REAL MONOMIAL BURNSIDE RINGS AND A DECOMPOSITION OF THE TOM DIECK MAP

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By

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July, 2009

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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This thesis is mainly concerned with a decomposition of the reduced tom Dieck map $\widetilde{\mathrm{die}} : A(\mathbb{R}G) \rightarrow B(G)^{\times}$ into two maps die^{+} and die^{-} of the real monomial Burnside ring. The key idea is to introduce a real Lefschetz invariant as an element of the real monomial Burnside ring and to generalize the assertion that the image of an $\mathbb{R}G$-module under the tom Dieck map coincides with the Lefschetz invariant of the sphere of the same module.

Keywords: Monomial Burnside rings, tom Dieck map, Lefschetz invariant.

ÖZET

TEKIL BURNSIDE HALKALARI VE TOM DIECK DÖNÜŞÜMÜNÜN AYRIŞIMI

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Bu tez, esas olarak tom Dieck dönüşümünün die^+ ve die^- olmak üzere gerçel tekil Burnside halkasında iki dönüşüme ayrıştırılması hakkındadır. Bunu yaparken, öncelikle gerçel Lefschetz değişmezini gerçel tekil Burnside halkasının bir elemanı olarak tanımladık. Sonra herhangi bir $\mathbb{R}G$-modülünün tom Dieck dönüşümü altındaki görüntüsünün bu modülün küresinin Lefschetz değişmeziyle aynı olduğu gerçeğini kullanarak die^+ ve die^- dönüşümlerinin gerçel tekil Burnside halkasına ait olduğunu ispatladık.

Anahtar sözcükler: Tekil Burnside halkaları, tom Dieck dönüşümü, Lefschetz değişmezi.

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Contents

Chapter 1

Introduction

This thesis aims to decompose the reduced tom Dieck map $\widetilde{\mathrm{die}}$ into two maps die^+ and die^-. We define these two maps in such a way that both of them lie in the monomial ghost ring, then we show that in fact both of them lie in the unit group of the monomial Burnside ring. We also give two applications of this result.

Since the 1980s, the tom Dieck map [4] has been studied in connection with permutation modules. We consider, more generally, a monomial scenario where the permutation sets are fibred by a cyclic group C. In particular, we concentrate on the isomorphism classes of these permutation sets. The monomial Burnside ring $B(C, G)$, introduced by A. Dress in [5], is a ring whose elements are the isomorphism classes with addition and multiplication defined in some natural way that will be explained. In Chapter 2, we introduce monomial Burnside rings. We give the basic properties of it, and we state important theorems about it which we need for the next sections.

Let $A(\mathbb{R}G)$ be the real representation ring of a finite group G, and $B(G)^\times$ the unit group of the Burnside ring $B(G)$. The tom Dieck map [4] $\mathrm{die} : A(\mathbb{R}G) \to$

$B(G)^\times$ counts dimensions of subspaces fixed by subgroups, thus

$$\mathrm{die}[M] = \bigoplus_{H \leq_G G} (-1)^{\dim(M^H)} e_H^G$$

where e_H^G is the primitive idempotent of $\mathbb{Q}B(G)$ associated with H-fixed points. The usual proof that this formula does indeed yield units in $B(G)$ (and not just in $\mathbb{Q}B(G)$) comes from a characterization in terms of the reduced Lefshcetz invariant

$$\widetilde{\Lambda}_G(S(M)) = -\bigoplus_{n=-1}^{\infty} (-1)^n [C_n]$$

as an element of the Burnside ring $B(G)$ which is introduced in [1]. Here, C_n is the set of n-simplices of a G-invariant triangulation of $S(M)$.

In Chapter 3, we introduce the Lefschetz invariant of an $\mathbb{R}G$-module M. Then we write this invariant in terms of the idempotent basis of $\mathbb{Q}B(G)$ which gives us an equivalent way to write the tom Dieck map die in Chapter 4. Indeed

$$\mathrm{die}[M] = -\widetilde{\Lambda}_G(S(M)).$$

When dealing with all of these, we give an algebraic proof of the formula of the Euler characteristic of an n-sphere which is

$$\tilde{\chi}(\mathbb{S}^n) = (-1)^n.$$

Moreover we generalize this formula to the case where the fibre group is an arbitrary cyclic group.

In Chapter 4, we deal with the exponential and tom Dieck maps of the monomial Burnside ring. Our ultimate goal, in this chapter, is to write the reduced tom Dieck map as the decomposition of two maps die^+ and die^- in $B(C_2, G)$. We state and prove a theorem which shows these two maps lie in the unit group of the monomial Burnside ring $B(C_2, G)^\times$. Out of this theorem we give two interesting corollaries.

Chapter 2

Monomial Burnside Rings

This chapter is concerned with the structure of the monomial Burnside ring introduced by A. Dress in [5]. The results and remarks in this section are taken from [2] and [3]. Throughout this chapter, let C be any cyclic group and G be a finite group. We start this chapter with the definition of the C-fibred G-sets.

A G-set S is said to be free if $\mathrm{stab}_G(s) = \{g \in G : gs = s\} = 1$ for any $s \in S$.

We write $C \times G = CG = \{cg : c \in C, g \in G\}$. A C-free CG-set is called a C-fibred G-set. A C-orbit of a C-fibred G-set S $\mathrm{orb}_C(s) = \{cs : c \in C\}$ is called a fibre. So S can be written as $S = CX = \{cx : c \in C, x \in X\}$ where X is a set of representatives of fibres.

Let CX and CY be two C-fibred G-sets. Then their coproduct $CX \sqcup CY$ is defined to be their disjoint union as sets

$$CX \sqcup CY = C(X \sqcup Y)$$

is a C-fibred G-set. Moreover C acts on the cartesian product $CX \times CY$ by $c(\xi, \eta) = (c\xi, c^{-1}\eta)$. Let $CX \otimes CY$ denote the set of C-orbits of the cartesian product $CX \times CY$. We denote $\xi \otimes \eta$ for the C-orbit containing $(\xi, \eta) \in CX \times CY$.

We let CG act on $CX \otimes CY$ as

$$cg(\xi \otimes \eta) = cg\xi \otimes g\eta.$$

We define the addition and multiplication of the isomorphism classes of C-fibred G-sets by

$$[CX] + [CY] = [CX \sqcup CY] = [C(X \sqcup Y)]$$

$$[CX][CY] = [CX \otimes CY].$$

These operations are well-defined, commutative, associative and multiplication is distributive over addition. So, the set of isomorphism classes of C-fibred G-sets forms a semiring. The monomial Burnside ring $B(C,G)$ is the Groethendieck ring associated with this semiring.

Let $C \setminus CX$ denote the set of fibres of CX. By [3] we know that CX is transitive as a CG-set if and only if $C \setminus CX$ is transitive as a G-set. In that case, CX is said to be transitive as a C fibred G-set. Thus as an abelian group, $B(C,G)$ is freely generated by the isomorphism classes of transitive C-fibred G-sets.

We define a C-character of G to be a group homomorphism $\nu : G \to C$. We define a C-subcharacter of G to be a pair (V,ν) where $V \leq G$ and ν is a C-character of V and we call $\mathrm{ch}(C,G)$ as the set of C-subcharacters of G. So we have

$$\mathrm{ch}(C,G) = \{(V,\nu) : V \leq G, \nu \in Hom(V,C)\}.$$

Then G acts on $\mathrm{ch}(C,G)$ by conjugation: ${}^g(V,\nu) = ({}^gV, {}^g\nu)$ where ${}^g\nu : {}^gV \to C$ is given by ${}^g\nu(gvg^{-1}) = \nu(v)$ for all $v \in V$. We have $(V,\nu) =_G (W,\omega)$ if (V,ν) and (W,ω) is in the same G-orbit of $\mathrm{ch}(C,G)$. Let $C_\nu G/V$ denote a transitive C-fibred G-set such that V is the stabilizer of a fibre Cx and $vx = \nu(v)x$ for all $v \in V$. The proofs of the following remarks can be found in [3].

Remark 2.1. *Given C-subcharacters (V,ν) and (W,ω) of G, then $C_\nu G/V$ is isomorphic to $C_\omega G/W$ if and only if (V,ν) is G-conjugate to (W,ω). Every transitive C-fibred G-set is isomorphic to a C-fibred G-set of the form $C_\nu G/V$.*

Remark 2.2. *As an abelian group*

$$B(C,G) = \bigoplus_{(V,\nu)\in_G \mathrm{ch}(C,G)} \mathbb{Z}[C_\nu G/V].$$

Let $O(G)$ denote the intersection of the kernels of the C-characters of G. Thus $O(G)$ is the minimal normal subgroup of G such that $G/O(G)$ is abelian with exponent dividing $|C|$. We define a C-subelement of G to be a pair $(H, hO(H))$ where $h \in H \leq G$. We usually write (H,h) rather than $(H, hO(H))$ for short. Two C-subelements (H,h) and (I,i) are equal if and only if $H = I$ and $hO(H) = iO(H)$. G acts on the C-subelements of G by conjugation: ${}^g(H,h) = ({}^gH, {}^gh)$. The G-set of the C-subelements of G is denoted by

$$\mathrm{el}(C,G) = \{(H, hO(H)) : H \leq G, hO(H) \in H/O(H)\}.$$

The species $s^G_{H,h}$ of the algebra $\mathbb{C}B(C,G)$ from $\mathbb{C}B(C,G)$ to the ground field $\mathbb{C}$ is defined to be

$$s^G_{H,h}[CX] = \sum_{Cx} \phi_x(h)$$

where Cx runs over the fibres in CX that are stabilized by H. Here ϕ_x is the C-character of H that satisfies $hx = \phi_x(h)x$ for all $h \in H$. Let CY be another C-fibred G-set. A fibre $Cx \otimes y \subseteq CX \bigotimes CY$ is stabilized by H if and only if the fibres $Cx \subseteq CX$ and $Cy \subseteq CY$ are stabilized by H. This means $\phi_{xy} = \phi_x\phi_y$. Therefore we have

$$s^G_{H,h}([CX])s^G_{H,h}([CY]) = s^G_{H,h}([CX][CY]).$$

This shows $s^G_{H,h}$ is a species. We have the following lemma from Dress.

Lemma 2.3 (Dress)**.** *Given subelements (H,h) and (I,i) of G, then $s^G_{H,h} = s^G_{I,i}$ if and only if $(H,h) =_G (I,i)$. Every species of $\mathbb{C}B(C,G)$ is of the form $s_{H,h'}$ and the species span the dual space of $\mathbb{C}B(C,G)$.*

By the lemma there exists a unique element $e^G_{H,h} \in \mathbb{C}B(C,G)$ such that

$$s^G_{I,i}(e^G_{H,h}) = \lfloor (I,i) =_G (H,h) \rfloor .$$

Also in the proof of the lemma, Dress uses the isomorphism between the algebra $\mathbb{C}B(C,G)$ and the direct sum of copies of $\mathbb{C}$. So each $e^G_{H,h}$ is a primitive idempotent and $\mathbb{C}e^G_{H,h} \cong \mathbb{C}$. Therefore we have

$$\mathbb{C}B(C,G) = \bigoplus_{(H,h)\in_G \mathrm{el}(C,G)} \mathbb{C}e^G_{H,h}.$$

So we now have the following immediate observation: With respect to the basis of the primitive idempotents any element $b \in \mathbb{C}B(C,G)$ can be written as

$$b = \sum_{(H,h)\in_G \mathrm{el}(C,G)} s^G_{H,h}(b) e^G_{H,h}.$$

The Burnside ring $B(G)$ is equal to the monomial Burnside ring with trivial fibre group, that is $B(G) = B(1,G)$. The Burnside ring can also be considered as a subset of the monomial one consisting of elements satisfying some condition. The following remark gives the characterization.

Remark 2.4. *An element $b \in \mathbb{C}B(C,G)$ belongs to $\mathbb{C}B(G)$ if and only if $s^G_{H,h}(b) = s^G_{H,1}(b)$ for all C-subelements (H,h) of G. In that case, $s^G_{H,h}(b) = s^G_H(b)$.*

Proof. Let S be a G-set, then $s^G_{H,h}[CS] = \sum_{Cs} \phi_s(h) = s^G_H[S]$ because $hs = \phi_s(h)s = s$ for all $s \in S$. So $\mathbb{C}B(G)$ is contained in the space of vectors satisfying that criterion. The reverse direction comes from counting dimensions. □

By Remarks 2.1 and 2.2 when applied to the trivial fibre group, we get G/V is isomorphic to G/W if and only if V and W are conjugate to each other. Also ch(1,G) becomes the set of conjugacy classes of subgroups of G, denoted by $\mathrm{Cl}(G)$. Hence $B(G)$ has a basis $\{[G/H] : [H] \in \mathrm{Cl}(G)\}$, and has the following $\mathbb{Z}$-module structure

$$B(G) = \bigoplus_{[H]\in \mathrm{Cl}(G)} \mathbb{Z}[G/H].$$

The monomial ghost ring $\beta(C,G)$ is defined to be the ring such that

$$\beta(C,G) = \bigoplus_{(H,h)\in \mathrm{el}(C,G)} \mathbb{Z}e^G_{H,h}$$

which contains the monomial Burnside ring. In this thesis we work with the monomial ghost ring with the fibre group C_2. And the unit group of this monomial ghost ring is

$$\beta(C_2,G)^\times = \bigoplus_{(H,h)\in \mathrm{el}(C_2,G)} \{\pm 1\}e^G_{H,h}$$

which is an elementary abelian 2-group. Moreover the unit group of the monomial Burnside ring $B(C_2,G)^\times$ is also an elementary abelian 2-group since $B(C_2,G)^\times \subseteq \beta(C_2,G)^\times$. Similarly the ghost ring of the Burnside ring $\beta(G)$ is defined to be the subring such that

$$\beta(G) = \bigoplus_{H\leq_G G} \mathbb{Z}e^G_H.$$

So the unit group of the Burnside ring is $B(G)^\times = B(G)\cap\beta(G)^\times$ which is again an elementary abelian 2-group with rank at most $|\mathrm{Cl}(G)|$.

Chapter 3

Lefschetz Invariant

In [4], Dieck pointed out the topological significance of the Burnside ring. He mentions that a finite G-simplicial complex with simplicial G-action is a combinatorial object built from finite G-sets. So one can expect some basic invariants to lie in the Burnside ring. In this chapter we shall introduce the Lefschetz invariant for the sphere of a module, which is an element of the Burnside ring. Also we will give an equivalent (topological) description of this invariant which will be the main subject of this chapter. Indeed, we are dealing with Lefschetz invariant to use it in view of the tom Dieck map of the monomial Burnside ring.

Definition 3.1. *Let M be a real vector space and $m \in M$, then the ray of m is defined to be*

$$[m] = \{\lambda m : \lambda > 0\}.$$

The sphere of M is defined to be

$$S(M) \simeq \{[m] : m \in M - \{0\}\}.$$

Definition 3.2. *Let M be an $n+1$-dimensional real vector space, choosing an inner product on M, the unit sphere is defined as $S(M) = \{m \in M : \|m\| = 1\}$.*

Actually when M is an $\mathbb{R}G$-module the two definitions determine the same objects since there is an evident G-homeomorphism between them. Moreover we have $S(M) \cong \mathbb{S}^n$.

Definition 3.3. *Let M be an $\mathbb{R}G$-module. Then the reduced Lefschetz invariant of M, denoted by $\widetilde{\Lambda_G}(S(M))$, is an element of $B(G)$ defined by*

$$\widetilde{\Lambda_G}(S(M)) = -\bigoplus_{n=-1}^{\infty}(-1)^n[C_n]$$

where C_n is the set of n-simplices of a triangulation of $S(M)$ and C_{-1} is the set with a single element which is the empty set.

A triangulation is said to be admissible if whenever an element of G stabilizes a simplex of the triangulation, then it stabilizes that simplex pointwise. Although we can use any triangulation to define Lefschetz invariant, the triangulation we will be using throughout the thesis will be the octahedral triangulation.

An $\mathbb{R}G$-module M is said to be monomial if there exists $\mathbb{R}$-vector space decomposition

$$M = M_0 \oplus \ldots \oplus M_n$$

into 1-dimensional subspaces where each M_i is permuted by G. Here the stabilizer of M_i acts on M_i as multiplication by ± 1 for each i. So if we have a monomial $\mathbb{R}G$-module M, then there is an evident triangulation $O(M)$ of $S(M)$ whose set of vertices is $S(M) \cap (M_0 \cup \ldots \cup M_n)$ which is called octahedral triangulation. Let $K(M)$ be the barycentric subdivision of $O(M)$, we will use $K(M)$ in our calculations. We pass to the barycentric subdivision because $O(M)$ is not an admissible triangulation, that is the stabilizer of a simplex does not fix the simplex. As pointed out in [1], with the triangulation $K(M)$, C_n becomes a permutation module. Although the permutation basis depends on a choice of orientations consistent over G-orbits, the isomorphism type of the permutation representation is well-defined and gives us a well-defined element $[C_n]$ of $B(G)$. So this explains why the Lefschetz invariant lies in $B(G)$.

We aim to give an equivalent way to define the reduced Lefschetz invariant in terms of the idempotent basis of $\mathbb{Q}B(G)$ which is in the following theorem.

Theorem 3.4. *The reduced Lefschetz invariant of an* $\mathbb{R}G$*-module* M *is*

$$\widetilde{\Lambda_G}(S(M)) = -\bigoplus_{H \leq_G G} (-1)^{\dim(M^H)} e_H^G.$$

This theorem reduces to the problem of calculating the species of $\widetilde{\Lambda_G}(S(M))$ associated with the trivial subgroup. That is, we can reduce to the following problem which does not involve any group actions. From now on, in our calculations we deal with the $\mathbb{R}G$-module M with $M = \mathbb{R}X$ where X is a finite G-set.

Lemma 3.5. *We have*

$$-\sum_{r=-1}^{\infty} (-1)^r |C_r| = (-1)^n$$

where C_r *is the set of* r*-simplices of the triangulation of* X *and* n *is the number of elements of* X*.*

Proving this lemma requires some technical argument and terminology. Let P be a finite poset, then it gives rise to a simplicial complex whose r-simplices are the r-chains $(x_0 < \ldots < x_r)$ with each $x_r \in P$. Let $\tilde{\mathrm{sd}}(P)$ denote the set of simplices in P with the -1 simplex $*$ allowed, and let $\mathrm{sd}(P)$ denote the set of simplices with $*$ disallowed. Thus we have $\tilde{\mathrm{sd}}(P) = \mathrm{sd}(P) \cup \{*\}$.
We define the **reduced Euler characteristic** $\tilde{\chi}(P)$ and the **unreduced Euler characteristic** $\chi(P)$ to be

$$\tilde{\chi}(P) = \sum_{r=-1}^{n} (-1)^r c_r(P), \qquad \chi(P) = \sum_{r=0}^{n} (-1)^r c_r(P)$$

where $c_r(P)$ is the number of r-chains. Another way of defining this is

$$\tilde{\chi}(P) = \sum_{x \in \tilde{\mathrm{sd}}(P)} (-1)^{l(x)}, \qquad \chi(P) = \sum_{x \in \mathrm{sd}(P)} (-1)^{l(x)}$$

where $l(x)$ is the length of the chain,that is if $x = (x_0, \ldots, x_r)$ then $l(x) = r$.

Lemma 3.5 is equivalent to saying the n-sphere has reduced Euler characteristic $(-1)^n$. Actually this has a well-known easy proof which we shall give below, but the reason is that we are trying to give a purely algebraic proof is to gain algebraic insight into the tom Dieck map of the monomial Burnside ring. Before handling this algebraic proof let us give the following.

Remark 3.6. *The n-sphere $\mathbb{S}^n$ has reduced Euler characteristic $(-1)^n$.*

Proof. We will use induction. Let X_n be a set with size n, and let $\tilde{K_n}$ be the simplicial complex such that the vertices are $\{S : \emptyset \subset S \subset X_n\}$ and the n simplexes are $S_0 \subset S_1 \subset \ldots \subset S_n$. So the set of r-simplexes is

$$C_r(\tilde{K_n}) = \{\emptyset \subset S_0 \subset \ldots \subset S_r \subset X_n\}.$$

This simplicial complex is homotopy equivalent to $\mathbb{S}^n$, so they must have the same Euler characteristic. Thus we have

$$\tilde{\chi}(\mathbb{S}^n) = \tilde{\chi}(\tilde{K_n}) = \sum_{r=-1}^{n-1} (-1)^r |C_r(\tilde{K_n})|.$$

Let us denote $c_{r,n} = |C_r(\tilde{K_n})|$ and $\tilde{\chi}(n) = -1 + \sum_{r=0}^{n-1}(-1)^r c_{r,n}$. Fix some S_r with size s where $1 \leq s \leq n-1$. Then the number of chains $\emptyset \subset S_0 \subset \ldots \subset S_{r-1} \subset S_r$ is $c_{r-1,s}$. Since we can choose $\binom{n}{s}$ different sets with size s, then the number of r-chains is

$$c_{r,n} = \sum_{s=1}^{n-1} \binom{n}{s} c_{r-1,s}$$

for $r \geq 0$. Then we have

$$\tilde{\chi}(n) = -1 + \sum_{r=0}^{n-1} \sum_{s=1}^{n-1} (-1)^r \binom{n}{s} c_{r-1,s}$$

$$= -1 - \sum_{r=-0}^{n-1} \sum_{s=1}^{n-1} (-1)^{r-1} \binom{n}{s} c_{r-1,s}$$

$$= -1 - \sum_{s=1}^{n-1} \binom{n}{s} \sum_{r=0}^{s} (-1)^{r-1} c_{r-1,s}$$

$$= -1 - \sum_{s=1}^{n-1} \binom{n}{s} (-1 + \sum_{q=-1}^{s-1} (-1)^q c_{q,s})$$

$$= -1 - \sum_{s=1}^{n-1} \binom{n}{s} \tilde{\chi}(s)$$

$$= -1 - \sum_{s=1}^{n-1} \binom{n}{s} (-1)^s = (-1)^n.$$

□

Now let us concentrate on the proof of Lemma 3.5. Let K_n be the poset of vectors $z = (z_0, \ldots, z_n)$ where each $z_i \in \{-1, 0, 1\}$ and some $z_i \neq 0$. The ordering relation is such that $z \leq z'$ provided $z_i = z_i{}'$ whenever $z_i \neq 0$.

Before proving Lemma 3.5 we need some properties of the reduced Euler characteristics of posets.

Remark 3.7. *Let P be a finite poset with a unique maximal element. Then $\tilde{\chi}(P) = 0$.*

Proof. Let m be the unique maximal element. We define a function $f : \tilde{\mathrm{sd}}(P) \to \tilde{\mathrm{sd}}(P)$ as follows. Consider a chain $x = (x_0 < \ldots < x_n)$. If $x_n \neq m$ let $f(x)$ be the $n+1$- chain $(x_0 < \ldots < x_n < m)$, that is, the chain obtained from x by inserting m. If $x_n = m$, let $f(x)$ be the $n-1$-chain $(x_0 < \ldots < x_{n-1})$ the chain obtained from x by deleting m. Then it is easy to see that $f^2(x) = x$. Thus we can pair each chain x with a chain $f(x)$. Also note that the 0-chain (m) is paired with the -1- chain $* = \emptyset$. Moreover the length of each chain is of opposite parity to the length of its partner. □

Corollary 3.8. *For a non-empty finite set X, let $P^+(X)$ be the poset of non-empty subsets of X, the ordering relation being inclusion. Then $\tilde{\chi}(P^+(X)) = 0$.*

Proof. This follows from Remark 3.6, because the poset $P^+(X)$ has a unique maximal element X. □

Proposition 3.9. *For a non-empty finite set X with size n, let $P_-^+(X) = \{Y : \emptyset < Y < X\}$, the poset of proper subsets of X, ordered by inclusion. Then $\tilde{\chi}(P_-^+(X)) = (-1)^n$.*

This proposition is equivalent to the formula $\tilde{\chi}(\mathbb{S}^n) = (-1)^n$, because $P_-^+(X)$ is the triangulation of an n-sphere. And actually it has the same proof as the previous Remark 3.6 about the Euler characteristic of a sphere. So we can omit the proof of it.

Proof of Lemma 3.5. We prove this by induction. The chains in $K(n)$ are the elements of $\mathrm{sd}(K(n))$. These chains have length between 0 and n. For $0 \leq r \leq n$, let us write any r-chain $\sigma = (\sigma^0, \ldots, \sigma^r)$ and $\sigma^j = (z_0^j, \ldots, z_n^j)$ where $z_i^j \in \{-1, 0, 1\}$ for each j.

Let A_* be the set of chains σ such that $z_i^r \neq 0$ for all i. That is, A_* consists of chains σ whose top element σ^r has the form $\sigma^r = (\pm 1, \ldots, \pm 1)$ with no zero coordinates.

For each $0 \leq i \leq n$, let A_i be the set of chains σ such that $z_i^r = 0$. Then each chain σ belongs either A_* or $A_0 \cup \ldots \cup A_n$. Since the sets A_* and $A_0 \cup \ldots \cup A_n$ are disjoint and their union is $\mathrm{sd}(K(n))$, so

$$\chi(\mathrm{sd}(K(n)) = \chi(n) = \chi(A_*) + \chi(A_0 \cup \ldots \cup A_n).$$

Let us first deal with A_*. Let us fix the top element $\sigma^r = (\pm 1, \ldots, \pm 1)$. Each r-chain with this top element σ^r corresponds to an $(r-1)$-chain in $P_-^+\{0, \ldots, n\}$, such as 0- chain (σ^r) corresponds to the -1-chain $\emptyset$. So the contribution to $\chi(A_*)$ from all the chains with that fixed top element σ^r is $\tilde{\chi}(P_-^+\{0, \ldots, n\}) = (-1)^n$, by Proposition 3.9. There are 2^{n+1} choices for such a top element σ^r, so

$$\chi(A_*) = (-1)^n 2^{n+1}.$$

Now let us deal with $\chi(A_0 \cup \ldots \cup A_n)$. Obviously the sets A_i intersect, so we must use the Inclusion-Exclusion Formula which gives,

$$\chi(A_0 \cup \ldots \cup A_n) = \chi(A_0) + \ldots + \chi(A_n) -$$

$$\chi(A_0 \cap A_1) + \ldots + \chi(A_{n-1} \cap A_n))$$

$$\chi(A_0 \cap A_1 \cap A_2) + \ldots + \chi(A_{n-2} \cap A_{n-1} \cap A_n)) -$$

$$\chi(A_0 \cap A_1 \cap A_2 \cap A_3) + \ldots + \chi(A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n)) + \ldots$$

The value of χ for the intersection of $s+1$ distinct sets A_i does not depend on s. Indeed, for mutually distinct indices $i_0, \ldots, i_s$ we have

$$\chi(A_{i_0} \cap \ldots \cap A_{i_s}) = \chi(A_0 \cap \ldots \cap A_s) = \chi(n-s).$$

Therefore we have,

$$\chi(A_0 \cup \ldots \cup A_n) = \binom{n+1}{1}\chi(n-1) - \binom{n+1}{2}\chi(n-2) + \ldots$$

$$+(-1)^{s-1}\binom{n+1}{s}\chi(n-s) + \ldots + (-1)^{n-1}\binom{n+1}{n}\chi(0).$$

Trivially, $\chi(0) = 2$. Suppose $n \geq 1$ then by the induction hypothesis we have $\chi(m) = 1 + (-1)^m$ for $0 \leq m < n$. Then

$$\chi(A_0 \cup \ldots \cup A_n) = \binom{n+1}{1} - \binom{n+1}{2} + \ldots + (-1)^{n-1}\binom{n+1}{n} +$$

$$(-1)^{n-1}(\binom{n+1}{1} + \binom{n+1}{2} + \ldots + (-1)^{n-1}\binom{n+1}{n})$$

$$= -(1-1)^{n+1} + \binom{n+1}{0} - (-1)^n\binom{n+1}{n+1} + (-1)^{n-1}((1+1)^{n+1} - \binom{n}{0} - \binom{n+1}{n+1})$$

$$= 1 + (-1)^n - (-1)^n 2^{n+1}.$$

So, as desired we have

$$\chi(n) = \chi(A_*) + \chi(A_0 \cup \ldots \cup A_n) = 1 + (-1)^n.$$

□

Now we can prove theorem 3.4.

Proof of Theorem 3.4. We would like to find $s_H^G(\widetilde{\Lambda_G})$ and since

$$s_H^G(\widetilde{\Lambda_G}) = -\sum_{n=-1}^{\infty} (-1)^n s_H([C_n])$$

$$= -\sum_{n=-1}^{\infty} (-1)^n |{C_n}^H| = f(|H \setminus X|)$$

because the number of n-simplices that are fixed by H depends on $H \setminus X$, the set of H-orbits in X. Moreover by Lemma we have

$$s_1^G(\widetilde{\Lambda_G}) = -\sum_{n=-1}^{\infty} (-1)^n |C_n| = (-1)^{|X|} = f(|X|).$$

So we have

$$s_H^G(\widetilde{\Lambda_G}) = f(|H \setminus X|) = (-1)^{|H \setminus X|}.$$

Now it remains to show that

$$\dim(M^H) = |H \setminus X|.$$

Let $m = \sum_{x \in X} \lambda_x x \in M^H$, then

$$h(\sum_{x \in X} \lambda_x x) = \sum_{x \in X} \lambda_x x$$

for any $h \in H$. Then

$$\sum_{x\in X} \lambda_x x = \sum_{x\in X} \lambda_x hx = \sum_{x\in X} \lambda_{h^{-1}x} x.$$

So $m \in M$ if and only if $\lambda_x = \lambda_{h^{-1}x}$ for all $h \in H$. So we have

$$\widetilde{\Lambda_G}(S(M)) = -\bigoplus_{H\leq_G G} (-1)^{\dim(M^H)} e_H^G.$$

□

We close this chapter with generalizing Lemma 3.5. Actually this gives us a hope of generalizing the result about decomposing the reduced tom Dieck map $\widetilde{\text{die}}$ to the monomial Burnside rings with an arbitrary fibre group.

As usual, let X be a finite G-set with n elements and $M = \mathbb{R}X$ and $S(M)$ be the unit sphere of M. And let C_m be the fibre group, the cyclic group of order m. Let K_n^m be the poset of vectors $z = (z_0, \cdots, z_n)$ where each $z_i \in \{0, c_1, \cdots, c_m\}$ where $c_i \in C_m$ for any i and some $z_i \neq 0$. The ordering relation is again the following: $z \leq z'$ provided $z_i = z_i{}'$ whenever $z_i \neq 0$.

Proposition 3.10. *The reduced Euler characteristic of K_n^m is*

$$\tilde{\chi}(K_n^m) = (-1)^n (m-1)^{n+1}.$$

Proof. $K(n)$ consists of the chains of length between 0 and n. For $0 \leq r \leq n$, let us write any r-chain $\sigma = (\sigma^0, \ldots, \sigma^r)$ and $\sigma^j = (z_0^j, \ldots, z_n^j)$ for each j. let A_* be the set of chains σ such that $z_i^r \neq 0$ for all i. And let A_i be the set of chains σ such that $z_i^r = 0$. Then each chain belongs either A_* or $A_0 \cup \ldots \cup A_n$. So denoting $\chi(K_n^m)$ as $\chi(n)$ we get

$$\chi(n) = \chi(A_*) + \chi(A_0 \cup \ldots \cup A_n).$$

Let us first deal with A_*. Fix a top element σ_r, then each r-chain with this top element corresponds to an $r-1$-chain in $P_-^+(\{0, \cdots, n\})$. So the contribution to

$\chi(A_*)$ with this fixed top element σ_r is $\tilde{\chi}(P_-^+(\{0,\cdots,n\})) = (-1)^n$ by Proposition 3.9. Since there are m^{n+1} choices for such a top element σ_r

$$\chi(A_*) = (-1)^n m^{n+1}.$$

Let us consider the term $\chi(A_0 \cup \ldots \cup A_n)$. By the Inclusion-Exclusion Formula we have

$$\chi(A_0 \cup \ldots \cup A_n) = \chi(A_0) + \ldots + \chi(A_n) -$$

$$\chi(A_0 \cap A_1) + \ldots + \chi(A_{n-1} \cap A_n))$$

$$\chi(A_0 \cap A_1 \cap A_2) + \ldots + \chi(A_{n-2} \cap A_{n-1} \cap A_n)) -$$

$$\chi(A_0 \cap A_1 \cap A_2 \cap A_3) + \ldots + \chi(A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n)) + \ldots$$

The value of χ for the intersection of $s+1$ distinct sets A_i does not depend on s. In fact, for mutually distinct indices $i_0, \ldots, i_s$, we have

$$\chi(A_{i_0} \cap \ldots \cap A_{i_s}) = \chi(A_0 \cap \ldots \cap A_s) = \chi(n-s).$$

Therefore we have,

$$\chi(A_0 \cup \ldots \cup A_n) = \binom{n+1}{1}\chi(n-1) - \binom{n+1}{2}\chi(n-2) + \ldots$$

$$+(-1)^{s-1}\binom{n+1}{s}\chi(n-s) + \ldots + (-1)^{n-1}\binom{n+1}{n}\chi(0).$$

By induction assumption we have $\chi(j) = 1 + (-1)^j (m-1)^{j+1}$ for $0 \leq j < n$. Then

$$\chi(A_0 \cup \ldots \cup A_n) = \binom{n+1}{1} - \binom{n+1}{2} + \ldots + (-1)^{n-1}\binom{n+1}{n} +$$

$$(-1)^{n-1}\left(\binom{n+1}{1}(m-1)^n + \binom{n+1}{2}(m-1)^{n-1} + \ldots + \binom{n+1}{n}(m-1)\right)$$

So substituting χ into the above equation we get,

$$\chi(A_0 \cup \ldots \cup A_n) = -(1-1)^{n+1} + 1 - (-1)^n +$$

$$(-1)^{n-1}\left((m-1+1)^{n+1} - \binom{n+1}{0}(m-1)^{n+1} - \binom{n+1}{n+1}\right).$$

We get

$$\chi(A_0 \cup \ldots \cup A_n) = 1 - (-1)^n + (-1)^{n-1}(m^{n+1} - (m-1)^{n+1} - 1).$$

So

$$\chi(A_0 \cup \ldots \cup A_n) = 1 + (-1)^{n-1}m^{n+1} + (-1)^n(m-1)^{n+1}.$$

Therefore

$$\chi(n) = \chi(A_*) + \chi(A_0 \cup \ldots \cup A_n) = (-1)^n m^{n+1} + 1 + (-1)^{n-1}m^{n+1} + (-1)^n(m-1)^{n+1}$$

and this gives $\chi(n) = 1 + (-1)^n(m-1)^{n+1}$ and so $\tilde{\chi}(n) = (-1)^n(m-1)^{n+1}$ as desired.

□

Chapter 4

The Exponential and tom Dieck maps

The aim of this chapter is to decompose the reduced tom Dieck map for $B(C_2, G)$ into two parts: the plus tom Dieck map die^+ and the minus tom Dieck map die^-. We begin by recalling the exp and die maps on the Burnside ring $B(G)$. Let us write $\mathrm{par}(m) = (-1)^m$ for $m \in \mathbb{Z}$ and $\mathrm{par}(S) = \mathrm{par}(|S|)$ for a finite set S.

Definition 4.1. *The combinatorial exponential map* $\exp : B(G) \to \beta(G)^\times$ *is defined as*

$$s_H^G(\exp[X]) = \mathrm{par}(H \setminus X)$$

for a G-set X.

Definition 4.2. *The tom Dieck map* $\mathrm{die} : A(\mathbb{R}G) \to \beta(G)^\times$ *is defined as*

$$s_H^G(\mathrm{die}[M]) = \mathrm{par}(\dim(M^H))$$

for an $\mathbb{R}G$-module M.

In fact, the tom Dieck map die and the exponential map exp maps into the unit group of the Burnside ring $B(G)^\times$. Indeed, for an $\mathbb{R}G$- module M we have

$$\mathrm{die}[M] = -\tilde{\Lambda}_G(S(M))$$

so $\mathrm{die}[M] \in B(G)$ and by definition it lies in the unit group of the ghost ring $\beta(G)^\times$, thus $\mathrm{die} \in \beta(G)^\times \cap B(G)$ which is $B(G)^\times$. Moreover since we have the following relation

$$\exp = \mathrm{die} \circ \mathrm{lin}$$

where $\mathrm{lin} : B(G) \to A(\mathbb{R}G)$ and for a G-set X, $\mathrm{lin}[X] = [\mathbb{R}X]$. Thus this formulation yields that $\mathrm{Im}(\exp) \subseteq B(G)^\times$.

It can be easily seen that both exp and die are additive-to-multiplicative maps.

Let M be an $\mathbb{R}G$-module. Given $H \leq G$, then H stabilizes $M^{O(H)}$ and any element $h \in H$ acts on $M^{O(H)}$ as an involution, so the eigenvalues are 1 and -1. Also, by Maschke's Theorem, applied to a group of order 2 any involution on a real vector space is diagonalizable. So

$$M^{O(H)} = M^+_{H,h} \oplus M^-_{H,h}$$

where $M^+_{H,h}$ and $M^-_{H,h}$ are the eigenspaces of h on $M^{O(H)}$ with eigenvalues 1 and -1, respectively.

Definition 4.3. *The elements* $\mathrm{die}^+[M]$ *and* $\mathrm{die}^-[M]$ *as elements of* $\beta(C_2, G)$ *are defined by*

$$s_{H,h}(\mathrm{die}^+[M]) = \mathrm{par}(\dim(M^+_{H,h}))$$

$$s_{H,h}(\mathrm{die}^-[M]) = \mathrm{par}(\dim(M^-_{H,h})).$$

Recall that for the exp and die maps, we give an equivalent way to define them with the Lefschetz invariant on the ordinary Burnside ring. We should generalize the Lefschetz invariant to the monomial Burnside ring in order to do this in the monomial case.

Definition 4.4. *Let K be a CG-invariant triangulation of $S(M)$, then the Lefschetz invariant which is an element of $B(C, G)$ is defined to be*

$$\Lambda_{CG}(S(M)) = \sum_{\sigma \in_{CG} \mathrm{sd}(K)} (-1)^{l(\sigma)} [\mathrm{Orb}_{CG}(\sigma)]$$

where $\mathrm{Orb}_{CG}(\sigma)$ *denotes the* CG*-orbit of* σ*.*

Given two $\mathbb{R}G$-modules M and M' then

$$\mathrm{die}^{+}[M \oplus M'] = \mathrm{par}(\dim(M \oplus M'^{+}_{H,h}))$$

$$= \mathrm{par}(\dim(M^{+}_{H,h}) + \dim((M'_{H,h})^{+})) = \mathrm{die}^{+}[M]\mathrm{die}^{+}[M']$$

so die^{+} has an additive-to-multiplicative structure, similarly die^{-} has this structure. Thus these two maps are defined as additive-to-multiplicative maps from $A(\mathbb{R}G)$ to $\beta(C_2, G)^{\times}$. But in fact we have a stronger result which we shall prove: the images are contained in $B(C_2, G)^{\times}$.

Remark 4.5. *Let* K *be an admissible* G*-simplicial complex and* $|K|$ *be the geometric realization of* K*. Then* $|K|^G$ *is the geometric realization of* K^G*.*

Proof. Let $x \in |K|^G$ then x is in the interior of a unique simplex σ. Then $gx \in g\sigma$ and $gx = x$ for all $g \in G$. So σ is stabilized by G and since K is admissible σ is fixed by G. □

Lemma 4.6. *We have*

$$s^G_{H,h}(\Lambda_{C_2G}) = s^H_{H,h}(\Lambda_{C_2H}(S(M^{+}_{H,h}))) + s^H_{H,h}(\Lambda_{C_2H}(S(M^{-}_{H,h})))$$

for every subelement (H, h) *of* G*.*

Proof. We have

$$\Lambda_{C_2G} = \sum_{r=0}^{\infty}(-1)^r[\mathrm{sd}_r(K)]$$

where $\mathrm{sd}_r(K)$ is the set of r-simplices. Let $\sigma \in \mathrm{sd}_r(K)$ and consider the fibre $\{\sigma, -\sigma\}$ in $\mathrm{sd}_r(K)$. Then this fibre makes a contribution to $s_{H,h}(\Lambda)$ if and only if H stabilizes this fibre. If h fixes σ and $-\sigma$ then $\{\sigma, -\sigma\} \subseteq S(M^{+}_{H,h})$. In that case $\{\sigma, -\sigma\}$ contributes $(-1)^r$ to $s^G_{H,h}(\Lambda_{C_2G})$. Moreover

$$\{\sigma \in K : O(H) \text{ fixes } \sigma \text{ and } h(\sigma) = \sigma\}$$

is a triangulation of $S(M^+_{H,h})$ by Remark 4.5 and σ contributes $(-1)^r$ to $s^H_{H,h}(\Lambda_{C_2H}(S(M^+_{H,h}))$. If h does not fix σ, then we have $h(\sigma) = -\sigma$. So $\{\sigma, -\sigma\} \subseteq S(M^-_{H,h}$, and $\{\sigma, -\sigma\}$ contributes $-(-1)^r$ to $s^G_{H,h}(\Lambda_{C_2G})$. Similarly

$$\{\sigma \in K : O(H) \text{ fixes } \sigma \text{ and } h(\sigma) = -\sigma\}$$

is a triangulation of $S(M^-_{H,h})$ by Remark 4.5 and σ contributes $-(-1)^r$ to $s^H_{H,h}(\Lambda_{C_2H}(S(M^-_{H,h}))$. As a result we get the desired equation. □

Theorem 4.7. *The images of* die$^+$ *and* die$^-$ *are contained in* $B(C_2, G)^\times$.

Proof. Let M be an $\mathbb{R}G$-module. It is enough to show that die$^+[M]$ and die$^-[M]$ belong to $B(C_2, G)$. We allow the fibre group C_2 to act on $S(M)$ as the antipodal map, that is it sends each vector to its negation. So $S(M)$ becomes a C_2-fibred G-space. Choosing C_2G- invariant triangulation K for $S(M)$, we can regard to Lefschetz invariant $\Lambda = \Lambda_{C_2G}(S(M))$ as a virtual C_2-fibred G-set.

Let us fix a subelement (H, h) of G and let $m^+ = \dim(M^+_{H,h})$ and $m^- = \dim(M^-_{H,h})$. Regarding $S(M^+_{H,h})$ and $S(M^-_{H,h})$ as C_2-fibred H-spaces by the Lemma 4.6,

$$s^G_{H,h}(\Lambda) = s^H_{H,h}(\Lambda_{C_2H}(S(M^+_{H,h}))) + s^H_{H,h}(\Lambda_{C_2H}(S(M^-_{H,h}))).$$

$M^+_{H,h}$ is the +1-eigenspace of h on $M^{O(H)}$, thus h acts trivially on $M^+_{H,h}$ and on $S(M^+_{H,h})$. Moreover we have

$$\mathrm{Orb}_{CH}(\sigma) \simeq H/H_\sigma$$

so the Lefschetz invariant becomes

$$\Lambda_{C_2H}(S(M^+_{H,h})) = \sum_{\sigma \in_H \mathrm{sd}(K)} (-1)^{l(\sigma)}[H/H_\sigma].$$

So we can calculate the species by counting the orbits. Thus

$$s_{H,h}(\Lambda_{C_2H}(S(M^+_{H,h}))) = \sum_{\sigma \in_H \mathrm{sd}(K)} (-1)^{l(\sigma)} s_{H,h}[H/H_\sigma]$$

$$= \frac{1}{2} \sum_{\sigma \in_H \mathrm{sd}(K)} (-1)^{l(\sigma)} |H/H_\sigma|$$

we divided by 2 ,the order of C_2, and that gives

$$= \frac{1}{2} \sum_{\sigma \in \mathrm{sd}(K)} \frac{1}{|H/H_\sigma|} (-1)^{l(\sigma)} |H/H_\sigma|$$

$$= \frac{1}{2} \sum_{\sigma \in \mathrm{sd}(K)} (-1)^{l(\sigma)}$$

$$= \frac{1 - (-1)^{m^+}}{2}.$$

Similarly, $M^-_{H,h}$ is the -1-eigenspace of h on $M^{O(H)}$, so h acts as reflection on $M^{O(H)}$.

$$s_{H,h}(\Lambda_{C_2 H}(S(M^-_{H,h}))) = \sum_{\sigma \in_H \mathrm{sd}(K)} (-1)^{l(\sigma)} s_{H,h}[H/H_\sigma]$$

$$= \frac{1}{2} \sum_{\sigma \in_H \mathrm{sd}(K)} (-1)^{l(\sigma)} (-1) \cdot |H/H_\sigma|$$

$$= -\frac{1}{2} \sum_{\sigma \in \mathrm{sd}(K)} (-1)^{l(\sigma)}$$

$$= -\frac{1 - (-1)^{m^-}}{2}.$$

So we obtain modulo 2 congruence condition

$$s^G_{H,h}(\Lambda) = \begin{cases} 1 & \text{if } (m^+, m^-) \equiv (1,0) \\ 0 & \text{if } m^+ \equiv m^- \\ -1 & \text{if } (m^+, m^-) \equiv (0,1) \end{cases}$$

Let us consider now that the Lefschetz invariant $\Gamma = \Lambda_{CG}(S(M \oplus \mathbb{R}))$ where $\mathbb{R}$ denotes the trivial $\mathbb{R}G$-module. Adding $\mathbb{R}$ to M replaces $M^+_{H,h}$ with $M^+_{H,h} \oplus \mathbb{R}$ while leaving $M^-_{H,h}$ unchanged. Thus this flips the parity of m^+ and preserves the parity of m^-. Hence we get

$$s^G_{H,h}(\Gamma) = \begin{cases} 1 & \text{if } (m^+, m^-) \equiv (0,0) \\ 0 & \text{if } m^+ \not\equiv m^- \\ -1 & \text{if } (m^+, m^-) \equiv (1,1) \end{cases}$$

Moreover we have

$$s^G_{H,h}(\Lambda^2 + \Lambda) = \begin{cases} 2 & \text{if } (m^+, m^-) \equiv (1,0) \\ 0 & \text{otherwise} \end{cases}$$

also

$$s^G_{H,h}(\Gamma^2 - \Gamma) = \begin{cases} 2 & \text{if } (m^+, m^-) \equiv (1,1) \\ 0 & \text{otherwise} \end{cases}$$

Therefore we have,

$$s^G_{H,h}(\Lambda^2 + \Lambda + \Gamma^2 - \Gamma) = \begin{cases} 2 & \text{if } m^+ \equiv 1 \\ 0 & \text{if } m^+ \equiv 0 \end{cases}$$

Consequently we get,

$$s^G_{H,h}(1 - \Lambda^2 - \Lambda - \Gamma^2 + \Gamma) = \operatorname{par}(m^+) = s^G_{H,h}(\operatorname{die}^+[M]).$$

Since (H, h) is arbitrary, $\operatorname{die}^+[M] = 1 - \Lambda^2 - \Lambda - \Gamma^2 + \Gamma$. Thus $\operatorname{die}^+[M] \in B(C_2, G)$.

Meanwhile we have,

$$s^G_{H,h}(\Gamma^2 - \Gamma) = \begin{cases} 2 & \text{if } (m^+, m^-) \equiv (0,1) \\ 0 & \text{otherwise} \end{cases}$$

And this gives

$$s^G_{H,h}(\Lambda^2 - \Lambda + \Gamma^2 - \Gamma) = \begin{cases} 2 & \text{if } m^- \equiv 1 \\ 0 & \text{if } m^- \equiv 0 \end{cases}$$

Similarly we get,

$$s^G_{H,h}(1 - \Lambda^2 + \Lambda - \Gamma^2 + \Gamma) = \mathrm{par}(m^-) = s^G_{H,h}(\mathrm{die}^-[M]).$$

and so $\mathrm{die}^-[M] = 1 - \Lambda^2 + \Lambda - \Gamma^2 + \Gamma$. Therefore, $\mathrm{die}^-[M] \in B(C_2, G)$. □

Corollary 4.8. *There is an additive-to-multiplicative map namely the reduced tom Dieck map* $\widetilde{\mathrm{die}} : A(\mathbb{R}G) \to B(G)^\times$ *such that given an* $\mathbb{R}G$*-module* M *and* $H \leq G$ *then*

$$s^G_H(\widetilde{\mathrm{die}}[M]) = \mathrm{par}(\dim(M^{O(H)})).$$

Proof. There is an additive-to-multiplicative map $A(\mathbb{R}G) \to \beta(G)^\times$ determined by the formula. We must show that the image lies in $B(G)$. Recalling that $B(G)$ is the subset of $B(C_2, G)$ consisting of the elements b satisfying

$$s^G_{H,h}(b) = s^G_{H,1}(b)$$

for all subelements (H, h) of G. So by the theorem above

$$s^G_{H,h}(\mathrm{die}^+[M]\mathrm{die}^-[M]) = par(m^+ + m^-) = s^G_H(\widetilde{\mathrm{die}}[M]).$$

Therefore we have, $\widetilde{\mathrm{die}}[M] = \mathrm{die}^+[M]\mathrm{die}^-[M]$. □

Corollary 4.9. *If* G *is a* 2*-group, then there is a group endomorphism* ϕ *of* $B(G)^\times$ *such that given* $b \in B(G)^\times$ *and* $H \leq G$*, we have*

$$s^G_H(\phi(b)) = s^G_{O(H)}(b).$$

Proof. There is a group endomorphism of $\beta(G)^\times$ given by the formula. We must show that $\phi(B(G)^\times) \subseteq B(G)^\times$. Moreover we know that die is surjective for 2-groups. Then for $b \in B(G)^\times$ we have $b = \mathrm{die}([M_1] - [M_2])$ for some $\mathbb{R}G$- modules M_1 and M_2. But die annihilates multiples of 2, so $b = \mathrm{die}[M]$ where $M = M_1 \oplus M_2$. So eventually we get

$$\phi(b) = \widetilde{\mathrm{die}}[M]$$

which belongs to $B(G)^\times$. □

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