}$, and $\acute{\pi}(3) \succ_3 \pi(3)$. So, there is no strongly Nash stable partition for this game. Hence, the top-coalition property does not guarantee the existence of a strongly Nash stable partition.

[4]This example is a modification of an example given in Bogomolnaia and Jackson (2002).

[the weak top-coalition property $\nRightarrow$ the top-choice property]

Let $G = (N, \succeq)$ where $N = \{1, 2, 3\}$ and players' preferences are as follows:[5]

$\{1, 2, 3\} \succ_1 \{1, 2\} \succ_1 \{1, 3\} \succ_1 \{1\}$,

$\{2, 3\} \succ_2 \{1, 2\} \succ_2 \{1, 2, 3\} \succ_2 \{2\}$,

$\{1, 3\} \succ_3 \{1, 2, 3\} \succ_3 \{2, 3\} \succ_3 \{3\}$.

This hedonic game satisfies the weak top-coalition property. A weak top-coalition of N is $\{1, 2, 3\}$ with corresponding partition $H^1 = \{1\}$, $H^2 = \{3\}$ and $H^3 = \{2\}$. A weak top-coalition of any $\widetilde{N}$ which is a strict subset of N is $\widetilde{N}$ with $H^1 = \widetilde{N}$. However, this game does not satisfy the top-choice property since $R(N) = \emptyset$.[6] □

We will show that preferences are descending separable and the weak top-choice properties are independent of each other.

Lemma 24 *Preferences being descending separable and the weak top-choice property are independent of each other.*

Proof **[preferences being descending separable $\nRightarrow$ the weak top-choice property]**[7]

Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and the profile of purely cardinal preferences generated by the following individual weights: $w(1) = 6$, $w(i) = 1$ for $i = 2, 3$, and $w(j) = -2$ for $j = 4, 5, 6, 7$. Players 1, 2 and 3 are each indifferent between coalitions $\{1, 2, 3, 4, 7\}$ and $\{1, 2, 3, 5, 6\}$. Now, modify these agents' preferences by making $\{1, 2, 3, 5, 6\} \succ_1 \{1, 2, 3, 4, 7\}$ and $\{1, 2, 3, 5, 6\} \succ_3 \{1, 2, 3, 4, 7\}$, and making $\{1, 2, 3, 4, 7\} \succ_2 \{1, 2, 3, 5, 6\}$, without changing any other relationships.

[5] This examle is taken from Bogomolnaia and Jackson (2002).

[6] The partition $\pi^\star = \{\{1, 2, 3\}\}$ is strongly Nash stable for this game. Hence, this example also shows that the top-choice property is not necessary for a hedonic game to have a strongly Nash stable partition.

[7] We will take the example of Burani and Zwicker (2003, page 43), and apply their methods (given in page 42) to this example.

The resulting preference profile is not purely cardinal,[8] but it is descending separable with the identity function p.

Now, the analysis of Burani and Zwicker (2003, page 43) given in the proof of their proposition 4 applies here, to show that there is no weak top-coalition of N. Hence, this game fails to satisfy the weak top-choice property.

[the weak top-choice property $\nRightarrow$ preferences being descending separable]

Let $G = (N, \succeq)$, where $N = \{1, 2, 3\}$ and players' preferences be as follows:

$\{1, 2, 3\} \succ_1 \{1, 3\} \succ_1 \{1, 2\} \succ_1 \{1\}$,

$\{1, 2\} \succ_2 \{1, 2, 3\} \succ_2 \{2, 3\} \succ_2 \{2\}$,

$\{2, 3\} \succ_3 \{1, 3\} \succ_3 \{1, 2, 3\} \succ_3 \{3\}$.

This hedonic game satisfies the weak top-choice property, i.e., $W(N) = \{\{1, 2, 3\}\}$ with $H^1 = \{1\}$, $H^2 = \{2\}$ and $H^3 = \{3\}$. Suppose that there exists a reference ranking p such that CRI is satisfied. Then, $\{2, 3\} \succ_3 \{1, 3\}$ implies that we have $2 > 1$ under p, and $\{1, 3\} \succ_1 \{1, 2\}$ implies that we have $3 > 2$ under p. So, p is such that $3 > 2 > 1$. However, for agent 2 we have $\{1, 2\} \succ_2 \{2, 3\}$, a contradiction. So, there is no p such that CRI is satisfied. Hence, players' preferences are not descending separable. □

We now provide an example showing that purely cardinal preferences are not necessary for a hedonic game to have a strongly Nash stable partition.

Example 21 Let $G = (N, \succeq)$ where $N = \{1, 2, 3\}$ and players' preferences are as follows:

$\{1, 2\} \succ_1 \{1, 3\} \succ_1 \{1, 2, 3\} \succ_1 \{1\}$,

$\{1, 2\} \succ_2 \{2\} \succ_2 \{1, 2, 3\} \succ_2 \{2, 3\}$,

$\{3\} \succ_3 \{2, 3\} \succ_3 \{1, 2, 3\} \succ_3 \{1, 3\}$.

[8] Since the resulting profile is not additively separable, i.e., for agent 2 we have $\{2, 3, 4, 7\} \sim_2 \{2, 3, 5, 6\}$ and also $\{1, 2, 3, 4, 7\} \succ_2 \{1, 2, 3, 5, 6\}$.

Now, $\pi^\star = \{\{1,2\},\{3\}\}$ is a strongly Nash stable partition. Suppose that players' preferences are purely cardinal. Since $\{1,3\} \succ_1 \{1\}$, we have $v(1,3) = w(1)+w(3) > 0 = v(1,1)$. Since $\{3\} \succ_3 \{1,3\}$, we have that $v(3,3) = 0 > w(1) + w(3) = v(1,3)$, a contradiction. So, players' preferences are not purely cardinal.[9]

We now show that preferences being purely cardinal and the weak top-choice property are independent of each other.

Lemma 25 *Purely cardinal preferences neither implies nor is implied by the weak top-choice property.*

Proof **[purely cardinal preferences $\not\Rightarrow$ the weak top-choice property]**

Let $N = \{1,2,3,4,5,6,7\}$ and players' preferences are purely cardinal with individual weights $w(1) = 6$, $w(i) = 1$ for $i = 2,3$, and $w(j) = -2$ for $j = 4,5,6,7$.[10] Now, additively separable and symmetric preferences is represented by the following functions $v = (v_i)_{i \in N}$:

	1	2	3	4	5	6	7
v_1	0	7	7	4	4	4	4
v_2	7	0	2	−1	−1	−1	−1
v_3	7	2	0	−1	−1	−1	−1
v_4	4	−1	−1	0	−4	−4	−4
v_5	4	−1	−1	−4	0	−4	−4
v_6	4	−1	−1	−4	−4	0	−4
v_7	4	−1	−1	−4	−4	−4	0

Each of the following partition is a top-segment partition and is strongly Nash stable:
$\pi_1 = \{\{1,2,3,4\},\{5\},\{6\},\{7\}\}$, $\pi_2 = \{\{1,2,3,5\},\{4\},\{6\},\{7\}\}$,
$\pi_3 = \{\{1,2,3,6\},\{4\},\{5\},\{7\}\}$, $\pi_4 = \{\{1,2,3,7\},\{4\},\{5\},\{6\}\}$.

[9] In fact, this hedonic game is not representable with additively separable and symmetric preferences, since for player 1 we have $\{1,3\} \succ_1 \{1\}$ and for player 3 we have $\{3\} \succ_3 \{1,3\}$.

[10] This example is taken from Burani and Zwicker (2003) which is used to show that a game with purely cardinal preferences need not satisfy the weak top-coalition property.

Burani and Zwicker (2003) showed that there is no weak top-coalition of N for this game, so we have that $W(N) = \emptyset$. Hence, this game fails to satisfy the weak top-choice property.

[the weak top-choice property $\not\Rightarrow$ purely cardinal preferences]

Let $G = (N, \succeq)$ where $N = \{1, 2, 3\}$ and players' preferences are as follows:

$\{1, 2, 3\} \succ_1 \{1\} \succ_1 \{1, 2\} \succ_1 \{1, 3\}$,

$\{1, 2\} \succ_2 \{1, 2, 3\} \succ_2 \{2\} \succ_2 \{2, 3\}$,

$\{2, 3\} \succ_3 \{1, 3\} \succ_3 \{1, 2, 3\} \succ_3 \{3\}$.

This game satisfies the weak top-choice property, i.e., $W(N) = \{\{1, 2, 3\}\}$ with $H^1 = \{1\}$, $H^2 = \{2\}$ and $H^3 = \{3\}$.

Suppose that players' preferences are purely cardinal. We have $\{1\} \succ_1 \{1, 2\}$, so $v(1, 1) = 0 > w(1) + w(2) = v(1, 2)$. Since $\{1, 2\} \succ_2 \{2\}$, we have $v(1, 2) = w(1) + w(2) > 0 = v(2, 2)$, a contradiction. So, players' preferences are not purely cardinal. □

We will now construct an example showing that neither the weak top-choice property nor the preferences being descending separable is necessary for a hedonic game to have a strongly Nash stable partition.

Example 22 Let $G = (N, \succeq)$, where $N = \{1, 2, 3, 4, 5\}$ and the preferences of players are as follows:

$\{1, 3, 5\} \succ_1 \{1, 2, 3\} \succ_1 \{1, 4\} \succ_1 \{1, 3\} \succ_1 \{1, 2\} \succ_1 \{1, 2, 5\} \succ_1 \{1\} \succ_1 \{N\} \succ_1 \ldots$,

$\{1, 2, 5\} \succ_2 \{1, 2\} \succ_2 \{N\} \succ_2 \{2, 3, 4, 5\} \succ_2 \{1, 2, 3\} \succ_2 \{2, 3\} \succ_2 \{2, 4, 5\} \succ_2 \{2, 4\} \succ_2 \{2\} \succ_2 \ldots$,

$\{1, 2, 3\} \succ_3 \{2, 3\} \succ_3 \{1, 3, 5\} \succ_3 \{2, 3, 4, 5\} \succ_3 \{1, 3\} \succ_3 \{3, 4\} \succ_3 \{3, 4, 5\} \succ_3 \{3\} \succ_3 \ldots$,

$\{2,3,4,5\} \succ_4 \{4,5\} \succ_4 \{N\} \succ_4 \{3,4\} \succ_4 \{2,4\} \succ_4 \{4\} \succ_4 \ldots,$

$\{N\} \succ_5 \{2,3,4,5\} \succ_5 \{1,2,5\} \succ_5 \{1,3,5\} \succ_5 \{2,4,5\} \succ_5 \{4,5\} \succ_5 \{5\} \succ_5 \ldots.$

Claim 1. The partition $\pi^\star = \{\{1,2,3\},\{4,5\}\}$ is strongly Nash stable.

Proof of Claim 1. Notice that $Ch_3(N) = \{1,2,3\} = \pi^\star(3)$, so player 3 is not a member of any subset of players H that strongly Nash blocks the partition $\pi^\star$.

Also note that $\pi^\star$ is Nash stable, that is for all $i \in N$ we have $\pi^\star(i) \succ_i H \cup \{i\}$ for any $H \in (\pi^\star \cup \{\emptyset\})$.

Now, we will show that $\pi^\star$ is core stable. Since $3 \notin H$ for any H that blocks $\pi^\star$, we have that neither the grand coalition N nor the coalition $\{2,3,4,5\}$ blocks the partition $\pi^\star$. Note that $Ch_4(N) = \{2,3,4,5\}$ and $\pi^\star(4) = \{4,5\}$ is the second best alternative for player 4. So, we have that $4 \notin H$ for any H that blocks $\pi^\star$. Remaining candidates of coalitions that block $\pi^\star$ are as follows:
$H = \{1,2,5\}$ cannot block $\pi^\star$, since we have $\{1,2,3\} \succ_1 \{1,2,5\}$ for player 1;
$H = \{1,2\}$ cannot block $\pi^\star$, since we have $\{1,2,3\} \succ_1 \{1,2\}$ for player 1;
$H = \{1,5\}$ cannot block $\pi^\star$, since we have $\{1,2,3\} \succ_1 \{1,5\}$ for player 1;
$H = \{2,5\}$ cannot block $\pi^\star$, since we have $\{4,5\} \succ_5 \{2,5\}$ for player 5.
So, there is no coalition which blocks $\pi^\star$, that is $\pi^\star$ is core stable.

Hence, the partition $\pi^\star$ is both Nash and core stable. Now, we will check other cases to show that $\pi^\star$ is also strongly Nash stable.
Players 1 and 2 cannot strongly Nash block the partition $\pi^\star$ by joining $\{4,5\}$, $\pi^\star \xrightarrow{\{1,2\}} \{\{1,2,4,5\},\{3\}\}$, since $\{1,2,3\} \succ_1 \{1,2,4,5\}$.
Players 4 and 5 cannot strongly Nash block the partition $\pi^\star$ by joining $\{1,2,3\}$, $\pi^\star \xrightarrow{\{4,5\}} \{\{N\}\}$, since $\{4,5\} \succ_4 \{N\}$.
Players 1 and 4 cannot strongly Nash block the partition $\pi^\star$ by exchanging their current coalitions, $\pi^\star \xrightarrow{\{1,4\}} \{\{1,5\},\{2,3,4\}\}$, since $\{4,5\} \succ_4 \{2,3,4\}$.
Players 1 and 5 cannot strongly Nash block the partition $\pi^\star$ by exchanging their current coalitions, $\pi^\star \xrightarrow{\{1,5\}} \{\{1,4\},\{2,3,5\}\}$, since $\{1,2,3\} \succ_1 \{1,4\}$.
Players 2 and 4 cannot strongly Nash block the partition $\pi^\star$ by exchanging their current

coalitions, $\pi^\star \xrightarrow{\{2,4\}} \{\{1,3,4\},\{2,5\}\}$, since $\{1,2,3\} \succ_2 \{2,5\}$.
Players 2 and 5 cannot strongly Nash block the partition $\pi^\star$ by exchanging their current coalitions, $\pi^\star \xrightarrow{\{2,5\}} \{\{1,3,5\},\{2,4\}\}$, since $\{1,2,3\} \succ_2 \{2,4\}$.
$H = \{1,2,4\}$ cannot strongly Nash block the partition $\pi^\star$ that players 1 and 2 leave from their current coalition and move to player 4's coalition, and player 4 leaves from her current coalition and moves to players 1 and 2's coalition, $\pi^\star \xrightarrow{\{1,2,4\}} \{\{1,2,5\},\{3,4\}\}$, since $\{4,5\} \succ_4 \{3,4\}$.
$H = \{1,2,5\}$ cannot strongly Nash block the partition $\pi^\star$ that players 1 and 2 leave from their current coalition and move to player 5's coalition, and player 5 leaves from her current coalition and moves to players 1 and 2's coalition, $\pi^\star \xrightarrow{\{1,2,5\}} \{\{1,2,4\},\{3,5\}\}$, since $\{4,5\} \succ_5 \{3,5\}$.
Hence, $\pi^\star$ is strongly Nash stable.

Claim 2. This hedonic game does not satisfy the weak top-choice property.

Proof of Claim 2. We will show that the weak top-coalitions of the grand coalition is not a partition of N.

Note that for player 1 we have $\{1\} \succ_1 \{N\}$, so N is not a weak top-coalition of itself for any ordered partition.

Also notice that $\{1,2,3\}$ is a weak top-coalition of N with $H^1 = \{3\}$, $H^2 = \{1\}$ and $H^3 = \{2\}$.

Now, we will show that $\{4,5\}$ is not a weak top-coalition of N for any ordered partition.
$\{4,5\}$ is not a weak top-coalition of N with $H^1 = \{4,5\}$, since for player 5 we have $Ch_5(N) \neq \{4,5\}$, i.e., $\{N\} \succ_5 \{4,5\}$;
$\{4,5\}$ is not a weak top-coalition of N with $H^1 = \{4\}$ and $H^2 = \{5\}$, since $Ch_4(N) \neq \{4,5\}$ and also notice that $\{1,2,5\} \subset N$, $\{1,2,5\} \succ_5 \{4,5\}$ and $\{1,2,5\} \cap \{4\} = \emptyset$;
$\{4,5\}$ is not a weak top-coalition of N with $H^1 = \{5\}$ and $H^2 = \{4\}$, since for player 5, we have $Ch_5(N) \neq \{4,5\}$.

So, $W(N) = \{\{1,2,3\}\}$ which is not a partition for N. Hence, this game does not satisfy the weak top-choice property.

Claim 3. None of the conditions of descending separable preferences is satisfied.

Proof of Claim 3. **Condition 1 (CRI)**. Suppose that there exists a reference ranking p such that CRI is satisfied. Under p, either $3 > 4$ or $4 > 3$ holds. $3 > 4$ does not hold, since we have $\{1,4\} \succ_1 \{1,3\}$ for agent 1. $4 > 3$ does not hold, since we have $\{2,3\} \succ_2 \{2,4\}$ for agent 2. So, there is no p such that CRI is satisfied.

Condition 2 (DD). Suppose that there exists a reference ranking p such that DD is satisfied. Then, either $1 > 2$ or $2 > 1$ holds under p. If $1 > 2$ holds, then $\{2,3,4,5\} \succ_2 \{2\}$ for agent 2 implies that $\{1,3,4,5\} \succ_1 \{1\}$ must hold for agent 1. However, for agent 1 we have $\{1\} \succ_1 \{1,3,4,5\}$. So, $1 > 2$ does not hold. If $2 > 1$ holds, then $\{1,3,5\} \succ_1 \{1\}$ for agent 1 implies that $\{2,3,5\} \succ_2 \{2\}$ must hold for agent 2. However, for agent 2 we have $\{2\} \succ_2 \{2,3,5\}$, so $2 > 1$ does not hold. Hence, there is no p such that DD is satisfied.

Condition 3 (SP). If SP holds, then $\{4,5\} \succ_5 \{5\}$ for agent 5 implies that $\{1,3,4,5\} \succ_5 \{1,3,5\}$. However, we have $\{1,3,5\} \succ_5 \{1,3,4,5\}$. Hence, SP is violated.

Condition 4 (GSP). If GSP holds, then $\{1,2,5\} \succ_2 \{2\}$ for agent 2 implies that $\{1,2,3,5\} \succ_2 \{2,3\}$. However, we have $\{2,3\} \succ_2 \{1,2,3,5\}$. Hence, GSP is violated.

Condition 5 (RESP). If RESP holds, then $\{1,2\} \succ_2 \{2,5\}$ for agent 2 implies that we must have $\{1,2,3,4\} \succ_2 \{2,3,4,5\}$. However, we have $\{2,3,4,5\} \succ_2 \{1,2,3,4\}$. Hence, RESP is violated.

Condition 6 (REP). Suppose that there exists a reference ranking p such that REP is satisfied. Then, either $3 > 5$ or $5 > 3$ holds under p. If $3 > 5$ holds, then $\{N\} \succ_5 \{5\}$ for agent 5 implies that $\{N\} \succ_3 \{3\}$ must hold for agent 3. However, we have $\{3\} \succ_3 \{N\}$. So, $3 > 5$ does not hold. If $5 > 3$ holds, then $\{3,4,5\} \succ_3 \{3\}$ for agent 3 implies that we must have $\{3,4,5\} \succ_5 \{5\}$ for agent 5. However, we have $\{5\} \succ_5 \{3,4,5\}$. So, $5 > 3$ does not hold. Hence, there is no p such that REP is satisfied.

We now provide omitted proofs of Chapter 3.

Proof of Lemma 10 Let F be an SCR which is implementable via Rechtsstaat $\omega = (\beta, \gamma)$. Let $i \in N$, $B \in \mathcal{P}_0(A)$ and $x \in B$.

(i) Let $B \neq A$ and $\{i\} \notin \gamma(x, y)$ for all $y \in (A \setminus B)$. We will show that $x \in Ess(F, i, B)$, i.e., we will show that there exists $R_N \in W(A)^N$ such that $L(x, R_i) \subseteq B$ and $x \in F(R_N)$. We consider profile R_N where for agent i, R_i is any weak ordering such that $L(x, R_i) = B$, and any agent $j \in (N \setminus \{i\})$ is indifferent among all alternatives under R_N, i.e., $top(R_j) = A$. We will show that $x \in F(R_N)$. Since $top(R_j) = A$ for all $j \in (N \setminus \{i\})$, for any $y \in (A \setminus B)$ we have $\beta(x, y; R_N) = \{i\}$. This fact, together with $\{i\} \notin \gamma(x, y)$ for all $y \in (A \setminus B)$, implies that $\beta(x, y; R_N) \cap \gamma(x, y) = \emptyset$ for any $y \in (A \setminus B)$. So, $x \in \varepsilon(\omega, R_N)$. As F is implementable via ω, we have $x \in F(R_N)$. Hence, $x \in Ess(F, i, B)$.

(ii) Let $B \neq A$ and $\{i\} \in \gamma(x, y)$ for some $y \in (A \setminus B)$. We will show that $x \notin Ess(F, i, B)$. Suppose not. Then, there exists $\widehat{R}_N \in W(A)^N$ such that $L(x, \widehat{R}_i) \subseteq B$ and $x \in F(\widehat{R}_N)$. Since $B \subsetneqq A$ and $L(x, \widehat{R}_i) \subseteq B$, for all $y \in (A \setminus B)$ we have $y \widehat{P}_i x$. So, $\{i\} \in \beta(x, y; \widehat{R}_N)$ for all $y \in (A \setminus B)$. This fact combined with $\{i\} \in \gamma(x, y)$ for some $y \in (A \setminus B)$ implies that there exists at least an alternative $y \in (A \setminus B)$ such that $\{i\} \in [\beta(x, y; \widehat{R}_N) \cap \gamma(x, y)]$, which is in contradiction with $x \in F(\widehat{R}_N)$. Hence, $x \notin Ess(F, i, B)$.

(iii) Let $B = A$. We will show that $Ess(F, i, B) = A$. Let $x \in A$. We consider profile R_N where $top(R_i) = \{x\}$ for agent i, and $top(R_j) = A$ for all $j \in (N \setminus \{i\})$. Now, $\beta(x, y; R_N) = \emptyset$ for all $y \in (A \setminus \{x\})$ implying that $x \in F(R_N)$. Hence $x \in Ess(F, i, B)$, and since x was arbitrary we have $Ess(F, i, B) = A$.

We now prove Corollary 1 which states that if an SCR is implementable via some Rechtsstaat satisfying $ETEA$, then it satisfies Condition μ of Moore and Repullo (1990).

Proof of Corollary 1: Let $F : W(A)^N \to \mathcal{P}_0(A)$ be an SCR that is implementable via some Rechtsstaat $\omega = (\beta, \gamma)$ satisfying $ETEA$. We will show that F satisfies Condition μ.

Let $R_N \in W(A)^N$ and $a \in A$ be such that $a \in F(R_N)$. We take $B = A$ and $C_i(a, R_N) = L(a, R_i)$, and show that conditions (i), (ii) and (iii) are satisfied.

(i) Let $\widehat{R}_N \in W(A)^N$ be such that $L(a, R_i) \subseteq L(a, \widehat{R}_i)$ for all $i \in N$. Then, $a \in F(\widehat{R}_N)$ since F satisfies Maskin monotonicity (by Proposition 9). Hence, condition (i) is satisfied.

(ii) Let $\widehat{R}_N \in W(A)^N$, $i \in N$ and $c \in A$ be such that $c \in L(a, R_i) \subseteq L(c, \widehat{R}_i)$, and $L(c, \widehat{R}_j) = A$ for all $j \in (N \setminus \{i\})$. We will show that $c \in F(\widehat{R}_N)$.

Let $\overline{R}_N \in W(A)^N$ be such that $c\overline{I}_i a$ for all $i \in N$, and for any $z, \acute{z} \in (A \setminus \{c\})$, $z\overline{R}_i\acute{z}$ if and only if $zR_i\acute{z}$. That is, for each $i \in N$, $\overline{R}_i$ is obtained from R_i by placing c indifferent to a without changing the ordering of other alternatives. Note that for all $i \in N$ we have $L(a, R_i) \cup \{c\} = L(a, \overline{R}_i)$, i.e., $L(a, R_i) \subset L(a, \overline{R}_i)$ for all $i \in N$. So, $a \in F(\overline{R}_N)$ since $a \in F(R_N)$ and F is Maskin monotonic (by Proposition 9). This result, together with $c\overline{I}_i a$ for all $i \in N$, implies that $c \in F(\overline{R}_N)$ since F satisfies $ETEA$.

We will now show that for all $i \in N$, $L(c, \overline{R}_i) \subseteq L(c, \widehat{R}_i)$. Since $L(c, \widehat{R}_j) = A$ for all $j \in (N \setminus \{i\})$, $L(c, \overline{R}_j) \subseteq L(c, \widehat{R}_j) = A$ is trivially satisfied for all $j \in (N \setminus \{i\})$. For the agent i we have $L(a, R_i) = L(c, \overline{R}_i)$ because $c \in L(a, R_i)$ and $c\overline{I}_i a$. This fact, together with $L(a, R_i) \subseteq L(c, \widehat{R}_i)$, implies that $L(c, \overline{R}_i) \subseteq L(c, \widehat{R}_i)$. Hence, for all $i \in N$, we have $L(c, \overline{R}_i) \subseteq L(c, \widehat{R}_i)$. Then, $c \in F(\widehat{R}_N)$ by Maskin monotonicity of F. Hence, condition (ii) is satisfied.

(iii) Let $\widehat{R}_N \in W(A)^N$ and $c \in A$ be such that $L(c, \widehat{R}_i) = A$ for all $i \in N$. Since $L(c, \widehat{R}_i) = A$ for all $i \in N$, for all $b \in (A \setminus \{c\})$ we have $\beta(c, b; \widehat{R}_N) = \emptyset$. So, $c \in \varepsilon(\omega, \widehat{R}_N)$. Hence, $c \in F(\widehat{R}_N)$, showing that condition (iii) is satisfied.

Hence, F satisfies Condition μ. So, F is Nash implementable by Moore and Repullo (1990)'s result when there are at least three agents in the society.

We now provide omitted proofs of Chapter 4.

We first provide an SCR which satisfies Young's cancellation property (Y-Ca) but violates degree equality (DE).

Example 23 Let $A = \{a, b, c\}$. The set of all weak preference orderings over A is given below.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}
a, b, c	a	a	b	b	c	c	a	b	c	a, b	a, c	b, c
	b	c	a	c	a	b	b, c	a, c	a, b	c	b	a
	c	b	c	a	b	a						

We define $\widetilde{F}$ as follows:

- for $\#N = 1$, $\widetilde{F}(R_k) = top(R_k)$ for all $k \in \{1, \ldots, 13\}$,
- for $\#N = 2$, $\widetilde{F}(R_{11} + R_{11}) = \{c\}$, $\widetilde{F}(R_{12} + R_{12}) = \{b\}$, $\widetilde{F}(R_{13} + R_{13}) = \{a\}$, and for any other profile $(R_j + R_k)$ we have $\widetilde{F}(R_j + R_k) = A$,
- for $\#N \geq 3$, $\widetilde{F}(R^N) = A$ for all $R^N \in W(A)^N$.

It is clear that $\widetilde{F}$ satisfies Young's cancellation property (Y-Ca). For all $x \in A$ we have $D(x, R_2 + R_4) = D(x, R_{11} + R_{11})$, however $\widetilde{F}(R_2 + R_4) \neq \widetilde{F}(R_{11} + R_{11})$. Hence, $\widetilde{F}$ violates degree equality (DE).[11]

Proof of Lemma 12 Let F be an SCR which satisfies faithfulness (F), reinforcement (RE) and degree equality (DE). Let $R^N \in W(A)^N$ and $x \in A$.

(i) Note that for any $R \in W(A)$ and any $x \in A$, we have $d(x, R) + d(x, \widehat{R}) = m+1$. So, for any $x \in A$ we have $D(x, R^N) + D(x, \widehat{R}^N) = (m+1)n$.

[11] We note that $\widetilde{F}$ also satisfies neutrality (N) and faithfulness (F), and violates reinforcement (RE).

We now consider the profile $\bar{R} \in W(A)$:

$\bar{R}$
A

For every $x \in A$, $d(x, \bar{R}) = \frac{m+1}{2}$. Consider $2n$ copies of $\bar{R}$, denoted by $\bar{R}^{2n}$. So, for every $x \in A$ we have $D(x, \bar{R}^{2n}) = (m+1)n$. So, for all $x \in A$, $D(x, R^N + \widehat{R}^N) = D(x, \bar{R}^{2n}) = (m+1)n$. Then, degree equality (DE) implies that $F(R^N + \widehat{R}^N) = F(\bar{R}^{2n})$. Faithfulness (F) and reinforcement (RE) imply that $F(\bar{R}^{2n}) = A$. Hence, $F(R^N + \widehat{R}^N) = A$.

(ii) We know that for any $x \in A$, $D(x, R^N + \widehat{R}^N) = (m+1)n$. We now consider the profile $\sum_{\tau \in \Psi_x} (R^N + \widehat{R}^N)^\tau = \sum_{\tau \in \Psi_x} (R^N)^\tau + \sum_{\tau \in \Psi_x} (\widehat{R}^N)^\tau = \mathcal{R}_x + \widehat{\mathcal{R}}_x$.

We find the total degree of every alternative at $\mathcal{R}_x + \widehat{\mathcal{R}}_x$. For alternative x, we have $D(x, \mathcal{R}_x + \widehat{\mathcal{R}}_x) = (m-1)!(m+1)n$, and for any $y \in (A \setminus \{x\})$ we have

$$
\begin{aligned}
D(y, \mathcal{R}_x + \widehat{\mathcal{R}}_x) &= (m-2)![m(m+1)n - (m+1)n] \\
&= (m-2)![(m+1)n(m-1)] \\
&= (m-1)!(m+1)n.
\end{aligned}
$$

Hence, for every $a \in A$ we have $D(a, \mathcal{R}_x + \widehat{\mathcal{R}}_x) = (m-1)!(m+1)n$.

Now, consider following profile, $\bar{R} \in W(A)$:

$\bar{R}$
A

The degree of x at $\bar{R}$ is $(m+1)/2$. Consider $2n$ copies of $\bar{R}$, denoted by $\bar{R}^{2n}$. We now consider the profile $\bar{\mathcal{R}}_x^{2n} = \sum_{\tau \in \Psi_x} (\bar{R}^{2n})^\tau$. For alternative x we have $D(x, \bar{\mathcal{R}}_x^{2n}) = (m-1)!2n\frac{m+1}{2} = (m-1)!n(m+1)$. For any $y \in (A \setminus \{x\})$, we have $D(y, \bar{\mathcal{R}}_x^{2n}) = (m-2)![m(m+1)n - n(m+1)] = (m-2)![(m+1)n(m-1)] = (m-1)!(m+1)n$.

Hence, for every $a \in A$ we have $D(a, \bar{\mathcal{R}}_x^{2n}) = (m-1)!(m+1)n$. So, for every $a \in A$ we have $D(a, \mathcal{R}_x + \widehat{\mathcal{R}}_x) = D(a, \bar{\mathcal{R}}_x^{2n})$. Then degree equality (DE) implies that $F(\mathcal{R}_x + \widehat{\mathcal{R}}_x) = F(\bar{\mathcal{R}}_x^{2n})$. Faithfulness (F) and reinforcement (RE) imply that $F(\bar{\mathcal{R}}_x^{2n}) = A$. So, $F(\mathcal{R}_x + \widehat{\mathcal{R}}_x) = A$.

Proof of Lemma 15 Let F be an SCR satisfying faithfulness (F), reinforcement (RE) and degree equality (DE). Let N be a finite set of voters, $R^N \in W(A)^N$ and $x \in A$. Consider two copies of R^N, denoted by R^{2N}. Let $D(x, R^{2N}) = \sum_{R_i \in R^{2N}} d(x, R_i) = t_x$ and $\mathcal{R}_x = \sum_{\tau \in \Psi_x} (R^{2N})^\tau$.

(i) Since $D(x, R^N) = \frac{(m+1)n}{2}$, we have $t_x = (m+1)n$. We will show that $F(\mathcal{R}_x) = A$. Let us first find the total degree of every alternative at the profile $\mathcal{R}_x$:
$D(x, \mathcal{R}_x) = (m-1)!t_x = (m-1)![(m+1)n]$,
and for any $y \in (A \setminus \{x\})$,

$$\begin{aligned} D(y, \mathcal{R}_x) &= (m-2)![m(m+1)n - t_x] \\ &= (m-2)![m(m+1)n - (m+1)n)] \\ &= (m-2)![(m+1)n(m-1)] \\ &= (m-1)![(m+1)n]. \end{aligned}$$

Hence, for all $a \in A$, $D(a, \mathcal{R}_x) = (m-1)![(m+1)n]$.

We now consider the profile $\bar{R} \in W(A)$:

$\bar{R}$
A

The degree of x at $\bar{R}$ is $\frac{m+1}{2}$. Consider $2n$ copies of $\bar{R}$, denoted by $\bar{R}^{2n}$. Consider the profile $\bar{\mathcal{R}}_x^{2n} = \sum_{\tau \in \Psi_x} (\bar{R}^{2n})^\tau$. Now,
$D(x, \bar{\mathcal{R}}_x^{2n}) = (m-1)!2n(\frac{m+1}{2}) = (m-1)!(nm+n) = (m-1)!(m+1)n$,
and for any $y \in (A \setminus \{x\})$,

$$\begin{aligned} D(y, \bar{\mathcal{R}}_x^{2n}) &= (m-2)![m(m+1)n - (nm+n)] \\ &= (m-2)![m(m+1)n - (m+1)n] \\ &= (m-1)!(m+1)n. \end{aligned}$$

Now, for all $a \in A$ we have $D(a, \mathcal{R}_x) = D(a, \bar{\mathcal{R}}_x^{2n}) = (m-1)!(m+1)n$. Then, degree equality (DE) implies that $F(\mathcal{R}_x) = F(\bar{\mathcal{R}}_x^{2n})$. Faithfulness (F) and reinforcement (RE) imply that $F(\bar{\mathcal{R}}_x^{2n}) = A$. Hence, $F(\mathcal{R}_x) = A$.

(ii) Since $D(x, R^N) < \frac{(m+1)n}{2}$, we have $t_x < (m+1)n$. We will show that $F(\mathcal{R}_x) = \{x\}$.

Let $k = (m+1)n - t_x$. Note that $k \in \mathbb{Z}_{++}$. We find the total degree of every alternative at $\mathcal{R}_x$,
$D(x, \mathcal{R}_x) = (m-1)!t_x = (m-1)![(m+1)n - k]$,
and for any $y \in (A \setminus \{x\})$,

$$\begin{aligned} D(y, \mathcal{R}_x) &= (m-2)![m(m+1)n - t_x] \\ &= (m-2)![m(m+1)n - ((m+1)n - k)] \\ &= (m-2)![(m+1)n(m-1) + k]. \end{aligned}$$

Now, consider following profile, $\bar{R} \in W(A)$:

$\bar{R}$
A

The degree of x at $\bar{R}$ is $\frac{m+1}{2}$, i.e., $d(x, \bar{R}) = \frac{m+1}{2}$. Consider $2k$ copies of $\bar{R}$, denoted by $\bar{R}^{2k}$. Consider the profile $\bar{\mathcal{R}}_x = \sum_{\tau \in \Psi_x} (\bar{R}^{2k})^\tau)$. Now,
$D(x, \bar{\mathcal{R}}_x) = (m-1)!2k(\frac{m+1}{2}) = (m-1)!(km + k)$,
and for any $y \in (A \setminus \{x\})$,
$D(y, \bar{\mathcal{R}}_x) = (m-2)![m(m+1)k - (km + k)]$.

We now consider the profile $\mathcal{R}_x + \bar{\mathcal{R}}_x$. Note that the total number of voters is $(m-1)!(2n + 2k)$ at the profile $\mathcal{R}_x + \bar{\mathcal{R}}_x$, and

$$\begin{aligned} D(x, \mathcal{R}_x + \bar{\mathcal{R}}_x) &= (m-1)![(m+1)n - k] + (m-1)!(km + k) \\ &= (m-1)![(m+1)n + km], \end{aligned}$$

and for any $y \in (A \setminus \{x\})$,

$$\begin{aligned} D(y, \mathcal{R}_x + \bar{\mathcal{R}}_x) &= (m-2)![(m+1)n(m-1) + k] + (m-2)![m(m+1)k - (km + k)] \\ &= (m-2)![(m+1)n(m-1) + m(m+1)k - km]. \end{aligned}$$

Consider following profile $\acute{R} \in W(A)$:

$\acute{R}$
$A \setminus \{y\}$
y

where $y \in (A \setminus \{x\})$. Now, the degree of x at $\acute{R}$ is $\frac{m}{2}$, i.e., $d(x, \acute{R}) = \frac{m}{2}$. Consider $2k$ copies of $\acute{R}$, denoted by $\acute{R}^{2k}$. Consider the profile $\acute{\mathcal{R}}_x = \sum_{\tau \in \Psi_x} (\acute{R}^{2k})^\tau)$. We have,
$D(x, \acute{\mathcal{R}}_x) = (m-1)!2k(\frac{m}{2}) = (m-1)!km$,
and for any $y \in (A \setminus \{x\})$,
$D(y, \acute{\mathcal{R}}_x) = (m-2)![m(m+1)k - km]$.

We again consider the profile $\bar{R} \in W(A)$:

$\bar{R}$
A

The degree of x at $\bar{R}$ is $\frac{m+1}{2}$. Consider $2n$ copies of $\bar{R}$, denoted by $\bar{R}^{2n}$. Consider the profile $\bar{\mathcal{R}}_x^{2n} = \sum_{\tau \in \Psi_x} (\bar{R}^{2n})^\tau$. Now,
$D(x, \bar{\mathcal{R}}_x^{2n}) = (m-1)!2n(\frac{m+1}{2}) = (m-1)!(nm+n) = (m-1)!(m+1)n$,
and for any $y \in (A \setminus \{x\})$,

$$\begin{aligned} D(y, \bar{\mathcal{R}}_x^{2n}) &= (m-2)![m(m+1)n - (nm+n)] \\ &= (m-2)![m(m+1)n - (m+1)n] \\ &= (m-2)!(m+1)n(m-1). \end{aligned}$$

We now consider the profile $\acute{\mathcal{R}}_x + \bar{\mathcal{R}}_x^{2n}$. Note that the total number of voters at $\acute{\mathcal{R}}_x + \bar{\mathcal{R}}_x^{2n}$ is $(m-1)!(2n+2k)$ which is equal to the number of voters at $\mathcal{R}_x + \bar{\mathcal{R}}_x$, and for every $a \in A$ we have $D(a, \acute{\mathcal{R}}_x + \bar{\mathcal{R}}_x^{2n}) = D(a, \mathcal{R}_x + \bar{\mathcal{R}}_x)$, that is,
for alternative x we have,

$$\begin{aligned} D(x, \acute{\mathcal{R}}_x + \bar{\mathcal{R}}_x^{2n}) &= D(x, \acute{\mathcal{R}}_x) + D(x, \bar{\mathcal{R}}_x^{2n}) \\ &= (m-1)!km + (m-1)!(m+1)n \\ &= (m-1)![(m+1)n + km] \\ &= D(x, \mathcal{R}_x + \bar{\mathcal{R}}_x), \end{aligned}$$

and for any $y \in (A \setminus \{x\})$ we have,

$$\begin{aligned} D(y, \acute{\mathcal{R}}_x + \bar{\mathcal{R}}_x^{2n}) &= D(y, \acute{\mathcal{R}}_x) + D(y, \bar{\mathcal{R}}_x^{2n}) \\ &= (m-2)![m(m+1)k - km] + (m-2)!(m+1)n(m-1) \\ &= (m-2)![m(m+1)k - km + (m+1)n(m-1)] \\ &= D(y, \mathcal{R}_x + \bar{\mathcal{R}}_x). \end{aligned}$$

Since F satisfies degree equality (DE), we have $F(\mathcal{R}_x + \bar{\mathcal{R}}_x) = F(\acute{\mathcal{R}}_x + \bar{\mathcal{R}}_x^{2n})$.

We now have that $F\big(\bar{\mathcal{R}}_x^{2n}\big) = A$ and $F\big(\acute{\mathcal{R}}_x\big) = \{x\}$[12] because F satisfies faithfulness (F) and reinforcement (RE). So, $F\big(\acute{\mathcal{R}}_x\big) \cap F\big(\bar{\mathcal{R}}_x^{2n}\big) = \{x\}$, then reinforcement (RE) implies that $F(\acute{\mathcal{R}}_x + \bar{\mathcal{R}}_x^{2n}) = \{x\}$.

Hence, $F(\mathcal{R}_x + \bar{\mathcal{R}}_x) = \{x\}$. By faithfulness (F) and reinforcement (RE), we have, $F\big(\bar{\mathcal{R}}_x\big) = A$. Then, $\{x\} = F(\mathcal{R}_x + \bar{\mathcal{R}}_x) = F(\mathcal{R}_x) \cap F\big(\bar{\mathcal{R}}_x\big) = F(\mathcal{R}_x) \cap A = F(\mathcal{R}_x)$.

(iii) Since $D(x, R^N) > \frac{(m+1)n}{2}$, we have $t_x > (m+1)n$. We will show that $F(\mathcal{R}_x) \subseteq A \setminus \{x\}$. Let $t_x = (m+1)n + k$. Note that $k \in \mathbb{Z}_{++}$. We calculate the total degree of every alternative at $\mathcal{R}_x$,
$D(x, \mathcal{R}_x) = (m-1)!t_x = (m-1)![(m+1)n + k]$,
and for any $y \in (A \setminus \{x\})$,

$$\begin{aligned} D(y, \mathcal{R}_x) &= (m-2)![m(m+1)n - t_x] \\ &= (m-2)![m(m+1)n - ((m+1)n + k)] \\ &= (m-2)![(m+1)n(m-1) - k]. \end{aligned}$$

Consider following profile $\acute{R} \in W(A)$:

$\acute{R}$
$A \setminus \{y\}$
y

where $y \in (A \setminus \{x\})$. Now, the degree of x at $\acute{R}$ is $\frac{m}{2}$. Consider $2k$ copies of $\acute{R}$, denoted by $\acute{R}^{2k}$. Consider the profile $\acute{\mathcal{R}}_x = \sum_{\tau \in \Psi_x} (\acute{R}^{2k})^{\tau})$. Now,
$D(x, \acute{\mathcal{R}}_x) = (m-1)!2k(\frac{m}{2}) = (m-1)!km$,
and for any $y \in (A \setminus \{x\})$,
$D(y, \acute{\mathcal{R}}_x) = (m-2)![m(m+1)k - km]$.

We now consider the profile $\mathcal{R}_x + \acute{\mathcal{R}}_x$. Note that the total number of voters is

[12] Since $x \in top\acute{R}$, for every $\tau \in \Psi_x$ we have $x \in top\acute{R}^{\tau}$, and for every $y \in (A \setminus \{x\})$ there exists $\overline{\tau} \in \Psi_x$ such that $y \notin top\acute{R}^{\overline{\tau}}$. Hence, $\bigcap_{\tau \in \Psi_x} top\acute{R}^{\tau} = \{x\}$.

$(m-1)!(2n+2k)$ at the profile $\mathcal{R}_x + \acute{\mathcal{R}}_x$, and

$$
\begin{aligned}
D(x, \mathcal{R}_x + \acute{\mathcal{R}}_x) &= (m-1)![(m+1)n + k] + (m-1)!km \\
&= (m-1)![(m+1)n + k + km],
\end{aligned}
$$

and for any $y \in (A \setminus \{x\})$,

$$
\begin{aligned}
D(y, \mathcal{R}_x + \acute{\mathcal{R}}_x) &= (m-2)![(m+1)n(m-1) - k] + (m-2)![m(m+1)k - km] \\
&= (m-2)![(m+1)n(m-1) - k + m(m+1)k - km] \\
&= (m-2)![(m+1)n(m-1) + m(m+1)k - k(m+1)] \\
&= (m-2)![(m+1)n(m-1) + k(m+1)(m-1)] \\
&= (m-2)![(m+1)(m-1)(n+k)].
\end{aligned}
$$

Now, consider following profile, $\bar{R} \in W(A)$:

$\bar{R}$
A

The degree of x at $\bar{R}$ is $\frac{m+1}{2}$. Consider $2n+2k$ copies of $\bar{R}$, denoted by $\bar{R}^{2n+2k}$. Consider the profile $\bar{\mathcal{R}}_x^{2n+2k} = \sum_{\tau \in \Psi_x} (\bar{R}^{2n+2k})^\tau$. Note that the total number of voters is $(m-1)!(2n+2k)$ at $\bar{\mathcal{R}}_x^{2n+2k}$ which is equal to the number of voters at $\mathcal{R}_x + \acute{\mathcal{R}}_x$, and for every $a \in A$ we have $D(a, \mathcal{R}_x + \acute{\mathcal{R}}_x) = D(a, \bar{\mathcal{R}}_x^{2n+2k})$, that is,
for alternative x we have,

$$
\begin{aligned}
D(x, \bar{\mathcal{R}}_x^{2n+2k}) &= (m-1)!(2n+2k)(\frac{m+1}{2}) \\
&= (m-1)!(n+k)(m+1) \\
&= (m-1)![nm + n + km + k] \\
&= (m-1)![(m+1)n + km + k] \\
&= D(x, \mathcal{R}_x + \acute{\mathcal{R}}_x),
\end{aligned}
$$

and for any $y \in (A \setminus \{x\})$ we have,

$$
\begin{aligned}
D(y, \bar{\mathcal{R}}_x^{2n+2k}) &= (m-2)![m(m+1)(n+k) - (n+k)(m+1)] \\
&= (m-2)![(m+1)(n+k)(m-1)] \\
&= D(y, \mathcal{R}_x + \acute{\mathcal{R}}_x).
\end{aligned}
$$

Since F satisfies degree equality (DE), we have $F(\mathcal{R}_x + \acute{\mathcal{R}}_x) = F(\bar{\mathcal{R}}_x^{2n+2k})$. Faithfulness (F) and reinforcement (RE) imply that $F(\bar{\mathcal{R}}_x^{2n+2k}) = A$. So, $F(\mathcal{R}_x + \acute{\mathcal{R}}_x) = A$. We have that $F(\acute{\mathcal{R}}_x) = \{x\}$ since F satisfies faithfulness (F) and reinforcement (RE).

If we suppose that $x \in F(\mathcal{R}_x)$, then $F(\mathcal{R}_x + \acute{\mathcal{R}}_x) = \{x\}$ which contradicts with that $F(\mathcal{R}_x + \acute{\mathcal{R}}_x) = A$. Hence, $x \notin F(\mathcal{R}_x)$, i.e., $F(\mathcal{R}_x) \subseteq A \setminus \{x\}$.

Proof of Lemma 16 Let F be an SCR satisfying faithfulness (F), reinforcement (RE) and degree equality (DE). Let N be a finite set of voters, $R^N \in W(A)^N$ and $x, y \in A$. Consider two copies of R^N, denoted by R^{2N}. Let $D(x, R^{2N}) = D(y, R^{2N}) = t < (m+1)n$ and $\mathcal{R}_{xy} = \sum_{\tau \in \Psi_{xy}} (R^{2N})^\tau$. We will show that $F(\mathcal{R}_{xy}) = \{x, y\}$.

Let $k = (m+1)n - t$. Note that $k \in \mathbb{Z}_{++}$. We calculate the total degree of every alternative at $\mathcal{R}_{xy}$,
$D(x, \mathcal{R}_{xy}) = D(y, \mathcal{R}_{xy}) = (m-2)!t = (m-2)![(m+1)n - k]$,
and for any $z \in (A \setminus \{x, y\})$,
$D(z, \mathcal{R}_{xy}) = (m-3)![m(m+1)n - 2t] = (m-3)![m(m+1)n - (2(m+1)n - 2k)] = (m-3)![(m+1)n(m-2) + 2k]$.

Now, consider following profile, $\bar{R} \in W(A)$:

$\bar{R}$
A

Note that $d(x, \bar{R}) = d(y, \bar{R}) = \frac{m+1}{2}$. Consider $2k$ copies of $\bar{R}$, denoted by $\bar{R}^{2k}$. Consider the profile $\bar{\mathcal{R}}_{xy} = \sum_{\tau \in \Psi_{xy}} (\bar{R}^{2k})^\tau$. We have,
$D(x, \bar{\mathcal{R}}_{xy}) = D(y, \bar{\mathcal{R}}_{xy}) = (m-2)!2k\frac{m+1}{2} = (m-2)!(km+k)$,
and for any $z \in (A \setminus \{x, y\})$,
$D(z, \bar{\mathcal{R}}_{xy}) = (m-3)![m(m+1)k - 2(km+k)]$.

We now consider the profile $\mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}$. See that the total number of voters is $(m-2)!(2n+2k)$ at the profile $\mathcal{R}_x + \bar{\mathcal{R}}_x$, and

$$
\begin{aligned}
D(x, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}) &= D(y, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}) \\
&= (m-2)![(m+1)n - k] + (m-2)!(km+k) \\
&= (m-2)![(m+1)n + km],
\end{aligned}
$$

for any $z \in (A \setminus \{x, y\})$,

$$\begin{aligned} D(z, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}) &= (m-3)![(m+1)n(m-2) + 2k] \\ &\quad +(m-3)![m(m+1)k - 2(km+k)] \\ &= (m-3)![(m+1)n(m-2) + 2k + m(m+1)k - 2km - 2k] \\ &= (m-3)![(m+1)n(m-2) + m(m+1)k - 2km]. \end{aligned}$$

Consider following profile $\acute{R} \in W(A)$:

$\acute{R}$
$A \setminus \{z\}$
z

where $z \in (A \setminus \{x, y\})$. Now, $d(x, \acute{R}) = d(y, \acute{R}) = \frac{m}{2}$. Consider $2k$ copies of $\acute{R}$, denoted by $\acute{R}^{2k}$. Consider the profile $\acute{\mathcal{R}}_{xy} = \sum_{\tau \in \Psi_{xy}} (\acute{R}^{2k})^\tau$. We have,
$D(x, \acute{\mathcal{R}}_{xy}) = D(y, \acute{\mathcal{R}}_{xy}) = (m-2)!2k\frac{m}{2} = (m-2)!km$,
and for any $z \in (A \setminus \{x, y\})$,
$D(z, \acute{\mathcal{R}}_{xy}) = (m-3)![m(m+1)k - 2km]$.

We again consider the profile $\bar{R} \in W(A)$:

$\bar{R}$
A

where, $d(x, \bar{R}) = d(y, \bar{R}) = \frac{m+1}{2}$. Consider $2n$ copies of $\bar{R}$, denoted by $\bar{R}^{2n}$. Consider the profile $\bar{\mathcal{R}}^{2n}_{xy} = \sum_{\tau \in \Psi_{xy}} (\bar{R}^{2n})^\tau$. Now,
$D(x, \bar{\mathcal{R}}^{2n}_{xy}) = D(y, \bar{\mathcal{R}}^{2n}_{xy}) = (m-2)!2n\frac{m+1}{2} = (m-2)!(nm+n) = (m-2)!(m+1)n$,
and for any $z \in (A \setminus \{x, y\})$,
$D(z, \bar{\mathcal{R}}^{2n}_{xy}) = (m-3)![m(m+1)n - 2(m+1)n] = (m-3)!(m+1)n[m-2]$.

We now consider the profile $\acute{\mathcal{R}}_{xy} + \bar{\mathcal{R}}^{2n}_{xy}$. Note that the total number of voters is $(m-2)!(2n+2k)$ which is equal to the number of voters at $\mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}$, and for every $a \in A$ we have $D(a, \acute{\mathcal{R}}_{xy} + \bar{\mathcal{R}}^{2n}_{xy}) = D(a, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy})$, that is, for alternatives x and y we have,

$$\begin{aligned} D(x, \acute{\mathcal{R}}_{xy} + \bar{\mathcal{R}}^{2n}_{xy}) &= D(y, \acute{\mathcal{R}}_{xy} + \bar{\mathcal{R}}^{2n}_{xy}) \\ &= (m-2)!km + (m-2)!(m+1)n \\ &= (m-2)![(m+1)n + km] \\ &= D(x, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}) \\ &= D(y, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}), \end{aligned}$$

and for any $z \in (A \setminus \{x, y\})$ we have,

$$\begin{aligned} D(z, \acute{\mathcal{R}}_{xy} + \bar{\mathcal{R}}^{2n}_{xy}) &= (m-3)![m(m+1)k - 2km] + (m-3)!(m+1)n[m-2] \\ &= (m-3)![(m+1)n(m-2) + m(m+1)k - 2km] \\ &= D(z, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}). \end{aligned}$$

Since F satisfies degree equality (DE), we have $F(\mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}) = F(\acute{\mathcal{R}}_{xy} + \bar{\mathcal{R}}^{2n}_{xy})$.

Now, $F\big(\bar{\mathcal{R}}^{2n}_{xy}\big) = A$ and $F\big(\acute{\mathcal{R}}_{xy}\big) = \{x, y\}$[13] because F satisfies faithfulness (F) and reinforcement (RE). So, $F\big(\acute{\mathcal{R}}_{xy}\big) \cap F\big(\bar{\mathcal{R}}^{2n}_{xy}\big) = \{x, y\}$, then reinforcement (RE) implies $F(\acute{\mathcal{R}}_{xy} + \bar{\mathcal{R}}^{2n}_{xy}) = \{x, y\}$.

Hence, $F(\mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}) = \{x, y\}$. Faithfulness (F) and reinforcement (RE) imply that $F\big(\bar{\mathcal{R}}_{xy}\big) = A$. So, $F(\mathcal{R}_{xy}) \cap F\big(\bar{\mathcal{R}}_{xy}\big) \neq \emptyset$. Then, $\{x, y\} = F(\mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy}) = F(\mathcal{R}_{xy}) \cap F\big(\bar{\mathcal{R}}_{xy}\big) = F(\mathcal{R}_{xy}) \cap A = F(\mathcal{R}_{xy})$.

Proof of Lemma 17 Let F be an SCR satisfying faithfulness (F), reinforcement (RE) and degree equality (DE). Let N be a finite set of voters, $R^N \in W(A)^N$ and $x, y \in A$.

Let $D(x, R^N) = t_x < \frac{(m+1)n}{2}$, $D(y, R^N) = t_y < \frac{(m+1)n}{2}$ and $t_x < t_y$. Let $k_x = \frac{(m+1)n}{2} - t_x$ and $k_y = \frac{(m+1)n}{2} - t_y$. Since $t_x < t_y$, we have $k_x > k_y$. Let $k = k_x - k_y$.

Consider two copies of R^N, denoted by R^{2N}, and consider $\mathcal{R}_{xy} = \sum_{\tau \in \Psi_{xy}} (R^{2N})^\tau$. We will show that $y \notin F(\mathcal{R}_{xy})$ and $x \in F(\mathcal{R}_{xy})$.

[13] Since $x, y \in top\acute{R}$, for every $\tau \in \Psi_{xy}$ we have $x, y \in top\acute{R}^\tau$, and for every $z \in (A \setminus \{x, y\})$ there exists $\overline{\tau} \in \Psi_{xy}$ such that $z \notin top\acute{R}^{\overline{\tau}}$. Hence, $\bigcap_{\tau \in \Psi_{xy}} top\acute{R}^\tau = \{x, y\}$.

We find the total degree of every alternative at $\mathcal{R}_{xy}$.

$D(x,\mathcal{R}_{xy}) = (m-2)!2t_x = (m-2)![(m+1)n-2k_x] = (m-2)![(m+1)n-2k_y-2k]$.

$D(y,\mathcal{R}_{xy}) = (m-2)!2t_y = (m-2)![(m+1)n-2k_y]$.

For any $z \in (A \setminus \{x,y\})$,

$D(z,\mathcal{R}_{xy}) = (m-3)![m(m+1)n - 2t_x - 2t_y] = (m-3)![m(m+1)n - 2(m+1)n + 4k_y + 2k]$.

Note that $2k \in \mathbb{Z}_{++}$ and $4k_y \in \mathbb{Z}_{++}$.

We now consider the profile $\bar{R} \in W(A)$, where

$\bar{R}$
A

Note that $d(x,\bar{R}) = d(y,\bar{R}) = \frac{m+1}{2}$. Consider $2k+4k_y$ copies of $\bar{R}$, denoted by $\bar{R}^{2k+4k_y}$. Consider the profile $\bar{\mathcal{R}}_{xy} = \sum_{\tau\in\Psi_{xy}} (\bar{R}^{2k+4k_y})^\tau$. We have,

$D(x,\bar{\mathcal{R}}_{xy}) = D(y,\bar{\mathcal{R}}_{xy}) = (m-2)!(2k+4k_y)\frac{m+1}{2} = (m-2)!(km+k+2k_ym+2k_y)$,

and for any $z \in (A \setminus \{x,y\})$,

$D(z,\bar{\mathcal{R}}_{xy}) = (m-3)![m(m+1)(k+2k_y) - 2(km+k+2k_ym+2k_y)]$.

We now consider the profile $\widetilde{R} \in W(A)$, where

$\widetilde{R}$
x, y
$A \setminus \{x,y\}$

Note that $d(x,\widetilde{R}) = d(y,\widetilde{R}) = \frac{3}{2}$. Consider $2k$ copies of $\widetilde{R}$, denoted by $\widetilde{R}^{2k}$. Consider the profile $\widetilde{\mathcal{R}}_{xy} = \sum_{\tau\in\Psi_{xy}} (\widetilde{R}^{2k})^\tau$. We have,

$D(x,\widetilde{\mathcal{R}}_{xy}) = D(y,\widetilde{\mathcal{R}}_{xy}) = (m-2)!(2k)\frac{3}{2} = (m-2)!3k$,

for any $z \in (A \setminus \{x,y\})$,

$D(z,\widetilde{\mathcal{R}}_{xy}) = (m-3)![m(m+1)k - 2(3k)]$.

We now consider the profile $\mathcal{R}_{xy}+\bar{\mathcal{R}}_{xy}+\widetilde{\mathcal{R}}_{xy}$. Note that the total number of voters is $(m-2)!(2n+4k+4k_y)$ at the profile $\mathcal{R}_{xy}+\bar{\mathcal{R}}_{xy}+\widetilde{\mathcal{R}}_{xy}$, and

$$\begin{aligned}
D(x,\mathcal{R}_{xy}+\bar{\mathcal{R}}_{xy}+\widetilde{\mathcal{R}}_{xy}) &= (m-2)![(m+1)n-2k_y-2k] \\
&\quad +(m-2)!(km+k+2k_ym+2k_y)+(m-2)!3k \\
&= (m-2)![(m+1)n+2k_ym+km+2k],
\end{aligned}$$

$$
\begin{aligned}
D(y, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy} + \widetilde{\mathcal{R}}_{xy}) &= (m-2)![(m+1)n - 2k_y] \\
&\quad +(m-2)!(km + k + 2k_y m + 2k_y) + (m-2)!3k \\
&= (m-2)![(m+1)n + 2k_y m + km + 4k],
\end{aligned}
$$

for any $z \in (A \setminus \{x, y\})$,

$$
\begin{aligned}
D(z, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy} + \widetilde{\mathcal{R}}_{xy}) &= (m-3)![m(m+1)n - 2(m+1)n + 4k_y + 2k] \\
&\quad +(m-3)![m(m+1)(k + 2k_y) \\
&\quad -2(km + k + 2k_y m + 2k_y)] \\
&\quad +(m-3)![m(m+1)k - 2(3k)] \\
&= (m-3)![(m+1)n(m-2) + m(m+1)(k + 2k_y) \\
&\quad -2(km + 2k_y m) + m(m+1)k - 6k].
\end{aligned}
$$

Note that faithfulness (F) and reinforcement (RE) imply that $F(\bar{\mathcal{R}}_{xy}) = A$ and $F(\widetilde{\mathcal{R}}_{xy}) = \{x, y\}$.

Now, we will construct another profile (with $(m-2)!(2n + 4k + 4k_y)$ number of voters) such that the total degree of every alternative is equal to its total degree at the profile $\mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy} + \widetilde{\mathcal{R}}_{xy}$ and such that under this profile alternative y is not chosen and alternative x is chosen by the SCR F.

Again, consider the profile $\bar{R} \in W(A)$:

$\bar{R}$
A

where $d(x, \bar{R}) = d(y, \bar{R}) = \frac{m+1}{2}$. Now, consider $2n$ copies of $\bar{R}$, denoted by $\bar{R}^{2n}$. Consider the profile $\bar{\mathcal{R}}^{2n}_{xy} = \sum_{\tau \in \Psi_{xy}} (\bar{R}^{2n})^{\tau}$. We have,
$D(x, \bar{\mathcal{R}}^{2n}_{xy}) = D(y, \bar{\mathcal{R}}^{2n}_{xy}) = (m-2)!(2n)\frac{m+1}{2} = (m-2)![(m+1)n]$,
and for any $z \in (A \setminus \{x, y\})$,
$D(z, \bar{\mathcal{R}}^{2n}_{xy}) = (m-3)![m(m+1)n - 2(m+1)n] = (m-3)!(m+1)n(m-2)$.

We now consider the profile $\breve{R} \in W(A)$:

$\breve{R}$
$A \setminus \{z\}$
z

where $z \in (A \setminus \{x, y\})$. Note that $d(x, \breve{R}) = d(y, \breve{R}) = \frac{m}{2}$. Consider $2k + 4k_y$ copies of $\breve{R}$, denoted by $\breve{R}^{2k+4k_y}$. Consider the profile $\breve{\mathcal{R}}_{xy} = \sum_{\tau \in \Psi_{xy}} (\breve{R}^{2k+4k_y})^{\tau}$. Now,
$D(x, \breve{\mathcal{R}}_{xy}) = D(y, \breve{\mathcal{R}}_{xy}) = (m-2)!(2k + 4k_y)\frac{m}{2} = (m-2)!(km + 2k_y m)$,
and for any $z \in (A \setminus \{x, y\})$,
$D(z, \breve{\mathcal{R}}_{xy}) = (m-3)![m(m+1)(k + 2k_y) - 2(km + 2k_y m)]$.

Consider the profile $\grave{R} \in W(A)$:

$\grave{R}$
x
y
$A \setminus \{x, y\}$

where $d(x, \grave{R}) = 1$ and $d(y, \grave{R}) = 2$. Consider $2k$ copies of $\grave{R}$, denoted by $\grave{R}^{2k}$. Consider the profile $\grave{\mathcal{R}}_{xy} = \sum_{\tau \in \Psi_{xy}} (\grave{R}^{2k})^{\tau}$. Now,
$D(x, \grave{\mathcal{R}}_{xy}) = (m-2)!2k$, $D(y, \grave{\mathcal{R}}_{xy}) = (m-2)!4k$,
and for any $z \in (A \setminus \{x, y\})$,
$D(z, \grave{\mathcal{R}}_{xy}) = (m-3)![m(m+1)k - (2k + 4k)] = (m-3)![m(m+1)k - 6k]$.

We now consider the profile $\bar{\mathcal{R}}^{2n}_{xy} + \breve{\mathcal{R}}_{xy} + \grave{\mathcal{R}}_{xy}$. Note that the total number of voters is $(m-2)!(2n + 4k + 4k_y)$ at the profile $\bar{\mathcal{R}}^{2n}_{xy} + \breve{\mathcal{R}}_{xy} + \grave{\mathcal{R}}_{xy}$ which is equal to the number of voters at the profile $\mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy} + \widetilde{\mathcal{R}}_{xy}$.

We now show that $D(a, \bar{\mathcal{R}}^{2n}_{xy} + \breve{\mathcal{R}}_{xy} + \grave{\mathcal{R}}_{xy}) = D(a, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy} + \widetilde{\mathcal{R}}_{xy})$ for every $a \in A$.
For alternative x,

$$\begin{aligned} D(x, \bar{\mathcal{R}}^{2n}_{xy} + \breve{\mathcal{R}}_{xy} + \grave{\mathcal{R}}_{xy}) &= (m-2)![(m+1)n + (km + 2k_y m) + 2k] \\ &= D(x, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy} + \widetilde{\mathcal{R}}_{xy}). \end{aligned}$$

For alternative y,

$$\begin{aligned} D(y, \bar{\mathcal{R}}^{2n}_{xy} + \breve{\mathcal{R}}_{xy} + \grave{\mathcal{R}}_{xy}) &= (m-2)![(m+1)n + (km + 2k_y m) + 4k] \\ &= D(y, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy} + \widetilde{\mathcal{R}}_{xy}). \end{aligned}$$

For any $z \in (A \setminus \{x, y\})$,

$$\begin{aligned} D(z, \bar{\mathcal{R}}^{2n}_{xy} + \breve{\mathcal{R}}_{xy} + \grave{\mathcal{R}}_{xy}) &= (m-3)![(m+1)n(m-2) + m(m+1)(k+2k_y) \\ &\quad -2(km + 2k_y m) + m(m+1)k - 6k] \\ &= D(z, \mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy} + \widetilde{\mathcal{R}}_{xy}). \end{aligned}$$

Hence, degree equality (DE) implies that $F(\mathcal{R}_{xy} + \bar{\mathcal{R}}_{xy} + \widetilde{\mathcal{R}}_{xy}) = F(\bar{\mathcal{R}}^{2n}_{xy} + \breve{\mathcal{R}}_{xy} + \grave{\mathcal{R}}_{xy})$.

Faithfulness (F) and reinforcement (RE) imply that $F(\bar{\mathcal{R}}^{2n}_{xy}) = A$, $F(\breve{\mathcal{R}}_{xy}) = \{x, y\}$ and $F(\grave{\mathcal{R}}_{xy}) = \{x\}$. Then, reinforcement (RE) implies that $F(\bar{\mathcal{R}}^{2n}_{xy} + \breve{\mathcal{R}}_{xy} + \grave{\mathcal{R}}_{xy}) = \{x\}$. Hence, $F(\mathcal{R}_x + \bar{\mathcal{R}}_x + \widetilde{\mathcal{R}}_{xy}) = \{x\}$. This result, together with the facts that $F(\bar{\mathcal{R}}_{xy}) = A$ and $F(\widetilde{\mathcal{R}}_{xy}) = \{x, y\}$, implies, $y \notin F(\mathcal{R}_{xy})$ and $x \in F(\mathcal{R}_{xy})$.

We now show that degree equality (DE) is stronger than cancellation property (CA).

Proof of Lemma 20 (i) Let F be an SCR which satisfies degree equality (DE). We will show that F satisfies cancellation property (CA).

Let $R^N \in W(A)^N$ be such that there exist $i, j \in N$ and $\alpha, \beta \in \{1, \ldots, m-1\}$ such that $r_\alpha(R_i) = r_{\beta+1}(R_j)$ and $r_{\alpha+1}(R_i) = r_\beta(R_j)$. Let $\acute{R}^N$ (derived from R^N) be as follows:

- for all voters $l \in (N \setminus \{i, j\})$, $\acute{R}_l = R_l$,
- for voter i, $r_\alpha(\acute{R}_i) = r_\alpha(R_i) \cup r_{\alpha+1}(R_i)$,
  for all $h < \alpha$, $r_h(\acute{R}_i) = r_h(R_i)$,
  for all $h > \alpha + 1$, $r_h(\acute{R}_i) = r_{h+1}(R_i)$,
- for voter j, $r_\beta(\acute{R}_j) = r_\beta(R_j) \cup r_{\beta+1}(R_j)$,
  for all $h < \beta$, $r_h(\acute{R}_j) = r_h(R_j)$,
  for all $h > \beta + 1$, $r_h(\acute{R}_j) = r_{h+1}(R_j)$.

In order to show that F satisfies cancellation property (CA), we need to show that $F(R^N) = F(\acute{R}^N)$. It is enough to show that for all $a \in A$ we have $D(a, R^N) = D(a, \acute{R}^N)$, since F satisfies degree equality (DE).

Let $x \in r_\alpha(R_i) = r_{\beta+1}(R_j)$ and $y \in r_{\alpha+1}(R_i) = r_\beta(R_j)$.

Since $\acute{R}_l = R_l$ for all $l \in (N \setminus \{i,j\})$, for any $a \in A$ we have $D(a, R^{N\setminus\{i,j\}}) = D(a, \acute{R}^{N\setminus\{i,j\}})$. So, we will show followings:

1- $d(x,R_i) + d(x,R_j) = d(x,\acute{R}_i) + d(x,\acute{R}_j)$,

2- $d(y,R_i) + d(y,R_j) = d(y,\acute{R}_i) + d(y,\acute{R}_j)$, and

3- for any $z \in [A \setminus \big(r_\alpha(R_i) \cup r_{\alpha+1}(R_i)\big)]$, $d(z,R_i)+d(z,R_j) = d(z,\acute{R}_i)+d(z,\acute{R}_j)$.

For voter i, we have $SU(x,R_i) = SU(x,\acute{R}_i)$, $U(x,\acute{R}_i) = U(x,R_i) \cup r_{\alpha+1}(R_i)$ and $U(x,R_i) \cap r_{\alpha+1}(R_i) = \emptyset$. For voter j, we have $U(x,R_j) = U(x,\acute{R}_j)$, $SU(x,R_j) = SU(x,\acute{R}_j) \cup r_\beta(R_j)$ and $SU(x,\acute{R}_j) \cap r_\beta(R_j) = \emptyset$. Now,

$$\begin{aligned} d(x,R_i) + d(x,R_j) &= (\underbrace{\#SU(x,R_i)}_{\#SU(x,\acute{R}_i)} + \underbrace{\#U(x,R_i)}_{\#U(x,\acute{R}_i)-\#r_{\alpha+1}(R_i)} +1)/2 \\ &\quad +(\underbrace{\#SU(x,R_j)}_{\#SU(x,\acute{R}_j)+\#r_\beta(R_j)} + \underbrace{\#U(x,R_j)}_{\#U(x,\acute{R}_j)} +1)/2, \\ &= [\#SU(x,\acute{R}_i) + \#U(x,\acute{R}_i) + 1]/2 \\ &\quad +[\#SU(x,\acute{R}_j) + \#U(x,\acute{R}_j) + 1]/2, \\ &= d(x,\acute{R}_i) + d(x,\acute{R}_j). \end{aligned}$$

For voter i, we have $U(y,R_i) = U(y,\acute{R}_i)$, $SU(y,R_i) = SU(y,\acute{R}_i) \cup r_\alpha(R_i)$ and $SU(y,\acute{R}_i) \cap r_\alpha(R_i) = \emptyset$. For voter j, we have $SU(y,R_j) = SU(y,\acute{R}_j)$, $U(y,\acute{R}_j) = U(y,R_j) \cup r_{\beta+1}(R_j)$ and $U(y,R_j) \cap r_{\beta+1}(R_j) = \emptyset$. So,

$$\begin{aligned} d(y,R_i) + d(y,R_j) &= (\underbrace{\#SU(y,R_i)}_{\#SU(y,\acute{R}_i)+\#r_\alpha(R_i)} + \underbrace{\#U(y,R_i)}_{\#U(y,\acute{R}_i)} +1)/2 \\ &\quad +(\underbrace{\#SU(y,R_j)}_{\#SU(y,\acute{R}_j)} + \underbrace{\#U(y,R_j)}_{\#U(y,\acute{R}_j)-\#r_{\beta+1}(R_j)} +1)/2, \\ &= [\#SU(y,\acute{R}_i) + \#U(y,\acute{R}_i) + 1]/2 \\ &\quad +[\#SU(y,\acute{R}_j) + \#U(y,\acute{R}_j) + 1]/2, \\ &= d(y,\acute{R}_i) + d(y,\acute{R}_j). \end{aligned}$$

Notice that for any $z \in [A \setminus \big(r_\alpha(R_i) \cup r_{\alpha+1}(R_i)\big)]$, we have $U(z,R_i) = U(z,\acute{R}_i)$,

$SU(z, R_i) = SU(z, \acute{R}_i)$, $U(z, R_j) = U(z, \acute{R}_j)$ and $SU(z, R_j) = SU(z, \acute{R}_j)$. So, for any $z \in [A \setminus \big(r_\alpha(R_i) \cup r_{\alpha+1}(R_i)\big)]$ we have $d(z, R_i) + d(z, R_j) = d(z, \acute{R}_i) + d(z, \acute{R}_j)$.

So, for all $a \in A$ we have $D(a, R^N) = D(a, \acute{R}^N)$. Then, we have $F(R^N) = F(\acute{R}^N)$ by degree equality (DE). Hence, F satisfies cancellation property (CA).

(ii) We now provide an SCR which satisfies cancellation property (CA) but violates degree equality (DE).

Let $A = \{a, b, c\}$. The set of all weak preference orderings over A is given below.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}
a, b, c	a	a	b	b	c	c	a	b	c	a, b	a, c	b, c
	b	c	a	c	a	b	b, c	a, c	a, b	c	b	a
	c	b	c	a	b	a						

We define $\grave{F}$ as follows:

- for $\#N = 1$, $\grave{F}(R_k) = A$ for all $k \in \{1, \ldots, 13\}$,
- for $\#N = 2$, $\grave{F}(R_2 + R_5) = \{c\}$, $\grave{F}(R_1 + R_4) = \{b\}$, and for any other profile $(R_j + R_k)$ we have $\grave{F}(R_j + R_k) = A$,
- for $\#N \geq 3$, $\grave{F}(R^N) = A$ for all $R^N \in W(A)^N$.

It is clear that $\grave{F}$ satisfies cancellation property (CA). For all $x \in A$ we have $D(x, R_2 + R_5) = D(x, R_1 + R_4)$, however $\grave{F}(R_2 + R_5) \neq \grave{F}(R_1 + R_4)$. Hence, $\grave{F}$ violates degree equality (DE).

The Algorithm $SPGAA$ for Example 6

We apply the algorithm $SPGAA$ to the problem given in Example 6.

Note that student 2 is not permitted to make offers to department A in the algorithm $SPGAA$, since $a_A^2 = 0$.

$t = 1$. a) Student 1 offers $b_B = 1075$ to department B, student 2 offers $b_B = 1075$ to department B and student 3 offers $b_B = 1075$ to department B, since for each student $s \in \{1, 3\}$ we have $(B, b_B)P_s(A, b_A)$ and student 2 can make offers only to department B.

b) Department A has no offer, so $S_1^A = T_1^A = \emptyset$. Department B has offers from the group students $S_1^B = \{1, 2, 3\}$ and it accepts student 2's offer and rejects the others, since $(\{2\}, b_B)P_B(\{s\}, b_B)$ for any $s \in (S_1^B \setminus \{2\})$. So, $T_1^B = \{2\}$.

c) Student 2 accepts department B's acceptance, students 1 and 3 have no acceptances.

So, we have following matching at the end of period 1:

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & \emptyset \\ 0 & 1075 & 0 \end{pmatrix}.$$

$t = 2$. a) Student 1 makes a new offer $m_{1B}(2) = b_B - 1 = 1074$ to department B, student 2 offers $b_B = 1075$ to department B and student 3 makes a new offer $m_{3B}(2) = b_B - 1 = 1074$ to department B. Note that there is no holding offer at period 2.

b) Department A has no offer, hence $S_2^A = T_2^A = \emptyset$. Department B has offers from $S_2^B = \{1, 2, 3\}$ and it accepts student 2's offer and rejects the others, so $T_2^B = \{2\}$.

c) Student 2 accepts department B's acceptance, students 1 and 3 have no acceptances, yielding the matching

$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & \emptyset \\ 0 & 1075 & 0 \end{pmatrix}.$$

Note that at further periods students 1 and 3 decrease their offers to department B, and student 2 continues to offer $b_B = 1075$ to department B. So, at some further period $t = l$ we have following:

$t = l$. a) Student 1 makes a new offer $m_{1B}(l) = 741$ to department B, student 2 offers $m_{2B}(l) = m_{2B}(1) = b_B = 1075$ to department B, and student 3 makes a new offer $m_{3B}(l) = 741$ to department B. Note that there is no holding offer at this period l.

b) Department A has no offer, so $S_l^A = T_l^A = \emptyset$. Department B has offers from $S_l^B = \{1, 2, 3\}$ and it accepts student 2's offer and rejects the others, hence $T_l^B = \{2\}$.

c) Student 2 accepts department B's acceptance, students 1 and 3 has no acceptances, which yields the matching

$$\mu_l = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & \emptyset \\ 0 & 1075 & 0 \end{pmatrix}.$$

$t = l + 1$. a) Student 1 makes a new offer $m_{1B}(l + 1) = m_{1B}(l) - 1 = 740$ to department B. Student 2 offers $m_{2B}(l + 1) = m_{2B}(1) = b_B = 1075$ to department B. Student 3 makes a new offer $m_{3A}(l + 1) = b_A = 440$ to department A, and her last new offer $m_{3B}(l) = 741$ made to department B and got rejected remains valid at this period as a holding offer $\ddot{m}_{3B}(l + 1) = m_{3B}(l) = 741$.[14]

[14] Let us make clear that why student 3 makes a new offer to department A but not to department B. Note that if student 3 would make a new offer to department B at this period the algorithm $SPGAA$ requires that it has to be $m_{3B}(l) - 1 = 740$. However, for student 3 we have $(A, 440)P_3(B, 740)$, so student 3 makes a new offer to department A and her last new offer made to department B and got rejected remains valid at this period as a holding offer.

b) Department A has an offer from student 3 and accepts her offer, i.e., $S^A_{l+1} = T^A_{l+1} = \{3\}$. Department B has offers from $S^B_{l+1} = \{1, 2, 3\}$ and it accepts student 2's offer and rejects the others, so $T^B_{l+1} = \{2\}$.

c) Student 1 has no acceptance, student 2 accepts department B's acceptance and student 3 accepts department A's acceptance.

So, at the end of period $l + 1$ we have matching

$$\mu_{l+1} = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & A \\ 0 & 1075 & 440 \end{pmatrix}.$$

At further periods, student 1 decreases her offers to department B, student 2 continues to offer $b_B = 1075$ to department B, and student 3 continues to make offers $b_A = 440$ to department A and her last new offer $m_{3B}(l) = 741$ made to department B and got rejected remains valid as a holding offer. Hence, at some further period h we have following:

$t = h$. a) Student 1 makes a new offer $m_{1A}(h) = b_A = 440$ to department A and her last new offer $m_{1B}(h - 1) = 340$ made to department B and got rejected remains valid as a holding offer $\ddot{m}_{1B}(h) = m_{1B}(h - 1) = 340$.[15] Student 2 offers $m_{2B}(h) = m_{2B}(1) = b_B = 1075$ to department B. Student 3 offers $m_{3A}(h) = m_{3A}(l + 1) = b_A = 440$ to department A, and her last new offer $m_{3B}(l) = 741$ made to department B and got rejected remains valid at this period as a holding offer $\ddot{m}_{3B}(h) = m_{3B}(l) = 741$.

b) Department A has offers from $S^A_h = \{1, 3\}$, and it accepts student 3's offer and rejects 1's offer, i.e., $T^A_h = \{3\}$. Department B has offers from $S^B_h = \{1, 2, 3\}$ and it accepts student 2's offer and rejects the others, so $T^B_h = \{2\}$.

c) Student 1 has no acceptance, student 2 accepts department B's acceptance and

[15] Student 1 makes a new offer to department A, since for student 1 we have $(A, 440)P_1(B, 339)$, where if she would make a new offer to department B at this period h the algorithm $SPGAA$ requires that $m_{1B}(h) = m_{1B}(h - 1) - 1 = 339$.

student 3 accepts department A's acceptance, yielding the matching

$$\mu_h = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & A \\ 0 & 1075 & 440 \end{pmatrix}.$$

At further periods student 1 decreases her offers to departments A and B, and we have following at some further period $\bar{k}$:

$t = \bar{k}$. a) Student 1 makes a new offer $m_{1A}(\bar{k}) = 435$ to department A and her last new offer $m_{1B}(\bar{k}-1) = 335$ made to department B and got rejected remains valid as a holding offer $\ddot{m}_{1B}(\bar{k}) = m_{1B}(\bar{k}-1) = 335$.[16] Student 2 offers $m_{2B}(\bar{k}) = m_{2B}(1) = b_B = 1075$ to department B. Student 3 offers $m_{3A}(\bar{k}) = m_{3A}(l+1) = b_A = 440$ to department A and her last new offer $m_{3B}(l) = 741$ made to department B and got rejected remains valid at this period as a holding offer $\ddot{m}_{3B}(\bar{k}) = m_{3B}(l) = 741$.

b) Department A has offers from $S^A_{\bar{k}} = \{1, 3\}$, and it accepts student 3's offer and rejects 1's offer, i.e., $T^A_{\bar{k}} = \{3\}$. Department B has offers from $S^B_{\bar{k}} = \{1, 2, 3\}$ and it accepts student 2's offer and rejects the others, so, $T^B_{\bar{k}} = \{2\}$.

c) Student 1 has no acceptance, student 2 accepts department B's acceptance and student 3 accepts department A's acceptance.

So, at the end of period $\bar{k}$ we have matching

$$\mu_{\bar{k}} = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & A \\ 0 & 1075 & 440 \end{pmatrix}.$$

$t = \bar{k} + 1$. a) Student 1 makes a new offer $m_{1B}(\bar{k}+1) = m_{1B}(\bar{k}-1) - 1 = 334$ to department B and her last new offer $m_{1A}(\bar{k}) = 435$ made to department A and got rejected remains valid as a holding offer $\ddot{m}_{1A}(\bar{k}+1) = m_{1A}(\bar{k}) = 435$. Student 2 offers $m_{2B}(\bar{k}+1) = m_{2B}(1) = b_B = 1075$ to department B. Student 3 offers $m_{3A}(\bar{k}+1) = m_{3A}(l+1) = b_A = 440$ to department A and she also has a holding offer $\ddot{m}_{3B}(\bar{k}+1) = m_{3B}(l) = 741$ to department B.

[16] Since for student 1 we have $(A, 435)P_1(B, 334)$, where if she would make a new offer to department B at this period $\bar{k}$ the algorithm $SPGAA$ requires that $m_{1B}(\bar{k}) = m_{1B}(\bar{k}-1) - 1 = 334$.

b) Department A has offers from $S^A_{\bar{k}+1} = \{1, 3\}$, and it accepts student 3's offer and rejects 1's offer, hence $T^A_{\bar{k}+1} = \{3\}$. Department B has offers from $S^B_{\bar{k}+1} = \{1, 2, 3\}$ and it accepts students 1 and 3's offers and rejects 2's offer, so $T^B_{\bar{k}+1} = \{1, 3\}$.

c) Student 1 accepts department B's acceptance. Student 2 has no acceptance. Student 3 has acceptances from departments A and B, and she accepts B's acceptance and rejects A's acceptance since $(B, 741)P_3(A, 440)$.

So, we have following matching at the end of period $\bar{k} + 1$:

$$\mu_{\bar{k}+1} = \begin{pmatrix} 1 & 2 & 3 \\ B & \emptyset & B \\ 334 & 0 & 741 \end{pmatrix}.$$

$t = \bar{k} + 2$. a) Student 1 makes a new offer $m_{1B}(\bar{k} + 1) = m_{1B}(\bar{k} - 1) - 1 = 334$ to department B and her last new offer $m_{1A}(\bar{k}) = 435$ made to department A and got rejected remains valid as a holding offer $\ddot{m}_{1A}(\bar{k} + 2) = m_{1A}(\bar{k}) = 435$. Student 2 has no new offer and his last offer $m_{2B}(\bar{k} + 1) = 1075$ made to department B and got rejected remains valid as a holding offer $\ddot{m}_{2B}(\bar{k} + 2) = m_{2B}(\bar{k} + 1) = 1075$.[17] Student 3 offers $m_{3B}(\bar{k} + 2) = m_{3B}(l) = 741$ to department B, and note that there is no holding offer for student 3 at this period.

b) Department A has an offer from student 1 and accepts her offer, i.e., $S^A_{\bar{k}+2} = T^A_{\bar{k}+2} = \{1\}$. Department B has offers from $S^B_{\bar{k}+2} = \{1, 2, 3\}$ and it accepts students 1 and 3's offers and rejects 2's offer, so $T^B_{\bar{k}+2} = \{1, 3\}$.

c) Student 1 accepts department A's acceptance and rejects B's acceptance, since for student 1 we have $(A, 435)P_1(B, 334)$. Student 2 has no acceptance. Student 3 accepts B's acceptance and rejects A's acceptance, which yields the matching

$$\mu_{\bar{k}+2} = \begin{pmatrix} 1 & 2 & 3 \\ A & \emptyset & B \\ 435 & 0 & 741 \end{pmatrix}.$$

[17]Note that at this period department B is a rejector of student 2, i.e., $F^2_{\bar{k}+2} = \{B\}$, since department B rejected student 2's offer because of the group of students $T^B_{\bar{k}+1} = \{1, 3\}$ and B matched with $T^B_{\bar{k}+1}$ at the end of period $\bar{k} + 1$.

$t = \bar{k} + 3$. a) Student 1 makes a new offer $m_{1A}(\bar{k} + 3) = m_{1A}(\bar{k}) = 435$ to department A, student 2 makes a new offer $m_{2B}(\bar{k} + 3) = m_{2B}(\bar{k} + 1) = 1075$ to department B,[18] and student 3 offers $m_{3B}(\bar{k} + 3) = 741$ to department B. Note that there is no holding offer at this period.

b) Department A has an offer from student 1 and accepts her offer, i.e., $S^A_{\bar{k}+3} = T^A_{\bar{k}+3} = \{1\}$. Department B has offers from $S^B_{\bar{k}+3} = \{2, 3\}$ and it accepts students 2's offer and rejects 3's offer, so $T^B_{\bar{k}+3} = \{2\}$.

c) Student 1 accepts department A's acceptance. Student 2 accepts B's acceptance. Student 3 has no acceptance.

So, at the end of period $\bar{k} + 3$ we have matching

$$\mu_{\bar{k}+3} = \begin{pmatrix} 1 & 2 & 3 \\ A & B & \emptyset \\ 435 & 1075 & 0 \end{pmatrix}.$$

$t = \bar{k} + 4$. a) Student 1 makes a new offer $m_{1A}(\bar{k} + 4) = m_{1A}(\bar{k}) = 435$ to department A. Student 2 makes a new offer $m_{2B}(\bar{k} + 4) = m_{2B}(\bar{k} + 1) = 1075$ to department B. Student 3 makes a new offer $m_{3A}(\bar{k} + 4) = m_{3A}(\bar{k} + 1) = 440$ to department A and her last new offer $m_{3B}(\bar{k} + 3) = 741$ made to department B and got rejected remains valid at this period as a holding offer $\ddot{m}_{3B}(\bar{k}+4) = m_{3B}(\bar{k}+3) = 741$.

b) Department A has offers from $S^A_{\bar{k}+4} = \{1, 3\}$, and it accepts student 3's offer and rejects 1's offer, hence $T^A_{\bar{k}+4} = \{3\}$. Department B has offers from $S^B_{\bar{k}+3} = \{2, 3\}$ and it accepts students 2's offer and rejects 3's offer, so $T^B_{\bar{k}+3} = \{2\}$.

c) Student 1 has no acceptance, student 2 accepts B's acceptance, and student 3 accepts department A's acceptance, which yields the matching

$$\mu_{\bar{k}+4} = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & A \\ 0 & 1075 & 440 \end{pmatrix}.$$

[18]Note that student 2 can make this offer to department B since $F^2_{\bar{k}+3} = \emptyset$.

$t = \bar{k} + 5$ a) Student 1 makes a new offer $m_{1B}(\bar{k}+5) = m_{1B}(\bar{k}+1) = 334$ to department B and her last new offer $m_{1A}(\bar{k}+4) = 435$ made to department A and got rejected remains valid as a holding offer $\ddot{m}_{1A}(\bar{k}+5) = m_{1A}(\bar{k}+4) = 435$. Student 2 offers $m_{2B}(\bar{k}+5) = 1075$ to department B. Student 3 offers $m_{3A}(\bar{k}+5) = 440$ to department A, and she also has a holding offer $\ddot{m}_{3B}(\bar{k}+5) = m_{3B}(\bar{k}+3) = 741$ to department B.

b) Department A has offers from $S^A_{\bar{k}+5} = \{1,3\}$, and it accepts student 3's offer and rejects 1's offer, so $T^A_{\bar{k}+5} = \{3\}$. Department B has offers from $S^B_{\bar{k}+5} = \{1,2,3\}$ and it accepts students 1 and 3's offers and rejects 2's offer, hence $T^B_{\bar{k}+5} = \{1,3\}$.

c) Student 1 accepts department B's acceptance. Student 2 has no acceptance. Student 3 accepts B's acceptance and rejects A's acceptance.

So, we have following matching at the end of period $\bar{k}+5$:

$$\mu_{\bar{k}+5} = \begin{pmatrix} 1 & 2 & 3 \\ B & \emptyset & B \\ 334 & 0 & 741 \end{pmatrix}.$$

Note that $\mu_{\bar{k}+5} = \mu_{\bar{k}+1}$ and if we continue we get following matchings at further periods: $\mu_{\bar{k}+6} = \mu_{\bar{k}+2}$, $\mu_{\bar{k}+7} = \mu_{\bar{k}+3}$, $\mu_{\bar{k}+8} = \mu_{\bar{k}+4}$, $\mu_{\bar{k}+9} = \mu_{\bar{k}+5} = \mu_{\bar{k}+1}$ and so on.

Hence, the algorithm $SPGAA$ does not stop and a cycle occurs consisting of following four matchings:

$$\mu_{\bar{k}+1} = \begin{pmatrix} 1 & 2 & 3 \\ B & \emptyset & B \\ 334 & 0 & 741 \end{pmatrix}, \mu_{\bar{k}+2} = \begin{pmatrix} 1 & 2 & 3 \\ A & \emptyset & B \\ 435 & 0 & 741 \end{pmatrix},$$

$$\mu_{\bar{k}+3} = \begin{pmatrix} 1 & 2 & 3 \\ A & B & \emptyset \\ 435 & 1075 & 0 \end{pmatrix}, \mu_{\bar{k}+4} = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & B & A \\ 0 & 1075 & 440 \end{pmatrix}.$$