ON SECTIONS AND TAILS OF POWER SERIES

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By
Natalya Zheltukhina

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Iossif V. Ostrovskii(Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Asst. Prof. Dr. C. Yalçın Yıldırım (Former Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. Turgay Kaptanoğlu

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Mefharet Kocatepe

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. Sinan Sertöz

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet Baray
Director of the Institute

ABSTRACT

ON SECTIONS AND TAILS OF POWER SERIES

Natalya Zheltukhina
Ph.D. in Mathematics
Supervisor: Prof. Dr. Iossif V. Ostrovskii
July, 2002

The thesis is devoted to the study of connections between properties of a power series and properties of its sections and tails.

Power series having sections or tails with multiply positive coefficients are considered and their growth estimates are obtained. Our results strengthen and supplement previous results in this direction, in particular, the well-known Pólya theorem on power series with sections having only negative zeros.

The asymptotic zero distribution of linear combinations of sections and tails of the Mittag-Leffler function $E_{1/\rho}$ of order $\rho > 1$ is studied. Our results generalize and supplement previous results in this direction, in particular, the well-known Szegö result on the linear combinations of sections and tails of the exponential function and the A Edrei, E.B. Saff and R.S. Varga results on sections of $E_{1/\rho}$.

Keywords: Laurent series, Mittag-Leffler functions, m-times positive sequence, power series, sections, tails, zero distribution.

ÖZET

KUVVET SERİLERİNİN KISMİ TOPLAMLARI VE KUYRUKLARI ÜZERİNE

Natalya Zheltukhina
Matematik Bölümü, Doktora
Tez Yöneticisi: Prof. Dr. Iossif V. Ostrovskii
Temmuz, 2002

Bu tez kuvvet serilerinin kısmi toplamlarının ve kuyruklarının özellikleri arasındaki bağlantıları çalışmaya adanmıştır.

Kuvvet serilerinin kısmi toplamlarının ya da kuyruklarının çoklu pozitif katsayılı olanları ele alınmış ve bunların artış hızı tahmin edilmiştir. Sonuçlarımız bu alanda daha önceden bulunmuş sonuçları, özellikle kuvvet serilerinin kısmi toplamları negatif katsayılı olanları üzerine olan ünlü Pólya teoremini, genişletmekte ve tamamlamaktadır.

Ayrıca, derecesi birden büyük olan Mittag-Leffler fonksiyonlarının kısmi toplamlarının lineer kombinasyonlarının sıfırlarının asimptotik dağılımı da çalışılmıştır. Sonuçlarımız bu alanda da daha önceki sonuçları, özellikle de Szegö'nün üstel fonksiyonun kısmi toplamlarının lineer kombinasyonları üzerine ve A. Edrei, E. B. Saff ile R. S. Varga'nın Mittag-Leffler fonksiyonlarının kısmi toplamları üzerine olanlarını, genelleştirmekte ve tamamlamaktadır.

Anahtar sözcükler: Laurent serisi, Mittag-Leffler fonksiyonları, m-katlı pozitif dizi, kuvvet serisi, kısmi toplamlar, kuyruklar, sıfır dağılımı.

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Contents

1

Introduction

Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_0 > 0, \tag{1.1}$$

be a formal power series. Denote by $R(f)$ its radius of convergence, and by

$$s_n(z; f) = \sum_{k=0}^{n} a_k z^k, \qquad n = 0, 1, 2, \ldots, \tag{1.2}$$

its sections. For $R(f) > 0$, denote by

$$t_n(z; f) = \sum_{k=n}^{\infty} a_k z^k, \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

the tails of the series (1.1), and set

$$M(r, f) = \max_{|z|=r} |f(z)|, \qquad 0 < r < R(f).$$

The study of the distribution of zeros of $s_n(z, f)$ was started at the end of 19th century. The first important result concerning the distribution of zeros was a work of A. Hurwitz (see [18]). Later on, in the first half of 20th century, deep and general results revealing connections among the asymptotic (as $n \to \infty$) behavior of zeros of $s_n(z, f)$, the radius of convergence $R(f)$ of (1.1) and the growth of function f in terms of $M(r, f)$ were obtained in works of G. Pólya [29],

R. Jentzsch [19], G. Szegö [35], P.C. Rosenbloom [32], F. Carlson [6], A. Dvoretzky [10] and others. For some widely applicable concrete entire functions (such as the exponential function, the trigonometric functions and some others) elegant and sharp asymptotics (as $n \to \infty$) for zeros of $s_n(z, f)$ were obtained by G. Szegö [36], J. Dieudonné [8] and P.C. Rosenbloom [32]. For a review of these results, as well as a detailed bibliography, the reader may consult the thesis of P.C. Rosenbloom [32].

During the second half of 20th century in works of T. Ganelius [15], [16], J.D. Buckholtz [3], [4], [5], J. Korevaar [21], A. Edrei [13] and other mathematicians some important unsettled problems have been solved and new general phenomena have been discovered. Obtained earlier by G. Szegö [36] and J. Dieudonné [8] asymptotics for the zeros of $s_n(z, f)$ of some concrete entire functions have been considerably improved by J.D. Buckholtz [2], A.J. Carpenter, R.S. Varga, J. Waldvogel [7], D.J. Newman, T.J. Rivlin [22] and others. In the work of A. Edrei, E.B. Saff and R.S. Varga [12] these asymptotics were extended to the Mittag-Leffler functions which generalize the exponential function.

Apparently, the first study of the zero distribution of tails $t_n(z, f)$ has been done by G. Szegö in [36] where even a more general problem of the asymptotic distribution of the zeros of the linear combination

$$I_n(z, \lambda, e^z) = (1 - \lambda)s_n(z, e^z) - \lambda t_n(z, e^z) \tag{1.4}$$

for any $\lambda \in \mathbb{C}$ was considered. Evidently, $I_n(z, 0, e^z) = s_n(z, e^z)$ and $I_n(z, 1, e^z) = t_n(z, e^z)$.

In the general case, the study of the distribution of the zeros of $t_n(z, f)$ of an *arbitrary* power series (1.1) with a positive radius of convergence was initiated by M. Pommiez [31] in 1960. In 1970s, J.D. Buckholtz and J.K. Shaw [5] continued this investigation and obtained complete solutions for some problems related to the distribution of the zeros of sections $s_n(z,f)$ and tails $t_n(z,f)$. In 1990s new results on the distribution of the zeros of $t_n(z,e^z)$ were obtained by C.Y. Yıldırım [37], [38], [39], who needed them for the evaluation of some terms that came up in certain mean-value estimate related to the ζ-function. Recently, I.V. Ostrovskii [23], [24] has found a new phenomenon on the distribution of the zeros of tails $t_n(z,f)$ in terms of the arguments of the zeros. A survey of investigations prior to 1997 on several aspects of the distribution of zeros of sections and tails is given by I.V. Ostrovskii in [25].

The present thesis is devoted to the study of some open problems concerning the asymptotic distribution of the zeros of sections $s_n(z,f)$ and tails $t_n(z,f)$.

In a previous work of I.V. Ostrovskii and the author [26] a new generalization of Pólya's Theorem [29] on formal power series whose sections have only negative zeros was found. This generalization was achieved by introducing some classes P_k of power series with multiply positive (in the sense of M. Fekete) coefficients. These classes form a decreasing sequence $P_1 \supset P_2 \supset \ldots$. Moreover the class $P_\infty := \cap_{k=1}^{\infty} P_k$ is exactly the class of power series that appears in Pólya's Theorem.

In [26] it was shown that the statement of Pólya's Theorem remains in force for the much larger than P_∞ class P_3, and hence for all the smaller classes P_k, $k \geq 4$ (it fails for the classes P_1 and P_2 as can be seen on simple examples). In [26] a sharp growth estimate for functions of P_3 has been obtained. This estimate certainly holds in all the classes P_k, $k \geq 4$. However, it was not known before whether this estimate is sharp in P_k, $k \geq 4$, or not. In Chapter 4, we give a negative answer to this problem and find an improvement of the estimate obtained in [26] for the classes P_k, $k \geq 4$.

In another work of I.V. Ostrovskii and the author [28], analogous results for tails $t_n(z,f)$ have been given. Similar to P_k, classes Q_k of power series with multiply positive coefficients have been considered in [28]. The classes Q_k also form a decreasing sequence $Q_1 \supset Q_2 \supset \ldots$. Results obtained in [28] are sharp in Q_3. However, it was unknown before whether they are sharp in the classes Q_k, $k \geq 4$, or not. In Chapter 5, we show that a growth estimate obtained in [28] is not sharp in Q_k, $k \geq 4$, and improve it for Q_k, $k \geq 4$.

A Laurent series can be viewed as a generalization of a power series. To the best of our knowledge, the distribution of the zeros of sections of Laurent series has not been studied before. In Chapter 6, we obtain a generalization of results of [26] to Laurent series.

In the work of A. Edrei, E.B. Saff and R.S. Varga [12], the distribution of the zeros of sections $s_n(z, f)$ of Mittag-Leffler functions

$$E_{1/\rho}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(1+\frac{k}{\rho}\right)}, \qquad \rho > 1,$$

was studied. When $\rho = 1$, the Mittag-Leffler function coincides with e^z. In [12], the authors generalized some results of [36] to the class of Mittag-Leffler functions. However, as mentioned before, in [36], the zeros not only of sections $s_n(z, e^z)$ but also, more generally, of the linear combinations (1.4) have been considered. The question arises about such a generalization of results of [12] that contains the complete result of G. Szegö on the linear combinations (1.4). In Chapter 7 we obtain such a generalization.

The results of this thesis have been accepted [40], [41] and submitted [42] for publication.

2

Statement of results

2.1 Power series with multiply positive coefficients

In 1913, G. Pólya proved the following theorem.

Theorem A ([29]). *If, for all sufficiently large n, sections (1.2) have real negative zeros only, then $R(f) = \infty$ and*

$$\log M(r, f) = O((\log r)^2). \tag{2.1}$$

This result shows that restrictions on argument of the zero distribution of sections can imply a rather serious growth restriction on the original power series. Since then, formal power series with restrictions on zeros of their sections have been deeply investigated by several mathematicians. The most far reaching generalization of Pólya's result was obtained by T. Ganelius in 1963.

Theorem B ([16]). *Assume that there exist $\alpha > 0$ and a sequence of real numbers $\{\gamma_n\}_{n=1}^{\infty}$ such that sections (1.2), for all sufficiently large n, do not vanish in the* *angle*

$$\{z : \gamma_n < \arg z < \gamma_n + \alpha\}.$$

Then $R(f) = \infty$ and (2.1) holds.

In the paper by I.V. Ostrovskii and the author [26], a generalization of Theorem A whose character is quite different from that of Theorem B has been obtained. It is based on concept of multiple positivity introduced by M. Fekete in 1912.

Let $\mathbf{R}^\infty$ be the set of all sequences of real numbers, and sequences $a = \{a_n\}_{n=-\infty}^{\infty}$ and $\{b_n\}_{n=-\infty}^{\infty}$ belong to $\mathbf{R}^\infty$. Let

$$a * b = \{(a_0 \cdot b_k + a_1 \cdot b_{k-1} + \ldots + a_k \cdot b_0)\}_{k=0}^{\infty}$$

be the convolution of the sequences a and b. Denote by $\nu[a]$ the number of changes of sign in the sequence a.

Definition 1. A sequence a is called *m-times positive*, $m \in \mathbb{N}$, if for all $b \in \mathbf{R}^\infty$ satisfying $\nu[b] \leq m - 1$, we have $\nu[a * b] \leq \nu[b]$. A sequence a is called *∞-times (or totally) positive* if $\nu[a * b] \leq \nu[b]$ for all $b \in \mathbf{R}^\infty$.

There is an equivalent but more convenient definition of multiply positive sequences (see [20]), even though its formulation is somewhat complicated:

Definition 2. A sequence $a = \{a_k\}_{k=0}^{\infty}$ of real numbers is said to be *m- times positive* for $m \in \mathbb{N} \cup \{\infty\}$ if all minors of orders less than $m + 1$ of the infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \ldots \\ 0 & a_0 & a_1 & a_2 & \ldots \\ 0 & 0 & a_0 & a_1 & \ldots \\ . & . & . & . & \ldots \end{pmatrix}$$

are non-negative.

Multiply positive sequences were introduced by M. Fekete [14] for the study of zeros of real polynomials and entire functions. Since then such sequences have been studied by several mathematicians and have found many applications (see, e.g. [1], [20]).

Denote by PF_m, $m \in N \cup \infty$, the set of all m-times positive sequences. If series (1.1) is a generating function of such a sequence, we shall also write $f \in PF_m$. Evidently,

$$PF_1 \supset PF_2 \supset \ldots \supset PF_\infty.$$

Clearly, the class PF_1 consists of all the sequences $\{a_k\}_{k=0}^{\infty}$ with positive terms. It is easy to see that the class PF_2 consists of all the sequences of the form

$$a_n = \exp\{-\psi(n)\}, \quad n = 0, 1, 2, \ldots,$$

where $\psi : \mathbb{N} \cup \{0\} \to (-\infty, +\infty]$, $\psi(0) < \infty$, is a convex function. The problem of the description of the classes PF_r for $3 \leq r < \infty$ is at present far from being solved. Nevertheless, the smallest class PF_∞ (contained in all $PF_m, m \in \mathbb{N} \cup \{\infty\}$) had been completely described by M. Aissen, A. Edrei, I.J. Shoenberg and A. Whitney.

Theorem C([20], p.412). *A function $f(z)$ belongs to PF_∞ if and only if*

$$f(z) = C \exp(qz) \prod_{i=1}^{\infty} \frac{(1 + \alpha_i z)}{(1 - \beta_i z)}, \tag{2.2}$$

where

$$C > 0; \quad q, \alpha_i, \beta_i \geq 0, \quad \sum_{i=1}^{\infty} (\alpha_i + \beta_i) < \infty. \tag{2.3}$$

Theorem C yields that an entire function (1.1) of genus 0 has purely negative zeros if and only if the sequence $\{a_k\}_{k=0}^{\infty}$ is totally positive. Applying this to section (1.2), we see that negativity of all its zeros is equivalent to total positivity of the truncated sequence

$$\{a_k\}_{k=0}^n = \{a_0, a_1, \ldots, a_n, 0, 0, \ldots\}.$$

Thus the condition of Pólya's Theorem A is equivalent to total positivity of the truncated sequences $\{a_k\}_{k=0}^n$ for all sufficiently large n.

Denote by P_m, $m \in \mathbb{N} \cup \{\infty\}$, the class of all power series (1.1) such that the truncated sequences $\{a_k\}_{k=0}^n$ are m-times positive for all sufficiently large n. Evidently,

$$P_1 \supset P_2 \supset P_3 \supset \ldots \supset P_\infty.$$

In [26] the following question was considered. Does the assertion of Theorem A remain in force if we replace total positivity of the truncated sequences $\{a_k\}_{k=0}^n$ by a weaker condition of their m-times positivity for some $m < \infty$? It is easy to see that the answer is negative for $m = 1$ and $m = 2$. For example, if $a_k = 1$ for any $k = 0, 1, 2, \ldots$, all the truncated sequences $\{a_k\}_{k=0}^n$ are 2-times positive, but the series (1.1) does not converge in the whole complex plane $\mathbb{C}$. It was proved in [26] that the situation changes if $m \geq 3$.

Theorem D([26]). *If a formal power series (1.1) belongs to P_m for some $m \geq 3$, then it converges in the whole complex plane and its sum $f(z)$ is an entire function of order* 0. *Moreover,*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log c}, \qquad c = \frac{1+\sqrt{5}}{2} = 1.613\ldots \tag{2.4}$$

The estimate (2.4) cannot be improved for $f \in P_3$.

The question arises whether the estimate (2.4) is unimprovable for $m \geq 4$.

It will be shown in Chapter 3 that following Pólya's own proof of Theorem A, one can improve the estimate (2.1) for functions satisfying the condition of Theorem A (which is equivalent to the condition $f \in P_\infty$). More precisely, the following theorem holds.

Theorem E. *If, for all sufficiently large n, sections (1.2) have real negative zeros only (or, equivalently, $f \in P_\infty$), then the bound (2.4) can be improved in the following way:*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \le \frac{1}{2\log 2}. \tag{2.5}$$

As far as we know, the question whether Pólya's estimate (2.5) is sharp has not yet been settled. In [30] (Part V., Problem 176), an example was constructed showing that $\log 2$ cannot be replaced by a constant greater than $\log 4$. O. M. Katkova and A. M. Vishnyakova kindly informed me that, by modifying this example, they showed that $\log 2$ cannot be replaced by a constant greater than $\log(3.5)$.

In the present work we show that an estimate better than (2.5) already holds in the class P_4 which is much larger than P_∞. Our first result is the following theorem.

Theorem 1 . *If $f \in P_m$, $m \ge 4$, then the bound (2.4) can be improved in the following way:*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \le \frac{1}{2\log d(P_4)}, \tag{2.6}$$

where the constant $d(P_4)$ is independent of f and

$$2.016 \le d(P_4) \le 2.087.$$

One may approach the problem of finding the sharp estimate in Theorem E by obtaining the sharp growth estimates for functions from P_m, $m \ge 4$. This problem remains open for any $m \ge 4$.

Note that for $m < \infty$ there is also a relation between m-times positivity of the sequence $\{a_k\}_{k=0}^{n}$ and zeros of the corresponding section (1.2) but this relation is a less strict one than for $m = \infty$. I. J. Schoenberg ([33], pp.397, 415) proved: (i) the necessary condition for $\{a_k\}_{k=0}^n$ to be m-times positive is the non-vanishing of (1.2) in the angle $\{z : |\arg z| \le (\pi m)/(m+n-1)\}$, (ii) the sufficient condition is its non-vanishing in the larger angle $\{z : |\arg z| \le (\pi m)/(m+1)\}$. Both conditions are best possible in terms of the sizes of angles.

As an immediate consequence of Theorem 1 and Schoenberg's result (ii) for $m = 4$ we obtain a generalization of Pólya's Theorem in terms of the zeros of sections (1.2).

Corollary 1 . *Let $f(z)$ be a formal power series (1.1) with real coefficients a_k. Assume that for all sufficiently large n, the zeros of sections (1.2) are located in the angle $\{z : |\arg z - \pi| < \pi/5\}$. Then the series (1.1) converges in the whole complex plane and the estimate (2.6) holds.*

As mentioned in the Introduction, the last years' interest in the study of tails of power series is considerably increasing. A natural question arises whether analogues of Theorems A and B exist for tails. Note that $z^{-n}t_n(z; e^z)$ does not have any zero in the half-plane $\{z : \mathrm{Re} z \le n-1\}$ ([30], Part V, Problem 179). So, there isn't a complete analogue of Theorem B for tails. Nevertheless, as was proved by I.V. Ostrovskii, an analogue of Theorem A is true.

Theorem F ([23]). *Assume $R(f) = \infty$. If all zeros of tails (1.3), for all sufficiently large n, are real non-positive, then (2.1) remains true.*

The question arose whether there exists an analogue of Theorem F for a series (1.1) with m–times positive coefficients of tails (1.3). In the joint work of I.V. Ostrovskii and the author [28], such an analogue has been obtained.

Denote by R_m, $m \in \mathbb{N} \cup \{\infty\}$, the class of all power series (1.1) such that the sequences $\{a_n, a_{n+1}, \ldots\}$ are m–times positive for all n large enough. Evidently,

$$R_1 \supset R_2 \supset R_3 \supset \ldots \supset R_\infty.$$

Series (1.1) belonging to R_2 may have singularities of rather arbitrary kind and location:

Example ([28]). Let (1.1) be the power series expansion of

$$f(z) = 1 + \frac{z}{(1-z)^2} + h(z),$$

where $h(z)$ is an arbitrary power series with real coefficients whose radius of convergence is strictly greater than 1. Evidently, for some $\varepsilon > 0$,

$$a_k = k + O((1-\varepsilon)^k), \quad k \to \infty.$$

Hence, $a_k^2 \geq a_{k-1}a_{k+1}$ for k large enough and therefore $\{a_n, a_{n+1}, \ldots\}$ is 2-times positive for all sufficiently large n.

The next theorem from [28] shows that for $m \geq 3$ the situation is quite different.

Theorem G ([28]). *If $f \in R_m$ for some $m \geq 3$, then: either*
(i) $R(f) = \infty$ and the bound (2.4) holds; or
(ii) $0 < R(f) < \infty$ and

$$f(z) = \frac{A}{1 - z/R(f)} - g(z),$$

where A is a positive constant and g is an entire function with non-negative coefficients (except at most a finite number) satisfying the condition

$$\log M(r,g) \leq \frac{\log r \cdot \log\log r}{\log 2} + O(\log r), \quad r \to \infty. \tag{2.7}$$

The bound (2.4) cannot be improved for entire functions from R_3. The bound (2.7) cannot be improved for non entire functions from R_m for any $m \geq 3$.

The last theorem means that a series (1.1) from R_m, $m \geq 3$, either represents an entire function satisfying (2.4) or has exactly one singularity (a simple pole) in the whole complex plane and satisfies the very restrictive condition (2.7).

The question arises whether the bound (2.4) can be improved for entire functions from R_m for $m \geq 4$. Here we show that, for *entire* functions belonging to R_m, $m \geq 4$, an estimate better than (2.4) holds.

Theorem 2 . *If $m \geq 4$, then an entire function $f \in R_m$ satisfies the condition*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log d(R_4)}, \tag{2.8}$$

where the constant $d(R_4) > 2$ is independent of f and

$$2.016 \leq d(R_4) \leq 2.087.$$

As a consequence of Theorem 2 and Schoenberg's result (ii) mentioned above we obtain the following

Corollary 2 . *Let $f(z)$ be an entire function (1.1) of order strictly less than 1 with real coefficients a_k. Assume that for all sufficiently large n, the zeros of tails (1.3) are located in the angle $\left\{z : |\arg z - \pi| < \frac{\pi}{5}\right\}$. Then the estimate (2.8) holds.*

2.2 Laurent series with multiply positive coefficients

A formal Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \qquad a_0 \neq 0, \tag{2.9}$$

can be considered as a generalization of a formal power series (1.1).

Let $\{a_n\}_{n=-\infty}^{\infty}$, $a_0 \neq 0$, be a two-sided sequence. Recall (see e.g. [33], [20], p.418, [34], [11]) that the sequence $\{a_n\}_{n=-\infty}^{\infty}$ is called *r–times positive*, $r \in \mathbb{N}\cup\infty$, if all minors of orders less than $r+1$ of the four-way infinite matrix

$$\begin{pmatrix} \dots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \dots & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ \dots & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & \dots \\ \dots & a_{-4} & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & \dots \\ \dots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix}$$

are non-negative. Usually, ∞–times positive sequences are called *totally positive* sequences.

Denote by $\tilde{PF}_r$ the class of all r–times ($r \in \mathbb{N} \cup \{\infty\}$) two-sided positive sequences. If Laurent series (2.9) is a generating function of r-times positive two-sided sequence, we shall write $f \in \tilde{PF}_r$. Evidently,

$$\tilde{PF}_1 \supset \tilde{PF}_2 \supset \tilde{PF}_3 \ldots \supset \tilde{PF}_\infty,$$

and also $PF_k \subset \tilde{PF}_k$, $k \in \mathbb{N} \cup \{\infty\}$ (any one-sided sequence $\{a_k\}_{k=0}^{\infty}$ can be considered as two-sided $\{a_n\}_{n=-\infty}^{\infty}$ by putting $a_k = 0$ for $k < 0$). Clearly, the class $\tilde{PF}_1$ consists of all the sequences $\{a_n\}_{n=-\infty}^{\infty}$ with non-negative terms. It is a simple matter to see that the class $\tilde{PF}_2$ consists of all the sequences of the form

$$a_n = \exp\{-\psi(n)\}, \quad n \in \mathbb{Z},$$

where $\psi : \mathbb{Z} \to (-\infty; +\infty], \psi(0) < \infty$, is a convex function. The problem of the description of the classes $\tilde{PF}_r$, $3 \leq r < \infty$, is at present far from being solved. Nevertheless, the smallest class $\tilde{PF}_\infty$ (among all $\tilde{PF}_m$, $m \in \mathbb{N} \cup \{\infty\}$) had been completely described by A. Edrei.

Theorem H ([20], p.427, [11]). *A Laurent series $f(z)$ with non-empty annulus* *of convergence belongs to $\tilde{PF}_\infty$ if and only if*

$$f(z) = C\exp(q_{-1}z^{-1} + q_1 z)\prod_{i=1}^{\infty}\frac{(1+\alpha_i z)(1+\delta_i z^{-1})}{(1-\beta_i z)(1-\gamma_i z^{-1})}, \tag{2.10}$$

where

$$C > 0; \quad q_{-1}, q_1, \alpha_i, \beta_i, \gamma_i, \delta_i \geq 0, \quad \sum_{i=1}^{\infty}(\alpha_i + \beta_i + \gamma_i + \delta_i) < \infty. \tag{2.11}$$

In view of Theorem H, I.J. Schoenberg [33] stated the problem of describing analytical properties of the generating function (2.9) of an r-times positive sequence $\{a_n\}_{n=-\infty}^{\infty}$. Here we restrict ourselves to some subclasses of $\tilde{PF}_r$, $3 \leq r \leq \infty$, containing infinite sequences. Mostly, we deal with the case $r = 3$.

Denote by $\tilde{Q}_r$, $r \in \mathbb{N} \cup \{\infty\}$, the class of all the sequences $\{a_n\}_{n=-\infty}^{\infty}$ such that all truncated sequences

$$\{a_k\}_{k=-n}^{n} := \{\ldots, 0, 0, a_{-n}, a_{-n+1}, \ldots, a_{n-1}, a_n, 0, 0, \ldots\}, \quad n = 1, 2, 3, \ldots$$

are r-times positive. Their subclasses $Q_r \subset \tilde{Q}_r$ consisting of all the one-sided sequences (with $a_n = 0$ for $n < 0$) were considered in [26] and [27].

Note that (2.10) gives the description of the class $\tilde{PF}_\infty$ in terms of *independent* parameters C, q_1, q_{-1}, α_i, β_i, δ_i, γ_i ($i = 1, 2, 3, \ldots$). This means: (i) arbitrary values of the parameters can be chosen under the conditions (2.11); (ii) there is one-to-one correspondence between collections of these parameters and sequences of $\tilde{PF}_\infty$.

To the best of our knowledge, the problem of the description of the classes $\tilde{PF}_m$, $3 \leq m < \infty$, in terms of independent parameters has not been solved until now. It is not clear even which kind of parameters could play the corresponding role (similar to that of C, q_1, q_{-1}, α_i, γ_i, δ_i and β_i in Theorem H).

In [27], a description of the subclass Q_3 of $\tilde{Q}_3$ in terms of independent parameters has been given. It turns out that the role of independent parameters describing Q_3 is played by the points of the set $(0,\infty)\times[0,\infty)\times U$, where

$$U=\left\{\{\alpha_k\}_{k=2}^{\infty}: \begin{array}{ll} (i) & 0\le\alpha_k\le 1, \\ (ii) & if \quad \exists j \quad with \quad a_j=0, \quad then \quad a_k=0, \quad \forall k\ge j \end{array}\right\}. \tag{2.12}$$

To give a precise statement of the result of [27], let us define the numbers

$$[\alpha_2]=1+\alpha_2,\quad [\alpha_2\alpha_3]=1+\alpha_3\sqrt{[\alpha_2]},\quad [\alpha_2\alpha_3\alpha_4]=1+\alpha_4\sqrt{[\alpha_2\alpha_3]},\quad \dots$$

$$[\alpha_2\alpha_3\dots\alpha_n]=1+\alpha_n\sqrt{[\alpha_2\alpha_3\dots\alpha_{n-1}]},\quad \dots \tag{2.13}$$

Theorem I ([27]). *A sequence $\{a_n\}_{n=0}^{\infty}$ belongs to Q_3 if and only if*

$$a_1=a_0\alpha,$$

$$a_n=\frac{a_0\alpha^n\alpha_2^{n-1}\alpha_3^{n-2}\dots\alpha_{n-1}^2\alpha_n}{[\alpha_2]^{n/2}[\alpha_2\alpha_3]^{(n-1)/2}\dots[\alpha_2\alpha_3\dots\alpha_{n-1}]^{3/2}[\alpha_2\alpha_3\dots\alpha_n]},\quad n\ge 2, \tag{2.14}$$

where

$$a_0>0,\quad \alpha>0,\quad \{\alpha_k\}_{k=2}^{\infty}\in U$$

and U is defined by (2.12).

In the present work we reduce the problem of characterization of the class $\tilde{Q}_3$ to that of Q_3 by proving the following theorem.

Theorem 3 . *A formal Laurent series (2.9) belongs to the class $\tilde{Q}_3$ if and only if both power series $f_1(z)=\sum_{k=0}^{\infty}a_{k-1}z^k$ and $f_2(z)=\sum_{k=0}^{\infty}a_{1-k}z^k$ belong to the class Q_3.*

It is natural to ask whether the class $\tilde{PF_3}$ itself has the property: both power series $f_1(z)=\sum_{k=0}^{\infty}a_{k-1}z^k$ and $f_2(z)=\sum_{k=0}^{\infty}a_{1-k}z^k$ belong to the class $\tilde{PF_3}$. The answer is negative as the following example shows. The Laurent series

$$f(z)=\sum_{n=-\infty}^{\infty}q^{-n^2}z^n$$

belongs to $\tilde{PF}_\infty$ for any $q > 1$ ([20], p.433). However, from the identity

$$I_k := \begin{vmatrix} q^{-(k-1)^2} & q^{-(k+2)^2} & q^{-(k+3)^2} \\ q^{-k^2} & q^{-(k+1)^2} & q^{-(k+2)^2} \\ 0 & q^{-k^2} & q^{-(k+1)^2} \end{vmatrix} = q^{-3k^2-6k-9}(q^6 - 2q^4 + 1), \quad k \in \mathbb{Z},$$

we conclude that $I_k < 0$ for $1 < q^2 < \frac{1+\sqrt{5}}{2}$ and hence

$$\sum_{n=k}^{\infty} q^{-n^2} z^n \notin PF_3 \quad \text{for} \quad 1 < q^2 < \frac{1+\sqrt{5}}{2} \quad \text{for any} \quad k \in \mathbb{Z}.$$

Combining Theorems 3 and I, we deduce a characterization of the class $\tilde{Q}_3$ in terms of independent parameters.

Theorem 4 . *A formal Laurent series (2.9) belongs to* $\tilde{Q}_3$ *if and only if*

$$a_0 = a_{-1}\alpha,$$

$$a_{n-1} = \frac{a_{-1}\alpha^n \alpha_2^{n-1}\alpha_3^{n-2}\dots\alpha_{n-1}^2\alpha_n}{[\alpha_2]^{n/2}[\alpha_2\alpha_3]^{(n-1)/2}\dots[\alpha_2\alpha_3\dots\alpha_{n-1}]^{3/2}[\alpha_2\alpha_3\dots\alpha_n]}, \quad n \geq 2, \quad (2.15)$$

$$a_{-n+1} = \frac{a_{-1}\alpha\beta^{n-1}\beta_2^{n-1}\beta_3^{n-2}\dots\beta_{n-1}^2\beta_n}{[\beta_2]^{n/2}[\beta_2\beta_3]^{(n-1)/2}\dots[\beta_2\beta_3\dots\beta_{n-1}]^{3/2}[\beta_2\beta_3\dots\beta_n]}, \quad n \geq 2, \quad (2.16)$$

where

$$\alpha > 0, \quad \{\alpha_k\}_{k=2}^{\infty} \in U, \quad \beta = \frac{1+\alpha_2}{\alpha\alpha_2}, \quad \beta_2 = \alpha_2, \quad \{\beta_{k+1}\}_{k=2}^{\infty} \in U$$

and U *is defined by (2.12).*

Using only the definition of the classes $\tilde{PF}_m$ it is rather difficult to construct functions belonging to $\tilde{PF}_m$, $m \geq 3$. Since $\tilde{Q}_3 \subset \tilde{PF}_3$, Theorem 4 provides a rich source of functions from $\tilde{PF}_3$. The important point to note here is that Theorem 4 allows to construct also functions from $\tilde{PF}_3 \backslash \tilde{PF}_4$.

Corollary 3 . *Let* U *be defined by (2.12). For any* α_2, $\frac{\sqrt{5}-1}{2} < \alpha_2 \leq 1$, *there exist such* α_3, β_3, $0 < \alpha_3, \beta_3 \leq 1$, *that for all* $\{\beta_{k+2}\}_{k=2}^{\infty} \in U$ *and* $\{\alpha_{k+2}\}_{k=2}^{\infty} \in U$, *the sequence* $\{a_n\}_{n=-\infty}^{\infty}$ *defined by (2.15) and (2.16) belongs to* $\tilde{PF}_3 \backslash \tilde{PF}_4$.

Theorems 3 and D allow us to derive the following result on growth estimates of functions from $\tilde{Q}_3$ being an analogue of Theorem D of [26] for Laurent series.

Theorem 5 . *Let a formal Laurent series (2.9) belong to* $\tilde{Q}_r$ *for some* $r \geq 3$. *Then it converges in* $\mathbb{C}\backslash\{0\}$ *and*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log\frac{1+\sqrt{5}}{2}}, \tag{2.17}$$

$$\limsup_{r\to 0} \frac{\log M(r,f)}{\left(\log\frac{1}{r}\right)^2} \leq \frac{1}{2\log\frac{1+\sqrt{5}}{2}}. \tag{2.18}$$

These estimates cannot be improved for $f \in \tilde{Q}_3$.

2.3 The distribution of the zeros of sections and tails of the Mittag–Leffler functions

Since e^z vanishes nowhere the zeros of the partial sum

$$s_n(z,e^z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

of e^z tend to infinity as $n \to \infty$ by the Hurwitz Theorem. A detailed study of the asymptotic behavior of the zeros was made by Szegö [36] who showed among many other things, that if $z_1^{(n)}$, $z_2^{(n)}$, ..., $z_n^{(n)}$ are the zeros of $s_n(z,e^z)$, then the point set $\{z_1^{(n)}/n, z_2^{(n)}/n, \ldots, z_n^{(n)}/n\}$ tends, as $n \to \infty$, to some simple closed curve that will be defined later. Evidently, points $z_1^{(n)}/n$, $z_2^{(n)}/n$, ..., $z_n^{(n)}/n$ are the zeros of $s_n(nz,e^z)$. Therefore, to study the zeros of $s_n(z,e^z)$, it is convenient to work with the *normalized* sections $s_n(nz,e^z)$.

Let $f(z)$ be an entire function having series representation (1.1). In the light of the preceding discussion, when considering the zero distribution of sections $s_n(z,f)$ and tails $t_n(z,f)$, it is convenient to study the behavior of the normalized sections $s_n(R_nz, f)$ and tails $t_n(R_nz, f)$, where $\{R_n\}_{n=1}^{\infty}$ is some suitable for f sequence of real numbers tending to ∞.

It turns out that it is convenient to choose as $\{R_n\}_{n=1}^{\infty}$ the sequence of discontinuity points of the central index of f. Recall that the central index $\nu(r)$, $0 \leq r < \infty$, is the value of n for which the $\max\{|a_n|r^n : n = 0, 1, \ldots\}$ is attained (see [30, pp.5–6]). If maximum value of $|a_n|r^n$ is attained for several n, then $\nu(r)$ denotes the largest of the corresponding n.

Let $\mathcal{M}_n(\lambda, f)$, $\lambda \in \mathbb{C}$, be the set of all roots of the equation $I_n(R_nz; \lambda, f) = 0$, where

$$I_n(R_nz; \lambda, f) = (1 - \lambda)s_n(R_nz, f) - \lambda t_{n+1}(R_nz, f). \tag{2.19}$$

In particular, $\mathcal{M}_n(0, f)$ ($\mathcal{M}_{n-1}(1, f)$) coincides with the zero set of $s_n(R_nz, f)$ ($t_n(R_{n-1}z, f)$). Define $\mathcal{M}(\lambda, f)$ to be the set of all accumulation points of $\cup_{n=1}^{\infty}\mathcal{M}_n(\lambda, f)$.

In 1924, G. Szegö [36] proved a remarkable theorem related to the asymptotic behavior of the roots of the equation $I_n(R_nz; \lambda, e^z) = 0$. Note that $R_n = n$ for $f(z) = e^z$. G. Szegö introduced the curve $S := \{z : |ze^{1-z}| = 1\}$ that is called the Szegö curve now.

Theorem J ([36]). *One has:*
(i) $\mathcal{M}(0, e^z) = S \cap \{z : |z| \leq 1\}$,
(ii) $\mathcal{M}(1, e^z) = S \cap \{z : |z| \geq 1\}$,
(iii) $\mathcal{M}(\lambda, e^z) = S$ *for* $\lambda \neq 0, 1$.

Theorem J asserts that the set of all zeros of $s_n(nz, e^z)$ is approximately equal to $S \cap \{z : |z| \leq 1\}$, the set of all zeros of $t_n(nz, e^z)$ is approximately equal to $S \cap \{z : |z| \geq 1\}$, and the set of all zeros of $(1 - \lambda)s_n(nz, e^z) - \lambda t_{n+1}(nz, e^z)$, $\lambda \neq 0, 1$, is approximately equal to S. It is worth mentioning that the main results of [36] were rediscovered by Dieudonné [8] in 1935 by a quite different method.

G. Szegö also investigated the accumulation density for the zeros of $I_n(nz, \lambda, e^z)$ to S. Let L be an open set on S such that 1 does not belong to the boundary of L. Let G_L be an open set in $\mathbb{C}$ such that $S \cap G_L = L$ and 1 does not belong to the boundary of G_L. Denote by $\nu_n(G, \lambda)$ the number of roots (counting multiplicities) of the equation $I_n(nz, \lambda, e^z) = 0$ in the region G_L. Denote by

$$\omega(z) = ze^{1-z}.$$

Let $l(L)$ be the length of $\omega(L)$ on $\{\omega : |\omega| = 1\}$ divided by 2π.

Theorem K ([36]). *We have*

$$\lim_{n\to\infty} \frac{\nu_n(G_L, \lambda)}{n} = \begin{cases} l(L \cap A), & for \quad \lambda = 0, \\ l(L \cap B), & for \quad \lambda = 1, \\ l(L), & for \quad \lambda \in \mathbb{C}\backslash\{0,1\}, \end{cases}$$

where

$$A = S \cap \{z : |z| \leq 1\} \qquad \text{and} \qquad B = S \cap \{z : |z| \geq 1\}$$

This theorem can be interpreted as "equidistribution" of the zeros of $I_n(nz, \lambda, e^z)$ along S. Nevertheless, Theorem K does not give good information about the zero distribution in the neighborhood of $z = 1$, because it is assumed that 1 is not contained in the closure of L. The following result of D.J. Newman and T.J. Rivlin [22] of 1972 complements Theorem J in this direction.

Theorem L ([22]). *We have, uniformly on compact subsets of the z-plane,*

$$\lim_{n\to\infty} \frac{s_n(n(1 + \frac{z}{\sqrt{n}}); e^z)}{\exp(n + z\sqrt{n})} = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{t^2}{2}} dt. \tag{2.20}$$

In particular, Theorem L implies that, as $n \to \infty$, the zeros of $s_n(n(1 + z/\sqrt{n}), e^z)$ approach the zeros of the error function

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-v^2} dv. \tag{2.21}$$

The validity of analogous result for tails $t_{n+1}(nz; e^z)$ is a consequence of (2.20) by virtue of the identity $s_n(z) + t_{n+1}(z) = e^z$. Several new properties of tails and sections of e^z were established by C. Y. Yıldırım [37], [38], [39] who utilized them for asymptotic estimation of quantities which came up in the theory of the Riemann zeta–function. We quote the result, which gives a relation between the zeros of sections and the zeros of the corresponding tails for e^z.

Theorem M ([37]). *Let ν_j be the zeros of $s_k(z, e^z)$, $j = 1, 2, \ldots, k$, and μ_l be the zeros of $t_{k+1}(z, e^z)$, $l = 1, 2, \ldots$. Then, for every $k \geq 2$,*

$$\sum_{j=1}^{k} e^{-\nu_j} - \sum_{l=1}^{\infty} e^{-\mu_l} = k + 1.$$

In 1983, A. Edrei, E.B. Saff and R.S. Varga [12] studied the distribution of the zeros of sections $s_n(R_n z, E_{1/\rho})$ of the Mittag–Leffler function of order $\rho > 1$:

$$E_{1/\rho}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1 + j/\rho)}, \qquad 1 < \rho < \infty. \tag{2.22}$$

This function is a natural generalization of e^z; evidently, $E_1(z) = e^z$. Mittag-Leffler functions play important role in analysis (see, e.g. [9], [17]).

A. Edrei, E.B. Saff and R.S. Varga [12] discovered that the zeros of $s_n(R_n z, E_{1/\rho})$ are related with the curve

$$S_1(\rho) = S'(\rho) \cup S''(\rho),$$

where

$$S'(\rho) = \{z = re^{i\phi} : r \leq 1, \quad |\phi| \leq \frac{\pi}{2\rho}, \quad r^\rho \cos(\phi\rho) - 1 - \rho \log r = 0\}, \tag{2.23}$$

$$S''(\rho) = \{z = re^{i\phi} : \frac{\pi}{2\rho} < \phi < 2\pi - \frac{\pi}{2\rho}, \quad r = e^{-\frac{1}{\rho}}\}. \tag{2.24}$$

It is easy to see that $S'(\rho)$ tends to $A \cap \{z : \mathrm{Re} z \geq 0\}$ as $\rho \to 1$, therefore the curve $S_1(\rho)$ can be viewed as an analogue of the part A of Szegö's curve S.

The arguments of the zeros of $E_{1/\rho}(z)$ for $1 < \rho < \infty$ tend to $\pm\pi/(2\rho)$ as $|z| \to \infty$. Hence, there are zeros of $s_n(R_n z, E_{1/\rho})$ whose arguments are close to $\pm\pi/(2\rho)$. Denote by

$$\mathcal{M}_0(\lambda, E_{1/\rho}) = \mathcal{M}(\lambda, E_{1/\rho}) \backslash \left\{ z : \arg z = \pm\frac{\pi}{2\rho} \right\}. \tag{2.25}$$

A. Edrei, E.B. Saff and R.S. Varga proved the following theorem which is an analogue of part (i) of Theorem J.

Theorem N ([12]). $\mathcal{M}_0(0, E_{1/\rho}) = S_1(\rho)$.

This theorem is a corollary of much more precise and complicated results of [12] related to the description of zero–free regions for $s_n(R_n z, E_{1/\rho})$.

The similarity between the zero distribution of sections of $E_{1/\rho}(z)$ (given by Theorem N) and the zero distribution of sections of e^z (given by Theorem J, part (i)) provokes the following questions. Does also an analogue of Szegö's result for $I_n(R_n z; \lambda, e^z)$ hold for $I_n(R_n z; \lambda, E_{1/\rho})$? What is the analogue of the part B of Szegö's curve S lying in the exterior of the unit disc? What is the description of zero–free regions for $I_n(R_n z; \lambda, E_{1/\rho})$ for arbitrary $\lambda \in \mathbb{C}$? Which are analogues for $I_n(R_n z; \lambda, E_{1/\rho})$ of other results of [12] related to asymptotic properties of $s_n(R_n z, E_{1/\rho}) = I_n(R_n z; 0, E_{1/\rho})$?

To give answers to these questions, denote by

$$S(\rho) = S_1(\rho) \cup S_2(\rho), \qquad \rho > 1,$$

where

$$S_2(\rho) = \{z = re^{i\phi} : \quad r \geq 1, \quad |\phi| \leq \frac{\pi}{2\rho}, \quad r^\rho \cos(\rho\phi) - 1 - \rho \log r = 0\}. \tag{2.26}$$

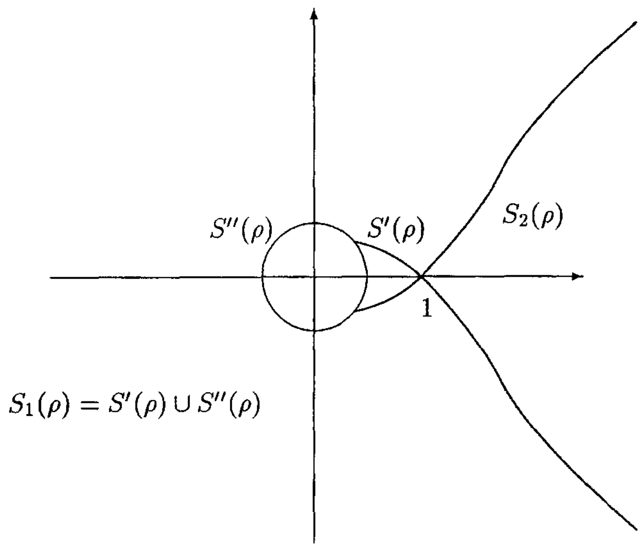

Figure 2.1: The generalized Szegö's curve $S(\rho)$.

It will be proved in Chapter 3 that the curve $S(\rho)$ has asymptotes $\arg z = \pm\pi/(2\rho)$ (meanwhile the original Szegö's curve S doesn't have any asymptote).

Our main result concerning the zero distribution of $I_n(R_n z, \lambda, E_{1/\rho})$ can be considered as a complete analogue of Szegö's Theorem J.

Theorem 6 . *One has:*

(i) $\mathcal{M}_0(0, E_{1/\rho}) = S_1(\rho)$,

(ii) $\mathcal{M}_0(1, E_{1/\rho}) = S_2(\rho)$,

(iii) $\mathcal{M}_0(\lambda, E_{1/\rho}) = S(\rho)$, *for* $\lambda \neq 0, 1$.

This theorem means that each point on the curve $S_1(\rho)$ is an accumulation point of the zeros of $s_n(R_n z, E_{1/\rho})$, each point on the curve $S_2(\rho)$ is an accumulation point of zeros of $t_{n+1}(R_n z, E_{1/\rho})$, and each point on the curve $S(\rho) = S_1(\rho) \cup S_2(\rho)$ is an accumulation point of the zeros of $I_n(R_n z, \lambda, E_{1/\rho})$, $\lambda \neq 0, 1$. Theorem 6 also answers the question how to continue the curve $S(\rho)$ into the exterior of the unit disc. Theorem 6 is an immediate corollary of Theorems 7, 8, 9 below.

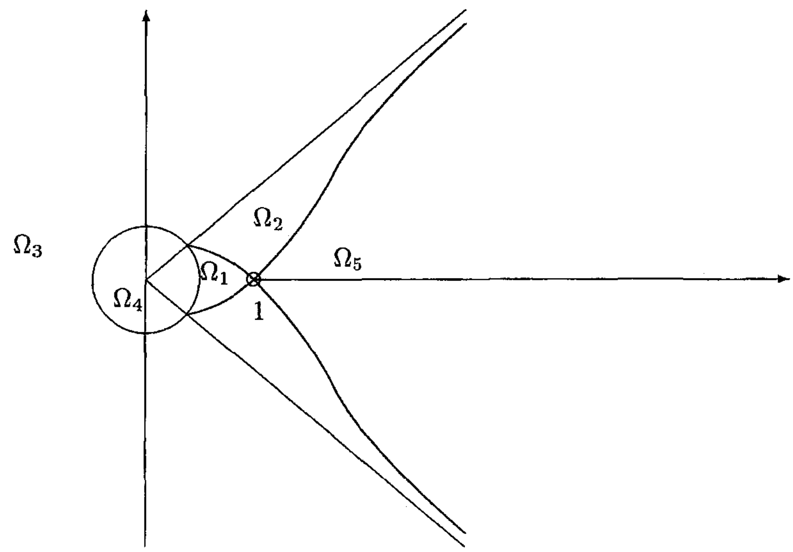

Figure 2.2: Regions Ω_i, $i = 1, 2, \ldots, 5$

For given sufficiently small $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ and $h > 0$, let us introduce the following regions.

$$\begin{aligned} \Omega_1 &= \{z = re^{i\phi} : \delta_3 \le r \le 1, |z-1| \ge \delta_1, |\phi| \le \tfrac{\pi}{2\rho} - \delta_2, \\ &\qquad\qquad r^\rho \cos(\rho\phi) - 1 - \rho \log r \ge 0\}, \\ \Omega_2 &= \{z = re^{i\phi} : |\phi| \le \tfrac{\pi}{2\rho} - \delta_2, r^\rho \cos(\rho\phi) - 1 - \rho\log r \le -h\}, \\ \Omega_3 &= \{z = re^{i\phi} : e^{-1/\rho} + h \le r, |\phi| \ge \tfrac{\pi}{2\rho} + \delta_2\}, \\ \Omega_4 &= \{z = re^{i\phi} : \delta_3 \le r \le e^{-1/\rho} - h, |\phi| \ge \tfrac{\pi}{2\rho} + \delta_2\}, \\ \Omega_5 &= \{z = re^{i\phi} : r \ge 1, |\phi| \le \tfrac{\pi}{2\rho} - \delta_2, r^\rho\cos(\rho\phi) - 1 - \rho\log r \ge h\}. \end{aligned} \tag{2.27}$$

The next theorem deals with the zero–free regions of $I_n(R_n z; \lambda, E_{1/\rho})$.

Theorem 7 . *Let δ_1, δ_2, δ_3 and h be given sufficiently small positive constants. Then, for all sufficiently large n, $I_n(R_n z; \lambda, E_{1/\rho})$ has no zeros in $\cup_{i=1}^5 \Omega_i$.*

Theorem 7 can be viewed as an extension of the following Theorem 5 from [12] that corresponds to the case $\lambda = 0$ (i.e. related to the zeros of $s_n(R_n z, E_{1/\rho}) = I_n(R_n z, 0, E_{1/\rho})$).

Theorem O ([12]). *Let δ_1, δ_2, δ_3 and h be given sufficiently small positive constants. Then, for all sufficiently large n, sections $s_n(R_n z, E_{1/\rho}) = I_n(R_n z, 0, E_{1/\rho})$ has no zeros outside the unit disc and in $\cup_{i=1}^4 \{\Omega_i \cap \{z : |z| \leq 1\}\}$.*

Theorem 7 implies that the *zeros* of $I_n(R_n z; \lambda, E_{1/\rho})$ may lie *only* in the vicinity of the curve $S(\rho)$ and two rays $\arg z = \pm\pi/(2\rho)$. Let us consider the functions $I_n(R_n z, \lambda, E_{1/\rho})$ in the neighborhood of points on the curve $S(\rho)$. The next theorem gives information about the behavior (and, in particular, the zero distribution) of $I_n(R_n z, \lambda, E_{1/\rho})$ in the neighborhood of the point $z = 1$ on $S(\rho)$.

Theorem 8 . *As $n \to \infty$, we have*

$$\left(1 + \left(\frac{2}{\rho n}\right)^{1/2} \zeta\right)^{-n} \left\{E_{1/\rho}(R_n)\right\}^{-1} I_n\left(R_n\left(1 + \left(\frac{2}{\rho n}\right)^{1/2} \zeta\right); \lambda, E_{\frac{1}{\rho}}\right)$$
$$\to e^{\zeta^2} \left\{\frac{erfc(\zeta)}{2} - \lambda\right\}$$

uniformly on every compact set of the ζ-plane.

By Hurwitz's Theorem, it follows that, as $n \to \infty$, the zeros of $I_n(R_n z, \lambda, E_{1/\rho})$ approach the zeros of $\frac{1}{2}\mathrm{erfc}(\zeta) - \lambda$. This theorem was first proved for $\rho = 1$, $\lambda = 0$ in [22] by D.J. Newman and T.J. Rivlin (see Theorem L before). It was proved in the case $\rho > 1$ and $\lambda = 0$ by A. Edrei, E.B. Saff and R.S. Varga (see [12], Theorem 1). Theorem 8 can be viewed as a generalization of all these results. More strict results were obtained in the case $\rho = 1$ and $\lambda = 1$ by C.Y. Yıldırım in [38].

Our next theorem is concerned with the behavior of $I_n(R_n z, \lambda, E_{1/\rho})$ near the points $\xi = \xi(\phi) \in S(\rho)$ distinct from the point $z = 1$.

Theorem 9 . *I. Let $\xi = \xi(\phi)$, $0 < \phi < \frac{\pi}{2\rho}$, be a fixed point on the generalized Szegö curve $S(\rho)$. Let $\tau = |\zeta|^\lambda \sin(\phi\rho) - \rho\phi$, and let the sequences $\{\tau_n\}_{n=1}^\infty$ and*

$\{\varepsilon_n(\zeta)\}_{n=1}^{\infty}$ *be defined by the conditions*

$$\tau_n \equiv \frac{\tau}{\rho} n (\bmod 2\pi), \qquad -\pi < \tau_n \leq \pi,$$

and

$$\varepsilon_n(\zeta) = \frac{\log n}{2(1-\xi^\rho)n} - \frac{\zeta - i\tau_n}{(1-\xi^\rho)n}.$$

Then, as $n \to \infty$,

$$I_n\left(R_n \xi \left(1 + \varepsilon_n(\zeta)\right); \lambda, E_{1/\rho}\right) \frac{\Gamma\left(1 + \frac{n}{\rho}\right)}{R_n^n \xi^n \left(1 + \varepsilon_n(\zeta)\right)^n}$$

$$\to \begin{cases} \alpha(\xi) e^{\zeta} - \dfrac{\xi}{1-\xi}, & \text{if} \quad |\xi| < 1, \\ -\beta(\xi) e^{\zeta} - \dfrac{\xi}{1-\xi}, & \text{if} \quad |\xi| > 1, \end{cases} \tag{2.28}$$

uniformly on every compact set of the ζ *- plane, where*

$$\alpha(\xi) = (1-\lambda)(2\pi\rho)^{\frac{1}{2}} e^{\frac{\rho+1}{2\rho}(\xi^\rho - 1)}$$

and

$$\beta(\xi) = \lambda(2\pi\rho)^{\frac{1}{2}} e^{\frac{\rho+1}{2\rho}(\xi^\rho - 1)}.$$

II. Let $\xi = e^{-\frac{1}{\rho}} e^{i\phi}$, $\frac{\pi}{2\rho} < \phi \leq \pi$, *be a fixed point on the circular portion of* $S(\rho)$, *and let the sequences* $\{\tau_n'\}_{n=1}^{\infty}$ *and* $\{\varepsilon_n'(\zeta)\}_{n=1}^{\infty}$ *be defined by the conditions*

$$\tau_n' \equiv (n+1)\phi (\ mod \quad 2\pi), \quad -\pi < \tau_n' \leq \pi,$$

and

$$\varepsilon_n'(\zeta) = \left(\frac{1}{2} - \frac{1}{\rho}\right) \frac{\log n}{n} - \frac{\zeta - i\tau_n'}{n+1}.$$

Then

$$I_n\left(R_n \xi \left(1 + \varepsilon_n'(\zeta)\right); \lambda, E_{1/\rho}\right) \frac{\Gamma\left(1 + \frac{n}{\rho}\right)}{R_n^n \xi^n \left(1 + \varepsilon_n'\right)^n} \to \gamma(\xi) e^{-\zeta} - \frac{\xi}{1-\xi} \tag{2.29}$$

uniformly on every compact set of the ζ *- plane, where*

$$\gamma(\xi) = \frac{(\lambda - 1)(2\pi e^{\frac{1-\rho}{\rho}})^{\frac{1}{2}}}{\rho^{\frac{1}{2}-\frac{1}{\rho}} \Gamma\left(1 - \frac{1}{\rho}\right)}.$$

As was mentioned before, by Theorem 7, the zeros of $I_n(R_n z; \lambda, E_{1/\rho})$ *may* lie only in the vicinity of the curve $S(\rho)$ and two rays $\arg z = \pm\pi/(2\rho)$. Theorem 9 implies that each point ξ on the curve $S(\rho)$ is an accumulation point of the zeros of $I_n(R_n z; \lambda, E_{1/\rho})$. Indeed, by Hurwitz's Theorem, the zeros of $I_n(R_n\xi(1+\varepsilon_n(\zeta)); \lambda, E_{1/\rho})$ approach the zeros of the limit functions in (2.28), if $\xi \in S'(\rho) \cup S_2(\rho)$, and the zeros of $I_n(R_n\xi(1+\varepsilon_n'(\zeta)); \lambda, E_{1/\rho})$ approach the zeros of the limit functions in (2.29), if $\xi \in S''(\rho)$.

The proof of Theorems 7 and 9 is based on the following theorem that deals with the asymptotic expressions for $I_n(R_n z; \lambda, E_{1/\rho})$ in different domains of $\mathbb{C}$.

Theorem 10 . *Let δ_1, δ_2, δ_3 be given sufficiently small positive constants, and $\rho > 1$. Then, as $n \to \infty$,*

$$\frac{I_n(R_n z; \lambda, E_{1/\rho})\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n z^n} = -\lambda\rho\frac{e^{R_n^\rho z^\rho}\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n z^n}(1+o(1)) - \frac{z}{1-z}(1+o(1)), \tag{2.30}$$

if $z \in \{z = re^{i\phi} : r \geq 1, |\phi| \leq \frac{\pi}{2\rho}, |z-1| \geq \delta_1\}$,

$$\frac{I_n(R_n z; \lambda, E_{1/\rho})\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n z^n} = (1-\lambda)\rho\frac{e^{R_n^\rho z^\rho}\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n z^n}(1+o(1)) - \frac{z}{1-z}(1+o(1)), \tag{2.31}$$

if $z \in \{z = re^{i\phi} : \delta_3 \leq r \leq 1, |\phi| \leq \frac{\pi}{2\rho}, |z-1| \geq \delta_1\}$,

$$\frac{I_n(R_n z; \lambda, E_{1/\rho})\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n z^n} = \frac{(\lambda-1)}{\Gamma\left(1-\frac{1}{\rho}\right)}\frac{\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^{n+1}z^{n+1}}(1+o(1)) - \frac{z}{1-z}(1+o(1)), \tag{2.32}$$

if $z \in \{z = re^{i\phi} : r \geq \delta_3, |\phi| \geq \frac{\pi}{2\rho} + \delta_2, \}$.
In all expressions above, $o(1)$ is uniform with respect to z.

We remark that, in the special case $\lambda = 1$, $|z| < 1$, one can find the asymptotic expression for $I_n(R_n z; 1, E_{1/\rho})$ in [12, Lemma 9.2].

3

Preliminaries

In this chapter we recall some definitions and results that we will need in the sequel.

3.1 One-sided multiply positive sequences

Let

$$\{a_n\}_{n=0}^{\infty}, \quad a_0 > 0, \tag{3.1}$$

be a sequence of real numbers. In this section we always assume that the sequence (3.1) is at least 2-times positive.

Lemma 1 *([26]). Let (3.1) be a 2–times positive sequence. Set $n = \min\{k : a_k = 0\}$. If n is finite, then $a_k = 0$ for any $k \geq n$.*

Proof. By the definition of 2-times positivity, we have, for any $k > n$,

$$\begin{vmatrix} a_n & a_k \\ a_{n-1} & a_{k-1} \end{vmatrix} \geq 0.$$

Since $a_n = 0$, $a_{n-1} > 0$, we conclude that $a_k = 0$. □

Therefore, a 2-times positive sequence (3.1) is either finite (has only finitely many nonzero terms), or is composed entirely of positive terms.

With (3.1), we associate the generating function (1.1). The fact that sequence (3.1) is 2–times positive implies that the radius of convergence $R(f)$ of series (1.1) is nonzero as the following lemma shows.

Lemma 2 *Let $f \in PF_2$. Then $R(f) > 0$.*

Proof. If the sequence (3.1) of coefficients of (1.1) has all nonzero terms – the only case to discuss due to Lemma 1 – we have

$$\begin{vmatrix} a_n & a_{n+1} \\ a_{n-1} & a_n \end{vmatrix} = a_n a_{n-1} \left(\frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n} \right) \geq 0.$$

Thus, $\left\{ \frac{a_{n+1}}{a_n} \right\}_{n=0}^{\infty}$ is a non increasing sequence of positive numbers that implies the existence of the limit

$$\frac{1}{R(f)} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}. \qquad \square$$

By Lemma 1, for all the functions (1.1) generating by 2-times positive *infinite* sequences (3.1), all their coefficients a_k are strictly positive, that allows us to introduce the positive numbers

$$\rho_k = \frac{a_{k-1}}{a_k}, \quad k = 1, 2, \ldots, \tag{3.2}$$

and

$$\delta_k = \frac{\rho_k}{\rho_{k-1}} = \frac{a_{k-1}^2}{a_k a_{k-2}}, \quad k = 2, 3, \ldots \tag{3.3}$$

The next formulas follow from (3.2) and (3.3)

$$\begin{aligned} a_k &= \frac{a_0}{\prod_{j=1}^{k} \rho_j}, \quad k = 1, 2, \ldots \quad \text{and} \\ a_k &= \frac{a_0}{\rho_1^k \delta_2^{k-1} \delta_3^{k-2} \ldots \delta_k}, \quad k = 2, 3, \ldots \end{aligned} \tag{3.4}$$

We have already mentioned that $\left\{\frac{a_{n+1}}{a_n}\right\}_{n=0}^{\infty}$ is a non increasing sequence. It implies that $\{\rho_n\}_{n=1}^{\infty}$ is a non decreasing sequence and hence

$$\delta_k \geq 1, \qquad k \geq 2. \tag{3.5}$$

It turns out that if sequence (3.1) of positive numbers satisfies a stronger condition than (3.5), namely, $\delta_n \geq A > 1$, then the corresponding generating function (1.1) is an entire function of order 0. More precisely, the following lemma holds.

Lemma 3 . *Let (1.1) be a formal power series with all positive coefficients $a_k > 0$. Assume that there exists a constant $A > 1$ such that*

$$\delta_k \geq A, \quad k \geq k_0, \tag{3.6}$$

where numbers δ_k are defined by (3.3). Then the series (1.1) converges in the whole complex plane $\mathbb{C}$, and

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log A}. \tag{3.7}$$

Proof. The proof of this lemma is contained in [26], but we present it here for the reader's convenience. By (3.4) and (3.6), we have

$$\log a_k = \log a_0 - k\log\rho_1 - \sum_{j=2}^{k}(k-j+1)\log\delta_j \leq \log a_0 - k\log\rho_1$$

$$-\sum_{j=2}^{k_0-1}(k-j+1)\log\delta_j - \sum_{j=k_0}^{k}(k-j+1)\log A = -\frac{k^2}{2}\log A + O(k), \quad k\to\infty.$$

Hence,

$$a_k \leq CD^k A^{-\frac{k^2}{2}}, \qquad k = 0,1,2,\ldots, \tag{3.8}$$

where C and D are positive constants not depending on k. Since $A > 1$, then $\lim_{k\to\infty} a_k^{1/k} = 0$. Therefore the series (1.1) converges in the whole plane. Further, using (3.8), we have

$$M(r,f) = \sum_{k=0}^{\infty} a_k r^k \leq C\sum_{k=0}^{\infty} A^{-\frac{k^2}{2}}(Dr)^k$$

$$= C\exp\left(\frac{(\log Dr)^2}{2\log A}\right)\sum_{k=0}^{\infty}\exp\left\{-\frac{\log A}{2}\left(k-\frac{\log(Dr)}{\log A}\right)^2\right\}$$
$$\le C\exp\left(\frac{(\log Dr)^2}{2\log A}\right)\sup_{-\infty<x<\infty}\sum_{k=-\infty}^{\infty}\exp\left\{-\frac{\log A}{2}(k-x)^2\right\}.$$

Since the sum of the series under the supremum sign is a periodic function of x (with period 1), its supremum is finite. Hence

$$M(r,f) = O\left(\exp\left\{\frac{(\log Dr)^2}{2\log A}\right\}\right), \quad r\to\infty,$$

and

$$\log M(r,f) \le \frac{(\log r)^2}{2\log A} + O(\log r), \quad r\to\infty. \qquad \square$$

Therefore, there is a close relation between estimates from below on δ_n and growth estimates of the corresponding entire functions of order 0.

The definition of multiple positivity is quite complicated. Therefore it is not easy to check whether or not a given sequence is m-times positive using the definition. Here we present the following test of m-times positivity of a finite positive sequences. It turns out that this test is rather effective.

Lemma 4 ***(Schoenberg's Theorem*** *([33])). Let $\{b_k\}_{k=0}^{n}$ be a finite sequence of numbers. Consider the m matrices*

$$B_k = \begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_n & 0 & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_{n-1} & b_n & 0 & \dots & 0 \\ 0 & 0 & b_0 & \dots & b_{n-2} & b_{n-1} & b_n & \dots & 0 \\ . & . & . & \dots & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & . & . & . & \dots & b_n \end{pmatrix}, \qquad k = 1,2,\dots .m,$$

where B_k consists of k rows and $n+k$ columns. Assume that the following condition is satisfied for $k = 1,2,\dots,m$: all $k\times k$–minors of B_k consisting of consecutive columns are strictly positive. Then the sequence $\{b_0, b_1, \dots, b_n, 0, 0, \dots\}$ is m–times positive.

3.1.1 One-sided sequences having sections with multiply positive terms

In this thesis we do not work with the classes PF_m, $m \geq 3$, themselves, but we study their subclasses $P_m \in PF_m$, $m \geq 3$. Recall that series (1.1) belongs to the class P_m, $m \geq 2$, if the truncated sequences $\{a_k\}_{k=0}^n$ are m-times positive for all sufficiently large n. It is easy to see that the class PF_2 coincides with the class P_2. Therefore, for all *infinite* sequences (3.1) from $P_m \subset P_2 = PF_2$, $m \geq 3$, the numbers δ_n, $n \geq 2$, are well defined. The next two lemmas from [26] give us information about sequences $\{\delta_n\}_{n=2}^\infty$ corresponding to the sequences $\{a_n\}_{n=0}^\infty$ from P_3.

Lemma 5 *([26]). Let* $\{a_k\}_{k=0}^n = \{a_0, a_1, \ldots, a_n, 0, 0, \ldots\}$, $a_0 > 0$, $a_n > 0$, $n \geq 2$, *be a* 3*–times positive sequence. Then*
(i) for $n = 2$, *we have* $\delta_2 \geq 2$;
(ii) for $n = 3$, *we have* $\delta_n > 1$ *and*

$$(\delta_n - 1)^2 \geq 1 - \frac{1}{\delta_{n-1}}. \tag{3.9}$$

Let sequence (3.1) with nonzero terms belong to the class P_3. Then there exists some $n_0 \geq 2$ such that, for each $n \geq n_0$, the truncated sequence $\{a_k\}_{k=0}^n$ is 3–times positive. By Lemma 5, we have $\delta_n > 1$ for all $n \geq n_0$, and therefore the numbers

$$y_n = \frac{1}{\delta_n - 1}, \quad n \geq n_0, \tag{3.10}$$

are well defined. Using (3.9), we obtain

$$y_{n+1}^2 \leq y_n + 1, \quad n \geq n_0.$$

Consider the sequence $\{z_n\}_{n=n_0}^\infty$ of positive numbers satisfying the recurrence equation

$$z_{n+1}^2 = z_n + 1, \quad n \geq n_0,$$

and the initial condition $z_{n_0} = y_{n_0}$. It is easy to see that

$$y_n \le z_n, \quad n \ge n_0. \tag{3.11}$$

Lemma 6 *([26]). Let a formal power series (1.1), which is not a polynomial, belong to P_3 and let for all $n \ge n_0$ the sequences $\{a_k\}_{k=0}^{n}$ be 3–times positive. Let $\{z_n\}_{n=n_0}^{\infty}$ be the sequence of positive numbers satisfying the recurrence relation*

$$z_{n+1}^2 = z_n + 1, \quad n \ge n_0, \tag{3.12}$$

and the initial condition

$$z_{n_0} = \frac{1}{\delta_{n_0} - 1}.$$

Then there exists the limit

$$\lim_{n\to\infty} z_n = \frac{1+\sqrt{5}}{2} := c,$$

and we have

$$\frac{1}{\delta_n - 1} \le z_n, \qquad n \ge n_0.$$

Moreover,

(i) if $z_{n_0} < c$, then the sequence $\{z_n\}_{n=n_0}^{\infty}$ increases;

(ii) if $z_{n_0} > c$, then the sequence $\{z_n\}_{n=n_0}^{\infty}$ decreases;

(iii) if $z_{n_0} = c$, then $z_n = c$ for any $n \ge n_0$.

Using (3.10), (3.11) and Lemma 6, we obtain that if sequence (3.1) belongs to P_3, then for any given $0 < \varepsilon < 1$,

$$\delta_n \ge c - \varepsilon, \qquad n \ge n(\varepsilon). \tag{3.13}$$

The estimate (2.4) for functions belonging to P_m, $m \ge 3$, follows from (3.13) and Lemma 3.

In general, it is difficult to check whether sequence (3.1) belong to PF_m, $m \ge 3$, or not. The next lemma from [27] gives a sufficient condition for (3.1) to belong to $Q_3 \subset PF_3$.

Lemma 7 *([27]). For a sequence (3.1) to belong to Q_3 it is sufficient to satisfy*

$$\delta_n \geq 2, \quad n \geq 2.$$

3.1.2 One-sided sequences having tails with multiply positive terms

Recall that series (1.1) belongs to the class R_m, $m \geq 2$, if the sequences $\{a_n, a_{n+1}, \ldots\}$ are m-times positive for all n large enough. By Lemma 2 such a series always have a positive radius of convergence by 2-times positivity of $\{a_n, a_{n+1}, \ldots\}$.

Assuming $f \in R_m$, $m \geq 2$, we choose n such that $\{a_n, a_{n+1}, \ldots\}$ is m-times positive and moreover $a_n > 0$. By Lemma 1, $a_k > 0$ for all $k \geq n$. This permits us to introduce the positive numbers

$$\rho_k = \frac{a_{k+1}}{a_k}, \quad k = n+1, n+2, \ldots \tag{3.14}$$

Evidently,

$$a_k = \frac{a_n}{\prod_{j=n+1}^{k} \rho_j}, \quad k = n+1, n+2, \ldots \tag{3.15}$$

Let us introduce the numbers

$$\delta_k = \frac{\rho_k}{\rho_{k-1}} = \frac{a_{k-1}^2}{a_k a_{k-2}}, \quad k = n+2, n+3, \ldots \tag{3.16}$$

Since the sequence $\{a_n, a_{n+1}, \ldots\}$ is 2-times positive, we have

$$\begin{vmatrix} a_{k-1} & a_k \\ a_{k-2} & a_{k-1} \end{vmatrix} \geq 0, \qquad k \geq n+2,$$

i.e. $a_{k-1}^2 \geq a_k a_{k-2}$, $k \geq n+2$. Therefore,

$$\delta_k \geq 1, \quad k = n+2, n+3, \ldots \tag{3.17}$$

It is easy to see that

$$a_k = \frac{a_n}{\rho_{n+1}^{k-n} \prod_{j=n+2}^{k} \delta_j^{k-j+1}}. \tag{3.18}$$

The next two lemmas from [28] give information about series (1.1) from R_m, $m \geq 3$.

Lemma 8 *([28]). Let $\{a_n, a_{n+1}, \ldots\}$ be a 3-times positive sequence with positive coefficients. Then*

$$(\delta_{n+2} - 1)^2 \geq 1 - \frac{1}{\delta_{n+3}}. \tag{3.19}$$

Lemma 9 *([28]). Let sequence (3.1) belong to R_3. Then there exists an integer p such that either*

(I) for any $n \geq p+2$, $\delta_n \geq c$, or

(II) there exists $q \geq p+2$, $\delta_q < c$.

In case (I) the assertion (i) of theorem G is valid while in the case (II) the assertion (ii) is.

We will need the following test of m-times positivity that is contained in [28] and which is due to I.J. Schoenberg (see [33]).

Lemma 10 ***(Schoenberg's Theorem*** *([33])). Let $\{b_k\}_{k=0}^{\infty}$ be a sequence of positive numbers. Consider the m matrices*

$$B_\nu = \begin{pmatrix} b_0 & b_1 & b_2 & \ldots & b_{\nu-1} & \ldots \\ 0 & b_0 & b_1 & \ldots & b_{\nu-2} & \ldots \\ 0 & 0 & b_0 & \ldots & b_{\nu-3} & \ldots \\ . & . & . & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & b_0 & \ldots \end{pmatrix} \qquad \nu = 1, 2, \ldots, m,$$

consisting of ν rows and infinitely many columns. Assume that for $\nu = 1, 2, \ldots, m$, the matrix B_ν satisfies the condition: all its $\nu \times \nu$–block-minors (i.e. minors consisting of ν consecutive rows and ν consecutive columns) are positive. Then the sequence $\{b_k\}_{k=0}^{\infty}$ is m–times positive.

Lemmas 4 and 10 are very similar. However, Lemma 4 deals with finite sequences $\{b_k\}_{k=0}^{n}$ while Lemma 10 deals with infinite sequences $\{b_k\}_{k=0}^{\infty}$.

3.2 Two-sided multiply positive sequences

Let

$$\{a_n\}_{n=-\infty}^{\infty}, \quad a_0 \neq 0, \tag{3.20}$$

be a two-sided infinite sequence of real numbers. In this section we always assume that (3.20) is at least 2-times positive.

Assume that the sequence (3.20) doesn't coincide with the trivial sequence $\{\rho^n\}_{n=-\infty}^{\infty}$ for some $\rho > 0$. It turns out (see [20], p.418) that such a 2-times positive sequence (3.20) generates Laurent series (2.9) that converges in a nonempty annulus $0 \leq 1/R_1 < |z| < R_2$.

Lemma 11 . *Let the sequence (3.20) be 2-times positive. Set*

$$k_1 = \min\{k > 0 : a_k = 0\}, \quad k_2 = \max\{k < 0 : a_k = 0\}.$$

Then $a_k = 0$ for all $k < k_2 + 1$ and $k > k_1 - 1$.

Proof. Let us show that $a_k = 0$ for all $k \leq k_2$. Assume $k_2 > -\infty$. We have

$$\begin{vmatrix} a_{k+1} & a_{k_2+1} \\ a_k & a_{k_2} \end{vmatrix} = -a_k a_{k_2+1} \geq 0$$

for all $k < k_2$. Hence, $a_k = 0$ for all $k \leq k_2$. That $a_k = 0$ for all $k \geq k_1$ can be proved similarly. □

Without loss of generality we can assume that $a_2 \neq 0$ and $a_{-2} \neq 0$, i.e. $k_1 > 2$ and $k_2 < -2$.

Lemma 11 allows us to introduce the positive numbers

$$\delta_k = \begin{cases} \dfrac{a_{k-1}^2}{a_k a_{k-2}}, & 0 < k < k_1, \\ \dfrac{a_{k+1}^2}{a_k a_{k+2}}, & k_2 < k < 0. \end{cases} \tag{3.21}$$

Note that $\delta_1 = \delta_{-1}$.

Lemma 12 . *Let sequence (3.20) be 2-times positive and the numbers k_1 and k_2 be defined as in Lemma 11. Then $\delta_k \geq 1$ for all k, $k_2 < k < k_1$.*

Proof. For any k , $k_2 < k < k_1$, we have

$$\begin{vmatrix} a_{k-1} & a_k \\ a_{k-2} & a_{k-1} \end{vmatrix} = a_k a_{k-2}(\delta_k - 1) \geq 0, \quad if \quad 1 \leq k < k_1,$$

and

$$\begin{vmatrix} a_{k+1} & a_{k+2} \\ a_k & a_{k+1} \end{vmatrix} = a_k a_{k+2}(\delta_k - 1) \geq 0, \quad if \quad k_2 < k \leq -1.$$

□

3.3 Power series with sections from PF_∞

Now we present the proof of Pólya' s Theorem E for the reader's convenience (see also [26]). The condition $f \in P_\infty$ means (see Chapter 2) that sections (1.2) have purely negative zeros for sufficiently large n, that is, for $n \geq n_1$ say. It is well known that the derivatives of a polynomial having purely real zeros have purely real zeros. Hence, the quadratic polynomial

$$\left(\frac{d}{dz}\right)^{n-2} s_n(z, f) = (n-2)! a_{n-2} + (n-1)! a_{n-1} z + \frac{1}{2} n! a_n z^2$$

does not have any complex zero. This means that the following inequality is valid:

$$(n-1) a_{n-1}^2 \geq 2n a_{n-2} a_n, \quad n \geq n_1.$$

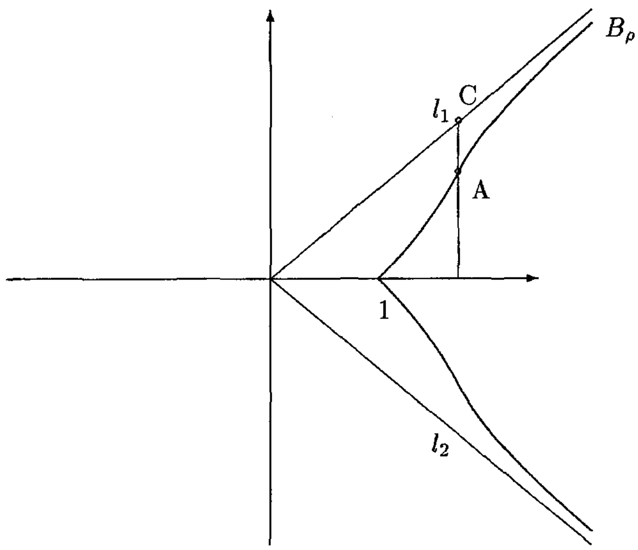

Figure 3.1: Asymptotes l_1 and l_2.

By the definition of δ_n, we can rewrite the last inequality in the form

$$\delta_n \geq \frac{2n}{n-1} > 2, \quad n \geq n_1.$$

Now Theorem E follows at once from Lemma 3. □

3.4 Some asymptotic relations for Mittag-Leffler functions

In Chapter 2 we have pointed out that the zeros of $I_n(R_n z, \lambda, E_{1/\rho})$, $\rho \geq 1$, are closely connected with the curve $S(\rho) = S_1(\rho) \cup S_2(\rho)$. It turns out that the curve $S_2(\rho)$ has asymptotes.

Lemma 13 . *The curve $S_2(\rho)$ has asymptotes* $l_j = \left\{ \arg z = (-1)^{j-1} \frac{\pi}{2\rho} \right\}$, $j = 1, 2$.

Proof. It suffices to consider only the upper half $B(\rho)$ of the curve $S_2(\rho)$ (see Figure 3.1). It is given by the equation

$$\cos\rho\phi = r^{-\rho}(1+\rho\log r),$$

which implies that the conditions $r\to\infty$ and $\phi\to\dfrac{\pi}{2\rho}$ are equivalent. Take a point $A=(r\cos\phi, r\sin\phi)$ on $B(\rho)$ and a point $C=\left(r\cos\phi, r\tan\dfrac{\pi}{2\rho}\cos\phi\right)$ on l_1. The distance $d(A,l_1)$ from the point A to l_1 does not exceed the distance between the points A and C, i.e.

$$d(A,l_1)\le r\tan\frac{\pi}{2\rho}\cos\phi - r\sin\phi \le \frac{r\sin\rho\left(\dfrac{\pi}{2\rho}-\phi\right)}{\cos\dfrac{\pi}{2\rho}}$$

$$=\frac{1+\rho\log r}{r^{\rho-1}\cos\dfrac{\pi}{2\rho}}\to 0 \quad \text{as} \quad r\to\infty. \qquad \square$$

Rewrite (2.19) as

$$I_n(R_nz;\lambda,E_{1/\rho}) = (1-\lambda)E_{1/\rho}(R_nz) - t_{n+1}(R_nz,E_{\frac{1}{\rho}}). \tag{3.22}$$

This formula shows that the study of the asymptotic behavior of $I_n(R_nz;\lambda,E_{1/\rho})$ can be reduced to asymptotic formulas for $E_{1/\rho}(R_nz)$ and $t_{n+1}(R_nz,E_{\frac{1}{\rho}})$. They will play a major role in the proof of Theorem 10. Let us present those formulas.

The function $E_{1/\rho}(z)$ have the following well-known asymptotic relations (see [17], p.114):

$$E_{1/\rho}(z) = \begin{cases} \rho e^{z^\rho} - \dfrac{1}{z\Gamma\left(1-\frac{1}{\rho}\right)} + O\left(\dfrac{1}{|z|^2}\right), & |\arg z|\le\dfrac{\pi}{2\rho}, \quad |z|\to\infty, \\ -\dfrac{1}{z\Gamma\left(1-\frac{1}{\rho}\right)} + O\left(\dfrac{1}{|z|^2}\right), & \dfrac{\pi}{2\rho}\le|\arg z|\le\pi, \quad |z|\to\infty. \end{cases} \tag{3.23}$$

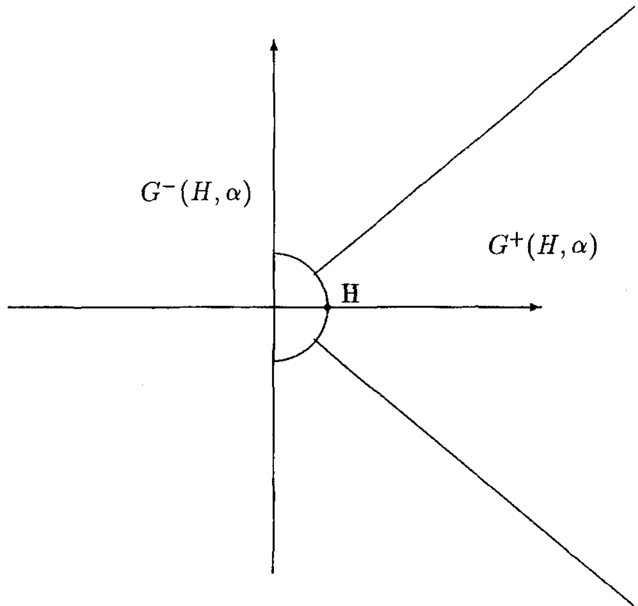

Figure 3.2: The contour $L(\alpha, H)$ and the regions $G^{\pm}(H, \alpha)$.

To write an asymptotic expression for $t_{n+1}(R_n z, E_{\frac{1}{\rho}})$, we will use so-called Mittag–Leffler type functions

$$E_{1/\rho}(z, \mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)}, \qquad \rho > 0, \quad \mu \in \mathbb{C},$$

introduced by M.M. Djrbashian (see [9, p.117]). Denote by $L(\alpha, H)$ ($H > 0$, $0 < \alpha \leq \pi$) a contour following nondecreasing direction of $\arg \zeta$ and consisting of two rays $\arg \zeta = \pm\alpha$, $|\zeta| \geq H$ and an arc $-\alpha \leq \arg \zeta \leq \alpha$ of a circle $|\zeta| = H$. By $G^-(H, \alpha)$ and $G^+(H, \alpha)$ we denote two regions lying respectively from the left and right sides of $L(\alpha, H)$ (see Figure 3.2). For $\rho > 1$ and $\pi/(2\rho) < \nu \leq \pi/\rho$, the following representations hold ([9, p.127])

$$E_{\frac{1}{\rho}}(z, \mu) = \begin{cases} \rho z^{\rho(1-\mu)} e^{z^\rho} + \dfrac{\rho}{2\pi i} \displaystyle\int_{L(\nu,H)} \frac{e^{\zeta^\rho} \zeta^{\rho(1-\mu)}}{\zeta - z} d\zeta, & z \in G^+(H, \nu), \\ \dfrac{\rho}{2\pi i} \displaystyle\int_{L(\nu,H)} \frac{e^{\zeta^\rho} \zeta^{\rho(1-\mu)}}{\zeta - z} d\zeta, & z \in G^-(H, \nu). \end{cases}$$

Since

$$t_{n+1}(R_n z, E_{\frac{1}{\rho}}) = \sum_{k=n+1}^{\infty} \frac{(R_n z)^k}{\Gamma(1 + \frac{k}{\rho})} = (R_n z)^{n+1} E_{\frac{1}{\rho}} \left(R_n z, 1 + \frac{n+1}{\rho} \right),$$

we have

$$t_{n+1}(R_n z, E_{\frac{1}{\rho}}) = \begin{cases} \rho e^{R_n^\rho z^\rho} + \dfrac{\rho(R_n z)^{n+1}}{2\pi i} \displaystyle\int_{L(\nu,H)} \dfrac{e^{\zeta^\rho}\zeta^{-(n+1)}}{\zeta - R_n z} d\zeta, & R_n z \in G^+(H,\nu), \\ \dfrac{\rho(R_n z)^{n+1}}{2\pi i} \displaystyle\int_{L(\nu,H)} \dfrac{e^{\zeta^\rho}\zeta^{-(n+1)}}{\zeta - R_n z} d\zeta, & R_n z \in G^-(H,\nu), \end{cases}$$

where ν and $H > 0$ are any constants but $\dfrac{\pi}{2\rho} < \nu \le \dfrac{\pi}{\rho}$. Then it follows from (3.22) and (3.23) that, as $n \to \infty$,

$$I_n(R_n z; \lambda, E_{1/\rho}) = -\lambda\rho e^{R_n^\rho z^\rho} + O\left(\frac{1}{R_n|z|}\right) - \frac{\rho(R_n z)^{n+1}}{2\pi i} \int_{L\left(\frac{\pi}{2\rho}+\frac{\delta_2}{2}, R_n\right)} \frac{e^{\zeta^\rho}\zeta^{-(n+1)}}{\zeta - R_n z} d\zeta, \tag{3.24}$$

if $|z| > 1$ and $|\arg z| \le \frac{\pi}{2\rho}$,

$$I_n(R_n z; \lambda, E_{1/\rho}) = \frac{(\lambda - 1)}{R_n z \Gamma\left(1 - \frac{1}{\rho}\right)} + O\left(\frac{1}{R_n^2|z|^2}\right) - \frac{\rho(R_n z)^{n+1}}{2\pi i} \int_{L\left(\frac{\pi}{2\rho}+\frac{\delta_2}{2}, R_n\right)} \frac{e^{\zeta^\rho}\zeta^{-(n+1)}}{\zeta - R_n z} d\zeta, \tag{3.25}$$

if $|z| > 0$ and $|\arg z| > \frac{\pi}{2\rho} + \frac{\delta_2}{2\rho}$

$$I_n(R_n z; \lambda, E_{1/\rho}) = (1-\lambda)\rho e^{R_n^\rho z^\rho} + O\left(\frac{1}{R_n|z|}\right) - \frac{\rho(R_n z)^{n+1}}{2\pi i} \int_{L\left(\frac{\pi}{2\rho}+\frac{\delta_2}{2}, R_n\right)} \frac{e^{\zeta^\rho}\zeta^{-(n+1)}}{\zeta - R_n z} d\zeta, \tag{3.26}$$

if $0 < |z| < 1$ and $|\arg z| \le \frac{\pi}{2\rho}$.

For $E_{1/\rho}(z)$ (see [30, p.26]), we have

$$R_n = \Gamma\left(1 + \frac{n}{\rho}\right) / \Gamma\left(1 + \frac{n-1}{\rho}\right).$$

Then, by Stirling's formula for $\Gamma(1+x)$, $x > 0$,

$$R_n = \left(\frac{n}{\rho}\right)^{1/\rho}\left(1 + \frac{\rho - 1}{2\rho n} + O(n^{-2})\right), \quad n \to \infty \tag{3.27}$$

4

Power series of sections with m-times positive coefficients

In this chapter we consider power series (1.1) from P_m, $m \geq 4$, i.e. such power series $\sum_{k=0}^{\infty} a_k z^k$ that the truncated sequences $\{a_k\}_{k=0}^{n}$ are m-times positive for all sufficiently large n. It was proved in [26] that if $f \in P_m$, $m \geq 3$, then $f(z)$ is an entire function satisfying

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log c}, \qquad c = \frac{1+\sqrt{5}}{2} < 2. \tag{2.4}$$

Moreover, estimate (2.4) cannot be improved for $f \in P_3$. Theorem E states that if $f \in P_\infty$, then f is an entire function satisfying

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log 2}. \tag{2.5}$$

Here we prove that an estimate better than (2.5) already holds in the class P_4 which is much larger than P_∞. Our first result is the following theorem.

Theorem 1 *If $m \geq 4$, then any formal power series of P_m converges in the whole* *complex plane and its sum $f(z)$ is an entire function satisfying*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log d(P_4)}, \tag{2.6}$$

where constant $d(P_4)$ is independent of f and

$$2.016 \leq d(P_4) \leq 2.087.$$

4.1 The lower bound for $d(P_4)$

The following lemma plays a basic role in the proof of the Theorem 1.

Lemma 14 . *Let $\{a_k\}_{k=0}^{n} = \{a_0, a_1, \ldots, a_n, 0, 0, \ldots\}$, $a_0 > 0$, $a_n > 0$, $n \geq 2$, be a 4–times positive sequence. Then*
(i) for $n = 2$, we have $\delta_2 \geq \dfrac{3+\sqrt{5}}{2}$;
(ii) for $n = 3$, we have

$$\left(\delta_3 - \frac{3}{2}\right)^2 \geq \frac{5}{4} - \frac{2}{\delta_2}; \tag{4.1}$$

(iii) for $n \geq 4$, we have

$$\left(\delta_n - \frac{3}{2}\right)^2 \geq \frac{5}{4} - \frac{2}{\delta_{n-1}} + \frac{1}{\delta_{n-2}\delta_{n-1}^2\delta_n}. \tag{4.2}$$

Proof. Since $\{a_k\}_{k=0}^{n}$ is 4–times positive, we have

$$I_n := \begin{vmatrix} a_{n-1} & a_n & 0 & 0 \\ a_{n-2} & a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix}$$

$$= a_{n-1}^4 - 3a_{n-1}^2 a_{n-2} a_n + 2a_{n-3}a_{n-1}a_n^2 - a_n^3 a_{n-4} + a_n^2 a_{n-2}^2 \geq 0,$$

where $a_k = 0$ for $k < 0$. In particular,

$$I_2 = a_1^4 - 3a_1^2 a_0 a_2 + a_2^2 a_0^2,$$

$$I_3 = a_2^4 - 3a_2^2a_1a_3 + 2a_0a_2a_3^2 + a_3^2a_1^2.$$

By (3.3),

$$\frac{a_{n-4}a_n}{a_{n-2}^2} = \frac{a_{n-4}a_{n-2}}{a_{n-3}^2} \cdot \frac{a_{n-3}^2a_{n-1}^2}{a_{n-2}^4} \cdot \frac{a_na_{n-2}}{a_{n-1}^2} = \frac{1}{\delta_{n-2}\delta_{n-1}^2\delta_n}.$$

Hence,

$$I_2 = a_0^2a_2^2\left(\left(\frac{a_1^2}{a_0a_2}\right)^2 - 3\frac{a_1^2}{a_0a_2} + 1\right) = a_0^2a_2^2\left(\delta_2^2 - 3\delta_2 + 1\right),$$

$$I_3 = a_1^2a_3^2\left(\left(\frac{a_2^2}{a_1a_3}\right)^2 - 3\frac{a_2^2}{a_1a_3} + 2\frac{a_0a_2}{a_1^2} + 1\right) = a_1^2a_3^2\left(\delta_3^2 - 3\delta_3 + \frac{2}{\delta_2} + 1\right),$$

$$I_n = a_{n-2}^2a_n^2\left(\left(\frac{a_{n-1}^2}{a_na_{n-2}}\right)^2 - 3\frac{a_{n-1}^2}{a_na_{n-2}} + 2\frac{a_{n-3}a_{n-1}}{a_{n-2}^2} - \frac{a_na_{n-4}}{a_{n-2}^2} + 1\right)$$

$$= a_n^2a_{n-2}^2\left(\delta_n^2 - 3\delta_n + \frac{2}{\delta_{n-1}} + 1 - \frac{1}{\delta_{n-2}\delta_{n-1}^2\delta_n}\right).$$

Since, $\delta_n \geq 1$ by (3.5), the assertions of the lemma are easy consequences of the condition $I_n \geq 0$. □

Corollary 4 . *Let a formal power series (1.1), which is not a polynomial, belong to Q_4. Then $\delta_k > 2$ for all $k \geq 2$.*

Proof. The inequality (4.2) implies

$$\left(\delta_n - \frac{3}{2}\right)^2 > \frac{5}{4} - \frac{2}{\delta_{n-1}}$$

for $n \geq 4$. Note that, by Lemma 14, we have $\delta_2 \geq \dfrac{3+\sqrt{5}}{2} > 2$ and

$$\left(\delta_3 - \frac{3}{2}\right)^2 \geq \frac{5}{4} - \frac{2}{\delta_2} > \frac{1}{4},$$

i.e. $\delta_3 > 2$. Then the proof can be completed by induction. □

Lemma 15 . *Let a formal power series (1.1), which is not a polynomial, belong to P_4. Then there exists N such that $\delta_n > c$ for all $n \geq N$, where $c = \dfrac{1+\sqrt{5}}{2}$.*

Proof. By Lemma 5, $\delta_n > 1$ for all sufficiently large n. Since 4–times positivity implies 3–times positivity, then, by Lemma 6, for any $\varepsilon > 0$, there exists such n_1 that $\dfrac{1}{\delta_n - 1} < c + \varepsilon$ for all $n \geq n_1$. Assuming that $\delta_n \leq c$ for all sufficiently large n, we get from (4.2)

$$\left(c - \frac{3}{2}\right)^2 \geq \frac{5}{4} - \frac{2}{c - \varepsilon} + \frac{1}{c^4}$$

for all sufficiently small ε, in contradiction with the equality

$$\left(c - \frac{3}{2}\right)^2 = \frac{5}{4} - \frac{2}{c}. \tag{4.3}$$

Hence, there exists such N that $\delta_N > c$.

Set $n_0 = N$ and construct the sequence $\{z_n\}_{n=N}^{\infty}$ by (3.12) and the initial condition $z_N = \dfrac{1}{\delta_N - 1}$. Since $\delta_N > c$, then $z_N < c$. It follows from Lemma 6 that $z_n < c$ for all $n \geq N$, whence $\delta_n > c$ for all $n \geq N$. □

Consider the sequence $\{k_n\}_{n=N}^{\infty}$ of positive numbers satisfying the recurrence equation

$$\left(k_n - \frac{3}{2}\right)^2 = \frac{5}{4} - \frac{2}{k_{n-1}}, \quad n > N, \tag{4.4}$$

and the initial condition

$$k_N = \delta_N. \tag{4.5}$$

The inequalities (4.1) and (4.2) imply

$$\delta_n > k_n, \quad n > N. \tag{4.6}$$

Since $k_N = \delta_N > c$ and (4.3) holds, we get from (4.4) by induction that

$$k_n > c, \quad n \geq N. \tag{4.7}$$

Lemma 16 . *We have*

$$\lim_{n \to \infty} k_n = 2.$$

Moreover,

(i) if $c < k_N < 2$, then the sequence $\{k_n\}_{n=N}^{\infty}$ increases;

(ii) if $k_N > 2$, then the sequence $\{k_n\}_{n=N}^{\infty}$ decreases;

(iii) if $k_N = 2$, then $k_n = 2$ for any $n \geq N$.

Proof. Note that the equation $(x-3/2)^2 = 5/4 - 2/x$ can be rewritten in the form $(x-2)(x^2-x-1) = 0$. Therefore, if $c < x < 2$, then $(x-3/2)^2 < 5/4-2/x$. Let $n \geq N$ and $c < k_n < 2$. Then, by (4.4) and the arguments before,

$$\left(k_{n+1} - \frac{3}{2}\right)^2 = \frac{5}{4} - \frac{2}{k_n} > \left(k_n - \frac{3}{2}\right)^2.$$

It implies $k_{n+1} > k_n$. That $c < k_{n+1} < 2$ follows from

$$\left(k_{n+1} - \frac{3}{2}\right)^2 = \frac{5}{4} - \frac{2}{k_n}.$$

The proof of part (i) can be completed by induction on n. The proof of parts (ii) and (iii) is similar to that of part (i) and can be omitted. □

By Lemma 16 and (4.6), we have for any $f \in P_4$

$$\delta(f) := \liminf_{n\to\infty} \delta_n \geq 2. \tag{4.8}$$

Define

$$\delta(P_4) = \inf\{\delta(f) : \quad f \in P_4\}. \tag{4.9}$$

Lemma 17 . *We have*

$$\delta(P_4) \geq 2 + \frac{1}{66}.$$

Proof. Assume that there exists $f \in P_4$ such that $\delta(f) \leq 2 + \varepsilon$, where $\varepsilon = \dfrac{1}{66}$. Then there exists a subsequence $\{n_k\}_{k=0}^{\infty}$ such that $\delta_{n_k} < 2 + A\varepsilon$, where $A = \dfrac{66}{65} > 1$. The inequality (4.2) implies

$$\delta_{n_k-2}\delta_{n_k-1}^2\left(2 + A\varepsilon - \frac{3}{2}\right)^2 + 2\delta_{n_k-2}\delta_{n_k-1} > \frac{5}{4}\delta_{n_k-2}\delta_{n_k-1}^2 + \frac{1}{2+A\varepsilon}, \tag{4.10}$$

which can be rewritten as

$$\delta_{n_k-2}\delta_{n_k-1}(2 - \delta_{n_k-1}(1 - A\varepsilon - (A\varepsilon)^2)) \geq \frac{1}{2+A\varepsilon},$$

whence

$$\delta_{n_k-1} \le \frac{2}{1 - A\varepsilon - (A\varepsilon)^2} \le 2 + 4A\varepsilon.$$

The last inequality holds for any ε such that $A\varepsilon < \frac{1}{4}$. By the repetition of argument,

$$\delta_{n_k-2} \le 2 + 16A\varepsilon.$$

Therefore, the inequality (4.2) implies

$$\frac{2}{\delta_{n_k-1}} \ge \frac{5}{4} - \left(\delta_{n_k} - \frac{3}{2}\right)^2 + \frac{1}{\delta_{n_k-2}\delta_{n_k-1}^2\delta_{n_k}} \ge 1 - A\varepsilon - (A\varepsilon)^2 + \frac{1}{(2+16A\varepsilon)^4}$$

$$> 1 - 2A\varepsilon + \frac{1}{16}(1 - 32A\varepsilon) = \frac{17}{16} - 4A\varepsilon = 1 + B,$$

where $B = \frac{1}{16 \cdot 65} > 0$. Hence, we get

$$\delta_{n_k-1} < \frac{2}{1+B} < 2,$$

in contradiction with (4.8). □

Define $d(f)$ and $d(P_4)$ by formulas

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} =: \frac{1}{2\log d(f)}, \tag{4.11}$$

and

$$d(P_4) := \inf\{d(f): \quad f \in P_4\}. \tag{4.12}$$

It follows from Lemma 3 that $d(f) \ge \delta(f)$ and hence, $d(P_4) \ge \delta(P_4)$. Then, by Lemma 17, $d(P_4) \ge 2 + \frac{1}{66} = 2.016\ldots$.

4.2 The upper bound for $d(P_4)$

Let

$$I_n := \begin{vmatrix} a_{n-1} & a_n & 0 & 0 \\ a_{n-2} & a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix}, \qquad J_n := \begin{vmatrix} a_{n-1} & a_n & a_{n+1} & 0 \\ a_{n-2} & a_{n-1} & a_n & a_{n+1} \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix},$$

$$L_n := \begin{vmatrix} a_{n-1} & a_n & a_{n+1} & a_{n+2} \\ a_{n-2} & a_{n-1} & a_n & a_{n+1} \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix},$$

where $a_k = 0$ for $k < 0$.

Lemma 18 . *Let $\{a_k\}_{k=0}^{\infty}$ be a 2–times positive sequence without zero terms. Assume $I_n > 0$ for all $n \geq 2$. Then $L_n > 0$ and $J_n > 0$ for all $n \geq 2$.*

Proof. By (3.5), $\delta_n \geq 1$ for all $n \geq 2$. Then, the inequality

$$I_2 = a_0^2 a_2^2(\delta_2^2 - 3\delta_2 + 1) > 0$$

implies $\delta_2 > \dfrac{3+\sqrt{5}}{2} > 2$. Suppose $\delta_{n-1} > 2$ for some $n \geq 3$. Then the inequalities (4.1) and (4.2), which correspond to $I_3 > 0$ and $I_n > 0$, $n \geq 4$, respectively, give

$$\left(\delta_n - \frac{3}{2}\right)^2 > \frac{5}{4} - 1 = \frac{1}{4},$$

i.e. $\delta_n > 2$. Thus, $\delta_n > 2$ for all $n \geq 2$. Hence, by Lemma 7, we have $\sum_{k=0}^{\infty} a_k z^k \in Q_3$.

We have

$$L_n = I_n + A_n + B_n,$$

where

$$A_n = \begin{vmatrix} 0 & 0 & a_{n+1} & a_{n+2} \\ a_{n-2} & a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix}$$

and

$$B_n = \begin{vmatrix} a_{n-1} & a_n & a_{n+1} & a_{n+2} \\ 0 & 0 & 0 & a_{n+1} \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix}.$$

For $n \geq 4$ we have

$$A_n = a_{n+1} \begin{vmatrix} a_{n-2} & a_{n-1} & 0 \\ a_{n-3} & a_{n-2} & a_n \\ a_{n-4} & a_{n-3} & a_{n-1} \end{vmatrix} - a_{n+2} \begin{vmatrix} a_{n-2} & a_{n-1} & a_n \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-4} & a_{n-3} & a_{n-2} \end{vmatrix}$$

$$= a_{n+1}a_{n-2}a_{n-3}a_{n-4} \begin{vmatrix} 1 & \dfrac{a_{n-1}}{a_{n-2}} & 0 \\ 1 & \dfrac{a_{n-2}}{a_{n-3}} & \dfrac{a_n}{a_{n-3}} \\ 1 & \dfrac{a_{n-3}}{a_{n-4}} & \dfrac{a_{n-1}}{a_{n-4}} \end{vmatrix}$$

$$-a_{n+2}a_{n-2}a_{n-3}a_{n-4} \begin{vmatrix} 1 & \dfrac{a_{n-1}}{a_{n-2}} & \dfrac{a_n}{a_{n-2}} \\ 1 & \dfrac{a_{n-2}}{a_{n-3}} & \dfrac{a_{n-1}}{a_{n-3}} \\ 1 & \dfrac{a_{n-3}}{a_{n-4}} & \dfrac{a_{n-2}}{a_{n-4}} \end{vmatrix}$$

$$= a_{n+1}a_{n-2}a_{n-3}a_{n-4}\cdot\frac{a_{n-1}}{a_{n-2}}\cdot\frac{a_{n+1}}{a_{n-2}}\begin{vmatrix}1 & 1 & 0\\ 1 & \dfrac{a_{n-2}^2}{a_{n-1}a_{n-3}} & \dfrac{a_na_{n-2}}{a_{n-3}a_{n+1}}\\ 1 & \dfrac{a_{n-3}a_{n-2}}{a_{n-4}a_{n-1}} & \dfrac{a_{n-1}a_{n-2}}{a_{n-4}a_{n+1}}\end{vmatrix}$$

$$-a_{n+2}a_{n-2}a_{n-3}a_{n-4}\cdot\frac{a_{n-1}}{a_{n-2}}\cdot\frac{a_n}{a_{n-2}}\begin{vmatrix}1 & 1 & 1\\ 1 & \dfrac{a_{n-2}^2}{a_{n-1}a_{n-3}} & \dfrac{a_{n-1}a_{n-2}}{a_{n-3}a_n}\\ 1 & \dfrac{a_{n-3}a_{n-2}}{a_{n-4}a_{n-1}} & \dfrac{a_{n-2}^2}{a_{n-4}a_n}\end{vmatrix}.$$

Since $a_k \neq 0$, $k = 0, 1, 2, \ldots$, and $\sum_{k=0}^{\infty} a_k z^k \in Q_3$, then all minors in the last equation are non-negative. Using $a_{n+2} \leq \dfrac{a_{n+1}^2}{a_n}$ (which is equivalent to $\delta_{n+2} \geq 1$), we get

$$A_n \geq a_{n-3}a_{n-4}a_{n-1}\cdot\frac{a_{n+1}^2}{a_{n-2}}\Delta_n,$$

where

$$\Delta_n = \begin{vmatrix}1 & 1 & 0\\ 1 & \dfrac{a_{n-2}^2}{a_{n-1}a_{n-3}} & \dfrac{a_na_{n-2}}{a_{n-3}a_{n+1}}\\ 1 & \dfrac{a_{n-3}a_{n-2}}{a_{n-4}a_{n-1}} & \dfrac{a_{n-1}a_{n-2}}{a_{n-4}a_{n+1}}\end{vmatrix} - \begin{vmatrix}1 & 1 & 1\\ 1 & \dfrac{a_{n-2}^2}{a_{n-1}a_{n-3}} & \dfrac{a_{n-1}a_{n-2}}{a_{n-3}a_n}\\ 1 & \dfrac{a_{n-3}a_{n-2}}{a_{n-4}a_{n-1}} & \dfrac{a_{n-2}^2}{a_{n-4}a_n}\end{vmatrix}.$$

By the second formula of (3.4),

$$\Delta_n = \begin{vmatrix}1 & 1 & 0\\ 1 & \delta_{n-1} & \delta_{n-1}\delta_n\delta_{n+1}\\ 1 & \delta_{n-1}\delta_{n-2} & \delta_{n+1}\delta_n^2\delta_{n-1}^2\delta_{n-2}\end{vmatrix} - \begin{vmatrix}1 & 1 & 1\\ 1 & \delta_{n-1} & \delta_{n-1}\delta_n\\ 1 & \delta_{n-1}\delta_{n-2} & \delta_n\delta_{n-1}^2\delta_{n-2}\end{vmatrix}$$

$$= \delta_{n+1}\delta_n^2\delta_{n-1}^3\delta_{n-2} + \delta_{n-1}\delta_n\delta_{n+1} - \delta_{n-1}^2\delta_{n-2}\delta_n\delta_{n+1} - \delta_{n+1}\delta_n^2\delta_{n-1}^2\delta_{n-2}$$

$$-\delta_n\delta_{n-1}^3\delta_{n-2} - \delta_{n-1}\delta_n - \delta_{n-1}\delta_{n-2} + \delta_{n-1} + \delta_{n-1}^2\delta_n\delta_{n-2} + \delta_n\delta_{n-1}^2\delta_{n-2}$$

$$= \delta_{n+1}\delta_n\delta_{n-1}^2\delta_{n-2}\left\{\frac{1}{4}\delta_n\delta_{n-1} - 1\right\} + \delta_{n+1}\delta_n^2\delta_{n-1}^2\delta_{n-2}\left\{\frac{1}{2}\delta_{n-1} - 1\right\}$$

$$+\delta_n\delta_{n-1}^3\delta_{n-2}\left\{\frac{1}{4}\delta_{n+1}\delta_n-1\right\}+\delta_{n-1}\delta_n\{\delta_{n+1}-1\}$$
$$+\delta_{n-1}\delta_{n-2}\{\delta_{n-1}\delta_n-1\}+\delta_{n-1}+\delta_n\delta_{n-1}^2\delta_{n-2}.$$

Since $\delta_n > 2$ for all $n \geq 2$, then $\Delta_n > 0$ for all $n \geq 4$, and so $A_n > 0$ for all $n \geq 4$.

As we mentioned before, $\sum_{k=0}^{\infty} a_k z^k \in Q_3$. Hence,

$$B_n = a_{n+1}\begin{vmatrix} a_{n-1} & a_n & a_{n+1} \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-4} & a_{n-3} & a_{n-2} \end{vmatrix} \geq 0$$

for all $n \geq 4$. Therefore, $L_n > 0$ for all $n \geq 4$.

The condition $\delta_n > 2$ also yields

$$L_2 = I_2 + a_0^3 a_4(\delta_4\delta_3\delta_2 - 1) + a_3 a_0^3 > 0$$

and

$$L_3 = I_3 + A_3 + a_4\begin{vmatrix} a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} > 0,$$

since $\delta_n > 2$ and, hence,

$$A_3 = a_0a_1a_2a_5\left\{\delta_3\delta_4\delta_5\left\{\frac{1}{2}\delta_2-1\right\}+\delta_4\delta_5\left\{\frac{1}{4}\delta_2\delta_3-1\right\}\right.$$
$$\left.+\delta_2\left\{\frac{1}{4}\delta_3\delta_4\delta_5-1\right\}+\left\{2-\frac{1}{\delta_2\delta_3}\right\}\right\} > 0.$$

Taking into account the equality

$$J_n = L_n + a_{n+2}\begin{vmatrix} a_{n-2} & a_{n-1} & a_n \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-4} & a_{n-3} & a_{n-2} \end{vmatrix}$$

and the fact that $\sum_{k=0}^{\infty} a_k z^k \in Q_3$, we obtain $J_n > 0$. □

Lemma 19 . *Let $\{a_k\}_{k=0}^{\infty}$ be a 2–times positive sequence without zero terms. Assume $I_n > 0$, for all $n \geq 2$. Then $f(z) \in P_4$.*

Proof. We shall use Schoenberg's test of m–times positivity (see Lemma 4). Fix any $k \geq 2$, and consider the three matrices

$$B_2 = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_k & 0 \\ 0 & a_0 & a_1 & \dots & a_{k-1} & a_k \end{pmatrix}$$

$$B_3 = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_k & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_{k-1} & a_k & 0 \\ 0 & 0 & a_0 & \dots & a_{k-2} & a_{k-1} & a_k \end{pmatrix}$$

$$B_4 = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_k & 0 & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_{k-1} & a_k & 0 & 0 \\ 0 & 0 & a_1 & \dots & a_{k-2} & a_{k-1} & a_k & 0 \\ 0 & 0 & 0 & \dots & a_{k-3} & a_{k-2} & a_{k-1} & a_k \end{pmatrix}.$$

The condition $I_n > 0$ for all $n \geq 2$ implies $\delta_n > 2$ for all $n \geq 2$. Then 2×2 minors of B_2,

$$\begin{vmatrix} a_{n-1} & a_n \\ a_{n-2} & a_{n-1} \end{vmatrix} = a_{n-2}a_n(\delta_n - 1)$$

and 3×3 minors of B_3,

$$\begin{vmatrix} a_{n-1} & a_n & 0 \\ a_{n-2} & a_{n-1} & a_n \\ a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix} = a_{n-3}a_n^2\delta_{n-1}\left(\delta_n^2 - 2\delta_n + \frac{1}{\delta_{n-1}}\right),$$

$$\begin{vmatrix} a_{n-1} & a_n & a_{n+1} \\ a_{n-2} & a_{n-1} & a_n \\ a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix} = \begin{vmatrix} a_{n-1} & a_n & 0 \\ a_{n-2} & a_{n-1} & a_n \\ a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix} + a_{n+1}a_{n-3}a_{n-1}(\delta_{n-1} - 1),$$

consisting of consecutive columns are strictly positive. By Lemma 18, all 4×4 minors of B_4, consisting of consecutive columns are also strictly positive . Hence, by Schoenberg's theorem, the sequence $\{a_0, a_1, \dots, a_k, 0, 0, \dots\}$ is 4–times positive for all $k \geq 2$, and hence $f(z) \in P_4$. □

Consider the sequence $\{d_n > 2\}_{n=2}^{\infty}$ of positive numbers satisfying the recurrence equation

$$\left(d_n - \frac{3}{2}\right)^2 = \frac{5}{4} - \frac{2}{d_{n-1}} + \frac{1}{d_n^4} \tag{4.13}$$

and the initial condition

$$d_2 = \frac{3+\sqrt{5}}{2} + \frac{1}{2}. \tag{4.14}$$

Denote

$$g(x,y) := x^6 - 3x^5 + \left(1 + \frac{2}{y}\right) x^4 - 1.$$

Note that if $y > 2$, then $g(2,y) < 0$ and hence there exists $x_y > 2$ such that $g(x_y, y) = 0$ for any $y > 2$. Since (4.13) can be rewritten as

$$g(d_n, d_{n-1}) = 0$$

and $d_2 > 2$, then the sequence $\{d_n > 2\}_{n=2}^{\infty}$ is well defined.

Let $a(= 2.08679\ldots)$ be the largest positive root of

$$g(x,x) = 0. \tag{4.15}$$

Lemma 20 . *The limit*

$$\lim_{n\to\infty} d_n = a,$$

exists, where a is the largest positive root of (4.15).

Proof. Let us show that $d_n > a$ for all $n \geq 2$. Indeed, $d_2 > a$. Assume $d_{n-1} > a$ for some $n \geq 3$. Then

$$\frac{5}{4} + \frac{1}{d_n^4} - \left(d_n - \frac{3}{2}\right)^2 = \frac{2}{d_{n-1}} < \frac{2}{a} = \frac{5}{4} + \frac{1}{a^4} - \left(a - \frac{3}{2}\right)^2.$$

Since $f(x) = \frac{5}{4} + \frac{1}{x^4} - \left(x - \frac{3}{2}\right)^2$ is a decreasing function on $[\frac{3}{2}, \infty)$, then the condition $f(d_n) < f(a)$ implies $d_n > a$.

Further, since $d_n > a$, we have $\dfrac{2}{d_{n-1}} = \dfrac{5}{4} + \dfrac{1}{d_n^4} - \left(d_n - \dfrac{3}{2}\right)^2 < \dfrac{2}{d_n}$. Then $\{d_n\}_{n=2}^{\infty}$ is a decreasing sequence. Moreover, it is bounded below by a. Hence, there exists $\lim_{n\to\infty} d_n =: b$ and, in view of (4.13), $b = a$. □

Lemma 21 . *We have*

$$d(P_4) \le a,$$

where a is the largest positive root of (4.15).

Proof. Consider

$$\phi(z) = 1 + z + \sum_{k=2}^{\infty} \frac{z^k}{d_2^{k-1} d_3^{k-2} \dots d_k},$$

where the sequence $\{d_k\}_{k=2}^{\infty}$ is defined by (4.13) and (4.14). Then

$$I_2 = a_2^2(d_2^2 - 3d_2 + 1) > 0,$$

$$I_3 = a_3^2\left(d_3^2 - 3d_3 + \frac{2}{d_2} + 1\right) = a_3^2\left(\left(d_3 - \frac{3}{2}\right)^2 - \left(\frac{5}{4} - \frac{2}{d_2}\right)\right) = \frac{a_3^2}{d_3^4} > 0.$$

Since $\{d_n\}_{n=2}^{\infty}$ is a decreasing sequence, then

$$\left(d_n - \frac{3}{2}\right)^2 = \frac{5}{4} - \frac{2}{d_{n-1}} + \frac{1}{d_n^4} > \frac{5}{4} - \frac{2}{d_{n-1}} + \frac{1}{d_{n-2} d_{n-1}^2 d_n}$$

for $n \ge 4$. So, $I_n > 0$ for all $n \ge 4$. Therefore, by Lemma 19, $\phi(z) \in P_4$.

It follows from Lemmas 20 and 3 that $d(\phi) \ge a$. On the other hand, by Cauchy's inequality, we have

$$\log M(r,\phi) \ge n \log r - \sum_{j=2}^{n-1} (n - j + 1) \log d_j, \qquad n \ge 3.$$

By Lemma 20, for any given $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that $d_j \le a + \varepsilon$ for all $n > n_0$. Therefore,

$$\begin{aligned} \log M(r,\phi) &= n \log r - \sum_{j=2}^{n_0} (n - j + 1) \log d_j - \sum_{j=n_0}^{n-1} (n - j + 1) \log d_j \\ &\ge n \log r - \frac{n^2}{2} \log(a + \varepsilon) + O(n), \qquad n \to \infty. \end{aligned}$$

Letting n to be equal to the integer part of $\log r/\log(a+\varepsilon)$, we get

$$\log M(r,\phi) \geq \frac{\log^2 r}{2\log(a+\varepsilon)} + O(\log r), \qquad r \to \infty.$$

Thus, $d(\phi) \leq a+\varepsilon$ for all $\varepsilon > 0$. Hence, $d(\phi) = a$ and, therefore, $d(P_4) \leq a$. □

5

Power series having tails with m-times positive coefficients

Recall that series $\sum_{k=0}^{\infty} a_k z^k$ belongs to the class R_m, $m \geq 1$, if the sequences $\{a_k\}_{k=n}^{\infty}$ are m-times positive for all sufficiently large n. In [28] it was shown that the series (1.1) from R_m, $m \geq 3$, either represents an entire function satisfying

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log c}, \qquad c = \frac{1+\sqrt{5}}{2} < 2. \tag{2.4}$$

or has exactly one singularity (a simple pole) in the whole complex plane and satisfies a much more restrictive condition at infinity than (2.4). Here we show that, for entire functions belonging to R_m, $m \geq 4$, the estimate (2.8) holds.

Theorem 2 *If $m \geq 4$, then entire function $f \in R_m$ satisfies the condition*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log d(R_4)}, \tag{2.8}$$

where constant $d(R_4) > 2$ is independent of f and

$$2.016 \leq d(R_4) \leq 2.087.$$

5.1 The lower bound for $d(R_4)$

Assuming $f \in R_m$, $m \geq 4$, we choose n_0 such that $\{a_n, a_{n+1}, \ldots\}$ are m–times positive for all $n \geq n_0$. The next analogue of Lemma 14 plays a basic role in the proof of Theorem 2. Without loss of generality we may assume that $a_k > 0$ for all $k \geq n_0$. Thus, the numbers ρ_k, $k \geq n_0 + 1$, and δ_k, $k \geq n_0 + 2$, defined by (3.14) and (3.16) respectively, are well defined.

Lemma 22 . *Let $\{a_k\}_{k=n}^{\infty}$, $a_n > 0$, $n \geq 1$, be a 4–times positive sequence. Then*

$$\left(\delta_{n+1} - \frac{3}{2}\right)^2 \geq \frac{5}{4} - \frac{2}{\delta_{n+2}} + \frac{1}{\delta_{n+1}\delta_{n+2}^2\delta_{n+3}}. \tag{5.1}$$

Proof. Using (3.16), we have

$$I_n = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} & a_{n+3} \\ a_{n-1} & a_n & a_{n+1} & a_{n+2} \\ 0 & a_{n-1} & a_n & a_{n+1} \\ 0 & 0 & a_{n-1} & a_n \end{vmatrix}$$

$$= a_n^2 a_{n-1} a_{n+1} \left(\frac{a_n^2}{a_{n-1}a_{n+1}} + 2\frac{a_{n-1}a_{n+2}}{a_n a_{n+1}} - 3 - \frac{a_{n-1}^2 a_{n+3}}{a_{n+1}a_n^2} + \frac{a_{n-1}a_{n+1}}{a_n^2} \right)$$

$$= a_n^2 a_{n-1} a_{n+1} \left(\delta_{n+1} + \frac{2}{\delta_{n+1}\delta_{n+2}} - 3 - \frac{1}{\delta_{n+1}^2\delta_{n+2}^2\delta_{n+3}} + \frac{1}{\delta_{n+1}} \right).$$

4–times positivity of $\{a_k\}_{k=n}^{\infty}$ implies $I_n \geq 0$, from which the inequality (5.1) follows. □

Lemma 23 . *Let an entire function (1.1) belong to R_4. Then $\delta_k > 2$ for all $k \geq n_0 + 2$.*

Proof. By Lemma 9, if a formal power series (1.1) belongs to R_3, i.e. there exists such n_0 that the sequences $\{a_k\}_{k=n}^{\infty}$ are 3–times positive for all $n \geq n_0$, then only in the case if all $\delta_k \geq c$, $k \geq n_0 + 2$, the series $f(z)$ is an entire function. Now, the entire function $f(z)$ belongs to R_4. Hence, $\delta_n \geq c$, $n \geq n_0 + 2$. Assume that there exists $n' \geq n_0 + 2$ such that $\delta_{n'} < 2$. Then, by (5.1),

$$\frac{1}{4} > \left(\delta_{n'} - \frac{3}{2}\right)^2 > \frac{5}{4} - \frac{2}{\delta_{n'+1}},$$

whence $\delta_{n'+1} < 2$. Applying the same procedure successively, we get $\delta_n < 2$ for all $n \geq n' + 1$. Since

$$\frac{5}{4} - \frac{2}{x} \geq \left(x - \frac{3}{2}\right)^2,$$

for $c \leq x \leq 2$, then for $n \geq n' + 1$ we have

$$\frac{5}{4} - \frac{2}{\delta_n} \geq \left(\delta_n - \frac{3}{2}\right)^2 > \frac{5}{4} - \frac{2}{\delta_{n+1}},$$

whence $\delta_{n+1} < \delta_n$. Therefore, there exists

$$\lim_{n\to\infty} \delta_n =: b \in [c, 2]. \tag{5.2}$$

On the other hand, this limit, by (5.1), satisfies the condition

$$\left(b - \frac{3}{2}\right)^2 \geq \frac{5}{4} - \frac{2}{b} + \frac{1}{b^4} > \frac{5}{4} - \frac{2}{b}$$

and, hence, $b \notin [c; 2]$ in contradiction with (5.2). □

Hence, by Lemma 23, for any function $f \in R_4$,

$$\delta(f) \geq 2,$$

where $\delta(f)$ is defined by (4.8). Set

$$\delta(R_4) = \inf\{\delta(f) : \quad f \in R_4\}, \quad d(R_4) = \inf\{d(f) : f \in R_4\}, \tag{5.3}$$

where $d(f)$ is defined by (4.11).

Lemma 24 . *We have*

$$\delta(R_4) \geq 2 + \varepsilon,$$

where $\varepsilon = \dfrac{1}{66}$. *Therefore,* $d(R_4) \geq 2 + \varepsilon$.

The proof repeats the proof of Lemma 17 with evident modifications.

It follows from Lemma 3 that $d(f) \geq \delta(f)$ and hence $d(R_4) \geq \delta(R_4)$. Then, by Lemma 24, $d(R_4) \geq 2 + \frac{1}{66} = 2.016\ldots$.

5.2 The upper bound for $d(R_4)$

Lemma 25 . *The entire function*

$$f_q(z) = \sum_{k=0}^{\infty} q^{-\frac{(k-1)k}{2}} z^k$$

belongs to R_4 for all $q > a$, where a is the biggest positive root of (4.15).

Proof. Consider the functions

$$f_q(z) = \sum_{k=0}^{\infty} q^{-\frac{k(k-1)}{2}} z^k.$$

We have,

$$t_n(z, f_q) = z^n \sum_{k=0}^{\infty} q^{-\frac{(k+n)(k+n-1)}{2}} z^k$$

$$= (q^{1/2}z)^n \sum_{k=0}^{\infty} q^{-\frac{k(k-1)}{2}} (zq^{-n})^k = (q^{1/2}z)^n f_q(q^{-n}z).$$

Thus, if for some q, the sequence $\left\{q^{-\frac{k(k-1)}{2}}\right\}_{k=0}^{\infty}$ is 4-times positive, then all the sequences $\left\{q^{-\frac{k(k-1)}{2}}\right\}_{k=n}^{\infty}$, $n \geq 1$, are also 4-times positive. Therefore, to show that $f_q(z) \in R_4$ for all $q > a$, it is enough to show that the sequences $\left\{q^{-\frac{k(k-1)}{2}}\right\}_{k=0}^{\infty}$ are 4-times positive for all $q > a$.

To do this we will use Lemma 10. Construct

$$B_1 := (b_0, b_1, b_2, b_3, \ldots)$$

$$B_2 = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \dots \\ 0 & b_0 & b_1 & b_2 & \dots \end{pmatrix}$$

$$B_3 = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \dots \\ 0 & b_0 & b_1 & b_2 & \dots \\ 0 & 0 & b_0 & b_1 & \end{pmatrix}$$

$$B_4 = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ 0 & b_0 & b_1 & b_2 & b_3 & \dots \\ 0 & 0 & b_0 & b_1 & b_2 & \dots \\ 0 & 0 & 0 & b_0 & b_1 & \dots \end{pmatrix}$$

Positivity of all 1×1-block-minors of B_1 is trivial. All 2×2-block-minors of B_2 are positive for all $q > 1$ and hence they are positive for $q > a > 2$. All 3×3-block-minors of B_3 are of one of the kind:

$$A_1 = \begin{vmatrix} 1 & 1 & q^{-1} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$A_2 = \begin{vmatrix} 1 & q^{-1} & q^{-3} \\ 1 & 1 & q^{-1} \\ 0 & 1 & 1 \end{vmatrix} = q^{-3}(q-1)(q^2-q-1),$$

$$A_n = \begin{vmatrix} q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} & q^{-\frac{(n+2)(n+1)}{2}} \\ q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} \\ q^{-\frac{(n-2)(n-3)}{2}} & q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} \end{vmatrix}$$

$$= q^{-\frac{n(n-1)+(n-1)(n-2)+(n-2)(n-3)}{2}} \begin{vmatrix} 1 & q^{-n} & q^{-2n-1} \\ 1 & q^{-n+1} & q^{-2n+1} \\ 1 & q^{-n+2} & q^{-2n+3} \end{vmatrix}$$

$$= q^{-\frac{n(n-1)+(n-1)(n-2)+(n-2)(n-3)}{2}-n-2n-1} \begin{vmatrix} 1 & 1 & 1 \\ 1 & q & q^2 \\ 1 & q^2 & q^4 \end{vmatrix}, \quad n \geq 3.$$

Thus, for all $n \geq 3$, $A_n > 0$, if $q > 1$ and $A_2 > 0$, if $q > \frac{1+\sqrt{5}}{2}$. Hence, if $q > \frac{1+\sqrt{5}}{2}$, then all 3×3 -block-minors of B_3 are positive.

All 4×4-block -minors of B_4 are one of the kind

$$C_1 = \begin{vmatrix} 1 & 1 & q^{-1} & q^{-3} \\ 0 & 1 & 1 & q^{-1} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$C_2 = \begin{vmatrix} 1 & q^{-1} & q^{-3} & q^{-6} \\ 1 & 1 & q^{-1} & q^{-3} \\ 0 & 1 & 1 & q^{-1} \\ 0 & 0 & 1 & 1 \end{vmatrix} = q^{-6}(q^6 - 3q^5 + q^4 + 2q^3 - 1),$$

$$C_n = \begin{vmatrix} q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} & q^{-\frac{(n+2)(n+1)}{2}} & q^{-\frac{(n+3)(n+2)}{2}} \\ q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} & q^{-\frac{(n+2)(n+1)}{2}} \\ q^{-\frac{(n-2)(n-3)}{2}} & q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} & q^{-\frac{(n+1)n}{2}} \\ q^{-\frac{(n-3)(n-4)}{2}} & q^{-\frac{(n-2)(n-3)}{2}} & q^{-\frac{(n-1)(n-2)}{2}} & q^{-\frac{n(n-1)}{2}} \end{vmatrix}$$

$$= q^{-\frac{n(n-1)+(n-1)(n-2)+(n-2)(n-3)+(n-3)(n-4)}{2}} \begin{vmatrix} 1 & q^{-n} & q^{-2n-1} & q^{-3n-3} \\ 1 & q^{-n+1} & q^{-2n+1} & q^{-3n} \\ 1 & q^{-n+2} & q^{-2n+3} & q^{-3n+3} \\ 1 & q^{-n+3} & q^{-2n+5} & q^{-3n+6} \end{vmatrix}$$

$$=: \lambda_n \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & q & q^2 & q^3 \\ 1 & q^2 & q^4 & q^6 \\ 1 & q^3 & q^6 & q^9 \end{vmatrix}, \quad n \geq 4,$$

where

$$\lambda_n = q^{-\frac{n(n-1)+(n-1)(n-2)+(n-2)(n-3)+(n-3)(n-4)}{2}-n-2n-1-3n-3}$$

and

$$C_3 = \begin{vmatrix} q^{-1} & q^{-3} & q^{-6} & q^{-10} \\ 1 & q^{-1} & q^{-3} & q^{-6} \\ 1 & 1 & q^{-1} & q^{-3} \\ 0 & 1 & 1 & q^{-1} \end{vmatrix} = q^{-2} \begin{vmatrix} 1 & q^{-2} & q^{-5} & q^{-9} \\ 1 & q^{-1} & q^{-3} & q^{-6} \\ 1 & 1 & q^{-1} & q^{-3}| \\ 0 & q & q & 1 \end{vmatrix}$$

$$= q^{-18} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & q & q^2 & q^3 \\ 1 & q^2 & q^4 & q^6 \\ 0 & q^3 & q^6 & q^9 \end{vmatrix} = \frac{q^{-18}}{\lambda_4} I_4 + q^{-18} \begin{vmatrix} 1 & 1 & 1 \\ q & q^2 & q^3 \\ q^2 & q^4 & q^6 \end{vmatrix}.$$

We have, $C_2 > 0$, if $q > a$. For all $n \geq 3$, $C_n > 0$, if $q > 1$. Thus all 4×4-block-minors of B_4 are positive if $q > a$.

By Lemma 10, the sequences $\left\{q^{-\frac{k(k-1)}{2}}\right\}_{k=0}^{\infty}$ are 4-times positive for all $q > a$.

□

Using Cauchy's inequality

$$M(r, f_q) \geq q^{-\frac{k(k-1)}{2}} r^k$$

with k equal to the integer part of $\log r / \log q$, we get

$$\log M(r, f_q) \geq \frac{\log^2 r}{2 \log q} + O(\log r), \quad r \to \infty.$$

Thus $d(f_q) \leq q$ for all $q > a$. On the other hand, by Lemma 3, $d(f_q) \geq q$ for all $q > a$. It implies $d(f_q) = q$ and hence, by Lemma 25, $d(R_4) \leq a$.

6

Laurent series with m-times positive coefficients

In this chapter we study Laurent series (2.9) belonging to $\tilde{Q}_r$, $r \geq 3$, i.e. such Laurent series $\sum_{k=-\infty}^{\infty} a_k z^k$ that all the truncated sequences $\{a_k\}_{k=-n}^{n}$ are r-times positive. The subclasses $Q_r \subset \tilde{Q}_r$ consisting of all one-sided sequences (with $a_n = 0$ for $n < 0$) were considered in [26] and [27]. In the present work we reduce the problem of characterization of the class $\tilde{Q}_3$ to that of Q_3 by proving the following theorem.

Theorem 3. *A Laurent series (2.9) belongs to the class $\tilde{Q}_3$ if and only if both power series $f_1(z) = \sum_{k=0}^{\infty} a_{k-1} z^k$ and $f_2(z) = \sum_{k=0}^{\infty} a_{1-k} z^k$ belong to the class Q_3.*

Combining Theorems 3 and I, we deduce a characterization of the class $\tilde{Q}_3$ in terms of independent parameters.

Theorem 4. *A Laurent series (2.9) belongs to $\tilde{Q}_3$ if and only if*

$$a_0 = a_{-1}\alpha,$$

$$a_{n-1} = \frac{a_{-1}\alpha^n \alpha_2^{n-1}\alpha_3^{n-2}\dots\alpha_{n-1}^2\alpha_n}{[\alpha_2]^{n/2}[\alpha_2\alpha_3]^{(n-1)/2}\dots[\alpha_2\alpha_3\dots\alpha_{n-1}]^{3/2}[\alpha_2\alpha_3\dots\alpha_n]}, \quad n \geq 2, \quad (2.15)$$

$$a_{-n+1} = \frac{a_{-1}\alpha\beta^{n-1}\beta_2^{n-1}\beta_3^{n-2}\dots\beta_{n-1}^2\beta_n}{[\beta_2]^{n/2}[\beta_2\beta_3]^{(n-1)/2}\dots[\beta_2\beta_3\dots\beta_{n-1}]^{3/2}[\beta_2\beta_3\dots\beta_n]}, \quad n \geq 2, \quad (2.16)$$

where

$$\alpha > 0, \quad \{\alpha_k\}_{k=2}^{\infty} \in U, \quad \beta = \frac{1+\alpha_2}{\alpha\alpha_2}, \quad \beta_2 = \alpha_2, \quad \{\beta_{k+1}\}_{k=2}^{\infty} \in U,$$

and U is defined by (2.12).

Theorem 4 provides a rich source of functions from $\tilde{PF}_3$. The important point to note here is that Theorem 4 allows to construct also functions from $\tilde{PF}_3 \backslash \tilde{PF}_4$.

Corollary 3. *Let U be defined by (2.12). For any* α_2, $\frac{\sqrt{5}-1}{2} < \alpha_2 \leq 1$, *there exist such* α_3, β_3, $0 < \alpha_3, \beta_3 \leq 1$, *that for all* $\{\beta_{k+2}\}_{k=2}^{\infty} \in U$ *and* $\{\alpha_{k+2}\}_{k=2}^{\infty} \in U$, *the sequence* $\{a_n\}_{n=-\infty}^{\infty}$ *defined by (2.15) and (2.16) belongs to* $\tilde{PF}_3 \backslash \tilde{PF}_4$.

Theorems 3 and D allow us to derive the following result on growth estimates of functions from $\tilde{Q}_3$.

Theorem 5. *Let a Laurent series (2.9) belong to* $\tilde{Q}_r$ *for some* $r \geq 3$. *Then it converges in* $\mathbb{C}\backslash\{0\}$ *and*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq \frac{1}{2\log\frac{1+\sqrt{5}}{2}}, \quad (2.17)$$

$$\limsup_{r\to 0} \frac{\log M(r,f)}{\left(\log\frac{1}{r}\right)^2} \leq \frac{1}{2\log\frac{1+\sqrt{5}}{2}}. \quad (2.18)$$

These estimates cannot be improved for $f \in \tilde{Q}_3$.

6.1 Connection between two-sided and one-sided multiply positive sequences

Lemma 26 . *Let a Laurent series (2.9) belong to $\tilde{Q}_3$. Then both $f_1(z) = \sum_{k=0}^{\infty} a_{k-1}z^k$ and $f_2(z) = \sum_{k=0}^{\infty} a_{1-k}z^k$ belong to Q_3.*

Proof. (i) Let us show that $f_1(z)$ belongs to Q_3. We shall use I.J. Schoenberg's test of m–times positivity (see Lemma 4).

Fix any n, $1 < n < k_1$, and consider the three matrices

$$A_1 = (a_{-1} \quad a_0 \quad a_1 \quad \dots \quad a_n - \varepsilon);$$

$$A_2 = \begin{pmatrix} a_{-1} & a_0 & a_1 & \dots & a_n - \varepsilon & 0 \\ 0 & a_{-1} & a_0 & \dots & a_{n-1} & a_n - \varepsilon \end{pmatrix};$$

$$A_3 = \begin{pmatrix} a_{-1} & a_0 & a_1 & \dots & a_n - \varepsilon & 0 & 0 \\ 0 & a_{-1} & a_0 & \dots & a_{n-1} & a_n - \varepsilon & 0 \\ 0 & 0 & a_{-1} & \dots & a_{n-2} & a_{n-1} & a_n - \varepsilon \end{pmatrix};$$

All minors of A_1 are positive for $0 < \varepsilon < a_n$. For all $m \in \mathbb{N} \cap \{m \leq n\}$,

$$\begin{vmatrix} a_{-m+1} & a_m & 0 \\ a_{-m} & a_{m-1} & a_m \\ 0 & a_{m-2} & a_{m-1} \end{vmatrix} = a_{-m+1}a_{m-2}a_m \left(\delta_m - 1 - \frac{a_{m-1}a_{-m}}{a_{m-2}a_{-m+1}}\right) \geq 0,$$

whence $\delta_m > 1$, $m \in \mathbb{N} \cap \{m \leq n\}$. Therefore, all 2×2 minors of A_2

$$\begin{vmatrix} a_k & a_{k+1} \\ a_{k-1} & a_k \end{vmatrix} = a_{k-1}a_{k+1}(\delta_{k+1} - 1), \quad 0 \leq k \leq n-1,$$

$$\begin{vmatrix} a_{-1} & a_0 \\ 0 & a_{-1} \end{vmatrix}, \quad \begin{vmatrix} a_n - \varepsilon & 0 \\ a_{n-1} & a_n - \varepsilon \end{vmatrix},$$

consisting of consecutive columns are strictly positive.

Consider all 3×3 minors of A_3, consisting of consecutive columns:

$$M_{-1} = \begin{vmatrix} a_{-1} & a_0 & a_1 \\ 0 & a_{-1} & a_0 \\ 0 & 0 & a_{-1} \end{vmatrix} > 0, \quad M_0 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_{-1} & a_0 & a_1 \\ 0 & a_{-1} & a_0 \end{vmatrix},$$

$$M_k = \begin{vmatrix} a_k & a_{k+1} & a_{k+2} \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix}, \quad 1 \leq k \leq n-1,$$

$$M_{n-2} = \begin{vmatrix} a_k & a_{k+1} & a_{k+2} - \varepsilon \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix},$$

$$M_{n-1}(\varepsilon) = \begin{vmatrix} a_{n-1} & a_n - \varepsilon & 0 \\ a_{n-2} & a_{n-1} & a_n - \varepsilon \\ a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix},$$

$$M_n(\varepsilon) = \begin{vmatrix} a_n - \varepsilon & 0 & 0 \\ a_{n-1} & a_n - \varepsilon & 0 \\ a_{n-2} & a_{n-1} & a_n - \varepsilon \end{vmatrix} > 0.$$

Since $\{\ldots, 0, 0, a_{-k-1}, a_{-k}, \ldots, a_k, a_{k+1}, \ldots\}$ are 3-times positive, and $\delta_k > 1$ for all $1 \leq k \leq n-2$, then

$$M_0 = \begin{vmatrix} a_0 & a_1 & 0 \\ a_{-1} & a_0 & a_1 \\ 0 & a_{-1} & a_0 \end{vmatrix} + a_2 a_{-1}^2 > 0$$

and

$$M_k \geq \begin{vmatrix} a_k & a_{k+1} & a_{k+2} \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix}$$

$$= \begin{vmatrix} a_k & a_{k+1} & 0 \\ a_{k-2} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix} + a_{k+2}a_{k-2}a_k(\delta_k - 1) > 0,$$

$$M_k \geq \begin{vmatrix} a_k & a_{k+1} & a_{k+2} - \varepsilon \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix}$$

$$= \begin{vmatrix} a_k & a_{k+1} & 0 \\ a_{k-2} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix} + (a_{k+2} - \varepsilon)a_{k-2}a_k(\delta_k - 1) > 0$$

for all sufficiently small ε.

We have

$$M_{n-1}(\varepsilon) = a_{n-1}^3 + (a_n - \varepsilon)^2 a_{n-3} - 2(a_n - \varepsilon)a_{n-1}a_{n-2},$$

$$M_{n-1}(0) \geq 0, \quad M'_{n-1}(0) = 2a_n a_{n-3}(\delta_{n-1}\delta_n - 1) > 0.$$

Hence, $M_{n-1}(\varepsilon) > 0$ for all sufficiently small $\varepsilon > 0$. So, all 3×3 minors of A_3 consisting of consecutive columns are strictly positive. By Schoenberg's theorem, $\{\ldots 0, 0, a_{-1}, \ldots, a_{n-1}, a_n - \varepsilon, 0, 0, \ldots\}$ is a 3-times positive sequence for all sufficiently small ε. Taking a limit as ε tends to 0, we have that $\{\ldots 0, 0, a_{-1}, \ldots, a_{n-1}, a_n, 0, 0, \ldots\}$ is a 3-times positive sequence for all $n \in \mathbb{N}$. To prove that $f_1(z) \in Q_3$, it suffices to prove that $\{\ldots, 0, 0, a_{-1}, a_0, 0, 0, \ldots\}$ and $\{\ldots, 0, 0, a_{-1}, a_0, a_1, 0, 0, \ldots\}$ are 3-times positive sequences. But the first sequence is even totally positive. And the second one is 3-times positive, that follows from $f \in \tilde{Q}_3$.

(ii) Note that $f \in \tilde{Q}_3$ implies $f(1/z) \in \tilde{Q}_3$. Then, by part (i), $f_2(z) \in Q_3$. □

Lemma 27 . *Let Laurent series $f_1(z) = \sum_{k=0}^{\infty} a_{k-1}z^k$ and $f_2(z) = \sum_{k=0}^{\infty} a_{1-k}z^k$ both belong to Q_3. Then the Laurent series (2.9) belongs to $\tilde{Q}_3$.*

Proof. Fix any $n > 1$. Let k_1 and k_2 be as in Lemma 11. Denote

$$n_1 = \begin{cases} n, & if \quad k_1 = \infty, \\ \min\{n, k_1 - 1\}, & if \quad k_1 < \infty. \end{cases}$$

$$n_2 = \begin{cases} -n, & if \quad k_2 = -\infty, \\ \max\{-n, k_2 + 1\}, & if \quad k_2 > -\infty. \end{cases}$$

Let us prove that

$$\{\ldots, 0, 0, a_{n_2} - \varepsilon, a_{n_2-1}, \ldots, a_{n_1-1}, a_{n_1} - \varepsilon, 0, 0, \ldots\}$$

is a 3-times positive sequence for any sufficiently small ε. Consider the three matrices

$$A_1 = (a_{n_2} - \varepsilon \quad a_{n_2-1} \quad \ldots \quad a_{n_1-1} \quad a_{n_1} - \varepsilon)$$

$$A_2 = \begin{pmatrix} a_{n_2} - \varepsilon & a_{n_2} - 1 & \ldots & a_{n_1} - \varepsilon & 0 \\ 0 & a_{n_2} - \varepsilon & \ldots & a_{n_1-1} & a_{n_1} - \varepsilon \end{pmatrix}$$

$$A_3 = \begin{pmatrix} a_{n_2} - \varepsilon & a_{n_2-1} & a_{n_2-2} & \ldots & a_{n_1} - \varepsilon & 0 & 0 \\ 0 & a_{n_2} - \varepsilon & a_{n_2-1} & \ldots & a_{n_1-1} & a_{n_1} - \varepsilon & 0 \\ 0 & 0 & a_{n_2} - \varepsilon & \ldots & a_{n_1-2} & a_{n_1-1} & a_{n_1} - \varepsilon \end{pmatrix}$$

All minors of A_1 are strictly positive for $0 < \varepsilon < \min\{a_{n_1}, a_{n_2}\}$. For $1 \le m \le n_1$,

$$\begin{vmatrix} a_0 & a_m & 0 \\ a_{-1} & a_{m-1} & a_m \\ 0 & a_{m-2} & a_{m-1} \end{vmatrix} = a_0 a_{m-2} a_m \left(\delta_m - 1 - \frac{a_{-1} a_{m-1}}{a_0 a_{m-2}} \right) \ge 0.$$

So, $\delta_m > 1$ for all m, $1 \le m \le n_1$. Similarly, $\delta_m > 1$ for all m, $n_2 \le m \le -1$. Therefore, all 2×2 minors of A_2

$$\begin{vmatrix} a_{n_2} - \varepsilon & a_{n_2+1} \\ 0 & a_{n_2} - \varepsilon \end{vmatrix}, \quad \begin{vmatrix} a_{k-1} & a_k \\ a_{k-2} & a_{k-1} \end{vmatrix} = a_k a_{k-2}(\delta_k - 1), \quad 1 \le k \le n_1,$$

$$\begin{vmatrix} a_{k-1} & a_k \\ a_{k-2} & a_{k-1} \end{vmatrix} = a_k a_{k-2}(\delta_{k-2} - 1), \quad n_2 + 2 \le k \le 1,$$

$$\begin{vmatrix} a_{n_1} - \varepsilon & 0 \\ a_{n_1-1} & a_{n_1} - \varepsilon \end{vmatrix}$$

consisting of consecutive columns are strictly positive. Consider all 3×3 minors of A_3 consisting of consecutive columns,

$$M_{n_2+1}(\varepsilon) = \begin{vmatrix} a_{n_2+1} & a_{n_2+2} & a_{n_2+3} \\ a_{n_2} - \varepsilon & a_{n_2} + 1 & a_{n_2+2} \\ 0 & a_{n_2} - \varepsilon & a_{n_2+1} \end{vmatrix},$$

$$M_{n_1+1}(\varepsilon) = \begin{vmatrix} a_{n_1-1} & a_{n_1} - \varepsilon & 0 \\ a_{n_1-2} & a_{n_1-1} & a_{n_1} - \varepsilon \\ a_{n_1-3} & a_{n_1-2} & a_{n_1-1} \end{vmatrix},$$

$$M_k = \begin{vmatrix} a_k & a_{k+1} & a_{k+2} \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix}, \quad n_2 + 2 \le k \le n_1 - 2.$$

Since $f_1(z) \in Q_3$, for $k \geq 1$ we have

$$M_k > \begin{vmatrix} a_k & a_{k+1} & 0 \\ a_{k-1} & a_k & a_{k+1} \\ a_{k-2} & a_{k-1} & a_k \end{vmatrix} \geq 0.$$

Since $f_2(z) \in Q_3$, for $k \leq -1$ we have

$$M_k > \begin{vmatrix} a_k & a_{k+1} & a_{k+2} \\ a_{k-1} & a_k & a_{k+1} \\ 0 & a_{k-1} & a_k \end{vmatrix} \geq 0.$$

Also

$$M_0 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_{-1} & a_0 & a_1 \\ a_{-2} & a_{-1} & a_0 \end{vmatrix} > \begin{vmatrix} a_0 & a_1 & a_2 \\ a_{-1} & a_0 & a_1 \\ 0 & a_{-1} & a_0 \end{vmatrix} \geq 0,$$

by $f_1(z) \in Q_3$. That $M_{n_1-1}(\varepsilon)$ and $M_{n_2+1}(\varepsilon)$ are strictly positive for all sufficiently small $\varepsilon > 0$ one can prove by the same method as in Lemma 26. So, all 3×3 minors of A_3 consisting of consecutive columns are strictly positive. By Schoenberg's theorem,

$$\{\ldots,0,0,a_{n_2}-\varepsilon,a_{n_2-1},\ldots,a_{n_1-1},a_{n_1}-\varepsilon,0,0,\ldots\}$$

is a 3-times positive sequence for any sufficiently small $\varepsilon > 0$. Taking a limit as ε tends to 0, we have

$$\{\ldots 0,0,a_{n_2},a_{n_2-1},\ldots,a_{n_1-1},a_{n_1},0,0,\ldots\}$$

is a 3-times positive sequence. □

6.2 Parametric representation of sequences from $\tilde{Q}_3$. Growth estimates

Proof of Theorem 4 .

(i) Let $f \in \tilde{Q}_3$. Then by Lemma 26, $\sum_{k=0}^{\infty} a_{k-1}z^k \in Q_3$, and Theorem I gives the representation (2.15). Also, by Lemma 26, $\sum_{k=0}^{\infty} a_{-k+1}z^k \in Q_3$ and Theorem I gives

$$a_0 = a_1\beta,$$
$$a_{-n+1} = \frac{a_1\beta^n\beta_2^{n-1}\beta_3^{n-2}\ldots\beta_{n-1}^2\beta_n}{[\beta_2]^{n/2}[\beta_2\beta_3]^{(n-1)/2}\ldots[\beta_2\beta_3\ldots\beta_{n-1}]^{3/2}[\beta_2\beta_3\ldots\beta_n]}, \quad n\geq 2. \qquad (6.1)$$

To prove (2.16) it is left to prove

$$\beta = \frac{1+\alpha_2}{\alpha\alpha_2}, \quad \beta_2 = \alpha_2.$$

The formula (2.15) for the coefficient a_1 implies $\dfrac{a_{-1}}{a_1} = \dfrac{1+\alpha_2}{\alpha_2\alpha^2}$. Formulas (2.15) and (6.1) for the coefficient a_0 give $a_{-1}\alpha = a_1\beta$, i.e $\beta = \dfrac{a_{-1}}{a_1}\alpha = \dfrac{1+\alpha_2}{\alpha_2\alpha}$. It follows from $\delta_1 = \delta_{-1}$ that $\alpha_2 = \beta_2$.

(ii) Consider a Laurent series

$$g_1(z) = \frac{a_{-1}}{z} + a_{-1}\alpha + \sum_{n=2}^{\infty} a_{n-1}z^{n-1},$$

where $a_{-1} > 0$, $\alpha \geq 0$ and coefficients a_{n-1}, $n \geq 2$ are given by (2.15). By Theorem I, the sequence $\{a_k\}_{k=-1}^{\infty}$ belongs to Q_3.

Consider a Laurent series

$$g_2(z) = a_1 z^{-1} + a_1\beta + \sum_{n=2}^{\infty} a_{-n+1} z^{n-1},$$

where

$$a_1 = \frac{a_{-1}\alpha^2\alpha_2}{1+\alpha_2}, \quad \beta = \frac{1+\alpha_2}{\alpha\alpha_2}, \quad \beta_2 = \alpha_2$$

and coefficients a_{-n+1}, $n \geq 2$, are given by (2.16). We have $a_1\beta = a_{-1}\alpha$ and hence,

$$a_0 = a_{-1}\alpha = a_1\beta,$$

$$a_{-n+1} = \frac{a_1\beta^n\beta_2^{n-1}\beta_3^{n-2}\ldots\beta_{n-1}^2\beta_n}{[\beta_2]^{n/2}[\beta_2\beta_3]^{(n-1)/2}\ldots[\beta_2\beta_3\ldots\beta_{n-1}]^{3/2}[\beta_2\beta_3\ldots\beta_n]}, \quad n \geq 2.$$

By Theorem I, the sequence $\{a_{-k}\}_{k=-1}^{\infty}$ belongs to Q_3. Therefore, $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \in \tilde{Q}_3$, by Theorem 3. □

Proof of Theorem 5 .

Consider

$$f_1(z) = \sum_{k=0}^{\infty} a_{k-1} z^k, \quad f_2(z) = \sum_{k=0}^{\infty} a_{1-k} z^k$$

By Lemma 26, both $f_1(z)$ and $f_2(z)$ belong to Q_3. Applying Theorem D to $f_1(z)$ and $f_2(z)$, we have

$$\limsup_{r\to\infty} \frac{\log M(r, f_i)}{(\log r)^2} \leq \frac{1}{2\log c}, \quad c = \frac{1+\sqrt{5}}{2}, \quad i = 1, 2.$$

Hence, the Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k = \frac{1}{z} f_1(z) + z f_2\left(\frac{1}{z}\right) - \frac{a_{-1}}{z} - a_0 - a_1 z$$

converges in $\mathbb{C}\backslash\{0\}$ and the estimates (2.17) and (2.18) hold. □

Proof of Corollary 3.

For a given $\alpha > 0$ and $\frac{\sqrt{5}-1}{2} < \alpha_2 \leq 1$, define the coefficients a_0, a_1, a_{-1}, a_2, a_{-2} by formulas (2.15) and (2.16). Taking appropriate α_n and β_n, $n \geq 3$, in the formulas (2.15) and (2.16), we will define a_n, $|n| \geq 3$, such that $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \in \tilde{PF}_3 \backslash \tilde{PF}_4$.

Since, by (2.15) and (2.16),

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{vmatrix} \leq \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ 0 & a_{-2} & a_{-1} & a_0 \end{vmatrix} =$$

$$a_{-1}^4 \begin{vmatrix} \alpha & \frac{\alpha^2\alpha_2}{[\alpha_2]} & \frac{\alpha^3\alpha_2^2\alpha_3}{[\alpha_2]^{3/2}[\alpha_2\alpha_3]} & 0 \\ 1 & \alpha & \frac{\alpha^2\alpha_2}{[\alpha_2]} & \frac{\alpha^3\alpha_2^2\alpha_3}{[\alpha_2]^{3/2}[\alpha_2\alpha_3]} \\ \frac{\alpha\beta^2\beta_2^2\beta_3}{[\beta_2]^{3/2}[\beta_2\beta_3]} & 1 & \alpha & \frac{\alpha^2\alpha_2}{[\alpha_2]} \\ 0 & \frac{\alpha\beta^2\beta_2^2\beta_3}{[\beta_2]^{3/2}[\beta_2\beta_3]} & 1 & \alpha \end{vmatrix} =$$

$$a_{-1}^4 \left(\alpha^4 - 3\frac{\alpha^4\alpha_2}{[\alpha_2]} + \frac{\alpha^4\alpha_2^2}{[\alpha_2]^2} + \frac{\alpha^8\beta^4\alpha_2^8\alpha_3^2\beta_3^2}{[\alpha_2]^6[\alpha_2\alpha_3]^2[\beta_2\beta_3]^2} + 2\frac{\alpha^6\beta^2\alpha_2^4\beta_3}{[\alpha_2]^{\frac{7}{2}}[\beta_2\beta_3]} \right.$$
$$\left. +2\frac{\alpha^4\alpha_2^2\alpha_3}{[\alpha_2]^{\frac{3}{2}}[\alpha_2\alpha_3]} - 2\frac{\alpha^6\beta^2\alpha_2^4\beta_3\alpha_3}{[\alpha_2]^3[\alpha_2\alpha_3][\beta_2\beta_3]} - 2\frac{\alpha^6\beta^2\alpha_2^5\alpha_3\beta_3}{[\alpha_2]^4[\alpha_2\alpha_3][\beta_2\beta_3]} \right) =: a_{-1}^4\alpha^4 J(\alpha_3,\beta_3),$$

the inequality $J(\alpha_3,\beta_3) < 0$ yields $f(z) \notin \tilde{PF}_4$.

Substituting $\beta = \frac{1+\alpha_2}{\alpha_2\alpha}$ into $J(\alpha_3,\beta_3)$, we have

$$J(\alpha_3,\beta_3) = 1 - 3\frac{\alpha_2}{[\alpha]} + \frac{\alpha_2^2}{[\alpha]^2} + \frac{\alpha_2^4\alpha_3^2\beta_3^2}{[\alpha_2]^2[\alpha_2\alpha_3]^2[\beta_2\beta_3]^2}$$
$$+2\frac{\alpha_2^2\beta_3}{[\alpha_2]^{\frac{3}{2}}[\beta_2\beta_3]} + 2\frac{\alpha_2^2\alpha_3}{[\alpha_2]^{\frac{3}{2}}[\alpha_2\alpha_3]} - 2\frac{\alpha_2^2\beta_3\alpha_3}{[\alpha_2][\alpha_2\alpha_3][\beta_2\beta_3]}$$

$$-2\frac{\alpha_2^3\alpha_3\beta_3}{[\alpha_2]^2[\alpha_2\alpha_3][\beta_2\beta_3]} =: J(0,0) + \alpha_3 J_1 + \beta_3 J_2.$$

In all fractions entering in the last equations numerators do not exceed 1 while denominators are greater than 1. Therefore, $|J_1| + |J_2| \leq 9$. The quantity $\alpha_3 J_1 + \beta_3 J_2$ may be made arbitrarily small by choosing sufficiently small $\alpha_3 > 0$ and $\beta_3 > 0$. Therefore, the signs of $J(\alpha_3, \beta_3)$ and $J(0,0)$ coincide for all sufficiently small $\alpha_3 > 0$ and $\beta_3 > 0$. Note that

$$J(0,0) = 1 - 3\frac{\alpha_2}{[\alpha_2]} + \frac{\alpha_2^2}{[\alpha_2]^2} = -\frac{\alpha_2^2 + \alpha_2 - 1}{[\alpha_2]^2} < 0,$$

since $\alpha_2 > \frac{-1+\sqrt{5}}{2}$. □

7

Zeros of sections and tails of Mittag-Leffler function

In [12], A. Edrei, E.B. Saff and R.S. Varga studied the zero distribution of sections $s_n(R_n z, E_{1/\rho})$ of the Mittag–Leffler functions of order $\rho > 1$:

$$E_{1/\rho}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+\frac{j}{\rho})}, \qquad 1 < \rho < \infty. \tag{2.22}$$

In this chapter we study the zero distribution of the linear combination

$$I_n(R_n z, \lambda, E_{1/\rho}) = (1-\lambda) s_n(R_n z, E_{1/\rho}) - \lambda t_{n+1}(R_n z, E_{1/\rho}), \qquad \lambda \in \mathbb{C}, \tag{2.19}$$

of sections and tails of the Mittag-Leffler function. Our main result concerning the zero distribution of $I_n(R_n z, \lambda, E_{1/\rho})$ can be considered as a complete analogue of Szegö's Theorem J.

Theorem 6. *One has:*

(i) $\mathcal{M}_0(0, E_{1/\rho}) = S_1(\rho)$,

(ii) $\mathcal{M}_0(1, E_{1/\rho}) = S_2(\rho)$,

(iii) $\mathcal{M}_0(\lambda, E_{1/\rho}) = S(\rho)$, *for* $\lambda \neq 0, 1$.

This theorem means that each point on the curve $S_1(\rho)$ is an accumulation point of the zeros of $s_n(R_n z, E_{1/\rho})$, each point on the curve $S_2(\rho)$ is an accumulation point of the zeros of $t_{n+1}(R_n z, E_{1/\rho})$, and each point on the curve $S(\rho) = S_1(\rho) \cup S_2(\rho)$ is an accumulation point of the zeros of $I_n(R_n z, \lambda, E_{1/\rho})$, $\lambda \neq 0, 1$.

The next theorem deals with the zero–free regions of $I_n(R_n z; \lambda, E_{1/\rho})$. Let us remind their definition. For given $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ and $h > 0$, region Ω_i, $i = 1, 2, \ldots, 5$, are defined as follows.

$$\begin{aligned}
\Omega_1 &= \{z = re^{i\phi} : \delta_3 \leq r \leq 1, |z-1| \geq \delta_1, |\phi| \leq \tfrac{\pi}{2\rho} - \delta_2 \\
&\qquad\qquad r^\rho \cos(\rho\phi) - 1 - \rho \log r \geq 0\}, \\
\Omega_2 &= \{z = re^{i\phi} : |\phi| \leq \tfrac{\pi}{2\rho} - \delta_2, r^\rho \cos(\rho\phi) - 1 - \rho \log r \leq -h\}, \\
\Omega_3 &= \{z = re^{i\phi} : e^{-1/\rho} + h \leq r, |\phi| \geq \tfrac{\pi}{2\rho} + \delta_2\}, \\
\Omega_4 &= \{z = re^{i\phi} : \delta_3 \leq r \leq e^{-1/\rho} - h, |\phi| \geq \tfrac{\pi}{2\rho} + \delta_2\}, \\
\Omega_5 &= \{z = re^{i\phi} : r \geq 1, |\phi| \leq \tfrac{\pi}{2\rho} - \delta_2, r^\rho \cos(\rho\phi) - 1 - \rho \log r \geq h\}.
\end{aligned} \tag{2.27}$$

Theorem 7. *Let δ_1, δ_2, δ_3 and h be given sufficiently small positive constants. Then, for all sufficiently large n, $I_n(R_n z; \lambda, E_{1/\rho})$ has no zeros in $\cup_{i=1}^5 \Omega_i$.*

Theorem 7 implies that the *zeros* of $I_n(R_n z; \lambda, E_{1/\rho})$ may lie *only* in the vicinity of the curve $S(\rho)$ and two rays $\arg z = \pm\pi/(2\rho)$. Therefore, it is natural to consider the function $I_n(R_n z, \lambda, E_{1/\rho})$ in the neighborhood of the points on the curve $S(\rho)$. The next theorem give information about the behavior (and, in particular, the zero distribution) of $I_n(R_n z, \lambda, E_{1/\rho})$ in the neighborhood of the point $z = 1$ on $S(\rho)$.

Theorem 8. *As $n \to \infty$, we have*

$$\left(1 + \left(\frac{2}{\rho n}\right)^{1/2} \zeta\right)^{-n} \left\{E_{1/\rho}(R_n)\right\}^{-1} I_n\left(R_n\left(1 + \left(\frac{2}{\rho n}\right)^{1/2} \zeta\right); \lambda, E_{\frac{1}{\rho}}\right)$$

$$\to e^{\zeta^2}\left\{\frac{erfc(\zeta)}{2}-\lambda\right\}$$

uniformly on every compact set of the ζ-plane.

Our next theorem is concerned with the behavior of $I_n(R_n z, \lambda, E_{1/\rho})$ near the points $\xi = \xi(\phi) \in S(\rho)$ distinct from the point $z = 1$.

Theorem 9. *I. Let $\xi = \xi(\phi)$, $0 < \phi < \frac{\pi}{2\rho}$, be a fixed point on the generalized Szegö curve $S(\rho)$. Let $\tau = |\zeta|^\lambda \sin(\phi\rho) - \rho\phi$, and let the sequences $\{\tau_n\}_{n=1}^\infty$ and $\{\varepsilon_n(\zeta)\}_{n=1}^\infty$ be defined by the conditions*

$$\tau_n \equiv \frac{\tau}{\rho} n (\operatorname{mod} 2\pi), \qquad -\pi < \tau_n \leq \pi,$$

and

$$\varepsilon_n(\zeta) = \frac{\log n}{2(1-\xi^\rho)n} - \frac{\zeta - i\tau_n}{(1-\xi^\rho)n}.$$

Then, as $n \to \infty$,

$$I_n\left(R_n\xi\left(1+\varepsilon_n(\zeta)\right);\lambda, E_{1/\rho}\right)\frac{\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n\xi^n\left(1+\varepsilon_n(\zeta)\right)^n}$$

$$\to \begin{cases} \alpha(\xi)e^\zeta - \dfrac{\xi}{1-\xi}, & if \quad |\xi| < 1, \\ -\beta(\xi)e^\zeta - \dfrac{\xi}{1-\xi}, & if \quad |\xi| > 1, \end{cases} \tag{2.28}$$

uniformly on every compact set of the ζ - plane, where

$$\alpha(\xi) = (1-\lambda)(2\pi\rho)^{\frac{1}{2}} e^{\frac{\rho+1}{2\rho}(\xi^\rho - 1)}$$

and

$$\beta(\xi) = \lambda(2\pi\rho)^{\frac{1}{2}} e^{\frac{\rho+1}{2\rho}(\xi^\rho - 1)}.$$

II. Let $\xi = e^{-\frac{1}{\rho}}e^{i\phi}$, $\frac{\pi}{2\rho} < \phi \leq \pi$, be a fixed point on the circular portion of $S(\rho)$, and let the sequences $\{\tau_n'\}_{n=1}^\infty$ and $\{\varepsilon_n'(\zeta)\}_{n=1}^\infty$ be defined by the conditions

$$\tau_n' \equiv (n+1)\phi(\ mod \quad 2\pi), \quad -\pi < \tau_n' \leq \pi,$$

and

$$\varepsilon_n'(\zeta) = \left(\frac{1}{2} - \frac{1}{\rho}\right)\frac{\log n}{n} - \frac{\zeta - i\tau_n'}{n+1}.$$

Then

$$I_n\left(R_n\xi\left(1+\varepsilon_n'(\zeta)\right);\lambda, E_{1/\rho}\right)\frac{\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n\xi^n\left(1+\varepsilon_n'\right)^n} \to \gamma(\xi)e^{-\zeta} - \frac{\xi}{1-\xi} \tag{2.29}$$

uniformly on every compact set of the ζ – plane, where

$$\gamma(\xi) = \frac{(\lambda-1)(2\pi e^{\frac{1-\rho}{\rho}})^{\frac{1}{2}}}{\rho^{\frac{1}{2}-\frac{1}{\rho}}\Gamma\left(1-\frac{1}{\rho}\right)}.$$

As it was mentioned before, by Theorem 7, the zeros of $I_n(R_nz;\lambda,E_{1/\rho})$ *may* lie only in the vicinity of the curve $S(\rho)$ and two rays $\arg z = \pm\pi/(2\rho)$. Theorem 9 implies that each point ξ on the curve $S(\rho)$ is an accumulation point of the zeros of $I_n(R_nz;\lambda,E_{1/\rho})$. Indeed, by Hurwitz's Theorem, the zeros of $I_n(R_n\xi(1+\varepsilon_n(\zeta));\lambda,E_{1/\rho})$ approach the zeros of the limit functions in (2.28), if $\xi \in S'(\rho)\cup S_2(\rho)$, and the zeros of $I_n(R_n\xi(1+\varepsilon_n'(\zeta));\lambda,E_{1/\rho})$ approach the zeros of the limit functions in (2.29), if $\xi \in S''(\rho)$.

The proof of Theorems 7 and 9 is based on the following theorem that deals with the asymptotic expressions for $I_n(R_nz;\lambda,E_{1/\rho})$ in different domains of $\mathbb{C}$.

Theorem 10. *Let δ_1, δ_2, δ_3 be given sufficiently small positive constants, and $\rho>1$. Then, as $n\to\infty$,*

$$\frac{I_n(R_nz;\lambda,E_{1/\rho})\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^nz^n} = -\lambda\rho\frac{e^{R_n^\rho z^\rho}\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^nz^n}(1+o(1))$$
$$-\frac{z}{1-z}(1+o(1)), \tag{2.30}$$

if $z\in\{z=re^{i\phi}: r\geq 1, |\phi|\leq\frac{\pi}{2\rho}, |z-1|\geq\delta_1\}$,

$$\frac{I_n(R_nz;\lambda,E_{1/\rho})\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^nz^n} = (1-\lambda)\rho\frac{e^{R_n^\rho z^\rho}\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^nz^n}(1+o(1))$$

$$-\frac{z}{1-z}(1+o(1)), \tag{2.31}$$

if $z \in \{z = re^{i\phi} : \delta_3 \le r \le 1, |\phi| \le \frac{\pi}{2\rho}, |z-1| \ge \delta_1\}$,

$$\frac{I_n(R_n z; \lambda, E_{1/\rho})\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n z^n} = \frac{(\lambda-1)}{\Gamma\left(1-\frac{1}{\rho}\right)} \frac{\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^{n+1} z^{n+1}}(1+o(1))$$

$$-\frac{z}{1-z}(1+o(1)), \tag{2.32}$$

if $z \in \{z = re^{i\phi} : r \ge \delta_3, |\phi| \ge \frac{\pi}{2\rho} + \delta_2, \}$.

In all expressions above, $o(1)$ *is uniform with respect to* z.

7.1 Asymptotic expressions for $I_n(R_n z; \lambda, E_{1/\rho})$

Set

$$K_n(z) := \int\limits_{L\left(\frac{\pi}{2\rho}+\frac{\delta_2}{2}, R_n\right)} \frac{e^{\zeta^\rho}\zeta^{-(n+1)}}{\zeta - R_n z} d\zeta. \tag{7.1}$$

To prove Theorem 10, by (3.24) - (3.26) and (7.1), it is enough to find asymptotic expression for the integral $K_n(z)$. We will do this in the following three steps:

1) change the contour of integration of $K_n(z)$;

2) show that the main contribution to $K_n(z)$ comes from the neighborhood of the point $\zeta = R_n$;

3) find asymptotic expression for $K_n(z)$ by using Laplace's Method for contour integrals.

To introduce new contours of integration we define the following curves. Denote by $T(\rho)$ a simple curve which is represented in polar coordinates as the set of all points

$$r^\rho = \frac{\rho\phi}{\sin \rho\phi}, \qquad -\frac{\pi}{\rho} < \phi < \frac{\pi}{\rho}.$$

Define

$$S(\rho, h) := \left\{ z = re^{i\phi} : r^\rho \cos(\rho\phi) - \rho \log r - 1 = -\frac{h}{2} \right\}.$$

By the same reason as in Lemma 13, $S(\rho, h)$ has two asymptotes $\arg z = \pm\dfrac{\pi}{2\rho}$. The curves $S(\rho, h)$ and $T(\rho)$ have two common points, $z_1 = de^{i\gamma}$ and $z_2 = de^{-i\gamma}$, say. Note that constants d and γ depend only on h, and as $h \to 0$ we see that $\gamma \sim \dfrac{\sqrt{h}}{\rho}$ and $d^\rho \sim \dfrac{\sqrt{h}}{\sin\sqrt{h}}$. Denote by $TS_1(\rho, h)$ the curve which consists of

(i) $T(\rho) \cap \{z : |z| \leq d\}$, where $d = d(h)$ is defined above;

(ii) $S(\rho, h) \cap \left\{z : |z| \geq d, |\arg z| \leq \frac{\pi}{2\rho} - \delta_2\right\}$;

(iii) $\{z : |z| = D_1, \frac{\pi}{2\rho} - \delta_2 \leq |\arg z| \leq \frac{\pi}{2\rho} + \frac{\delta_2}{2}\}$, where D_1 is such that $D_1^\rho \sin(\rho\delta_2) - \rho \log D_1 - 1 = -\dfrac{h}{2}$;

(iv) $\{z : \arg z = \pm\left(\frac{\pi}{2\rho} + \frac{\delta_2}{2}\right), \quad |z| \geq D_1\}$, where D_1 was defined in part (iii).

Denote by $TS_2(\rho, h)$ the curve which consists of

(i) defined above;

(ii') $S(\rho, h) \cap \left\{z : |z| \leq d, |\arg z| \leq \frac{\pi}{2\rho} + \frac{\delta_2}{2}\right\}$, where d is the same as in (i);

(iii') $\{z : \arg z = \pm\left(\frac{\pi}{2\rho} + \frac{\delta_2}{2}\right), |z| \geq D_2\}$, where D_2 is such that $D_2^\rho \sin(\rho\frac{\delta_2}{2}) + \rho \log D_2 + 1 = \frac{h}{2}$.

Lemma 28 . *Let*

$$z \in \Omega := \mathbb{C} \backslash \left\{ \quad \left\{z : \frac{\pi}{2\rho} \leq |\arg z| \leq \frac{\pi}{2\rho} + \delta_2\right\} \cup \{z : |z - 1| \leq \delta_1\} \quad \right\}$$

Then, for sufficiently small $h > 0$,

$$K_n(z) = \frac{e^{n/\rho}}{R_n^{n+1}} \int_{(i)} \frac{\exp\left\{\dfrac{n}{\rho}(t^\rho - \rho \log t - 1) + \dfrac{\rho - 1}{2\rho} t^\rho\right\}}{(t - z)t} \left(1 + O\left(\frac{1}{n}\right) t^\rho\right) dt$$

$$+ O\left(\frac{\exp\{\frac{n}{\rho}(1 - \frac{h}{2})\}}{R_n^{n+1}}\right),$$

where $K_n(z)$ is defined by (7.1).

Proof. Let $z \in G^-(1, \frac{\pi}{2\rho} + \frac{\delta_2}{2}) \cap \Omega$. For sufficiently small $h > 0$, using (3.27), we have

$$K_n(z) = \frac{1}{R_n^{n+1}} \int_{L\left(1, \frac{\pi}{2\rho}+\frac{\delta_2}{2}\right)} \frac{e^{R_n^\rho t^\rho} t^{-(n+1)}}{t-z} dt = \frac{1}{R_n^{n+1}} \int_{TS_1(\rho,h)} \frac{e^{R_n^\rho t^\rho} t^{-n}}{(t-z)t} dt$$

$$= \frac{1}{R_n^{n+1}} \int_{TS_1(\rho,h)} \frac{e^{\frac{n}{\rho}t^\rho} t^{-n} e^{\frac{\rho-1}{2\rho}t^\rho}}{(t-z)t} \left(1 + O\left(\frac{1}{n}\right) t^\rho\right) dt.$$

Since there exists such constant $A_1 > 0$ that $|t-z| > A_1$ for all $z \in G^-(1, \frac{\pi}{2\rho} + \frac{\delta_2}{2}) \cap \Omega$ while $t \in TS_1(\rho, h)$, for any number l we have

$$\left| \int_{(ii)} \frac{e^{\frac{n}{\rho}t^\rho} t^{l-n} e^{\frac{\rho-1}{2\rho}t^\rho}}{t-z} dt \right| \le B_1 \int_{(ii)} \exp\left\{\frac{n}{\rho}|t|^\rho \cos(\rho\phi)\right\} |t|^{-n} d|t|$$

$$= O\left(\exp\left\{\frac{n}{\rho}(1-\frac{h}{2})\right\}\right);$$

$$\left| \int_{(iii)} \frac{e^{\frac{n}{\rho}t^\rho} t^{l-n} e^{\frac{\rho-1}{2\rho}t^\rho}}{t-z} dt \right| \le B_2 \int_{(iii)} \exp\left\{\frac{n}{\rho}D_1^\rho \sin(\rho\delta_2)\right\} D_1^{-n} d|t|$$

$$= O\left(\exp\left\{\frac{n}{\rho}(1-\frac{h}{2})\right\}\right);$$

$$\left| \int_{(iv)} \frac{e^{\frac{n}{\rho}t^\rho} t^{l-n} e^{\frac{\rho-1}{2\rho}t^\rho}}{t-z} dt \right| \le B_3 \int_{(iv)} \exp\left\{-\frac{n}{\rho}|t|^\rho \sin\left(\frac{\rho\delta_2}{2}\right)\right\} d|t| = o(1);$$

where B_1, B_2, B_3 are constants not depending on z. The last three inequalities prove the lemma if $z \in G^-(1, \frac{\pi}{2\rho} + \frac{\delta_2}{2}) \cap \Omega$. The same proof works for $z \in G^+(1, \frac{\pi}{2\rho} + \frac{\delta_2}{2}) \cap \Omega$ if we change the contour of integration in the integral $K_n(z)$ by $TS_2(\rho, h)$. It follows that Lemma 28 holds for $z \in \Omega$. □

Lemma 29 . *Let $|z-1| \ge \delta_1$ and l be a fixed real number. Then, for sufficiently small $h > 0$,*

$$\int_{(i)} \frac{\exp\left\{\frac{n}{\rho}(t^\rho - \rho \log t - 1) + \frac{\rho-1}{2\rho}t^\rho\right\} t^l}{t-z} dt = \frac{i\sqrt{2\pi} e^{\frac{\rho-1}{2\rho}} (1+o(1))}{\rho(1-z)(n/\rho)^{1/2}}.$$

Proof. Rewrite

$$\int_{(i)} \frac{\exp\left\{\frac{n}{\rho}(t^\rho - \rho\log t - 1) + \frac{\rho-1}{2\rho}t^\rho\right\}t^l}{t-z}dt = \int_{\mathcal{D}} e^{-\frac{n}{\rho}v} f(v)dv,$$

where $v = -t^\rho + \rho\log t + 1$, $f(v)dv = \frac{e^{\frac{\rho-1}{2\rho}t^\rho}t^l}{t-z}dt$, or equivalently, $f(v) = \frac{e^{\frac{\rho-1}{2\rho}t^\rho}t^{l+1}}{\rho(t-z)(1-t^\rho)}$, and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where $\mathcal{D}_1$ is the upper side of the segment $[0; d_1]$ following the direction of the decrease of v and $\mathcal{D}_2$ is the lower side of the segment $[0; d_1]$ following the direction of increase of v. Here $d_1 = \frac{h}{2} = -d^\rho\cos(\rho\gamma) + \rho\log d + 1$, where d and γ are the polar coordinates of a point of intersection of $T(\rho)$ and $S(\rho, h)$.

Note that the function $v = -t^\rho + \rho\log t + 1$ maps the region on the right side of $T(\rho)$ conformally onto the whole v–plane cut along the positive ray. In particular, the image of the curve (i) is the segment $[0; d_1]$ traced twice. The transformation $w = \sqrt{v}$ maps v–plane cut along the positive ray onto the upper half plane. We have $w^2 = -t^\rho + \rho\log t + 1 = -\frac{\rho^2}{2}(t-1)^2\psi(t)$, where $\psi(t)$ is an analytic function in some neighborhood of $t = 1$ and $\psi(1) = 1$. Then $w = \frac{\rho}{\sqrt{2}}i(t-1)\psi_1(t)$, where $\psi_1(t)$ is an analytic function in some neighborhood of $t = 1$ and $\psi_1(1) = 1$. Since w is analytic in a neighborhood of $t = 1$ and $w'(1) = \frac{\rho i}{\sqrt{2}} \neq 0$, then its inverse function $t(w)$ is analytic in a neighborhood of $w = 0$, and hence the following function

$$\begin{aligned} g(w) := wf(w^2) &= \frac{i(t-1)\psi_1(t)e^{\frac{\rho-1}{2\rho}t^\rho}t^{l+1}}{\sqrt{2}(t-z)(1-t^\rho)} \\ &= -\frac{i\psi_1(t)e^{\frac{\rho-1}{2\rho}t^\rho}t^{l+1}}{\sqrt{2}\rho(t-z)}(1+o(1)), \quad |t| \to 1, \end{aligned}$$

is analytic in some neighborhood of $w = 0$, say $|w| \le C$, where C is a constant not depending on z. Let $|w| < C/2$, then

$$g(w) = \frac{1}{2\pi i}\int_{|\zeta|=C} \frac{g(\zeta)}{\zeta - w}d\zeta = \frac{1}{2\pi i}\int_{|\zeta|=C} \frac{g(\zeta)}{\zeta}d\zeta + \frac{w}{2\pi i}\int_{|\zeta|=C} \frac{g(\zeta)}{\zeta(\zeta-w)}d\zeta$$

$$= g(0) + w\alpha(w) = -\frac{ie^{\frac{\rho-1}{2\rho}}}{\sqrt{2}\rho(1-z)} + w\alpha(w),$$

where $\alpha(w)$ is a function analytic in $|w| < C/2$, and

$$|\alpha(w)| \le \frac{2\pi C \max_{|\zeta|=C} |g(\zeta)|}{2\pi C^2/2} \le C_3,$$

where C_3 is a constant not depending on z and n. This implies that $f(v) = g(0)v^{-1/2} + \alpha(v^{1/2})$ in some neighborhood of $v = 0$ cut along the positive ray. Let h be so small that $d_1 < \frac{C^4}{16}$. Then

$$\int_{\mathcal{D}} e^{-\frac{n}{\rho}v} f(v)dv = g(0)\int_{\mathcal{D}} e^{-\frac{n}{\rho}v} v^{-1/2}dv + \frac{1}{2\pi i}\int_{\mathcal{D}} e^{-\frac{n}{\rho}v}\alpha(v^{1/2})dv$$

$$=: g(0)I_1 + \frac{1}{2\pi i}I_2.$$

Note that

$$I_2 = \int_{\mathcal{D}} e^{-\frac{n}{\rho}v}\alpha(v^{1/2})dv = O\left(\frac{1}{n}\right)$$

and

$$I_1 = \int_{\mathcal{D}} e^{-\frac{n}{\rho}v} v^{-1/2}dv = \frac{1}{(n/\rho)^{1/2}}\int_{\frac{n}{\rho}\mathcal{D}} e^{-u}u^{-1/2}du = -\frac{2\Gamma\left(\frac{1}{2}\right)(1+o(1))}{(n/\rho)^{1/2}}.$$

Thus,

$$\int_{\mathcal{D}} e^{-\frac{n}{\rho}v} f(v)dv = \frac{i\sqrt{2\pi}e^{\frac{\rho-1}{2\rho}}(1+o(1))}{\rho(1-z)\left(\frac{n}{\rho}\right)^{1/2}}. \qquad \square$$

Lemmas 28 and 29 imply that, as $n \to \infty$,

$$K_n(z) = \frac{e^{\frac{n}{\rho}}}{R_n^{n+1}}\frac{i\sqrt{2\pi}e^{\frac{\rho-1}{2\rho}}(1+o(1))}{\rho(1-z)(n/\rho)^{1/2}}, \quad z \in \Omega,$$

and hence, by (3.27), (7.1) and Stirling's formula for $\Gamma(x)$, $x > 0$,

$$\frac{\rho(R_n z)^{n+1}}{2\pi i}K_n(z) = \frac{R_n^n z^n}{\Gamma\left(1+\frac{n}{\rho}\right)}\frac{z}{1-z}(1+o(1)), \quad n \to \infty, \quad z \in \Omega. \tag{7.2}$$

Then Theorem 10 follows from (3.24)-(3.26), (7.1) and (7.2).

7.2 Zero-free regions of $I_n(R_n z; \lambda, E_{1/\rho})$

Proof of Theorem 7. Denote by

$$J_1' := \frac{e^{R_n^\rho z^\rho}\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n z^n}; \qquad J_2' := \frac{\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^{n+1} z^{n+1}}. \tag{7.3}$$

Lemma 30 . *Let $z \neq 0$, then for all sufficiently large n,*

$$|J_1'| \geq C_1 \left(\frac{n}{\rho}\right)^{1/2}, \quad \text{if} \quad z \in \Omega_1 \cup \Omega_5,$$

$$|J_1'| = o(1), \quad \text{if} \quad z \in \Omega_2,$$

$$|J_2'| = o(1), \quad \text{if} \quad z \in \Omega_3,$$

$$|J_2'| \geq (1 + (he^{1/\rho})/2)^n, \quad \text{if} \quad z \in \Omega_4,$$

where C_1 is some constant not depending on n and z.

Proof. Using Stirling's formula for $\Gamma(x)$, $x > 0$, and (3.27), we have

$$\begin{aligned} |J_1'| &= \frac{\exp\{\frac{n}{\rho}(1+\frac{\rho-1}{2\rho}+O(1/n^2))r^\rho\cos(\rho\phi)\}\,(n/\rho)^{n/\rho+1/2}\sqrt{2\pi}e^{-n/\rho}}{(n/\rho)^{n/\rho}\,e^{\frac{\rho-1}{2\rho}}\,r^n}(1+o(1)) \\ &= \sqrt{2\pi}(1+o(1))\exp\left\{\frac{\rho-1}{2\rho}((1+o(1))r^\rho\cos(\rho\phi)-1)\right\} \\ &\quad \times \exp\left\{\frac{n}{\rho}\{r^\rho\cos(\rho\phi)-1-\rho\log r\}\right\}\left(\frac{n}{\rho}\right)^{1/2}, \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} |J_2'| &= \frac{\left(\frac{n}{\rho}\right)^{n/\rho+1/2}\sqrt{2\pi}e^{-n/\rho}}{\left(\frac{n}{\rho}\right)^{n/\rho+1/\rho}e^{\frac{\rho-1}{2\rho}}r^{n+1}}(1+o(1)) \\ &= \frac{\sqrt{2\pi}e^{1/\rho}}{e^{\frac{\rho-1}{2\rho}}}(1+o(1))\left(\frac{e^{-1/\rho}}{r}\right)^{n+1}\left(\frac{n}{\rho}\right)^{1/2-1/\rho}. \end{aligned} \tag{7.5}$$

Equations (7.4) and (7.5) make Lemma 30 obvious. □

According to Theorem 10 and (7.3), we have

$$\frac{I_n(R_n z; \lambda, E_{1/\rho})\Gamma\left(1+\frac{n}{\rho}\right)}{R_n z^n} = -\lambda\rho(1+o(1))J_1' - (1+o(1))\frac{z}{1-z},$$

if $z \in \Omega_5 \cup \{\Omega_2 \cap \{z : |z| \geq 1\}\}$;

$$\frac{I_n(R_n z; \lambda, E_{1/\rho})\Gamma\left(1+\frac{n}{\rho}\right)}{R_n z^n} = (1-\lambda)\rho(1+o(1))J_1' - (1+o(1))\frac{z}{1-z},$$

if $z \in \Omega_1 \cup \{\Omega_2 \cap \{z : |z| \leq 1\}\}$;

$$\frac{I_n(R_n z; \lambda, E_{1/\rho})\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n z^n} = \frac{(\lambda-1)}{\Gamma\left(1-\frac{1}{\rho}\right)}J_2' - (1+o(1))\frac{z}{1-z},$$

if $z \in \Omega_3 \cup \Omega_4$.

The last three expressions and Lemma 30 completes the proof of Theorem 7.

7.3 The asymptotic behavior of $I_n(R_n z; \lambda, E_{1/\rho})$ in the neighborhood of the generalized Szegö curve

Proof of Theorem 8. We can rewrite (2.19) as

$$I_n(R_n z; \lambda, E_{1/\rho}) = s_n(R_n z, E_{\frac{1}{\rho}}) - \lambda E_{\frac{1}{\rho}}(R_n z). \tag{7.6}$$

Theorem 1 of [12] implies that

$$\frac{s_n\left(R_n\left(1+\left(\frac{2}{\rho n}\right)^{1/2}\zeta\right); E_{\frac{1}{\rho}}\right)}{\left(1+\left(\frac{2}{\rho n}\right)^{1/2}\zeta\right)^n E_{\frac{1}{\rho}}(R_n)} \to \frac{1}{2}e^{\zeta^2}\operatorname{erfc}(\zeta), \tag{7.7}$$

as $n \to \infty$, uniformly on every compact set of ζ – plane.

By (3.23), we have

$$\frac{E_{\frac{1}{\rho}}\left(R_n\left(1+\left(\frac{2}{\rho n}\right)^{1/2}\zeta\right)\right)}{\left(1+\left(\frac{2}{\rho n}\right)^{1/2}\zeta\right)^n E_{\frac{1}{\rho}}(R_n)} = \exp\left\{\left(\frac{n}{\rho}+\frac{\rho-1}{2\rho}\right)\left(1+\left(\frac{2}{\rho n}\right)^{1/2}\zeta\right)^\rho - \right.$$

$$\left. n\log\left(1+\left(\frac{2}{\rho n}\right)^{1/2}\zeta\right) - \left(\frac{n}{\rho}+\frac{\rho-1}{2\rho}\right) + o(1)\right\} = e^{\zeta^2} + o(1), \quad n \to \infty. \tag{7.8}$$

Theorem 8 follows from (7.6) – (7.8).

Proof of Theorem 9.

I. By Theorem 10 and (3.23), we have,

$$\frac{I_n\left(R_n\xi\left(1+\frac{\log n}{2(1-\xi^\rho)n}-\frac{\zeta-i\tau_n}{(1-\xi^\rho)n}\right);\lambda,E_{1/\rho}\right)\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n\xi^n\left(1+\frac{\log n}{2(1-\xi^\rho)n}-\frac{\zeta-i\tau_n}{(1-\xi^\rho)n}\right)^n}$$

$$=\mu\rho(1+o(1))\frac{\exp\{R_n^\rho\xi^\rho\left(1+\frac{\log n}{2(1-\xi^\rho)n}-\frac{\zeta-i\tau_n}{(1-\xi^\rho)n}\right)^\rho\}\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n\xi^n\left(1+\frac{\log n}{2(1-\xi^\rho)n}-\frac{\zeta-i\tau_n}{(1-\xi^\rho)n}\right)^n}$$

$$-\frac{\xi}{1-\xi}(1+o(1)) =: \mu\rho A_n(1+o(1))-\frac{\xi}{1-\xi}+o(1), \tag{7.9}$$

where $\mu=1-\lambda$, if $|\xi|\le 1$, and $\mu=-\lambda$, if $|\xi|>1$. Using Sitirling's formula and (3.27), it is easy to obtain

$$A_n=\frac{\exp\{R_n^\rho\xi^\rho\left(1+\frac{\log n}{2(1-\xi^\rho)n}-\frac{\zeta-i\tau_n}{(1-\xi^\rho)n}\right)^\rho\}\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n\xi^n\left(1+\frac{\log n}{2(1-\xi^\rho)n}-\frac{\zeta-i\tau_n}{(1-\xi^\rho)n}\right)^n}$$

$$=\exp\left\{\frac{n}{\rho}(|\xi|^\rho\cos(\rho\phi)-1-\rho\log|\xi|)+\frac{in}{\rho}(|\xi|^\rho\sin(\rho\phi)-\rho\phi)\right.$$

$$\left.+\zeta-i\tau_n+\frac{\rho+1}{2\rho}(\xi^\rho-1)-\frac{\log\rho}{2}+\frac{1}{2}\log 2\pi+O\left(\frac{\log^2 n}{n}\right)\right\}$$

By the definition of the numbers τ and τ_n and the fact that the point $\xi=|\xi|e^{i\phi}\in S'(\rho)\cup S_2(\rho)$, we get

$$A=\left(\frac{2\pi}{\rho}\right)^{1/2}e^{\frac{\rho+1}{2\rho}(\xi^\rho-1)}e^{\zeta}(1+o(1)),\quad n\to\infty. \tag{7.10}$$

Combining (7.9) with (7.10), we obtain (2.28).

II. By Theorem 10 and (3.23), we have,

$$\frac{I_n\left(R_n\xi\left(1+\left(\frac{1}{2}-\frac{1}{\rho}\right)\frac{\log n}{n}-\frac{\zeta-i\tau_n'}{n+1}\right);\lambda,E_{1/\rho}\right)\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^n\xi^n\left(1+\left(\frac{1}{2}-\frac{1}{\rho}\right)\frac{\log n}{n}-\frac{\zeta-i\tau_n'}{n+1}\right)^n}$$

$$=\frac{(\lambda-1)(1+o(1))}{\Gamma\left(1-\frac{1}{\rho}\right)}\frac{\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^{n+1}\xi^{n+1}\left(1+\frac{\log n}{2(1-\xi^\rho)n}-\frac{\zeta-i\tau_n'}{(1-\xi^\rho)n}\right)^{n+1}}$$

$$-\frac{\xi}{1-\xi}(1+o(1)) =: \frac{(\lambda-1)}{\Gamma\left(1-\frac{1}{\rho}\right)}B_n(1+o(1))-\frac{\xi}{1-\xi}+o(1). \tag{7.11}$$

Using Stirling's formula and (3.27), it is easy to obtain for $\xi = e^{-1/\rho}e^{i\phi}$,

$$B_n = \frac{\Gamma\left(1+\frac{n}{\rho}\right)}{R_n^{n+1}\xi^{n+1}\left(1+\frac{\log n}{2(1-\xi^\rho)n}-\frac{\zeta - i\tau_n'}{(1-\xi^\rho)n}\right)^{n+1}} = \exp\left\{\frac{1}{2}\log(2\pi) - \frac{\rho-1}{2\rho} + \right.$$

$$\left. \left(\frac{1}{\rho}-\frac{1}{2}\right)\log\rho - \zeta + i(\tau_n' - (n+1)\phi) + O\left(\frac{1}{n}\right)\right\}.$$

By the definition of numbers τ_n', we get

$$B_n = \frac{(2\pi e^{\frac{1-\rho}{\rho}})^{1/2}}{\rho^{\frac{1}{2}-\frac{1}{\rho}}} e^{-\zeta}(1+o(1)), \quad n\to\infty. \tag{7.12}$$

Combining (7.11) with (7.12), we obtain (2.29).

Bibliography

[1] Baker, G.A., and Graves–Morris, P., *Padé Approximations*, Addison–Wesley, London, 1981.

[2] Buckholtz, J.D., *A characterization of the exponential series*, Amer. Math. Monthly, 73 (1966), 121-123.

[3] Buckholtz, J.D., *Zeros of partial sums of power series*, II, Michigan Math. J., 17(1970), 5-14.

[4] Buckholtz, J.D., and Frank, J.L., *Whittaker constants*, Proc. Lond. Math. Soc. (3) 23 (1971), 348-370.

[5] Buckholtz, J.D., and Shaw, J.K., *Zeros of partial sums and remainders of power series*, Trans. Amer. Math. Soc., 166(1972), 269-284.

[6] Carlson, F., *Sur les fonctions entières*, Arkiv för mat., astr. och fys. 35A No. 14 (1948).

[7] Carpenter, A.J., Varga, R.S., and Waldvogel, J., *Asymptotics for the zeros of the partial sums of* e^z, Rocky Mountain Journal of Mathematics, Vol. 21, Number 1, 99-120, 1991.

[8] Dieudonné, J., *Sur les zéros des polynomes-sections de* e^x, Bull. Soc. Math. France, 70 (1935), 333-351.

[9] Djrbashian, M.M., *Integral transforms and representations of functions in the complex domain*(Russian) Nauka, Moskow, 1966.

[10] Dvoretzky, A., *On sections of power series*, Annals of Mathematics, Vol. 51, No. 3, May, 1950.

[11] Edrei, A., *On the generating function of a doubly infinite totally positive sequence*, Trans. Amer. Math. Soc., 74 (1953), 376–383.

[12] Edrei, A., Saff, E.B., Varga R.S., *Zeros of sections of power series*, Lecture Notes in Math., 1002 (1983), 1-115.

[13] Edrei, A., *Zeros of sections of power series representing entire functions admissible in the sense of Hayman*, Contemporary Math., 25 (1983), 19-35.

[14] Fekete, M., Polya, G. *Über ein Problem von Laguerre*, Rendiconti Circ. Math. Palermo, 34 (1912), 89-120.

[15] Ganelius, T., *Sequences of analytic functions and their zeros*, Arkiv för Matematik, Bd 3 nr 1, 1953

[16] Ganelius, T., *The zeros of the partial sums of power series*, Duke Math. J., 30 (1963), 533-540.

[17] Gol'dberg, A.A. and Ostrovskii, I.V., *Distribution of values of meromorphic functions* (Russian), Nauka, Moskow, 1970.

[18] Hurwitz, A., *Über die Nullstellen der Besselschen Funktionen*, Math. Ann. 33, pp.246-266 (1889).

[19] Jentzsch, R., *Untersuchhungen zur Theorie der Folgen analytischen Functionen*, Dissertation, University of Berlin, 1914.

[20] Karlin, S., *Total Positivity*, Stanford University Press, Stanford, 1968.

[21] Korevaar, J., *The zeros of approximating polynomials and the canonical representation of an entire function*, Duke Math. J., 18(1951), 573-592.

[22] Newman, D.J., and Rivlin, T.J., *The zeros of the partial sums of the exponential function*, J. Approx. Theory, 5 (1972), 405-412.

[23] Ostrovskii, I.V., *Les séries des puissances dont les restes ont seulement des zéros non positifs*, Comptes Rendus Acad. Sci., Paris, 325 (1997), 1257-1262.

[24] Ostrovskii, I.V., *On the zeros of tails of power series*, Operator Theory: Advances and Applications, Vol 113 (2000), 279-285.

[25] Ostrovskii, I.V., *On zero distribution of sections and tails of power series*, Israel Math. Conferences Proceedings, 15 (2001), 297-310.

[26] Ostrovskii, I.V., and Zheltukhina, N.A., *On power series having sections with multiply positive coefficients and a theorem of Pólya*, J. London Math. Soc. (2) 58 (1998), 97-110.

[27] Ostrovskii, I.V. and Zheltukhina, N.A., *Parametric representation of a class of multiply positive sequences*, Complex Variables, Vol 37 (1998), 457-469.

[28] Ostrovskii, I.V., and Zheltukhina, N.A., *On power series having tails with multiply positive coefficients*, in Complex Analysis and Difference Equations. (eds. C. Kiselman, A. Vretblad) Uppsala, Uppsala University, 1999, 263-280.

[29] Pólya, G., *Über Annäherung durch Polynome mit lauter reelle Wurzeln*, Rend. Circ. Math. Palermo, 36 (1913), 279-295.

[30] Pólya, G., and Szegö, G., *Problems and Theorems in Analysis, II*, Springer-Verlag, Berlin-Heidelberg, 1976.

[31] Pommiez, M., *Sur les restes successifs des séries de Taylor*, Ann. Fac. Sci. Univ. de Toulouse, (4) 24 (1960), 77-165.

[32] Rosenbloom, P.C., *On sequences of polynomials, especially sections of power series*, Thesis, Stanford University, 1944.

[33] Schoenberg, I.J., *On the zeros of generating functions of multiply positive sequences and functions*, Ann. Math., 62(1955), 447–471.

[34] Schoenberg, I.J., *Some analytic aspects of the problem of smoothing*, Courant Anniversary Volume, New York, 1948, 351–370.

[35] Szegö, G., *Über die Nullstellen von Polynomen, die in einem Kreise gleichmässig konvergieren* Berlin Math. Ges. Sitzungsber. 21 (1922), 59-64.

[36] Szegö, G., *Über eine Eigenschaft der Exponentialreihe*, Sitzungsberichte der Berliner Math. Gesellschaft, 23 (1924), 50-64.

[37] Yıldırım, C.Y., *On the tails of the exponential series*, Canad. Math. Bull. Vol.37(2), 1994, 276-286.

[38] Yıldırım, C.Y., *A sum over the zeros of partial sums of* e^x, J. Ramanujan Math. Soc. 6 Nos. 1 & 2 (1991), 51-66.

[39] Yıldırım, C.Y., *On the zeros of the sections of the exponential function*, Doga - Turk. J. Math. 16 (1992), 177-182.

[40] Zheltukhina, N.A., *On power series having sections with multiply positive coefficients*, Complex Variables (accepted, February, 2002).

[41] Zheltukhina, N.A., *Asymptotic distribution of zeros of sections and tails of Mittag-Leffler functions*, Comptes Rendus (accepted, May, 2002).

[42] Zheltukhina, N.A., *On Laurent series with multiply positive coefficients*, (submitted, June, 2001).