SCHUBERT CALCULUS, ADJOINT REPRESENTATION AND MOMENT POLYTOPES

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By

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June, 2012

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

SCHUBERT CALCULUS, ADJOINT REPRESENTATION AND MOMENT POLYTOPES

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Let $\mathcal{H}^\nu$ denote the irreducible representation of the special unitary group $\mathrm{SU}(n)$ corresponding to Young diagram ν and let $\mathfrak{g} = \mathfrak{su}(n)$ denote the Lie algebra of $\mathrm{SU}(n)$. One can show that $\mathcal{H}^\nu$ appears in the symmetric algebra $\mathrm{S}^* \mathfrak{g}$ if and only if n divides the size of the Young diagram ν. Kostant's problem asks what is the least number N such that $\mathcal{H}^\nu$ appear in $\mathrm{S}^N \mathfrak{g}$. The *moment polytope* of the adjoint representation is the polytope generated by the normalized weights $\tilde{\nu}$ such that $\mathcal{H}^\nu$ appears in $\mathrm{S}^* \mathfrak{g}$ and it helps to put lower bounds on number N in the Kostant's problem. In this thesis, we compute the moment polytope of the adjoint representation of $\mathrm{SU}(n)$ for $n \leq 9$ using the solutions of the classical spectral problem and so-called ν-representability problem.

Keywords: Kostant's problem, level of a representation, moment polytope.

ÖZET

SCHUBERT CALCÜLÜSÜ, EŞLENİK TEMSİL VE MOMENT POLİTOPLARI

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ν bir Young diyagramı olsun ve $\mathcal{H}^\nu$ özel uniter grubun ν'ye karşılık gelen indirgenemez temsilini göstersin. Ayrıca, $\mathfrak{g} = \mathfrak{su}(n)$ de $\mathrm{SU}(n)$'nin Lie cebrini göstersin. Gösterilebilir ki $\mathcal{H}^\nu$'nin $\mathfrak{g}$'nin simetrik cebri $\mathrm{S}^* \mathfrak{g}$'da görünmesi için gerek ve yeter bir koşul n'nin ν'nün boyutunu bölmesidir. Kostant'ın problemi, $\mathcal{H}^\nu$'in $\mathrm{S}^N \mathfrak{g}$'de görüldüğü en küçük N değerinin ne olduğunu sorar. Eşlenik temsilin moment politopu, $\mathcal{H}^\nu$'nün $\mathrm{S}^* \mathfrak{g}$'de görüldüğü ν'lerin normalize edilmiş halleriyle gerilir. Moment politopu, Kostant'ın probleminde bahsedilen N sayısı için bir alt sınır koymaya yardımcı olur. Bu tezde klasik spektral probleminin ve ν-temsil edilebilirlik problemi adıyla bilinen bir diğer spektral problemin çözümlerini kullanarak $n \leq 9$ için moment politoplarını hesaplıyoruz.

Anahtar sözcükler: Kostant problemi, bir temsilin seviyesi, moment politopu.

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Contents

Chapter 1

Introduction

A very essential tool to study groups is their representations. A representation of a group G is a linear action of G on a vector space and it allows to see the elements of G as matrices. Then one has the extensive tools of linear algebra to investigate the structure of G.

It is an elementary fact in representation theory that any ordinary irreducible representation of a finite group G appears as a subrepresentation of the group algebra $\mathbb{C}G$. It is desirable to find analogues of the group algebra for a Lie group G. The algebra of analytical functions on G is an analague but let us focus on the special unitary group $\mathrm{SU}(n)$ and consider the symmetric algebra $\mathrm{S}^* \mathfrak{g}$ of the *adjoint representation* $\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{su}(n)$ is the Lie algebra of $\mathrm{SU}(n)$. The action of $\mathrm{SU}(n)$ on $\mathfrak{g}$ is the differential of the action of $\mathrm{SU}(n)$ on itself by conjugation. It is well known that all irreducible representations of $\mathrm{SU}(n)$ is are parametrized by Young diagrams ν having less than n rows. See Section (2.3.3.2) for this parametrization. The irreducible representation of $\mathrm{SU}(n)$ corresponding to a given Young diagram ν is denoted by $\mathcal{H}^\nu$, where the n-dimensional Hilbert space $\mathcal{H}$ is the *standard representation* of $\mathrm{SU}(n)$ on which $\mathrm{SU}(n)$ acts in the obvious way.

One can prove that a necessary and sufficient condition for a representation $\mathcal{H}^\nu$ of $\mathrm{SU}(n)$ to appear in $\mathrm{S}^* \mathfrak{g}$ is that n divides the size $|\nu|$ of the Young diagram ν.

Problem 1.0.1. *What is the smallest integer N such that $\mathcal{H}^\nu$ appears in $\mathrm{S}^N \mathfrak{g}$?*

This is known as Kostant's problem in the literature. See [10] and [6] for more information.
What is also related to this is the *level* k of the representation $\mathcal{H}^\nu$, which is defined by $k = 1 + \frac{Nn-|\nu|}{n}$.
A seemingly unrelated problem concerns the possible spectrum of a commutator $[A, A^*] = AA^* - A^*A$ where A is a square matrix normalized with $\mathrm{tr}(AA^*) = 1$.
The two problems turn out to be related as we will see now. Let H^λ be an irreducible representation that appear in $S^N\mathfrak{g}$. Adding columns of length n does not change the representation since a column of length n corresponds to the determinant. So, we may assume that size of λ is Nn. Normalize λ by $\tilde{\lambda} = \lambda/N$ so that it has trace n. In this setting, consider the following theorem, which is a special case of the theorem (3.4.3).

Theorem 1.0.2. *Every $\tilde{\lambda}$ is a spectrum of an operator $I + [A, A^*]$ such that spec$(I + [A, A^*])$ is in non-decreasing order and A is a unit complex matrix, i.e., $\mathrm{tr}(AA^*) = 1$.*
The convex hull of $\tilde{\lambda}$ in the space $\mathbb{R}^n$ is a rational convex polytope.
Furthermore, every spectrum $\lambda =$ spec$(I + [A, A^*])$ (in non-decreasing order) can be approximated by the $\tilde{\lambda}$.

The polytope mentioned in the theorem is called the moment polytope of the adjoint representation of $\mathrm{SU}(n)$ and we denote it by P_n.
Caution. In the tables of the moment polytopes we supply in the Appendix B, the points in the polytopes are shifted by $(-1, -1, \ldots, -1)$ so that the sum of the components of a point is 0. This is due to the fact that we consider the spectrum of $[A, A^*]$ instead of the spectrum of $[A, A^*]$ for the sake of simplicity.
Given a representation $\mathcal{H}^\nu$ of $\mathrm{SU}(n)$, if we have the moment polytope P_n then we have some information of the smallest integer N such that $\mathcal{H}^\nu$ appears in $\mathrm{S}^N \mathfrak{g}$: Since adding columns of length n to ν does not change $\mathcal{H}^\nu$, we can add such columns to ν until it is contained in P_n. Then, the number of columns added gives a lower bound for $k - 1$.

Example 1.0.3. Let $n = 3$ and consider the representation V with Young diagram $\square\square\square$. After adding 2 columns of length 3, the corresponding spectrum enters the moment polytope. This gives the lower bound $\frac{2.3+3}{3} = 3$ for N and V actually appears in $\mathrm{S}^3\,\mathfrak{g}$.

Conjecture 1.0.4. *For an arbitrary n and the irreducible representation V of $\mathrm{SU}(n)$ corresponding to a row diagram with n boxes, V appears in $\mathrm{S}^N\,\mathfrak{g}$, where $N = \binom{n}{2}$.*

Hence, the computation of the P_n becomes important and it is the heart of this thesis.

As the definition suggests, the first step to compute P_n is to compute symmetric powers of the adjoint representation. Thanks to the software package *LIE* this can be accomplished. However, due the complexity of the function which computes the symmetric powers, it is hard to compute P_n for $n \geq 7$ on an ordinary computer of today's standards.

As Theorem (1.0.2) suggests, one can try to find all possible spectra of the matrices of the form $[A, A^*] = AA^* - A^*A$. This is an instance of classical spectral problem since AA^* and A^*A are Hermitian matrices. We can use the recursively defined inequalities conjectured by Horn or we can make use of hive model invented by Knutson and Tao. However, due to the complexity of algorithms we can compute P_n only for $n \leq 6$. We can still proceed by the virtue of a theorem of M. Altunbulak and A. Klyachko. The fact that P_n consists of spectra of $I + AA^* - A^*A$ makes the problem an instance of another spectral problem, the so-called ν-representability problem, the solution of which makes it possible to determine if a given linear inequality determining a halfspace in the space $\mathbb{R}^n$ determines a facet of P_n or not. Checking every inequality would be impossible so it makes sense to approximate P_n from inside by a polytope Q_n and check if all facets of Q_n is genuine or not.

To approximate P_n from inside, one can set A to be $\sqrt{D}P^{-1}$ where D is a nonnegative diagonal matrix and P is a permutation matrix, so that $[A, A^*] = D - PDP^{-1}$. Thus we deal with a spectra of the form $(d - \sigma d)_i = d_i - d_{\sigma(i)}$ where $\sigma \in S_n$. In this way, one gets the polytopes P_3, P_4 and P_6 but cannot get P_5, for

instance. Then, we also use the decomposition of the symmetric powers of the adjoint representation. In this way, we could compute P_5 and P_7. When $n = 8$, we were stuck again but by guessing some points in some particular directions and proving that they are inside P_8 by virtue of the solutions of classical spectral theory, we obtained the moment polytope.
When $n = 9$, we could not make a sufficient approximation due to the complexity of algorithms being employed. However, by the fact that the convex hull of the vertices on a facet of a moment polytope forms the moment polytope of the adjoint representation of another group, the genuine facets we obtained enabled us to get the whole moment polytope P_9. Due to seemingly exponential complexity of the problem and the limitations on the speed of current computers, it does not seem easy at all to compute P_n for $n \geq 10$.

In Chapter 2, we give a digest of the Lie theory and representation theory.
In Chapter 3, we describe the solution of the so-called ν-representability problem, which is due to Klyachko.
In Chapter 4, we discuss Kostant's problem and describe our method to compute the moment polytope P_n for $n \leq 9$.
Chapter 5 is devoted to the classical spectral problem, which waited a long time to be given a complete solution by Klyachko. We state Klyachko's solution to the problem in terms Schubert calculus. Furthermore we discuss other answers to the problem including the recursive inequlities conjectured by Horn and Knutson and Tao's hive model. We also describe another problem concerning the invariant factors of matrices which have exactly the same solutions as the classical spectral problem.

Chapter 2

A Digest of Lie Groups and Lie Algebras

2.1 Lie Groups

A real Lie group is a smooth manifold with a smooth structure given by multiplication $\times : G \times G \to G$ and inversion $i : G \to G$.

Example 2.1.1. **(1)** Additive $\mathbb{C}^+$ and multiplicative $\mathbb{C}^* = \mathbb{C} - \{0\}$ groups of the complex field $\mathbb{C}$.
(2) The unit circle $\mathbb{S}^1 = \{z \in \mathbb{C}^* : |z| = 1\}$ in $\mathbb{C}^*$.
(3) The unit sphere $\mathbb{S}^3 = \{a + bi + cj + dk \in \mathbb{H} : |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1\}$ in the division ring of quaternions $\mathbb{H}$.
(4) The unit quaternions $\{q \in \mathbb{H}^* : |q| = 1\}$.
In fact, an element of $\mathbb{H}$ can be considered as a 2×2 complex matrix, identifying i with $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, j with $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and k with $\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$. So, $x + iy + jz + kt$ can be interpreted as a matrix $\left(\begin{smallmatrix} u & v \\ -\bar{v} & \bar{u} \end{smallmatrix}\right)$. This establishes an isomorphism between SU(2) and $\mathbb{S}^3$.
$\mathbb{S}^1$ and $\mathbb{S}^3$ are the only spheres which admit a group structure.
(5) Special linear groups, $\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : det(A) = 1\}$ and
$\mathrm{SL}_n(\mathbb{C}) = \{A \in \mathrm{GL}_n(\mathbb{C}) : det(A) = 1\}$.

(6) $\mathrm{SU}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) : det(A) = 1, AA^* = A^*A = I\}$, the group of unitary transformations of $\mathbb{C}^n$ having determinant 1.
(7) $\mathrm{O}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : AA^t = A^tA = I\}$, the group of orthogonal transformations of $\mathbb{R}^n$.
(8) $\mathrm{SO}(n)$, the group of isometries of $\mathbb{E}^n$ with determinant 1.
(9) $\mathrm{SP}(2n)$, the group of $2n \times 2n$ real matrices M satisfying $M^T \Omega M = \Omega$ where $\Omega = \left(\begin{smallmatrix} O & -I_n \\ I_n & 0 \end{smallmatrix}\right)$.

2.2 Lie Algebras

Given a Lie group G, consider the left invariant smooth differential operators X on functions $f : G \to \mathbb{R}$. This means Leibnez rule $Xfh = fXh + (Xf)h$ is satisfied and $Xf(gx) = (Xf)(t)|_{t=gx}$. Such operators X and Y are closed under *Lie brackets* $[X, Y] = XY - YX$ and form the *Lie Algebra* $\mathrm{Lie}\, G$ of the group G, sometimes denoted by $\mathfrak{g}$. Given X, Y and Z in $\mathfrak{g}$ we

$$(i)[X, X] = 0, \forall X \in \mathfrak{g}$$
$$(ii)[[X, Y], Z] + [[Y, Z], X]] + [[Z, X], Y] = 0, \forall X, Y, Z \in \mathfrak{g}$$

The second property is known as *Jacobi identity.*
A smooth differentiation operator is locally given by the usual differentiation of a function in a tangent direction τ. Since a smooth differentiation operator X can be identified with a smooth vector field on G and X is taken to be left invariant in our setting, X can be taken to be a left invariant smooth vector field or rather an element of the tangent space T_e at the identity e of G. For this reason, Lie algebra of G is often called the *tangent algebra* of G.

Theorem 2.2.1 (Fundamental Theorem of Lie Theory)**.** *The functor $G \mapsto LieG = \mathfrak{g}$ establishes an equivalence of the category of the simply connected Lie groups and the category of real finite dimensional Lie algebras.*

The *exponential map* $\exp : \mathrm{Lie}\, G \to G$ is defined by sending an element $s\tau \in \mathrm{Lie}\, G$ to the end point of the trajectory of length s starting at $e \in G$ which is determined by the differentiation operator X corresponding to τ. Here, $s \in \mathbb{R}$ and τ is a unit vector in $\operatorname{Lie} G$.

Example 2.2.2. (1) Set $G = \mathrm{SU}(\mathcal{H})$. Then, an operator X on $\mathcal{H}$ is in $\mathfrak{su}(\mathcal{H})$ if and only if $\exp X = \exp X$ is in $\mathrm{SU}(\mathcal{H})$. This amounts to saying X is a traceless skew-Hermitian operator.
(2) Set $G = \mathrm{U}(\mathcal{H})$. Then, an operator X on $\mathcal{H}$ is in $\mathfrak{u}(\mathcal{H})$ if and only if $\exp X = \exp X$ is in $\mathrm{U}(\mathcal{H})$. In this case, X is a skew-Hermitian operator.
(3) Set $G = \mathrm{SL}(\mathcal{H})$. Then, an operator X on $\mathcal{H}$ is in $\mathfrak{sl}(\mathcal{H})$ if and only if $\exp X = \exp X$ is in $\mathrm{SL}(\mathcal{H})$. This means X is a traceless operator.
(4) Set $G = \mathrm{SO}(\mathcal{H})$. Then, an operator X on $\mathcal{H}$ is in $\mathfrak{so}(\mathcal{H})$ if and only if $\exp X = \exp X$ is in $\mathrm{SO}(\mathcal{H})$. This means X is a traceless anti-symmetric operator.

A non-abelian Lie algebra $\mathfrak{g}$ is called *simple* if its only ideals are 0 and itself. A simple Lie algebra $\mathfrak{g}$ either belong to the three families corresponding to the families of Lie groups SL, SO and SP or $\mathfrak{g}$ is one of the exceptional Lie groups E_6, E_7, E_8, F_4, and G_2.
Lie algebra $\mathfrak{g}$ is called semisimple if it has no nonzero *abelian* ideals. All semisimple Lie algebras are direct sums of the simple ones.
All finite dimensional representations of semisimple Lie algebras and Lie groups are completely reducible. This follows from the fact that representations of compact Lie groups are always completely reducible which can be proven by the averaging argument in the proof of Maschke's theorem for finite groups. The difference in the proof is that one integrates over G instead of summing over the elements of G.
Given a real Lie group G with the Lie algebra L, one may consider the *complexification* $L \otimes_{\mathbb{R}} \mathbb{C}$ of L, which becomes a complex Lie algebra. Then by the virtue of the fundamental theorem, one defines the complexification $G^{\mathbb{C}}$ of the Lie group G to be the complex Lie group with the Lie algebra $L \otimes_{\mathbb{R}} \mathbb{C}$.
For instance, we have the classical reductive groups $\mathrm{U}(n)^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{SO}(n)^{\mathbb{C}} = \mathrm{SO}(n, \mathbb{C})$ and $\mathrm{SU}(n)^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$.

Example 2.2.3. (1) Consider the Lie algebra $\mathfrak{u}(n)$ of the unitary group $\mathrm{U}(n)$, which consists of skew-Hermitian matrices. Its complexification is the full matrix

algebra. Indeed, every complex matrix $X \in \mathrm{Mat}(n,\mathbb{C}) = \mathrm{Lie}\,\mathrm{GL}(n,\mathbb{C})$ can be written as $X = A + iB$ for Hermitian matrices $A = \frac{1}{2}(X + X^*)$ and $B = \frac{1}{2i}(X - X^*)$. Therefore we have $\mathrm{U}(n) \otimes \mathbb{C} = \mathrm{GL}(n,\mathbb{C})$.
(2) By the same argument above, $\mathrm{SU}(n) \otimes \mathbb{C} = \mathrm{SL}(n,\mathbb{C})$.
(3) Since any element $X \in \mathfrak{so}(n,\mathbb{C})$ can be written as $X = A + iB$, where A and B are in $\mathfrak{so}(n)$, it follows that $\mathrm{SO}(n) \otimes \mathbb{C} = \mathrm{SO}(n,\mathbb{C})$.

2.3 Group representations

To study the properties of a given group, it is usually quite useful to study its *representations*.
A representation of a group G is a linear action of G on a finite dimensional vector space V over a field k (in our study $k = \mathbb{R}$ or $\mathbb{C}$). In the case G is a Lie group, we also want the resulting homomorphism $G \to \mathrm{GL}(V)$ to be smooth.

2.3.1 Tensor product of representations

Given two groups G and K and representations V and W of G and K, $V \otimes W$ becomes a representation of the direct product $G \times K$. The action of $G \times K$ on $V \otimes W$ is given by

$$(g,k).(v \otimes w) = g.v \otimes k.w$$

In fact, every irreducible representation of $G \times K$ is of the form $V \otimes W$, where V is an irreducible representation of G and W is an irreducible representation of K.
If we take $G = K$ then $V \otimes W$ becomes a representation of G as well as a representation of $G \otimes G$, considering the diagonal embedding $g \mapsto (g,g)$ of G into $G \times G$. Taking $V = W$ and proceeding inductively, we obtain tensor powers $V^{\otimes n}$ of V which are again representations of G. The tensor power $V^{\otimes n}$ also becomes a representation of the symmetric group S_n, where an element $\sigma \in S_n$ acts by permuting the tensor factors:

$$\sigma.(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(n)}$$

2.3.2 Symmetric and alternating powers of a representation.

The elements of $V^{\otimes n}$ fixed under the action of S_n form a subrepresentation $\mathrm{S}^n V$ and is called the n^{th} *symmetric power* of V.
The elements v of $V^{\otimes n}$ satisfying $\sigma v = \mathrm{sgn}(\sigma)v$ for all $\sigma \in S_n$ form a subrepresentation $\wedge^n V$ and is called the n^{th} *symmetric power* of V.
This shows that $V^{\otimes n}$ is not irreducible for $n > 1$.

2.3.3 Representations of the symmetric and special unitary groups

2.3.3.1 Representations of the symmetric groups

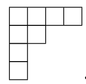
A *partition* ν of a natural number n, denoted $\nu \vdash n$, is a non-increasing sequence $\nu_1, \nu_2, \ldots$ of natural numbers such that $\sum_i \nu_i = n$. The condition $\sum_i \nu_i = n$ forces ν to stabilize at 0 after some point k, thus we may write ν as a k-tuple. Sometimes we write $\nu = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ and it is understood that the number of occurences of a number m in the sequence $(\nu_1, \nu_2, \ldots, \nu_k)$ is α_m.
The *cycle type* of an element $\sigma \in S_n$ is the non-increasing sequence of the lengths of the cycles in the decomposition of σ into disjoint cycles, which is a partition of n.

Example 2.3.1. $\nu = (4, 3, 2)$ is the cycle type of $(1234)(567)(89) \in S_9$.

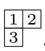
Under conjugacy, the cycle type does not change and any two elements having the same cycle type are conjugate. Therefore, the conjugacy classes of a symmetric group S_n are determined by the *cycle types* of the elements.

A *Young diagram* D is a visualization of a partition $\nu \vdash n$. i^{th} row of D has ν_i boxes.

Example 2.3.2. The Young diagram corresponding to the partition $(4, 2, 1, 1)$ of 8 is


.

A nice property of S_n is that its irreducible representations have a nice description in terms of the conjugacy classes, or equivalently, in terms of the partitions ν of n:
Let $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$ be a partition of n. Consider the Young diagram D of shape ν filled with the numbers $1, 2, \ldots, n$ such that when the numbers are read row by row starting from the first row to the last one, the sequence obtained is $1, 2, \ldots, n$.

Example 2.3.3. The Young diagram corresponding to the partition $\nu = (2, 1)$ is 

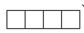
.

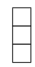
Let P (resp. Q) be the subgroup of S_n consisting of the permutations which fix each row (resp. column) setwise. Let $\mathbb{C}S_n$ denote the group algebra, $a_\nu = \sum_{\sigma \in P} \sigma$, $b_\nu = \sum_{\sigma \in Q} sgn(\sigma)\sigma$ and $c_\nu = a_\nu b_\nu$. c_ν is called the Young symmetrizer.
The following theorem gives a characterization of the irreducible representations of the symmetric groups.

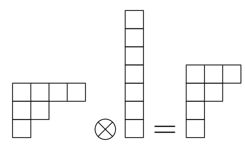
Theorem 2.3.4. *Given a partition ν of n, the image $S^\nu = \mathbb{C}S_n c_\nu$ of the element c_ν is an irreducible representation of S_n and every irreducible representation of S_n occurs this way.*

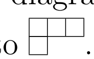
If the Young diagram is filled with the numbers $1, 2, \ldots, n$ in another way, one gets a representation isomorphic to S^ν.
S^ν has the following characterization. S^ν is the unique (up to isomorphism) irreducible representation of S_n such that

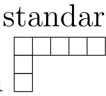
(i) S^ν contains an invariant element under the action of P,
(ii) S^ν contains a skew-symmetric element x under the action of Q, i.e., the action of an element $\sigma \in Q$ on x is multiplication by $sgn(\sigma)$.

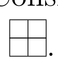
Example 2.3.5. **(1)** If $\nu = (n)$, i.e., the Young diagram D is a row diagram (for $n = 4$, ), then S^ν is the trivial representation.

(2) If the diagram is taken to be a column diagram  with n boxes, then S_ν is the signed representation. The tensor product of an irreducible representation S^ν with the signed representation is S^{ν^t}, where ν^t stands for the transpose of ν. For example,

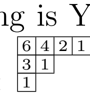


(3) Let U be an n-dimensional vector space with the basis $\{e_i\}_{1\leq i\leq n}$ and consider the action of S_n on U by permuting the basis elements and extended linearly. In this way U becomes a representation of S_n and $V = \{\sum_{i=1}^{n} c_i e_i : \sum_{i=1}^{n} c_i = 0\}$ is an irreducible subrepresentation of U. V is called the *standard representation* of S_n and corresponds to the Young diagram of shape $(n-1, 1)$. The standard representation V of S_4 corresponds to .

(4) k^{th} exterior power $\wedge^k V$ of the standard representation S_n is again an irreducible one and has Young diagram  with shape $(1^N, (n-N)^1)$.

(5) Consider the irreducible representation of S_4 corresponding to Young diagram . It has dimension 2 by the hook length formula below and comes from the standard representation of $S_3 \cong S_4/V_4$, where V_4 is the subgroup of S_4 which consists of double transpositions and the identity.

Define the *hook length* of a box B in a Young diagram of shape ν is the number of boxes which are either on the right of B or below B, including B.

Example 2.3.6. Let $\nu = (4, 2, 1)$. Then the following is Young diagram where the number appearing in each box is its Hook length: 

.

The dimension of the representation S^ν can be calculated via the *Hook length formula.*

Theorem 2.3.7.

$$\dim S^\nu = \frac{n!}{\prod \text{Hook lengths}}$$

Example 2.3.8. For $\nu = (4, 2, 1)$, $\dim S^\nu = \frac{7!}{6.4.2.1.3.1.1}$. For $\nu = (n-1, 1)$, $\dim S^\nu = \frac{n!}{n.(n-2)!} = n - 1$, which is consistent with the definition of the standard representation of S_n.

2.3.3.2 Representations of $\mathrm{SU}(\mathcal{H})$

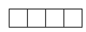
Let $\mathcal{H}$ be a finite dimensional Hilbert space. Then, studying the representations of $\mathrm{SU}(\mathcal{H})$ amounts to studying the complexified Lie algebra $\mathfrak{su}(\mathcal{H}) \otimes \mathbb{C} = sl(\mathcal{H})$, which are simply the traceless complex $n \times n$ matrices, where n is the dimension of the space $\mathcal{H}$.

$\mathfrak{sl}(\mathcal{H})$ can be decomposed as $\mathfrak{sl}(\mathcal{H}) = D \oplus T^+ \oplus T^-$, where D is the Cartan subalgebra consisting of traceless diagonal matrices and T^+ and T^- are the upper-triangular and lower-triangular matrices, respectively.

Irreducible representations of $\mathrm{SU}(\mathcal{H})$ are parametrized by Young diagrams having less than n rows and the irreducible representation corresponding to a Young diagram ν is denoted by $\mathcal{H}^\nu$.

Given an irreducible representation $\mathcal{H}^\nu$ of $\mathrm{SU}(\mathcal{H})$, one can recover the Young diagram ν as follows. The space $V = \{x \in \mathcal{H}^\nu : T^+x = 0\}$ is one-dimensional and is invariant under the action of $D \oplus T^+$. A non-zero element $x \in V$ is called a *highest vector*. Since V is one-dimensional, action of an element $d \in D$ is multiplication by a scalar $\omega(d)$, where ω is a weight. The point is that ω can be written as a linear combination $\sum_{i=1}^{n-1} a_i\omega_i$ of fundamental weights ω_i. Then, the a_i determine ν via $\nu_i - \nu_{i-1} = a_i$.

Example 2.3.9. **(1)** The action of $\mathrm{SU}(\mathcal{H})$ on $\mathcal{H}$ in the obvious way makes $\mathcal{H}$ a representation of $\mathrm{SU}(\mathcal{H})$, which is called the *standard representation.* The standard representation corresponds to the Young diagram containing a single box, $\square$.

(2) A row diagram  with k boxes describes the symmetric power $\mathrm{S}^k\,\mathcal{H}$ of the standard representation $\mathcal{H}$.

(3) A column diagram  having k boxes corresponds to the exterior power $\wedge^k\mathcal{H}$ of $\mathcal{H}$.

(4) The irreducible representation of $\mathrm{SU}(n)$ corresponding to the Young diagram  describes the space of Riemann curvature tensors $R(i,j;k,l)$ in space $\mathcal{H}$, which satisfy

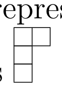
$$
\begin{aligned}
&R(\alpha,\beta;\gamma,\delta) = -R(\beta,\alpha;\gamma,\delta)\\
&R(\alpha,\beta;\gamma,\delta) = R(\gamma,\delta;\beta,\alpha)\\
&R(\alpha,\beta;\gamma,\delta) + R(\alpha,\gamma;\delta,\beta) + R(\alpha,\delta;\beta,\gamma) = 0
\end{aligned}
$$

We examine the adjoint representation of $\mathrm{SU}(n)$ in a separate section.

2.3.3.3 Adjoint representation of $\mathrm{SU}(n)$

The Lie group $\mathrm{SU}(\mathcal{H})$ acts on itself by conjugation and the differential of this action is the *adjoint representation* of $\mathrm{SU}(n)$ on its Lie algebra $\mathfrak{g} = \mathfrak{su}(n)$ The respective representation of $\mathfrak{g}$ is given by

$$\mathrm{ad}\, y : x \mapsto [y,x]$$

and is also named the adjoint representation. The Young diagram corresponding to the adjoint representation is  and the corresponding partition is $(1^{n-2}, 2^1)$. The adjoint representation is irreducible if and only if the Lie algebra $\mathfrak{g}$ is simple.

2.3.3.4 Schur's duality

As discussed above, the tensor power $\mathcal{H}^{\otimes N}$ of the standard representation $\mathcal{H}$ of $\mathrm{SU}(n)$ is a representation of both $\mathrm{SU}(n)$ and the symmetric group S_n, whose actions commute with each other. Hence, $\mathcal{H}^{\otimes N}$ is a direct sum of irreducible representations of the direct product $\mathrm{SU}(n) \times S_n$. The explicit decomposition is given by Schur's duality.

$$\mathcal{H}^{\otimes N} = \sum_{|\nu|=N} \mathcal{H}^{\nu} \otimes S^{\nu}$$

Chapter 3

Representations and spectra

Having discussed the classical spectral problem in the previous chapter, we now discuss the so-called ν-representability problem.

The problem has its roots in physics and is a generalization of N-representability problem, which is concerned with the relationship between a state in an N-fermion system and its reduced state. In 2008, M. Altunbulak and A. Klyachko [1] gave a solution to this problem using the machinaries of algebraic geometry, representation theory and geometric invariant theory.

To state the problem in mathematical language, let us first introduce *tensor contraction*. Firstly, let us start with the space $\mathcal{H} \otimes \mathcal{H}^*$, where $\mathcal{H}$ is a Hilbert space and $\mathcal{H}^*$ stands for its dual. Using the isomorphism $\mathcal{H} \otimes \mathcal{H}^* \cong \mathrm{Hom}(\mathcal{H}, \mathcal{H})$ and fixing a basis of $\mathcal{H}$, an element A of $\mathcal{H}^* \otimes \mathcal{H}$ (a rank $(1,1)$ tensor) can be considered as a matrix (a_i^j), where the lower index i corresponds to rows and the upper index j corresponds the columns. In this case, The contraction of $A = (a_i^j)$ is $\sum_i a_i^i = \mathrm{tr}(A)$ (a $(0,0)$ tensor). Sometimes one writes simply a_i^i in lieu of $\sum_i a_i^i$ using Einstein's convention: It is understood that we sum up over the repeating indices.

More generally, fixing a basis of $\mathcal{H}$, if we have a rank (m,n)-tensor, i.e., an element A of $(\mathcal{H}^*)^{\otimes m} \otimes \mathcal{H}^{\otimes n}$ we can consider A as a "multidimensional" matrix $(a_{i_1 i_2 \dots i_m}^{j_1 j_2 \dots j_n})$. Then given $1 \leq m_0 \leq m$ and $1 \leq n_0 \leq n$, the contraction of A on the indices m_0 and n_0 is defined similarly. Contraction is defined by fixing a basis of $\mathcal{H}$ but it can be shown that it is independent of this choice.

Example 3.0.10. Consider the Ricci tensor $R(\vec{A}, \vec{B})$ obtained by the contraction of the Riemann tensor $R(\tilde{e}^{\kappa}, \vec{A}, e_{\lambda}, \vec{B})$. This can be stated as $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$.

3.1 Statement of the ν-Representability Problem

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite dimensional complex Hilbert spaces and set $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\rho : \mathcal{H}_{AB} \to \mathcal{H}_{AB}$ be an Hermitian operator. Then, the *reduced operators* $\rho_A : \mathcal{H}_A \to \mathcal{H}_A$ and $\rho_B : \mathcal{H}_B \to \mathcal{H}_B$ are defined to be the contractions of ρ_{AB}: Considering ρ_{AB} as a matrix (a^{jl}_{ik}), $\rho_A = a^{jk}_{ik}$ and $\rho_B = a^{il}_{ik}$.
Now, consider *Schur's duality*

$$\mathcal{H}^{\otimes N} = \sum_{|\nu|=N} \mathcal{H}^{\nu} \otimes \mathcal{S}^{\nu}$$

where the sum is over all the Young diagrams ν of size N and $\mathcal{S}^{\nu}$ is the irreducible representation of the symmetric group S_N corresponding to the Young diagram ν. An Hermitian operator ρ^{ν} on $\mathcal{H}^{\nu}$ can be considered as an operator on $\mathcal{H}^{\otimes N}$ which equals $(\rho^{\nu} \otimes 1)/\dim \mathcal{S}^{\nu}$ on $\mathcal{H}^{\nu} \otimes \mathcal{S}^{\nu}$ and zero on the other components. Let $\rho = \rho_i : \mathcal{H} \to \mathcal{H}$ be the i^{th} reduced state. ρ turns out to be independent of i since $\rho^{\nu} \otimes 1$ and the action of S_N commutes.
In *ν-representability problem*, one studies the relationship between the spectrum μ of an operator ρ^{ν} and the spectrum λ of its reduced operator ρ. The spectrum λ is called the *occupation numbers*.

3.2 Invariant theory

Let k be an algebraically closed field and V a finite dimensional vector space over k. Let $V^* = \mathrm{Hom}_k(V, k)$ be the dual of V. Then V^* generates a subalgebra

$S(V) = S$ of the k-algebra of all k-valued functions on V.
Then there is an isomorphism of k-algebras $\phi : S(V) \to k[T_1, \dots, T_n]$ such that $\phi(f_i) = T_i$.
$S(V)$ is a representation of G via

$$(g.f)(v) = f(g^{-1}.v)$$

Then, V^* is a subrepresentation of $S(V)$. We call $S(V)^G := \{f \in S(V) : g.f = f\}$ the G-invariant polynomial functions on V. In invariant theory, one studies the properties of $S(V)^G$. Note that the elements of $S(V)^G$ are the elements of $S(V)$ which are constants the orbits in V under the action of G. An invariant is said to *separate* two orbits if it takes different values on those orbits.
Since a polynomial $S(V)$ is determined by its roots, which are points in $\mathbb{P}k$, one can regard invariants as geometric configurations which stay unchanged under the action of the given group.
In fact, according to the Erlangen program of Felix Klein, the study of invariant theory is nothing but the study of geometry. Invariant quantities of a geometric object are those which do not depend on the choice of a coordinate system for the space.

Example 3.2.1. (1) The volume of a parallelepiped formed by n vectors in the space $\mathbb{R}^n$ is given by the determinant $det(A)$, where A is the matrix whose columns are the given vectors. The fact that special linear group respects volume is reflected by $det(A) = det(gA)$. Also, $det(gAg^{-1}) = det(A)$ says the volume of the parallelepiped formed by the column vectors of a remains unchanged if the basis is changed, provided the volume of the parallelepiped formed by the vectors in the basis is taken to be 1 after the change of basis.
(2) Another invariant associated to an endomorphism of a vectors space is its trace. It is related to the determinant via $det(e^A) = e^{\mathrm{tr}(A)}$. Setting $\mathfrak{g} = su(\mathcal{H})$, define an inner product on $su(\mathfrak{g})$ by $(a, b) = \mathrm{tr}(ab)$. Then $([x, a], b) + (a, [x, b]) = 0$ and the metric induced by this inner product is the so called *invariant metric* in adjoint representation.

As noted above, to a d-dimensional subspace W of $\mathcal{H}$ corresponds a unique point (up to a nonzero scalar) in $\wedge^d \mathcal{H}$. Generalizing this, we can consider a configuration of subspaces W_i in $\mathcal{H}$ as a point in $X = \otimes_\alpha \wedge^{d_\alpha \mathcal{H}}$. Consider the orbit of a point $\phi \in X$. If its closure contains 0, then we can find no invariant polynomial on X which is nonzero at ϕ. We call such ϕ *unstable configurations*. If the orbit of a non-zero ϕ is closed, we call ϕ a *stable configuration*. The remaining configurations are called *semistable*.

3.3 Geometric Stability Criteria

In this section we discuss the geometric stability criteria which is needed in the solution of the ν-representability problem. Let $SL(\mathcal{H})$ be the group of transformations on $\mathcal{H}$ in the following discussion.
The following theorem is due to Mumford.

Theorem 3.3.1 ([15]). *A configuration of subspaces $\mathcal{F}_\alpha$ in $\mathcal{H}$ is semistable if and only if for any proper subspace $E \subset \mathcal{H}$ the following inequality is satisfied.*

$$\frac{1}{\dim E} \sum_\alpha \dim(\mathcal{F}_\alpha \cap E) \leq \frac{1}{\dim \mathcal{H}} \sum_\alpha \dim \mathcal{F}_\alpha \tag{3.1}$$

If inequalities are strict, then the configuration is stable.

Roughly speaking, the theorem says that a semistable configuration is not concentrated in a subspace E of $\mathcal{H}$.

Theorem 3.3.2 (Kempf-Ness Unitary Trick). *Fix a metric in $\mathcal{H}$ and the induced metric on $\otimes_\alpha \wedge^{d_\alpha \mathcal{H}}$. Then, the following conditions on a Plücker vector $\phi \in \otimes_\alpha \wedge^{d_\alpha \mathcal{H}}$ are equivalent:*
(1) ϕ is stable.
(2) The orbit of ϕ contains a unique (up to a unitary transformation) vector ϕ_0 of minimal length.

In other words, a stable configuration ϕ determines a unique (up to) proportionality matric on $\mathcal{H}$ such that ϕ has minimal length in its orbit.
The stability of ϕ can be restated as in terms of the projection operators P_α onto

$\mathcal{F}_\alpha$ via
$$\sum_\alpha P_\alpha = \text{scalar}$$

Given a Hermitian operator X_α with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_k$, one can consider the filtration $\mathcal{F}_\alpha$ given by

$$\mathcal{F}_\alpha(s) = \text{ the sum of eigenspaces of } X_\alpha \text{ with eigenvalues} \geq s$$

instead of subspaces $\mathcal{F}_\alpha$ of $\mathcal{H}$. This can be considered as a flag

$$0 \subset \mathcal{F}_\alpha(\lambda_1) \subset \mathcal{F}_\alpha(\lambda_2) \subset \ldots \subset \mathcal{F}_\alpha(\lambda_k) = \mathcal{H}$$

For simplicity, assume X_α is a nonnegative operator. Note that X_α can be recovered via

$$\begin{aligned} X_\alpha &= \int_0^\infty P_\alpha(s)\,ds \\ &= (\lambda_1^\alpha - \lambda_2^\alpha)P_\alpha(\lambda_1^\alpha) + (\lambda_2^\alpha - \lambda_3^\alpha)P_\alpha(\lambda_2^\alpha) + \ldots + (\lambda_{k-1}^\alpha - \lambda_k^\alpha)P_\alpha(\lambda_{k-1}^\alpha) + \lambda_k^\alpha P_\alpha(\lambda_k) \end{aligned}$$

where $P_\alpha(s)$ is the projection operator onto the subspace $\mathcal{F}_\alpha(s)$. Considering $\mathcal{F}_\alpha(\lambda_i)$ as a subspace of multiplicity $\lambda_i^\alpha - \lambda_{i+1}^\alpha$ we get the following stability criterion.

Theorem 3.3.3. *A system of filtrations $\mathcal{F}_\alpha$ is semistable if and only if for any proper subspace $E \subset \mathcal{H}$ the following inequality is satisfied.*

$$\frac{1}{\dim E} \sum_\alpha \int_0^\infty \dim(\mathcal{F}_\alpha(s) \cap E)\,ds \leq \frac{1}{\dim \mathcal{H}} \sum_\alpha \int_0^\infty \dim \mathcal{F}_\alpha(s)\,ds \qquad (3.2)$$

If the inequalities are strict, then the system of filtrations is stable.

Stability have a metric characterization again.

Theorem 3.3.4. *A system of filtrations $\mathcal{F}_\alpha$ is stable if and only if there exists a metric on $\mathcal{H}$ such that*

$$\sum_\alpha X_\alpha = \textit{scalar}$$

where $X_\alpha = \int_0^\infty P_\alpha(s)ds$ and $P_\alpha(s)$ is the projection operator onto $\mathcal{F}_\alpha(s)$.

The geometric stability criterion (3.2)) can be restated for a test filtration instead of a test space.

Theorem 3.3.5. *A system of filtrations $\mathcal{F}_\alpha$ is semistable if and only if for any test filtration $E(t)$ the following inequality is satisfied.*

$$\sum_\alpha \int\int (\dim(\mathcal{F}_\alpha(s) \cap E(t)) - \frac{\dim \mathcal{F}_\alpha \dim E(t)}{\dim \mathcal{H}})\, ds\, dt \leq 0 \tag{3.3}$$

where the integration is over the whole (s,t)-plane.
If the inequalities are strict, then the system is filtrations is stable.

Let $G \subset SU(\mathcal{H})$ be a connected subgroup and $\mathfrak{g} \subset su(\mathcal{H})$ be its Lie algebra. Here, $su(\mathcal{H})$ consists of traceless Hermitian operators with the Lie bracket $[X,Y] = i(XY - YX)$. Let $G^{\mathrm{c}} \subset SL(\mathcal{H})$ be the complexification of G. Then, we have the following geometric stability criterion.

Theorem 3.3.6. *A system of filtrations $\mathcal{F}_\alpha$ is G^c-semistable if and only if for any non-zero operator $x \in \mathfrak{g}$ with spectral filtration $E_x(t)$ the following inequality is satisfied.*

$$\sum_\alpha \int\int (\dim(\mathcal{F}_\alpha(s) \cap E_x(t)) - \frac{\dim \mathcal{F}_\alpha \dim E_x(t)}{\dim \mathcal{H}})\, ds\, dt \leq 0 \tag{3.4}$$

where the integration is over the whole (s,t)-plane.
If the inequalities are strict, then the system is filtrations G^c-stable.

The metric characterization of stability is as follows.

Theorem 3.3.7. *A system of filtrations $\mathcal{F}_\alpha$ is G^c-stable if and only if there exists a metric such that*

$$\sum_\alpha X_\alpha \in \mathfrak{g}^\perp$$

where $X_\alpha = \int_0^\infty P_\alpha(s)ds$, $P_\alpha(s)$ is the projection operator onto $\mathcal{F}_\alpha(s)$ and $\mathfrak{g}^\perp$ is the orthogonal complement of $\mathfrak{g}$ in the space of all Hermitian operators with the trace norm $(X,Y) = \mathrm{tr}(XY)$.

Suppose $\mathcal{F}(s)$ and $E(t)$ are complete filtrations, i.e., the dimension drops are at most one at any point. Then,

$$\int\int \dim(\mathcal{F}(s) \cap E(t))\, ds\, dt = \sum t_i s_{w(i)} \tag{3.5}$$

where s_i, t_j are eigenvalues of the respective operators and w is the permutation which describes the *relative position* of the respective flags

$$0 \subset \mathcal{F}_\alpha(s_1) \subset \mathcal{F}_\alpha(s_2) \subset \ldots \subset \mathcal{F}_\alpha(s_k) = \mathcal{H}$$

and

$$0 \subset E_\alpha(t_1) \subset E_\alpha(t_2) \subset \ldots \subset E_\alpha(t_k) = \mathcal{H}$$

When w is considered as a permutation matrix, it is the unique permutation matrix such that the rank of its principal ij^{th} submatrix equals $\dim(\mathcal{F}_\alpha(s_i) \cap E_\alpha(t_j))$.
Flags in a position w with respect to a reference flag form a Schubert cell s_w. In the geometric setting (3.5)), $E_x = \cap_\alpha s_{w_\alpha} \neq \emptyset$.

When the filtrations $\mathcal{F}_\alpha$ are in generic position, only the topologically inevitable intersection survives. This leads to the constraint on the corresponding Schubert cocycles $\sigma_w = [\overline{s_w}]$

$$\cap_\alpha \phi_x^* s_{w_\alpha} \neq \emptyset$$

where $\phi_x : \mathcal{F}_x(\mathcal{H}) \to \mathcal{F}_x(\mathfrak{g})$ is the natural inclusion of the flag varieties and $\phi_x^* : H^*(\mathcal{F}_x(\mathfrak{g})) \to H^*(\mathcal{F}_x(\mathcal{H}))$ is the respective map between the cohomologies.

Now, let us consider a representation $\mathcal{H}^\nu$ of $SU(N)$ and an operator X^ν on $\mathcal{H}^\nu$ with its projection X into $su(\mathcal{H}) \subset su(\mathcal{H}^\nu)$. Then, (3.4) enhanced by (3.5) gives all constraints on the spectra of X and X^ν.

Let $x \in su(\mathcal{H})$ and let x^ν denote the operator on $\mathcal{H}^\nu$ induced by x. Denote by a and a^ν the respective spectra. Then, we may consider the flag varieties $\mathcal{F}_a$ and $\mathcal{F}_{a_\nu}$ consisting of Hermitian operators of spectra a and a^ν, respectively. We have maps $\phi_a : \mathcal{F}_a \to \mathcal{F}_{a^\nu}$, sending x to x^ν, and $\phi_a^* : H^*(\mathcal{F}_{a^\nu}) \to H^*(\mathcal{F}_a)$ such that $\phi_a^* : \sigma(w) \mapsto \sum_v c_w^v(a)\sigma(v)$. The following theorem is from [2] which was put in another form in [1] and we state the modified version.

Theorem 3.3.8. *All constraints on the spectra* $\lambda = spec(X^\nu)$ *and* $\mu = spec(X)$ *are given by the inequalities*

$$\sum_i a_i \mu_{v(i)} \leq \sum_j a_j^\nu \lambda_{w(j)} \quad (awv)$$

where v *and* w *are permutations and* a *is a test spectrum such that* $c_w^v(a) \neq 0$.

3.4 Solution of the ν-Representability Problem

Given a diagonal operator $z = diag(z_1, z_2, \ldots, z_r) \in U(\mathcal{H}_r)$, its character on $\mathcal{H}_r^\nu$ is the *Schur's function* $S_\nu(z_1, z_2, \ldots, z_n)$ which has a nice combinatorial description: By a *semistandard tableau of shape* ν, we mean a Young diagram of shape ν filled with the numbers $1, 2, \ldots, r$ such that the numbers weakly increase in rows and strictly increase in columns.

Example 3.4.1. $\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&3&3\\ \cline{1-3} 3\\ \cline{1-1}\end{array}$ is a semistandard tableau of shape $(4, 3, 1)$ whereas $\begin{array}{|c|c|c|c|}\hline 1&2&3&3\\ \hline 2&3&3\\ \cline{1-3} 3\\ \cline{1-1}\end{array}$ is not since the numbers in the third column do not strictly increase.

Then, Schur's function is the sum of monomials

$$S_\nu(z) = \sum_T z_T \tag{3.6}$$

where the sum is over all the semistandard tableaux of shape ν and z_T stands for $\prod_{i \in T} z_i$.

Example 3.4.2. Let $\nu = (2, 1)$ and $r = 3$. Then, the semistandard tableaux of shape ν are

$$\begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array}\,,\begin{array}{|c|c|}\hline 1&1\\ \hline 3\\ \cline{1-1}\end{array}\,,\begin{array}{|c|c|}\hline 1&2\\ \hline 2\\ \cline{1-1}\end{array}\,,\begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\,,\begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\,,\begin{array}{|c|c|}\hline 1&3\\ \hline 3\\ \cline{1-1}\end{array}\,.$$

Hence, in this case $S_\nu(z) = z_1^2 z_2 + z_1^2 z_3 + z_1 z_2^2 + 2 z_1 z_2 z_3 + z_1 z_3^2$.

Actually, the monomials appearing in (3.6) are weights of the representation $\mathcal{H}_r^\nu$, which means

$$z.e_T = z^T e_T \tag{3.7}$$

for a basis e_T of $\mathcal{H}_r^\nu$ indexed by the semistandard tableaux of shaped ν. Let $\mathfrak{t}$ and $\mathfrak{t}^\nu$ denote the Cartan subalgebras of $u(\mathcal{H})$ and $u(\mathcal{H}_r^\nu)$, which consist of real diagonal operators. Then the differential of the group action (3.7) gives the morphism

$$f_* : \mathfrak{t} \to \mathfrak{t}^\nu, \quad f_*(a) : e_T \mapsto a_T e_T \tag{3.8}$$

where by a_T we mean $\sum_{i \in T} a_i$. Given a spectrum $a = a_1 \geq a_2 \geq \ldots \geq a_r$ and a semistandard tableau T of shape ν, denote by a^ν the spectrum consisting of the quantites a_T arranged in non-increasing order:

$$a^\nu = \{a_T : \text{T is a semistandard tableau of shape } \nu\}^\downarrow \tag{3.9}$$

We have morphisms

$$\begin{aligned} &\phi_a : \mathcal{F}(\mathcal{H}_r) \to \mathcal{F}(\mathcal{H}_r^\nu) \\ &\phi_a^* : H^*(\mathcal{F}(\mathcal{H}_r^\nu)) \to H^*(\mathcal{F}(\mathcal{H}_r)) \\ &\phi_a^* : \sigma(w) \mapsto \sum_v c_w^v(a) \sigma(v) \end{aligned}$$

so we have the following:

Theorem 3.4.3. *In the above notations, all constraints on the spectra λ and ν are given by the inequalities*

$$\sum_i a_i \lambda_{v_i} \leq \sum_k a_k^\nu \mu_{w(k)}$$

for all spectra a and permutations v and w with the topological coefficient $c_w^v(a) \neq 0$.

3.4.1 Calculation of the topological coefficients $c_w^v(a)$.

The coefficients $c_w^v(a)$ also has a combinatorial description which uses the *Schubert polynomials* which were first introduced in [12].

Fix some positive integer n and consider the so-called *divided difference operators* ∂_i on the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$, which are defined by

$$\partial_i : f \mapsto \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}$$

Note that ∂_i satisfies
(i) $\partial_i(mf+g) = m\partial_i(f) + \partial_i(g)$ for $m \in \mathbb{Z}$ and polynomials f and g,
(ii) $\partial_i(fg) = \partial_i(f)g + f^{\sim i}\partial_i g$
where $f^{\sim i}$ denotes the polynomial $f(\ldots, x_{i+1}, x_i, \ldots)$.
Let $w \in S_n$ be a permutation and let s_i denote the transposition $(i, i+1)$. Write w as a product of minimal number of transpositions

$$w = s_{i_1} s_{i_2} \ldots s_{i_{l(w)}} \tag{3.10}$$

$l(w)$ is called the *length* of the permutation w. It turns out that the operator $\partial_w := \partial_{i_1}\partial_{i_2}\ldots\partial_{i_{l(w)}}$ is well-defined, i.e., it remains the same if one chooses a different minimal decomposition instead of (3.10)).
Let $w_0 = (1,n)(2,n-1)(3,n-2)\ldots$ be the permutation of maximal length. Then, the Schubert polynomial $S_w(x_1, x_2, \ldots, x_n)$ is defined to be

$$S_w(x_1, x_2, \ldots, x_n) = \partial_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2}\ldots x_{n-1}) \tag{3.11}$$

Then S_w has positive coefficients and has degree $l(w)$. See [11] for more details. The following is a list of Schubert polynomials for $n = 4$. The permutation $1 \mapsto i, 2 \mapsto j, 3 \mapsto k, 4 \mapsto l$ is denoted by $ijkl$ for the sake of simplicity of the table. x, y and z stands for x_1, x_2 and x_3. The list is borrowed from [11], where it appears as a lattice of polynomials. Two polynomials are connected in the lattice if one is obtained from the other by an operator ∂_i. In the list below, the permutations $ijkl$ are ordered lexicographically.

1234	0	2134	x	3124	x^2	4123	x^3
1243	$x+y+z$	2143	$x^2+xy+yz$	3142	x^2y+x^2z	4132	x^3y+x^3z
1324	$x+y$	2314	xy	3214	x^2y	4213	x^3y
1342	$xy+xz+yz$	2341	xyz	3241	x^2yz	4231	x^3yz
1423	x^2+xy+y^2	2413	x^2y+xy^2	3412	x^2y^2	4312	x^3y^2
1432	$x^2y+x^2z+xy^2$ $+xyz+y^2z$	2431	x^2yz+xy^2z	3421	x^2y^2z	4321	x^3y^2z

Example 3.4.4. Let $n = 4$ and $w = 1432 = (24)$. Then $w^{-1}w_0 = (1234) = s_1s_2s_3$ and $\partial_{w^{-1}w_0}(x^3y^2z) = \partial_1\partial_2\partial_3(x^3y^2z) = \partial_1\partial_2(x^3y^2) = \partial_1(x^3(y+z)) = \partial_1(x^3y+x^3z) = xy(x+y) + z(x^2+xy+y^2) = x^2y+x^2z+xy^2+xyz+y^2z$.

Theorem 3.4.5 ([1]). *The topological coefficients* $c^v_w(a)$ *in the* ν*-representability problem is given by the formula*

$$c^w_v(a) = \partial_v(S_w(x^\nu)|_{x^\nu_k \mapsto x_T}) \tag{3.12}$$

where T *is a tableau such that* $a^\nu_k = a_T$.

Remark. The right hand side of (3.12) is independent of the tableaux which satisfy $a^\nu_k = a_T$.

Chapter 4

Kostant's Problem

In this chapter, $\wedge^* \mathfrak{g}$ and $S^* \mathfrak{g}$ will denote the exterior and symmetric algebras of the adjoint representation of the group $SU(n)$, respectively.
Note that the center Z of the group $\mathrm{SU}(n)$ consists of the multiples of the identity ξI such that $\xi^n = 1$. Also note that Z acts on $\mathfrak{g} = \mathfrak{su}(n)$ trivially. Then, its action on a symmetric power $\mathrm{S}^N \mathfrak{g}$ must be also trivial. This implies n must divide $|\nu|$ since the action of ξI on $\mathcal{H}^\nu$ is multiplication by $\xi^{|\nu|}$ and ξ can be taken to have order n. In fact, the converse is also true, hence this characterizes the irreducible representations of $\mathrm{SU}(n)$ which can appear in $S^* \mathfrak{g}$.

Problem 4.0.6. *Kostant's problem asks what is the smallest integer N such that a given irreducible representation of $\mathcal{H}^\nu$ appears in $\mathrm{S}^N \mathfrak{g}$.*

Given a representation $\mathcal{H}^\nu$ where $|\nu|$ is a multiple of n, adding ν columns of length n does not change the representation ν since those columns corresponds to the determinant of elements in $\mathrm{SU}(n)$, which is just 1. Then ν can be taken to have at most $n-1$ rows and one defines the *level* of $\mathcal{H}^\nu$ as $k = 1 + \frac{Nn - |\nu|}{n}$.

4.1 Method

In the introduction, we discussed the difficulties while trying to compute the moment polytopes. Now, we restate our current method to obtain the moment polytope.

The idea is to produce many points which are known to be in the polytope and obtain a polytope $Q_n \subset P_n$ such that it has at least one correct facet, by which we mean a facet whose corresponding linear inequality has a nonzero topological coefficient c_w^v in Theorem (3.4.3). Then we find all vertices of P_n lying on this facet (see section (4.1.6)) and add them to R_n, then repeat this procedure until we obtain P_n. Depending on the initial inner approximation R_n, this method may take some time. We have obtained P_n for $n \leq 9$.

4.1.1 Points from smaller polytopes

Notice that if a point $p_{n-1} = (p_1 \geq p_2 \geq \ldots \geq p_{n-1})$ is in P_{n-1}, then adding a component with value 0 to p_{n-1} without disturbing the order of the components we get a point in P_n. The same is true for P_{n-i}, where one adds i zeros.

The reason is that if p_{n-i} is the spectrum of $[A, A^*]$ then p_n is the spectrum of the commutator of $[B, B^*]$ where B is the matrix with i additional rows and columns.

$$\begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

.

4.1.2 Points of the form $d - \sigma d$

Since AA^* is Hermitian, it is unitarily similar to a diagonal matrix D. So, let us write $AA^* = UDU^*$ for some unitary matrix U. Since AA^* must have nonnegative eigenvalues, $D = diag(d_1, d_2, \ldots, d_n)$ for some nonnegative $(d_i)_{1\leq i\leq n}$ with $\sum_{i=1}^{n} d_i = 1$. Then letting $\sqrt{D} = diag(\sqrt{d_1}, \ldots, \sqrt{d_n})$ and $B = \sqrt{D}U^*$ for an arbitrary unitary matrix U, we have $[B, B^*] = D - UDU^*$. So we have a subset of matrices of type $[A, A^*]$.

Generating all matrices of the form $D - UDU^*$ by a computer is obviously impossible since there are infinitely many unitary matrices. However, we can consider the finite subset of $U(n)$ consisting of permutation matrices, i.e., the matrices which have exactly one 1 in each row and column and 0 elsewhere. So we deal with the diagonal matrices $D - PDP^{-1}$ whose spectra are of the form $d - \sigma d$ where $d - \sigma d$ is defined by $(d - \sigma d)_i = d_i - d_{\sigma(i)}$.

But still there are infinitely many spectra d and we want $d - \sigma d$ to be in nondecreasing order. To solve this problem, firstly note that the image of the convex hull of points $p_1, \ldots, p_k$ in the space under a linear map L is nothing but the convex hull of the points $L(p_1), \ldots, L(p_k)$. In other words, the operations of taking convex hull of points and applying a linear operator commute with each other. We fix a permution σ and the corresponding permutation matrix P. We set $L = I - P$, $R_P = \{(d_1, \ldots, d_n) : d_i \leq 0, \sum_{i=1}^{n} d_i = 1\}$ and $R'_P = L(R)$. Taking the convex hull of the intersections of the R'_P with the positive Weyl chamber $\{(x_1, \ldots, x_n) : x_1 \geq x_2 \geq \ldots \geq x_n\}$ as P ranges over all permutation matrices, we obtain an inner approximation to the polytope P_n.

4.1.3 Decomposition of symmetric powers of $\mathfrak{g}$

By the very definition of the problem, we can decompose the symmetric powers of $\mathfrak{g}$ to get points in the moment polytope P_n until it stabilizes. This can be done thanks to LIE, a software package for Lie group theoretical computations. However, this has fairly high computational complexity and even worse, one does

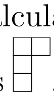
not know when to stop without the aid of ν-representability problem.
Nevertheless, this helps to approximate the moment polytope from inside.

4.1.4 Computation of moment polytopes as an instance of additive spectral problem

For small values of n we can compute P_n by treating the problem as an additive spectral problem since the matrices AA^* and $-A^*A$ are Hermitian and we are interested in the constraints on the spectra of their sum $[A, A^*]$. One can either use the inequalities in the sets $T_{r,n}$ for $1 \leq r < n$ or the inequalities supplied by the Hive model to produce a polytope Q_n which keeps the information of constraints on all of the variables appear in the inequalities, and obtain P_n by projecting Q_n to the subspace spanned by the variables corresponding to the spectra of $[A, A^*]$. On a typical PC, this allows one to compute P_n up to $n = 6$ via the former inequalities and up to $n = 5$ via the latter. For $n \geq 7$, we estimate that it would take at least weeks to compute the polytope. This is possibly due to exponential growth of the cardinality of $\cup_r T_{r,n}$ and quadratic increase of the number of variables in the inequalities given by Hive model.
Therefore, we need another way to proceed.

4.1.5 Computation of moment polytopes as an instance of ν-representability problem

Note that the problem described above becomes a ν-representability problem. Let $A \in sl(\mathcal{H})$ and consider the projection operator $\rho^\nu = |A\rangle\langle A|$. Then the contraction of ρ^ν turns out to be the commutator $[A, A^*]$.

Now we illustrate how the topological coefficients $c_w^v(a)$ related to a facet (genuine or not) can be calculated. Note that the Young diagram corresponding to adjoint representation is , where the first column has length $n-1$. Also note that the spectrum μ in Theorem (3.4.3) is $(1, 0, 0, \ldots)$ since we have a projection operator $|A\rangle\langle A|$.

Example 4.1.1. Consider the facet of P_3 given by the inequality $\lambda_1 - \lambda_2 \leq 1$. We want to put it in the form

$$\sum_i a_i \lambda_{v(i)} \leq \sum_k a_k^\nu \mu_{w(k)}$$

So, the test spectrum a in the Theorem (3.4.3) is $(1, 0, -1)$ and v must be (23). All semistandard tableaux of shape ν are

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

Hence $a^\nu = (2, 1, 1, 0, 0, -1, -1, -2)$. We want v and w have the same length, so we must have $w = (12)$. We can read $S_w(x^\nu)$ from the table in the previous chapter as x_1^ν. Now we replace x_1^ν with $2x_1 + x_2$ as dictated by Theorem (3.4.5). Upon applying ∂_v we get $c_w^v(a) = 1$. Hence, $\lambda_1 - \lambda_2 \leq 1$ is a true inequality.

4.1.6 Finding the vertices on a genuine facet

Here is a precise description of a facet of the moment polytope for representation $\mathcal{H}^\nu$ via a moment polytope of a representation of a subgroup $G \subset \mathrm{SU}(\mathcal{H})$. Let a genuine facet be given by the equation

$$c_1\lambda_1 + c_2\lambda_2 + \cdots + c_n\lambda_n = b. \tag{4.1}$$

Consider the tensor algebra $T\mathcal{H} = \sum_{n\geq 0} \mathcal{H}^{\otimes n}$ and for every $\alpha \in \mathcal{H}$ define the *annihilation operator*

$$\alpha : x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto \sum_{1\leq i\leq n} (\alpha, x_i) \cdot x_1 \otimes x_2 \otimes \cdots \otimes \widehat{x_i} \otimes \cdots \otimes x_n$$

and its conjugate *creation operator*

$$\alpha^\dagger : x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto \sum_{0\leq i\leq n} x_1 \otimes x_2 \otimes \cdots \otimes x_{i-1} \otimes \alpha \otimes x_i \otimes \cdots \otimes x_n.$$

The composition

$$\alpha^\dagger\alpha : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$$

commutes with the action of the symmetric group S_n and via the Schur's decomposition $\mathcal{H}^{\otimes n} = \sum_{\nu \vdash n} \mathcal{H}^\nu \otimes \mathcal{S}^\nu$ induces an action in $\mathcal{H}^\nu$.
Now, let (α_i) be an orthonormal basis of $\mathcal{H}$ and consider the operator $A = c_1\alpha_1^\dagger\alpha_1 + c_2\alpha_2^\dagger\alpha_2 + \cdots + c_n\alpha_n^\dagger\alpha_n$ where c_i are coefficients of the equation (4.1). Then the centralizer $\mathfrak{g}_A = \{x \in \mathfrak{su}(\mathcal{H}) \mid Ax = xA\}$ acts in the space $\mathcal{H}_A^\nu = \{\psi \in \mathcal{H}^\nu \mid A\psi = b\psi\}$, where b is the right hand side of the equation (4.1). In this setting, we have the following theorem, due to Klyachko (private communication).

Theorem 4.1.2. *The facet (4.1) is the part of the moment polytope of the representation $\mathfrak{g}_A : \mathcal{H}_A^\nu$ cut out by the ordering condition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (the ordering conditions in $\mathfrak{g}_A$ are weaker than that of $\mathfrak{su}(\mathcal{H})$).*

Let us see how this works for the adjoint representation. To recover vertices of a facet from the coefficients of its equation

$$a_1\lambda_1 + a_2\lambda_2 + .. + a_n\lambda_n = 1 \tag{4.2}$$

we have to consider the diagonal matrix $A = \mathrm{diag}(a_1, a_2, .., a_n)$ in the space $M = \mathfrak{g}_A$ of all $n \times n$ matrices X s.t. $AX - XA = X$. They form a two-sided module M over the centralizer Z_A of A. Clearly, Z_A is a direct sum of matrix algebras whose ranks are equal to multiplicities of the coefficients in the sequence $(a_1, a_2, \ldots, a_n)$ and M is spanned by the matrix units E_{ij} s.t. $a_i - a_j = 1$. We observe the experimental fact that for a genuine constraint (4.2) the successive terms of the sequence a_i arranged in non-increasing order either coincide or differ by 1. According to the above theorem, the facet is a part of the moment polytope of the adjoint action $X \mapsto ZX - XZ$, of Z_A in the two-sided module M. The respective moment polytope is formed by spectra of commutators $[X, X^*]$, $X \in M$, $\operatorname{Tr} XX^* = 1$ arranged in decreasing order within every block. This imposes the ordering conditions only on the eigenvalues λ_i, λ_j in (4.2) with equal coefficient $a_i = a_j$. The facet itself is obtained from the moment polytope by imposing the global ordering condition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Example 4.1.3. Let us consider the inequality $\lambda_2 + \lambda_3 - \lambda_6 \leq 1$ of a facet F of P_6. The multiplicity of the coefficients 1, 0 and -1 are 2, 3 and 1, respectively.

Then M consists of block matrices X of the form

$$\begin{pmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix}$$

where A is a 2×3 matrix and B is a 3×1 matrix.

Then, $XX^* - X^*X = \begin{pmatrix} AA^* & 0 & 0 \\ 0 & BB^* - A^*A & 0 \\ 0 & 0 & -B^*B \end{pmatrix}$ So we need to deal with an additive spectral problem with the matrices having dimension 3 to find the vertices of P_6 in F.

4.2 The absolute values of the coordinates of the points in moment polytopes

Note that we use the normalization $\mathrm{tr}(AA^*) = 1$ and the eigenvalues of a matrix of the form AA^* are nonnegative. Thus, AA^* has spectrum

$$1 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$$

and $-A^*A$ has spectrum

$$0 \geq -\lambda_n \geq -\lambda_{n-1} \geq \ldots \geq -\lambda_1 \geq -1$$

since AA^* and A^*A are isospectral matrices.
Then, the inequality (5.2) in the next chapter says that the largest eigenvalue of $[A, A^*]$ is not larger than 1. By symmetry, the smallest eigenvalue of $[A, A^*]$ is not less than -1. Thus we have

Theorem 4.2.1. *The absolute values of the coordinates of the points in the moment polytopes are not larger than* 1.

4.3 A related conjecture by Kostant

We finish this section by the following remark. One may also ask which irreducible representations of $SU(n)$ appear in $\wedge^*\mathfrak{g}$ and $S^*\mathfrak{g}$. B. Kostant conjectured that an irreducible representation of weight μ appear in $\wedge^*\mathfrak{g}$ if and only if $2\rho - \mu$ is a linear combination of simple roots, where ρ stands for half sum of all positive roots of $\mathfrak{g}$.
This condition can be restated as $\mu \preceq 2\rho$ where $\alpha \preceq \beta$ is a shorthand notation for the majorization inequalities

$$\begin{gathered}\alpha_1 \leq \beta_1 \\ \alpha_1 + \alpha_2 \leq \beta_1 + \beta_2 \\ \alpha_1 + \alpha_2 + \alpha_3 \leq \beta_1 + \beta_2 + \beta_3 \\ \vdots \\ \alpha_1 + \alpha_2 + \ldots + \alpha_n \leq \beta_1 + \beta_2 + \ldots + \beta_n\end{gathered}$$

This result was proven by Berenstein and Zelevinsy for the group $SL(n)$ in [3].

Chapter 5

Classical Spectral Problem

One may ask the following question: Given Hermitian square matrices A, B and C in $\mathbb{C}^{n\times n}$ with $C = A + B$, what are the constraints on the spectra of the matrices?
Firstly note that Hermitian matrices have real spectrum. Hence we can write the spectra of A, B and C as $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, $\mu = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, $\nu = \nu_1 \geq \nu_2 \geq \ldots \geq \nu_n$, respectively.
An immediate observation is regarding the traces of the matrices:

$$\sum_i \nu_i = \sum_i \lambda_i + \sum_i \mu_i \tag{5.1}$$

Another observation is the inequality (5.2) below. Note that if $A \in \mathbb{C}^{n\times n}$ is Hermitian and $v \in \mathbb{C}^n$ is a column vector then the entry of the 1×1 matrix v^*Av is real. Therefore, v^*Av can be regarded as a real number. Then, one may try to find the value $\sup\limits_{|v|=1} v^*Av$. Since A is Hermitian, A is unitarily diagonalizable, hence there exists a unitary matrix U such that $D := U^*AU = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. We may assume $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ by multiplying U by a permutation matrix, if necessary. Since U is unitary, it respects the inner product $(v|w) = v^*w$ so multiplication by U on the set of unit column vectors is a bijection. Hence,

$$\sup_{|v|=1} v^*Av = \sup_{|v|=1} v^*Dv = \sup_{\substack{c_i\geq 0 \\ c_1+c_2+\ldots+c_n=1}} c_1\lambda_1 + c_2\lambda_2 + \ldots + c_n\lambda_n = \lambda_1$$

Thus, the largest eigenvalue of an Hermitian square matrix A is $\sup_{|v|=1} v^* A v$. So, the inequality

$$\nu_1 \leq \lambda_1 + \mu_1 \tag{5.2}$$

follows. Note that the supremum $\sup_{|v|=1} v^* A v$ is attained exactly when v is a unit eigenvector of A associated to the eigenvalue λ_1. So, (5.2) becomes an equality if and only if the intersection of the eigenspaces of matrices A and B associated to the eigenvalues λ_1 and μ_1 is nonzero.

In 1912 H.Weyl proved that

$$\nu_{i+j-1} \leq \lambda_i + \mu_j \tag{5.3}$$

holds if $i + j - 1 \leq n$. Note that the inequality (5.2) is among this inequalities.

Later, other inequalities of the type

$$\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \qquad (*IJK)$$

were found, where $I,\ J,\ K \subset \{1, 2, \ldots, n\}$ have the same cardinality r for some $r < n$.

In 1949, K. Fan found inequalities of the form

$$\sum_{i=1}^{r} \nu_r \leq \sum_{i=1}^{r} \lambda_r + \sum_{i=1}^{r} \mu_r \tag{5.4}$$

where $1 \leq r < n$.

In 1950, V.B. Lindskii found a geometric condition on the spectra: When a spectrum $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ is considered as a point $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in the space $\mathbb{R}^n$, ν must be inside the convex hull of the points $\lambda + \sigma\nu$ where σ is a permutation of the n indices and $(\sigma\nu)_i = \nu_{\sigma(i)}$. Later H. Wielandt proved that this condition was similar to the ones above, since it could be rephrased as

$$\sum_{i \in I} \nu_r \leq \sum_{i \in I} \lambda_r + \sum_{i=1}^{r} \mu_r \tag{5.5}$$

where $1 \leq r < n$ and I is a subset of $\{1, 2, \ldots, n\}$ of cardinality r.

In 1962, A.Horn conjectured that all the necessary and sufficient constraints other than the trace inequality must be of this form and must satisfy a recurrence

relation as n grows. We will describe these recurrence relations below.
In 1998, A.Klyachko gave a complete solution of this problem in [8] using the machinary of Schubert calculus, vector bundles and representation theory.
In 1999, A.Knutson and T.Tao proved Horn's conjecture.
To draw attention to the importance of the problem, we quote them:

> "...Weyl's partial answers to this problem have since had many direct applications to perturbation theory, quantum measurement theory, and spectral theory of self-adjoint operators. The purpose of this article is to describe the complete resolution to this problem, based on recent breakthroughs [Kl], [HR], [KT], [KTW]."

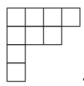
A.Knutson and T.Tao also provided two other methods to solve the problem. One is called the *hive model* and the other is called *honeycomb model*. We will discuss how one can use the hive model to get all the necessary constraints on the spectra of A, B and C.
Another matter of interest is the possible values of the k^{th} largest eigenvalue ν_k of C. The following inequality, which is proven in [7], fully decribes the possible values.

$$\max_{i+j=n+k} \lambda_i + \mu_j \leq \nu_k \leq \min_{i+j=k+1} \lambda_i + \mu_j$$

5.1 Representations of $\mathrm{GL}_n(\mathbb{C})$, Littlewood-Richardson Coefficients and Spectral Problem

The group $G := \mathrm{GL}_n(\mathbb{C})$ has the canonical representation $V := \mathbb{C}^n$, called the *standard representation* and this representation is obviously irreducible. One can tensor V by itself a number of times to get new representations $V^{\otimes n}$ of G but there is no reason for them to be irreducible. However, given a partition λ of n, on can apply the *Schur functor* to $V^{\otimes n}$ to get an irreducible representation of G.

A partition $\lambda = \lambda_1 \geq \ldots \geq \lambda_k$ of n can be interpreted as a Young diagram with rows of length $\lambda_1, \ldots, \lambda_k$.

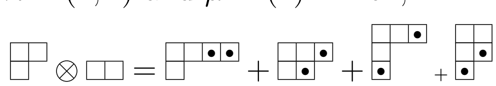
Example 5.1.1. The Young diagram corresponding to the partition $9 = 4+3+1+1$ is  .

Note that the symmetric group S_n naturally acts on a Young diagram D with n boxes. Let a_λ be the sum $\sum_{\sigma \in P} \sigma$ and b_λ be the sum $\sum_{\sigma \in Q} sgn(\sigma)\sigma$ in the group algebra $\mathbb{C}S_n$, where P consists of the elements of S_n fixing each row of D and Q consists of the elements of S_n fixing each column of D. Then, V_λ is given by $V_\lambda = a_\lambda b_\lambda V^{\otimes n}$. So the Schur functor symmetrizes $V^{\otimes n}$ with respect to each row and antisymmetrizes it with respect to each column. In particular, row diagrams correspond to symmetric powers and column diagrams correspond to alternating powers.

It turns out that the V_λ are all of the irreducible representations of G. In fact, λ is nothing but the highest weight of the representation V_λ.

Since $G = \mathrm{GL}_n(\mathbb{C})$ is reductive, the representations of G must be written as a direct sum of irreducible ones. In particular, $V_\lambda \otimes V_\mu = \sum_\nu c^\nu_{\lambda\mu} V_\nu$ must hold for some nonnegative integers $c^\nu_{\lambda\mu}$. The $c^\nu_{\lambda\mu}$ are called the *Littlewood-Richardson coefficients* and has a nice combinatorial description. So, let us write $\lambda \otimes \mu = \sum_\nu c^\nu_{\lambda\mu} \nu$ referring to this description which is explained below. Let us first consider the case when μ is a row diagram with m boxes. Then, the *Pieri rule* states that $c^\nu_{\lambda\mu}$ is the number of ways to add m boxes to the Young diagram of λ to get the Young diagram of ν such that none of the new boxes are in the same column. So this number is either 1 or 0.

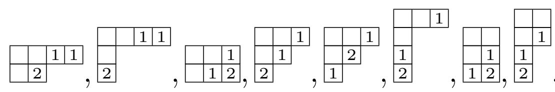
Example 5.1.2. Let $\lambda = (2, 1)$ and $\mu = (2)$. Then,

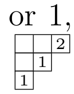


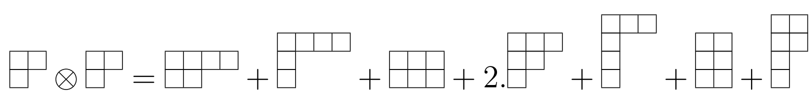
where the boxes labeled with a $\bullet$ are the newly added ones according to the Pieri rule.

For an arbitrary μ, the description of $c^\nu_{\lambda\mu}$ is a little bit more complicated. By a *μ-expansion* of λ, we mean the process of adding μ_1 boxes to the Young diagram of λ obeying the Pieri rule described above (let us call this the first step) and continuing this with the remaining rows of μ (let us call the i^{th} step if the i^{th} row of μ is the row that is processed). By a *strict μ-expansion* of λ, we mean a μ expansion of λ such that the boxes added in the i^{th} step is labelled with the integer i and when the labels of the newly added boxes are listed as a sequence read from the diagram row-by-row from the top to the bottom, at no point of the sequence the number of occurences of a number p so far is larger than that of a smaller number $q < p$ so far.

Example 5.1.3. Let $\lambda = \mu = (2, 1)$. The strict μ-expansions of λ are



Note that there are no boxes labelled with 1 (resp. 2) in the same column and the sequences read from the diagrams are either $1, 1, 2$ or $1, 2, 1$. In both cases, the number of occurences of 2 never exceeds that of 1.  is a μ-expansion of λ but it is not a strict one since the first occurence of 2 is before that of 1. We may write



The *Littlewood-Richardson rule* says that $c^{\nu}_{\lambda\mu}$ is the number of ways the Young diagram of ν can be obtained from the Young diagram of λ by a strict μ-expansion.

5.2 Schubert Calculus

In the progress of the solution of the classical spectral problem, a subject called *Schubert calculus* played an extremely important role. Let us give a brief exposition of Schubert calculus in this section.

Let $G(n, V)$ denote the set of n-dimensional subspaces of an m-dimensional vector space V over K. $G(n, V)$ is called a *Grassmannian*. We want to see $G(n, V)$ as a subvariety of $\mathbb{P}(\wedge^n V)$. Let W be an n-dimensional subspace of V and fix a basis $(w_i)_{1\leq i\leq n}$ of W. Then $w_1 \wedge w_2 \wedge \ldots \wedge w_n$ is a point in $\wedge^n V$ and it is determined up to a nonzero scalar in K since it is multiplied by the determinant of the matrix of the change of basis when another basis of W is chosen. Conversely, when $w = w_1 \wedge w_2 \wedge \ldots \wedge w_n$ corresponding to a W is given, W can be recovered by $W = \{v \in V : v \wedge w = 0 \in \wedge^* V\}$. Hence, $G(n, V)$ can be embedded into $\mathbb{P}(\wedge^n V)$. This is called the *Plücker embedding* and the homogeneous coordinates on $\mathbb{P}(\wedge^n V)$ are called the *Plücker coordinates*. If M_W is the $n \times m$ matrix whose rows are the w_i considered as row vectors in K^m, then the Plücker coordinates of W corresponds to the *maximal minors* of the matrix M_W. In this case, the maximal minors are nothing but the determinants of $n \times n$ submatrices of M_W.

Now fix a basis $\mathcal{B} = \{e_i\}_{1\leq i\leq m}$ for V. Write each e_i is written as a row vector $(0, \ldots, 0, 1, 0 \ldots, 0)$ where 1 appears in the i^{th} position. Then, we have a complete flag

$$0 = F_0 \subset F_1 \subset \ldots \subset F_m = V$$

where F_i denotes the span of the first i basis vectors.
For any subset $P = \{p_1 \leq p_2 \leq \ldots \leq p_n\}$ of $\{1, 2, \ldots, m\}$ having cardinality n,

$$\Omega_P^0(F) = \{W \in X : \dim(F_{p_i} \cap W) = i,\ i = 1, 2, \ldots, n\}$$

is called a *Schubert cell*.
Its closure is

$$\Omega_P(F) = \{W \in X : \dim(F_{p_i} \cap W) \geq i,\ i = 1, 2, \ldots, n\}$$

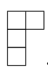
and is called a *Schubert variety*.
Let W be an n-dimensional subspace of V and choose a basis $\mathcal{B}'$ of W. Write the elements of $\mathcal{B}'$ as row vectors R_i with respect to the basis $\mathcal{B}$ and consider the $n \times m$ matrix M having the R_i as its rows. It is a fundamental fact in linear algebra that M is row equivalent to a matrix of the form

$$\begin{pmatrix} * & \ldots & * & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ * & \ldots & * & 0 & * & \ldots & * & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \ldots & * & 0 & * & \ldots & * & 0 & * & \ldots & * & 1 & 0 & \ldots & 0 \end{pmatrix}$$

Therefore, n-dimensional subspaces of V are in one-to-one correspondence with the matrices in the form above. Letting p_i be the position of the last nonzero entry in i^{th} row, we see that the subspaces of V in the same Schubert cell are those whose corresponding matrices have the same position indices p_i.
Corresponds to the Schubert variety $\Omega_P(F)$, a Young diagram $\alpha = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ where $\alpha_i = m - n + i - p_i$. The size of the Young diagram is the codimension of the Schubert cell in the Grassmanian.

Example 5.2.1. The Young diagram corresponding to the Schubert variety determined by the row reduced matrix

$$\begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 \\ * & 0 & * & 0 & 1 & 0 \end{pmatrix}$$

is  .

Let σ_α denote the cocyle corresponding to $\Omega_P(F)$ in the cohomology ring. Then the σ_α form a $\mathbb{Z}$-basis for the cohomology ring. Hence, a product $\sigma_\alpha.\sigma_\beta$ has a decomposition

$$\sigma_\alpha.\sigma_\beta = \sum d^\gamma_{\alpha\beta}\sigma_\gamma$$

where the sum is taken over all γ such that $\sum \alpha_i + \sum \beta_i = \sum \gamma_i$. The coefficients $d^\gamma_{\alpha\beta}$ are nonnegative by the virtue of the transitive action of $\mathrm{GL}_m(\mathbb{C})$ on X.
In 1947, L. Lesieur proved in [13] that the coefficients $d^\gamma_{\alpha\beta}$ are nothing but the Littlewood-Richardson coefficients $c^\gamma_{\alpha\beta}$.
Now, we state Klyachko's theorem, which gives a complete solution to the classical spectral problem.

Theorem 5.2.2. *Consider a triple of subsets $I, J, K \subset \{1, 2, \ldots, n\}$ such that the Schubert cycle σ_K is a component of $\sigma_I.\sigma_K$. Then,*
*(1) The inequality (*IJK) holds.*
(2) Together with the trace identity, this inequalities form a complete set of restrictions on spectra of A, B and $A + B$.

An application of Schubert Calculus.

Consider 4 lines l_1, l_2, l_3 and l_4 in $\mathbb{R}^3$ in general position. What is the number of

the lines in $\mathbb{R}^3$ intersecting all l_i?
Firstly, considering the embedding of $\mathbb{R}^3$ in $\mathbb{P}^3$, we can interpret a line in $\mathbb{R}^3$ as a plane through origin in $\mathbb{R}^4$. Then, if two lines in $\mathbb{R}^3$ intersect at one point, the codimension of their span is one. Therefore the set of lines intersecting a given line at one point form a Schubert cell with Young diagram $\square$.
To find the intersection of Schubert cells corresponding to the lines l_i, we can look at the product of the corresponding Scubert cocycles in the cohomology ring: Noting that a Young diagram corresponding to a Schubert cell must be contained in $2.\boxplus$ we get

$$\square \otimes \square \otimes \square \otimes \square = 2.\boxplus$$

using Pieri's rule. But since the size of $\boxplus$ is 4, the corresponding Schubert cell has codimension 4, which means it describes a point in $\mathbb{R}^4$. Therefore, we conclude that there are two lines in $\mathbb{R}^3$ which intersect given 4 lines in $\mathbb{R}^3$ in general position.

5.3 Horn's Conjecture

The linear inequalities mentioned above has the form

$$\sum_{l=1}^{r} \nu_{k_l} \leq \sum_{l=1}^{r} \lambda_{i_l} + \sum_{l=1}^{r} \mu_{j_l} \tag{5.6}$$

where $1 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_r \leq n$, $1 \leq \mu_1 < \mu_2 < \ldots < \mu_r \leq n$, $1 \leq \nu_1 < \nu_2 < \ldots < \nu_r \leq n$ and $1 \leq r < n$. For a fixed $1 \leq r < n$, let $T_{r,n}$ denote the set of such indices such that the inequality (5.6) is a necessary condition.
Horn showed that an element in $T_{r,n}$ satisfies the following trace condition:

$$\sum_{l=1}^{r} i_l + \sum_{l=1}^{r} j_l = \sum_{l=1}^{r} k_l + \frac{r(r+1)}{2} \tag{5.7}$$

Horn's conjecture was settled by A. Klyachko ([8]), A. Knutson and T. Tao ([9]) and provides a recurrence relation between the sets $T_{r,n}$:

Theorem 5.3.1 (Horn's conjecture). *$T_{r,n}$ is the set of all indices $1 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_r \leq n$, $1 \leq \mu_1 < \mu_2 < \ldots < \mu_r \leq n$ and $1 \leq \nu_1 < \nu_2 < \ldots < \nu_r \leq n$ which satisfy* (5.7) *and the condition*

$$\sum_{l=1}^{s} i_{a_l} + \sum_{l=1}^{s} j_{b_l} = \sum_{l=1}^{s} k_{c_l} + \frac{s(s+1)}{2} \tag{5.8}$$

for any $1 \leq s < r$ and for any element $1 \leq a_1 < a_2 < \ldots < a_s \leq r$, $1 \leq b_1 < b_2 < \ldots < b_s \leq r$ and $1 \leq c_1 < c_2 < \ldots < c_s \leq r$ in $T_{s,r}$.

The inequalities in the set $T_{1,2}$ are

$$\lambda_1 + \mu_1 \geq \nu_1$$
$$\lambda_1 + \mu_2 \geq \nu_2$$
$$\lambda_2 + \mu_1 \geq \nu_1$$

The inequalities in the sets $T_{1,3}$ and $T_{2,3}$ are

$$\lambda_1 + \mu_1 \geq \nu_1$$
$$\lambda_1 + \mu_2 \geq \nu_2$$
$$\lambda_1 + \mu_3 \geq \nu_3$$
$$\lambda_2 + \mu_1 \geq \nu_2$$
$$\lambda_2 + \mu_2 \geq \nu_3$$
$$\lambda_3 + \mu_1 \geq \nu_3$$

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 \geq \nu_1 + \nu_2$$
$$\lambda_1 + \lambda_2 + \mu_1 + \mu_3 \geq \nu_1 + \nu_3$$
$$\lambda_1 + \lambda_2 + \mu_2 + \mu_3 \geq \nu_2 + \nu_3$$
$$\lambda_1 + \lambda_3 + \mu_1 + \mu_2 \geq \nu_1 + \nu_3$$
$$\lambda_1 + \lambda_3 + \mu_1 + \mu_3 \geq \nu_2 + \nu_3$$
$$\lambda_2 + \lambda_3 + \mu_1 + \mu_2 \geq \nu_2 + \nu_3$$

5.4 Hive Model

Another answer to the problem can be given by using the Hive Model. This answer consists of $3\binom{n}{2}$ linear inequalities in terms of $\binom{n+2}{2}$ variables as well as the trace condition (5.1) stated in an equivalent form.

To state the theorem, let us first change the setting a little bit. Let us consider Hermitian matrices A, B and C with $A + B + C = 0$ so that the situation is symmetric with respect to the three matrices. Again, let $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, $\mu = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, $\nu = \nu_1 \geq \nu_2 \geq \ldots \geq \nu_n$ be the spectra of A, B and C respectively.

Roughly speaking, there exists Hermitian matrices A, B, C satisfying $A+B+C = 0$ with spectra λ, μ and ν if and only if there is a "convex dome standing on the arc formed by the λ, μ and ν". Let us make this statement more rigorous.

Consider the set $H_n = \{(i, j, k) : 0 \leq i, j, k \leq n, i + j + k = n\}$ consisting of $\binom{n+2}{2}$ points. Let T be an equilateral triangle in the plane with height n. Then, an element $P = (i, j, k)$ of H_n can be seen as a point in the plane considering i, j and k as the distances of P to the three sides of T. Let $H'_n \subset H_n$ be the set of points whose coordinates are not all nonzero. So H'_n consists of points in H_n which are on one side of the triangle. Define a function $f'_{\lambda,\mu,\nu} : H'_n \to \mathbb{R}$ by

$$(0, n-i, i) \mapsto \sum_{l=0}^{i} \lambda_l$$

$$(i, 0, n-i) \mapsto \mathrm{tr}(A) + \sum_{l=0}^{i} \mu_l$$

$$(n-i, i, 0) \mapsto \mathrm{tr}(A) + \mathrm{tr}(B) + \sum_{l=0}^{i} \nu_l$$

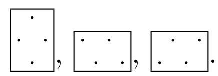
i.e., the partial sums of the sequence $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n$ are the images of the elements of H'_n when they are traversed clockwise starting from the point $(0, n, 0)$.

Definition 5.4.1. *Call a function $f : H_n \to \mathbb{R}$ a convex n-dome if it satisfies the following condition which has cyclic symmetry in terms of the coordinates of elements of H_n:*

Whenever $2 \leq k \leq n$ and $0 \leq i \leq n - k$,

(i) $f(i, j+1, k-1) + f(i+1, j, k-1) \geq f(i, j, k) + f(i+1, j+1, k-2)$
(ii) $f(k-1, i, j+1) + f(k-1, i+1, j) \geq f(k, i, j) + f(k-2, i+1, j+1)$
(iii) $f(j+1, k-1, i) + f(j, k-1, i+1) \geq f(j, k, i) + f(j+1, k-2, i+1)$

This condition can be restated as follows: The sum of values of f at the two points with acute angles must not exceed the sum of the values of f at the two points with obtuse angles whenever we consider the subsets of H_n of shapes



Now we are ready to rigorously state the answer.

Theorem 5.4.2. *Given n-tuples of real numbers* $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, $\mu = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ *and* $\nu = \nu_1 \geq \nu_2 \geq \ldots \geq \nu_n$*; there exists Hermitian matrices A, B, C satisfying* $A + B + C = 0$ *with spectra* λ*,* μ *and* ν *if and only if* $f'_{\lambda,\mu,\nu}$ *is defined and can be extended to an n-dome* $f : H_n \to \mathbb{R}$*.*

5.5 A Digression of Invariant Factors

In this section, we describe a problem which has exactly the same solution as the additive spectral problem we described above, although it is still not clear why this is the case.
Firstly we discuss the elementary properties of discrete valuation rings to state the problem.

5.5.1 A Digression of Discrete Valuation Rings

Given a field K, a group epimorphism $\nu : K^* \to \mathbb{Z}$ is called a *discrete valuation* on K if it satisfies

$$\nu(a+b) \geq min\{\nu(a), \nu(b)\} \text{ for all } a, b \in K^* \text{ with } a+b \neq 0.$$

Then, $\{0\} \cup \nu^{-1}(\mathbb{Z}_{\geq 0})$ is a subring of K and is called the *valuation ring* of ν. A valuation ring is an integral domain being a subring of a field.

An integral domain R is called a *discrete valuation ring* if there is a discrete valuation ν on its field of fractions such that R is the valuation ring of ν. If this is the case, then

(1) R is a Euclidean Domain with the norm N which is given by $N(0) = 0$ and $N(x) = \nu(x)$ for $x \neq 0$.

(2) R is a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Furthermore, $\mathfrak{m}$ is principal, say $\mathfrak{m} = (\pi)$, and any nonzero ideal of R is some power of $\mathfrak{m}$. In fact, $\mathfrak{m}^k = \{x \in R : \nu(x) \geq k\}$. Any generator of $\mathfrak{m}$ is called a *uniformizing parameter.* The set of uniformizing parameters are exactly the set $\nu^{-1}(\{1\})$.

(3) Any nonzero element of R can be uniquely written as $c\pi^k$ where $c \in R$ is a unit and k is a nonnegative integer.

Without dealing with possible valuations on its field of fractions, it may be possible to decide if a commutative ring with identity is a discrete valuation ring by the virtue of the following theorem, which is borrowed from [4].

Theorem 5.5.1. *For a commutative ring R with identity the following are equivalent.*

(1) *R is a discrete valuation ring.*

(2) *R is a P.I.D. with a unique nonzero maximal ideal.*

(3) *R is a U.F.D. with a unique (up to associates) irreducible element.*

(4) *R is a Noetherian integral domain which is at the same time a local ring whose maximal ideal is nonzero and principal.*

(5) *R is an integrally closed Noetherian integral domain which is at the same time a local ring with Krull dimension 1.*

Example 5.5.2. **(1)** Given a prime p, the ring of p-adic integers is a discrete valuation ring which has p as a uniformizing parameter.

(2) The ring of convergent power series over the complex numbers is a discrete valuation ring, having the function z as a uniformizing parameter.

5.5.2 Invariant factors of a matrix over a discrete valuation ring

In this section we introduce a problem described in [5]. Now, if A is an invertible $n \times n$ matrix with entries in R, then A can be reduced to a diagonal matrix $diag(\pi^{\alpha_1}, \pi^{\alpha_2}, \ldots, \pi^{\alpha_n}) = PAQ$ by elementary row and column operations such that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ are nonnegative integers. Elementary row operations are interchanging two rows, multiplying a row by a unit in R or adding a multiple of a row to another one. Elementary column operations are described similarly. Hence, the determinants of the elementary matrices corresponding to the elementary row and column operations are units in R.
To see the existence of such $(\alpha_i)_{1\leq i\leq n}$, let $A_{ij} = c\pi^{\alpha}$ be a nonzero entry of A such that α is minimal. Then, all the entries of A which are in the same column or in the same row as A_{ij}, excluding A_{ij}, can be made 0 by elementary row and column operations of the third type explained above. Then, by interchanging the n^{th} row with the i^{th} one, interchanging the n^{th} column with the j^{th} one and finally multiplying the bottom row by c^{-1}, we get a block matrix $\left(\begin{smallmatrix} A' & 0 \\ 0 & \pi^{\alpha} \end{smallmatrix}\right)$. Now, one argues inductively to obtain the desired diagonal matrix.
To see the uniqueness of $(\alpha_i)_{1\leq i\leq n}$, consider P, A and Q as elements of the endomorphism ring $End_R(R^n)$ of a free R-module of rank n. We can apply "The Five Lemma" in [14] to the commutative diagram below with exact rows, where the π stand for the projections and ϕ is the map sending the coset $[x]$ to the coset $[Px]$.

$$\begin{array}{ccccccccc}
R^n & \xrightarrow{A} & R^n & \xrightarrow{\pi} & R^n/AR^n & \longrightarrow & 0 & \longrightarrow & 0 \\
\uparrow{\scriptstyle Q} & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow \\
R^n & \xrightarrow{PAQ} & R^n & \xrightarrow{\pi} & R^n/PAQR^n & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The Five Lemma says ϕ is an isomorphism. Then, the theory of modules over P.I.D.s gives the uniqueness of $(\alpha_i)_{1\leq i\leq n}$. $(\alpha_i)_{1\leq i\leq n}$ are called the *invariant factors* of A.
Note that α_1 is given by

$$\alpha_1 = \min\{\alpha \in \mathbb{Z}_{\geq 0} : p^{\alpha} coker A = 0\} \tag{5.9}$$

One can ask how the invariant factors $(\alpha_i)_{1\leq i\leq n}$, $(\beta_i)_{1\leq i\leq n}$ and $(\gamma_i)_{1\leq i\leq n}$ of $n\times n$ matrices A, B and C over R are related if $C = AB$. Note that in this case we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & R^n & \xrightarrow{B} & R^n & \longrightarrow & coker(B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle A} & & \downarrow & & \\
0 & \longrightarrow & R^n & \xrightarrow{id} & R^n & \longrightarrow & 0 & &
\end{array}$$

Applying the "Snake Lemma", Lemma II.5.2. in [14], we obtain an exact sequence

$$0 \longrightarrow coker(B) \xrightarrow{i} coker(C) \xrightarrow{\pi} coker(A) \longrightarrow 0$$

which gives an isomorphism $coker(C) \big/ i(coker(B)) \cong coker(A)$. Now, we employ (5.9). p^{α_1} maps $coker(C)$ into $i(coker(B))$. Since p^{β_1} annihilates $coker(B)$, $p^{\alpha_1+\beta_1}$ annihilates $coker(C)$. Therefore, we have

$$\gamma_1 \leq \alpha_1 + \beta_1 \tag{5.10}$$

analogous to (5.3). Note that this is one of the inequalities of the type (*IJK) in the additive spectral problem discussed above. It turns out that all the necessary and sufficient constraints on the triple $(\alpha_i)_{1\leq i\leq n}$, $(\beta_i)_{1\leq i\leq n}$ and $(\gamma_i)_{1\leq i\leq n}$ are Horn's inequalities and the same trace condition 5.11 as (5.3) proven below, so the two problems have exactly the same solutions.
From $det(C) = det(A)det(B)$, it follows that

$$c\pi^{\sum_i \alpha_i} = \pi^{\sum_i \beta_i + \sum_i \gamma_i}$$

for some unit $c \in R$. This is possible only if $c = 1$ and

$$\sum_i \alpha_i = \sum_i \beta_i + \sum_i \gamma_i \tag{5.11}$$

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Appendix A

Numbers of the vertices & the facets of the moment polytopes

In this appendix, we provide the data of the moment polytopes P_n of the adjoint representation of $\mathrm{SU}(n)$ for $n \leq 9$.
Here are the numbers of the vertices and facets of the polytopes which seem to grow exponentially in terms of n:

n	Number of vertices	Number of facets
3	4	4
4	7	7
5	15	16
6	20	27
7	48	62
8	69	105
9	139	220

Appendix B

Facets & vertices of the moment polytopes

An inequality $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b$ in terms of the coordinates $x_1, \ldots, x_n$ is denoted by $[a_1, a_2, \ldots, a_n] \leq b$.

Facets of P_3		
[-1,1,0] $\leq$ 0	[0,-1,1] $\leq$ 0	[0,1,-1] $\leq$ 1
[1,-1,0] $\leq$ 1]		

Vertices of P_3		
(0, 0, 0)	(1, 0, -1)	(1/3, 1/3, -2/3)
(2/3, -1/3, -1/3)		

Facets of P_4		
[-1, 1, 0, 0] $\leq$ 0	[0, -1, 1, 0] $\leq$ 0	[0, 0, -1, 1] $\leq$ 0
[0, 1, 0, -1] $\leq$ 1	[1, 0, -1, 0] $\leq$ 1	[0, 1, 2, -1] $\leq$ 1
[2, -1, 0, 1] $\leq$ 1		

Vertices of P_4		
(0, 0, 0, 0)	(1, 0, 0, -1)	(1/3, 1/3, 0, -2/3)
(2/3, 0, -1/3, -1/3)	(1/2, 1/2, -1/2, -1/2)	(1/6, 1/6, 1/6, -1/2)
(1/2, -1/6, -1/6, -1/6)		

Facets of P_5		
[-1, 1, 0, 0, 0] $\leq$ 0	[0, -1, 1, 0, 0] $\leq$ 0	[0, 0, -1, 1, 0] $\leq$ 0
[0, 0, 0, -1, 1] $\leq$ 0	[-1, 0, 1, 2, -2] $\leq$ 1	[2, -2, -1, 0, 1] $\leq$ 1
[0, 1, 1, 0, -1] $\leq$ 1	[1, 1, -1, 0, 0] $\leq$ 1	[1, 0, 1, -1, 0] $\leq$ 1
[2, 0, -1, 0, 1] $\leq$ 1	[0, 1, 2, -1, 0] $\leq$ 1	[0, 1, 2, 1, -1] $\leq$ 1
[1, 2, -1, 0, 1] $\leq$ 1	[2, 1, 0, 0, 1] $\leq$ 1	[1, 2, 0, 1, 0] $\leq$ 1
[1, 1, 2, 0, 0] $\leq$ 1		

Vertices of P_5		
(1, 0, 0, 0, -1)	(1/2, 1/2, 0, -1/2, -1/2)	(1/3, 1/3, 0, 0, -2/3)
(2/3, 0, 0, -1/3, -1/3)	(1/4, 1/4, 1/4, -1/4, -1/2)	(1/2, 1/4, -1/4, -1/4, -1/4)
(2/5, 2/5, -1/5, -1/5, -2/5)	(2/5, 1/5, 1/5, -2/5, -2/5)	(0, 0, 0, 0, 0)
(1/6, 1/6, 1/6, 0, -1/2)	(1/2, 0, -1/6, -1/6, -1/6)	(2/9, 2/9, 2/9, -1/3, -1/3)
(1/3, 1/3, -2/9, -2/9, -2/9)	(1/10, 1/10, 1/10, 1/10, -2/5)	(2/5, -1/10, -1/10, -1/10, -1/10)

Facets of P_6		
[-1, 1, 0, 0, 0, 0] $\leq$ 0	[0, -1, 1, 0, 0, 0] $\leq$ 0	[0, 0, -1, 1, 0, 0] $\leq$ 0
[0, 0, 0, -1, 1, 0] $\leq$ 0	[0, 0, 0, 0, -1, 1] $\leq$ 0	[0, 0, 1, 1, -1, -1] $\leq$ 1
[1, 1, -1, -1, 0, 0] $\leq$ 1	[0, 1, 1, 0, 0, -1] $\leq$ 1	[1, 1, 0, -1, 0, 0] $\leq$ 1
[1, 0, 1, 0, -1, 0] $\leq$ 1	[2, 1, 1, 0, 0, 1] $\leq$ 1	[1, 2, 1, 0, 1, 0] $\leq$ 1
[1, 1, 2, 1, 0, 0] $\leq$ 1	[-1, 0, 1, 2, 1, -2] $\leq$ 1	[2, 0, -1, -1, 0, 1] $\leq$ 1
[2, 1, -1, 0, 0, 1] $\leq$ 1	[0, 1, 1, 2, 0, -1] $\leq$ 1	[0, 1, 2, 0, 1, -1] $\leq$ 1
[2, 0, 1, -1, 0, 1] $\leq$ 1	[0, 1, 2, 1, -1, 0] $\leq$ 1	[1, 0, 1, 2, -1, 0] $\leq$ 1
[1, 2, 0, -1, 0, 1] $\leq$ 1	[1, 2, -1, 0, 1, 0] $\leq$ 1	[-1, 0, 1, 2, 3, -2] $\leq$ 1
[3, -2, -1, 0, 1, 2] $\leq$ 1	[0, 1, 2, 2, 1, -1] $\leq$ 1	[3, 0, -1, 0, 1, 2] $\leq$ 1

Vertices of P_6	
(1, 0, 0, 0, 0, -1)	(1/2, 1/2, 0, 0, -1/2, -1/2)
(1/3, 1/3, 1/3, -1/3, -1/3, -1/3)	(1/3, 1/3, 0, 0, 0, -2/3)
(2/3, 0, 0, 0, -1/3, -1/3)	(1/4, 1/4, 1/4, 0, -1/4, -1/2)
(1/2, 1/4, 0, -1/4, -1/4, -1/4)	(1/6, 1/6, 1/6, 1/6, -1/3, -1/3)
(1/3, 1/3, -1/6, -1/6, -1/6, -1/6)	(2/5, 2/5, 0, -1/5, -1/5, -2/5)
(2/5, 1/5, 1/5, 0, -2/5, -2/5)	(1/6, 1/6, 1/6, 0, 0, -1/2)
(1/2, 0, 0, -1/6, -1/6, -1/6)	(1/7, 1/7, 1/7, 1/7, -1/7, -3/7)
(3/7, 1/7, -1/7, -1/7, -1/7, -1/7)	(0, 0, 0, 0, 0, 0)
(1/10, 1/10, 1/10, 1/10, 0, -2/5)	(2/5, 0, -1/10, -1/10, -1/10, -1/10)
(1/15, 1/15, 1/15, 1/15, 1/15, -1/3)	(1/3, -1/15, -1/15, -1/15, -1/15, -1/15)

Facets of P_7			
[0, 0, 0, -1, 1, 0, 0] $\leq$ 0	[0, 0, -1, 1, 0, 0, 0] $\leq$ 0	[0, 2, 1, 1, 1, 1, 1] $\leq$ 0	[0, 0, 0, 0, 0, -1, 1] $\leq$ 0
[0, -1, 1, 0, 0, 0, 0] $\leq$ 0	[0, 0, 0, 0, -1, 1, 0] $\leq$ 0	[0, 0, -3, -2, -2, -1, -1] $\leq$ 1	[0, 0, 1, 1, 2, -1, -1] $\leq$ 1
[0, 0, 1, -2, -1, 0, -1] $\leq$ 1	[0, -1, 0, 1, -2, -1, -1] $\leq$ 1	[0, -2, -4, -3, -3, -2, -1] $\leq$ 1	[0, 1, 2, 2, 3, 1, -1] $\leq$ 1
[0, 1, 1, 2, 0, -1, 0] $\leq$ 1	[0, 1, 0, -2, -1, -1, 0] $\leq$ 1	[0, -1, -1, 0, -2, -2, -1] $\leq$ 1	[0, 1, 1, -1, 0, 0, -1] $\leq$ 1
[0, 1, 0, 1, -1, 0, -1] $\leq$ 1	[0, -1, 0, -2, -1, -2, -1] $\leq$ 1	[0, 0, 1, 1, 0, -1, -1] $\leq$ 1	[0, 0, -1, -2, -2, -1, -1] $\leq$ 1
[0, 1, -1, -1, -2, -1, 0] $\leq$ 1	[0, 1, 2, 1, 1, -1, 0] $\leq$ 1	[0, 1, 1, 1, 0, 0, -1] $\leq$ 1	[0, 1, 1, 2, 1, 0, -1] $\leq$ 1
[0, 1, 2, 3, 1, 2, -1] $\leq$ 1	[0, 1, 2, 1, 0, 1, -1] $\leq$ 1	[0, 1, 2, 2, 1, 1, -1] $\leq$ 1	[0, -1, -2, -3, -2, -2, -1] $\leq$ 1
[0, -1, -1, -2, -2, -2, -1] $\leq$ 1	[0, -3, -2, -4, -3, -2, -1] $\leq$ 1	[0, -2, -2, -3, -3, -2, -1] $\leq$ 1	[0, -2, -1, -2, -3, -2, -1] $\leq$ 1
[0, 1, -1, -2, -1, 0, -1] $\leq$ 1	[0, 1, 0, -1, -1, 0, -1] $\leq$ 1	[0, -1, 0, 1, 0, -2, -1] $\leq$ 1	[0, -1, 0, 0, -1, -2, -1] $\leq$ 1
[0, 0, 1, -1, 0, -1, -1] $\leq$ 1	[0, 0, 0, -2, -1, -1, -1] $\leq$ 1	[0, 1, 2, 0, 1, -1, 0] $\leq$ 1	[0, 0, 0, 1, -1, -1, -1] $\leq$ 1
[0, 0, -1, 0, -2, -1, -1] $\leq$ 1	[0, 1, -1, 0, -2, -1, 0] $\leq$ 1	[0, 1, 2, 0, 0, 1, -1] $\leq$ 1	[0, -2, -1, -1, -3, -2, -1] $\leq$ 1
[0, -1, -3, -3, -2, -2, -1] $\leq$ 1	[0, 1, 1, 2, 2, 0, -1] $\leq$ 1	[0, 1, -2, -2, -1, 0, -1] $\leq$ 1	[0, -1, 0, 1, 1, -2, -1] $\leq$ 1
[0, 1, 2, 3, 4, 3, -1] $\leq$ 1	[0, 1, 2, 3, 3, 2, -1] $\leq$ 1	[0, -4, -5, -4, -3, -2, -1] $\leq$ 1	[0, -3, -4, -4, -3, -2, -1] $\leq$ 1
[0, 0, 1, 2, -1, -1, 0] $\leq$ 1	[0, 1, 1, -2, -1, 0, 0] $\leq$ 1	[0, 1, 2, 1, 2, -1, 0] $\leq$ 1	[0, 1, -2, -1, -2, -1, 0] $\leq$ 1
[0, 1, 2, 3, -1, 0, 1] $\leq$ 1	[0, 1, 2, -2, -1, 0, 1] $\leq$ 1	[0, -6, -5, -4, -3, -2, -1] $\leq$ 1	[0, 1, 2, 3, 4, 5, -1] $\leq$ 1
[0, 1, -3, -2, -1, -1, 0] $\leq$ 1	[0, 1, 1, 2, 3, -1, 0] $\leq$ 1		

Vertices of P_7	
(0, 0, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0, -1)
(1/2, 1/2, 0, 0, 0, -1/2, -1/2)	(1/3, 1/3, 1/3, 0, -1/3, -1/3, -1/3)
(2/3, 0, 0, 0, 0, -1/3, -1/3)	(1/3, 1/3, 0, 0, 0, 0, -2/3)
(1/6, 1/6, 1/6, 0, 0, 0, -1/2)	(1/2, 0, 0, 0, -1/6, -1/6, -1/6)
(1/2, 1/4, 0, 0, -1/4, -1/4, -1/4)	(1/4, 1/4, 1/4, 0, 0, -1/4, -1/2)
(2/5, 1/5, 1/5, 0, 0, -2/5, -2/5)	(2/5, 2/5, 0, 0, -1/5, -1/5, -2/5)
(2/9, 2/9, 2/9, 0, 0, -1/3, -1/3)	(1/3, 1/3, 0, 0, -2/9, -2/9, -2/9)
(2/5, 0, 0, -1/10, -1/10, -1/10, -1/10)	(1/10, 1/10, 1/10, 1/10, 0, 0, -2/5)
(1/3, 1/3, 0, -1/6, -1/6, -1/6, -1/6)	(1/6, 1/6, 1/6, 1/6, 0, -1/3, -1/3)
(3/7, 1/7, 0, -1/7, -1/7, -1/7, -1/7)	(1/7, 1/7, 1/7, 1/7, 0, -1/7, -3/7)
(1/3, 0, -1/15, -1/15, -1/15, -1/15, -1/15)	(1/15, 1/15, 1/15, 1/15, 1/15, 0, -1/3)
(1/5, 1/5, 1/5, 1/5, -1/5, -1/5, -2/5)	(2/5, 1/5, 1/5, -1/5, -1/5, -1/5, -1/5)
(1/3, 1/3, 1/6, -1/6, -1/6, -1/6, -1/3)	(1/3, 1/6, 1/6, 1/6, -1/6, -1/3, -1/3)
(2/7, 2/7, 2/7, -1/7, -1/7, -2/7, -2/7)	(2/7, 2/7, 1/7, 1/7, -2/7, -2/7, -2/7)
(1/4, 1/4, 1/4, -1/8, -1/8, -1/8, -3/8)	(3/8, 1/8, 1/8, 1/8, -1/4, -1/4, -1/4)
(1/3, 2/9, -1/9, -1/9, -1/9, -1/9, -1/9)	(1/9, 1/9, 1/9, 1/9, 1/9, -2/9, -1/3)
(3/10, 3/10, -1/10, -1/10, -1/10, -1/10, -1/5)	(1/5, 1/10, 1/10, 1/10, 1/10, -3/10, -3/10)
(3/11, 3/11, 2/11, -2/11, -2/11, -2/11, -2/11)	(2/11, 2/11, 2/11, 2/11, -2/11, -3/11, -3/11)
(1/11, 1/11, 1/11, 1/11, 1/11, -1/11, -4/11)	(4/11, 1/11, -1/11, -1/11, -1/11, -1/11, -1/11)
(1/4, 1/6, 1/6, 1/6, -1/4, -1/4, -1/4)	(1/4, 1/4, 1/4, -1/6, -1/6, -1/6, -1/4)
(2/13, 2/13, 2/13, 1/13, 1/13, -4/13, -4/13)	(4/13, 4/13, -1/13, -1/13, -2/13, -2/13, -2/13)
(1/6, 1/6, 1/6, 1/6, -2/9, -2/9, -2/9)	(2/9, 2/9, 2/9, -1/6, -1/6, -1/6, -1/6)
(2/7, -1/21, -1/21, -1/21, -1/21, -1/21, -1/21)	(1/21, 1/21, 1/21, 1/21, 1/21, 1/21, -2/7)
(5/19, 5/19, -2/19, -2/19, -2/19, -2/19, -2/19)	(2/19, 2/19, 2/19, 2/19, 2/19, -5/19, -5/19)

Facets of P_8			
[0, 0, 0, -1, 1, 0, 0, 0] $\leq$ 0	[0, -1, 1, 0, 0, 0, 0, 0] $\leq$ 0	[0, 0, 0, 0, 0, -1, 1, 0] $\leq$ 0	[0, 2, 1, 1, 1, 1, 1, 1] $\leq$ 0
[0, 0, 0, 0, 0, 0, -1, 1] $\leq$ 0	[0, 0, -1, 1, 0, 0, 0, 0] $\leq$ 0	[0, 0, 0, 0, -1, 1, 0, 0] $\leq$ 0	[0, -2, -2, -1, -3, -3, -2, -1] $\leq$ 1
[0, 1, 2, 2, 0, 1, 1, -1] $\leq$ 1	[0, 1, 1, 2, 2, -1, 0, 0] $\leq$ 1	[0, 1, 0, 1, 2, -1, -1, 0] $\leq$ 1	[0, 0, 1, -2, -2, -1, -1, 0] $\leq$ 1
[0, 1, 1, -2, -1, 0, -1, 0] $\leq$ 1	[0, -3, -5, -4, -4, -3, -2, -1] $\leq$ 1	[0, 1, 2, 3, 3, 4, 2, -1] $\leq$ 1	[0, -1, -4, -3, -3, -2, -2, -1] $\leq$ 1
[0, 1, 1, 2, 2, 3, 0, -1] $\leq$ 1	[0, 0, -1, 0, 1, -2, -1, -1] $\leq$ 1	[0, 0, 1, -2, -1, 0, -1, -1] $\leq$ 1	[0, -1, 0, -3, -2, -2, -1, -1] $\leq$ 1
[0, 0, 1, 1, 2, -1, 0, -1] $\leq$ 1	[0, 1, 1, -1, -2, -1, 0, 0] $\leq$ 1	[0, 0, 1, 2, 1, -1, -1, 0] $\leq$ 1	[0, 0, -2, -3, -2, -2, -1, -1] $\leq$ 1
[0, 0, -3, -3, -2, -2, -1, -1] $\leq$ 1	[0, 1, -1, -2, -1, -2, -1, 0] $\leq$ 1	[0, 0, 1, 1, 2, 1, -1, -1] $\leq$ 1	[0, 0, 1, 1, 2, 2, -1, -1] $\leq$ 1
[0, 1, 2, 1, 2, 1, -1, 0] $\leq$ 1	[0, -1, 0, 1, 1, 0, -2, -1] $\leq$ 1	[0, 1, -1, -2, -2, -1, 0, -1] $\leq$ 1	[0, 1, -1, 0, -1, -2, -1, 0] $\leq$ 1
[0, 1, 2, 1, 0, 1, -1, 0] $\leq$ 1	[0, -1, -1, 0, -1, -2, -2, -1] $\leq$ 1	[0, -1, -1, 0, 0, -2, -2, -1] $\leq$ 1	[0, -2, -1, -1, -2, -3, -2, -1] $\leq$ 1
[0, -1, 0, 1, 0, -2, -1, -1] $\leq$ 1	[0, 1, 1, 0, -1, 0, 0, -1] $\leq$ 1	[0, 1, 1, -1, -1, 0, 0, -1] $\leq$ 1	[0, 1, 2, 1, 0, 0, 1, -1] $\leq$ 1
[0, 0, 1, -1, -2, -1, 0, -1] $\leq$ 1	[0, -2, -3, -4, -3, -3, -2, -1] $\leq$ 1	[0, 1, 2, 2, 3, 2, 1, -1] $\leq$ 1	[0, -1, -2, -3, -3, -2, -2, -1] $\leq$ 1
[0, 1, 1, 2, 2, 1, 0, -1] $\leq$ 1	[0, -1, -1, -3, -2, -2, -2, -1] $\leq$ 1	[0, 0, 0, -2, -2, -1, -1, -1] $\leq$ 1	[0, 1, 1, 1, 2, 0, 0, -1] $\leq$ 1
[0, 0, 0, 1, 1, -1, -1, -1] $\leq$ 1	[0, -1, -2, -1, -3, -2, -2, -1] $\leq$ 1	[0, 1, 1, 2, 0, 1, 0, -1] $\leq$ 1	[0, 0, 1, 1, 0, 0, -1, -1] $\leq$ 1
[0, 0, 1, 1, 1, 0, -1, -1] $\leq$ 1	[0, 0, -1, -1, -2, -2, -1, -1] $\leq$ 1	[0, 0, -1, -2, -2, -2, -1, -1] $\leq$ 1	[0, 1, 0, 1, 0, -1, 0, -1] $\leq$ 1
[0, -1, 0, -1, -2, -1, -2, -1] $\leq$ 1	[0, 1, 1, 1, 0, 0, 0, -1] $\leq$ 1	[0, -1, -1, -1, -2, -2, -2, -1] $\leq$ 1	[0, 0, 1, 0, -1, 0, -1, -1] $\leq$ 1
[0, 0, -1, 0, -1, -2, -1, -1] $\leq$ 1	[0, 1, 0, -2, -1, -1, 0, -1] $\leq$ 1	[0, 1, 0, -1, -2, -1, -1, 0] $\leq$ 1	[0, 1, -1, 0, -2, -1, 0, -1] $\leq$ 1
[0, 1, 0, 0, -1, -1, 0, -1] $\leq$ 1	[0, 0, 0, -1, -2, -1, -1, -1] $\leq$ 1	[0, 1, 1, 2, 1, 0, -1, 0] $\leq$ 1	[0, -1, 0, 1, -1, 0, -2, -1] $\leq$ 1
[0, -1, 0, 0, 1, -1, -2, -1] $\leq$ 1	[0, 0, 0, 1, 0, -1, -1, -1] $\leq$ 1	[0, -1, 0, 0, -1, -1, -2, -1] $\leq$ 1	[0, 1, 2, 1, 2, 0, 1, -1] $\leq$ 1
[0, 1, 2, 2, 2, 1, 1, -1] $\leq$ 1	[0, -2, -1, -3, -2, -3, -2, -1] $\leq$ 1	[0, -2, -2, -3, -3, -3, -2, -1] $\leq$ 1	[0, 1, 2, 3, 2, 1, 2, -1] $\leq$ 1
[0, -3, -2, -3, -4, -3, -2, -1] $\leq$ 1	[0, 1, 0, 1, 1, -1, 0, -1] $\leq$ 1	[0, -1, 0, -2, -2, -1, -2, -1] $\leq$ 1	[0, 1, 2, 3, 3, 2, 2, -1] $\leq$ 1
[0, -3, -3, -4, -4, -3, -2, -1] $\leq$ 1	[0, 1, 2, -1, 0, 0, 1, -1] $\leq$ 1	[0, -2, -1, -1, 0, -3, -2, -1] $\leq$ 1	[0, 1, -3, -2, -2, -1, 0, -1] $\leq$ 1
[0, -1, 0, 1, 1, 2, -2, -1] $\leq$ 1	[0, 1, 1, 2, 3, 2, -1, 0] $\leq$ 1	[0, 1, 2, 1, 2, 3, -1, 0] $\leq$ 1	[0, 1, -3, -2, -1, -2, -1, 0] $\leq$ 1
[0, 1, -2, -3, -2, -1, -1, 0] $\leq$ 1	[0, 1, 2, 3, 4, 2, 3, -1] $\leq$ 1	[0, -4, -3, -5, -4, -3, -2, -1] $\leq$ 1	[0, 1, 2, -1, 0, 1, -1, 0] $\leq$ 1
[0, 1, -1, 0, 1, -2, -1, 0] $\leq$ 1	[0, 1, 2, 2, 3, 3, 1, -1] $\leq$ 1	[0, -2, -4, -4, -3, -3, -2, -1] $\leq$ 1	[0, 1, 2, 3, 4, 4, 3, -1] $\leq$ 1
[0, -4, -5, -5, -4, -3, -2, -1] $\leq$ 1	[0, 1, 2, 3, 1, 1, 2, -1] $\leq$ 1	[0, -3, -2, -2, -4, -3, -2, -1] $\leq$ 1	[0, -5, -6, -5, -4, -3, -2, -1] $\leq$ 1
[0, 1, 2, 3, 4, 5, 4, -1] $\leq$ 1	[0, -7, -6, -5, -4, -3, -2, -1] $\leq$ 1	[0, 1, 2, 3, 4, 5, 6, -1] $\leq$ 1	[0, 1, 2, -2, -1, -1, 0, 1] $\leq$ 1
[0, 1, 2, 2, 3, -1, 0, 1] $\leq$ 1			

Vertices of P_8	
(1, 0, 0, 0, 0, 0, 0, -1)	(1/2, 1/2, 0, 0, 0, 0, -1/2, -1/2)
(1/3, 1/3, 1/3, 0, 0, -1/3, -1/3, -1/3)	(1/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4, -1/4)
(1/3, 1/3, 0, 0, 0, 0, 0, -2/3)	(2/3, 0, 0, 0, 0, 0, -1/3, -1/3)
(1/4, 1/4, 1/4, 0, 0, 0, -1/4, -1/2)	(1/2, 1/4, 0, 0, 0, -1/4, -1/4, -1/4)
(1/5, 1/5, 1/5, 1/5, 0, -1/5, -1/5, -2/5)	(2/5, 1/5, 1/5, 0, -1/5, -1/5, -1/5, -1/5)
(2/5, 2/5, 0, 0, 0, -1/5, -1/5, -2/5)	(2/5, 1/5, 1/5, 0, 0, 0, -2/5, -2/5)
(1/7, 1/7, 1/7, 1/7, 1/7, -1/7, -2/7, -2/7)	(2/7, 2/7, 1/7, -1/7, -1/7, -1/7, -1/7, -1/7)
(1/6, 1/6, 1/6, 1/6, 0, 0, -1/3, -1/3)	(1/3, 1/3, 0, 0, -1/6, -1/6, -1/6, -1/6)
(1/6, 1/6, 1/6, 0, 0, 0, 0, -1/2)	(1/2, 0, 0, 0, 0, -1/6, -1/6, -1/6)
(2/7, 2/7, 2/7, 0, -1/7, -1/7, -2/7, -2/7)	(2/7, 2/7, 1/7, 1/7, 0, -2/7, -2/7, -2/7)
(1/4, 1/4, 1/4, -1/8, -1/8, -1/8, -1/8, -1/4)	(1/4, 1/8, 1/8, 1/8, 1/8, -1/4, -1/4, -1/4)
(1/3, 1/3, 1/6, 0, -1/6, -1/6, -1/6, -1/3)	(1/3, 1/6, 1/6, 1/6, 0, -1/6, -1/3, -1/3)
(1/7, 1/7, 1/7, 1/7, 0, 0, -1/7, -3/7)	(3/7, 1/7, 0, 0, -1/7, -1/7, -1/7, -1/7)
(1/8, 1/8, 1/8, 1/8, 1/8, -1/8, -1/8, -3/8)	(3/8, 1/8, 1/8, -1/8, -1/8, -1/8, -1/8, -1/8)
(1/10, 1/10, 1/10, 1/10, 0, 0, 0, -2/5)	(2/5, 0, 0, 0, -1/10, -1/10, -1/10, -1/10)
(2/9, 2/9, 1/9, 1/9, 1/9, -2/9, -2/9, -1/3)	(1/3, 2/9, 2/9, -1/9, -1/9, -1/9, -2/9, -2/9)
(1/9, 1/9, 1/9, 1/9, 1/9, 0, -2/9, -1/3)	(1/3, 2/9, 0, -1/9, -1/9, -1/9, -1/9, -1/9)
(1/12, 1/12, 1/12, 1/12, 1/12, 1/12, -1/4, -1/4)	(1/4, 1/4, -1/12, -1/12, -1/12, -1/12, -1/12, -1/12)
(1/11, 1/11, 1/11, 1/11, 1/11, 0, -1/11, -4/11)	(4/11, 1/11, 0, -1/11, -1/11, -1/11, -1/11, -1/11)
(1/4, 1/4, 1/4, 0, -1/8, -1/8, -1/8, -3/8)	(3/8, 1/8, 1/8, 1/8, 0, -1/4, -1/4, -1/4)
(2/9, 2/9, 2/9, 0, 0, 0, -1/3, -1/3)	(1/3, 1/3, 0, 0, 0, -2/9, -2/9, -2/9)
(3/10, 3/10, 0, -1/10, -1/10, -1/10, -1/10, -1/5)	(1/5, 1/10, 1/10, 1/10, 1/10, 0, -3/10, -3/10)
(2/13, 2/13, 2/13, 1/13, 1/13, 0, -4/13, -4/13)	(4/13, 4/13, 0, -1/13, -1/13, -2/13, -2/13, -2/13)
(1/15, 1/15, 1/15, 1/15, 1/15, 0, 0, -1/3)	(1/3, 0, 0, -1/15, -1/15, -1/15, -1/15, -1/15)
(2/13, 2/13, 2/13, 2/13, 1/13, -3/13, -3/13, -3/13)	(3/13, 3/13, 3/13, -1/13, -2/13, -2/13, -2/13, -2/13)
(2/15, 2/15, 2/15, 2/15, 2/15, -1/5, -1/5, -4/15)	(4/15, 1/5, 1/5, -2/15, -2/15, -2/15, -2/15, -2/15)
(1/4, 1/4, 1/4, -1/12, -1/12, -1/12, -1/4, -1/4)	(1/4, 1/4, 1/12, 1/12, 1/12, -1/4, -1/4, -1/4)
(1/13, 1/13, 1/13, 1/13, 1/13, 1/13, -2/13, -4/13)	(4/13, 2/13, -1/13, -1/13, -1/13, -1/13, -1/13, -1/13)
(1/16, 1/16, 1/16, 1/16, 1/16, 1/16, -1/16, -5/16)	(5/16, 1/16, -1/16, -1/16, -1/16, -1/16, -1/16, -1/16)
(4/17, 4/17, 4/17, -2/17, -2/17, -2/17, -3/17, -3/17)	(3/17, 3/17, 2/17, 2/17, 2/17, -4/17, -4/17, -4/17)
(0, 0, 0, 0, 0, 0, 0, 0)	(2/13, 2/13, 2/13, 2/13, -1/13, -1/13, -1/13, -5/13)
(5/13, 1/13, 1/13, 1/13, -2/13, -2/13, -2/13, -2/13)	(2/7, 0, -1/21, -1/21, -1/21, -1/21, -1/21, -1/21)
(1/21, 1/21, 1/21, 1/21, 1/21, 1/21, 0, -2/7)	(1/4, -1/28, -1/28, -1/28, -1/28, -1/28, -1/28, -1/28)
(1/28, 1/28, 1/28, 1/28, 1/28, 1/28, 1/28, -1/4)	(5/24, 5/24, 5/24, -1/8, -1/8, -1/8, -1/8, -1/8)
(1/8, 1/8, 1/8, 1/8, 1/8, -5/24, -5/24, -5/24)	

Facets of P_9			
[0, 0, -1, 1, 0, 0, 0, 0, 0] ≤ 0	[0, 0, 0, 0, 0, -1, 1, 0, 0] ≤ 0	[0, -1, 1, 0, 0, 0, 0, 0, 0] ≤ 0	[0, 0, 0, 0, 0, 0, -1, 1, 0] ≤ 0
[0, 2, 1, 1, 1, 1, 1, 1, 1] ≤ 0	[0, 0, 0, 0, 0, 0, 0, -1, 1] ≤ 0	[0, 0, 0, -1, 1, 0, 0, 0, 0] ≤ 0	[0, 0, 0, 0, -1, 1, 0, 0, 0] ≤ 0
[0, 0, 1, 1, 2, 0, 1, -1, -1] ≤ 1	[0, 0, 1, 1, 1, 0, 0, -1, -1] ≤ 1	[0, 0, -2, -1, -3, -2, -2, -1, -1] ≤ 1	[0, 0, -1, -1, -2, -2, -2, -1, -1] ≤ 1
[0, 0, 1, 1, -1, 0, 0, -1, -1] ≤ 1	[0, 0, -1, -1, 0, -2, -2, -1, -1] ≤ 1	[0, 0, -4, -3, -3, -2, -2, -1, -1] ≤ 1	[0, 0, 1, 1, 2, 2, 3, -1, -1] ≤ 1
[0, 0, -2, -3, -3, -2, -2, -1, -1] ≤ 1	[0, 0, 1, 1, 2, 2, 1, -1, -1] ≤ 1	[0, 0, 1, 1, 1, 2, 0, -1, -1] ≤ 1	[0, 0, -1, -3, -2, -2, -2, -1, -1] ≤ 1
[0, 0, 0, 1, 1, 0, -1, -1, -1] ≤ 1	[0, 0, 0, -1, -2, -2, -1, -1, -1] ≤ 1	[0, 1, 1, 0, -1, -1, 0, 0, -1] ≤ 1	[0, -1, -1, 0, 0, -1, -2, -2, -1] ≤ 1
[0, 1, 1, 1, 2, 2, 0, 0, -1] ≤ 1	[0, -1, -1, -3, -3, -2, -2, -2, -1] ≤ 1	[0, 1, 1, 1, -1, 0, 0, 0, -1] ≤ 1	[0, -1, -1, -1, 0, -2, -2, -2, -1] ≤ 1
[0, -1, -1, 0, 1, -2, -2, -1, -1] ≤ 1	[0, 0, 1, -2, -2, -1, 0, -1, -1] ≤ 1	[0, 0, -1, 0, 1, 1, -2, -1, -1] ≤ 1	[0, 0, 1, 1, -2, -1, 0, 0, -1] ≤ 1
[0, 0, 1, 2, 1, 1, -1, -1, 0] ≤ 1	[0, 1, 1, -1, -1, -2, -1, 0, 0] ≤ 1	[0, 0, 1, 1, 2, 0, -1, -1, 0] ≤ 1	[0, 1, 1, 2, 2, 1, -1, 0, 0] ≤ 1
[0, 1, 1, 0, -2, -1, -1, 0, 0] ≤ 1	[0, 0, 1, -1, -2, -2, -1, -1, 0] ≤ 1	[0, -1, -1, 0, 0, 1, -2, -2, -1] ≤ 1	[0, 1, 1, -2, -1, -1, 0, 0, -1] ≤ 1
[0, 0, 1, 0, 1, -1, 0, -1, -1] ≤ 1	[0, 0, -1, 0, -2, -1, -2, -1, -1] ≤ 1	[0, -1, 0, 0, -2, -1, -1, -2, -1] ≤ 1	[0, 1, 0, 0, 1, -1, -1, 0, -1] ≤ 1
[0, 1, 2, 2, 3, 3, 2, 1, -1] ≤ 1	[0, -2, -3, -4, -4, -3, -3, -2, -1] ≤ 1	[0, -1, -2, -2, -3, -3, -2, -2, -1] ≤ 1	[0, 1, 1, 2, 2, 1, 1, 0, -1] ≤ 1
[0, -2, -2, -1, -1, -3, -3, -2, -1] ≤ 1	[0, 1, 2, 2, 0, 0, 1, 1, -1] ≤ 1	[0, -1, 0, -1, -2, -2, -1, -2, -1] ≤ 1	[0, -1, -1, 0, -2, -1, -2, -2, -1] ≤ 1
[0, 1, 0, 1, 1, 0, -1, 0, -1] ≤ 1	[0, 1, 1, 0, 1, -1, 0, 0, -1] ≤ 1	[0, -1, -2, -3, -3, -3, -2, -2, -1] ≤ 1	[0, 1, 1, 2, 2, 2, 1, 0, -1] ≤ 1
[0, -2, -1, -1, 0, -3, -2, -2, -1] ≤ 1	[0, 1, 1, 2, -1, 0, 0, 1, -1] ≤ 1	[0, 1, 1, 1, 2, 0, 0, -1, 0] ≤ 1	[0, 1, 0, 0, -2, -1, -1, -1, 0] ≤ 1
[0, -1, 0, 0, 1, -1, -2, -1, -1] ≤ 1	[0, 0, 1, 0, -2, -1, -1, 0, -1] ≤ 1	[0, 0, 0, 1, -2, -1, 0, -1, -1] ≤ 1	[0, 0, -1, 0, 1, -2, -1, -1, -1] ≤ 1
[0, 1, 2, 2, 3, 3, 4, 1, -1] ≤ 1	[0, -2, -5, -4, -4, -3, -3, -2, -1] ≤ 1	[0, 1, 1, 2, 2, 3, 2, 0, -1] ≤ 1	[0, -1, -3, -4, -3, -3, -2, -2, -1] ≤ 1
[0, -2, -2, -4, -3, -3, -3, -2, -1] ≤ 1	[0, 1, 2, 2, 2, 3, 1, 1, -1] ≤ 1	[0, -1, 0, 0, 1, 1, -1, -2, -1] ≤ 1	[0, 1, 0, -2, -2, -1, -1, 0, -1] ≤ 1
[0, 1, 1, 1, 2, 1, 0, 0, -1] ≤ 1	[0, -1, -1, -2, -3, -2, -2, -2, -1] ≤ 1	[0, 0, 1, -1, -2, -1, 0, -1, -1] ≤ 1	[0, 0, -1, 0, 1, 0, -2, -1, -1] ≤ 1
[0, -1, 0, -1, 0, -2, -1, -2, -1] ≤ 1	[0, 1, 0, 1, -1, 0, -1, 0, -1] ≤ 1	[0, 0, 0, -2, -2, -2, -1, -1, -1] ≤ 1	[0, 0, 0, 1, 1, 1, -1, -1, -1] ≤ 1
[0, 1, 1, 2, 2, 3, 3, 0, -1] ≤ 1	[0, -1, -4, -4, -3, -3, -2, -2, -1] ≤ 1	[0, 1, 2, 2, -2, -1, 0, 1, 1] ≤ 1	[0, -1, 0, 1, 2, -2, -1, -1, 0] ≤ 1
[0, 1, 1, 2, -2, -1, 0, 1, 0] ≤ 1	[0, 0, 1, 2, 0, 1, -1, -1, 0] ≤ 1	[0, 1, 1, -1, 0, -2, -1, 0, 0] ≤ 1	[0, 1, 2, 1, -2, -1, 0, 0, 1] ≤ 1
[0, 0, 1, 2, 3, -1, -1, 0, 1] ≤ 1	[0, 1, 1, 2, 3, 0, -1, 0, 1] ≤ 1	[0, 1, 1, 2, 2, 0, 1, -1, 0] ≤ 1	[0, 1, -1, 0, -2, -2, -1, -1, 0] ≤ 1
[0, 1, 0, -1, 0, -2, -1, -1, 0] ≤ 1	[0, 1, 0, -1, -1, -2, -1, -1, 0] ≤ 1	[0, 1, 0, -2, -1, -2, -1, -1, 0] ≤ 1	[0, -1, 0, 1, -1, 0, -2, -1, -1] ≤ 1
[0, -1, 0, 1, 0, 0, -2, -1, -1] ≤ 1	[0, -1, 0, 1, 0, 1, -2, -1, -1] ≤ 1	[0, 1, 2, 1, 2, 1, 1, -1, 0] ≤ 1	[0, 1, 2, 1, 1, 0, 1, -1, 0] ≤ 1
[0, 1, -1, -1, -2, -1, -2, -1, 0] ≤ 1	[0, 1, -1, 0, -1, -1, -2, -1, 0] ≤ 1	[0, 1, 2, 3, 4, 4, 5, 3, -1] ≤ 1	[0, -4, -6, -5, -5, -4, -3, -2, -1] ≤ 1
[0, 1, 2, 3, 3, 4, 3, 2, -1] ≤ 1	[0, -3, -4, -5, -4, -4, -3, -2, -1] ≤ 1	[0, 1, 2, 3, 2, 1, 1, 2, -1] ≤ 1	[0, 1, 2, 3, 3, 1, 2, 2, -1] ≤ 1
[0, -3, -2, -2, -3, -4, -3, -2, -1] ≤ 1	[0, -3, -3, -2, -4, -4, -3, -2, -1] ≤ 1	[0, 1, 2, 2, 3, 1, 2, 1, -1] ≤ 1	[0, -2, -3, -2, -4, -3, -3, -2, -1] ≤ 1
[0, 1, 0, 1, 2, 1, -1, -1, 0] ≤ 1	[0, 0, 1, 1, 2, 1, -1, 0, -1] ≤ 1	[0, 1, 1, -1, -2, -1, 0, -1, 0] ≤ 1	[0, -1, 0, -2, -3, -2, -2, -1, -1] ≤ 1
[0, -2, -1, -1, -1, -2, -3, -2, -1] ≤ 1	[0, -2, -1, -1, -2, -2, -3, -2, -1] ≤ 1	[0, -2, -2, -1, -2, -3, -3, -2, -1] ≤ 1	[0, 1, 2, 1, 0, 0, 0, 1, -1] ≤ 1
[0, 1, 2, 1, 1, 0, 0, 1, -1] ≤ 1	[0, 1, 2, 2, 1, 0, 1, 1, -1] ≤ 1	[0, -1, 0, 1, 1, 0, 0, -2, -1] ≤ 1	[0, 1, -1, -1, -2, -2, -1, 0, -1] ≤ 1
[0, 1, 1, 2, 0, 0, 1, 0, -1] ≤ 1	[0, 1, 1, 2, 1, 0, 1, 0, -1] ≤ 1	[0, -1, -2, -1, -1, -3, -2, -2, -1] ≤ 1	[0, -1, -2, -1, -2, -3, -2, -2, -1] ≤ 1
[0, -1, 0, 0, 1, 0, -1, -2, -1] ≤ 1	[0, 1, 0, -1, -2, -1, -1, 0, -1] ≤ 1	[0, 1, -1, 0, 0, -1, -2, -1, 0] ≤ 1	[0, 1, 2, 1, 0, 0, 1, -1, 0] ≤ 1
[0, 1, 1, 2, 0, 1, 0, -1, 0] ≤ 1	[0, 0, 1, -1, 0, -2, -1, 0, -1] ≤ 1	[0, 0, 1, -1, -1, -2, -1, 0, -1] ≤ 1	[0, 1, 1, 2, 1, 1, 0, -1, 0] ≤ 1
[0, 1, -1, -2, -2, -1, -2, -1, 0] ≤ 1	[0, 1, 2, 1, 2, 2, 1, -1, 0] ≤ 1	[0, 1, -1, -2, -2, -2, -1, 0, -1] ≤ 1	[0, -1, 0, 1, 1, 1, 0, -2, -1] ≤ 1
[0, 0, 1, -2, -1, -2, -1, 0, -1] ≤ 1	[0, -1, 0, -3, -2, -2, -1, -2, -1] ≤ 1	[0, 1, 1, 2, 1, 2, 0, -1, 0] ≤ 1	[0, 1, 0, 1, 1, 2, -1, 0, -1] ≤ 1
[0, -3, -5, -5, -4, -4, -3, -2, -1] ≤ 1	[0, 1, 2, 3, 3, 4, 4, 2, -1] ≤ 1	[0, 1, 1, 2, 3, 1, 2, -1, 0] ≤ 1	[0, 1, 1, 2, 3, 2, 2, -1, 0] ≤ 1
[0, 1, 1, 2, 3, 2, 3, -1, 0] ≤ 1	[0, 1, -2, -1, -3, -2, -1, -1, 0] ≤ 1	[0, 1, -2, -2, -3, -2, -1, -1, 0] ≤ 1	[0, 1, -3, -2, -3, -2, -1, -1, 0] ≤ 1
[0, 1, 1, 2, 2, 3, 4, -1, 0] ≤ 1	[0, 1, -4, -3, -2, -2, -1, -1, 0] ≤ 1	[0, -2, -1, -1, 0, 0, -3, -2, -1] ≤ 1	[0, 1, 2, -1, -1, 0, 0, 1, -1] ≤ 1
[0, 0, 0, 1, 2, -1, -1, -1, 0] ≤ 1	[0, 1, 0, 1, 2, -1, 0, -1, 0] ≤ 1	[0, 1, 1, 1, -2, -1, 0, 0, 0] ≤ 1	[0, 1, 0, 1, -2, -1, 0, -1, 0] ≤ 1
[0, 1, 1, 2, -1, 0, 1, -1, 0] ≤ 1	[0, 1, -1, 0, 1, -2, -1, -1, 0] ≤ 1	[0, -1, 0, 1, -1, -1, 0, -2, -1] ≤ 1	[0, 1, -1, 0, 0, -2, -1, 0, -1] ≤ 1
[0, 0, -1, 0, 0, -1, -2, -1, -1] ≤ 1	[0, 0, 1, 0, -1, -1, 0, -1, -1] ≤ 1	[0, 1, 0, 0, -1, -1, -1, 0, -1] ≤ 1	[0, -1, 0, 0, 0, -1, -1, -2, -1] ≤ 1
[0, 0, 0, 1, -1, 0, -1, -1, -1] ≤ 1	[0, 0, 0, -1, 0, -2, -1, -1, -1] ≤ 1	[0, 1, -1, 0, 1, 0, -2, -1, 0] ≤ 1	[0, 1, 2, 0, -1, 0, 1, -1, 0] ≤ 1
[0, -1, 0, 1, 0, -1, 0, -2, -1] ≤ 1	[0, 1, -1, 0, -1, -2, -1, 0, -1] ≤ 1	[0, 1, 1, 1, 1, 0, 0, 0, -1] ≤ 1	[0, 0, 0, 0, 1, -1, -1, -1, -1] ≤ 1
[0, 0, 0, 0, -2, -1, -1, -1, -1] ≤ 1	[0, -1, -1, -1, -2, -2, -2, -2, -1] ≤ 1	[0, 1, 2, 0, -1, 0, 0, 1, -1] ≤ 1	[0, -2, -1, -1, 0, -1, -3, -2, -1] ≤ 1
[0, -2, -2, -2, -3, -3, -3, -2, -1] ≤ 1	[0, -2, -1, -3, -3, -2, -3, -2, -1] ≤ 1	[0, -2, -1, -2, -3, -2, -3, -2, -1] ≤ 1	[0, 1, 2, 2, 2, 1, 1, 1, -1] ≤ 1
[0, 1, 2, 1, 2, 2, 0, 1, -1] ≤ 1	[0, 1, 2, 1, 2, 1, 0, 1, -1] ≤ 1	[0, -3, -2, -2, -1, -4, -3, -2, -1] ≤ 1	[0, 1, 2, 3, 0, 1, 1, 2, -1] ≤ 1
[0, 1, 2, 3, 3, 3, 2, 2, -1] ≤ 1	[0, 1, 1, 2, 2, 3, 0, 1, -1] ≤ 1	[0, 1, 2, 3, 2, 3, 1, 2, -1] ≤ 1	[0, -3, -3, -4, -4, -4, -3, -2, -1] ≤ 1
[0, -2, -1, -4, -3, -3, -2, -2, -1] ≤ 1	[0, -3, -2, -4, -3, -4, -3, -2, -1] ≤ 1	[0, 1, 1, 2, 2, 3, 1, -1, 0] ≤ 1	[0, 1, 2, 1, 2, 3, 3, -1, 0] ≤ 1
[0, 1, 2, 1, 2, 3, 2, -1, 0] ≤ 1	[0, -1, 0, 1, 1, 2, 1, -2, -1] ≤ 1	[0, 1, -1, -3, -2, -2, -1, -1, 0] ≤ 1	[0, 1, -3, -3, -2, -1, -2, -1, 0] ≤ 1
[0, 1, -2, -3, -2, -1, -2, -1, 0] ≤ 1	[0, 1, -2, -3, -2, -2, -1, 0, -1] ≤ 1	[0, 1, -3, -3, -2, -2, -1, 0, -1] ≤ 1	[0, -1, 0, 1, 1, 2, 2, -2, -1] ≤ 1
[0, 1, 2, 3, 4, 2, 2, 3, -1] ≤ 1	[0, 1, 2, 3, 4, 3, 2, 3, -1] ≤ 1	[0, 1, 2, 3, 4, 4, 3, 3, -1] ≤ 1	[0, -4, -3, -3, -5, -4, -3, -2, -1] ≤ 1
[0, -4, -3, -4, -5, -4, -3, -2, -1] ≤ 1	[0, -4, -4, -5, -5, -4, -3, -2, -1] ≤ 1	[0, -5, -4, -6, -5, -4, -3, -2, -1] ≤ 1	[0, 1, 2, 3, 4, 5, 3, 4, -1] ≤ 1
[0, -5, -6, -6, -5, -4, -3, -2, -1] ≤ 1	[0, 1, 2, 3, 4, 5, 5, 4, -1] ≤ 1	[0, 1, 2, 3, 4, 5, 6, 5, -1] ≤ 1	[0, -6, -7, -6, -5, -4, -3, -2, -1] ≤ 1
[0, -8, -7, -6, -5, -4, -3, -2, -1] ≤ 1	[0, 1, 2, 3, 4, 5, 6, 7, -1] ≤ 1	[0, 1, -1, 0, 1, 1, -2, -1, 0] ≤ 1	[0, 1, 2, -1, -1, 0, 1, -1, 0] ≤ 1

Vertices of P_9	
(1/2, 1/2, 0, 0, 0, 0, 0, -1/2, -1/2)	(1/3, 1/3, 1/3, 0, 0, 0, -1/3, -1/3, -1/3)
(1/4, 1/4, 1/4, 0, 0, 0, 0, -1/4, -1/2)	(2/5, 1/5, 1/5, 0, 0, 0, 0, -2/5, -2/5)
(2/9, 2/9, 2/9, 0, 0, 0, 0, -1/3, -1/3)	(2/7, 2/7, 2/7, 0, 0, -1/7, -1/7, -2/7, -2/7)
(1/4, 1/4, 1/4, 0, -1/12, -1/12, -1/12, -1/4, -1/4)	(1/2, 1/4, 0, 0, 0, 0, -1/4, -1/4, -1/4)
(2/5, 2/5, 0, 0, 0, 0, -1/5, -1/5, -2/5)	(1/3, 1/3, 0, 0, 0, 0, -2/9, -2/9, -2/9)
(2/7, 2/7, 1/7, 1/7, 0, 0, -2/7, -2/7, -2/7)	(1/4, 1/4, 1/12, 1/12, 1/12, 0, -1/4, -1/4, -1/4)
(1, 0, 0, 0, 0, 0, 0, 0, -1)	(1/3, 1/3, 0, 0, 0, 0, 0, 0, -2/3)
(1/6, 1/6, 1/6, 0, 0, 0, 0, 0, -1/2)	(1/10, 1/10, 1/10, 1/10, 0, 0, 0, 0, -2/5)
(1/15, 1/15, 1/15, 1/15, 1/15, 0, 0, 0, -1/3)	(1/21, 1/21, 1/21, 1/21, 1/21, 1/21, 0, 0, -2/7)
(2/3, 0, 0, 0, 0, 0, 0, -1/3, -1/3)	(1/2, 0, 0, 0, 0, 0, -1/6, -1/6, -1/6)
(2/5, 0, 0, 0, 0, -1/10, -1/10, -1/10, -1/10)	(1/3, 0, 0, 0, -1/15, -1/15, -1/15, -1/15, -1/15)
(2/7, 0, 0, -1/21, -1/21, -1/21, -1/21, -1/21, -1/21)	(3/11, 1/22, -1/22, -1/22, -1/22, -1/22, -1/22, -1/22, -1/22)
(1/22, 1/22, 1/22, 1/22, 1/22, 1/22, 1/22, -1/22, -3/11)	(1/16, 1/16, 1/16, 1/16, 1/16, 1/16, 0, -1/16, -5/16)
(5/16, 1/16, 0, -1/16, -1/16, -1/16, -1/16, -1/16, -1/16)	(1/12, 1/12, 1/12, 1/12, 1/12, 1/12, -1/12, -1/12, -1/3)
(1/3, 1/12, 1/12, -1/12, -1/12, -1/12, -1/12, -1/12, -1/12)	(1/11, 1/11, 1/11, 1/11, 1/11, 0, 0, -1/11, -4/11)
(1/7, 1/7, 1/7, 1/7, 0, 0, 0, -1/7, -3/7)	(1/13, 1/13, 1/13, 1/13, 1/13, 1/13, 0, -2/13, -4/13)
(1/18, 1/18, 1/18, 1/18, 1/18, 1/18, 1/18, -1/9, -5/18)	(4/11, 1/11, 0, 0, -1/11, -1/11, -1/11, -1/11, -1/11)
(3/7, 1/7, 0, 0, 0, -1/7, -1/7, -1/7, -1/7)	(4/13, 2/13, 0, -1/13, -1/13, -1/13, -1/13, -1/13, -1/13)
(5/18, 1/9, -1/18, -1/18, -1/18, -1/18, -1/18, -1/18, -1/18)	(3/8, 1/8, 1/8, 0, -1/8, -1/8, -1/8, -1/8, -1/8)
(1/8, 1/8, 1/8, 1/8, 1/8, 0, -1/8, -1/8, -3/8)	(1/3, 1/3, 0, 0, 0, -1/6, -1/6, -1/6, -1/6)
(1/4, 1/4, 0, -1/12, -1/12, -1/12, -1/12, -1/12, -1/12)	(1/3, 2/9, 0, 0, -1/9, -1/9, -1/9, -1/9, -1/9)
(1/4, 3/16, -1/16, -1/16, -1/16, -1/16, -1/16, -1/16, -1/16)	(1/6, 1/6, 1/6, 1/6, 0, 0, 0, -1/3, -1/3)
(1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 0, -1/4, -1/4)	(1/9, 1/9, 1/9, 1/9, 1/9, 0, 0, -2/9, -1/3)
(1/16, 1/16, 1/16, 1/16, 1/16, 1/16, 1/16, -3/16, -1/4)	(1/5, 1/10, 1/10, 1/10, 1/10, 0, 0, -3/10, -3/10)
(2/17, 1/17, 1/17, 1/17, 1/17, 1/17, 1/17, -4/17, -4/17)	(3/10, 3/10, 0, 0, -1/10, -1/10, -1/10, -1/10, -1/5)
(4/17, 4/17, -1/17, -1/17, -1/17, -1/17, -1/17, -1/17, -2/17)	(3/11, 3/11, 1/11, -1/11, -1/11, -1/11, -1/11, -1/11, -2/11)
(4/13, 4/13, 0, 0, -1/13, -1/13, -2/13, -2/13, -2/13)	(2/7, 2/7, 1/14, -1/14, -1/14, -1/14, -1/7, -1/7, -1/7)
(1/4, 1/4, -1/20, -1/20, -1/20, -1/20, -1/10, -1/10, -1/10)	(3/10, 1/5, 1/10, -1/10, -1/10, -1/10, -1/10, -1/10, -1/10)
(2/9, 2/9, 2/9, -1/9, -1/9, -1/9, -1/9, -1/9, -1/9)	(2/7, 2/7, 1/7, 0, -1/7, -1/7, -1/7, -1/7, -1/7)
(2/11, 1/11, 1/11, 1/11, 1/11, 1/11, -1/11, -3/11, -3/11)	(2/13, 2/13, 2/13, 1/13, 1/13, 0, 0, -4/13, -4/13)
(1/7, 1/7, 1/7, 1/14, 1/14, 1/14, -1/14, -2/7, -2/7)	(1/10, 1/10, 1/10, 1/20, 1/20, 1/20, 1/20, -1/4, -1/4)
(1/10, 1/10, 1/10, 1/10, 1/10, 1/10, -1/10, -1/5, -3/10)	(1/9, 1/9, 1/9, 1/9, 1/9, 1/9, -2/9, -2/9, -2/9)
(1/7, 1/7, 1/7, 1/7, 1/7, 0, -1/7, -2/7, -2/7)	(2/5, 1/5, 1/5, 0, 0, -1/5, -1/5, -1/5, -1/5)
(5/14, 1/7, 1/7, -1/14, -1/14, -1/14, -1/7, -1/7, -1/7)	(1/5, 1/5, 1/5, 1/5, 0, 0, -1/5, -1/5, -2/5)
(1/7, 1/7, 1/7, 1/14, 1/14, 1/14, -1/7, -1/7, -5/14)	(1/6, 1/6, 1/6, 1/6, -1/12, -1/12, -1/12, -1/12, -1/3)
(2/13, 2/13, 2/13, 2/13, 0, -1/13, -1/13, -1/13, -5/13)	(1/3, 1/12, 1/12, 1/12, 1/12, -1/6, -1/6, -1/6, -1/6)
(5/13, 1/13, 1/13, 1/13, 0, -2/13, -2/13, -2/13, -2/13)	(2/9, 2/9, 1/9, 1/9, 1/9, 0, -2/9, -2/9, -1/3)
(1/3, 1/6, 1/6, 1/6, 0, 0, -1/6, -1/3, -1/3)	(1/4, 1/8, 1/8, 1/8, 1/8, 0, -1/4, -1/4, -1/4)
(2/11, 2/11, 2/11, 2/11, 0, 0, -2/11, -3/11, -3/11)	(1/3, 2/9, 2/9, 0, -1/9, -1/9, -1/9, -2/9, -2/9)
(1/3, 1/3, 1/6, 0, 0, -1/6, -1/6, -1/6, -1/3)	(1/4, 1/4, 1/4, 0, -1/8, -1/8, -1/8, -1/8, -1/4)
(1/4, 1/4, 1/4, 0, 0, -1/8, -1/8, -1/8, -3/8)	(1/4, 1/4, 1/4, 0, 0, -1/6, -1/6, -1/6, -1/4)
(1/6, 1/6, 1/6, 1/6, 1/6, -1/6, -1/6, -1/6, -1/3)	(1/3, 1/6, 1/6, 1/6, -1/6, -1/6, -1/6, -1/6, -1/6)
(1/3, 1/9, 1/9, 1/9, 1/9, -1/9, -2/9, -2/9, -2/9)	(3/8, 1/8, 1/8, 1/8, 0, 0, -1/4, -1/4, -1/4)
(2/9, 2/9, 2/9, 1/9, -1/9, -1/9, -1/9, -1/9, -1/3)	(1/4, 1/4, 1/4, 1/4, 0, -1/4, -1/4, -1/4, -1/4)
(2/7, 2/7, 1/7, 1/7, -1/7, -1/7, -1/7, -1/7, -2/7)	(1/4, 1/4, 1/4, 1/8, -1/8, -1/8, -1/8, -1/4, -1/4)
(2/9, 2/9, 2/9, 2/9, -1/9, -1/9, -2/9, -2/9, -2/9)	(2/7, 1/7, 1/7, 1/7, 1/7, -1/7, -1/7, -2/7, -2/7)
(1/4, 1/4, 1/8, 1/8, 1/8, -1/8, -1/4, -1/4, -1/4)	(2/9, 2/9, 2/9, 1/9, 1/9, -2/9, -2/9, -2/9, -2/9)
(3/10, 1/5, 1/10, 1/10, 1/10, -1/5, -1/5, -1/5, -1/5)	(1/5, 1/5, 1/5, 1/5, -1/10, -1/10, -1/10, -1/5, -3/10)
(2/25, 2/25, 2/25, 2/25, 2/25, 1/25, 1/25, -6/25, -6/25)	(6/25, 6/25, -1/25, -1/25, -2/25, -2/25, -2/25, -2/25, -2/25)
(3/11, 3/11, 1/11, 1/11, 1/11, -2/11, -2/11, -2/11, -3/11)	(3/11, 2/11, 2/11, 2/11, -1/11, -1/11, -1/11, -3/11, -3/11)
(3/10, 3/10, 1/10, 1/10, -1/10, -1/10, -1/5, -1/5, -1/5)	(4/15, 4/15, 1/15, 1/15, -2/15, -2/15, -2/15, -2/15, -2/15)
(1/5, 1/5, 1/5, 1/10, 1/10, -1/10, -1/10, -3/10, -3/10)	(2/15, 2/15, 2/15, 2/15, 2/15, -1/15, -1/15, -4/15, -4/15)
(1/4, 1/6, 1/6, 1/6, 0, 0, -1/4, -1/4, -1/4)	(3/11, 3/11, 2/11, 0, 0, -2/11, -2/11, -2/11, -2/11)
(1/5, 1/5, 2/15, 2/15, 2/15, -1/5, -1/5, -1/5, -1/5)	(1/5, 1/5, 1/5, 1/5, -2/15, -2/15, -2/15, -1/5, -1/5)
(3/14, 3/14, 3/14, 1/7, -1/7, -1/7, -1/7, -1/7, -3/14)	(3/14, 1/7, 1/7, 1/7, 1/7, -1/7, -3/14, -3/14, -3/14)
(3/13, 3/13, 2/13, 2/13, -2/13, -2/13, -2/13, -2/13, -2/13)	(2/13, 2/13, 2/13, 2/13, 2/13, -2/13, -2/13, -3/13, -3/13)
(3/14, 3/14, 3/14, 1/14, -1/7, -1/7, -1/7, -1/7, -1/7)	(1/7, 1/7, 1/7, 1/7, 1/7, -1/14, -3/14, -3/14, -3/14)
(3/16, 3/16, 3/16, 3/16, -1/8, -1/8, -1/8, -1/8, -1/4)	(5/19, 5/19, 0, 0, -2/19, -2/19, -2/19, -2/19, -2/19)
(2/19, 2/19, 2/19, 2/19, 2/19, 0, 0, -5/19, -5/19)	(1/4, 1/8, 1/8, 1/8, 1/8, -3/16, -3/16, -3/16, -3/16)
(5/29, 5/29, 5/29, 5/29, -4/29, -4/29, -4/29, -4/29, -4/29)	(0, 0, 0, 0, 0, 0, 0, 0, 0)
(2/11, 2/11, 2/11, 2/11, -3/22, -3/22, -3/22, -3/22, -2/11)	(2/33, 2/33, 2/33, 2/33, 2/33, 2/33, 2/33, -7/33, -7/33)
(3/17, 3/17, 3/17, 3/17, 0, 0, -4/17, -4/17, -4/17)	(7/33, 7/33, -2/33, -2/33, -2/33, -2/33, -2/33, -2/33, -2/33)
(4/17, 4/17, 4/17, 0, 0, -3/17, -3/17, -3/17, -3/17)	(4/21, 4/21, 4/21, 1/7, -1/7, -1/7, -1/7, -1/7, -1/7)
(2/11, 3/22, 3/22, 3/22, 3/22, -2/11, -2/11, -2/11, -2/11)	(4/29, 4/29, 4/29, 4/29, 4/29, -5/29, -5/29, -5/29, -5/29)
(1/7, 1/7, 1/7, 1/7, 1/7, -1/7, -4/21, -4/21, -4/21)	(2/7, 1/14, 1/14, 1/14, 1/14, 1/14, -3/14, -3/14, -3/14)
(3/14, 3/14, 3/14, -1/14, -1/14, -1/14, -1/14, -1/14, -2/7)	(1/10, 1/10, 1/10, 1/10, 1/10, -1/20, -1/20, -1/20, -7/20)
(7/20, 1/20, 1/20, 1/20, -1/10, -1/10, -1/10, -1/10, -1/10)	(1/4, 0, -1/28, -1/28, -1/28, -1/28, -1/28, -1/28, -1/28)
(2/9, -1/36, -1/36, -1/36, -1/36, -1/36, -1/36, -1/36, -1/36)	(1/28, 1/28, 1/28, 1/28, 1/28, 1/28, 1/28, 0, -1/4)
(1/36, 1/36, 1/36, 1/36, 1/36, 1/36, 1/36, 1/36, -2/9)	(4/17, 4/17, 1/17, 1/17, 1/17, 1/17, -4/17, -4/17, -4/17)
(4/17, 4/17, 4/17, -1/17, -1/17, -1/17, -1/17, -4/17, -4/17)	