CHARACTERISTIC LIE ALGEBRA AND CLASSIFICATION OF SEMI-DISCRETE MODELS

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

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In this thesis, we studied a differential-difference equation of the following form

$$t_x(n+1,x) = f(t(n,x), t(n+1,x), t_x(n,x)), \tag{1}$$

where the unknown $t = t(n,x)$ is a function of two independent variables: discrete n and continuous x. The equation (1) is called a Darboux integrable equation if it admits nontrivial x- and n-integrals. A function $F(x,t,t_{\pm 1}, t_{\pm 2}, ...)$ is called an x-integral if $D_x F = 0$, where D_x is the operator of total differentiation with respect to x. A function $I(x,t,t_x,t_{xx},...)$ is called an n-integral if $DI = I$, where D is the shift operator: $Dh(n) = h(n+1)$.

In this work, we introduced the notion of characteristic Lie algebra for semi-discrete hyperbolic type equations. We used characteristic Lie algebra as a tool to classify Darboux integrability chains and finally gave the complete list of Darboux integrable equations in the case when the function f in the equation (1) is of the special form $f = t_x(n,x) + d(t(n,x), t(n+1,x))$.

Keywords: Darboux integrability; Characteristic Lie Algebra; First Integrals.

ÖZET

KARAKTERİSTİK LIE CEBİRİ VE YARI-AYRIK MODELLERİN SINIFLANDIRILMASI

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Bu tezde

$$t_x(n+1,x) = f(t(n,x), t(n+1,x), t_x(n,x)), \tag{1}$$

halindeki diferansiyel-fark denklemi üzerinde çalıştık. Burada $t = t(n,x)$ ayrık n ve sürekli x bağımsız değişkenlerinin bir fonksiyonudur. Denklem (1), eğer basit olmayan x- ve n-integrallerini kabul ediyorsa, Darboux integrallenebilir denklem olarak adlandırılır. $F(x, n, t, t_{\pm 1}, t_{\pm 2}, ...)$ fonksiyonu eğer $D_x F = 0$ koşulunu sağlıyorsa denklem (1)'in x-integrali olarak isimlendirilir. Burada D_x, x'e göre toplam türev operatörüdür. $I(x, n, t, t_x, t_{xx}, ...)$ fonksiyonu eğer $DI = I$ şartını sağlıyorsa denklem (1)'in n-integrali olarak adlandırılır. Burada D, $Dh(n) = h(n+1)$ şeklindeki denklem (1)'in kaydırma operatörüdür.

Bu çalışmada, yarı-ayrık hiperbolik tipindeki denklemler için karakteristik Lie cebir mefhumunu tanıttık. Karakteristik Lie cebirini Darboux integrallenebilir zincir denklemlerini sınıflandırmak için kullandık ve son olarak, (1) denklemindeki f fonksiyonunun, $f = t_x(n,x) + d(t(n,x), t(n+1,x))$ özel haline sahip olduğu durumdaki Darboux integrallenebilir denklemlerin tam listesini verdik.

Anahtar sözcükler: Darboux integrallanebilirliği; Karakteristik Lie Cebiri; Birinci İntegraller.

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Contents

Chapter 1

Introduction

In the literature, there are various definitions for integrability. Different approaches and methods are applied for classifying different types of integrable equations (see [1], [2]-[5], [6], [7], [8] and [9]).

Investigation of the class of hyperbolic type differential equations of the form

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{1.1}$$

has also a very long history. There are various approaches to seek for particular and general solutions of these kind equations. In the literature we can find several definitions of integrability of the equation (1.1). According to one given by G. Darboux (see [10], [11]), equation (1.1) is called integrable if it reduces to a pair of ordinary (generally nonlinear) differential equations or, more exactly if there exist functions $F(x, y, u, u_x, u_{xx}, ..., D_x^m u)$ and $G(x, y, u, u_y, u_{yy}, ..., D_y^n u)$ such that arbitrary solution of (1.1) satisfies $D_y F = 0$ and $D_x G = 0$, where D_x and D_y are operators of differentiation with respect to x and y. Functions F and G are called y- and x-integrals of the equation (1.1) respectively. The famous Liuoville equation $u_{xy} = e^u$ provides an illustrative example of the Darboux integrable hyperbolic type differential equation.

An effective criterion of Darboux integrability has been proposed by G. Darboux himself. Equation (1.1) is integrable if and only if the Laplace sequence of the linearized equation terminates at both ends. The definition of the Laplace sequence and the proof of the criterion can be found in [12], [13]. A complete list of the Darboux integrable equations of the form (1.1) is given in [14].

In the beginning of the 80's, A. B. Shabat and R. I. Yamilov developed an alternative method to the classification problem based on the notion of the characteristic Lie algebra of hyperbolic type systems in [15],[16]. In these articles, an algebraic criterion of Darboux integrability property has been formulated. An important classification result was obtained in [15] for the exponential system

$$u^i_{xy} = \exp(a_{i1}u^1 + a_{i2}u^2 + ...a_{in}u^n), \quad i = 1, 2, ...n. \tag{1.2}$$

It was proved that system (1.2) is Darboux integrable if and only if the matrix $A = (a_{ij})$ is the Cartan matrix of a semi-simple Lie algebra. Properties of the characteristic Lie algebras of the hyperbolic systems

$$u^i_{xy} = c^i_{jk}u^j u^k, \quad i, j, k = 1, 2, ...n \tag{1.3}$$

have been studied in [17],[18]. The idea of adopting the characteristic Lie algebras to the problem of classification of the hyperbolic type equations of the form $u_{xy} = f(u, u_x)$, which are integrated by means of the inverse scattering transforms method is discussed by A. V. Zhiber and R. D. Murtazina in [19].

The method of characteristic Lie algebras studied in this thesis is closely connected with the symmetry approach [6] which is proved to be very effective tool to classify integrable nonlinear equations of evolutionary type [8], [7], [20], [5] (see also the survey [9] and references therein). However this method meets very serious difficulties when applied to hyperbolic type models. After the papers [21] and [22] it became clear that this case needs alternative methods.

In 2005, I. Habibullin introduced the notion of characteristic Lie algebra for fully discrete hyperbolic equations in [23]. In our later works with I. Habibullin and Natalya Zheltukhina (see [24, 25, 26]), an algorithm of classification of integrable semi-discrete chains is studied based on the notion of characteristic Lie algebras of the semi-discrete chains of the form

$$t_x(n + 1, x) = f(t(n, x), t(n + 1, x), t_x(n, x)). \tag{1.4}$$

Efficiency of the algorithm is approved by applying to a particular case of chain (1.4):

$$t_x(n+1,x) = t_x(n,x) + d(t(n,x), t(n+1,x)). \tag{1.5}$$

This thesis is completely based on four articles of us which are [24, 25, 26] and [27] that is not published yet.

The thesis is organized as follows. In Chapter 2, we basically gave the notion of characteristic Lie algebra. In Section 2.1, we introduced characteristic Lie algebras for hyperbolic type differential equations having continuous variables. Section 2.2 is devoted to explain characteristic Lie algebras for semi-discrete hyperbolic type equations having independent variables: one continuous x and one discrete n. There is also a subsection here, which gives a special case: equation with characteristic Lie algebras of the minimal possible dimensions.

Semi-discrete hyperbolic type equations are Darboux integrable if and only if their characteristic Lie algebras in both direction n and x are of finite dimension or equivalently, they have both nontrivial n- and x-integrals. Hence in Chapter 3, we found the equations which are admitting nontrivial x-integrals. In Chapter 4, we have analyzed these equations one by one and checked whether they also admit nontrivial n-integrals or under what conditions they have nontrivial n-integrals. Finally, we gave the complete list of Darboux integrable equations of the form (1.5).

Chapter 2

Characteristic Lie Algebra

2.1 Characteristic Lie Algebras for Continuous Case

Almost all the materials in this Chapter comes from [25].

The integrability of hyperbolic type differential equations having continuous variables of the form

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{2.1}$$

has been discussed for so many years. According to G. Darboux's integrability definition, equation (2.1) is called integrable if it is reduced to a pair of ordinary (generally nonlinear) differential equations, or more exactly, if its any solution satisfies the equations of the form [10], (see also [11])

$$F(x, y, u, u_x, u_{xx}, ..., D_x^m u) = a(x), \quad G(x, y, u, u_y, u_{yy}, ..., D_y^n u) = b(y), \tag{2.2}$$

for appropriately chosen functions $a(x)$ and $b(y)$. Here D_x and D_y are operators of differentiation with respect to x and y, $u_x = D_x u$, $u_{xx} = D_x^2 u$, $u_y = D_y u$, $u_{yy} = D_y^2 u$ and so on. Functions F and G are called y- and x-integrals of the equation respectively. They are also called as "first integral"s of the equation (2.1).

Let us give a brief explanation of the notion of characteristic Lie algebra by using y- and x-integrals. We begin with the basic property of the first integrals. Clearly, each y-integral satisfies the condition

$$D_y F(x, y, u, u_x, u_{xx}, ..., D_x^m u) = 0.$$

We take the derivative by applying the chain rule and we define a vector field X_1 such that

$$X_1 F = \Big(\frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + f \frac{\partial}{\partial u_x} + D_x(f) \frac{\partial}{\partial u_{xx}} + ...\Big) F = 0. \qquad (2.3)$$

Hence the vector field X_1 solves the equation $X_1 F = 0$. Note that the function F does not depend on u_y. Hence F should satisfy one more equation $X_2 F = 0$, where

$$X_2 = \frac{\partial}{\partial u_y}.$$

The commutator of these two operators will also annulate F. Moreover, for any operator X from the Lie algebra generated by X_1 and X_2, we get $XF = 0$. This Lie algebra is called characteristic Lie algebra of the equation (2.1) in the direction of y. We can define characteristic Lie algebra in the x-direction in a similar way. Now by virtue of the famous Jacobi theorem, equation (2.1) is Darboux integrable if and only if both of its characteristic Lie algebras are of finite dimension. Equivalently, equation (2.1) is Darboux integrable if it has nontrivial x- and y-integrals. The best known examples of Darboux integrable equations are the wave equation $u_{xy} = 0$ with x-integral $G = u_y$ and y-integral $F = u_x$ and the Liouville equation $u_{xy} = e^u$ with x-integral $G = u_{yy} - \frac{u_y^2}{2}$ and y-integral $F = u_{xx} - \frac{u_x^2}{2}$. In [15] and [16], the characteristic Lie algebras for the systems of nonlinear hyperbolic equations and their applications are studied.

In the following section, we will define characteristic Lie algebras for the semi-discrete hyperbolic type equations.

2.2 Characteristic Lie Algebras for Semi-Discrete Case

Here we will study semi-discrete chains of the following form

$$t_x(n+1,x) = f(t(n,x), t(n+1,x), t_x(n,x)) \tag{2.4}$$

from the Darboux integrability point of view. The unknown $t = t(n,x)$ is a function depending on two independent variables: one discrete n and one continuous x. Chain (2.4) can also be interpreted as an infinite system of ordinary differential equations for the sequence of the variables $\{t(n)\}_{n=-\infty}^{\infty}$. Here $f = f(t(n,x), t(n+1,x), t_x(n,x))$ is assumed to be locally analytic function of three variables satisfying at least locally the condition

$$\frac{\partial f}{\partial t_x} \neq 0. \tag{2.5}$$

Subindex denotes a shift or a derivative, for instance, $t_k = t(n+k,x)$ and $t_x = \frac{\partial}{\partial x} t(n,x)$. Below we use D to denote the shift operator and D_x to denote the x-derivative: $Dh(n,x) = h(n+1,x)$ and $D_x h(n,x) = \frac{\partial}{\partial x} h(n,x)$. For the iterated shifts we use the subindex $D^j h = h_j$. Set of all the variables $\{t_k\}_{k=-\infty}^{\infty}$, $\{D_x^m t\}_{m=1}^{\infty}$ constitutes the set of dynamical variables. Below we consider the dynamical variables as independent ones.

Let us give the definition of Darboux integrability for semi-discrete hyperbolic type equations. Before that we should introduce the notions of the first integrals i.e. x- and n-integrals for the semi-discrete chain (2.4). The x-integral is defined similar to the continuous case. We call a function $F = F(x, t, t_{\pm1}, t_{\pm2}, ...)$ depending on a finite number of shifts x-integral of the chain (2.4), if $D_x F = 0$. We also define n-integral similarly. We call a function $I = I(x, t, t_x, t_{xx}, ...)$ n-integral of the chain (2.4) if it is in the kernel of the difference operator: $(D-1)I = 0$ i.e. n-integral should not change under the action of the shift operator $DI = I$, (see also [28]). Each solution of the integrable chain (2.4) satisfies following two equations:

$$I(x, n, t, t_x, t_{xx}, ...) = p(x), \quad F(x, n, t, t_{\pm1}, t_{\pm2}, ...) = q(n)$$

with properly chosen functions $p(x)$ and $q(n)$.

Definition. Chain (2.4) is called integrable (Darboux integrable) if it admits a nontrivial n-integral and a nontrivial x-integral.

Darboux integrability implies the so-called C-integrability(solvability via an appropriate change of variables). All Darboux integrable chains of the form (2.4) are reduced to the d'Alembert wave equation $w_{1x}-w_x=0$ by a Cole-Hopf type differential substitution $w=F+I$. Indeed, $(D-1)D_x(w)=(D-1)D_xF+D_x(D-1)I=0$.

Now let us turn back to x-integral $F=F(x,n,t,t_{\pm1},t_{\pm2},...)$ to introduce characteristic Lie algebra in the direction x. Since F satisfies $D_xF=0$, we can expand this equation by using the chain rule, and we get $K_0F=0$, where

$$K_0=\frac{\partial}{\partial x}+t_x\frac{\partial}{\partial t}+f\frac{\partial}{\partial t_1}+g\frac{\partial}{\partial t_{-1}}+f_1\frac{\partial}{\partial t_2}+g_{-1}\frac{\partial}{\partial t_{-2}}+\ldots. \tag{2.6}$$

Note that the function F does not depend on the variable t_x. Hence F should also satisfy $XF=0$ where

$$X=\frac{\partial}{\partial t_x}. \tag{2.7}$$

Vector fields K_0 and X as well as any vector field from the Lie algebra generated by them annulate F. This algebra is called the characteristic Lie algebra L_x of the chain (2.4) in the x-direction. The following result is essential, its proof can be found in [15].

Theorem 2.1 *Equation (2.4) admits a nontrivial x-integral if and only if its Lie algebra L_x is of finite dimension.*

Now we will examine the n-integral $I=I(x,n,t,t_x,t_{xx},...)$ to introduce characteristic Lie algebra in the direction n. By the definition we know that $DI=I$. We can write it in an enlarged form

$$I(x,n+1,t_1,f,f_x,f_{xx},...)=I(x,n,t,t_x,t_{xx},...). \tag{2.8}$$

Notice that equation (2.8) is a functional equation, the unknown is taken at two different "points". This causes the main difficulty in studying discrete chains.

Such problems occur when we try to apply the symmetry approach to discrete equations (see [29], [30]). However the notion of the characteristic Lie algebra provides an effective tool to investigate chains.

We introduce vector fields in the following way. We focus on the equation (2.8). The left hand side of the equation contains the variable t_1 while the right hand side does not. Hence, the total derivative of the function DI with respect to t_1 should vanish. In other words, the n-integral is in the kernel of the operator $Y_1 := D^{-1}\frac{\partial}{\partial t_1}D$. Similarly the function I is also in the kernel of the operator $Y_2 := D^{-2}\frac{\partial}{\partial t_1}D^2$. It is because the right hand side of the equation $D^2I = I$ which immediately follows from (2.8) does not depend on t_1, so the derivative of the function D^2I with respect to t_1 vanishes. If we proceed this way, we can easily prove that the operator $Y_j = D^{-j}\frac{\partial}{\partial t_1}D^j$ solves the equation $Y_jI = 0$ for any natural j. It is clear that we have the relation $Y_{j+1} = D^{-1}Y_jD$ for any natural j.

So far we have shifted the argument n forward, but we can also shift it backward and use the equation (2.8) written as $D^{-1}I = I$. We rewrite the original equation (2.4) in the form

$$t_{-1x} = g(t, t_{-1}, t_x). \tag{2.9}$$

We can do this because of the condition $\frac{\partial f}{\partial t_x} \neq 0$ assumed at the beginning of the section. We again enlarge the equation $D^{-1}I = I$ and get

$$I(x, n-1, t_{-1}, g, g_x, g_{xx}, ...) = I(x, n, t, t_x, t_{xx}, ...). \tag{2.10}$$

We use the similar approach as before. The left hand side of the last equation depends on t_{-1}, but the right hand side does not. Therefore the total derivative of $D^{-1}I$ with respect to t_{-1} is zero, i.e. the operator $Y_{-1} := D\frac{\partial}{\partial t_{-1}}D^{-1}$ solves the equation $Y_{-1}I = 0$. Moreover, the operators $Y_{-j} = D^j\frac{\partial}{\partial t_{-1}}D^{-j}$ also satisfy similar conditions $Y_{-j}I = 0$ for any natural number j.

If we summarize the reasonings above we can conclude that the n-integral I is annulated by any operator from the Lie algebra $\tilde{L}_n$ generated by the operators [23]

$$\{..., Y_{-2}, Y_{-1}, Y_{-0}, Y_0, Y_1, Y_2, ...\} \tag{2.11}$$

where $Y_0 = \frac{\partial}{\partial t_1}$ and $Y_{-0} = \frac{\partial}{\partial t_{-1}}$. It is clear that we have $Y_0 I = 0$ and $Y_{-0} I = 0$ since the function I depends on neither t_1 nor t_{-1}.

The algebra $\tilde{L}_n$ consists of the operators from the sequence (2.11), all their possible commutators, and linear combinations with coefficients depending on n and x. Obviously equation (2.4) admits a nontrivial n-integral only if the dimension of the characteristic Lie algebra $\tilde{L}_n$ is finite. But it is not clear that the finiteness of dimension $\tilde{L}_n$ is essential for existence of nontrivial n-integrals. Because of this we introduce another Lie algebra called the characteristic Lie algebra of the equation (2.4) in the direction n. First we define differential operators

$$X_j = \frac{\partial}{\partial_{t_{-j}}}$$

for $j = 1, 2, ...$ in addition to the operators $Y_1, Y_2, ...$.

The following theorem defines the characteristic Lie algebra in the direction n.

Theorem 2.2 *Equation (2.4) admits a nontrivial n-integral if and only if the following two conditions hold:*
1) Linear envelope of the operators $\{Y_j\}_1^\infty$ is of finite dimension, denote this dimension N;
2) Lie algebra L_n generated by the operators $Y_1, Y_2, ..., Y_N, X_1, X_2, ..., X_N$ is of finite dimension. We call L_n the characteristic Lie algebra of (2.4).

Remark 2.3 *It is easy to prove that if dimension of $\{Y_j\}_1^\infty$ is N then the set $\{Y_j\}_1^N$ constitute a basis in the linear envelope of $\{Y_j\}_1^\infty$.*

In the next two sections, we will analyze the characteristic Lie algebras L_n and L_x by giving some properties of these algebras. In the Section 2.2.1, which is devoted to characteristic Lie algebra L_n, we will give the proof of Theorem 2.2 in detail.

2.2.1 Characteristic Lie Algebra L_n

In this section we study some properties of the characteristic Lie algebra L_n introduced in the Theorem 2.2. We will firstly begin with the proof of the Remark 2.3. It immediately follows from the following Lemma.

Lemma 2.4 *If for some integer N the operator Y_{N+1} is a linear combination of the operators with less indices:*

$$Y_{N+1} = \alpha_1 Y_1 + \alpha_2 Y_2 + ... + \alpha_N Y_N \tag{2.12}$$

then for any integer $j > N$, we have a similar expression

$$Y_j = \beta_1 Y_1 + \beta_2 Y_2 + ... + \beta_N Y_N. \tag{2.13}$$

Proof. We apply the property $Y_{k+1} = D^{-1} Y_k D$ to the expression (2.12) and get

$$Y_{N+2} = D^{-1}(\alpha_1)Y_2 + D^{-1}(\alpha_2)Y_3 + ... + D^{-1}(\alpha_N)(\alpha_1 Y_1 + ... + \alpha_N Y_N). \tag{2.14}$$

By using mathematical induction we can easily complete the proof of the Lemma. $\square$

Lemma 2.5 *The following commutativity relations take place:*

$$[Y_0, Y_{-0}] = 0, \quad [Y_0, Y_1] = 0, \quad [Y_{-0}, Y_{-1}] = 0.$$

Proof. Recall that $Y_0 = \frac{\partial}{\partial t_1}$ and $Y_{-0} = \frac{\partial}{\partial t_{-1}}$. The first of the relations is obvious. In order to prove the others we should find the coordinate representation of the operators Y_1 and Y_{-1} acting in the class of locally smooth functions of the variables $x, n, t, t_x, t_{xx}, \dots$. By applying Y_1 to a function H depending on these variables, we get

$$\begin{aligned} Y_1 H &= D^{-1} \frac{d}{dt_1} D H(t, t_x, t_{xx}, ...) \\ &= D^{-1} \frac{d}{dt_1} H(t_1, f, f_x, ...) \\ &= \Big\{ \frac{\partial}{\partial t} + D^{-1}\Big(\frac{\partial f}{\partial t_1}\Big) \frac{\partial}{\partial t_x} + D^{-1}\Big(\frac{\partial f_x}{\partial t_1}\Big) \frac{\partial}{\partial t_{xx}} + ... \Big\} H(t, t_x, t_{xx}, ...), \end{aligned}$$

which yields

$$Y_1 = \frac{\partial}{\partial t} + D^{-1}\Big(\frac{\partial f}{\partial t_1}\Big)\frac{\partial}{\partial t_x} + D^{-1}\Big(\frac{\partial f_x}{\partial t_1}\Big)\frac{\partial}{\partial t_{xx}} + D^{-1}\Big(\frac{\partial f_{xx}}{\partial t_1}\Big)\frac{\partial}{\partial t_{xxx}} + \dots . \quad (2.15)$$

Now note that all of the functions f, f_x, $f_{xx}, ...$ depend on the variables $t_1, t, t_x, t_{xx}, ...$ and do not depend on t_2 hence the coefficients of the vector field Y_1 do not depend on t_1. Therefore the operators Y_1 and Y_0 commute. In a similar way we find the coordinate representation of Y_{-1} as

$$Y_{-1} = \frac{\partial}{\partial t} + D\Big(\frac{\partial g}{\partial t_{-1}}\Big)\frac{\partial}{\partial t_x} + D\Big(\frac{\partial g_x}{\partial t_{-1}}\Big)\frac{\partial}{\partial t_{xx}} + D\Big(\frac{\partial g_{xx}}{\partial t_{-1}}\Big)\frac{\partial}{\partial t_{xxx}} + \dots , \quad (2.16)$$

and clearly $[Y_{-0}, Y_{-1}] = 0.\Box$

The following Lemma is very important since we will use it for several times while studying the characteristic Lie algebra L_n.

Lemma 2.6 *(1) Suppose that the vector field*

$$Y = \alpha(0)\frac{\partial}{\partial t} + \alpha(1)\frac{\partial}{\partial t_x} + \alpha(2)\frac{\partial}{\partial t_{xx}} + ...,$$

where $\alpha_x(0) = 0$, *solves the equation* $[D_x, Y] = 0$, *then* $Y = \alpha(0)\frac{\partial}{\partial t}$.
(2) Suppose that the vector field

$$Y = \alpha(1)\frac{\partial}{\partial t_x} + \alpha(2)\frac{\partial}{\partial t_{xx}} + \alpha(3)\frac{\partial}{\partial t_{xxx}} + ...$$

solves the equation $[D_x, Y] = hY$, *where* h *is a function of variables* t, t_x, t_{xx}, $\dots$, $t_{\pm 1}$, $t_{\pm 2}$, $\dots$, *then* $Y = 0$.

Proof. The proof of Lemma 2.6 can be easily derived from the following formula

$$\begin{aligned} [D_x, Y] &= -(\alpha(0)f_t + \alpha(1)f_{t_x})\frac{\partial}{\partial t_1} + (\alpha_x(0) - \alpha(1))\frac{\partial}{\partial t} \\ &+ (\alpha_x(1) - \alpha(2))\frac{\partial}{\partial t_x} + (\alpha_x(2) - \alpha(3))\frac{\partial}{\partial t_{xx}} + \dots . \end{aligned} \quad (2.17)$$

Let us just give the proof of part (1). Since the condition $[D_x, Y] = 0$ holds, the terms before the partial differentials should be zero. In part (1), we have also the condition $\alpha_x(0) = 0$. Hence from the terms before $\frac{\partial}{\partial t}$, we have $\alpha(1) = 0$.

Since $\alpha(1) = 0$, the terms before $\frac{\partial}{\partial t_1}$ gives us $\alpha(0) = 0$. We proceed in this way and we get all $\alpha(i) = 0$, $i = 0, 1, 2, \ldots$. Hence the vector field $Y = 0$ in part (1). Similarly, we can prove the second part of the Lemma. $\square$

In the formula (2.15) we have already given the coordinate representation of the operator Y_1. We can check that the operator Y_2 is a vector field of the form

$$Y_2 = D^{-1}(Y_1(f))\partial_{t_x} + D^{-1}(Y_1(f_x))\partial_{t_{xx}} + D^{-1}(Y_1(f_{xx}))\partial_{t_{xxx}} + \ldots . \tag{2.18}$$

It immediately follows from the equation $Y_2 = D^{-1}Y_1D$ and the coordinate representation of Y_1. By induction we can prove similar formulas for arbitrary j:

$$Y_{j+1} = D^{-1}(Y_j(f))\partial_{t_x} + D^{-1}(Y_j(f_x))\partial_{t_{xx}} + D^{-1}(Y_j(f_{xx}))\partial_{t_{xxx}} + \ldots . \tag{2.19}$$

Lemma 2.7 *For any $n \geq 0$, we have*

$$[D_x, Y_n] = -\sum_{j=0}^{n} D^{-j}(Y_{n-j}(f))Y_j \, . \tag{2.20}$$

In particular,

$$[D_x, Y_0] = -Y_0(f)Y_0 \quad , \quad [D_x, Y_1] = -Y_1(f)Y_0 - D^{-1}(Y_0(f))Y_1. \tag{2.21}$$

Proof. We have,

$$\begin{aligned} [D_x, Y_0]H(t, t_1, t_x, t_{xx}, \ldots) &= D_x H_{t_1} - Y_0 D_x H \\ &= (H_{tt_1}t_x + H_{t_1t_1}t_{1x} + \ldots) - \frac{\partial}{\partial_{t_1}}(H_t t_x + H_{t_1}t_{1x} + \ldots) \\ &= -H_{t_1}f_{t_1} = -Y_0(f)Y_0H, \end{aligned}$$

i.e. the first equation of (2.21) holds. By (2.15), (2.17) and $[D_x, Y_0] = -Y_0(f)Y_0$, we calculate $[D_x, Y_1]$ as

$$\begin{aligned} &[D_x, Y_1] = -Y_1(f)\frac{\partial}{\partial t_1} - D^{-1}(Y_0(f))\frac{\partial}{\partial t} + D^{-1}[D_x, Y_0]f\frac{\partial}{\partial t_x} + D^{-1}[D_x, Y_0]f_x\frac{\partial}{\partial t_{xx}} + \ldots \\ &= -Y_1(f)Y_0 - D^{-1}(Y_0(f))\frac{\partial}{\partial t} - D^{-1}(Y_0(f)Y_0(f))\frac{\partial}{\partial t_x} - D^{-1}(Y_0(f)Y_0(f_x))\frac{\partial}{\partial t_{xx}} - \ldots \\ &= -Y_1(f)Y_0 - D^{-1}(Y_0(f))Y_1. \end{aligned}$$

It is easy to see by mathematical induction that on the space of smooth functions of $t, t_1, t_x, t_{xx}, ...$ we have

$$[D_x, Y_n] = -\sum_{j=0}^{n} D^{-j}(Y_{n-j}(f))Y_j$$

for any integer $n \geq 0$. Hence the Lemma is proved. $\square$

Lemma 2.8 *Lie algebra generated by the operators $Y_1, Y_2, Y_3, ...$ is commutative.*

Proof. By Lemma 2.5, $[Y_1, Y_0] = 0$. As we said in the proof of this Lemma, the reason for this equality is that the coefficients of the vector field Y_1 do not depend on the variable t_1. They might depend only on t_{-1}, t, t_x, t_{xx}, t_{xxx}, $\ldots$. The coefficients of the vector field Y_2 being of the form $D^{-1}(Y_1(D_x^j f))$ which is seen in (2.18) also do not depend on the variable t_1. They might depend only on t_{-2}, t_{-1}, t, t_x, t_{xx}, t_{xxx}, $\ldots$. Therefore, we have $[Y_2, Y_0] = 0$. Continuing in that way we see that for any $n \geq 1$ the commutativity relation $[Y_n, Y_0] = 0$ holds. Consider now the commutator $[Y_n, Y_{n+m}]$, $n \geq 1$, $m \geq 1$. We have,

$$\begin{aligned}
[Y_n, Y_{n+m}] &= [D^{-n}Y_0D^n, D^{-(n+m)}Y_0D^{n+m}] \\
&= [D^{-n}Y_0D^n, D^{-n}D^{-m}Y_0D^mD^n] \\
&= [D^{-n}Y_0D^n, D^{-n}Y_mD^n] \\
&= D^{-n}[Y_0, Y_m]D^n = 0,
\end{aligned}$$

that finishes the proof of the Lemma. $\square$

Lemma 2.9 *If the operator $Y_2 = 0$ then $[X_1, Y_1] = 0$.*

Proof. By the coordinate representation of Y_2 given in (2.18), $Y_2 = 0$ implies that $Y_1(f) = 0$. Due to (2.15), $Y_1(f) = 0$ means that $f_t + D^{-1}(f_{t_1})f_{t_x} = 0$. Hence $D^{-1}(f_{t_1})$ does not depend on t_{-1} i.e. $X_1(D^{-1}(f_{t_1})) = 0$. By using Lemma 2.7 and the fact that $[D_x, X_1] = 0$, we conclude that

$$[D_x, [X_1, Y_1]] = -[X_1, D^{-1}(f_{t_1})Y_1] = -D^{-1}(f_{t_1})[X_1, Y_1],$$

which means $[D_x, [X_1, Y_1]] = -D^{-1}(f_{t_1})[X_1, Y_1]$. By Lemma 2.7, part (2), it follows that $[X_1, Y_1] = 0$. $\square$

Lemma 2.10 *The operator* $Y_2 = 0$ *if and only if we have*

$$f_t + D^{-1}(f_{t_1})f_{t_x} = 0. \tag{2.22}$$

Proof. Assume $Y_2 = 0$. By (2.18), $Y_1(f) = 0$. Due to (2.15), equality $Y_1(f) = 0$ is indeed another way of writing the equation (2.22).

Conversely, assume (2.22) holds, i.e. $Y_1(f) = 0$. It follows from (2.18) that $Y_2(f) = 0$. Due to Lemma 2.7, we have $[D_x, Y_2] = -D^{-2}(Y_0(f))Y_2$ that implies, by Lemma 2.6, part (2), that $Y_2 = 0$. $\square$

Corollary 2.11 *The dimension of the Lie algebra* L_n *associated with* n*-integral is equal to* 2 *if and only if (2.22) holds, or the same* $Y_2 = 0$.

Proof. By Theorem 2.2, the dimension of L_n is 2 if and only if $Y_2 = \lambda_1 X_1 + \mu_1 Y_1$ and $[X_1, Y_1] = \lambda_2 X_1 + \mu_2 Y_1$ for some λ_i, μ_i, $i = 1, 2$.

Assume the dimension of L_n is 2. Then $Y_2 = \lambda_1 X_1 + \mu_1 Y_1$. Since among X_1, Y_1, Y_2 differentiation by t_{-1} is used only in X_1, differentiation by t is used only in Y_1, then $\lambda_1 = \mu_1 = 0$. Therefore, $Y_2 = 0$, or the same, by Lemma 2.10, (2.22) holds.

Conversely, assume (2.22) holds, that is $Y_2 = 0$. By Lemma 2.9, $[X_1, Y_1] = 0$. Since Y_2 and $[X_1, Y_1]$ are trivial linear combinations of X_1 and Y_1 then the dimension of L_n is 2. $\square$

Now we can pass to the proof of Theorem 2.2.

Proof of Theorem 2.2. Suppose that there exists a nontrivial n-integral $I = I(t, t_x, ..., t_{[N]})$ for the equation (2.4), here $t_{[j]} = D_x^j t$ for any $j \geq 0$. Then all the vector fields from the Lie algebra M generated by $\{Y_j, X_k\}$ for $j = 1, 2, ...$ and $k = 1, ..., N_2$, where N_2 is arbitrary constant satisfying $N_2 \geq N$ annulate I. We will show that dimension of the Lie algebra M is finite. We consider first the projection of the algebra M given by the operator P_N:

$$P_N\Big(\sum_{i=-N_2}^{-1} x(i)\partial_{t_i} + \sum_{i=0}^{\infty} x(i)\partial_{t_{[i]}}\Big) = \sum_{i=-N_2}^{-1} x(i)\partial_{t_i} + \sum_{i=0}^{N} x(i)\partial_{t_{[i]}}. \tag{2.23}$$

Let $L_n(N)$ be the projection of the algebra M. Then evidently, for any Z_0 in $L_n(N)$ the equation $Z_0 I = 0$ is satisfied. Obviously, dim $L_n(N) < \infty$. Let the set $\{Z_{01}, Z_{02}, ..., Z_{0N_1}\}$ form a basis in $L_n(N)$. Hence we can represent any Z_0 in $L_n(N)$ as a linear combination

$$Z_0 = \alpha_1 Z_{01} + \alpha_2 Z_{02} + ... + \alpha_{N_1} Z_{0N_1}. \tag{2.24}$$

Suppose that the vector fields $Z, Z_1, ..., Z_{N_1}$ in M are connected with the operators $Z_0, Z_{01}, ..., Z_{0N_1}$ in $L_n(N)$ by the formulas $P_N(Z) = Z_0, P_N(Z_1) = Z_{01}, ..., P_N(Z_{N_1}) = Z_{0N_1}$. We have to prove that Z can be presented as a linear combination

$$Z = \alpha_1 Z_1 + \alpha_2 Z_2 + ... + \alpha_{N_1} Z_{N_1}. \tag{2.25}$$

In the proof, we will use the following Lemma.

Lemma 2.12 *Let $I_1 = D_x I$ and I is an n-integral. Then for each Z in M we have $ZI_1 = 0$.*

Proof. We should show that $I_1 = D_x I$ is also an n-integral for the algebra M. Really

$$DI_1 = DD_x I = D_x DI = D_x I = I_1.$$

Since I_1 is also n-integral then for each Z in M we have $ZI_1 = 0$. $\square$

We apply the operator $(Z - \alpha_1 Z_1 - \alpha_2 Z_2 - ... - \alpha_{N_1} Z_{N_1})$ to the function $I_1 = I_1(t, t_x, t_{xx}, ..., t_{[N+1]})$,

$$(Z - \alpha_1 Z_1 - \alpha_2 Z_2 - ... - \alpha_{N_1} Z_{N_1}) I_1 = 0. \tag{2.26}$$

We can write (2.26) as

$$\begin{aligned} & (Z_0 - \alpha_1 Z_{01} - \alpha_2 Z_{02} - ... - \alpha_{N_1} Z_{0N_1}) I_1 + (X(N+1) - \alpha_1 X_1(N+1) \\ - & \alpha_2 X_2(N+1) - ... - \alpha_{N_1} X_{N_1}(N+1)) \frac{\partial}{\partial t_{[N+1]}} I_1 = 0, \end{aligned} \tag{2.27}$$

where $X(N+1), X_1(N+1), ..., X_{N_1}(N+1)$ are the coefficients before $\partial_{t_{[N+1]}}$ of the vector fields $Z, Z_1, Z_2, ..., Z_{N_1}$. The first summand in (2.27) vanishes by (2.24).

In the second one the factor $\frac{\partial}{\partial t_{[N+1]}}I_1 = \frac{\partial}{\partial t_{[N]}}I$ is not zero. So we end up with the equation

$$X(N+1) = \alpha_1 X_1(N+1) + \alpha_2 X_2(N+1) + ... + \alpha_{N_1} X_{N_1}(N+1). \tag{2.28}$$

Equation (2.28) shows that

$$P_{N+1}(Z) = \alpha_1 P_{N+1}(Z_1) + \alpha_2 P_{N+1}(Z_2) + ... + \alpha_{N_1} P_{N+1}(Z_{N_1}). \tag{2.29}$$

So by applying mathematical induction, we can prove the formula (2.25). Thus the Lie algebra M is of finite dimension. Now we construct the characteristic algebra L_n by using M. Since dim $M < \infty$, the linear envelope of the vector fields $\{Y_j\}_1^\infty$ is of finite dimension. We choose a basis in this linear space consisting of $Y_1, Y_2, ..., Y_S$ for $S \leq N \leq N_2$. Then the algebra generated by $Y_1, Y_2, ..., Y_S, X_1, X_2, ..., X_S$ is of finite dimension, because it is a subalgebra of M. This algebra is just characteristic Lie algebra of the equation (2.4).

Suppose that conditions (1) and (2) of the Theorem 2.2 are satisfied. So there exists a finite dimensional characteristic Lie algebra L_n for the equation (2.4). We show that in this case equation (2.4) admits a nontrivial n-integral. Let N_1 is the dimension of L_n and N is the dimension of the linear envelope of the vector fields $\{Y_j\}_{j=1}^\infty$. We take the projection $L_n(N_2)$ of the Lie algebra L_n defined by the operator P_{N_2} defined by the formula (2.23)with N_2 instead of N. Evidently, $L_n(N_2)$ consists of the finite sums $Z_0 = \sum_{i=-N}^{-1} x(i)\partial_{t_i} + \sum_{i=0}^{N_2} x(i)\partial_{t_{[i]}}$ where $N = N_1 - N_2$. Let $Z_{01}, ..., Z_{0N_1}$ form a basis in $L_n(N_2)$. Then we have $N_1 = N + N_2$ equations $Z_{0j}G = 0$, $j = 1, ..., N_1$, for a function G depending on $N + N_2 + 1 = N_1 + 1$ independent variables. Then due to the well-known Jacobi theorem, there exists a function $G = G(t_{-N}, t_{-N+1}, ..., t_{-1}, t, t_x, t_{xx}, ..., t_{[N_2]})$, which satisfies the equation $ZG = 0$ for any Z in L_n. But really it does not depend on t_{-N}, ..., t_{-1} because $X_1G = 0$, $X_2G = 0$, ..., $X_NG = 0$. Thus the function G is $G = G(t, t_x, t_{xx}, ..., t_{[N_2]})$. Such a function is not unique but any other solution of these equations, depending on the same set of the variables, can be represented as $h(G)$ for some function h.

Note one more property of the algebra L_n. Let π be a map which assigns to each Z in L_n its conjugation $D^{-1}ZD$. Evidently, the map π acts from the algebra

L_n into its central extension $L_n \oplus \{X_{N+1}\}$, because for the generators of L_n we have $D^{-1}Y_jD = Y_{j+1}$ and $D^{-1}X_jD = X_{j+1}$. Evidently, $[X_{N+1}, Y_j] = 0$ and $[X_{N+1}, X_j] = 0$ for any integer $j \leq N$. Moreover $X_{N+1}F = 0$ for the function $G = G(t, t_x, ..., t_{[N_2]})$ mentioned above. This fact implies that $ZG_1 = 0$ for $G_1 = DG$ and for any vector field Z in L_n. Really, for any Z in L_n one has a representation of the form $D^{-1}ZD = \tilde{Z} + \lambda X_{N+1}$ where $\tilde{Z}$ in L_n and λ is a function. So

$$ZG_1 = ZDG = D(D^{-1}ZDG) = D(\tilde{Z} + \lambda X_{N+1})G = 0. \tag{2.30}$$

Therefore $G_1 = h(G)$ or $DG = h(G)$. In other words function $G = G(n)$ satisfies an ordinary difference equation of the first order. Its general solution can be written as $G = H(n, c)$ where H is a function of two variables and c is an arbitrary constant. By solving the equation $G = H(n, c)$ with respect to c one gets $c = F(G, n)$. The function $F = F(G, n)$ found is just n-integral searched. Actually, $DF(G, n) = Dc = c = F(G, n)$. So $DF = F$. This completes the proof of the Theorem 2.2. $\square$

2.2.2 Characteristic Lie Algebra L_x

Here we study some properties of the characteristic Lie algebra L_x. Consider an infinite sequence of the vector fields defined as follows,

$$K_1 = [X, K_0], \quad K_2 = [X, K_1], \quad \ldots, \quad K_{n+1} = [X, K_n], \quad n \geq 1\,, \tag{2.31}$$

where K_0 and X are defined by (2.6) and (2.7).

It is easy to see that

$$K_1 = \frac{\partial}{\partial t} + X(f)\frac{\partial}{\partial t_1} + X(g)\frac{\partial}{\partial t_{-1}} + X(f_1)\frac{\partial}{\partial t_2} + X(g_{-1})\frac{\partial}{\partial t_{-2}} + \ldots\,, \tag{2.32}$$

$$K_n = \sum_{j=1}^{\infty}\left\{X^n(f_{j-1})\frac{\partial}{\partial t_j} + X^n(g_{-j+1})\frac{\partial}{\partial t_{-j}}\right\}, \quad n \geq 2, \tag{2.33}$$

where $f_0 := f$ and $g_0 := g$.

Lemma 2.13 *We have,*

$$DXD^{-1}=\frac{1}{f_{t_x}}X,\qquad DK_0D^{-1}=K_0-\frac{t_xf_t+ff_{t_1}}{f_{t_x}}X, \tag{2.34}$$

$$DK_1D^{-1}=\frac{1}{f_{t_x}}K_1-\frac{f_t+f_{t_x}f_{t_1}}{f_{t_x}^2}X,\qquad DK_2D^{-1}=\frac{1}{f_{t_x}^2}K_2-\frac{f_{t_xt_x}}{f_{t_x}^3}K_1+\frac{f_{t_xt_x}f_t}{f_{t_x}^4}X, \tag{2.35}$$

$$DK_3D^{-1}=\frac{1}{f_{t_x}^3}K_3-3\frac{f_{t_xt_x}}{f_{t_x}^4}K_2+\left(3\frac{f_{t_xt_x}^2}{f_{t_x}^5}-\frac{f_{t_xt_xt_x}}{f_{t_x}^4}\right)K_1-\frac{f_t}{f_{t_x}}\left(3\frac{f_{t_xt_x}^2}{f_{t_x}^5}-\frac{f_{t_xt_xt_x}}{f_{t_x}^4}\right)X. \tag{2.36}$$

Proof. In the proof of this Lemma we will use A and A^* to denote the functions $A(x,t,t_x,t_1,t_{-1},t_2,t_{-2},\ldots)$ and $D^{-1}A=A(x,t_{-1},g,t,t_{-2},t_1,t_{-3},\ldots)$ respectively.

Since

$$DXD^{-1}A=DA^*_{t_x}=D\left\{g_{t_x}\frac{\partial A^*}{\partial g}\right\}=D(g_{t_x})\frac{\partial A}{\partial t_x}=D(g_{t_x})XA$$

and $D(g_{t_x})=\dfrac{1}{f_{t_x}}$ then $DXD^{-1}A=\frac{1}{f_{t_x}}XA$.

Since

$$\begin{aligned}DK_0D^{-1}A&=D\left(\frac{\partial}{\partial x}+t_x\frac{\partial}{\partial t}+f\frac{\partial}{\partial t_1}+g\frac{\partial}{\partial t_{-1}}+f_1\frac{\partial}{\partial t_2}+g_{-1}\frac{\partial}{\partial t_{-2}}+\ldots\right)A^*\\&=D\left(\frac{\partial A^*}{\partial x}+t_xg_t\frac{\partial A^*}{\partial g}+t_x\frac{\partial A^*}{\partial t}+f\frac{\partial A^*}{\partial t_1}+g\frac{\partial A^*}{\partial t_{-1}}+gg_{t_{-1}}\frac{\partial A^*}{\partial g}+\ldots\right)\\&=\left(\frac{\partial A}{\partial x}+t_x\frac{\partial A}{\partial t}+f\frac{\partial A}{\partial t_1}+f_1\frac{\partial A}{\partial t_2}+\ldots\right)+(t_xD(g_{t_{-1}})+fD(g_t))\frac{\partial A}{\partial t_x}\end{aligned}$$

and $D(g_{t_{-1}})=-\frac{f_t}{f_{t_x}}$, $D(g_t)=-\frac{f_{t_1}}{f_{t_x}}$ then $DK_0D^{-1}A=K_0A-\frac{t_xf_t+ff_{t_1}}{f_{t_x}}XA$.
Using formulas (2.34) for DXD^{-1}, DK_0D^{-1} and the definition (2.31) of K_1 we have

$$\begin{aligned}DK_1D^{-1}&=[DXD^{-1},DK_0D^{-1}]=\left[\frac{1}{f_{t_x}}X,K_0-\frac{t_xf_t+ff_{t_1}}{f_{t_x}}X\right]\\&=\frac{1}{f_{t_x}}K_1-K_0\left(\frac{1}{f_{t_x}}\right)X-\frac{1}{f_{t_x}}X\left(\frac{t_xf_t+ff_{t_1}}{f_{t_x}}\right)X+\frac{t_xf_t+ff_{t_1}}{f_{t_x}}\left(-\frac{f_{t_xt_x}}{f_{t_x}^2}\right)X\\&=\frac{1}{f_{t_x}}K_1-\frac{f_t+f_{t_x}f_{t_1}}{f_{t_x}^2}X.\end{aligned}$$

Using formulas (2.34) and (2.35) for DXD^{-1}, DK_1D^{-1} and the definition (2.31) of K_2 we have

$$
\begin{aligned}
DK_2D^{-1} = [DXD^{-1}, DK_1D^{-1}] &= \left[\frac{1}{f_{t_x}}X, \frac{1}{f_{t_x}}K_1 - \frac{f_t + f_{t_x}f_{t_1}}{f_{t_x}^2}X\right] \\
= \quad & -\frac{f_{t_xt_x}}{f_{t_x}^3}K_1 - \frac{1}{f_{t_x}}K_1\left(\frac{1}{f_{t_x}}\right)X + \frac{1}{f_{t_x}^2}K_2 - \frac{1}{f_{t_x}}X\left(\frac{f_t + f_{t_x}f_{t_1}}{f_{t_x}^2}\right)X \\
& -\frac{f_t + f_{t_x}f_{t_1}}{f_{t_x}^2}\frac{f_{t_xt_x}}{f_{t_x}^2}X \\
= \quad & \frac{1}{f_{t_x}^2}K_2 - \frac{f_{t_xt_x}}{f_{t_x}^3}K_1 + \frac{f_{t_xt_x}f_t}{f_{t_x}^4}X.
\end{aligned}
$$

Using formulas (2.34) and (2.35) for DXD^{-1}, DK_2D^{-1} and the definition (2.31) of K_3 we have

$$
\begin{aligned}
DK_3D^{-1} = [DXD^{-1}, DK_2D^{-1}] &= \Big[\frac{1}{f_{t_x}}X, \frac{1}{f_{t_x}^2}K_2 - \frac{f_{t_xt_x}}{f_{t_x}^3}K_1 + \frac{f_{t_xt_x}f_t}{f_{t_x}^4}X\Big] \\
= \quad & \frac{1}{f_{t_x}^3}K_3 - \frac{f_{t_xt_x}}{f_{t_x}^4}K_2 - \frac{2f_{t_xt_x}}{f_{t_x}^4}K_2 - \frac{1}{f_{t_x}}X\Big(\frac{f_{t_xt_x}}{f_{t_x}^3}\Big)K_1 + X\Big\{\frac{1}{f_{t_x}}X\Big(\frac{f_{t_xt_x}f_t}{f_{t_x}^4}\Big) \\
& -\frac{1}{f_{t_x}^2}K_2\Big(\frac{1}{f_{t_x}}\Big) + \frac{f_{t_xt_x}}{f_{t_x}^3}K_1\Big(\frac{1}{f_{t_x}}\Big) - \frac{f_{t_xt_x}f_t}{f_{t_x}^4}X\Big(\frac{1}{f_{t_x}}\Big)\Big\} \\
= \quad & \frac{1}{f_{t_x}^3}K_3 - \frac{3f_{t_xt_x}}{f_{t_x}^4}K_2 - \frac{f_{t_xt_xt_x}f_{t_x} - 3f_{t_xt_x}^2}{f_{t_x}^5}K_1 - \frac{f_t}{f_{t_x}}\frac{f_{t_xt_xt_x}f_{t_x} - 3f_{t_xt_x}^2}{f_{t_x}^5}X. \square
\end{aligned}
$$

Lemma 2.14 *For any $n \geq 1$ we have,*

$$
DK_nD^{-1} = a_n^{(n)}K_n + a_{n-1}^{(n)}K_{n-1} + a_{n-2}^{(n)}K_{n-2} + \ldots + a_1^{(n)}K_1 + b^{(n)}X, \qquad (2.37)
$$

where coefficients $b^{(n)}$ and $a_k^{(n)}$ are functions that depend only on variables t, t_1 and t_x for all k, $1 \leq k \leq n$. Moreover,

$$
\begin{aligned}
a_n^{(n)} = \frac{1}{f_{t_x}^n}, \quad n \geq 1, \qquad & a_{n-1}^{(n)} = -\frac{n(n-1)}{2}\frac{f_{t_xt_x}}{f_{t_x}^{n+1}}, \quad n \geq 2, \\
b^{(n)} \quad = \quad & -\frac{f_t}{f_{t_x}}a_1^{(n)}, \quad n \geq 2, \qquad (2.38)
\end{aligned}
$$

$$
a_{n-2}^{(n)} = \frac{(n-2)(n^2-1)n}{4}\frac{f_{t_xt_x}^2}{2f_{t_x}^{n+2}} - \frac{(n-2)(n-1)n}{3}\frac{f_{t_xt_xt_x}}{2f_{t_x}^{n+1}}, \quad n \geq 3\,. \qquad (2.39)
$$

Proof. We use the mathematical induction to prove the Lemma. As Lemma 2.13 shows the base of mathematical induction holds.

Assume the representation (2.37) for DK_nD^{-1} is true and all coefficients $a_k^{(n)}$ are functions of t, t_1, t_x only. Consider $DK_{n+1}D^{-1}$. We have,

$$\begin{aligned}
DK_{n+1}D^{-1} &= [DXD^{-1}, DK_nD^{-1}] \\
&= \left[\frac{1}{f_{t_x}}X, a_n^{(n)}K_n + a_{n-1}^{(n)}K_{n-1} + a_{n-2}^{(n)}K_{n-2} + \ldots + a_1^{(n)}K_1 + b^{(n)}X\right] \\
&= a_{n+1}^{(n+1)}K_{n+1} + a_n^{(n+1)}K_n + a_{n-1}^{(n+1)}K_{n-1} + \ldots + a_1^{(n+1)}K_1 + b^{(n+1)}X,
\end{aligned}$$

where

$$\begin{aligned}
a_{n+1}^{(n+1)} &= \frac{1}{f_{t_x}}a_n^{(n)}, \\
a_{n-k}^{(n+1)} &= \frac{1}{f_{t_x}}X(a_{n-k}^{(n)}) + \frac{1}{f_{t_x}}a_{n-k-1}^{(n)}, \quad 0 \le k \le n-2, \\
a_1^{(n+1)} &= \frac{1}{f_{t_x}}X(a_1^{(n)}).
\end{aligned}$$

It is easy to see then $a_{n-k}^{(n+1)}$, $0 \le k \le n-2$ are functions of t, t_1, t_x only.

Assuming formulas (2.38) and (2.39) for $a_n^{(n)}$, $a_{n-1}^{(n)}$ and $a_{n-2}^{(n)}$ are true, the following equality

$$\begin{aligned}
DK_{n+1}D^{-1} &= a_{n+1}^{(n+1)}K_{n+1} + a_n^{(n+1)}K_n + a_{n-1}^{(n+1)}K_{n-1} + \ldots + a_1^{(n+1)}K_1 + b^{(n+1)}X \\
&= \left[\frac{1}{f_{t_x}}X, \frac{1}{f_{t_x}^n}K_n + a_{n-1}^{(n)}K_{n-1} + a_{n-2}^{(n)}K_{n-2} + \ldots + a_1^{(n)}K_1 + b^{(n)}X\right]
\end{aligned}$$

implies that

$$a_{n+1}^{(n+1)} = \frac{1}{f_{t_x}^{n+1}},$$

$$\begin{aligned}
a_n^{(n+1)} &= \frac{1}{f_{t_x}}X\left(\frac{1}{f_{t_x}^n}\right) + \frac{1}{f_{t_x}}a_{n-1}^{(n)} \\
&= -\frac{nf_{t_xt_x}}{f_{t_x}^{n+2}} - \frac{n(n-1)f_{t_xt_x}}{2f_{t_x}^{n+2}} = -\frac{n(n+1)f_{t_xt_x}}{2f_{t_x}^{n+2}},
\end{aligned}$$

$$\begin{aligned}
a_{n-1}^{(n+1)} &= \frac{1}{f_{t_x}}X(a_{n-1}^{(n)}) + \frac{1}{f_{t_x}}a_{n-2}^{(n)} \\
&= \frac{(n-1)n(n+1)(n+2)}{4}\frac{f_{t_xt_x}^2}{2f_{t_x}^{n+3}} - \frac{(n-1)n(n+1)}{3}\frac{f_{t_xt_xt_x}}{2f_{t_x}^{n+2}}.
\end{aligned}$$

Using the same notation for A and A^* as in Lemma 2.13, we have (for $n \geq 2$),

$$
\begin{aligned}
& DYK_nD^{-1}A = D\left\{X^n(f)\frac{\partial}{\partial t_1} + X^n(g)\frac{\partial}{\partial t_{-1}} + X^n(f_1)\frac{\partial}{\partial t_2} + \ldots\right\}A^* \\
& = D\left\{X^n(f)\frac{\partial A^*}{\partial t_1} + X^n(g)\frac{\partial A^*}{\partial t_{-1}} + X^n(g)g_{t_{-1}}\frac{\partial A^*}{\partial g} + X^n(f_1)\frac{\partial A^*}{\partial t_2} + \ldots\right\} \\
& = D(X^n(f))\frac{\partial A}{\partial t_2} + D(X^n(g))\frac{\partial A}{\partial t} + D(X^n(g))D(g_{t_{-1}})\frac{\partial A}{\partial t_x} + D(X^n(f_1))\frac{\partial A}{\partial t_3} + \ldots \\
& = D(X^n(g))\frac{\partial A}{\partial t} - \frac{f}{f_{t_x}}D(X^n(g))XA + \sum_{k=1}^{\infty}\left(\alpha_k^{(n)}\frac{\partial}{\partial t_k} + \beta_k^{(n)}\frac{\partial}{\partial t_{-k}}\right)A \\
& = (b^{(n)}X + a_1^{(n)}K_1 + a_2^{(n)}K_2 + \ldots + a_n^{(n)}K_n)A.
\end{aligned}
$$

Since among X, K_i, $1 \leq i \leq n$ differentiation by t_x is used only in X, differentiation by t is used only in K_1 and then $a_1^{(n)} = D(X^n(g))$ and $b^{(n)} = -\frac{f_t}{f_{t_x}}D(X^n(g))$, which yields that $b^{(n)} = -\frac{f_t}{f_{t_x}}a_1^{(n)}$. The fact that $b^{(n)}$ is a function of t, t_1 and t_x follows from the similar result for $a_1^{(n)}$. □

Lemma 2.15 *Suppose that the vector field*

$$
K = \sum_{j=1}^{\infty}\left\{\alpha(k)\frac{\partial}{\partial t_k} + \alpha(-k)\frac{\partial}{\partial t_{-k}}\right\}
$$

solves the equation $DKD^{-1} = hK$, where h is a function of variables t, $t_{\pm 1}$, $t_{\pm 2}$, $\ldots$, t_x, t_{xx}, $\ldots$, then $K = 0$.

Proof. The proof of Lemma 2.15 can be derived from the following formula

$$
\begin{aligned}
DKD^{-1} &= -\frac{f_t}{f_{t_x}}D(\alpha(-1))X + D(\alpha(-1))\frac{\partial}{\partial t} + D(\alpha(-2))\frac{\partial}{\partial t_{-1}} \\
&+ \sum_{j=2}^{\infty}\left\{D(\alpha(j-1))\frac{\partial}{\partial t_j} + D(\alpha(-j-1))\frac{\partial}{\partial t_{-j}}\right\}. \qquad (2.40)
\end{aligned}
$$

If the function $h = 0$ then the vector field $K = 0$ automatically. Assume that $h \neq 0$. Since there is no differentiation with respect to t in K, the coefficient before $\frac{\partial}{\partial t}$ in the formula (2.40) should be zero which gives $\alpha(-1) = 0$. This yields that the coefficient before $\frac{\partial}{\partial t_{-1}}$ in K is zero. Hence the coefficient before $\frac{\partial}{\partial t_{-1}}$ in (2.40) which is $D(\alpha(-2)) = 0$ i.e. $\alpha(-2) = 0$. Proceeding in this way, we get all

$\alpha(i) = 0$ for any integer $i \neq 0$. Thus the vector field $K = 0$. □

Consider the linear space L^* generated by X and K_n, $n \geq 0$. It is a subset in the finite dimensional Lie algebra L_x. Therefore, there exists a natural number N such that

$$K_{N+1} = \mu X + \lambda_0 K_0 + \lambda_1 K_1 + \ldots + \lambda_N K_N, \tag{2.41}$$

and X, K_n, $0 \leq n \leq N$ are linearly independent. The coefficients μ, λ_i, $0 \leq i \leq N$, are functions depending on a finite number of the dynamical variables. Since among $X, K_0, ..., K_{N+1}$ we have differentiation with respect to t_x only in X, differentiation with respect to x only in K_0, we get $\mu = \lambda_0 = 0$. In this case, we have differentiation with respect to t only in K_1, hence $\lambda_1 = 0$. Since $\mu = \lambda_0 = \lambda_1 = 0$, then the equality (2.41) should be studied only if $N \geq 2$, or the same, if the dimension of L_x is 4 or more. Case, when the dimension of L_x is equal to 3 must be considered separately.

Assume $N \geq 2$. Then

$$\begin{aligned} DK_{N+1}D^{-1} &= D(\lambda_2)DK_2D^{-1} + D(\lambda_3)DK_3D^{-1} + \ldots + D(\lambda_{N-1})DK_{N-1}D^{-1} \\ &+ D(\lambda_N)DK_ND^{-1}. \end{aligned}$$

Rewriting DK_iD^{-1} in the last equation for each i, $2 \leq i \leq N+1$, using formulas (2.37), and K_{N+1} as a linear combination (2.41) allows us to compare coefficients before K_i, $2 \leq i \leq N$ and obtain the following system of equations.

$$\begin{array}{l} a_{N+1}^{(N+1)}\lambda_N + a_N^{(N+1)} = D(\lambda_N)a_N^{(N)} \\ a_{N+1}^{(N+1)}\lambda_{N-1} + a_{N-1}^{(N+1)} = D(\lambda_{N-1})a_{N-1}^{(N-1)} + D(\lambda_N)a_{N-1}^{(N)} \\ \ldots \\ a_{N+1}^{(N+1)}\lambda_i + a_i^{(N+1)} = D(\lambda_i)a_i^{(i)} + D(\lambda_{i+1})a_i^{(i+1)} + \ldots + D(\lambda_N)a_i^{(N)}, \end{array} \tag{2.42}$$

for $2 \leq i \leq N$. Since the coefficients λ_i, $2 \leq i \leq N$, depend on a finite number of arguments, it is clear that all of them are functions of only variables t and t_x.

Lemma 2.16 $K_2 = 0$ *if and only if* $f_{t_xt_x} = 0$.

Proof. Assume $K_2 = 0$. By representation (2.33) we have $X^2(f) = 0$, that is $f_{t_x t_x} = 0$.

Conversely, assume that $f_{t_x t_x} = 0$. By (2.35) we have $DK_2D^{-1} = \frac{1}{f_{t_x}^2}K_2$ that implies, by Lemma 2.15, that $K_2 = 0$. □

Now introduce

$$Z_2 = [K_0, K_1]. \tag{2.43}$$

Lemma 2.17 *We have,*

$$DZ_2D^{-1} = \frac{1}{f_{t_x}}Z_2 - \frac{t_x f_t + f f_{t_1}}{f_{t_x}^2}K_2 + CK_1 - \frac{f_t}{f_{t_x}}CX, \tag{2.44}$$

where $C = -\frac{t_x f_{t_x t}}{f_{t_x}^2} - \frac{f f_{t_x t_1}}{f_{t_x}^2} + \frac{f_t}{f_{t_x}^2} + \frac{f_{t_1}}{f_{t_x}} + \frac{t_x f_t f_{t_x t_x}}{f_{t_x}^3} + \frac{f f_{t_1} f_{t_x t_x}}{f_{t_x}^3}$.

Proof. Using the formulas (2.34) and (2.35) for DK_0D^{-1}, DK_1D^{-1} and the definition (2.43) of Z_2 we have,

$$\begin{aligned}
& DZ_2D^{-1} = [DK_0D^{-1}, DK_1D^{-1}] = [K_0 - AX, \frac{1}{f_{t_x}}K_1 - BX] \\
= \quad & K_0\left(\frac{1}{f_{t_x}}\right)K_1 + \frac{1}{f_{t_x}}Z_2 - K_0(B)X + BK_1 - AX\left(\frac{1}{f_{t_x}}\right)K_1 \\
& -A\frac{1}{f_{t_x}}K_2 + AX(B)X - BX(A)X \\
= \quad & \frac{1}{f_{t_x}}Z_2 - A\frac{1}{f_{t_x}}K_2 + \left(K_0\left(\frac{1}{f_{t_x}}\right) + B - AX\left(\frac{1}{f_{t_x}}\right)\right)K_1 \\
& +(AX(B) - BX(A) - K_0(B))X,
\end{aligned}$$

where

$$A = \frac{t_x f_t + f f_{t_1}}{f_{t_x}}, \qquad B = \frac{f_t + f_{t_x} f_{t_1}}{f_{t_x}^2}.$$

The coefficient before K_1 is

$$\begin{aligned}
K_0\left(\frac{1}{f_{t_x}}\right) + B - AX\left(\frac{1}{f_{t_x}}\right) \quad &= \quad -t_x\frac{f_{t_x t}}{f_{t_x}^2} - f\frac{f_{t_x t_1}}{f_{t_x}^2} + \frac{f_t + f_{t_x} f_{t_1}}{f_{t_x}^2} + \frac{f_{t_x t_x}}{f_{t_x}^2}\frac{t_x f_t + f f_{t_1}}{f_{t_x}} \\
&:= \quad C.
\end{aligned}$$

Note that the coefficient before X is $-\frac{f_t}{f_{t_x}}$ times the coefficient before K_1. To prove it we note that $Z_2 = (a_1\frac{\partial}{\partial t_1} + a_{-1}\frac{\partial}{\partial t_{-1}}) + (a_2\frac{\partial}{\partial t_2} + a_{-2}\frac{\partial}{\partial t_{-2}}) + \ldots$ for some functions a_i, $i = \pm 1, \pm 2, \ldots$, and then compare coefficients before XH and K_1H in $DZ_2D^{-1}H$ in the same way as we did for $DK_nD^{-1}H$ in Lemma 2.14. $\Box$

Lemma 2.18 *The dimension of the Lie algebra L_x generated by X and K_0 is equal to 3 if and only if*

$$f_{t_x t_x} = 0 \tag{2.45}$$

and

$$-\frac{t_x f_{t_x t}}{f_{t_x}^2} - \frac{f f_{t_x t_1}}{f_{t_x}^2} + \frac{f_t}{f_{t_x}^2} + \frac{f_{t_1}}{f_{t_x}} = 0\,. \tag{2.46}$$

Proof. Assume the dimension of the Lie algebra L_x generated by X and K_0 is equal to 3. It means that the algebra consists of X, K_0 and K_1 only, and

$$\begin{aligned} K_2 &= \lambda_1 X + \lambda_2 K_0 + \lambda_3 K_1, \\ Z_2 &= \mu_1 X + \mu_2 K_0 + \mu_3 K_1 \end{aligned}$$

for some functions λ_i and μ_i. Since among X, K_0, K_1, K_2 and Z_2 we have differentiation by t_x only in X, differentiation by x only in K_0, then $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$. Therefore, $K_2 = \lambda_3 K_1$ and $Z_2 = \mu_3 K_1$. Also, among K_1, K_2 and Z_2 we have differentiation by t only in K_1 then $\lambda_3 = \mu_3 = 0$. We have proved that if the dimension of the Lie algebra L_x is 3 then $K_2 = 0$ and $Z_2 = 0$. By Lemma 2.16, condition (2.45) is satisfied. It follows from (2.44) that

$$0 = DZ_2D^{-1} = \frac{1}{f_{t_x}} Z_2 - \frac{t_x f_t + f f_{t_1}}{f_{t_x}^2} K_2 + CK_1 - \frac{f_t}{f_{t_x}} CX = CK_1 - \frac{f_t}{f_{t_x}} CX.$$

Since X and K_1 are linearly independent then equality $CK_1 - \frac{f_t}{f_{t_x}} CX = 0$ implies $C = 0$. Equality (2.46) follows from (2.45) and $C = 0$.

Conversely, assume that properties (2.45) and (2.46) are satisfied. To prove that the dimension of the Lie algebra L_x is equal to 3 it is enough to show that $K_2 = 0$ and $Z_2 = 0$. It follows from (2.45) and Lemma 2.16 that $K_2 = 0$. From the formula (2.44) for DZ_2D^{-1}, property (2.46) and since $K_2 = 0$ we have that $DZ_2D^{-1} = \frac{1}{f_{t_x}} Z_2$, which implies, by Lemma 2.15, that $Z_2 = 0$. $\Box$

2.2.3 Special Case: Equations with Characteristic Lie Algebras of the Minimal Possible Dimensions.

Corollary 2.19 *If Lie algebras for $n-$ and $x-$ integrals have dimensions 2 and 3 respectively, then equation $t_{1x} = f(t, t_1, t_x)$ can be reduced to $t_{1x} = t_x + t_1 - t$.*

Proof. By Lemma 2.18 and Corollary 2.11, the dimensions of the characteristic Lie algebras L_n and L_x are 2 and 3 correspondingly means equations (2.22), (2.45), and (2.46) are satisfied. It follows from property (2.45) that $f(t, t_1, t_x) = A(t, t_1)t_x + B(t, t_1)$ for some functions $A(t, t_1)$ and $B(t, t_1)$. By (2.22), $A_t t_x + B_t + \{D^{-1}(A_{t_1} t_x + B_{t_1})\}A = 0$, that is

$$D^{-1}(A_{t_1} t_x + B_{t_1}) = -\frac{A_t}{A} t_x - \frac{B_t}{A}\,. \tag{2.47}$$

Note that $t_{1x} = At_x + B$ implies $t_x = D^{-1}(A)t_{-1x} + D^{-1}(B)$ and, therefore, $t_{-1x} = \frac{1}{D^{-1}(A)} t_x - \frac{D^{-1}(B)}{D^{-1}(A)}$. We continue with (2.47) and obtain the following equality

$$D^{-1}\left(\frac{A_{t_1}}{A}\right) t_x - D^{-1}\left(\frac{A_{t_1} B}{A}\right) + D^{-1}(B_{t_1}) = -\frac{A_t}{A} t_x - \frac{B_t}{A}$$

which gives to two equations

$$D^{-1}\left(\frac{A_{t_1}}{A}\right) = -\frac{A_t}{A}, \qquad D^{-1}\left(B_{t_1} - \frac{A_{t_1} B}{A}\right) = -\frac{B_t}{A}\,. \tag{2.48}$$

By the first equation of (2.48), we see that $\frac{A_t}{A}$ is a function that depends only on variable t, even though functions A and A_t depend on variables t and t_1. Let us denote $a(t) := \frac{A_t}{A}$. Then $\frac{A_{t_1}}{A} = -a(t_1)$. The last two equations imply that $A = T_1(t_1)e^{\tilde{a}(t)} = T_2(t)e^{-\tilde{a}(t_1)}$ for some functions $T_1(t_1)$ and $T_2(t)$ and $\tilde{a}(t) = \int_0^t a(\tau)d\tau$. We notice that $T_1(t_1)e^{\tilde{a}(t_1)} = T_2(t)e^{-\tilde{a}(t)}$ then we conclude that $A_1(t_1)e^{\tilde{a}(t_1)}$ is a constant. We denote $\gamma := A_1(t_1)e^{\tilde{a}(t_1)}$ and have $A_1(t) := e^{-\tilde{a}(t)}$ we have

$$A(t, t_1) = \gamma \frac{A_1(t_1)}{A_1(t)} \qquad \text{and therefore} \qquad f(t, t_1, t_x) = \gamma \frac{A_1(t_1)}{A_1(t)} t_x + B. \tag{2.49}$$

The second equation of (2.48) implies that

$$\frac{B_t}{A} = -\mu(t) \quad \text{and} \quad B_{t_1} - \frac{A_{t_1} B}{A} = \mu(t_1), \tag{2.50}$$

for some function $\mu(t)$. By using (2.49), the second equation in (2.50) can be rewritten as $B_{t_1} - \frac{A_1'(t_1)B}{A_1(t_1)} = \mu(t_1)$, or the same $\left\{\frac{B(t,t_1)}{A_1(t_1)}\right\}_{t_1} = \frac{\mu(t_1)}{A_1(t_1)}$. It means that

$$B(t,t_1) = A_1(t_1)B_1(t_1) + A_1(t_1)B_2(t), \tag{2.51}$$

for some functions $B_1(t_1)$ and $B_2(t)$. We substitute $B(t,t_1)$ from (2.51), $A(t,t_1)$ from (2.49) into the second equation of (2.50) and make all cancellations we have,

$$A_1(t_1)B_1'(t_1) = \mu(t_1), \quad \text{or the same,} \quad A_1(t)B_1'(t) = \mu(t)\,. \tag{2.52}$$

By substituting $A(t,t_1)$ from (2.49) and $B(t,t_1)$ from (2.51) into the first equation of (2.50) we have,

$$B_2'(t)A_1(t) = -\gamma\mu(t)\,. \tag{2.53}$$

We combine together (2.52) and (2.53), and we obtain that $B_2'(t)A_1(t) = -\gamma A_1(t)B_1'(t)$, or the same, $B_2'(t) = -\gamma B_1'(t)$, or $(B_2(t)+\gamma B_1(t))' = 0$, which implies that $B_2(t) = -\gamma B_1(t) + \eta$ for some constant η. Hence,

$$f(t,t_1,t_x) = \gamma\frac{A_1(t_1)}{A_1(t)}t_x + A_1(t_1)B_1(t_1) - \gamma A_1(t_1)B_1(t) + \eta A_1(t_1)\,. \tag{2.54}$$

Note that up to now we have only used properties (2.45) and (2.22). By using (2.46) and (2.22) we have $0 = \frac{\gamma A_1(t_1)}{A_1(t)}\{-B_1'(t)A_1(t) + A_1(t_1)B_1'(t_1)\}$ i.e. $-B_1'(t)A_1(t) + A_1(t_1)B_1'(t_1) = 0$. This implies that $B_1'(t)A_1(t) = c$, where c is some constant. We substitute $A_1(t) = \frac{c}{B_1'(t)}$ into (2.54) and get

$$f(t,t_1,t_x) = \gamma\frac{B_1'(t)}{B_1'(t_1)}t_x + c\frac{B_1(t_1)}{B_1'(t_1)} - \gamma c\frac{B_1(t)}{B'(t_1)} + \eta\frac{c}{B_1'(t_1)}\,. \tag{2.55}$$

Now use substitution $s = B_1(t)$, and equation (2.55) is reduced to $s_{1x} = \gamma s_x + cs_1 - c\gamma s + \eta c$. We introduce $\tilde{x} = cx$ to rewrite the last equation as $s_{1\tilde{x}} = \gamma s_{\tilde{x}} + s_1 - \gamma s + \eta$. If $\gamma = 1$ substitution $s = \tau - n\eta$ reduces the equation to $\tau_{1\tilde{x}} = \tau_{\tilde{x}} + \tau_1 - \tau$. If $\gamma \neq 1$, substitution $s = \gamma^n\tau + \eta\frac{\gamma^n - 1}{1-\gamma}$ reduces the equation to $\tau_{1\tilde{x}} = \tau_{\tilde{x}} + \tau_1 - \tau$. This finishes the proof of the Corollary. $\square$

Chapter 3

Equations Admitting Nontrivial x-integral

Almost all the materials in this Chapter comes from [26].

From now on we will study on a particular case of chain (2.4):

$$t_{1x} = f(t, t_1, t_x) = t_x + d(t, t_1). \tag{3.1}$$

The main result of this section, which is given by Theorem 3.1 below, is the complete list of chains (3.1) admitting nontrivial x-integrals.

Theorem 3.1 *Chain (3.1) admits a nontrivial x-integral if and only if $d(t, t_1)$ is one of the kind:*

(1) $d(t, t_1) = A(t - t_1)$,

(2) $d(t, t_1) = c_1(t - t_1)t + c_2(t - t_1)^2 + c_3(t - t_1)$,

(3) $d(t, t_1) = A(t - t_1)e^{\alpha t}$,

(4) $d(t, t_1) = c_4(e^{\alpha t_1} - e^{\alpha t}) + c_5(e^{-\alpha t_1} - e^{-\alpha t})$,

where $A = A(t - t_1)$ *is a function of* $\tau = t - t_1$ *and* c_1, c_2, c_3, c_4, c_5 *are some constants with* $c_1 \neq 0$, $c_4 \neq 0$, $c_5 \neq 0$, *and* α *is a nonzero constant. Moreover,* x*-integrals in each of the cases are*

i) $F = x + \int^{\tau} \frac{du}{A(u)}$, *if* $A(u) \neq 0$,
$F = t_1 - t$, *if* $A(u) = 0$,

ii) $F = \frac{1}{(-c_2-c_1)} \ln|(-c_2 - c_1)\frac{\tau_1}{\tau_2} + c_2| + \frac{1}{c_2} \ln|c_2 \frac{\tau_1}{\tau} - c_2 - c_1|$ *for* $c_2(c_2+c_1) \neq 0$,
$F = \ln \tau_1 - \ln \tau_2 + \frac{\tau_1}{\tau}$ *for* $c_2 = 0$,
$F = \frac{\tau_1}{\tau_2} - \ln \tau + \ln \tau_1$ *for* $c_2 = -c_1$,

iii) $F = \int^{\tau} \frac{e^{-\alpha u} du}{A(u)} - \int^{\tau_1} \frac{du}{A(u)}$,

iv) $F = \frac{(e^{\alpha t} - e^{\alpha t_2})(e^{\alpha t_1} - e^{\alpha t_3})}{(e^{\alpha t} - e^{\alpha t_3})(e^{\alpha t_1} - e^{\alpha t_2})}$.

3.1 The first integrability condition

In this section we use properly chosen sequence of multiple commutators to make a very rough classification about the function $d(t, t_1)$. Now let us see the process.

We define a class **F** of locally analytic functions each of which depends only on a finite number of dynamical variables. In particular we assume that the function $f(t, t_1, t_x) \in \mathbf{F}$. We will consider vector fields given as infinite formal series of the form

$$Y = \sum_{k=-\infty}^{\infty} y_k \frac{\partial}{\partial t_k} \tag{3.2}$$

with coefficients $y_k \in \mathbf{F}$. We introduce notions of linearly dependent and independent sets of the vector fields (3.2). We denote through P_N the projection operator acting according to the rule

$$P_N(Y) = \sum_{k=-N}^{N} y_k \frac{\partial}{\partial t_k}. \tag{3.3}$$

First we consider finite vector fields as

$$Z = \sum_{k=-N}^{N} z_k \frac{\partial}{\partial t_k}. \tag{3.4}$$

We say that a set of finite vector fields Z_1, Z_2, ..., Z_m is linearly dependent in some open region $\mathbf{U}$, if there is a set of functions μ_1, μ_2, ..., μ_m defined on $\mathbf{U}$ such that the function $|\mu_1|^2 + |\mu_2|^2 + ... + |\mu_m|^2$ does not vanish identically and the condition

$$\mu_1 Z_1 + \mu_2 Z_2 + ... + \mu_m Z_m = 0 \tag{3.5}$$

holds for each point of region $\mathbf{U}$.

We call a set of the vector fields Y_1, Y_2, ..., Y_m of the form (3.2) linearly dependent in the region $\mathbf{U}$ if for each natural N the following set of finite vector fields $P_N(Y_1)$, $P_N(Y_2)$, ..., $P_N(Y_m)$ is linearly dependent in this region. Otherwise we call the set Y_1, Y_2, ..., Y_m linearly independent in $\mathbf{U}$.

The following proposition is very useful, its proof is almost evident.

Proposition 3.2 *If a vector field Y is expressed as a linear combination*

$$Y = \mu_1 Y_1 + \mu_2 Y_2 + ... + \mu_m Y_m, \tag{3.6}$$

where the set of vector fields Y_1, Y_2, ..., Y_m is linearly independent in $\boldsymbol{U}$ and the coefficients of all the vector fields Y, Y_1, Y_2, ..,. Y_m belonging to $\boldsymbol{F}$ are defined in $\boldsymbol{U}$ then the coefficients μ_1, μ_2, ..., μ_m are in $\boldsymbol{F}$.

Below we focus on the class of chains of the form (3.1). For this special case the characteristic Lie algebra L_x splits down into a direct sum of two subalgebras. Indeed, since $f = t_x + d$ and $g = t_x - d_{-1}$ we get $f_k = t_x + d + \sum_{j=1}^{k} d_j$ and $g_{-k} = t_x - \sum_{j=1}^{k+1} d_{-k}$, for $k \geq 1$, where $d = d(t, t_1)$ and $d_j = d(t_j, t_{j+1})$. Due to this observation the vector field K_0 can be rewritten as $K_0 = t_x \tilde{X} + Y$, with

$$\tilde{X} = \frac{\partial}{\partial t} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_{-1}} + \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_{-2}} + \dots , \tag{3.7}$$

and

$$Y = \frac{\partial}{\partial x} + d\frac{\partial}{\partial t_1} - d_{-1}\frac{\partial}{\partial t_{-1}} + (d + d_1)\frac{\partial}{\partial t_2} - (d_{-1} + d_{-2})\frac{\partial}{\partial t_{-2}} + \dots .$$

Due to the relations $[X,\tilde{X}]=0$ and $[X,Y]=0$ we have $\tilde{X}=[X,K_0]\in L_x$, hence $Y\in L_x$. Therefore $L_x=\{X\}\bigoplus L_{x1}$, where L_{x1} is the Lie algebra generated by the operators $\tilde{X}$ and Y.

Lemma 3.3 *If equation (3.1) admits a nontrivial x-integral then it admits a nontrivial x-integral F such that* $\dfrac{\partial F}{\partial x}=0$.

Proof. Assume that a nontrivial x-integral of (3.1) exists. Then the Lie algebra L_{x1} is of finite dimension. We can choose a basis of L_{x1} in the form

$$T_1=\frac{\partial}{\partial x}+\sum_{k=-\infty}^{\infty}a_{1,k}\frac{\partial}{\partial t_k}\,,$$

$$T_j=\sum_{k=-\infty}^{\infty}a_{j,k}\frac{\partial}{\partial t_k}\,,\qquad 2\le j\le N.$$

Hence, there exists an x-integral F depending on the variables x, t, t_1, ..., t_{N-1} satisfying the system of equations

$$\frac{\partial F}{\partial x}+\sum_{k=0}^{N-1}a_{1,k}\frac{\partial F}{\partial t_k}=0\,,$$

$$\sum_{k=0}^{N-1}a_{j,k}\frac{\partial F}{\partial t_k}=0\,,\qquad 2\le j\le N.$$

Due to the famous Jacobi Theorem [16] there is a change of variables $\theta_j=\theta_j(t,t_1,\ldots,t_{N-1})$ that reduces the system to the form

$$\frac{\partial F}{\partial x}+\sum_{k=0}^{N-1}\tilde{a}_{1,k}\frac{\partial F}{\partial \theta_k}=0\,,$$

$$\frac{\partial F}{\partial \theta_k}=0\,,\qquad 2\le j\le N-2,$$

which is equivalent to

$$\frac{\partial F}{\partial x}+\tilde{a}_{1,N-1}\frac{\partial F}{\partial \theta_{N-1}}=0$$

for $F=F(x,\theta_{N-1})$.

Hence there are two possibilities:

1) $\tilde{a}_{1,N-1} = 0$,

2) $\tilde{a}_{1,N-1} \neq 0$.

In case 1), we automatically have $\dfrac{\partial F}{\partial x} = 0$. In case 2), we have $F = x + B(\theta_{N-1}) = x + B(t, t_1, \ldots, t_{N-1})$ for some function B. Evidently, $F_1 = DF = x + B(t_1, t_2, \ldots, t_N)$ is also an x-integral, and $F_1 - F$ is a nontrivial x-integral, which is not depending on the variable x. Hence $\frac{\partial F_1 - F}{\partial x} = 0$. This finishes the proof of the Lemma. □

Note that below we look for x-integrals F depending on dynamical variables t, $t_{\pm 1}$, $t_{\pm 2}$, $\ldots$ only (not depending on x). In other words, we study Lie algebra generated by vector fields $\tilde{X}$ and $\tilde{Y}$, where

$$\tilde{Y} = d\frac{\partial}{\partial t_1} - d_{-1}\frac{\partial}{\partial t_{-1}} + (d + d_1)\frac{\partial}{\partial t_2} - (d_{-1} + d_{-2})\frac{\partial}{\partial t_{-2}} + \ldots . \tag{3.8}$$

We can prove that the linear operator $Z \to DZD^{-1}$ defines an automorphism of the characteristic Lie algebra L_x. This automorphism is important for all of our further considerations. Further we refer to it as the shift automorphism. For instance, we have

$$D\tilde{X}D^{-1} = \tilde{X}, \qquad D\tilde{Y}D^{-1} = -d\tilde{X} + \tilde{Y}\,. \tag{3.9}$$

The proof of these statements are simple. Denote H and H^* as the functions $H(..., t_{-1}, t, t_1, ...)$ and $D^{-1}H = H(..., t_{-2}, t_{-1}, t, ...)$ correspondingly. We have

$$\begin{aligned} D\tilde{X}D^{-1}H &= D\tilde{X}H^* \\ &= D(H^*_t + H^*_{t_1} + H^*_{t_{-1}} + ...) \\ &= \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + ...\Big)H \\ &= \tilde{X}H, \end{aligned}$$

and similarly

$$\begin{aligned}
D\tilde{Y}D^{-1}H &= D\tilde{Y}H^* \\
&= D(dH^*_{t_1} - d_{-1}H^*_{t_{-1}} + H^*_t + (d+d_1)H^*_{t_2} + ...) \\
&= d_1H_{t_2} - dH_t + (d_1+d_2)H_{t_3} - (d+d_{-1})H_{t_{-1}} + ... \\
&\quad +(dH_{t_1} - dH_{t_1} + dH_{t_2} - dH_{t_2} + ...) \\
&= \Big\{\Big(d\frac{\partial}{\partial t_1} - d_{-1}\frac{\partial}{\partial t_{-1}} + (d+d_1)\frac{\partial}{\partial t_2} + ...\Big) - d\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_{-1}} + ...\Big)\Big\}H \\
&= -d\tilde{X} + \tilde{Y}.
\end{aligned}$$

Lemma 3.4 *Suppose that a vector field of the form $Z = \sum a(j)\frac{\partial}{\partial t_j}$ with the coefficients $a(j) = a(j, t, t_{\pm 1}, t_{\pm 2}, ...)$ depending on a finite number of the dynamical variables solves an equation of the form $DZD^{-1} = \lambda Z$. If for some $j = j_0$ we have $a(j_0) \equiv 0$ then $Z = 0$.*

Proof. We apply the shift automorphism to the vector field Z and we get $DZD^{-1} = \sum D(a(j))\frac{\partial}{\partial t_{j+1}}$. Now, we compare the coefficients of $\frac{\partial}{\partial t_j}$ in the equation $\sum D(a(j))\frac{\partial}{\partial t_{j+1}} = \lambda \sum a(j)\frac{\partial}{\partial t_j}$. If $\lambda = 0$, the vector field $Z = 0$ automatically. Assume that $\lambda \neq 0$. Then we have $D(\alpha(j)) = \lambda\alpha(j+1)$ for any j. Clearly, if for some $j = j_0$ we have $\alpha(j_0) = 0$, then all $\alpha(j) = 0$ for any j. Hence $Z = 0$. □

We construct an infinite sequence of multiple commutators of the vector fields $\tilde{X}$ and $\tilde{Y}$

$$\tilde{Y}_1 = [\tilde{X}, \tilde{Y}], \qquad \tilde{Y}_k = [\tilde{X}, \tilde{Y}_{k-1}] \quad \text{for} \quad k \geq 2\,. \tag{3.10}$$

Lemma 3.5 *We have,*

$$D\tilde{Y}_kD^{-1} = -\tilde{X}^k(d)\tilde{X} + \tilde{Y}_k, \quad k \geq 1. \tag{3.11}$$

Proof. We prove the statement by induction on k. The statement is true for $k = 1$. Indeed, by (3.9) and (3.10), we have

$$D\tilde{Y}_1D^{-1} = D[\tilde{X}, \tilde{Y}]D^{-1} = [D\tilde{X}D^{-1}, D\tilde{Y}D^{-1}] = [\tilde{X}, -d\tilde{X} + \tilde{Y}] = -\tilde{X}(d)\tilde{X} + \tilde{Y}_1.$$

Assume the equation (3.11) holds for $k = n-1$. We have

$$D\tilde{Y}_n D^{-1} = [D\tilde{X}D^{-1}, D\tilde{Y}_{n-1}D^{-1}] = [\tilde{X}, -\tilde{X}^{n-1}(d)\tilde{X} + \tilde{Y}_{n-1}] = -\tilde{X}^n(d)\tilde{X} + \tilde{Y}_n\,,$$

that finishes the proof of the Lemma. $\square$

Since vector fields X, $\tilde{X}$ and $\tilde{Y}$ are linearly independent, then the dimension of Lie algebra L_x is at least 3. By (3.11), if $\tilde{Y}_1 = 0$, we have $\tilde{X}(d) = 0$ i.e. $d_t + d_{t_1} = 0$ that implies $d = A(t - t_1)$, where $A(\tau)$ is an arbitrary differentiable function of one variable $\tau = t - t_1$.

Assume equation (3.1) admits a nontrivial x-integral and $\tilde{Y}_1 \neq 0$. Consider the sequence of the vector fields $\{\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \ldots\}$. Since L_x is of finite dimension, then there exists a natural number N such that

$$\tilde{Y}_{N+1} = \gamma_1\tilde{Y}_1 + \gamma_2\tilde{Y}_2 + \ldots + \gamma_N\tilde{Y}_N, \quad N \geq 1, \tag{3.12}$$

and $\tilde{Y}_1$, $\tilde{Y}_2$, …, $\tilde{Y}_N$ are linearly independent. Therefore, if we apply shift automorphism to both sides of (3.12) we get

$$D\tilde{Y}_{N+1}D^{-1} = D(\gamma_1)D\tilde{Y}_1D^{-1} + D(\gamma_2)D\tilde{Y}_2D^{-1} + \ldots + D(\gamma_N)D\tilde{Y}_ND^{-1}, \quad N \geq 1\,.$$

Due to Lemma 3.5 and the equation (3.12), the last equation can be rewritten as

$$-\tilde{X}^{N+1}(d)\tilde{X} + \gamma_1\tilde{Y}_1 + \gamma_2\tilde{Y}_2 + \ldots + \gamma_N\tilde{Y}_N =$$

$$= D(\gamma_1)(-\tilde{X}(d)\tilde{X} + \tilde{Y}_1) + D(\gamma_2)(-\tilde{X}^2(d)\tilde{X} + \tilde{Y}_2) + \ldots + D(\gamma_N)(-\tilde{X}^N(d)\tilde{X} + \tilde{Y}_N)\,.$$

We compare the coefficients before linearly independent vector fields $\tilde{X}$, $\tilde{Y}_1$, $\tilde{Y}_2$, … , $\tilde{Y}_N$, and we obtain the following system of equations

$$\begin{aligned}
&\tilde{X}^{N+1}(d) = D(\gamma_1)\tilde{X}(d) + D(\gamma_2)\tilde{X}^2(d) + \ldots + D(\gamma_N)\tilde{X}^N(d)\,,\\
&\gamma_1 = D(\gamma_1), \quad \gamma_2 = D(\gamma_2), \quad \ldots, \quad \gamma_N = D(\gamma_N)\,.
\end{aligned}$$

Since the coefficients of the vector fields $\tilde{Y}_j$ depend only on the variables $t, t_{\pm 1}, t_{\pm 2}, ...$ the factors γ_j might depend only on these variables by Proposition 3.2. Hence the system of equations implies that all coefficients γ_k, $1 \leq k \leq N$, are constants, and $d = d(t, t_1)$ is a function that satisfies the following differential equation

$$\tilde{X}^{N+1}(d) = \gamma_1\tilde{X}(d) + \gamma_2\tilde{X}^2(d) + \ldots + \gamma_N\tilde{X}^N(d)\,, \tag{3.13}$$

where $\tilde{X}(d) = d_t + d_{t_1}$. We use the substitution $s = t$ and $\tau = t - t_1$, so equation (3.13) can be rewritten as

$$\frac{\partial^{N+1} d}{\partial s^{N+1}} = \gamma_1 \frac{\partial d}{\partial s} + \gamma_2 \frac{\partial^2 d}{\partial s^2} + \ldots + \gamma_N \frac{\partial^N d}{\partial s^N}\,, \tag{3.14}$$

which implies

$$d(t, t_1) = \sum_k \left(\sum_{j=0}^{m_k - 1} \lambda_{k,j}(t - t_1) t^j \right) \mathrm{e}^{\alpha_k t}\,, \tag{3.15}$$

for some functions $\lambda_{k,j}(t - t_1)$, where α_k are roots of multiplicity m_k for characteristic equation of (3.14).

Let $\alpha_0 = 0$, α_1, …, α_i be the distinct roots of the characteristic equation (3.13). Equation (3.13) can be rewritten as

$$\Lambda(\tilde{X})d := \tilde{X}^{m_0}(\tilde{X} - \alpha_1)^{m_1}(\tilde{X} - \alpha_2)^{m_2} \ldots (\tilde{X} - \alpha_i)^{m_i} d = 0\,, \tag{3.16}$$

and $m_0 + m_1 + \ldots + m_i = N + 1$, $m_0 \geq 1$.

Initiated by the formula (3.8), we define a map $h \to Y_h$, which assigns to any function $h = h(t, t_{\pm 1}, t_{\pm 2}, ...)$ a vector field

$$Y_h = h\frac{\partial}{\partial t_1} - h_{-1}\frac{\partial}{\partial t_{-1}} + (h + h_1)\frac{\partial}{\partial t_2} - (h_{-1} + h_{-2})\frac{\partial}{\partial t_{-2}} + \ldots .$$

For any polynomial with constant coefficients $P(\lambda) = c_0 + c_1\lambda + ... + c_m\lambda^m$ we have a formula

$$P(ad_{\tilde{X}})\tilde{Y} = Y_{P(\tilde{X})h}, \quad \text{where} \quad ad_X Y = [X, Y], \tag{3.17}$$

which defines an isomorphism between the linear space V of all solutions of equation (3.14) and the linear space $\tilde{V} = \text{span}\{\tilde{Y}, \tilde{Y}_1, ..., \tilde{Y}_N\}$ of the corresponding vector fields.

Represent the function (3.15) as a sum $d(t, t_1) = P(t, t_1) + Q(t, t_1)$ of the polynomial part $P(t, t_1) = \sum_{j=0}^{m_0 - 1} \lambda_{0,j}(t - t_1)t^j$ and the "exponential" part $Q(t, t_1) = \sum_{k=1}^{s} \left(\sum_{j=0}^{m_k - 1} \lambda_{k,j}(t - t_1)t^j \right) \mathrm{e}^{\alpha_k t}$. The following Lemma proves that the function $d(t, t_1)$ is either in the form $P(t, t_1)$ or $Q(t, t_1)$.

Lemma 3.6 *Assume equation (3.1) admits a nontrivial x-integral. Then one of the functions $P(t,t_1)$ and $Q(t,t_1)$ vanishes.*

Proof. Assume in contrary that neither of the functions vanish. Firstly, we prove that in this case algebra L_x contains vector fields $T_0 = Y_{A(\tau)\mathrm{e}^{\alpha_k t}}$ and $T_1 = Y_{B(\tau)}$ for some functions $A(\tau)$ and $B(\tau)$, $\tau = t - t_1$. Let us take $T_0 := \Lambda_0(ad_{\tilde{X}})\tilde{Y} = Y_{\Lambda_0(\tilde{X})d} \in L_x$, where

$$\Lambda_0(\lambda) = \frac{\Lambda(\lambda)}{\lambda - \alpha_k} = \lambda^{m_0}(\lambda-\alpha_1)^{m_1}...(\lambda-\alpha_k)^{m_k-1}(\lambda-\alpha_i)^{m_i}.$$

Clearly, the function $\tilde{A}(t,t_1) = \Lambda_0(\tilde{X})d$ solves the equation $(\tilde{X}-\alpha_k)\tilde{A}(t,t_1) = \Lambda(\tilde{X})d = 0$ hence $\tilde{A}(t,t_1) = A(\tau)\mathrm{e}^{\alpha_k t}$. Now take $T_1 := \Lambda_0'(ad_{\tilde{X}})\tilde{Y} = Y_{\Lambda_0'(\tilde{X})d} \in L_x$, where

$$\Lambda_0'(\lambda) = \frac{\Lambda(\lambda)}{\lambda} = \lambda^{m_0-1}(\lambda-\alpha_1)^{m_1}...(\lambda-\alpha_i)^{m_i}.$$

Evidently, the function $\tilde{B}(t,t_1) = \Lambda_0'(\tilde{X})d = 0$ solves the equation $\tilde{X}\tilde{B}(t,t_1) = \Lambda(\tilde{X})d = 0$, which implies $\tilde{B}(t,t_1) = B(\tau)$. Note that due to our assumption the functions $A(\tau)$ and $B(\tau)$ cannot vanish identically.

Consider an infinite sequence of the vector fields defined as follows

$$T_2 = [T_0, T_1], \quad T_3 = [T_0, T_2], \quad \dots, \quad T_n = [T_0, T_{n-1}], \quad n \geq 2,$$

where T_0 and T_1 are written explicitly as

$$T_0 = A(\tau)e^{\alpha_k t}\frac{\partial}{\partial t_1} - A(\tau_{-1})e^{\alpha_k t_{-1}}\frac{\partial}{\partial t_{-1}} + \{A(\tau)e^{\alpha_k t} + A(\tau_1)e^{\alpha_k t_1}\}\frac{\partial}{\partial t_2} + ...,$$

$$T_1 = B(\tau)\frac{\partial}{\partial t_1} - B(\tau_{-1})\frac{\partial}{\partial t_{-1}} + \{B(\tau) + B(\tau_1)\}\frac{\partial}{\partial t_2} + ...$$

We can show that

$$[\tilde{X}, T_0] = \alpha_k T_0, \quad [\tilde{X}, T_1] = 0, \quad [\tilde{X}, T_n] = \alpha_k(n-1)T_n, \quad n \geq 2,$$

and

$$DT_0D^{-1} = -A\mathrm{e}^{\alpha_k t}\tilde{X} + T_0, \quad DT_1D^{-1} = -B\tilde{X} + T_1,$$

$$DT_nD^{-1} = T_n - \frac{(n-1)(n-2)}{2}\alpha_k A\mathrm{e}^{\alpha_k t}T_{n-1} + b_n\tilde{X} + \sum_{k=0}^{n-2} a_k^{(n)}T_k, \quad n \geq 2.$$

For the reader's convenience let us prove the statements. First two equalities from the first group are clear indeed. The third one can be proved by induction. Base of induction holds. Assume that it is satisfied for any $n \geq 2$ and prove it for $n+1$. We have

$$\begin{aligned}
[\tilde{X}, T_{n+1}] &= [\tilde{X}, [T_0, T_n]] \\
&= -[T_0, [T_n, \tilde{X}]] - [T_n, [\tilde{X}, T_0]] \\
&= [T_0, \alpha_k(n-1)T_n] - [T_n, \alpha_k T_0] \\
&= \alpha_k n T_{n+1}.
\end{aligned}$$

Now we prove the second group of the equations. Use the same notation for H and H^* as before. We have

$$\begin{aligned}
DT_0D^{-1}H &= DT_0H^* \\
&= D\{A(\tau)e^{\alpha_k t}H^*_{t_1} - A(\tau_{-1})e^{\alpha_k t_{-1}}H^*_{t_{-1}} + ...\} \\
&= A(\tau_1)e^{\alpha_k t_1}H_{t_2} - A(\tau)e^{\alpha_k t}H_t + ... \\
&\quad +(A(\tau)e^{\alpha_k t}H_{t_1} - A(\tau)e^{\alpha_k t}H_{t_1} + A(\tau)e^{\alpha_k t}H_{t_2} - A(\tau)e^{\alpha_k t}H_{t_2} + ...) \\
&= -A(\tau)e^{\alpha_k t}\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + ...\Big) + T_0 \\
&= -A(\tau)e^{\alpha_k t}\tilde{X} + T_0.
\end{aligned}$$

In a similar way, we can prove that $DT_1D^{-1} = -B\tilde{X} + T_1$. By mathematical induction it is also easy to prove the last equality of second group.

Since algebra L_x is of finite dimension then there exists number N such that

$$T_{N+1} = \lambda\tilde{X} + \mu_0T_0 + \mu_1T_1 + \ldots + \mu_NT_N, \tag{3.18}$$

and vector fields $\tilde{X}$, T_0, T_1, $\ldots$, T_N are linearly independent. We apply shift automorphism to both sides of (3.18) and we have

$$\begin{aligned}
DT_{N+1}D^{-1} &= D(\lambda)\tilde{X} + D(\mu_0)\{-A\mathrm{e}^{\alpha_k t}\tilde{X} + T_0\} + \ldots \\
&\quad +D(\mu_N)\Big\{T_N - \frac{(N-1)(N-2)}{2}\alpha_k A\mathrm{e}^{\alpha_k t}T_{N-1} + \ldots\Big\}.
\end{aligned}$$

We compare the coefficients before T_N in the last equation we get

$$\mu_N - \frac{N(N-1)}{2}\alpha_k A(\tau)\mathrm{e}^{\alpha_k t} = D(\mu_N). \tag{3.19}$$

It follows that μ_N is a function of variable t only. Also, we apply $ad_{\tilde{X}}$ to both sides of the equation (3.18), we get

$$N\alpha_k T_{N+1} = [\tilde{X}, T_{N+1}] = \tilde{X}(\lambda)\tilde{X}+(\tilde{X}(\mu_0)+\mu_0\alpha_k)T_0+\ldots+(\tilde{X}(\mu_N)+\mu_N(N-1)\alpha_k)T_N.$$

Again, by comparing coefficients before T_N, we have

$$N\alpha_k\mu_N = \tilde{X}(\mu_N) + (N-1)\alpha_k\mu_N, \quad \text{i.e.,} \quad \tilde{X}(\mu_N) = \alpha_k\mu_N.$$

Therefore, $\mu_N = A_1 e^{\alpha_k t}$, where A_1 is some nonzero constant, and thus from (3.19) we get $A(\tau)e^{\alpha_k t} = A_2 e^{\alpha_k t} - A_2 e^{\alpha_k t_1}$. Here A_2 is some constant. We have, $T_0 = A_2 e^{\alpha_k t}\tilde{X} - A_2 S_0$, where

$$S_0 = \sum_{j=-\infty}^{\infty} e^{\alpha_k t_j}\frac{\partial}{\partial t_j} = ... + e^{\alpha_k t_{-1}}\frac{\partial}{\partial t_{-1}} + e^{\alpha_k t}\frac{\partial}{\partial t} + e^{\alpha_k t_1}\frac{\partial}{\partial t_1} + ...$$

It is clear that we have

$$[\tilde{X}, S_0] = \alpha_k S_0, \quad DS_0D^{-1} = S_0. \tag{3.20}$$

Consider a new sequence of vector fields

$$P_1 = S_0, \quad P_2 = [T_1, S_0], \quad P_3 = [T_1, P_2], \quad P_n = [T_1, P_{n-1}], \quad n \geq 3\,.$$

By induction we can prove the following equalities.

$$[\tilde{X}, P_n] = \alpha_k P_n, \quad DP_nD^{-1} = P_n - \alpha_k(n-1)BP_{n-1} + b_n\tilde{X} + a_nS_0 + \sum_{j=2}^{n-2} a_j^{(n)}P_j, \quad n \geq 2\,.$$

Since algebra L_x is of finite dimension, then there exists number M such that

$$P_{M+1} = \lambda^*\tilde{X} + \mu_2^*P_2 + \ldots + \mu_M^*P_M, \tag{3.21}$$

and $\tilde{X}$, P_2, …, P_M are linearly independent. We apply shift automorphism to both sides of (3.21) and we have

$$DP_{M+1}D^{-1} = D(\lambda^*)\tilde{X} + D(\mu_2^*)\{P_2+\ldots\}+\ldots+D(\mu_M^*)\{P_M - \alpha_k(M-1)BP_{M-1}+\ldots\}.$$

We compare the coefficients before P_M in the last equation and get

$$\mu_M^* - M\alpha_k B(\tau) = D(\mu_M^*), \tag{3.22}$$

which implies that μ_M^* is a function of variable t only. Also, we apply $ad_{\tilde{X}}$ to both sides of (3.21) and get

$$\alpha_k P_{M+1} = [\tilde{X}, P_{M+1}] = \tilde{X}(\lambda^*)\tilde{X} + (\tilde{X}(\mu_2^*) + \alpha_k\mu_2^*)P_2 + \ldots + (\tilde{X}(\mu_M^*) + \alpha_k\mu_M^*)P_M.$$

Again, we compare the coefficients before P_M and have

$$\alpha_k\mu_M^*(t) = \tilde{X}(\mu_M^*(t)) + \alpha_k\mu_M^*(t),$$

which yields that μ_M^* is a constant. It follows then from (3.22) that the function $B(\tau) = 0$. This contradiction shows that our assumption that both functions are not identically zero was wrong. This finishes the proof of the Lemma. □

3.2 Multiple zero root

In this section we assume that equation (3.1) admits a nontrivial x-integral and that $\alpha_0 = 0$ is a root of the characteristic polynomial $\Lambda(\lambda)$. Then, due to Lemma 3.6, zero is the only root and therefore $\Lambda(\lambda) = \lambda^{m+1}$. From the formula (3.15) with multiplicity $m_0 = m + 1$, we have

$$d(t, t_1) = a(\tau)t^m + b(\tau)t^{m-1} + \ldots, \quad m = m_0 - 1 \geq 0.$$

If $m = 0$, then we get a very simple equation $t_{1x} = t_x + A(t - t_1)$, which is easily solved in quadratures. So we concentrate on the case $m \geq 1$. For this case the characteristic Lie algebra L_x contains a vector field $T = Y_{\tilde{\kappa}}$ with

$$\tilde{\kappa} = a(\tau)t + \frac{1}{m}b(\tau).$$

Indeed,

$$T = \frac{1}{m!}ad_{\tilde{X}}^{m-1}\tilde{Y} = Y_{\tilde{\kappa}} = \tilde{\kappa}\frac{\partial}{\partial t_1} - \tilde{\kappa}_{-1}\frac{\partial}{\partial t_{-1}} + (\tilde{\kappa} + \tilde{\kappa}_1)\frac{\partial}{\partial t_2} + \ldots \in L_x\,. \quad (3.23)$$

Consider a sequence of multiple commutators defined as follows

$$T_0 = \tilde{X}, \quad T_1 = [T, T_0] = Y_{-a(\tau)}, \quad T_{k+1} = [T, T_k], \quad k \geq 0, \quad T_{k,0} = [T_0, T_k].$$

Note that $T_{1,0} = 0$. We will see below that the linear space spanned by this sequence is not invariant under the action of the shift automorphism $Z \to DZD^{-1}$

introduced above. We extend the sequence to provide the invariance property. We define T_α with the multi-index α. For any sequence $\alpha = k, 0, i_1, i_2, \ldots, i_{n-1}, i_n$, where k is any natural number, $i_j \in \{0; 1\}$, denote

$$T_\alpha = \begin{cases} \left[T_0, T_{k,0,i_1,\ldots,i_{n-1}}\right], & \text{if} \quad i_n = 0; \\ \\ \left[T, T_{k,0,i_1,\ldots,i_{n-1}}\right], & \text{if} \quad i_n = 1; \end{cases}$$

$$m(\alpha) = \begin{cases} k, & \text{if} \quad \alpha = k; \\ k, & \text{if} \quad \alpha = k, 0; \\ k + i_1 + \ldots + i_n, & \text{if} \quad \alpha = k, 0, i_1, \ldots, i_n; \end{cases}$$

$$l(\alpha) = k + n + 1 - m(\alpha).$$

The multi-index α is characterized by two quantities $m(\alpha)$ and $l(\alpha)$ which allow to order partially the sequence $\{T_\alpha\}$.

Lemma 3.7 *We have,*

$$DT_0D^{-1} = T_0, \quad DTD^{-1} = T - \tilde{\kappa}T_0, \quad DT_1D^{-1} = T_1 + aT_0.$$

The first equality has been proved in Section 3.1. The others are also straightforward.

We can prove by induction on k that

$$DT_kD^{-1} = T_k + aT_{k-1} - \tilde{\kappa} \sum_{m(\beta)=k-1} T_\beta + \sum_{m(\beta)\le k-2} \eta(k,\beta)T_\beta\,. \tag{3.24}$$

In general, for any α,

$$DT_\alpha D^{-1} = T_\alpha + \sum_{m(\beta)\le m(\alpha)-1} \eta(\alpha,\beta)T_\beta\,. \tag{3.25}$$

We can choose a system P of linearly independent vector fields in the following way.

1) T and T_0 are linearly independent. We take them into P.

2) We check whether T, T_0 and T_1 are linearly independent or not. If they are dependent then $P = \{T, T_0\}$ and $T_1 = \mu T + \lambda T_0$ for some functions μ and λ.

3) If T, T_0, T_1 are linearly independent then we check whether T, T_0, T_1, T_2 are linearly independent or not. If they are dependent, then $P = \{T, T_0, T_1\}$.
4) If T, T_0, T_1, T_2 are linearly independent, we add vector fields T_β, $m(\beta) = 2$, $\beta \in I_2$, (actually, by definition I_2 is the collection of such β) in such a way that $J_2 := \{T, T_0, T_1, T_2, \cup_{\beta \in I_2} T_\beta\}$ is a system of linearly independent vector fields and for any T_γ with $m(\gamma) \leq 2$ we have $T_\gamma = \sum\limits_{T_\beta \in J_2} \mu(\gamma, \beta) T_\beta$.
5) We check whether $T_3 \cup J_2$ is a linearly independent system. If it is not, then P consists of all elements from J_2, and $T_3 = \sum\limits_{T_\beta \in J_2} \mu(\gamma, \beta) T_\beta$. If it is, then to the system $T_3 \cup J_2$ we add vector fields T_β, $m(\beta) = 3$, $\beta \in I_3$, in such a way that $J_3 := \{T_3, J_2, \cup_{\beta \in I_3} T_\beta\}$ is a system of linearly independent vector fields and for any T_γ with $m(\gamma) \leq 3$ we have $T_\gamma = \sum\limits_{T_\beta \in J_3} \mu(\gamma, \beta) T_\beta$.
We continue the construction of the system P. Since L_x is of finite dimension, then there exists such a natural number N that
(i) $T_k \in P$, $k \leq N$;
(ii) $m(\beta) \leq N$ for any $T_\beta \in P$;
(iii) for any T_γ with $m(\gamma) \leq N$ we have $T_\gamma = \sum\limits_{T_\beta \in P, m(\beta) \leq m(\gamma)} \mu(\gamma, \beta) T_\beta$ and also

$$T_{N+1} = \mu(N+1, N) T_N + \sum_{T_\beta \in P, m(\beta) \leq N} \mu(N+1, \beta) T_\beta.$$

It follows that
(iv) for any vector field T_α with $m(\alpha) = N$, that does not belong to P, the coefficient $\mu(\alpha, N)$ before T_N in the expansion

$$T_\alpha = \mu(\alpha, N) T_N + \sum_{T_\beta \in P} \mu(\alpha, \beta) T_\beta \tag{3.26}$$

is constant. Indeed, by (3.25),

$$DT_\alpha D^{-1} = T_\alpha + \sum_{m(\beta) \leq N-1} \eta(\alpha, \beta) T_\beta = \mu(\alpha, N) T_N + \sum_{T_\beta \in P} \mu(\alpha, \beta) T_\beta + \sum_{m(\beta) \leq N-1} \eta(\alpha, \beta) T_\beta\,.$$

From (3.26) we have also

$$\begin{aligned} DT_\alpha D^{-1} &= D(\mu(\alpha, N)) DT_N D^{-1} + \sum_{T_\beta \in P} D(\mu(\alpha, \beta)) DT_\beta D^{-1} \\ &= D(\mu(\alpha, N)) \{T_N + \ldots\} + \sum_{T_\beta \in P} D(\mu(\alpha, \beta)) \{T_\beta + \ldots\}\,. \end{aligned}$$

We compare the coefficients before T_N in these two expressions for $DT_\alpha D^{-1}$, we have

$$\mu(\alpha, N) = D(\mu(\alpha, N)),$$

which implies that $\mu(\alpha, N)$ is a constant.

Lemma 3.8 *We have, $a(\tau) = c_0\tau + c_1$, where c_0 and c_1 are some constants.*

Proof. Since

$$T_{N+1} = \mu(N+1, N)T_N + \sum_{T_\beta \in P} \mu(N+1, \beta)T_\beta\,,$$

then

$$DT_{N+1}D^{-1} = D(\mu(N+1, N))\{T_N + \ldots\} + \sum_{T_\beta \in P} D(\mu(N+1, \beta))\{T_\beta + \ldots\}.$$

From (3.25), we also have

$$DT_{N+1}D^{-1} = T_{N+1} + aT_N - \tilde{\kappa} \sum_{m(\beta)=N} T_\beta + \sum_{m(\beta)\leq N-1} \eta(N+1, \beta)T_\beta\,.$$

We compare the coefficients before T_N in the last two expressions. For $N \geq 0$ the equation is

$$\mu(N+1, N) + a - \tilde{\kappa} \sum_{T_\beta \in P, m(\beta)=N} \mu(\beta, N) = D(\mu(N+1, N))\,. \tag{3.27}$$

Denote by $c = -\sum_{T_\beta \in P, m(\beta)=N} \mu(\beta, N)$ and by $\mu_N = \mu(N+1, N)$. By property (iv), c is a constant. It follows from (3.27) that μ_N is a function of variables t and n only. Hence,

$$a(\tau) + c\left(a(\tau)t + \frac{1}{m}b(\tau)\right) = \mu_N(t_1, n+1) - \mu_N(t, n).$$

We differentiate both sides of the equation with respect to t and then t_1, we have

$$-a''(\tau) - c\left(a''(\tau)t + a'(\tau) + \frac{1}{m}b''(\tau)\right) = 0,$$

which implies that $a''(\tau)=0$, or the same, $a(\tau)=c_0\tau+c_1$ for some constants c_0 and c_1. $\square$

We rewrite the vector fields T_1 and T in new variables as

$$T_1=\sum_{j=-\infty}^{\infty} a(\tau_j)\frac{\partial}{\partial\tau_j}=\sum_{j=-\infty}^{\infty}(c_0\tau_j+c_1)\frac{\partial}{\partial\tau_j}\,, \tag{3.28}$$

$$\begin{aligned} T=-\sum_{j=-\infty}^{\infty}\{a(\tau_j)t_j+\frac{1}{m}b(\tau_j)\}\frac{\partial}{\partial\tau_j} &= -\sum_{j=-\infty}^{\infty}\{a(\tau_j)(t+\rho_j)+\frac{1}{m}b(\tau_j)\}\frac{\partial}{\partial\tau_j} \\ &= -tT_1-\sum_{j=-\infty}^{\infty}\{a(\tau_j)\rho_j+\frac{1}{m}b(\tau_j)\}\frac{\partial}{\partial\tau_j}\,, \end{aligned} \tag{3.29}$$

where

$$\rho_j=\begin{cases} -\tau-\tau_1-\ldots-\tau_{j-1}, & \text{if} \quad j\geq 1; \\ 0, & \text{if} \quad j=0; \\ \tau_{-1}+\tau_{-2}+\ldots+\tau_j, & \text{if} \quad j\leq -1\,. \end{cases}$$

The following two Lemmas will be very useful for us.

Lemma 3.9 *If the Lie algebra generated by the vector fields* $S_0=\sum\limits_{j=-\infty}^{\infty}\frac{\partial}{\partial w_j}$ *and* $P=\sum\limits_{j=-\infty}^{\infty}c(w_j)\frac{\partial}{\partial w_j}$ *is of finite dimension then* $c(w)$ *is one of the forms*
(1) $c(w)=c_2+c_3\mathrm{e}^{\lambda w}+c_4\mathrm{e}^{-\lambda w}$, $\lambda\neq 0$;
(2) $c(w)=c_2+c_3w+c_4w^2$, *where* c_2, c_3, c_4 *are some constants.*

Proof. Introduce vector fields

$$S_1=[S_0,P], \quad S_2=[S_0,S_1], \quad \ldots, \quad S_n=[S_0,S_{n-1}], \quad n\geq 3.$$

Clearly, we have

$$S_n=\sum_{j=-\infty}^{\infty}c^{(n)}(w_j)\frac{\partial}{\partial w_j}, \quad n\geq 1. \tag{3.30}$$

Since all vector fields S_n are elements of L_x, and L_x is of finite dimension, then there exists a natural number N such that

$$S_{N+1} = \mu_N S_N + \mu_{N-1} S_{N-1} + ... + \mu_1 S_1 + \mu_0 P + \mu S_0, \tag{3.31}$$

and $S_0, P, S_1, ..., S_N$ are linearly independent. (Note that we may assume S_0 and P are linearly independent). We have

$$DS_0D^{-1} = S_0, DPD^{-1} = P \text{and} DS_nD^{-1} = S_n$$

for any $n \geq 1$. Then it follows from (3.31) that

$$S_{N+1} = D(\mu_N)S_N + D(\mu_{N-1})S_{N-1} + ... + D(\mu_1)S_1 + D(\mu_0)P + D(\mu)S_0.$$

But we know the expression for S_{N+1} by (3.31). So the above equation gives that $\mu, \mu_0, \mu_1, ..., \mu_N$ are all constants.

We compare the coefficients before $\frac{\partial}{\partial w}$ in (3.31) we get, with the help of (3.30), the following equality

$$c^{(N+1)}(w) = \mu_N c^{(N)}(w) + ... + \mu_1 c'(w) + \mu_0 c(w) + \mu.$$

Thus, $c(w)$ is a solution of the nonhomogeneous linear differential equation with constant coefficient whose characteristic polynomial is

$$\Lambda(\lambda) = \lambda^{N+1} - \mu_N \lambda^N - ... - \mu_1 \lambda - \mu_0.$$

Denote by $\beta_1, \beta_2, ..., \beta_t$ characteristic roots and by $m_1, m_2, ..., m_t$ their multiplicities. There are four possibilities:

(i) There exists a nonzero characteristic root, say β_1, and its multiplicity $m_1 \geq 2$,

(ii) There exists zero characteristic root, say β_1, and $m_1 \geq 3$, $\mu = 0$ or $m_1 \geq 2$, $\mu \neq 0$,

(iii) There are two distinct characteristic roots, say β_1 and β_2 with $\beta_1 \neq 0$, $\beta_2 = 0$,

(iv) There are two nonzero distinct characteristic roots, say β_1 and β_2.

Now we will analyze these cases.

In case (i), consider

$$\Lambda_1(\lambda) = \frac{\Lambda(\lambda)}{\lambda - \beta_1} \quad \text{and} \quad \Lambda_1^{(2)}(\lambda) = \frac{\Lambda(\lambda)}{(\lambda - \beta_1)^2}.$$

Then $\Lambda_1(S_0)c(w) = \alpha_1 \mathrm{e}^{\beta_1 w} + \alpha_2$ and $\Lambda_1^{(2)}(S_0)c(w) = (\alpha_3 w + \alpha_4)\mathrm{e}^{\beta_1 w} + \alpha_5$, where α_j, $1 \le j \le 5$, are some constants with $\alpha_1 \ne 0$, $\alpha_3 \ne 0$. We have,

$$\begin{aligned}
\Lambda_1(ad_{S_0})P &= \sum_{j=-\infty}^{\infty} (\alpha_1 \mathrm{e}^{\beta_1 w_j} + \alpha_2)\frac{\partial}{\partial w_j} \\
&= \alpha_1\Big(\sum_{j=-\infty}^{\infty} \mathrm{e}^{\beta_1 w_j}\frac{\partial}{\partial w_j}\Big) + \alpha_2 S_0 = \alpha_1 P_1 + \alpha_2 S_0, \\
\Lambda_1^{(2)}(ad_{S_0})P &= \sum_{j=-\infty}^{\infty} ((\alpha_3 w_j + \alpha_4)e^{\beta_1 w_j} + \alpha_5)\frac{\partial}{\partial w_j} = \alpha_3\Big(\sum_{j=-\infty}^{\infty} w_j \mathrm{e}^{\beta_1 w_j}\frac{\partial}{\partial w_j}\Big) \\
&\quad + \alpha_4 P_1 + \alpha_5 S_0 \\
&= \alpha_3 P_2 + \alpha_4 P_1 + \alpha_5 S_0
\end{aligned}$$

are in L_x and therefore vector fields $P_1 = \sum_{j=-\infty}^{\infty} \mathrm{e}^{\beta_1 w_j}\frac{\partial}{\partial w_j}$ and $P_2 = \sum_{j=-\infty}^{\infty} w_j \mathrm{e}^{\beta_1 w_j}\frac{\partial}{\partial w_j}$ belong to L_x. Since P_1 and P_2 generate an infinite dimensional Lie algebra L_x then case (i) fails to be true.

In case (ii), consider

$$\Lambda_1^{(3)}(\lambda) = \frac{\Lambda(\lambda)}{\lambda^3} \quad \text{and} \quad \Lambda_1^{(2)}(\lambda) = \frac{\Lambda(\lambda)}{\lambda^2}, \quad \text{if} \quad \mu = 0,$$

or

$$\Lambda_1^{(3)}(\lambda) = \frac{\Lambda(\lambda)}{\lambda^2} \quad \text{and} \quad \Lambda_1^{(2)}(\lambda) = \frac{\Lambda(\lambda)}{\lambda}, \quad \text{if} \quad \mu \ne 0.$$

We have

$$\Lambda_1^{(3)}(S_0)c(w) = \alpha_1 w^3 + \alpha_2 w^2 + \alpha_3 w + \alpha_4 \quad \text{and} \quad \Lambda_1^{(2)}(S_0)c(w) = \alpha_5 w^2 + \alpha_6 w + \alpha_7,$$

where α_j, $1 \le j \le 7$, are some constants with $\alpha_1 \ne 0$, $\alpha_5 \ne 0$. Direct calculations show that vector fields

$$\Lambda_1^{(3)}(ad_{S_0})P = \sum_{j=-\infty}^{\infty} (\alpha_1 w_j^3 + \alpha_2 w_j^2 + \alpha_3 w_j + \alpha_4)\frac{\partial}{\partial w_j},$$

and

$$\Lambda_1^{(2)}(ad_{S_0})P = \sum_{j=-\infty}^{\infty}(\alpha_5 w_j^2 + \alpha_6 w_j + \alpha_7)\frac{\partial}{\partial w_j}$$

generate an infinite dimensional Lie algebra. It proves that case (ii) fails to be true.

In case (iii), consider

$$\Lambda_1(\lambda) = \frac{\Lambda(\lambda)}{\lambda - \beta_1} \quad \text{and} \quad \Lambda_2(\lambda) = \frac{\Lambda(\lambda)}{\lambda}.$$

We have

$$\Lambda_1 c(w) = \alpha_1 \mathrm{e}^{\beta_1 w} + \alpha_2 \quad \text{and} \quad \Lambda_2 c(w) = \alpha_3 w + \alpha_4, \quad \text{if} \quad \mu = 0,$$

or

$$\Lambda_1(S_0)c(w) = \alpha_1 \mathrm{e}^{\beta_1 w} + \alpha_2 \quad \text{and} \quad \Lambda_2(S_0)c(w) = \alpha_5 w^2 + \alpha_6 w + \alpha_7, \quad \text{if} \quad \mu \neq 0,$$

where α_j, $1 \le j \le 7$, are constants with $\alpha_1 \neq 0$, $\alpha_3 \neq 0$, $\alpha_5 \neq 0$. Since vector fields $\Lambda_1(ad_{S_0})P$ and $\Lambda_2(ad_{S_0})P$ generate an infinite dimensional Lie algebra, then case (iii) also fails to exist.

In case (iv), consider

$$\Lambda_1(\lambda) = \frac{\Lambda(\lambda)}{\lambda - \beta_1} \quad \text{and} \quad \Lambda_2(\lambda) = \frac{\Lambda(\lambda)}{\lambda - \beta_2}.$$

We have, $\Lambda_1(S_0)c(w) = \alpha_1 \mathrm{e}^{\beta_1 w} + \alpha_2$, $\Lambda_2(S_0)c(w) = \alpha_3 \mathrm{e}^{\beta_2 w} + \alpha_4$, where $\alpha_1 \neq 0$, α_2, $\alpha_3 \neq 0$, α_4 are some constants. Note that

$$\Lambda_1(ad_{S_0})P = \alpha_1\Big(\sum_{j=-\infty}^{\infty} \mathrm{e}^{\beta_1 w_j}\frac{\partial}{\partial w_j}\Big) + \alpha_2 S_0,$$

and

$$\Lambda_2(ad_{S_0})P = \alpha_3\Big(\sum_{j=-\infty}^{\infty} \mathrm{e}^{\beta_2 w_j}\frac{\partial}{\partial w_j}\Big) + \alpha_4 S_0,$$

and vector fields $\sum_{j=-\infty}^{\infty} \mathrm{e}^{\beta_1 w_j}\frac{\partial}{\partial w_j}$ and $\sum_{j=-\infty}^{\infty} \mathrm{e}^{\beta_2 w_j}\frac{\partial}{\partial w_j}$ generate an infinite dimensional Lie algebra if $\beta_1 + \beta_2 \neq 0$.

It follows from (i), (ii), (iii), (iv) that $c(w)$ can only be one of the forms
(1) $c(w) = c_2 + c_3\mathrm{e}^{\lambda w} + c_4\mathrm{e}^{-\lambda w}$, $\lambda \neq 0$;
(2) $c(w) = c_2 + c_3 w + c_4 w^2$, where c_2, c_3, c_4 are some constants. $\square$

Lemma 3.10 *If the Lie algebra generated by the vector fields* $S_0 = \sum\limits_{j=-\infty}^{\infty} \frac{\partial}{\partial w_j}$, $Q = \sum\limits_{j=-\infty}^{\infty} q(w_j)\frac{\partial}{\partial w_j}$ *and* $S_1 = \sum\limits_{j=-\infty}^{\infty} \{\tilde{\rho}_j + \tilde{b}(w_j)\}\frac{\partial}{\partial w_j}$ *is of finite dimension then* $q(w)$ *is a constant function.*

Proof. It follows from Lemma 3.9 that for the function $q(w)$ we have two possibilities:

(1) $q(w) = c_2 + c_3 w + c_4 w^2$, or

(2) $q(w) = c_2 + c_3 \mathrm{e}^{\lambda w} + c_4 \mathrm{e}^{-\lambda w}$, $\lambda \neq 0$,

where c_2, c_3, c_4 are some constants.

Consider case (1). We have,

$$[S_0, Q] = c_3 \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial w_j} + 2c_4 \sum_{j=-\infty}^{\infty} w_j \frac{\partial}{\partial w_j} = c_3 S_0 + 2c_4 \sum_{j=-\infty}^{\infty} w_j \frac{\partial}{\partial w_j}.$$

If $c_4 \neq 0$, then $\sum_{j=-\infty}^{\infty} w_j \frac{\partial}{\partial w_j} \in L_x$ and $\sum_{j=-\infty}^{\infty} w_j^2 \frac{\partial}{\partial w_j} \in L_x$.

If $c_4 = 0$, $c_3 \neq 0$, then $\sum_{j=-\infty}^{\infty} w_j \frac{\partial}{\partial w_j} = \frac{1}{c_3}(Q - c_2 S_0) \in L_x$.

If $c_3 = c_4 = 0$, then $q(w) = c_2$ and there is nothing to prove.

Assume $c_4^2 + c_3^2 \neq 0$. Denote by $P = \sum_{j=-\infty}^{\infty} w_j \frac{\partial}{\partial w_j}$. Construct the vector fields

$$P_1 = [P, S_1],\ P_n = [P, P_{n-1}], \quad n \geq 2.$$

We have,

$$\begin{aligned} DS_0D^{-1} &= S_0, \\ DS_1D^{-1} &= S_1 - (\mathrm{e}^w - \tilde{c})S_0, \\ DPD^{-1} &= P, \\ DP_1D^{-1} &= P_1 + (-w\mathrm{e}^w + \mathrm{e}^w - \tilde{c})S_0, \\ DP_2D^{-1} &= P_2 + (-w^2\mathrm{e}^w + w\mathrm{e}^w - \mathrm{e}^w + \tilde{c})S_0. \end{aligned}$$

In general,

$$DP_nD^{-1} = P_n + (-w^n e^w + R_{n-1}(w)e^w + c_n)S_0, \quad n \geq 3,$$

where R_{n-1} is a polynomial of degree $n-1$, and c_n is a constant. Since the algebra L_x is of finite dimension, then there exists a natural number N such that

$$P_{N+1} = \mu_N P_N + ... + \mu_1 P_1 + \mu_0 S_0,$$

and $S_0, P_1, ..., P_N$ are linearly independent. Thus

$$DP_{N+1}D^{-1} = D(\mu_N)DP_ND^{-1} + ... + D(\mu_1)DP_1D^{-1} + D(\mu_0)S_0,$$

or the same,

$$\begin{aligned}
&\mu_N P_N + ... + \mu_1 P_1 + \mu_0 S_0 + (-w^{N+1}e^w + R_N(w)e^w + c_{N+1})S_0 \\
&= D(\mu_N)\{P_N + (-w^N e^w + R_{N-1}(w)e^w + c_N)S_0\} + ... \\
&+ D(\mu_1)\{P_1 + (-we^w + e^w - \tilde{c})S_0\} + D(\mu_0)S_0.
\end{aligned}$$

We compare the coefficients before the vector fields $P_N, ..., P_1$ we have

$$\mu_N = D(\mu_N), \; ... \, , \; \mu_1 = D(\mu_1),$$

which implies that $\mu_N, ..., \mu_1$ are all constants. We also compare the coefficients before S_0 and we have

$$\begin{aligned}
\mu_0 - w^{N+1}e^w + R_N(w)e^w + c_{N+1} \;&=\; \mu_N(-w^N e^w + R_{N-1}(w)e^w + c_N) \\
&\quad + ... + \mu_1(-we^w + e^w - \tilde{c}) + D(\mu_0).
\end{aligned}$$

The last equality shows that $D(\mu_0) - \mu_0$ is a function of w only. But this is possible only if $D(\mu_0) - \mu_0$ is a constant, denote it by d_0. The last equality becomes a contradictory one:

$$\begin{aligned}
w^{N+1}e^w \;&=\; R_N(w)e^w + c_{N+1} - \mu_N(-w^N e^w + R_{N-1}(w)e^w + c_N) \\
&\quad - ... - \mu_1(-we^w + e^w - \tilde{c}) - d_0.
\end{aligned}$$

As it is seen clearly that on the left hand side we have $(N+1)$-th power of w but on the right we do not. This contradiction proves that $c_3^2 + c_4^2 = 0$, i.e. $c_3 = c_4 = 0$ in case (1). Therefore, $q(w) = c_2$.

Now consider case (2). Since

$$[S_0, Q] = \lambda c_3 \sum_{j=-\infty}^{\infty} \mathrm{e}^{\lambda w_j} \frac{\partial}{\partial w_j} - \lambda c_4 \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\lambda w_j} \frac{\partial}{\partial w_j},$$

$$[S_0, [S_0, Q]] = \lambda^2 c_3 \sum_{j=-\infty}^{\infty} \mathrm{e}^{\lambda w_j} \frac{\partial}{\partial w_j} + \lambda^2 c_4 \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\lambda w_j} \frac{\partial}{\partial w_j},$$

then vector fields $Q_\lambda = c_3 \sum_{j=-\infty}^{\infty} \mathrm{e}^{\lambda w_j} \frac{\partial}{\partial w_j}$ and $Q_{-\lambda} = c_4 \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\lambda w_j} \frac{\partial}{\partial w_j}$ both belong to L_x. We have, $DQ_\lambda D^{-1} = Q_\lambda$, $DQ_{-\lambda} D^{-1} = Q_{-\lambda}$.

Assume $c_3 \neq 0$. Construct vector fields

$$Q_1 = [Q_\lambda, S_1], \quad Q_n = [Q_\lambda, Q_{n-1}], \quad n \geq 2.$$

Straightforward calculations show that

$$\begin{aligned} DQ_1 D^{-1} &= Q_1 - c_3 \mathrm{e}^{(1+\lambda)w} S_0 + (\mathrm{e}^w - \tilde{c})\lambda Q_\lambda, \\ DQ_2 D^{-1} &= Q_2 - c_3^2 (1+\lambda) \mathrm{e}^{(1+2\lambda)w} S_0 + 2\lambda c_3 \mathrm{e}^{(1+\lambda)w} Q_\lambda. \end{aligned}$$

It can be proved by induction on n that

$$DQ_n Q^{-1} = Q_n - p_n S_0 + q_n Q_\lambda, \quad n \geq 2, \tag{3.32}$$

where

$$\begin{aligned} p_n &= c_3^n (1+\lambda)(1+2\lambda)...(1+(n-1)\lambda) \mathrm{e}^{(1+n\lambda)w}, \\ q_n &= n c_3^{n-1} \lambda (1+\lambda)...(1+(n-2)\lambda) \mathrm{e}^{(1+(n-1)\lambda)w}. \end{aligned}$$

Since L_x is of finite dimension then there exists a natural number N that

$$Q_{N+1} = \mu_N Q_N + ... + \mu_1 Q_1 + \mu_\lambda Q_\lambda + \mu_0 S_0,$$

and $S_0, Q_\lambda, Q_1, ..., Q_N$ are linearly independent. Then

$$DQ_{N+1} D^{-1} = D(\mu_N) DQ_N D^{-1} + ... + D(\mu_0) DS_0 D^{-1},$$

or by using (3.32)

$$\begin{aligned} &\mu_N Q_N + ... + \mu_1 Q_1 + \mu_\lambda Q_\lambda + \mu_0 S_0 - p_{N+1} S_0 + q_{N+1} Q_\lambda \\ &= D(\mu_N)\{Q_N - p_N S_0 + q_N Q_\lambda\} + ... + D(\mu_1)\{Q_1 - p_1 S_0 + q_1 Q_\lambda\} \\ &+ D(\mu_\lambda) Q_\lambda + D(\mu_0) S_0. \end{aligned}$$

We compare the coefficients before $Q_N, ..., Q_1$ and we have that μ_k, $1 \le k \le N$, are all constants. We also compare the coefficients before S_0 and get

$$\mu_0 - p_{N+1} = -\mu_N p_N - ... - \mu_2 p_2 - \mu_1 p_1 + D(\mu_0). \tag{3.33}$$

Since p_k, $1 \le k \le N+1$, depend on w only, then $D(\mu_0) - \mu_0$ is a function of w, and therefore $D(\mu_0) - \mu_0$ is a constant, denote it by d_0.

If $\lambda \neq -\frac{1}{r}$ for all $r \in \mathbb{N}$, then $p_k \neq 0$ for all $k \in \mathbb{N}$, and equation (3.33) fails to be true.

Consider case when $\lambda = -\frac{1}{r}$ for some $r \in \mathbb{N}$. Substitution $u_j = \mathrm{e}^{-\lambda w_j}$ transforms vector fields $\frac{-1}{\lambda c_3} Q_\lambda$, $\frac{-1}{\lambda} S_1$, $\frac{-1}{\lambda} S_0$ into vector fields

$$\begin{aligned} Q^*_\lambda &= \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial u_j}, \\ S^*_1 &= \sum_{j=-\infty}^{\infty} \{\tilde{\rho}^*_j + \tilde{b}^*(u_j)\} u_j \frac{\partial}{\partial u_j}, \\ S^*_0 &= \sum_{j=-\infty}^{\infty} u_j \frac{\partial}{\partial u_j}, \end{aligned}$$

where

$$\tilde{\rho}^*_j = \begin{cases} \sum\limits_{k=0}^{j-1} (u_k^r - \tilde{c}), & \text{if} \quad j \ge 1; \\ 0, & \text{if} \quad j = 0; \\ -\sum\limits_{k=j}^{-1} (u_k^r - \tilde{c}), & \text{if} \quad j \le -1, \end{cases} \quad , \quad \tilde{b}^*(u_j) = \tilde{b}(r \ln u_j) \, .$$

First consider the case $r = 1$. We have,

$$\begin{aligned} T := & \; [Q^*_\lambda, S^*_1] = \sum_{j=-\infty}^{\infty} \{j u_j + \tilde{\rho}^*_j + \tilde{b}^*(u_j) + u_j \tilde{b}^{*\prime}(u_j)\} \frac{\partial}{\partial u_j}, \\ K := & \; \frac{1}{2}[Q^*_\lambda, T] = \sum_{j=-\infty}^{\infty} \{j + c(u_j)\} \frac{\partial}{\partial u_j}, \end{aligned}$$

where $c(u_j) = \tilde{b}^{*\prime}(u_j) + \frac{1}{2} u_j \tilde{b}^{*\prime\prime}(u_j)$,

$$\begin{aligned} T_1 &= [T, K] = \gamma_1 \sum_{j=-\infty}^{\infty} \{j^2 + j g^{(j)}_{1,1}(u_j) + g^{(j)}_{1,0}(u, u_1, ..., u_j)\} \frac{\partial}{\partial u_j}, \\ T_2 &= [T, T_1] = \gamma_2 \sum_{j=-\infty}^{\infty} \{j^3 + j^2 g^{(j)}_{2,2}(u_j) + j g^{(j)}_{2,1}(u, u_1, ..., u_j) + g^{(j)}_{2,0}(u, u_1, ..., u_j)\} \frac{\partial}{\partial u_j}, \end{aligned}$$

where $\gamma_1 = -\frac{3}{2}$ and $\gamma_2 \neq 0$.

Construct vector fields, $T_n = [T, T_{n-1}]$, $n \geq 3$. Direct calculations show that

$$T_n = \gamma_n \sum_{j=0}^{\infty} \Big\{ j^{n+1} + j^n g_{n,n}(u_j) + \sum_{k=0}^{n-1} j^k g_{n,k}(u, u_1, ..., u_j) \Big\} \frac{\partial}{\partial u_j} + \sum_{j=-\infty}^{-1} a_j \frac{\partial}{\partial u_j}, \quad n \geq 1.$$

Since $\{T_n\}_{n=1}^{\infty}$ is an infinite sequence of linearly independent vector fields from L_x, then case $r = 1$ fails to exist.

Consider case $r \geq 2$. We have,

$$ad_{Q_\lambda^*} S_1^* = [Q_\lambda^*, S_1^*] = \sum_{j=-\infty}^{\infty} \Big\{ sgn(j) r \Big(\sum_{k=0}^{j-1} u_k^{r-1} \Big) u_j + \tilde{\rho}_j^* + \tilde{b}^*(u_j) + u_j \tilde{b}^{*'}(u_j) \Big\} \frac{\partial}{\partial u_j},$$

and

$$ad^r_{Q_\lambda^*} S_1^* = \sum_{j=-\infty}^{\infty} \Big\{ r! j u_j + sgn(j) r! \sum_{k=0}^{j-1} u_k + d(u_j) \Big\}$$

for some function d,

$$ad^{r+1}_{Q_\lambda^*} S_1^* = \sum_{j=-\infty}^{\infty} \Big\{ 2r! j + d'(u_j) \Big\} \frac{\partial}{\partial u_j}.$$

Note that vector fields $ad^r_{Q_\lambda^*} S_1^*$ and $ad^{r+1}_{Q_\lambda^*} S_1^*$ have coefficients of the same kind as vector fields T and K (from case $r = 1$) have. It means that $ad^r_{Q_\lambda^*} S_1^*$ and $ad^{r+1}_{Q_\lambda^*} S_1^*$ generate an infinite dimensional Lie algebra. This contradiction implies that case $r \geq 2$ also fails to exist.

Thus, $c_3 = 0$. By interchanging λ with $-\lambda$, we obtain that $c_4 = 0$ also. Hence $c_3 = c_4 = 0$ and $q(w) = c_2$. □

We already know that $a(\tau) = c_0 \tau + c_1$. The next Lemma shows that $c_0 \neq 0$.

Lemma 3.11 *c_0 is a nonzero constant.*

Proof. Assume contrary that $c_0 = 0$. Then $a(\tau) = c_1$ and $c_1 \neq 0$, vector fields (3.28) and (3.29) become

$$T_1 = c_1 \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial \tau_j} = c_1 \tilde{T}_1,$$

and

$$T = -tT_1 - c_1 \sum_{j=-\infty}^{\infty} \{\rho_j + \frac{1}{mc_1} b(\tau_j)\} \frac{\partial}{\partial \tau_j} = -c_1 t \tilde{T}_1 - c_1 \tilde{T},$$

where

$$\tilde{T}_1 = \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial \tau_j}, \qquad \tilde{T} = \sum_{j=-\infty}^{\infty} \{\rho_j + \frac{1}{mc_1} b(\tau_j)\} \frac{\partial}{\partial \tau_j}.$$

Since

$$[\tilde{T}_1, [\tilde{T}_1, \tilde{T}]] = \frac{1}{mc_1} \sum_{j=-\infty}^{\infty} b''(\tau_j) \frac{\partial}{\partial \tau_j}$$

and $\tilde{T}_1$ both belong to a finite dimensional L_x, then, by Lemma 3.9, we have two possibilities for the function $b''(\tau)$:

$$1) b''(\tau) = \tilde{C}_1 + \tilde{C}_2 e^{\lambda\tau} + \tilde{C}_3 e^{-\lambda\tau} \text{or} 2) b''(\tau) = \tilde{C}_1 + \tilde{C}_2 \tau + \tilde{C}_3 \tau^2$$

for some constants $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$.

In case 1), the function $b(\tau) = C_1 + C_2 e^{\lambda\tau} + C_3 e^{-\lambda\tau} + C_4 \tau^2 + C_5 \tau$ and

$$[\tilde{T}_1, [\tilde{T}_1, \tilde{T}]] - \lambda^2 \tilde{T} - \frac{2C_4 - \lambda^2 C_1}{mc_1} \tilde{T}_1 = -\lambda^2 \sum_{j=-\infty}^{\infty} \left\{\rho_j + \frac{C_4 \tau_j^2 + C_5 \tau_j}{mc_1}\right\} \frac{\partial}{\partial \tau_j}$$

is an element in L_x.

In case 2), the function $b(\tau) = C_1 + C_2 \tau + C_3 \tau^2 + C_4 \tau^3 + C_5 \tau^4$ and

$$\tilde{T} - \frac{C_1}{mc_1} \tilde{T}_1 = \sum_{j=-\infty}^{\infty} \left\{\rho_j + \frac{C_2 \tau_j + C_3 \tau_j^2 + C_4 \tau_j^3 + C_5 \tau_j^4}{mc_1}\right\} \frac{\partial}{\partial \tau_j}$$

belongs to L_x.

To finish the proof of the Lemma it is enough to show that vector fields

$$\tilde{T}_2 := \sum_{j=-\infty}^{\infty} \{\rho_j + C_2 \tau_j + C_3 \tau_j^2 + C_4 \tau_j^3 + C_5 \tau_j^4\} \frac{\partial}{\partial \tau_j},$$

and

$$\tilde{T}_1 = \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial \tau_j}$$

produce an infinite dimensional Lie algebra L_x for any fixed constants C_2, C_3, C_4 and C_5. We can prove it by showing that L_x contains vector fields $\sum\limits_{j=-\infty}^{\infty} j^k \frac{\partial}{\partial \tau_j}$, for all $k = 1, 2, \ldots$. Note that

$$[\tilde{T}_1, \tilde{T}_2] = \sum_{j=-\infty}^{\infty} (-j + C_2 + 2C_3\tau_j + 3C_4\tau_j^2 + 4C_5\tau_j^3)\frac{\partial}{\partial \tau_j}.$$

There are four cases: a) $C_5 \neq 0$ and b) $C_5 = 0, C_4 \neq 0$, c) $C_5 = C_4 = 0$, $C_3 \neq 0$ and d) $C_5 = C_4 = C_3 = 0$.

In case a),

$$[\tilde{T}_1, [\tilde{T}_1, [\tilde{T}_1, \tilde{T}_2]]] - 6C_4\tilde{T}_1 = \sum_{j=-\infty}^{\infty} 24C_5\tau_j\frac{\partial}{\partial \tau_j} = 24C_5P_1 \in L_x, \quad P_1 = \sum_{j=-\infty}^{\infty} \tau_j \frac{\partial}{\partial \tau_j},$$

$$[\tilde{T}_1, [\tilde{T}_1, \tilde{T}_2]] = \sum_{j=-\infty}^{\infty} \{2C_3 + 6C_4\tau_j + 12C_5\tau_j^2\}\frac{\partial}{\partial \tau_j} \in L_x,$$

and therefore,

$$P_2 := \sum_{j=-\infty}^{\infty} \tau_j^2 \frac{\partial}{\partial \tau_j} \in L_x,$$

and

$$\tilde{T}_3 := [\tilde{T}_1, \tilde{T}_2] - C_2\tilde{T}_1 - 2C_3P_1 - 3C_4P_2 = \sum_{j=-\infty}^{\infty} (-j + 4C_5\tau_j^3)\frac{\partial}{\partial \tau_j} \in L_x.$$

We have,

$$J_1 := -\frac{1}{3}([\tilde{T}_3, P_1] + 2\tilde{T}_3) = \sum_{j=-\infty}^{\infty} j\frac{\partial}{\partial \tau_j} \in L_x.$$

Now,

$$[J_1, [J_1, P_2]] = \frac{1}{2}\sum_{j=-\infty}^{\infty} j^2 \frac{\partial}{\partial \tau_j} \in L_x.$$

Assuming $J_k = \sum\limits_{j=-\infty}^{\infty} j^k \frac{\partial}{\partial \tau_j} \in L_x$ we have that

$$J_{k+1} := \frac{1}{2}[J_1, [J_k, P_2]] = \sum_{j=-\infty}^{\infty} j^{k+1}\frac{\partial}{\partial \tau_j} \in L_x.$$

In case b) we have

$$P_1 := \frac{1}{6C_4}\{[\tilde{T}_1, [\tilde{T}_1, \tilde{T}_2]] - 2C_3\tilde{T}_1\} = \sum_{j=-\infty}^{\infty} \tau_j \frac{\partial}{\partial \tau_j} \in L_x$$

and

$$\tilde{T}_3 = [\tilde{T}_1, \tilde{T}_2] - C_2\tilde{T}_1 - 2C_3P_1 = \sum_{j=-\infty}^{\infty} (-j + 3C_4\tau_j^2)\frac{\partial}{\partial \tau_j} \in L_x.$$

We have,

$$J_1 := -\frac{1}{2}([\tilde{T}_3, P_1] + \tilde{T}_3) = \sum_{j=-\infty}^{\infty} j\frac{\partial}{\partial \tau_j} \in L_x,$$

and

$$P_2 = \frac{1}{6C_4}(\tilde{T}_3 - [\tilde{T}_3, P_1]) = \sum_{j=-\infty}^{\infty} \tau_j^2 \frac{\partial}{\partial \tau_j} \in L_x.$$

As it was shown in the proof of case a), J_1 and P_2 produce an infinite dimensional Lie algebra.

In case c),

$$\tilde{T}_3 = [\tilde{T}_1, \tilde{T}_2] - C_2\tilde{T}_1 = \sum_{j=-\infty}^{\infty} (-j + 2C_3\tau_j)\frac{\partial}{\partial \tau_j} \in L_x,$$

$$\tilde{T}_4 = [\tilde{T}_3, \tilde{T}_2] = \sum_{j=-\infty}^{\infty} (\frac{j(j-1)}{2} - jC_2 - 2C_3 j\tau_j + 2C_3^2\tau_j^2)\frac{\partial}{\partial \tau_j} \in L_x.$$

Also,

$$\tilde{T}_5 = [\tilde{T}_3, \tilde{T}_4] = 2C_3 \sum_{j=-\infty}^{\infty} \left(\frac{j(j+1)}{2} + C_2 j - 2C_3 j\tau_j + 2C_3^2\tau_j^2\right)\frac{\partial}{\partial \tau_j} \in L_x.$$

Since $\tilde{T}_4$ and $\tilde{T}_5$ both belong to L_x then either

$$c)(i) \quad J_1 = \sum_{j=-\infty}^{\infty} j\frac{\partial}{\partial \tau_j} \in L_x, \qquad \tilde{T}_6 = \sum_{j=-\infty}^{\infty} (\frac{j^2}{2} - 2C_3 j\tau_j + 2C_3^2\tau_j^2)\frac{\partial}{\partial \tau_j} \in L_x,$$

or

$$c)(ii) \quad C_2 = -\frac{1}{2}, \quad \tilde{T}_6 = \sum_{j=-\infty}^{\infty} (\frac{j^2}{2} - 2C_3 j\tau_j + 2C_3^2\tau_j^2)\frac{\partial}{\partial \tau_j} \in L_x.$$

In case c) (i),

$$P_1 = \frac{1}{4C_3^2}\{[\tilde{T}_1, \tilde{T}_6] + 2C_3J_1\} = \sum_{j=-\infty}^{\infty} \tau_j \frac{\partial}{\partial \tau_j} \in L_x.$$

Since

$$[P_1, \tilde{T}_6] = \sum_{j=-\infty}^{\infty} (-\frac{j^2}{2} + 2C_3^2\tau_j^2)\frac{\partial}{\partial \tau_j},$$

and

$$[P_1, [P_1, \tilde{T}_6]] = \sum_{j=-\infty}^{\infty} (\frac{j^2}{2} + 2C_3^2\tau_j^2)\frac{\partial}{\partial \tau_j}$$

both belong to L_x then

$$J_2 = \sum_{j=-\infty}^{\infty} j^2\frac{\partial}{\partial \tau_j} \in L_x, \qquad P_2 = \sum_{j=-\infty}^{\infty} \tau_j^2\frac{\partial}{\partial \tau_j} \in L_x,$$

P_2 and J_1 generate an infinite dimensional Lie algebra.

In case c) (ii),

$$\tilde{T}_1 = \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial \tau_j}, \quad \tilde{T}_2 = \sum_{j=-\infty}^{\infty} \Big(C_3\tau_j^2 - \frac{1}{2}\tau_j + \rho_j\Big)\frac{\partial}{\partial \tau_j}.$$

Note that the Lie algebra generated by the vector fields

$$\tilde{T}_2^* = \tilde{T}_2 - \Big(C_3\tau^2 - \frac{1}{2}\tau\Big)\tilde{T}_1 = d(\tau,\tau_1)\frac{\partial}{\partial \tau_1} - d(\tau_{-1},\tau)\frac{\partial}{\partial \tau_{-1}} + (d(\tau,\tau_1)+d(\tau_1,\tau_2))\frac{\partial}{\partial \tau_2} + \ldots$$

and

$$\tilde{T}_1 = \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial \tau_j}$$

is infinite dimensional. It can be proved by comparing this algebra with the infinite dimensional characteristic Lie algebra of the chain

$$t_{1x} = t_x + C_3(t_1^2 - t^2) - \frac{1}{2}(t_1 + t). \tag{3.34}$$

Indeed, the Lie algebra L_{x1} for (3.34) is generated by the operators (3.7) and (3.8) with $d(t,t_1) = C_3(t_1^2 - t^2) - \frac{1}{2}(t_1 + t)$. To keep standard notations we put $a(\tau) = -2C_3\tau - 1$ and $b(\tau) = C_3\tau^2 + \frac{1}{2}\tau$. Note that since $C_3 \neq 0$ function $a(\tau)$ is not a constant. It follows from Theorem 3.1 proved below that the characteristic Lie algebras L_x (and therefore algebra L_{x1}) for equation (3.34) is of infinite dimension. Thus, in case c) (ii) we also have an infinite dimensional Lie algebra L_x.

In case d),

$$\tilde{T}_2 = \sum_{j=-\infty}^{\infty} (-\tau - \tau_1 - \ldots - \tau_{j-1} + C_2\tau_j)\frac{\partial}{\partial \tau_j} \in L_x.$$

Then

$$J_1 = c_2\tilde{T}_1 - [\tilde{T}_1, \tilde{T}_2] = \sum_{j=-\infty}^{\infty} j\frac{\partial}{\partial \tau_j} \in L_x,$$

and

$$J_2 = -2\Big([J_1, \tilde{T}_2] - \Big(\frac{1}{2} + C_2\Big)J_1\Big) = \sum_{j=-\infty}^{\infty} j^2\frac{\partial}{\partial \tau_j} \in L_x.$$

Assuming that J_k, $1 \le k \le n$ belong to L_x, by considering $[J_n, \tilde{T}_2]$ we may show that $J_{n+1} = \sum\limits_{j=-\infty}^{\infty} j^{k+1}\frac{\partial}{\partial \tau_j} \in L_x$. It implies L_x is of infinite dimension. $\square$

Let us introduce new variables

$$w_j = \ln\Big(\tau_j + \frac{c_1}{c_0}\Big).$$

We can rewrite the vector fields T_1 and T in variables w_j as

$$T_1 = c_0\sum_{j=-\infty}^{\infty}\frac{\partial}{\partial w_j} = c_0S_0\,,$$

$$T = -tc_0S_0 + c_0\sum_{j=-\infty}^{\infty}\{\tilde{\rho}_j + \tilde{b}(w_j)\}\frac{\partial}{\partial w_j} = -c_0tS_0 + c_0S_1,$$

where

$$S_0 = \sum_{j=-\infty}^{\infty}\frac{\partial}{\partial w_j}, \quad S_1 = \sum_{j=-\infty}^{\infty}\{\tilde{\rho}_j + \tilde{b}(w_j)\}\frac{\partial}{\partial w_j},$$

$$\tilde{\rho}_j = \begin{cases} \sum\limits_{k=0}^{j-1}(\mathrm{e}^{w_k} - \tilde{c}), & \text{if} \quad j \ge 1; \\ 0, & \text{if} \quad j = 0; \\ -\sum\limits_{k=j}^{-1}(\mathrm{e}^{w_k} - \tilde{c}), & \text{if} \quad j \le -1, \end{cases} \qquad \tilde{c} = \frac{c_1}{c_0}, \quad \tilde{b}(w_j) = -\frac{1}{m}\Big(\frac{b(\tau_j)}{c_0\tau_j + c_1}\Big).$$

We have

$$DS_0D^{-1} = S_0, \quad DS_1D^{-1} = S_1 - (\mathrm{e}^{w} - \tilde{c})S_0.$$

These equalities can be proved by applying DS_0D^{-1} and DS_1D^{-1} to the functions depending on $.., w_{-1}, w, w_1, w_2, \dots$.

The above Lemmas allow us to prove the following Theorem.

Theorem 3.12 *If equation*

$$t_{1x} = t_x + a(\tau)t^m + b(\tau)t^{m-1} + \ldots, \quad m \geq 1$$

admits a nontrivial x-integral, then
(1) $a(\tau) = c_0\tau$, $b(\tau) = c_2\tau^2 + c_3\tau$, where c_0, c_2, c_3 are some constants.
(2) $m = 1$.

Proof. Consider the case (1). Define vector field

$$Q = [S_0, [S_0, S_1]] - [S_0, S_1] = \sum_{j=-\infty}^{\infty} (\tilde{b''}(w_j) - \tilde{b'}(w_j))\frac{\partial}{\partial w_j}.$$

By Lemma 3.10, $\tilde{b''}(w) - \tilde{b'}(w) = C$ for some constant C. Thus, $\tilde{b}(w) = C_0 + C_1\mathrm{e}^w + C_2 w$ for some constants C_1, C_2, C_0. Consider vector fields

$$P = (C_2 - C_0)S_0 + S_1 - [S_0, S_1] = \sum_{j=-\infty}^{\infty} (C_2 w_j + \tilde{c}j)\frac{\partial}{\partial w_j},$$

$$\begin{aligned} R = [S_0, [S_0, S_1]] \quad &= \quad \sum_{j=1}^{\infty}\left\{\left(\sum_{k=1}^{j}\mathrm{e}^{w_k}\right) + C_1\mathrm{e}^{w_j}\right\}\frac{\partial}{\partial w_j} + C_1\mathrm{e}^{w}\frac{\partial}{\partial w} \\ &\quad - \sum_{j=-\infty}^{-1}\left\{\left(\sum_{k=j}^{-1}\mathrm{e}^{w_k}\right) + C_1\mathrm{e}^{w_j}\right\}\frac{\partial}{\partial w_j}, \end{aligned}$$

$$R_1 = [P, R], \quad R_{n+1} = [P, R_n], \quad n \geq 1.$$

Then

$$\begin{aligned} R_n \quad &= \quad \sum_{j\geq 0}\{\mathrm{e}^{w_j}(C_1 C_2^n w_j^n + P_{n,j}) + r_{n,j}(w, w_1, \ldots, w_{j-1})\}\frac{\partial}{\partial w_j} \\ &+ \quad \sum_{j\leq -1}\{\mathrm{e}^{w_j}((C_1 - 1)C_2^n w_j^n + P_{n,j}) + r_{n,j}(w_{-1}, w_{-2}, \ldots, w_{j+1})\}\frac{\partial}{\partial w_j}, \end{aligned}$$

where $P_{n,j} = P_{n,j}(w_j, j)$ is a polynomial of degree $n-1$ whose coefficients depend on j, and $r_{n,j}$ are the functions that do not depend on w_j. Since all vector fields R_n belong to a finite dimensional Lie algebra L_x then $C_1 C_2 = (C_1 - 1)C_2 = 0$, or the same $C_2 = 0$. Therefore,

$$\tilde{b}(w) = C_0 + C_1\mathrm{e}^w\,.$$

Since $C_2 = 0$, then
$$P = \tilde{c} \sum_{j=-\infty}^{\infty} j \frac{\partial}{\partial w_j},$$
$$R = \sum_{j=1}^{\infty} \left\{ \left(\sum_{k=1}^{j} \mathrm{e}^{w_k} \right) + C_1 \mathrm{e}^{w_j} \right\} \frac{\partial}{\partial w_j} + C_1 \mathrm{e}^{w} \frac{\partial}{\partial w} - \sum_{j=-\infty}^{-1} \left\{ \left(\sum_{k=j}^{-1} \mathrm{e}^{w_k} \right) + C_1 \mathrm{e}^{w_j} \right\} \frac{\partial}{\partial w_j}$$
and
$$\begin{aligned} R_n &= \tilde{c}^n \sum_{j=1}^{\infty} \{ \mathrm{e}^{w_1} + 2^n \mathrm{e}^{w_2} + (j-1)^n \mathrm{e}^{w_{j-1}} + j^n C_1 \mathrm{e}^{w_j} \} \frac{\partial}{\partial w_j} \\ &- \tilde{c}^n \sum_{j=-\infty}^{-1} \{ (-1)^n \mathrm{e}^{w_{-1}} + (-2)^n \mathrm{e}^{w_{-2}} + (j)^n \mathrm{e}^{w_j} + j^n C_1 \mathrm{e}^{w_j} \} \frac{\partial}{\partial w_j}. \end{aligned}$$
Again, vector fields R_n belong to a finite dimensional Lie algebra only if $\tilde{c} = 0$, or the same $c_1 = 0$ since $\tilde{c} = \frac{c_1}{c_0}$. It implies that
$$a(\tau) = c_0 \tau, \quad b(\tau) = c_2 \tau^2 + c_3 \tau.$$
Consider the case (2). Assume contrary, that is $m \geq 2$. Then the following vector field
$$\begin{aligned} &\frac{1}{m!} ad_{\tilde{X}}^{m-2}(\tilde{Y}) = Y_{\frac{1}{2}a(\tau)t^2 + \frac{1}{m}b(\tau)t + \frac{1}{m(m-1)}c(\tau)} \\ &= - \sum_{j=-\infty}^{\infty} (\frac{1}{2} a(\tau_j) t_j^2 + \frac{1}{m} b(\tau_j) t_j + \frac{1}{m(m-1)} c(\tau_j)) \frac{\partial}{\partial \tau_j} \\ &= - \sum_{j=-\infty}^{\infty} (\frac{1}{2} a(\tau_j)(t+\rho_j)^2 + \frac{1}{m} b(\tau_j)(t+\rho_j) + \frac{1}{m(m-1)} c(\tau_j)) \frac{\partial}{\partial \tau_j} - \frac{t^2}{2} \sum_{j=-\infty}^{\infty} a(\tau_j) \frac{\partial}{\partial \tau_j} \\ &-t \sum_{j=-\infty}^{\infty} \{ a(\tau_j)\rho_j + \frac{1}{m} b(\tau_j) \} \frac{\partial}{\partial \tau_j} - \sum_{j=-\infty}^{\infty} \{ \frac{1}{2} a(\tau_j) \rho_j^2 + \frac{1}{m} b(\tau_j) + \frac{1}{m(m-1)} c(\tau_j) \} \frac{\partial}{\partial \tau_j} \end{aligned}$$
is in L_x. In variables $w_j = \ln \tau_j$,
$$\frac{1}{m!} ad_{\tilde{X}}^{m-2}(\tilde{Y}) = -\frac{t^2}{2} c_0 S_0 + t c_0 S_1 - c_0 S_2,$$
where
$$S_2 = \sum_{j=-\infty}^{\infty} \{ \frac{1}{2} \tilde{\rho}_j^2 - \tilde{b}(w_j) \tilde{\rho}_j + \tilde{c}(w_j) \} \frac{\partial}{\partial w_j}, \quad \tilde{c}(w_j) = \frac{c(\tau_j)}{m(m-1)\tau_j}.$$

The vector fields S_0 and S_1 are as in Lemma 3.10. We have,

$$[S_0, S_2] = 2S_2 + C_0S_1 + P, \quad P = \sum_{j=-\infty}^{\infty} r(w_j)\frac{\partial}{\partial w_j}, \quad r(w) = \tilde{c}'(w) - 2\tilde{c}(w) - C_0\tilde{b}(w).$$

Construct the sequence

$$S_3 = [S_1, S_2], \quad S_{n+1} = [S_1, S_n], \quad n \geq 2.$$

We can prove by induction on n that

$$[S_0, S_n] = nS_n + \sum_{k=0}^{n-1} \nu_{n,k}S_k,$$

and

$$DS_nD^{-1} = S_n + \left\{\frac{n(n-1)}{2} - 1\right\} \mathrm{e}^w S_{n-1} + \sum_{k=0}^{n-2} \eta(n,k)S_k, \quad n \geq 3. \qquad (3.35)$$

Since L_x is of finite dimension then there exists a natural number N such that

$$S_{N+1} = \mu_N S_N + \mu_{N-1}S_{N-1} + \ldots + \mu_0 S_0.$$

Then

$$DS_{N+1}D^{-1} = D(\mu_N)DS_ND^{-1} + D(\mu_{N-1})DS_{N-1}D^{-1} + \ldots + D(\mu_0)DS_0D^{-1}.$$

On the other hand, by the formula (3.35) we have

$$DS_{N+1}D^{-1} = S_{N+1} + \left\{\frac{(N+1)N}{2} - 1\right\} \mathrm{e}^w S_N + \ldots .$$

We compare the coefficients before S_N and have two equations.

$$D(\mu_N) = \mu_N + \left\{\frac{(N+1)N}{2} - 1\right\} \mathrm{e}^w, \quad N \geq 2,$$

and

$$D(\mu_1) = \mu_1 + \mathrm{e}^w, \quad N = 1.$$

Both equation are contradictory. Therefore, our assumption that $m \geq 2$ was wrong. This finishes the proof of the Theorem. $\square$

3.3 Nonzero root

In this section, we prove that if the equation 3.1) admits a nontrivial x-integral and if the function $d(t, t_1)$ contains terms with $\lambda(t-t_1)t^j e^{\alpha_k t}$, $\alpha_k \neq 0$, then $j = 0$.

Lemma 3.13 *Assume equation (3.1) admits a nontrivial x-integral. Then the characteristic polynomial of the equation (3.14) can have only simple nonzero roots.*

Proof. Assume that $m_1 \geq 2$. Introduce polynomials

$$\Lambda^{(2)}_{\alpha_1}(\lambda) = \frac{\Lambda(\lambda)}{(\lambda-\alpha_1)^2} = \lambda^{m_0}(\lambda-\alpha_1)^{m_1-2}...(\lambda-\alpha_i)^{m_i},$$

$$\Lambda_{\alpha_1}(\lambda) = \frac{\Lambda(\lambda)}{(\lambda-\alpha_1)} = \lambda^{m_0}(\lambda-\alpha_1)^{m_1-1}...(\lambda-\alpha_i)^{m_i}.$$

Consider vector fields

$$S_0^* = \Lambda^{(2)}_{\alpha_1}(ad_{\tilde{X}})Y_d = Y_{A(\tau)\mathrm{e}^{\alpha_1 t}} = A(\tau)e^{\alpha_1 t}\frac{\partial}{\partial t_1} + \{A(\tau)e^{\alpha_1 t} + A(\tau_1)e^{\alpha_1 t_1}\}\frac{\partial}{\partial t_2} + \dots,$$

$$S_1^* = \Lambda_{\alpha_1}(ad_{\tilde{X}})Y_d = Y_{(A(\tau)t+B(\tau))\mathrm{e}^{\alpha_1 t}} = (A(\tau)t + B(\tau))e^{\alpha_1 t}\frac{\partial}{\partial t_1} + ...$$

from the the Lie algebra L_x.

In variables $\tau_j = t_j - t_{j+1}$, we have $\frac{\partial}{\partial t_j} = -\frac{\partial}{\partial \tau_{j-1}} + \frac{\partial}{\partial \tau_j}$ and so the vector fields S_0^* and S_1^* become

$$S_0^* = -\mathrm{e}^{\alpha_1 t}\sum_{j=-\infty}^{\infty} A(\tau_j)\mathrm{e}^{\alpha_1\rho_j}\frac{\partial}{\partial \tau_j} = -\mathrm{e}^{\alpha_1 t}S_0,$$

$$S_1^* = -t\mathrm{e}^{\alpha_1 t}S_0 - \mathrm{e}^{\alpha_1 t}\sum_{j=-\infty}^{\infty}\{A(\tau_j)\rho_j + B(\tau_j)\}\mathrm{e}^{\alpha_1\rho_j}\frac{\partial}{\partial \tau_j} = -t\mathrm{e}^{\alpha_1 t}S_0 - \mathrm{e}^{\alpha_1 t}S_1,$$

with $S_0 = \sum\limits_{j=-\infty}^{\infty} A(\tau_j)\mathrm{e}^{\alpha_1\rho_j}\frac{\partial}{\partial \tau_j}$ and $S_1 = \sum\limits_{j=-\infty}^{\infty}\{A(\tau_j)\rho_j + B(\tau_j)\}\mathrm{e}^{\alpha_1\rho_j}\frac{\partial}{\partial \tau_j}$.

We have

$$DS_0D^{-1} = \mathrm{e}^{\alpha_1\tau}S_0, \quad DS_1D^{-1} = \mathrm{e}^{\alpha_1\tau}S_1 + \tau\mathrm{e}^{\alpha_1\tau}S_0.$$

These equalities can be found easily by applying DS_0D^{-1} and DS_1D^{-1} to the functions depending on $..., \tau_{-1}, \tau, \tau_1, ...$. Define the sequence

$$S_2 = [S_0, S_1], \quad S_{n+1} = [S_0, S_n], \quad n \geq 2.$$

We can easily show that

$$\begin{aligned} DS_2D^{-1} &= D[S_0, S_1]D^{-1} = [e^{\alpha_1\tau}S_0, e^{\alpha_1\tau}S_1 + \tau e^{\alpha_1\tau}S_0] \\ &= \mathrm{e}^{2\alpha_1\tau}S_2 + \alpha_1\mathrm{e}^{2\alpha_1\tau}A(\tau)S_1 + \mathrm{e}^{2\alpha_1\tau}(A(\tau) - \alpha_1 B(\tau))S_0 \,. \end{aligned}$$

It can be proved by induction on n that

$$DS_nD^{-1} = \mathrm{e}^{n\alpha_1\tau}S_n + \alpha_1\frac{n(n-1)}{2}\mathrm{e}^{n\alpha_1\tau}A(\tau)S_{n-1} + \sum_{k=0}^{n-2}\gamma(n,k)S_k \,. \tag{3.36}$$

Since the dimension of L_x is finite and S_0, S_1, $\ldots$ belongs to L_x then there exists a natural number N such that

$$S_{N+1} = \mu_N S_N + \mu_{N-1}S_{N-1} + \ldots + \mu_0 S_0,$$

and S_0, S_1, $\ldots$, S_N are linearly independent. Therefore,

$$DS_{N+1}D^{-1} = D(\mu_N)DS_ND^{-1} + D(\mu_{N-1})DS_{N-1}D^{-1} + \ldots + D(\mu_0)DS_0D^{-1}.$$

On the other hand, by the formula (3.36) we have

$$DS_{N+1}D^{-1} = \mathrm{e}^{(N+1)\alpha_1\tau}S_{N+1} + \alpha_1\frac{(N+1)N}{2}\mathrm{e}^{(N+1)\alpha_1\tau}A(\tau)S_N + \sum_{k=0}^{N-1}\gamma(N+1,k)S_k.$$

We compare the coefficients before S_N in the last two equations and we have

$$\mathrm{e}^{(N+1)\alpha_1\tau}\mu_N + \frac{\alpha_1(N+1)N}{2}\mathrm{e}^{(N+1)\alpha_1\tau}A(\tau) = D(\mu_N)\mathrm{e}^{N\alpha_1\tau}.$$

It follows that μ_N is a constant and then

$$A(\tau) = C(\mathrm{e}^{-\alpha_1\tau} - 1), \quad C = \frac{2\mu_N}{\alpha_1 N(N+1)}.$$

Let us construct a new infinite sequence of vector fields, which are elements of L_x, enumerated by a multi-index.

$$T_0 := S_1, \quad T_1 := S_0, \quad T_2 = [S_1, T_1], \quad T_{n+1} = [S_1, T_n], \quad n \geq 2, \quad T_{n,0} = [S_0, T_n],$$

$$T_{n,0,i_1,\dots,i_{n-1},i_n} = [S_{i_n}, T_{n,0,i_1,\dots,i_{n-1}}], \quad i_j \in \{0;1\}.$$

Direct calculations show that

$$DT_2D^{-1} = \mathrm{e}^{2\alpha_1\tau}T_2 + \mathrm{e}^{2\alpha_1\tau}(\alpha_1 B - A)T_1 - \alpha_1\mathrm{e}^{2\alpha_1\tau}AT_0,$$

$$DT_3D^{-1} = \mathrm{e}^{3\alpha_1\tau}T_3 + \mathrm{e}^{3\alpha_1\tau}(3\alpha_1 B - A + 3\alpha_1\tau A)T_2 + \tau\mathrm{e}^{3\alpha_1\tau}T_{2,0} + \sum_{m(\beta)<2}\nu(3,\beta)T_\beta\,.$$

Here and below we use functions $m = m(\beta)$ and $l = l(\beta)$ defined in previous Section. It can be proved by induction on n that

$$\begin{aligned} DT_nD^{-1} &= \mathrm{e}^{n\alpha_1\tau}T_n + \mathrm{e}^{n\alpha_1\tau}\{c_nB - A + c_n\tau A\}T_{n-1} \\ &\quad +\tau\mathrm{e}^{n\alpha_1\tau}\sum_{m(\beta)=n-1,l(\beta)=1}\nu^*(n,\beta)T_\beta + \sum_{m(\beta)\le n-2}\nu(n,\beta)T_\beta\,, \end{aligned}$$

where

$$c_n = \frac{\alpha_1 n(n-1)}{2},$$

and $\nu^*(n,\beta)$ are constants for any β with $m(\beta) = n-1$ and $l(\beta) = 1$.

In general, for any γ,

$$DT_\gamma D^{-1} = \mathrm{e}^{(m(\gamma)+l(\gamma))\alpha_1\tau}T_\gamma + \sum_{m(\beta)\le m(\gamma)-1}\nu(\gamma,\beta)T_\beta\,.$$

Among the vector fields T_β we choose a system P of linearly independent vector fields in such a way that for some natural number N,

(i) $T_k \in P, \quad k \le N$,

(ii) $m(\beta) \le N$ for any $T_\beta \in P$.

(iii) for any T_γ with $m(\gamma) \le N$ we have $T_\gamma = \sum_{T_\beta\in P, m(\beta)\le m(\gamma)}\mu(\gamma,\beta)T_\beta$. Also

$$T_{N+1} = \mu(N+1,N)T_N + \sum_{T_\beta\in P}\mu(N+1,\beta)T_\beta.$$

(iv) for any $T_\gamma \notin P$ with $m(\gamma) = N$ and $l(\gamma) = 1$, we have $\mu(\gamma,N) = 0$.

Indeed,

$$DT_\gamma D^{-1} = D(\mu(\gamma,N))DT_ND^{-1} + \sum_{T_\beta\in P,\beta\ne N}D(\mu(\gamma,\beta))DT_\beta D^{-1}\,.$$

On the other hand,

$$\begin{aligned} DT_\gamma D^{-1} &= \mathrm{e}^{(m(\gamma)+l(\gamma))\alpha_1\tau}T_\gamma + \sum_{m(\beta)\le N-1}\nu(\gamma,\beta)T_\beta \\ &= \mathrm{e}^{(N+1)\alpha_1\tau}\{\mu(\gamma,N)T_N + \sum_{T_\beta\in P,m(\beta)\le N,\beta\ne N}\mu(\gamma,\beta)T_\beta\} + \sum_{m(\beta)\le N-1}\nu(\gamma,\beta)T_\beta. \end{aligned}$$

We compare the coefficients before T_N and we have

$$\mathrm{e}^{(N+1)\alpha_1\tau}\mu(\gamma,N) = D(\mu(\gamma,N))\mathrm{e}^{N\alpha_1\tau}$$

which proves that $\mu(\gamma,N)=0$ for any γ with $m(\gamma)=N$ and $l(\gamma)=1$. We have,

$$T_{N+1} = \mu_N T_N + \sum_{T_\beta\in P}\mu(N+1,\beta)T_\beta,$$

here $\mu_N=\mu(N+1,N)$. Then

$$DT_{N+1}D^{-1} = D(\mu_N)DT_ND^{-1} + \sum_{T_\beta\in P} D(\mu(N+1,\beta))DT_\beta D^{-1}.$$

We continue and have,

$$\begin{aligned}
&\mathrm{e}^{(N+1)\alpha_1\tau}\{\mu_N T_N + \sum_{T_\beta\in P}\mu(N+1,\beta)T_\beta\} + \mathrm{e}^{(N+1)\alpha_1\tau}\{c_{N+1}B - A + c_{N+1}\tau A\}T_N\\
&+\tau\mathrm{e}^{(N+1)\alpha_1\tau}\sum_{m(\beta)=N,l(\beta)=1}\nu^*(N+1,\beta)T_\beta + \sum_{m(\beta)\le N-1}\nu(N+1,\beta)T_\beta\\
&= D(\mu_N)\{\mathrm{e}^{N\alpha_1\tau}T_N + \sum_{m(\beta)\le N-1}\nu(N,\beta)T_\beta\}\\
&+\sum_{T_\beta\in P}D(\mu(N+1,\beta))\{\mathrm{e}^{(m(\beta)+l(\beta))\alpha_1\tau}T_\beta + \sum_{m(r)\le N-1}\nu(\beta,r)T_r\}.
\end{aligned}$$

We compare the coefficients before T_N and get

$$\mathrm{e}^{(N+1)\alpha_1\tau}\mu_N + \mathrm{e}^{(N+1)\alpha_1\tau}\{c_{N+1}B - A + c_{N+1}\tau A\} = \mathrm{e}^{N\alpha_1\tau}D(\mu_N).$$

Note that, by property (iv), we do not have term $\tau\mathrm{e}^{(N+1)\alpha_1\tau}$ in the left hand side of the last equality. Thus, using the expression for $A(\tau)=C(\mathrm{e}^{-\alpha_1\tau}-1)$ and the fact that μ_N is a constant, we have

$$B(\tau) = C_1A + C_2\tau A = C_1(\mathrm{e}^{-\alpha_1\tau}-1) + C_2\tau(\mathrm{e}^{-\alpha_1\tau}-1)\,,$$

where

$$C_1 = \frac{\mu_N}{Cc_{N+1}} + \frac{1}{c_{N+1}},\quad C_2=-1.$$

We introduce new vector fields

$$\tilde{S}_0 = \frac{1}{C}S_0 = (\mathrm{e}^{-\alpha_1\tau}-1)\frac{\partial}{\partial\tau}+\ldots,\quad \tilde{S}_1 = \frac{1}{C}S_1 + \frac{C_1}{C}S_0 = \tau(\mathrm{e}^{-\alpha_1\tau}-1)\frac{\partial}{\partial\tau}+\ldots.$$

$$\tilde{S}_2 = [\tilde{S}_0, \tilde{S}_1], \quad \tilde{S}_{n+1} = [\tilde{S}_0, \tilde{S}_n], \quad n \geq 2.$$

We have,

$$D\tilde{S}_0 D^{-1} = \mathrm{e}^{\alpha_1\tau}\tilde{S}_0, \quad D\tilde{S}_1 D^{-1} = \mathrm{e}^{\alpha_1\tau}\tilde{S}_1 - \tau\mathrm{e}^{\alpha_1\tau}\tilde{S}_0,$$
$$D\tilde{S}_n D^{-1} = \sum_{k=0}^{n}\tilde{\gamma}(n,k)\tilde{S}_k, \quad \tilde{\gamma}(n,n) = \mathrm{e}^{n\alpha_1\tau},$$

where $\tilde{\gamma}(n,k)$ are functions of τ only. Since all vector fields $\tilde{S}_k$ belong to a finite dimensional Lie algebra L_x, then there exists a natural number M that

$$\tilde{S}_{M+1} = \tilde{\mu}_M\tilde{S}_M + \tilde{\mu}_{M-1}\tilde{S}_{M-1} + \ldots + \tilde{\mu}_0\tilde{S}_0, \tag{3.37}$$

and $\tilde{S}_M$, $\tilde{S}_{M-1}$, ..., $\tilde{S}_0$ are linearly independent. Then

$$D\tilde{S}_{M+1}D^{-1} = D(\tilde{\mu}_M)D\tilde{S}_M D^{-1} + \ldots + D(\tilde{\mu}_0)D\tilde{S}_0 D^{-1},$$

and

$$\tilde{\gamma}(M+1,M+1)\{\tilde{\mu}_M\tilde{S}_M+\ldots+\tilde{\mu}_0\tilde{S}_0\}+\sum_{k=0}^{M}\tilde{\gamma}(M+1,k)\tilde{S}_k = D(\tilde{\mu}_N)\{\tilde{\gamma}(M,M)\tilde{S}_M+\ldots\}+\ldots.$$

We compare the coefficients before $\tilde{S}_M$ and we have

$$\mathrm{e}^{(M+1)\alpha_1\tau}\tilde{\mu}_M + \tilde{\gamma}(M+1,M) = D(\tilde{\mu}_M)\mathrm{e}^{M\alpha_1\tau},$$

which implies that $\tilde{\mu}_M$ is a constant. In the same way, by comparing the coefficients before $\tilde{S}_{M-1}$, and then before $\tilde{S}_{M-2}$, and so on, we can show that all coefficients $\tilde{\mu}_k$ are constants.

We can show by induction on n that for $n \geq 2$,

$$\tilde{S}_n = \{\alpha_1^{n-2}(-1)^{n-2}(n-2)!\mathrm{e}^{-n\alpha_1\tau} + \sum_{k=0}^{n-1} r(n,k)\mathrm{e}^{-\alpha_1 k\tau}\}\frac{\partial}{\partial\tau} + \ldots,$$

where $r(n,k)$ are some constants. Return to equality (3.37) with constant coefficients $\tilde{\mu}_k$ and compare the coefficients before $\frac{\partial}{\partial\tau}$:

$$\alpha_1^{M-1}(-1)^{M-1}(M-1)!\mathrm{e}^{-(M+1)\alpha_1\tau} + \sum_{k=0}^{M} r(M+1,k)\mathrm{e}^{-\alpha_1 k\tau}$$
$$= \tilde{\mu}_M\Big(\alpha_1^{M-2}(-1)^{M-2}(M-2)!\mathrm{e}^{-M\alpha_1\tau} + \sum_{k=0}^{M-1} r(M,k)\mathrm{e}^{-\alpha_1 k\tau}\Big) + \ldots + \tilde{\mu}_0(\mathrm{e}^{-\alpha_1\tau} - 1).$$

The last equality fails to be true since on the left hand side we have the factor $e^{-(M+1)\alpha_1\tau}$ but on the right hand side we do not. It shows that our assumption that multiplicity m_1 of a nonzero root α_1 can be 2 or more was wrong. This finishes the proof of the Lemma. □

If the characteristic polynomial of (3.14) has only one nonzero root α, then $d(t, t_1) = A(t - t_1)e^{\alpha t}$. In this case equation (3.1) admits a nontrivial x-integral as seen in Theorem 3.1. In the next section we consider a case when the characteristic polynomial of (3.14) has at least two nonzero roots.

3.4 Two nonzero roots

In this section we prove that if the equation (3.1) admits a nontrivial x-integral and if the function $d(t, t_1)$ contains terms with $e^{\alpha_k t}$ and $e^{\alpha_j t}$ having nonzero exponents then $\alpha_k = -\alpha_j$.

Let α and β be two nonzero roots. Consider the vector fields

$$S_0 = \sum_{j=-\infty}^{\infty} A(\tau_j)\mathrm{e}^{\alpha\rho_j}\frac{\partial}{\partial\tau_j}, \quad S_1 = \sum_{j=-\infty}^{\infty} B(\tau_j)\mathrm{e}^{\beta\rho_j}\frac{\partial}{\partial\tau_j}$$

from the Lie algebra L_x, and construct a new sequence of vector fields

$$S_2 = [S_0, S_1], \quad S_{n+1} = [S_0, S_n], \quad n \geq 1.$$

We have,

$$DS_0D^{-1} = \mathrm{e}^{\alpha\tau}S_0, \quad DS_1D^{-1} = \mathrm{e}^{\beta\tau}S_1\,,$$

$$DS_2D^{-1} = \mathrm{e}^{(\alpha+\beta)\tau}S_2 + \beta A\mathrm{e}^{(\alpha+\beta)\tau}S_1 - \alpha B\mathrm{e}^{(\alpha+\beta)\tau}S_0\,.$$

In general, for any $n \geq 3$,

$$\begin{aligned} DS_nD^{-1} &= \mathrm{e}^{((n-1)\alpha+\beta)\tau}\{S_n + (c_n\alpha + d_n\beta)AS_{n-1} \\ &\quad +(p_nA' + q_nA)AS_{n-2} + \sum_{k=0}^{n-2}\nu(n,k)S_k\}\,, \end{aligned} \tag{3.38}$$

where

$$c_n = \frac{(n-1)(n-2)}{2}, \quad d_n = n-1, \quad p_{n+1} = \frac{n(n-1)}{2}\left\{\frac{n-2}{3}\alpha + \beta\right\}, \quad n \geq 2,$$

$$q_{n+1} = \frac{n(n-2)(n-1)(3n-1)}{24}\alpha^2 + \frac{(n-1)^2 n}{2}\alpha\beta + \frac{n(n-1)}{2}\beta^2, \quad n \geq 2.$$

Let us consider a particular case when

$$S_2 = \mu_0 S_0 + \mu_1 S_1. \tag{3.39}$$

We have,

$$\begin{aligned} DS_2D^{-1} &= D(\mu_0)\mathrm{e}^{\alpha\tau}S_0 + D(\mu_1)\mathrm{e}^{\beta\tau}S_1 = \mathrm{e}^{(\alpha+\beta)\tau}S_2 + \beta A\mathrm{e}^{(\alpha+\beta)\tau}S_1 - \alpha B\mathrm{e}^{(\alpha+\beta)\tau}S_0 \\ &= \mathrm{e}^{(\alpha+\beta)\tau}\{\mu_0 S_0 + \mu_1 S_1\} + \beta A\mathrm{e}^{(\alpha+\beta)\tau}S_1 - \alpha B\mathrm{e}^{(\alpha+\beta)\tau}S_0. \end{aligned}$$

We compare the coefficients before S_0 and S_1 and we have the following two equations

$$\mathrm{e}^{(\alpha+\beta)\tau}\mu_0 - \alpha B\mathrm{e}^{(\alpha+\beta)\tau} = D(\mu_0)\mathrm{e}^{\alpha\tau}, \quad \mathrm{e}^{(\alpha+\beta)\tau}\mu_1 + \beta A\mathrm{e}^{(\alpha+\beta)\tau} = D(\mu_1)\mathrm{e}^{\beta\tau}.$$

It follows that μ_0, μ_1 are constants and

$$B(\tau) = -\frac{\mu_0}{\alpha}(\mathrm{e}^{-\beta\tau} - 1), \quad A(\tau) = \frac{\mu_1}{\beta}(\mathrm{e}^{-\alpha\tau} - 1).$$

And finally, we compare the coefficients before $\frac{\partial}{\partial\tau}$ in equation (3.39) and it implies that $\alpha = -\beta$.

Let us return to the general case. Since L_x is of finite dimension then there exists a natural number N such that

$$S_{N+1} = \mu_N S_N + \mu_{N-1}S_{N-1} + \ldots + \mu_0 S_0,$$

and S_0, S_1, …, S_N are linearly independent

Then

$$DS_{N+1}D^{-1} = D(\mu_N)DS_ND^{-1} + D(\mu_{N-1})DS_{N-1}D^{-1} + \ldots + D(\mu_0)DS_0D^{-1}.$$

By using the formula (3.38) we have

$$\mathrm{e}^{(N\alpha+\beta)\tau}\{(\mu_N S_N+\mu_{N-1}S_{N-1}+\ldots)+A(c_{N+1}\alpha+d_{N+1}\beta)S_N+A(p_{N+1}A'+q_{N+1}A)S_{N+1}+\ldots\} =$$

$$D(\mu_N)\{\mathrm{e}^{((N-1)\alpha+\beta)\tau}(S_N+A(c_N\alpha+d_N\beta)S_{N-1}+\ldots)\}+D(\mu_{N-1})\{\mathrm{e}^{((N-2)\alpha+\beta)\tau}S_{N-1}+\ldots\}+\ldots.$$

We compare the coefficients before S_N and it gives

$$\mathrm{e}^{(N\alpha+\beta)\tau}\{\mu_N + A(c_{N+1}\alpha + d_{N+1}\beta)\} = D(\mu_N)\mathrm{e}^{((N-1)\alpha+\beta)\tau}.$$

It follows that μ_N is a constant and then

$$A(c_{N+1}\alpha + d_{N+1}\beta) = \mu_N(\mathrm{e}^{-\alpha\tau} - 1).$$

If $c_{N+1}\alpha + d_{N+1}\beta = N\left\{\frac{N-1}{2}\alpha + \beta\right\} \neq 0$, then

$$A(\tau) = C_1(\mathrm{e}^{-\alpha\tau} - 1),$$

for some constant C_1.

If $c_{N+1}\alpha + d_{N+1}\beta = N\left\{\frac{N-1}{2}\alpha + \beta\right\} = 0$ (in this case $\mu_N = 0$) we compare coefficients before S_{N-1} and have

$$\mathrm{e}^{(N\alpha+\beta)\tau}\{\mu_{N-1} + A(p_{N+1}A' + q_{N+1}A)\} = D(\mu_{N-1})\mathrm{e}^{((N-2)\alpha+\beta)\tau}.$$

It follows that μ_{N-1} is a constant and

$$p_{N+1}AA' + q_{N+1}A^2 = \mu_{N-1}(\mathrm{e}^{-2\alpha\tau} - 1).$$

Note that if $c_{N+1}\alpha+d_{N+1}\beta = N\left\{\frac{N-1}{2}\alpha + \beta\right\} = 0$ then $p_{N+1} = -\frac{N(N-1)(N+1)}{12}\alpha \neq 0$ and $q_{N+1} = -\frac{(N-1)N(N+1)}{24}\alpha^2 \neq 0$ for $N \geq 2$. Therefore, $\left(\frac{2}{q_{N+1}}\right)p_{N+1} = \alpha$. Case $N = 1$ should be studied separately ($S_2 = \mu_1 S_1 + \mu_0 S_0$). But we have already studied this case. Let us solve the equation

$$p_{N+1}AA' + q_{N+1}A^2 = \mu_{N-1}(\mathrm{e}^{-2\alpha\tau} - 1).$$

Denote by $y = A^2$. We have,

$$y' + \alpha y = k_1\mathrm{e}^{-2\alpha\tau} - k_1$$

for some constant k_1. The solution is

$$A^2(\tau) = K_1(\mathrm{e}^{-2\alpha\tau} + K_2\mathrm{e}^{-\alpha\tau} + 1)$$

for some constants K_1 and K_2.

Construct new sequence of vector fields

$$S_2^* = [S_1, S_0], \quad S_{n+1}^* = [S_1, S_n^*], \quad n \geq 2.$$

Note that $S_2^* = -S_2$. Since L_x is of finite dimension then there exists a natural number M such that S_0, S_1, $\ldots$, S_M^* are linearly independent and

$$S_{M+1}^* = \mu_M^* S_M^* + \mu_{M-1}^* S_{M-1}^* + \ldots + \mu_0^* S_0.$$

There are the following possibilities.

$$1) \begin{cases} A(\tau) = K_1(\mathrm{e}^{-\alpha\tau} - 1), \\ B(\tau) = K_3(\mathrm{e}^{-\beta\tau} - 1), \end{cases}$$

$$2) \begin{cases} A(\tau) = K_1(\mathrm{e}^{-\alpha\tau} - 1), \\ B^2(\tau) = K_3^2(\mathrm{e}^{-2\beta\tau} + K_4\mathrm{e}^{-\beta\tau} + 1), \\ S_{M+1}^* = \mu_M^* S_M^* + \mu_{M-1}^* S_{M-1}^* + \ldots + \mu_0^* S_0, \quad \frac{M-1}{2}\beta + \alpha = 0, \end{cases}$$

$$3) \begin{cases} B(\tau) = K_3(\mathrm{e}^{-\beta\tau} - 1), \\ A^2(\tau) = K_1^2(\mathrm{e}^{-2\alpha\tau} + K_2\mathrm{e}^{-\alpha\tau} + 1), \\ S_{N+1} = \mu_N S_N + \mu_{N-1} S_{N-1} + \ldots + \mu_0 S_0, \quad \frac{N-1}{2}\alpha + \beta = 0, \end{cases}$$

$$4) \begin{cases} A^2(\tau) = K_1^2(\mathrm{e}^{-2\alpha\tau} + K_2\mathrm{e}^{-\alpha\tau} + 1), \\ S_{N+1} = \mu_N S_N + \mu_{N-1} S_{N-1} + \ldots + \mu_0 S_0, \quad \frac{N-1}{2}\alpha + \beta = 0, \\ B^2(\tau) = K_3^2(\mathrm{e}^{-2\beta\tau} + K_4\mathrm{e}^{-\beta\tau} + 1), \\ S_{M+1}^* = \mu_M^* S_M^* + \mu_{M-1}^* S_{M-1}^* + \ldots + \mu_0^* S_0, \quad \frac{M-1}{2}\beta + \alpha = 0, \end{cases}$$

where K_1, $K_2 \neq -2$, K_3, $K_4 \neq -2$ are some constants, $M, N \geq 2$.

In case 1), vector fields S_0 and S_1 generate an infinite dimensional Lie algebra L_x unless $\alpha + \beta = 0$.

In case 2), we make a substitution $1-\mathrm{e}^{\alpha\tau}=\mathrm{e}^{-\alpha w}$. Vector fields S_0 and S_1 become

$$S_0=K_1\frac{\partial}{\partial w}+\dots,$$

$$S_1=\{K_3^2((1-\mathrm{e}^{-\alpha w})^{-\frac{2\beta}{\alpha}}+K_4(1-\mathrm{e}^{-\alpha w})^{-\frac{\beta}{\alpha}}+1)\}^{1/2}\frac{\partial}{\partial w}+\dots=g(w)\frac{\partial}{\partial w}+\dots.$$

Note that if

$$S^*_{M+1}=\mu^*_M S^*_M+\mu^*_{M-1}S^*_{M-1}+\dots+\mu^*_0 S_0,$$

then all coefficients μ^*_k are constants. We compare the coefficients before $\frac{\partial}{\partial w}$ in both sides of the last equation and we obtain that $g(w)$ is a solution of linear differential equation with constant coefficients, that is

$$g(w)=\{K_3^2((1-\mathrm{e}^{-\alpha w})^{-\frac{2\beta}{\alpha}}+K_4(1-\mathrm{e}^{-\alpha w})^{-\frac{\beta}{\alpha}}+1)\}^{1/2}=\sum_k R_k(w)\mathrm{e}^{\nu_k w}\,, \quad (3.40)$$

where $R_k(w)$ are some polynomials. We can show that equality (3.40) holds only if $B(\tau)=K_3(\mathrm{e}^{\alpha\tau}+1)$. It can be shown that in case 3) $A(\tau)=K_1(\mathrm{e}^{\beta\tau}+1)$. In case 4) we make substitution $\mathrm{e}^{\alpha\tau}+\frac{K_1}{2}+\sqrt{\mathrm{e}^{2\alpha\tau+K_1\mathrm{e}^{\alpha\tau}+1}}=\mathrm{e}^{\alpha w}$. Then

$$\begin{aligned}
S_0 &= K_1\frac{\partial}{\partial w}+\dots,\\
S_1 &= \Big\{K_3^2\Big(\frac{1}{2}\mathrm{e}^{\alpha w}-\frac{K_1}{2}+\Big(\frac{K_1^2}{8}-\frac{1}{2}\Big)\mathrm{e}^{-\alpha w}\Big)^{-\frac{2\beta}{\alpha}}\\
&+ K_4\Big(\frac{1}{2}\mathrm{e}^{\alpha w}-\frac{K_1}{2}+\Big(\frac{K_1^2}{8}-\frac{1}{2}\Big)\mathrm{e}^{-\alpha w}\Big)^{-\frac{\beta}{\alpha}}+1\Big)\Big\}^{1/2}\frac{\partial}{\partial w}+\dots\\
&= g(w)\frac{\partial}{\partial w}+\dots
\end{aligned}$$

For function $g(w)$ to be of the form $\sum\limits_k R_k(w)\mathrm{e}^{\nu_k w}$, where $R_k(w)$ are polynomials, function $B(\tau)$ has to be of the form $B(\tau)=K_3(\mathrm{e}^{\alpha\tau}+1)$. Then, by case 3), $A(\tau)=K_1(\mathrm{e}^{-\alpha\tau}+1)$.

It has been proved that in cases 1), 2), 3), 4) we have

$$1^*)\begin{cases} A(\tau)=K_1(\mathrm{e}^{-\alpha\tau}-1),\\ B(\tau)=K_3(\mathrm{e}^{\alpha\tau}-1),\end{cases}$$

$$2^*)\begin{cases} A(\tau)=K_1(\mathrm{e}^{-\alpha\tau}-1),\\ B(\tau)=K_3(\mathrm{e}^{\alpha\tau}+1),\end{cases}$$

$$3^*)\begin{cases} A(\tau) = K_1(\mathrm{e}^{-\alpha\tau}+1), \\ B(\tau) = K_3(\mathrm{e}^{\alpha\tau}-1), \end{cases}$$

$$4^*)\begin{cases} A(\tau) = K_1(\mathrm{e}^{-\alpha\tau}+1), \\ B(\tau) = K_3(\mathrm{e}^{\alpha\tau}+1). \end{cases}$$

In case 1^*) function $d(t,t_1)$ in (3.1) has a form $d(t,t_1) = c_4(e^{\alpha t_1}-e^{\alpha t})+c_5(e^{-\alpha t_1}-e^{-\alpha t})$, where c_4 and c_5 are some constants. Equation (3.1) with such function $d(t,t_1)$ admits a nontrivial x-integral as seen in Theorem 3.1.

In the next two sections we show that Cases 3^*) and 4^*) both correspond to infinite dimensional Lie algebra L_x. Case 2^*) also produces an infinite dimensional Lie algebra L_x. It can be proved in the same way as it is proved for case 3^*).

3.5 Characteristic Lie Algebra L_x of the chain

$t_{1x} = t_x + A_1(\mathrm{e}^{\alpha t_1}+\mathrm{e}^{\alpha t}) - A_2(\mathrm{e}^{-\alpha t}-\mathrm{e}^{-\alpha t_1})$

Since $A(\tau) = A_1(\mathrm{e}^{-\alpha\tau}+1)$ and $B(\tau) = A_2(\mathrm{e}^{\alpha\tau}-1)$ then

$$A(\tau)\mathrm{e}^{\alpha t} + \sum_{j=1}^{k} A(\tau_j)\mathrm{e}^{\alpha t_j} = A_1\Big(\mathrm{e}^{\alpha t} + \Big(2\sum_{j=1}^{k-1}\mathrm{e}^{\alpha t_j}\Big) + \mathrm{e}^{\alpha t_k}\Big),$$

and

$$B(\tau)\mathrm{e}^{-\alpha t} + \sum_{j=1}^{k} B(\tau_j)\mathrm{e}^{-\alpha t_j} = A_2(\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\alpha t_k}).$$

We have,

$$\begin{aligned}\frac{1}{A_1}S_0 \;&=\; (\mathrm{e}^{\alpha t}+\mathrm{e}^{\alpha t_1})\frac{\partial}{\partial t_1} + \sum_{k=1}^{\infty}\Big(\mathrm{e}^{\alpha t} + \Big(2\sum_{j=1}^{k-1}\mathrm{e}^{\alpha t_j}\Big) + \mathrm{e}^{\alpha t_k}\Big)\frac{\partial}{\partial t_k} \\ &\quad + \sum_{k=1}^{\infty}\Big(\mathrm{e}^{\alpha t} + \Big(2\sum_{j=1}^{k-1}\mathrm{e}^{\alpha t_{-j}}\Big) + \mathrm{e}^{\alpha t_{-k}}\Big)\frac{\partial}{\partial t_{-k}},\end{aligned}$$

and

$$\frac{1}{A_2}S_1 = \mathrm{e}^{-\alpha\tau}\tilde{X} - \sum_{k=-\infty}^{\infty}\mathrm{e}^{-\alpha t_k}\frac{\partial}{\partial t_k} = \mathrm{e}^{-\alpha\tau}\tilde{X} - \tilde{S}_1,$$

where

$$\tilde{S}_1 = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\alpha t_k} \frac{\partial}{\partial t_k}.$$

Introduce new variables $w_j = \frac{1}{\alpha}\mathrm{e}^{\alpha t_j}$ and so vector fields $\tilde{S}_1$ and $\frac{1}{A_1}S_0$ can be rewritten as

$$\tilde{S}_1 = \sum_{k=-\infty}^{\infty} \frac{\partial}{\partial w_j},$$

$$\frac{1}{A_1}S_0 = \alpha^2 \sum_{k=1}^{\infty}\{w_k(w+2\sum_{j=1}^{k-1} w_j)+w_k^2\}\frac{\partial}{\partial w_k}+\alpha^2\sum_{k=1}^{\infty}\{w_{-k}(w+2\sum_{j=1}^{k-1} w_{-j})+w_{-k}^2\}\frac{\partial}{\partial w_{-k}}.$$

We have

$$T_1 = [\tilde{S}_1, [\tilde{S}_1, \frac{1}{\alpha^2 A_1}S_0]] = 4\sum_{k=-\infty}^{\infty} k\frac{\partial}{\partial w_k} = 4\tilde{T}_1, \quad \tilde{T}_1 = \sum_{k=-\infty}^{\infty} k\frac{\partial}{\partial w_k},$$

$$T_2 = [\tilde{S}_1, [\tilde{T}_1, \frac{1}{\alpha^2 A_1}S_0]] = 3\sum_{k=1}^{\infty}\{k^2-k+1\}(\frac{\partial}{\partial w_k}+\frac{\partial}{\partial w_{-k}}) = 3\tilde{T}_2 - 3\tilde{T}_1 + 3\tilde{S}_1,$$

$$\tilde{T}_2 = \sum_{k=-\infty}^{\infty} k^2\frac{\partial}{\partial w_k}.$$

Assume that $\tilde{T}_m = \sum\limits_{k=-\infty}^{\infty} k^m \frac{\partial}{\partial w_k}$, $m = 1, 2\ldots, n$, are vector fields from L_x. Then

$$\begin{aligned}
T_{m+1} &= [\tilde{S}_1, [\tilde{T}_m, \frac{1}{\alpha^2 A_1}S_0]] \\
&= \sum_{k=1}^{\infty}\{2(1+2^m+3^m+\ldots+k^m)+2k^{m+1}-k^m\}\Big(\frac{\partial}{\partial w_k}+\frac{\partial}{\partial w_{-k}}\Big) \\
&= \sum_{k=1}^{\infty}\Big\{2\Big(\frac{k^{m+1}}{m+1}+d_{m,m+1}k^m+\ldots+d_{0,m+1}\Big)+2k^{m+1}-k^m\Big\}\Big(\frac{\partial}{\partial w_k}+\frac{\partial}{\partial w_{-k}}\Big)
\end{aligned}$$

and therefore, $\tilde{T}_{m+1} = \sum\limits_{k=-\infty}^{\infty} k^{m+1}\frac{\partial}{\partial w_k} \in L_x$. It shows that $\tilde{T}_n = \sum\limits_{k=-\infty}^{\infty} k^n\frac{\partial}{\partial w_k} \in L_x$ for all $n = 1, 2, 3, \ldots$, and L_x is of infinite dimension.

3.6 Characteristic Lie Algebra L_x of the chain

$$t_{1x} = t_x + A_1(\mathrm{e}^{\alpha t_1} + \mathrm{e}^{\alpha t}) + A_2(\mathrm{e}^{-\alpha t} + \mathrm{e}^{-\alpha t_1})$$

It is seen in previous studies (see, for instance, [16]) that S-integrable models have the characteristic Lie algebra of finite growth. The chain studied in this section can be easily reduced to the semi-discrete sine-Gordon model $t_{1x} = t_x + \sin t + \sin t_1$, which belongs to the S-integrable class. It is remarkable that its characteristic Lie algebra L_x is of finite growth. Or, more exactly, the dimension of the linear space of multiple commutators grows linearly with the multiplicity. Below we prove that the linear space V_n of all commutators of multiplicity $\leq n$ has a basis $\{P_1, P_2, P_3, ...P_{2k}; Q_2, Q_4, ...Q_{2k}\}$ for $n = 2k$ and a basis $\{P_1, P_2, P_3, ...P_{2k+1}; Q_2, Q_4, ...Q_{2k}\}$ for $n = 2k+1$, where the operators P_j and Q_j are defined consecutively

$$
\begin{array}{ll}
P_1 = [S_0, S_1] + \alpha S_0 + \alpha S_1, & Q_1 = P_1, \\
P_2 = [S_1, P_1], & Q_2 = [S_0, Q_1], \\
P_3 = [S_0, P_2] + \alpha P_2, & Q_3 = [S_1, Q_2] - \alpha Q_2, \\
P_{2n} = [S_1, P_{2n-1}], & Q_{2n} = [S_0, Q_{2n-1}], \\
P_{2n+1} = [S_0, P_{2n}] + \alpha P_{2n}, & Q_{2n+1} = [S_1, Q_{2n}] - \alpha Q_{2n},
\end{array}
$$

for $n \geq 1$. Direct calculations show that

$$
\begin{array}{rcl}
DP_1D^{-1} & = & P_1 - 2\alpha(S_0 + S_1), \\
DP_2D^{-1} & = & \mathrm{e}^{-\alpha\tau}(P_2 + 2\alpha P_1 - 2\alpha^2(S_0 + S_1)), \\
DP_3D^{-1} & = & P_3 + 2\alpha Q_2 - 2\alpha P_2 - 4\alpha^2 P_1 + 4\alpha^3(S_0 + S_1), \\
DP_4D^{-1} & = & \mathrm{e}^{-\alpha\tau}(P_4 + 2\alpha Q_3 - 4\alpha^2 P_2 + 4\alpha^2 Q_2 - 4\alpha^3 P_1 + 4\alpha^4(S_0 + S_1)), \\
DQ_2D^{-1} & = & \mathrm{e}^{\alpha\tau}(Q_2 - 2\alpha P_1 + 2\alpha^2(S_0 + S_1)), \\
DQ_3D^{-1} & = & Q_3 + 2\alpha Q_2 - 2\alpha P_2 - 4\alpha^2 P_1 + 4\alpha^3(S_0 + S_1), \\
DQ_4D^{-1} & = & \mathrm{e}^{\alpha\tau}(Q_4 - 2\alpha P_3 + 2\alpha^2(P_2 - Q_2) + 4\alpha^3 P_1 - 4\alpha^4(S_0 + S_1)), \\
P_3 = Q_3 & , & [S_1, P_2] = -\alpha P_2, [S_0, Q_2] = \alpha Q_2, [S_1, P_4] = -\alpha P_4, [S_0, Q_4] = \alpha Q_4.
\end{array}
\tag{3.41}
$$

The coefficient before $\dfrac{\partial}{\partial \tau}$ in all vector fields DP_iD^{-1}, DQ_iD^{-1}, $1 \leq i \leq 4$ is zero.

Lemma 3.14 *For $n \geq 1$ we have,*

(1) $DP_{2n+1}D^{-1} + 2\alpha \mathrm{e}^{\alpha\tau} DP_{2n}D^{-1} = P_{2n+1} + 2\alpha Q_{2n}$,

(2) $e^{\alpha\tau}DP_{2n+2}D^{-1} - \alpha DP_{2n+1}D^{-1} = P_{2n+2} + \alpha Q_{2n+1}$,

(3) $DQ_{2n+1}D^{-1} - 2\alpha e^{-\alpha\tau}DQ_{2n}D^{-1} = Q_{2n+1} - 2\alpha P_{2n}$,

(4) $e^{-\alpha\tau}DQ_{2n+2}D^{-1} + \alpha DQ_{2n+1}D^{-1} = Q_{2n+2} - \alpha P_{2n+1}$,

(5) $P_{2n+1} = Q_{2n+1}$,

(6) $[S_1, P_{2n+2}] = -\alpha P_{2n+2}$,

(7) $[S_0, Q_{2n+2}] = \alpha Q_{2n+2}$.

Moreover, the coefficient before $\dfrac{\partial}{\partial\tau}$ *in all vector fields* DP_kD^{-1}, DQ_kD^{-1} *is zero.*

Proof. We prove the Lemma by induction on n. It follows from (3.41) that the base of induction holds for $n = 1$. Assume (1) – (7) are true for all n, $1 \le n \le k$. Let us prove that (1) is true for $n = k + 1$.

$$
\begin{aligned}
&DP_{2n+3}D^{-1} = D([S_0, P_{2n+2}] + \alpha P_{2n+2})D^{-1} = [e^{\alpha\tau}S_0, DP_{2n+2}D^{-1}] + \alpha DP_{2n+2}D^{-1}\\
&= [e^{\alpha\tau}S_0, \alpha e^{-\alpha\tau}DP_{2n+1}D^{-1} + e^{-\alpha\tau}P_{2n+2} + \alpha e^{-\alpha\tau}Q_{2n+1}] + \alpha DP_{2n+2}D^{-1}\\
&= -\alpha^2(1 + e^{-\alpha\tau})DP_{2n+1}D^{-1} + \alpha e^{-\alpha\tau}[e^{\alpha\tau}S_0, DP_{2n+1}D^{-1}] - \alpha(1 + e^{-\alpha\tau})P_{2n+2}\\
&\quad -\alpha^2(1 + e^{-\alpha\tau})Q_{2n+1} + P_{2n+3} - \alpha P_{2n+2} + \alpha Q_{2n+2} + \alpha DP_{2n+2}D^{-1}\\
&= -\alpha^2(1 + e^{-\alpha\tau})DP_{2n+1}D^{-1} + \alpha e^{-\alpha\tau}D[S_0, Q_{2n+1}]D^{-1} - \alpha(2 + e^{-\alpha\tau})P_{2n+2}\\
&\quad -\alpha^2(1 + e^{-\alpha\tau})Q_{2n+1} + P_{2n+3} + \alpha Q_{2n+2} + \alpha DP_{2n+2}D^{-1}\\
&= -\alpha^2(1 + e^{-\alpha\tau})DP_{2n+1}D^{-1} + \alpha Q_{2n+2} - \alpha^2 P_{2n+1} - \alpha^2 DQ_{2n+1}D^{-1} - \alpha(2 + e^{-\alpha\tau})P_{2n+2}\\
&\quad -\alpha^2(1 + e^{-\alpha\tau})Q_{2n+1} - 2\alpha^2 Q_{2n+1} - 2\alpha P_{2n+2} + P_{2n+3}\\
&= -2\alpha^2 DP_{2n+1}D^{-1} + 2\alpha Q_{2n+2} - 2\alpha^2 Q_{2n+1} - 2\alpha P_{2n+2} + P_{2n+3}\\
&= 2\alpha P_{2n+2} + 2\alpha^2 Q_{2n+1} - 2\alpha e^{\alpha\tau}DP_{2n+2}D^{-1} + 2\alpha Q_{2n+2} - 2\alpha^2 Q_{2n+1} - 2\alpha P_{2n+2} + P_{2n+3}\\
&= -2\alpha e^{\alpha\tau}DP_{2n+2}D^{-1} + 2\alpha Q_{2n+2} + P_{2n+3}.
\end{aligned}
$$

The proof of (3) is the same as the proof of (1). Let us show that (5) is true for $n = k + 1$. We have,

$$
\begin{aligned}
DP_{2n+3}D^{-1} &= -2\alpha e^{\alpha\tau}DP_{2n+2}D^{-1} + 2\alpha Q_{2n+2} + P_{2n+3}\\
&= -2\alpha(\alpha DP_{2n+1}D^{-1} + P_{2n+2} + \alpha Q_{2n+1}) + 2\alpha Q_{2n+2} + P_{2n+3},
\end{aligned}
$$

and

$$\begin{aligned} DQ_{2n+3}D^{-1} &= 2\alpha \mathrm{e}^{-\alpha\tau} DQ_{2n+2}D^{-1} - 2\alpha P_{2n+2} + Q_{2n+3} \\ &= 2\alpha(-\alpha DQ_{2n+1}D^{-1} + Q_{2n+2} - \alpha P_{2n+1}) - 2\alpha P_{2n+2} + Q_{2n+3}. \end{aligned}$$

By (5), $P_{2n+1} = Q_{2n+1}$ and therefore

$$D(P_{2n+3} - Q_{2n+3})D^{-1} = -2\alpha P_{2n+2} - 2\alpha Q_{2n+2} + 2\alpha Q_{2n+2} + 2\alpha P_{2n+2} = 0.$$

Hence, $P_{2n+3} = Q_{2n+3}$.

Let us prove (2) is true for $n = k + 1$. We have,

$$\begin{aligned} &\mathrm{e}^{\alpha\tau} DP_{2n+1}D^{-1} = \mathrm{e}^{\alpha\tau} D[S_1, P_{2n+3}]D^{-1} = \mathrm{e}^{\alpha\tau}[\mathrm{e}^{-\alpha\tau} S_1, DP_{2n+3}D^{-1}] \\ &= \mathrm{e}^{\alpha\tau}[\mathrm{e}^{-\alpha\tau} S_1, -2\alpha \mathrm{e}^{\alpha\tau} DP_{2n+2}D^{-1} + 2\alpha Q_{2n+2} + P_{2n+3}] \\ &= \mathrm{e}^{\alpha\tau}(-2\alpha^2(1 + \mathrm{e}^{\alpha\tau})DP_{2n+2}D^{-1}) - 2\alpha \mathrm{e}^{2\alpha\tau}[\mathrm{e}^{-\alpha\tau} S_1, DP_{2n+2}D^{-1}] + P_{2n+4} \\ &+2\alpha Q_{2n+3} + 2\alpha^2 Q_{2n+2} \\ &= -2\alpha^2(\mathrm{e}^{\alpha\tau} + \mathrm{e}^{2\alpha\tau})DP_{2n+2}D^{-1} + 2\alpha^2 \mathrm{e}^{2\alpha\tau} DP_{2n+2}D^{-1} + P_{2n+4} \\ &+2\alpha Q_{2n+3} + 2\alpha^2 Q_{2n+2} \\ &= -2\alpha^2 \mathrm{e}^{\alpha\tau} DP_{2n+2}D^{-1} + P_{2n+4} + 2\alpha Q_{2n+3} + 2\alpha^2 Q_{2n+2} \\ &= \alpha DP_{2n+3}D^{-1} - \alpha P_{2n+3} - 2\alpha^2 Q_{2n+2} + P_{2n+4} + 2\alpha Q_{2n+3} + 2\alpha^2 Q_{2n+2} \\ &= \alpha DP_{2n+3}D^{-1} + \alpha Q_{2n+3} + P_{2n+4}. \end{aligned}$$

The proof of (4) is similar to the proof of (2).

Let us prove that (6) is true for $n = k + 1$.

$$\begin{aligned}
&D[S_1, P_{2n+4}]D^{-1} = [\mathrm{e}^{-\alpha\tau}S_1, \alpha\mathrm{e}^{-\alpha\tau}DP_{2n+3}D^{-1} + \mathrm{e}^{-\alpha\tau}P_{2n+4} + \alpha\mathrm{e}^{-\alpha\tau}Q_{2n+3}]\\
&= [\mathrm{e}^{-\alpha\tau}S_1, \alpha\mathrm{e}^{-\alpha\tau}(-2\alpha\mathrm{e}^{\alpha\tau}DP_{2n+2}D^{-1} + P_{2n+3} + 2\alpha Q_{2n+2}) + \mathrm{e}^{-\alpha\tau}P_{2n+4} + \alpha\mathrm{e}^{-\alpha\tau}Q_{2n+3}]\\
&= [\mathrm{e}^{-\alpha\tau}S_1, -2\alpha^2DP_{2n+2}D^{-1} + 2\alpha\mathrm{e}^{-\alpha\tau}P_{2n+3} + 2\alpha^2\mathrm{e}^{-\alpha\tau}Q_{2n+2} + \mathrm{e}^{-\alpha\tau}P_{2n+4}]\\
&= -2\alpha^2D[S_1, P_{2n+2}]D^{-1} - 2\alpha^2\mathrm{e}^{-2\alpha\tau}(1 + \mathrm{e}^{\alpha\tau})P_{2n+3} - 2\alpha^3\mathrm{e}^{-2\alpha\tau}(1 + \mathrm{e}^{\alpha\tau})Q_{2n+2}\\
&\quad +2\alpha\mathrm{e}^{-2\alpha\tau}P_{2n+4} + 2\alpha^2\mathrm{e}^{-2\alpha\tau}Q_{2n+3} + 2\alpha^3\mathrm{e}^{-2\alpha\tau}Q_{2n+2} - \alpha\mathrm{e}^{-2\alpha\tau}(1 + \mathrm{e}^{\alpha\tau})P_{2n+4}\\
&\quad +\mathrm{e}^{-2\alpha\tau}[S_1, P_{2n+4}]\\
&= 2\alpha^3DP_{2n+2}D^{-1} - 2\alpha^2\mathrm{e}^{-\alpha\tau}P_{2n+3} + \alpha(\mathrm{e}^{-2\alpha\tau} - \mathrm{e}^{-\alpha\tau})P_{2n+4}\\
&\quad -2\alpha^3\mathrm{e}^{-\alpha\tau}Q_{2n+2} + \mathrm{e}^{-2\alpha\tau}[S_1, P_{2n+4}]\\
&= \alpha^2\mathrm{e}^{-\alpha\tau}P_{2n+3} + 2\alpha^3\mathrm{e}^{-\alpha\tau}Q_{2n+2} - \alpha^2\mathrm{e}^{-\alpha\tau}DP_{2n+3}D^{-1} - 2\alpha^2\mathrm{e}^{-\alpha\tau}P_{2n+3}\\
&\quad +\alpha(\mathrm{e}^{-2\alpha\tau} - \mathrm{e}^{-\alpha\tau})P_{2n+4} - 2\alpha^3\mathrm{e}^{-\alpha\tau}Q_{2n+2} + \mathrm{e}^{-2\alpha\tau}[S_1, P_{2n+4}]\\
&= -\alpha^2\mathrm{e}^{-\alpha\tau}P_{2n+3} + \alpha(\mathrm{e}^{-2\alpha\tau} - \mathrm{e}^{-\alpha\tau})P_{2n+4} - \alpha DP_{2n+4}D^{-1} + \alpha\mathrm{e}^{-\alpha\tau}P_{2n+4}\\
&\quad +\alpha^2\mathrm{e}^{-\alpha\tau}Q_{2n+3} + \mathrm{e}^{-2\alpha\tau}[S_1, P_{2n+4}].
\end{aligned}$$

Thus,

$$\begin{aligned}
&D[S_1, P_{2n+4}]D^{-1} = \mathrm{e}^{-2\alpha\tau}[S_1, P_{2n+4}] + \alpha\mathrm{e}^{-2\alpha\tau}P_{2n+4} - \alpha DP_{2n+4}D^{-1}\\
&D([S_1, P_{2n+4}] + \alpha P_{2n+4})D^{-1} = \mathrm{e}^{-2\alpha\tau}([S_1, P_{2n+4}] + \alpha P_{2n+4}).
\end{aligned}$$

Hence, $[S_1, P_{2n+4}] = -\alpha P_{2n+4}$.

Proof of (7) is similar to the proof of (6). $\square$

Corollary 3.15 *We have,*

$$\begin{aligned}
&\mathrm{e}^{-\alpha\tau} DQ_{2n}D^{-1} + \mathrm{e}^{\alpha\tau} DP_{2n}D^{-1} = Q_{2n} + P_{2n},\\
&DP_{2n+1}D^{-1} = P_{2n+1} + \sum_{k=1}^{n}(\mu_{2k}^{(2n+1)} P_{2k} + \nu_{2k}^{(2n+1)} Q_{2k}) + \sum_{k=0}^{n-1} \mu_{2k+1}^{(2n+1)} P_{2k+1}\\
&\qquad\qquad +\mu_0^{(2n+1)} S_0 + \nu_0^{(2n+1)} S_1,\\
&DP_{2n}D^{-1} = \mathrm{e}^{-\alpha\tau}\Big(P_{2n} + \sum_{k=1}^{n-1}(\mu_{2k}^{(2n)} P_{2k} + \nu_{2k}^{(2n)} Q_{2k})\\
&\qquad\qquad + \sum_{k=0}^{n-1} \mu_{2k+1}^{(2n)} P_{2k+1} + \mu_0^{(2n)} S_0 + \nu_0^{(2n)} S_1\Big),\\
&DQ_{2n}D^{-1} = \mathrm{e}^{\alpha\tau}\Big(Q_{2n} - \sum_{k=1}^{n-1}(\mu_{2k}^{(2n)} P_{2k} + \nu_{2k}^{(2n)} Q_{2k})\\
&\qquad\qquad - \sum_{k=0}^{n-1} \mu_{2k+1}^{(2n)} P_{2k+1} - \mu_0^{(2n)} S_0 - \nu_0^{(2n)} S_1\Big).
\end{aligned}$$

Moreover, $\mu_{2n}^{(2n+1)} = -2\alpha$, $\nu_{2n}^{(2n+1)} = 2\alpha$, $\mu_{2n-1}^{(2n)} = 2\alpha$.

Assume L_x is of finite dimension. There are three possibilities:

1) $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, ..., P_{2n-1}$ are linearly independent and $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, ..., P_{2n-1}, P_{2n}$ are linearly dependent,

2) $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, ..., P_{2n-1}, P_{2n}$ are linearly independent and $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, ..., P_{2n-1}, P_{2n}, Q_{2n}$ are linearly dependent,

3) $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, ..., P_{2n}, Q_{2n}$ are linearly independent and $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, ..., P_{2n}, Q_{2n}, P_{2n+1}$ are linearly dependent.

In case 1),

$$P_{2n} = \gamma_{2n-1}P_{2n-1} + \gamma_{2n-2}P_{2n-2} + \eta_{2n-2}Q_{2n-2} + ...$$

and

$$DP_{2n}D^{-1} = D(\gamma_{2n-1})DP_{2n-1}D^{-1} + D(\gamma_{2n-2})DP_{2n-2}D^{-1} + D(\eta_{2n-2})DQ_{2n-2}D^{-1} + \dots \tag{3.42}$$

We use Corollary 3.15 to compare the coefficients before P_{2n-1} in (3.42) and have the contradictory equality,

$$e^{-\alpha\tau}(\gamma_{2n-1} + 2\alpha) = D(\gamma_{2n-1}).$$

It shows that case 1) is impossible to have.

In case 2),

$$Q_{2n} = \gamma_{2n}P_{2n} + \gamma_{2n-1}P_{2n-1} + \eta_{2n-2}Q_{2n-2} + ...$$

and

$$DQ_{2n}D^{-1} = D(\gamma_{2n})DP_{2n}D^{-1} + D(\gamma_{2n-1})DP_{2n-1}D^{-1} + D(\eta_{2n-2})DQ_{2n-2}D^{-1} + \dots . \tag{3.43}$$

We use Corollary 3.15 to compare the coefficients before P_{2n-1} in (3.43) and have the contradictory equation,

$$e^{\alpha\tau}(\gamma_{2n-1} - 2\alpha) = D(\gamma_{2n-1}).$$

It shows that case 2) is impossible to have.

In case 3),

$$P_{2n+1} = \eta_{2n}Q_{2n} + \gamma_{2n}P_{2n} + ...$$

and

$$DP_{2n+1}D^{-1} = D(\eta_{2n})DQ_{2n}D^{-1} + D(\gamma_{2n})DP_{2n}D^{-1} + \dots . \tag{3.44}$$

We use Corollary 3.15 to compare the coefficients before P_{2n} in (3.44) and have the contradictory equation,

$$(\gamma_{2n} - 2\alpha) = D(\gamma_{2n})e^{-\alpha\tau}.$$

It shows that case 3) also fails to be true. Therefore, characteristic Lie algebra L_x is of infinite dimension.

3.7 Finding x-integrals

In this section, we will complete the proof of Theorem 3.1, given in the beginning of the Chapter. In the previous sections we proved that if chain (3.1) admits a nontrivial x-integral then it is one of the forms $(1)-(4)$. The list $i)-iv)$ allows us to prove the inverse statement: each of the equations from the list admits indeed a nontrivial x-integral.

Let us explain briefly how we found the list $i)-iv)$. Since for each equation $(1)-(4)$ we have constructed the related characteristic Lie algebra to find x-integral F we have to solve the corresponding system of the first order partial differential equations. Below we illustrate the method with the case (2), for which the basis of the characteristic Lie algebra L_x is given by the vector fields

$$\tilde{Y}=\partial_x+Y_{a(\tau)t+b(\tau)},\quad T_1=Y_{-a(\tau)},\quad \tilde{X}=\frac{\partial}{\partial t}+\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_{-1}}+\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_{-2}}+\dots,$$

where $a(\tau)=c_0\tau$ and $b(\tau)=c_2\tau^2+c_3\tau$. Note that x-integral F of (2) should satisfy the equations $\tilde{Y}F=0$, $T_1F=0$ and $\tilde{X}F=0$. Introduce new variables t,w,$w_{\pm1},\dots$ where $w_j=\ln(\tau_j)$ and $\tau_j=t_j-t_{j+1}$. We can rewrite the vector fields $\tilde{X}$, T_1, $\tilde{Y}$ in new variables as

$$\begin{aligned}
\tilde{X} &= \frac{\partial}{\partial t},\quad T_1=\sum_{j=-\infty}^{\infty}c_0\frac{\partial}{\partial w_j},\\
\tilde{Y} &= \frac{\partial}{\partial x}-t\sum_{j=-\infty}^{\infty}c_0\frac{\partial}{\partial w_j}+c_0\sum_{j=-\infty}^{\infty}\{\tilde{\rho}_j+\tilde{b}(w_j)\}\frac{\partial}{\partial w_j}\\
&= \frac{\partial}{\partial x}-tT_1+c_0\sum_{j=-\infty}^{\infty}\{\tilde{\rho}_j+\tilde{b}(w_j)\}\frac{\partial}{\partial w_j},
\end{aligned}$$

where

$$\tilde{\rho}_j=\begin{cases}\sum\limits_{k=0}^{j-1}\mathrm{e}^{w_k}, & \text{if}\quad j\geq 1;\\ 0, & \text{if}\quad j=0;\\ -\sum\limits_{k=j}^{-1}\mathrm{e}^{w_k}, & \text{if}\quad j\leq -1,\end{cases}\qquad \tilde{b}(w_j)=-\frac{1}{c_0}(c_2\mathrm{e}^{w_j}+c_3)\,.$$

Note that since we have $\tilde{X}F=0$, F does not depend on t. Now let us consider the vector field

$$\tilde{Y} + tT_1 = A = \frac{\partial}{\partial x} + c_0 \sum_{j=-\infty}^{\infty} \{\tilde{\rho}_j + \tilde{b}(w_j)\} \frac{\partial}{\partial w_j}.$$

We can write the vector field A explicitly as

$$\begin{aligned} A &= \frac{\partial}{\partial x} + \sum_{j=-\infty}^{\infty} \left\{ \left(c_0 \sum_{k=0}^{j-1} \mathrm{e}^{w_k} \right) - c_2 \mathrm{e}^{w_j} - c_3 \right\} \frac{\partial}{\partial w_j} \\ &= \frac{\partial}{\partial x} - \frac{c_3}{c_0} T_1 + \sum_{j=-\infty}^{\infty} \left\{ \left(c_0 \sum_{k=0}^{j-1} \mathrm{e}^{w_k} \right) - c_2 \mathrm{e}^{w_j} \right\} \frac{\partial}{\partial w_j}. \end{aligned}$$

The commutator $[T_1, A]$ gives

$$[T_1, A] = c_0 A - c_0 \frac{\partial}{\partial x} + c_3 T_1.$$

Thus we have three vector fields

$$\begin{aligned} A - \frac{\partial}{\partial x} + \frac{c_3}{c_0} T_1 := \tilde{A} &= \sum_{j=-\infty}^{\infty} \left\{ \left(c_0 \sum_{k=0}^{j-1} \mathrm{e}^{w_k} \right) - c_2 \mathrm{e}^{w_j} \right\} \frac{\partial}{\partial w_j}, \\ \frac{T_1}{c_0} := \tilde{T}_1 &= \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial w_j}, \quad \tilde{X}_1 = \frac{\partial}{\partial x}, \end{aligned}$$

which solve $\tilde{A}F = 0$, $\tilde{T}_1 F = 0$, $\tilde{X}_1 F = 0$. Note that $[\tilde{T}_1, \tilde{A}] = \tilde{A}$. Since $\tilde{X}_1 F = 0$, F does not depend on x. Hence we end up with two equations. By Jacobi theorem the system of equations has a nontrivial solution $F(w, w_1, w_2)$ depending on three variables. Therefore we need first three terms of $\tilde{A}$ and $\tilde{T}_1$;

$$\begin{aligned} \tilde{A} &= -c_2 w \frac{\partial}{\partial w} + (c_0 \mathrm{e}^{w} - c_2 \mathrm{e}^{w_1}) \frac{\partial}{\partial w_1} + (c_0 \mathrm{e}^{w} + c_0 \mathrm{e}^{w_1} - c_2 \mathrm{e}^{w_2}) \frac{\partial}{\partial w_2}, \\ \tilde{T}_1 &= \frac{\partial}{\partial w} + \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}. \end{aligned}$$

Now we again introduce new variables $w = \epsilon, \quad w - w_1 = \epsilon_1, \quad w_1 - w_2 = \epsilon_2$. We can rewrite the vector fields $\tilde{A}$ and $\tilde{T}_1$ in new variables as

$$\begin{aligned} \tilde{A} &= \mathrm{e}^{\epsilon} \Big\{ -c_2 \frac{\partial}{\partial \epsilon} + ((-c_2 - c_0) + c_2 \mathrm{e}^{-\epsilon_1}) \frac{\partial}{\partial \epsilon_1} + ((-c_2 - c_0) \mathrm{e}^{-\epsilon_1} + c_2 \mathrm{e}^{-\epsilon_1 - \epsilon_2}) \frac{\partial}{\partial \epsilon_2} \Big\}, \\ \tilde{T}_1 &= \frac{\partial}{\partial \epsilon}. \end{aligned}$$

We find the x-integral *ii)* in Theorem 3.1 by solving (use the characteristic method) the equation

$$\Big\{ ((-c_2 - c_0) + c_2 \mathrm{e}^{-\epsilon_1}) \frac{\partial}{\partial \epsilon_1} + \mathrm{e}^{-\epsilon_1} ((-c_2 - c_0) + c_2 \mathrm{e}^{-\epsilon_2}) \frac{\partial}{\partial \epsilon_2} \Big\} F = 0.$$

Chapter 4

Equations Admitting Both x- and n-integrals

Here we analyze the equations given in Theorem 3.1. We check whether these equations having nontrivial x-integrals also have nontrivial n-integrals.

4.1 Case 1) $t_{1x} = t_x + A(t - t_1)$

Introduce $\omega = t_1 - t$ and also to express the equation in a simpler form write $B(t_1 - t)$ instead of $A(t - t_1)$. We can do this since A is an arbitrary function of $t - t_1$. Hence we can rewrite the equation as $\omega_x = B(\omega)$. We study the question when this equation admits a nontrivial n-integral or the same when the corresponding Lie algebra L_n is of finite dimension.

Since in this case

$$Y_0 f = B'(\omega)\omega_{t_1} = D_\omega B(\omega),$$

$$Y_0 f_x = B''(\omega)B(\omega) + B'(\omega)B'(\omega) = D_\omega B(\omega) D_\omega B(\omega),$$

and $Y_0 D_x^k f = (D_\omega B(\omega))^{k+1}$, we can write Y_1 as

$$Y_1 = \frac{\partial}{\partial t} + \sum_{k=1}^{\infty} D^{-1}(D_\omega B(\omega))^k \frac{\partial}{\partial D_x^k t}. \tag{4.1}$$

Now let us introduce new variables: $\omega_+ = t, \omega = t_1 - t, \omega_{-1} = t - t_{-1}, \omega_j = t_{j+1} - t_j$. Since

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \omega_+} - \frac{\partial}{\partial \omega} + \frac{\partial}{\partial \omega_{-1}},$$

then the expression (4.1) for Y_1 becomes

$$Y_1 = \frac{\partial}{\partial \omega_+} - \frac{\partial}{\partial \omega} + \frac{\partial}{\partial \omega_{-1}} + \sum_{k=1}^{\infty} D^{-1}(D_\omega B(\omega))^k \frac{\partial}{\partial D_x^k \omega_+}. \tag{4.2}$$

We can ignore the term containing $\frac{\partial}{\partial \omega}$ since coefficients in the vector fields used below do not depend on ω.

We multiply Y_1 by $B(\omega_{-1})$,

$$B(\omega_{-1})Y_1 = B(\omega_{-1})\frac{\partial}{\partial \omega_+} + B(\omega_{-1})\frac{\partial}{\partial \omega_{-1}} + \sum_{k=1}^{\infty} B(\omega_{-1})D^{-1}(D_\omega B(\omega))^k \frac{\partial}{\partial D_x^k \omega_+}. \tag{4.3}$$

Introduce

$$p(\theta) = B(\omega_{-1}(\theta)), \qquad \text{where} \qquad d\theta = \frac{d\omega_{-1}}{B(\omega_{-1})}. \tag{4.4}$$

The equation (4.3) becomes

$$B(\omega_{-1})Y_1 = p(\theta)\frac{\partial}{\partial \omega_+} + \frac{\partial}{\partial \theta} + \sum_{k=1}^{\infty} D_x^k(p(\theta))\frac{\partial}{\partial D_x^k \omega_+}. \tag{4.5}$$

Now instead of $X_1 = \frac{\partial}{\partial t_{-1}}$, define

$$\tilde{X}_1 = B(\omega_{-1})X_1 = -B(\omega_{-1})\frac{\partial}{\partial \omega_{-1}} + B(\omega_{-1})\frac{\partial}{\partial \omega_{-2}}.$$

It is indeed with new variables

$$\tilde{X}_1 = -\frac{\partial}{\partial \theta} + \frac{p(\theta)}{p(\theta_{-1})}\frac{\partial}{\partial \theta_{-1}}. \tag{4.6}$$

Note that $[D_x, \tilde{X}_1] = D_x\left(\frac{p(\theta)}{p(\theta_{-1})}\right)W_1$, where $W_1 = \frac{\partial}{\partial \theta_{-1}}$. Since $[D_x, X_1] = -X_1(g)X_1 - X_1(g_{-1})X_2$, then $[D_x, \tilde{X}_1] \in L_n$. Therefore, we have two possibilities;

i) $D_x\left(\frac{p(\theta)}{p(\theta_{-1})}\right) = 0$, or

ii) $W_1 \in L_n$.

First let us consider case i). We have

$$D_x\Big(\frac{p(\theta)}{p(\theta_{-1})}\Big) = \frac{p'(\theta)p(\theta_{-1}) - p(\theta)p'(\theta_{-1})}{p^2(\theta_{-1})} = 0.$$

The solution of this equation is $p(\theta) = B(\omega_{-1}(\theta)) = \mu e^{\lambda\theta}$, $\mu \neq 0$ and λ are some constants. Since $\frac{d\theta}{d\omega_{-1}} = \frac{1}{B(\omega_{-1})}$, we have $B(\omega) = \lambda\omega + c$, c is a constant.

Now we concentrate on case ii). Since $D_x\Big(\frac{p(\theta)}{p(\theta_{-1})}\Big)W_1 \in L_n$, then $W_1 \in L_n$ and, due to (4.6), $W = \frac{\partial}{\partial\theta} \in L_n$.

Lemma 4.1 *If equation $\omega_x = B(\omega)$ admits a nontrivial n-integral then function $p(\theta)$, defined by (4.4), is a quasi-polynomial.*

Proof. Instead of Y_1, X_1, we take the pair of the operators $W = \frac{\partial}{\partial\theta}$ and

$$Z = B(\omega_{-1})Y_1 - W = p(\theta)\frac{\partial}{\partial\omega_+} + D_x p(\theta)\frac{\partial}{\partial\omega_{+x}} + D_x^2(p(\theta))\frac{\partial}{\partial\omega_{+xx}} + \dots. \quad (4.7)$$

We construct a sequence of the operators

$$C_1 = [W, Z], \quad C_2 = [W, C_1], \quad C_k = [W, C_{k-1}], \quad k \geq 2. \quad (4.8)$$

Since algebra L_n is of finite dimension then there exists a natural number N such that

$$C_{N+1} = \mu_0 Z + \mu_1 C_1 + ... + \mu_N C_N, \quad (4.9)$$

and Z, C_1, …, C_N are linearly independent.
Direct calculations show that $[D_x, W] = [D_x, Z] = 0$. Therefore, we have $[D_x, C_j] = 0$ for all j. It follows from (4.9) that

$$0 = D_x(\mu_0)Z + D_x(\mu_1)C_1 + ... + D_x(\mu_N)C_N,$$

which implies $D_x(\mu_j) = 0$. Clearly $\mu_j = \mu_j(\theta)$ and $D_x(\mu_j) = \mu_j'(\theta) = 0$. Hence μ_j is constant for all $j \geq 0$.
Terms before $\frac{\partial}{\partial\omega_+}$ in (4.9) give the equation

$$\mu_0 p(\theta) + \mu_1 p'(\theta) + ... + \mu_N p^{(N)}(\theta) = p^{(N+1)}(\theta). \quad (4.10)$$

This means $p(\theta)$ is a quasi-polynomial, i.e. it takes the form

$$p(\theta) = \sum_{j=1}^{s} q_j(\theta) e^{\lambda_j \theta}. \tag{4.11}$$

□

Lemma 4.2 *Let $p(\theta)$ be an arbitrary quasi-polynomial solving a differential equation of the form (4.10) and which does not solve any equation of this form of less order. Then the equation $t_{1x} = t_x + B(t_1 - t)$ with B found from the conditions*

$$B(\omega_{-1}) = p(\theta),$$
$$\omega_{-1} = \int_0^\theta p(\tilde{\theta}) d\tilde{\theta}$$

admits a nontrivial n-integral.

Proof. Introduce

$$L(D_x) = D_x^{N+1} - \mu_N D_x^N - \mu_{N-1} D_x^{N-1} - ... - \mu_1 D_x - \mu_0.$$

Equation (4.10) can be rewritten as $L(D_x)p(\theta) = 0$. However $L(D_x)p(\theta) = L(D_x)B(\omega_{-1})$. Since $L(D_x)t_{1x} = L(D_x)t_x + L(D_x)B(\omega)$ and $L(D_x)B(\omega) = 0$, we have $L(D_x)t_{1x} = L(D_x)t_x$. But $L(D_x)t_{1x} = DL(D_x)t_x$, therefore $DL(D_x)t_x = L(D_x)t_x$. Denote $L(D_x)t_x = I$ so we have $DI = I$. Hence $L(D_x)t_x$ is an n-integral.□

Therefore the condition (4.11) is necessary and sufficient for our equation to have nontrivial n-integral.

Example Take $p(\theta) = \frac{1}{2}e^\theta + \frac{1}{2}e^{-\theta} = \cosh\theta$, then

$$\begin{aligned} B(\omega_{-1}) &= \cosh\theta \\ \omega_{-1} &= \sinh\theta + c, \end{aligned}$$

or $B(\omega_{-1})^2 - (\omega_{-1} - c)^2 = 1$ which gives $B(\omega_{-1}) = \sqrt{1 + (\omega_{-1} - c)^2}$. So $t_{1x} = t_x + \sqrt{1 + (t_1 - t - c)^2}$, where c is arbitrary constant, is Darboux integrable. Moreover, its general solution is given by $t(n, x) = G(x) + nc + \sum_{k=0}^{n-1} \sinh(x + c_k)$, where $G(x)$ is arbitrary function depending on x, and c_k are arbitrary constants.

4.2 Case 2) $t_{1x} = t_x + c_1(t - t_1)t + c_2(t - t_1)^2 + c_3(t - t_1)$

Since c_1, c_2 and c_3 are arbitrary constants, let us express the equation as $t_{1x} = t_x + \bar{c}_1(t_1 - t)t + \bar{c}_2(t_1 - t)^2 + \bar{c}_3(t_1 - t)$ for simplicity. We have the following relation between $\bar{c}_1$ and $\bar{c}_2$.

Lemma 4.3 *If equation $t_{1x} = t_x + d(t, t_1) = t_x + \bar{c}_1(t_1 - t)t + \bar{c}_2(t_1 - t)^2 + \bar{c}_3(t_1 - t)$ admits a nontrivial n-integral, then there exists a natural number k such that*

$$(k + 1)\bar{c}_2 - k\bar{c}_1 = 0 . \tag{4.12}$$

Proof. Introduce vector fields $T_1 = [X_1, Y_1]$, $T_n = [X_1, T_{n-1}]$, $n \geq 2$. Direct calculations show that

$$[D_x, T_1] = (-\bar{c}_1 + 2\bar{c}_2)X_1 + (-\bar{c}_1 + 2\bar{c}_2)Y_1 + (d_{t_{-1}}(t_{-1}, t) - d_t(t_{-1}, t))T_1,$$

$$[D_x, T_n] = -A^{(n)}_{n-1}T_{n-1} - A^{(n)}_n T_n, \tag{4.13}$$

where

$$A^{(n)}_j = X_1^{n-j}\{-C(n, j-1)d_{t_{-1}}(t_{-1}, t) + C(n, j)d_t(t_{-1}, t)\}, \qquad C(n, k) = \frac{n!}{k!(n-k)!}.$$

Since algebra L_n is of finite dimension then there exists a natural number M such that

$$T_{M+1} = \mu_1 T_1 + \mu_2 T_2 + \ldots + \mu_M T_M,$$

and T_1, T_2, …, T_M are linearly independent. We have,

$$[D_x, T_{M+1}] = [D_x, \mu_1 T_1 + \mu_2 T_2 + \ldots + \mu_M T_M],$$

that can be rewritten by (4.13) in the following form:

$$\begin{aligned} &-A^{(M+1)}_M T_M - A^{(M+1)}_{M+1}\{\mu_1 T_1 + \mu_2 T_2 + \ldots + \mu_M T_M\} = (-\bar{c}_1 + 2\bar{c}_2)\mu_1(X_1 + Y_1) \\ &+\{D_x(\mu_1) - \mu_1 A^{(1)}_1 - \mu_2 A^{(2)}_1\}T_1 + \ldots + \{D_x(\mu_{N-1}) - \mu_{N-1}A^{(N-1)}_{N-1} - \mu_N A^{(N)}_{N_1}\}T_{N-1} \\ &+\{D_x(\mu_N) - \mu_N A^{(N)}_N\}T_N . \end{aligned} \tag{4.14}$$

We can prove equation (4.12) by comparing the coefficients before linearly independent vector fields X_1, Y_1, T_k, $1 \leq k \leq M$ in the equality (4.14). $\square$

Now introduce $\omega = t_1 - t$. Hence we can rewrite the equation $t_{1x} = t_x + \overline{c}_1(t_1 - t)t + \overline{c}_2(t_1 - t)^2 + \overline{c}_3(t_1 - t)$ as

$$\omega_x = \overline{c}_1\omega t + \overline{c}_2\omega^2 + \overline{c}_3\omega.$$

In this case we have two important relations;

1) $Y_0 f = D_x \ln H$, where

$$H = \frac{\omega\theta^{1/\epsilon}}{(\theta+\epsilon)^{1/\epsilon}}, \qquad \theta = \frac{\omega_1}{\omega}, \qquad \epsilon = \frac{\overline{c}_1}{\overline{c}_2} - 1\,. \tag{4.15}$$

2) $Y_1 f = D_x \ln RH_{-1}$, where

$$H_{-1} = D^{-1}H, \qquad R = \frac{\theta}{\omega(\theta+\epsilon)}, \qquad \text{when} \qquad \epsilon \neq 0.$$

Remark 4.4 *The case $\epsilon = 0$, i.e $\overline{c}_1 = \overline{c}_2$, is not realized due to Lemma 4.3. The case $\overline{c}_2 = 0$, due to Lemma 4.3, leads to $\overline{c}_1 = 0$, and the equation becomes $t_{1x} = t_x + \overline{c}_3(t_1 - t)$ with an n-integral $I = t_x - \overline{c}_3 t$.*

These two relations allow us to simplify the basis operators Y_0, Y_1, X_1. Really, we take

$$\tilde{Y}_1 = H_{-1}Y_1, \quad \tilde{Y}_0 = HY_0,$$

and get $[D_x, \tilde{Y}_0] = 0$ and $[D_x, \tilde{Y}_1] = \Lambda\tilde{Y}_0$, where $\Lambda = -\frac{H_{-1}}{H} D_x \ln(RH_{-1})$.

First we will restrict the set of the variables as follows: $t_1, t, t_{-1}, t_x, t_{xx}, ...$ and change the variables $t^+ = t$, $\omega_{-1} = t - t_{-1}$ keeping the other variables unchanged. Then some of the differentiations will change

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^+} + \frac{\partial}{\partial \omega_{-1}}, \quad \frac{\partial}{\partial t_{-1}} = -\frac{\partial}{\partial \omega_{-1}}.$$

So we have $X_1 = -\frac{\partial}{\partial \omega_{-1}} = -\hat{X}_1$ and

$$\tilde{Y}_1 = H_{-1}\Big(\frac{\partial}{\partial t^+} + \frac{\partial}{\partial \omega_{-1}}\Big) + \sum_{k=1}^{\infty} H_{-1}D^{-1}(Y_0 D_x^{k-1} f)\frac{\partial}{\partial D_x^k t}\,.$$

Since $[D_x, \hat{X}_1] = D_x(\ln R_{-1})\hat{X}_1$, we can introduce $\tilde{X}_1 = \frac{1}{R_{-1}}\hat{X}_1$ and get $[D_x, \tilde{X}_1] = 0$. Here $R_{-1} = D^{-1}R$.

We introduce vector fields $C_2 = [\tilde{X}_1, \tilde{Y}_1]$, $C_3 = [\tilde{X}_1, C_2]$, $C_k = [\tilde{X}_1, C_{k-1}]$, $k \geq 3$. We have,

$$[D_x, C_{j+1}] = \tilde{X}_1^j(\Lambda)\tilde{Y}_0, \quad j \geq 1.$$

Since the algebra L_n is of finite dimension then there is a natural number N such that

$$C_{N+1} = \mu_N C_N + ... + \mu_2 C_2 + \mu_1 \tilde{Y}_1, \tag{4.16}$$

where $\tilde{Y}_1$, C_1, C_2, $\ldots$ are linearly independent.

Applying the commutator with D_x we get $D_x(\mu_j) = 0$ for $j = 1, ..., N$ and

$$(\tilde{X}_1^N - \mu_N \tilde{X}_1^{N-1} - ... - \mu_1)\Lambda = 0. \tag{4.17}$$

All the operators in our sequence have coefficients depending on ω, ω_{-1}, t. So μ_j, $j = 1, ..., N$ also depend on these variables. But the relation $D_x\mu_j(\omega, \omega_{-1}, t) = 0$ shows that $\frac{\partial \mu_j}{\partial t} = 0$ i.e. $\mu_j = \mu_j(\omega, \omega_{-1})$. Since the minimal x-integral for an equation in case 2)(see Theorem 3.1) depends on variables t, t_1, t_2, t_3, the relation $D_x(\mu_j) = 0$ implies that μ_j is constant for all j.

Now we introduce new variables $\tilde{t}_1$, $\tilde{t}$, η as

$$\tilde{t}_1 = t_1, \qquad \tilde{t} = t^+,$$

$$\eta = \ln\Big(\frac{\omega_{-1}}{\omega_{-1} + \frac{1}{\epsilon}(t_1 - t)}\Big) \quad ; \quad \text{or the same} \qquad \omega_{-1} = \frac{\omega}{\epsilon}\Big(\frac{e^\eta}{1 - e^\eta}\Big). \tag{4.18}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial \omega_{-1}} &= \frac{\partial \eta}{\partial \omega_{-1}}\frac{\partial}{\partial \eta}, \\
\frac{\partial}{\partial t^+} &= \frac{\partial}{\partial \tilde{t}} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}, \\
\frac{\partial}{\partial t_1} &= \frac{\partial}{\partial \tilde{t}_1} + \frac{\partial \eta}{\partial t_1}\frac{\partial}{\partial \eta}.
\end{aligned}$$

In these new variables $\tilde{X}_1$ takes the form

$$\tilde{X}_1 = \frac{\omega_{-1}(\theta_{-1}+\epsilon)}{\theta_{-1}}\frac{\partial}{\partial\omega_{-1}} = \frac{\partial}{\partial\eta},$$

and equation (4.17) becomes

$$\Big(\frac{d^N}{d\eta^N} - \mu_N\frac{d^{N-1}}{d\eta^{N-1}} - \dots - \mu_1\Big)\Lambda = 0, \tag{4.19}$$

where

$$\begin{aligned}\Lambda &= -\frac{H_{-1}}{H}(\omega_x \ln R + D_x \ln H_{-1}) = -\frac{H_{-1}}{H}\Big(\frac{\partial f}{\partial t} + D^{-1}\frac{\partial f}{\partial t_1}\Big)\\ &= -\frac{H_{-1}}{H}(\overline{c}_1 - 2\overline{c}_2)(\omega - \omega_{-1}).\end{aligned} \tag{4.20}$$

Let us show that $\overline{c}_1 - 2\overline{c}_2 = 0$. Assume contrary. It follows from (4.19) and (4.20) that both functions H_{-1} and $\omega_{-1}H_{-1}$ should solve the linear differential equation with constant coefficients:

$$\Big(\frac{d^N}{d\eta^N} - \mu_N\frac{d^{N-1}}{d\eta^{N-1}} - \dots - \mu_1\Big)y(\eta) = 0.$$

Therefore, both functions H_{-1} and $\omega_{-1}H_{-1}$ must be quasi-polynomials in η.

Due to (4.15) and (4.18), we have

$$H_{-1} = \frac{\omega}{\epsilon}e^{\eta}(1-e^{\eta})^{\frac{1}{\epsilon}-1}$$

and

$$\omega_{-1}H_{-1} = \frac{\omega^2}{\epsilon^2}e^{2\eta}(1-e^{\eta})^{\frac{1}{\epsilon}-2}.$$

To be quasi-polynomials in η it is necessary that $\epsilon = \frac{1}{m}$ for some natural $m \geq 2$. We rewrite our vector fields $\tilde{X}_1, \tilde{Y}_1$ in the new variables;

$$\begin{aligned}\tilde{X}_1 &= \frac{\partial}{\partial\eta},\\ \tilde{Y}_1 &= H_{-1}\frac{\partial}{\partial\tilde{t}} + H_{-1}\Big(\frac{\partial\eta}{\partial t^+} + \frac{\partial\eta}{\partial\omega_{-1}}\Big)\frac{\partial}{\partial\eta} + \dots\end{aligned}$$

and study the projection on the direction $\frac{\partial}{\partial\eta}$.
The operators $\tilde{X}_1 = \frac{\partial}{\partial\eta}$ and $H_{-1}\Big(\frac{\partial\eta}{\partial t^+} + \frac{\partial\eta}{\partial\omega_{-1}}\Big)\frac{\partial}{\partial\eta}$ generate a finite dimensional Lie algebra over the field of constants. Due to Lemma 3.9 from Chapter 3, in this case the coefficient $H_{-1}\frac{\partial\eta}{\partial t}$ should be of one of the forms

$$\tilde{c}_1e^{\tilde{\alpha}\eta} + \tilde{c}_2e^{-\tilde{\alpha}\eta} + \tilde{c}_3 \quad \text{or} \quad \tilde{c}_1\eta^2 + \tilde{c}_2\eta + \tilde{c}_3, \tag{4.21}$$

but we have

$$H_{-1}\Big(\frac{\partial\eta}{\partial t^+}+\frac{\partial\eta}{\partial\omega_{-1}}\Big)=\Big(1+\Big(\frac{1}{\epsilon}-1\Big)e^{\eta}\Big)(1-e^{\eta})^{\frac{1}{\epsilon}},$$

with $\frac{1}{\epsilon}=m\geq 2$ and it is never of the form (4.21). This contradiction shows that $\overline{c}_1-2\overline{c}_2=0.\square$

4.3 Case 3) $t_{1x}=t_x+A(t-t_1)e^{\alpha t}$

Introduce $\omega=t_1-t$ and also to express the equation in a simpler form write $B(t_1-t)$ instead of $A(t-t_1)$. We can do this since A is an arbitrary function of $t-t_1$. Hence we can rewrite the equation as $\omega_x=B(\omega)e^{\alpha t}$. We will find out when the equation admits a nontrivial n-integral or the same when the corresponding Lie algebra L_n is of finite dimension.

Instead of the vector fields $Y_0=\frac{\partial}{\partial t_1}$ and $Y_1=\frac{\partial}{\partial t}+D^{-1}\Big(\frac{\partial f}{\partial t_1}\Big)\frac{\partial}{\partial t_x}+D^{-1}\Big(\frac{\partial f_x}{\partial t_1}\Big)\frac{\partial}{\partial t_{xx}}+$ ..., we will use the vector fields $\tilde{Y}_0=B(\omega)Y_0$ and $\tilde{Y}_1=B(\omega_{-1})Y_1$. They are more convenient since they satisfy more simple relations:

$$[D_x,\tilde{Y}_0]=0,\quad [D_x,\tilde{Y}_1]=\lambda_1\tilde{Y}_0$$

as operators acting on the enlarged set $t_1,t,t_{-1},t_{-2},...\ ;\ t_x,t_{xx},t_{xxx},...$. Here the coefficient λ_1 is

$$\lambda_1=\frac{B(\omega_{-1})}{B(\omega)}\Big(B'(\omega)-\alpha B(\omega)-B'(\omega_{-1})e^{-\alpha\omega_{-1}}\Big)e^{\alpha t}.$$

Since the equation is represented as $\omega_x=B(\omega)e^{\alpha t}$ it is reasonable to introduce new variables as $\omega_+=t,\omega_{-1}=t-t_{-1},\omega_{-2}=t_{-1}-t_{-2}$, such that

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial\omega_+}+\frac{\partial}{\partial\omega_{-1}},\quad \frac{\partial}{\partial t_{-1}}=-\frac{\partial}{\partial\omega_{-1}}+\frac{\partial}{\partial\omega_{-2}},\quad \frac{\partial}{\partial t_{-2}}=-\frac{\partial}{\partial\omega_{-2}}.$$

Instead of the operators $X_1=\frac{\partial}{\partial t_{-1}}$ and $X_2=\frac{\partial}{\partial t_{-2}}$ we use new ones $\tilde{X}_1=B(\omega_{-1})e^{-\alpha\omega_{-1}}\frac{\partial}{\partial\omega_{-1}}$ and $\tilde{X}_2=B(\omega_{-2})e^{-\alpha\omega_{-2}}\frac{\partial}{\partial\omega_{-2}}$. They satisfy relations $[D_x,\tilde{X}_2]=0$ and $[D_x,\tilde{X}_1]=\mu\tilde{X}_2$. Here the coefficient μ is

$$\mu=\alpha B(\omega_{-1})e^{-2\alpha\omega_{-1}+\alpha t}.$$

We construct a sequence by taking $\tilde{X}_1, \tilde{Y}_1, C_2 = [\tilde{X}_1, \tilde{Y}_1], C_3 = [\tilde{X}_1, C_2], C_k = [\tilde{X}_1, C_{k-1}]$ for $k \geq 3$. We can easily check that

$$[D_x, C_2] = -\tilde{Y}_1(\mu)\tilde{X}_2 + \tilde{X}_1(\lambda_1)\tilde{Y}_0 = b_2\tilde{X}_2 + \tilde{X}_1(\lambda_1)\tilde{Y}_0,$$

$$[D_x, C_3] = \tilde{X}_1^2(\lambda_1)\tilde{Y}_0 - (C_2 + \tilde{X}_1\tilde{Y}_1)(\mu)\tilde{X}_2 = \tilde{X}_1^2(\lambda_1)\tilde{Y}_0 + b_3\tilde{X}_2,$$

and for any k we have

$$[D_x, C_k] = \tilde{X}_1^{k-1}(\lambda_1)\tilde{Y}_0 + b_k\tilde{X}_2,$$

which can be proved by induction.

Since the characteristic Lie algebra L_n is of finite dimension then there is a number N such that

$$C_{N+1} = \mu_N C_N + ... + \mu_1\tilde{Y}_1 + \mu_0\tilde{X}_1, \tag{4.22}$$

where $\tilde{X}_1$, $\tilde{Y}_1$, C_1, C_2, $\ldots$ are linearly independent.

We commute both sides of (4.22) with D_x and get

$$\begin{aligned}\tilde{X}_1^N(\lambda_1)\tilde{Y}_0 + b_{N+1}\tilde{X}_2 \quad &= \quad D_x(\mu_N)C_N + ... + D_x(\mu_1)\tilde{Y}_1 + D_x(\mu_0)\tilde{X}_1 \\ &\quad +\mu_N\tilde{X}_1^{N-1}(\lambda_1)\tilde{Y}_0 + ... + \mu_1\lambda_1\tilde{Y}_0 + \{\sum_{k=2}^{N} b_k\mu_k\}\tilde{X}_2.\end{aligned}$$

We collect the coefficients before the operators and get $D_x(\mu_j) = 0$ for $j = 0, 1, ..., N$, and

$$(\tilde{X}_1^N - \mu_N\tilde{X}_1^{N-1} - \mu_{N-1}\tilde{X}_1^{N-2} - ... - \mu_1)\lambda_1 = 0. \tag{4.23}$$

Introduce new variables η, η_{-1} as solutions of the following ordinary differential equations

$$\frac{d\omega_{-1}}{d\eta} = B(\omega_{-1})e^{-\alpha\omega_{-1}}, \quad \frac{d\omega_{-2}}{d\eta_{-1}} = B(\omega_{-2})e^{-\alpha\omega_{-2}}. \tag{4.24}$$

Thus our vector fields are rewritten as

$$\tilde{X}_1 = \frac{\partial}{\partial\eta}, \quad \tilde{X}_2 = \frac{\partial}{\partial\eta_{-1}}, \quad \tilde{Y}_0 = B(\omega)\frac{\partial}{\partial t_1},$$

$$\tilde{Y}_1 = e^{\alpha\omega_{-1}}\frac{\partial}{\partial\eta} + B(\omega_{-1})\frac{\partial}{\partial\omega_+} + D_x(B(\omega_{-1}))\frac{\partial}{\partial t_x} + \ldots .$$

By looking at the projection on $\frac{\partial}{\partial \eta}$ we get an algebra generated by $\frac{\partial}{\partial \eta}$ and $e^{\alpha\omega_{-1}}\frac{\partial}{\partial \eta}$ containing all possible commutators and all possible linear combinations with constant coefficients. Due to Lemma 3.9 from Chapter 3, we get that $e^{\alpha\omega_{-1}}$ can be only one of the forms

a) $e^{\alpha\omega_{-1}} = c_1 e^{\beta\eta} + c_2 e^{-\beta\eta} + c_3,$

b) $e^{\alpha\omega_{-1}} = c_1\eta^2 + c_2\eta + c_3,$

where β, c_1, c_2, c_3 are some constants.

The equation $B(\omega_{-1}) = \frac{1}{\alpha}\frac{d}{d\eta}e^{\alpha\omega_{-1}}$ implies that
in case a) we have $B(\omega_{-1}) = (\beta/\alpha)(c_1 e^{\beta\eta} - c_2 e^{-\beta\eta})$, or the same

$$B^2(\omega) = \frac{\beta^2}{\alpha^2}\{(e^{\alpha\omega} - c_3)^2 - 4c_1c_2\}\,, \tag{4.25}$$

and
in case b) we have $B(\omega_{-1}) = (1/\alpha)(2c_1\eta + c_2)$, or the same,

$$B^2(\omega) = \frac{4c_1}{\alpha^2}e^{\alpha\omega} + \frac{c_2^2 - 4c_1c_3}{\alpha^2}. \tag{4.26}$$

In addition to the operators $\tilde{X}_1, \tilde{X}_2, \tilde{Y}_0, \tilde{Y}_1$ introduced above we will use $\tilde{Y}_2 = B(\omega_{-2})D^{-1}(Y_1 f)\partial_{t_x} + B(\omega_{-2})D^{-1}(Y_1 f_x)\partial_{t_{xx}} + \ldots$ defined as $\tilde{Y}_2 = B(\omega_{-2})Y_2$. It satisfies the commutativity relation

$$[D_x, \tilde{Y}_2] = \lambda\tilde{Y}_1 + \xi\tilde{Y}_0 + \nu\tilde{X}_1, \tag{4.27}$$

where

$$\begin{aligned}
\xi &= -\frac{B(\omega_{-2})}{B(\omega)}D^{-1}(Y_1 f) \\
&= -\frac{B(\omega_{-2})}{B(\omega)}\{(-B'(\omega_{-1}) + \alpha B(\omega_{-1}))e^{-\alpha\omega_{-1}} + B'(\omega_{-2})e^{-\alpha\omega_{-2}-\alpha\omega_{-1}}\}e^{\alpha t}.
\end{aligned} \tag{4.28}$$

$$\lambda = -\frac{B(\omega_{-2})}{B(\omega_{-1})}D^{-1}(Y_1 f) \qquad \text{and} \qquad \nu = -\lambda e^{\alpha\omega_{-1}}\,.$$

Lemma 4.5 *(1) Equation $t_{1x} = t_x + \frac{\beta}{\alpha}(e^{\alpha\omega} - c_3)e^{\alpha t}$ admits a nontrivial n-integral if and only if $c_3 = \pm 1$.*
(2) Equation $t_{1x} = t_x + c_5 e^{\alpha t}$, $c_5 \neq 0$ does not admit a nontrivial n-integral.

Proof. In this case the equation $\omega_x = B(\omega)e^{\alpha t}$ is reduced by evident scaling of x and t to

$$t_{1x} = t_x + e^t, \qquad \text{or} \qquad t_{1x} = t_x + e^{t_1} + \varepsilon e^t.$$

By induction on n we can easily see that for the equation $t_{1x} = t_x + e^t$, the basic vector fields Y_n are

$$Y_1 = \frac{\partial}{\partial t},$$

$$Y_n = e^{t-(n-1)}\frac{\partial}{\partial t_x} + e^{t-(n-1)}(t_x - e^{t-(n-1)})\frac{\partial}{\partial t_{xx}} + \dots .$$

Since these vector fields Y_n, $n \geq 1$, are linearly independent then equation $t_{1x} = t_x + e^t$ does not admit a nontrivial n-integral.

For equation $t_{1x} = t_x + e^{t_1} + \varepsilon e^t$, the basic vector fields Y_n are

$$Y_1 = \frac{\partial}{\partial t} + e^t\frac{\partial}{\partial t_x} + e^t(t_x + e^t)\frac{\partial}{\partial t_{xx}} + \dots,$$

$$Y_n = (\varepsilon+1)e^{t-(n-1)}\frac{\partial}{\partial t_x} + (\varepsilon+1)e^{t-(n-1)}(t_x + (1-\varepsilon)e^{t-(n-1)})\frac{\partial}{\partial t_{xx}} + \dots$$

We can see that vector fields Y_n, $n \geq 1$, are linearly independent if $\varepsilon \neq \pm 1$. Therefore, if $\varepsilon \neq \pm 1$, equation $t_{1x} = t_x + e^{t_1} + \varepsilon e^t$ does not admit a nontrivial n-integral. If $\varepsilon = -1$, the equation becomes $t_{1x} = t_x + e^{t_1} - e^t$, and one of its n-integrals is $I = t_x - e^t$. If $\varepsilon = 1$, the equation becomes $t_{1x} = t_x + e^{t_1} + e^t$, and one of its n-integrals is $I = 2t_{xx} - t_x^2 - e^{2t}$. □

Lemma 4.6 *Let equation $t_{1x} = t_x + B(t_1 - t)e^{\alpha t}$ with*
(a) $B^2(\omega) = \frac{\beta^2}{\alpha^2}\{(e^{\alpha\omega} - c_3)^2 - 4c_1c_2\}$, or
(b) $B^2(\omega) = \frac{4c_1}{\alpha^2}e^{\alpha\omega} + \frac{c_2^2 - 4c_1c_3}{\alpha^2}$,
admit a nontrivial n-integral. Then
in case (a), we have, $B(t_1 - t) = \frac{\beta}{\alpha}\sqrt{(e^{\alpha(t_1-t)} - c_3)^2 - c_3^2 + 1}$, where c_3 is an arbitrary constant, and
in case (b), we have, $B(t_1 - t) = ce^{\frac{\alpha}{2}(t_1-t)}$, where c is an arbitrary constant.

In cases (a) and (b) the corresponding n-integrals are $I = \frac{\alpha}{2}t_x^2 - t_{xx} + \frac{\alpha}{2}e^{2\alpha t}$ and $I = -\frac{\alpha}{2}t_x^2 + t_{xx}$.

Proof. Note that

$$D_x\rho = \lambda, \qquad \text{where} \qquad \rho = -\frac{B(\omega_{-2})}{B(\omega_{-1})} - e^{\alpha\omega_{-2}}.$$

This implies that the vector field

$$R_2 = \tilde{Y}_2 - \rho\tilde{Y}_1,$$

satisfies very simple and convenient relation

$$[D_x, R_2] = \tilde{\xi}\tilde{Y}_0 + \nu\tilde{X}_1\,, \quad \tilde{\xi} = -\frac{B(\omega_{-2})}{B(\omega)}D^{-1}(Y_1 f) - \rho\lambda_1, \quad \nu = e^{\alpha\omega_{-1}}\frac{B(\omega_{-2})}{B(\omega_{-1})}D^{-1}(Y_1 f).$$

Study now the sequence

$$R_{j+1} = [\hat{X}, R_j], \quad j \geq 2, \qquad \text{where} \qquad \hat{X} = \tilde{X}_1 + e^{-\alpha\omega_{-1}}\tilde{X}_2\,.$$

Direct calculations show that

$$[D_x, R_n] = \hat{X}^{(n-2)}(\tilde{\xi})\tilde{Y}_0 + \hat{X}^{(n-2)}(\tilde{\nu})\tilde{X}_1 + b_n\tilde{X}_2\,. \tag{4.29}$$

Since $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{Y}_0$, R_2 are linearly independent, then there exists a number $N \geq 2$ such that

$$R_{N+1} = \mu_N R_N + \mu_{N-1}R_{N-1} + \ldots \mu_2 R_2 + \mu_1\tilde{X}_1,$$

and

$$[D_x, R_{N+1}] = [D_x, \mu_N R_N + \mu_{N-1}R_{N-1} + \ldots \mu_2 R_2 + \mu_1\tilde{X}_1]\,. \tag{4.30}$$

We use $[D_x, \tilde{X}_1] = \alpha B(\omega_{-1})e^{-2\alpha\omega_{-1}+\alpha t}\tilde{X}_2$, $[D_x, \tilde{X}_2] = 0$ and (4.29) to compare the coefficients before linearly independent vector fields R_k and $\tilde{Y}_0$ in (4.30). We have, $D_x(\mu_k) = 0$, $k = 2, 3, \ldots, N$, and

$$\hat{X}^{(N-1)}(\tilde{\xi}) = \mu_N\hat{X}^{(N-2)}(\tilde{\xi}) + \ldots + \mu_2\tilde{\xi}\,. \tag{4.31}$$

Under the change of variables

$$\eta = z, \qquad \eta_{-1} = z_{-1} - q(z), \qquad \frac{\partial q(z)}{\partial z} = -e^{-\alpha\omega_{-1}}\,,$$

equation (4.31) is reduced to

$$(D_z^{N-1} - \mu_N D_z^{N-2} - \ldots - \mu_2)\tilde{\xi} = 0, \tag{4.32}$$

where $\mu_k = \mu_k(\omega_{-1}, \omega_{-2}) = \mu_k(z, z_{-1})$. Since $D_x(z_{-1}) = 0$, $D_x(z) = e^{\alpha t} \neq 0$ and $0 = D_x(\mu_k) = D_{z_{-1}}(\mu_k)D_x(z_{-1}) + D_z(\mu_k)D_x(z)$, then coefficients μ_k do not depend on variable z . Since, due to (4.32),

$$\tilde{\xi} = -\frac{B(\omega_{-2})}{B(\omega)} e^{-\alpha\omega_{-1}} e^{\alpha t}\{-B'(\omega_{-1}) + \alpha B(\omega_{-1}) + B'(\omega_{-2})e^{-\alpha\omega_{-2}}\}$$

$$+\frac{B(\omega_{-2})}{B(\omega)} e^{\alpha t}\{B'(\omega) - \alpha B(\omega) - B'(\omega_{-1})e^{-\alpha\omega_{-1}}\}$$

$$+\frac{B(\omega_{-1})}{B(\omega)} e^{\alpha\omega_{-2}} e^{\alpha t}\{B'(\omega) - \alpha B(\omega) - B'(\omega_{-1})e^{-\alpha\omega_{-1}}\}$$

is a quasi-polynomial in $z = \eta$ for any ω and t, then $\frac{d}{d\omega}(\tilde{\xi} B(\omega) e^{-\alpha t})$ is a quasi-polynomial as well. Hence we have,

$$(B''(\omega) - \alpha B'(\omega))\{B(\omega_{-2}) + B(\omega_{-1})e^{\alpha\omega_{-2}}\}$$

is a quasi-polynomial in z, which is possible only if

$$B''(\omega) - \alpha B'(\omega) = 0, \mathrm{or} B(\omega_{-2}) + B(\omega_{-1})e^{\alpha\omega_{-2}}$$

is a quasi-polynomial in z.
In case (a) we have,

$$B''(\omega) - \alpha B'(\omega) = -\alpha\beta c_4 \frac{e^{2\alpha\omega}}{(\sqrt{(e^{\alpha\omega} - c_3)^2 - c_4})^3}, \qquad c_4 = 4c_1c_2,$$

and in case (b) we have

$$B''(\omega) - \alpha B'(\omega) = -4c_1^2\alpha^{-2}e^{2\alpha\omega}\left(\frac{4c_1}{\alpha^2}e^{\alpha\omega} + \frac{c_2^2 - 4c_1c_3}{\alpha^2}\right)^{-3/2}$$

Therefore, $B''(\omega) - \alpha B'(\omega) = 0$ if $c_1c_2 = 0$ in case (a) and if $c_1 = 0$ in case (b). Both these cases are considered in Lemma 4.5.

It follows from $\frac{dq}{dz} = -e^{-\alpha\omega_{-1}}$ that, in case (a), if $r = \sqrt{c_3^2 - 4c_1c_2} \neq 0$, then

$$q(\eta) = -\frac{1}{\beta r}\ln\left|\frac{e^{\beta\eta} - p_1}{e^{\beta\eta} - p_2}\right|, \qquad p_1 = \frac{-c_3 + r}{2c_1}, \quad p_2 = \frac{-c_3 - r}{2c_1},$$

and if $r = \sqrt{c_3^2 - 4c_1c_2} = 0$, then

$$q(\eta) = \frac{1}{c_1\beta(e^{\beta\eta} - p_1)}.$$

In case (b), if $r_1 = \sqrt{c_2^2 - 4c_1c_3} \neq 0$, then

$$q(\eta) = -\frac{1}{\beta r_1} \ln\left|\frac{\eta - p_1^*}{\eta - p_2^*}\right|, \qquad p_1^* = \frac{-c_2 + r_1}{2c_1}, \quad p_2^* = \frac{-c_2 - r_1}{2c_1},$$

and if $r_1 = \sqrt{c_2^2 - 4c_1c_3} = 0$, then

$$q(\eta) = \frac{1}{c_1\beta(\eta - p_1^*)}.$$

In case (a) we have,

$$\frac{\alpha}{\beta}(B(\omega_{-2})+B(\omega_{-1})e^{\alpha\omega_{-2}}) = c_1e^{\beta\eta_{-1}}-c_2e^{-\beta\eta_{-1}}+(c_1e^{\beta\eta}-c_2e^{-\beta\eta})(c_1e^{\beta\eta_{-1}}+c_2e^{-\beta\eta_{-1}}+c_3)$$

$$= c_1e^{\beta\eta_{-1}}(c_1e^{\beta\eta} - c_2e^{-\beta\eta} + 1) + c_2e^{-\beta\eta_{-1}}(c_1e^{\beta\eta} - c_2e^{-\beta\eta} - 1) + c_3c_1e^{\beta\eta} - c_3c_2e^{-\beta\eta}$$

$$= c_1e^{\beta z_{-1}-\beta q(z)}(c_1e^{\beta z}-c_2e^{-\beta z}+1)+c_2e^{-\beta z_{-1}+\beta q(z)}(c_1e^{\beta z}-c_2e^{-\beta z}-1)+c_3c_1e^{\beta z}-c_3c_2e^{-\beta z}.$$

We can see that $B(\omega_{-2}) + B(\omega_{-1})e^{\alpha\omega_{-2}}$ is a quasi-polynomial in case (a) only if $r = \sqrt{c_3^2 - 4c_1c_2} = \pm 1$. If $r = \pm 1$, function $B(t_1 - t)$ becomes $\frac{\beta}{\alpha}\sqrt{(e^{\alpha(t_1-t)} - c_3)^2 - c_3^2 + 1}$, where c_3 is an arbitrary constant, and one of n-integrals for $t_{1x} = t_x + \frac{\beta}{\alpha}e^{\alpha t}\sqrt{(e^{\alpha(t_1-t)} - c_3)^2 - c_3^2 + 1}$ is $I = \frac{\alpha}{2}t_x^2 - t_{xx} + \frac{\alpha}{2}e^{2\alpha t}$.

In case (b) direct calculations show that,

$$B(\omega_{-2}) + B(\omega_{-1})e^{\alpha\omega_{-2}} = Q(z) + P(z, z_{-1}) + J(z, z_{-1}),$$

where $Q(z)$ is some function depending only on z, $P(z, z_{-1})$ is a polynomial function of two variables, and

$$J(z, z_{-1}) = -\frac{2c_1}{\alpha}z_{-1}q(z)(2c_1z + c_2).$$

Since $B(\omega_{-2}) + B(\omega_{-1})e^{\alpha\omega_{-2}} - P(z, z_{-1}) = Q(z) + J(z, z_{-1})$ is a quasi-polynomial in z, then

$$\frac{\partial(Q(z) + J(z, z_{-1}))}{\partial z_{-1}} = \frac{2c_1}{\alpha}q(z)(2c_1z + c_2)$$

is also a quasi-polynomial in z, which is possible only if $r_1 = \sqrt{c_2^2 - 4c_1c_3} = 0$. If $r_1 = 0$ we have $B(t_1 - t) = ce^{\frac{\alpha}{2}(t_1-t)}$, where c is an arbitrary constant, and the corresponding n-integral is $I = -\frac{\alpha}{2}t_x^2 + t_{xx}$. $\square$

4.4 Case 4) $t_{1x} = t_x + c_4(e^{\alpha t_1} - e^{\alpha t}) + c_5(e^{-\alpha t_1} - e^{-\alpha t})$

It is clear that this equation has a nontrivial n-integral without any additional condition. For this equation n-integral is $I = t_x - c_4 e^{\alpha t} + c_5 e^{-\alpha t}$. It satisfies the equation $DI = I$ since $DI = t_{1x} - c_4 e^{\alpha t_1} + c_5 e^{-\alpha t_1} = I$.

4.5 List of Darboux Integrable Semi-discrete Equations

Summarizing the reasonings given in the previous sections of Chapter 3 and Chapter 4, we give the following Theorem.

Theorem 4.7 *Chain (3.1) admits nontrivial x- and n-integrals if and only if $d(t, t_1)$ is one of the kind:*

(1) $d(t, t_1) = B(t_1 - t)$, *where* $B(t_1 - t)$ *is given in implicit form* $B(t_1 - t) = \frac{d}{d\theta} P(\theta)$, $t_1 - t = P(\theta)$, $P(\theta)$ *is a quasi-polynomial on* θ,

(2) $d(t, t_1) = C_1(t_1^2 - t^2) + C_2(t_1 - t)$

(3) $d(t, t_1) = \sqrt{C_3 e^{2\alpha t_1} + C_4 e^{\alpha(t_1+t)} + C_3 e^{2\alpha t}}$,

(4) $d(t, t_1) = C_5(e^{\alpha t_1} - e^{\alpha t}) + C_6(e^{-\alpha t_1} - e^{-\alpha t})$,

where $\alpha \neq 0$, C_i, $1 \leq 1 \leq 6$, *are arbitrary constants. Moreover, some nontrivial x-integrals F and n-integrals I in each of the cases are*

i) $F = x - \int^{t_1 - t} \frac{ds}{B(s)}$, $I = L(D_x) t_x$, *where* $L(D_x)$ *is a differential operator which annihilates* $\frac{d}{d\theta} P(\theta)$ *where* $D_x \theta = 1$.

ii) $F = \frac{(t_3 - t_1)(t_2 - t)}{(t_3 - t_2)(t_1 - t)}$, $I = t_x - C_1 t^2 - C_2 t$, *iii)* $F = \int^{t_1-t} \frac{\mathrm{e}^{-\alpha s}ds}{\sqrt{C_3\mathrm{e}^{2\alpha s}+C_4\mathrm{e}^{\alpha s}+C_3}} - \int^{t_2-t_1} \frac{ds}{\sqrt{C_3\mathrm{e}^{2\alpha s}+C_4\mathrm{e}^{\alpha s}+C_3}}$, $I = 2t_{xx} - \alpha t_x^2 - \alpha C_3\mathrm{e}^{2\alpha t}$,

iv) $F = \frac{(\mathrm{e}^{\alpha t}-\mathrm{e}^{\alpha t_2})(\mathrm{e}^{\alpha t_1}-\mathrm{e}^{\alpha t_3})}{(\mathrm{e}^{\alpha t}-\mathrm{e}^{\alpha t_3})(\mathrm{e}^{\alpha t_1}-\mathrm{e}^{\alpha t_2})}$, $I = t_x - C_5e^{\alpha t} - C_6e^{-\alpha t}$.

Chapter 5

Conclusion

In this thesis we studied the problem of classification of Darboux integrable nonlinear semi-discrete chains of hyperbolic type. We used an approach based on the notion of characteristic Lie algebra. At first, we gave the properties of characteristic Lie algebras for the equation $t_{1x} = f(t_x, t, t_1)$ and passed to analyze the special form of this equation which is

$$t_{1x} = t_x + d(t, t_1). \tag{5.1}$$

We found out all equations of this form, which are Darboux integrable. To be Darboux integrable, equation (5.1) should admit nontrivial x- and n-integrals or equivalently characteristic Lie algebras of it should be of finite dimensions. Hence we firstly find equations admitting nontrivial x-integrals and then analyzed these equations whether they have also nontrivial n-integrals. Finally, we gave a complete list of Darboux integrable hyperbolic type chains (5.1). We showed that the method of characteristic Lie algebras provides an effective tool to classify integrable discrete chains. This method did not get much attention in the literature. As we know, there are only two studies (see [15] and [19]), where the characteristic Lie algebras are used to solve the classification problem for the partial differential equations and systems. It is interesting that the first paper was published in 1981 and the second one twenty five years later.

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