FUSION SYSTEMS IN GROUP REPRESENTATION THEORY

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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Results on the Mackey category $\mathcal{M}_{\mathcal{F}}$ corresponding to a fusion system $\mathcal{F}$ and fusion systems defined on p-permutation algebras are our main concern.

In the first part, we give a new proof of semisimplicity of $\mathcal{M}_{\mathcal{F}}$ over $\mathbb{C}$ by using a different method than the method used by Boltje and Danz. Following their work in [8], we construct the ghost algebra corresponding to the quiver algebra of $\mathcal{M}_{\mathcal{F}}$ which is isomorphic to the quiver algebra. We then find a formula for the centrally primitive mutually orthogonal idempotents of this ghost algebra. Then we use this formula to give an alternative proof of semisimplicity of the quiver algebra of $\mathcal{M}_{\mathcal{F}}$ over the complex numbers.

In the second part, we focus on finding classes of p-permutation algebras which give rise to a saturated fusion system which has been studied by Kessar-Kunugi-Mutsihashi in [16]. By specializing to a particular p-permutation algebra and using a result of [16], the question is reduced to finding Brauer indecomposable p-permutation modules. We show for some particular cases of fusion systems we have Brauer indecomposability.

In the last part, we study real representations using the real monomial Burnside ring. We deduce a relation on the dimensions of the subgroup-fixed subspaces of a real representation.

Keywords: fusion system, Mackey category, semisimplicity, p-permutation algebra, Brauer indeomposability, monomial Burnside ring.

ÖZET

GRUP TEMSİL TEORİSİNDE FÜZYON SİSTEMLERİ

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Füzyon sistemleri teorisi grup temsil teorisi alanında önemli bir çalisma alani haline gelmiştir. $\mathcal{F}$ bir füzyon sistemi olsun, $\mathcal{M}_{\mathcal{F}}$ ise bu füzyon sistemine karşılık gelen Mackey kategorisi olsun. Bu Mackey kategorisi ve p-permütasyon cebirlerinin füzyon sistemleri temel ilgi alanımızı oluşturmaktadır.

Tezin ilk bölümünde, $\mathcal{M}_{\mathcal{F}}$ kategorisinin kompleks sayılar üzerinde yarıbasit olduğuna dair olan ispatı Boltje-Danz'ın yaptığından farklı bir şekilde yaptık. [8]'de yapılanları takip ederek, $\mathcal{M}_{\mathcal{F}}$'in quiver cebirine karşılık gelen bir hayalet cebiri oluşturduk. Daha sonra bu hayalet cebirinin, merkezi, birbirine dik, ilkel eşgüçlü elemanları için bir formül bulduk. Bu formülü, $\mathcal{M}_{\mathcal{F}}$'in kompleks quiver cebirinin yarıbasitliğini göstermek için alternatif bir ispat olarak kullandık.

Tezin ikinci bölümünde, doymuş füzyon sistemlerine olanak sağlayan p-permütasyon cebirlerinin bulunması problemine yoğunlaştık. Bu problem, Kessar-Kunugi-Mutsihashi tarafından [16]'de çalışılmıştı. Bu makalede, sözünü ettiğimiz problem Brauer-parçalanamaz özelliğine sahip modüller bulmaya indirgendi. Biz de bazı farklı özel füzyon sistemleri durumunda, Brauer-parçalanamaz modüller bulunduğunu gösterdik.

Son bölümde, gerçel tek terimli Burnside halkasını kullanarak gerçel temsilleri çalıştık. Bir gerçel temsilin altgruplar tarafından sabitlenen alt uzaylarıyla ilgili bir ilişki bulduk.

Anahtar sözcükler: füzyon sistemi, Mackey kategorisi, yarıbasit, p-permütasyon cebiri, Brauer parçalanamazlığı, tek terimli Burnside halkası.

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Contents

Chapter 1

Introduction

The theory of fusion systems became an important topic in the study of representation theory. The term "fusion" was introduced by Richard Brauer in $1950'$s, but the notion of fusion had been of interest before. For example in 1897, Burnside published a paper including the proof of the result that if P is an abelian Sylow p-subgroup of a finite group G, then the normalizer of P in G controls fusion in P. (A subgroup H of G is said to control fusion in P if for any pair of elements in P that are conjugate in G are also conjugate in H.)

In 1990s, Lluis Puig introduced the notion of a fusion system defined on a p-subgroup of G, by discarding G. He gave an axiomatic definition and called them Frobenius categories. Nowadays, these categories are referred to as "saturated fusion systems". Other people have taken up his approach and have extended his axiomatic definition to fusion systems.

This thesis is mainly based on results related to fusion systems. The first and third part consist of results related to representation theory in characteristic zero, whereas the second part contains results related to modular representation theory.

1.1 On Mackey category corresponding to a fusion system

The theory of Mackey functors is an important theory in the study of representations of finite groups. Representation rings, group cohomology, Burnside rings are some important Mackey functors.

Mackey functors are introduced by Green in early seventies. Then many mathematicians including Boltje, Bouc, Dress, Thévenaz and Webb become interested in this theory. Thévenaz and Webb identified Mackey functors with modules of a certain algebra in [24]. Later, Bouc gave an alternative definition for Mackey functors in terms of additive functors from a suitable category which we define below briefly.

Let $\mathcal{P}$ be a set of finite groups closed under taking subgroups up to isomorphism. Following Bouc [9], the category $\mathcal{M}^{\mathcal{P},\triangle}$ is defined where

- $\mathrm{Obj}(\mathcal{M}^{\mathcal{P},\triangle}) = \mathcal{P}$

- Given $P, Q \in \mathcal{P}$, $\mathrm{Hom}_{\mathcal{M}^{\mathcal{P},\triangle}}(P,Q) = B^{\triangle}(P,Q)$ where $B^{\triangle}(P,Q)$ is the Grothendieck group of bifree P-Q-bisets (we call this group the bifree double Burnside group). For details, see Chapter 2.

A Mackey functor on $\mathcal{P}$ is an additive functor from the category $\mathcal{M}^{\mathcal{P},\triangle}$ to the category of left $\mathbb{Z}$-modules. We can extend the coefficients to $\mathbb{C}$ in a straightforward way as explained in Chapter 4. In the case where we have a fusion system defined on $\mathcal{P}$, the results in [8] imply semisimplicity of the corresponding category denoted by $\mathcal{M}^{\mathcal{P},\triangle_{\mathcal{F}}}$ ($\mathrm{Hom}_{\mathcal{M}^{\mathcal{P},\triangle}}(P,Q) = B^{\triangle_{\mathcal{F}}}(P,Q)$) which we denote by $\mathcal{M}_{\mathcal{F}}$ for short. Díaz and Park, in [14], gave a parametrization and an explicit description for the simple Mackey functors for a fusion system in terms of seeds.

In Chapter 4, Theorem 4.1.1, we show semisimplicity of $\mathcal{M}_{\mathcal{F}}$ over $\mathbb{C}$ and hence semisimplicity of the quiver algebra by using a different method than the method used by Boltje and Danz. Following [8], we construct the ghost algebra corresponding to the quiver algebra of $\mathcal{M}_{\mathcal{F}}$ which is isomorphic to the quiver algebra. We then find a formula for the centrally primitive mutually orthogonal idempotents of this ghost algebra. Then we use this formula to give an alternative proof of semisimplicity.

1.2 On fusion systems defined on p-permutation algebras

Let G be a finite group, p a prime number dividing the order of G and k a field of characteristic p. In $1930'$s, Richard Brauer initiated the systematic study of the representations of G over k. In contrast to $\mathbb{C}G$, the modular group algebra is not a direct sum of simple algebras; whereas the indecomposable subalgebras of kG called blocks has a rich representation theory.

In 1979, Alperin and Broué, in [1], introduced the G-poset of Brauer pairs corresponding to a block b of kG. This G-poset consists of pairs (Q, e) where Q is a p-subgroup of G and e is a block of $kC_G(Q)$ in Brauer correspondence with b. They showed that there is a G-conjugation structure on the set of Brauer pairs which has similar properties with the G-poset of p-subgroups of G. These similarities led Puig to introduce a fusion pattern on an abstractly defined category. For any maximal b-Brauer pair (P, e), the category $\mathcal{F}$ is defined to be a category whose objects are subgroups of P and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ consist of morphisms induced by conjugations in the G-subposet of Brauer pairs contained in (P, e). Alperin and Broué's results can be interpreted as a statement that a fusion system defined on a maximal b-Brauer pair is saturated.

More generally, the theory of Brauer pairs can be extended to to primitive idempotents of G-fixed subalgebras of p-permutation algebras. As in the group algebra case, for a p-permutation G-algebra A and a primitive idempotent b of the subalgebra of fixed points A^G, there is associated a fusion system defined on a maximal (A, b, G)-Brauer pair. These fusion systems are not always saturated. In [16], a sufficient condition for saturation is given, it is a condition that suggests the triple (A, b, G) to be saturated. Hence, having found a saturated triple, we have a saturated fusion system. For the definition of saturated triples see Chapter 5.

Finding classes of p-permutation algebras which give rise to a saturated fusion system has been studied by Kessar-Kunugi-Mutsihashi in [16]. They posed the following question:

Given a saturated fusion system $\mathcal{F}$ on a finite p-group P, does there exist a finite group G, a p-permutation G-algebra A and a primitive idempotent b of A^G such that (A, b, G) is a saturated triple and $\mathcal{F} = \mathcal{F}_{(P,e_P)}(A, b, G)$ for some maximal (A, b, G)-Brauer pair (P, e_P)?

In the same paper, they construct a p-permutation G-algebra $A = \mathrm{End}_k(M)$ where M is an indecomposable p-permutation kG-module, and establish a necessary and sufficient condition for the triple $(A, 1_A, G)$ to be saturated. The condition implies that M is Brauer indecomposable. Moreover, they suggest that a good candidate for M is a Scott kG-module with vertex P. They prove Brauer indecomposability of Scott kG-modules with vertex P when P is an abelian p-group and $\mathcal{F}$ is a saturated fusion system on P.

In Chapter 5, Theorems 5.3.2 and 5.3.6, we prove Brauer indecomposability of Scott modules for some other particular cases where P is not necessarily abelian. Hence, for some new classes of saturated fusion systems $\mathcal{F}$, we have proved an affirmative answer to the question above and we have exhibited some saturated triples for $\mathcal{F}$.

1.3 On real representation spheres and real monomial Burnside ring

This chapter focuses on real representations, or equivalently finite dimensional $\mathbb{R}G$-modules. We deduce a relation on the dimensions of the subgroup-fixed subspaces of them using real monomial Burnside rings as well as Lefschetz invariant of spheres of real representations.

For a finite group G, the ordinary Burnside ring $B(G)$ is defined to be the $\mathbb{Z}-$module having a basis $\{[\frac{G}{H}] \mid H \leq_G G\}$ where addition is disjoint union and multiplication is Cartesian product. The real monomial Burnside ring $B_{\mathbb{R}}(G)$, or the monomial Burnside ring with fibre group $\{\pm 1\}$, is the $\mathbb{Z}-$module having a basis set as isomorphism classes of $\{\pm 1\}-$subcharacters of G. There exists a ghost ring $\beta(G)$ of the Burnside ring such that the algebras $\mathbb{Q}B(G)$ and $\mathbb{Q}\beta(G)$ become isomorphic. For detailed explanation on them, see Section 6.1.

There is a $\mathbb{Q}$-linear map $\overline{\mathrm{bol}}_G : A_{\mathbb{R}}(G) \to \beta^{\times}(G)$ where $A_{\mathbb{R}}(G)$ denotes real representation ring for G. This map happens to be modulo 2 reduction of the map $\mathrm{bol}_G^{\{\pm 1\},\mathbb{R}} : \mathbb{R}A_{\mathbb{R}}(G) \to \mathbb{R}B_{\mathbb{R}}(G)$ (see the paragraph before Theorem 6.3.4). In Theorem 6.1.1, we deduce a relation for the image of restriction of the map $\overline{\mathrm{bol}}_G$ to the subalgebra $\mathbb{Z}_{(2)}A_{\mathbb{R}}(G)$. We use the theory of Lefschetz invariants corresponding to an $\mathbb{R}G$-module together with the properties of group of the units of Burnside ring to prove this theorem.

Let $O(G)$ denote the smallest normal subgroup of G such that $G/O(G)$ is an elementary abelian 2-group and $O^2(G)$ denote the smallest normal subgroup of G such that $G/O^2(G)$ is a 2 group. Using Theorem 6.1.1 and Dress's characterization for the idempotents in $\mathbb{Q}B(G)$, we deduce the result on modulo 2 equivalence between the dimensions of $O(G)$ and $O^2(G)$-fixed subspaces of an $\mathbb{R}G$-module. This is stated in Theorem 6.1.2.

For the particular case when G is a 2-group, using a theorem of Tornehave we deduce a result which gives a constraint on the units of the Burnside ring $B(G)$. This is given in Theorem 6.1.3.

Chapter 2 and 3 contain the background that is needed to state the results of the remaining chapters.

Chapter 2

Fusion systems and bisets

In this chapter, we introduce fusion systems, the concept of bisets and Burnside rings. Further, we recall the concept of characteristic bisets which unifies the theory of bisets and saturated fusion systems. The theory of characteristic bisets led Park to introduce Park groups in [19]. The last part of this chapter is on Park groups.

2.1 Fusion systems

Let $\mathcal{P}$ be a set of finite groups closed under taking subgroups up to isomorphism. A **fusion system** $\mathcal{F}$ on $\mathcal{P}$ is defined to be a category where

- $\mathrm{Obj}(\mathcal{F}) = \mathcal{P}$
- Given $P, Q \in \mathcal{P}$, $\mathrm{Hom}_{\mathcal{F}}(Q, P)$ satisfies the following axioms:

A1. Every morphism in $\mathrm{Hom}_{\mathcal{F}}(Q, P)$ is an injective group homomorphism.

A2. For every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$, we have $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, \varphi(Q))$ as well as $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(Q), Q)$.

A3. For $Q \leq P$ and $u \in P$, then $c_u : Q \to P$ such that $c_u(v) = {}^{u}v$ is in

$\mathrm{Hom}_{\mathcal{F}}(Q, P)$. The composition of morphisms in $\mathcal{F}$ is the usual composition of functions.

Let P be a finite group. If $\mathcal{P}$ is the set of all subgroups of P, then we will say that $\mathcal{F}$ is a fusion system defined on P.

Let $\mathcal{F}$ be a fusion system defined on a finite group P. A subgroup Q of P is called **fully $\mathcal{F}$-centralized** if for every $R \leq P$ with $R =_{\mathcal{F}} Q$ we have $|C_P(R)| \leq |C_P(Q)|$. A subgroup Q of P is called **fully $\mathcal{F}$-normalized** if for every $R \leq P$ with $R =_{\mathcal{F}} Q$ we have $|N_P(R)| \leq |N_P(Q)|$. For any morphism $\varphi : Q \to P$ in $\mathcal{F}$, set the subgroup N_φ as

$$N_\varphi = \{u \in N_P(Q) \mid \exists y \in N_P(\varphi(Q)) \text{ such that } \varphi({}^u v) =^y \varphi(v) \text{ for all } v \in Q\}.$$

Among the fusion systems, there is an interesting class of fusion systems called saturated fusion systems. Let P be a finite p-group. There are equivalent definitions for saturated fusion systems. In [11], Definition 1.3, the following definition is given. A fusion system $\mathcal{F}$ on P is called a **saturated fusion system**, if the following axioms are satisfied:

(Sylow) $\mathrm{Aut}_P(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(Extension) Every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ such that $\varphi(Q)$ is fully $\mathcal{F}$-centralized extends to a morphism $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, P)$.

2.2 Bisets

A P-Q-**biset** is a set X equipped with a left P-action and a right Q-action such that

$$u.(x.v) = (u.x).v$$

for all elements $u \in P$, $v \in Q$ and $x \in X$. A P-Q-biset X is called **transitive** if for any elements $x, y \in X$, there exists an element $u \in P$ and an element $v \in Q$ such that $y = u.x.v$. Every P-Q-biset can be regarded as a $P \times Q$-set via

$$(u, v).x := u.x.v^{-1}$$

for all $u \in P, v \in Q$ and $x \in X$. Hence there is a bijective correspondence between

- the set of isomorphism classes of transitive P-Q-bisets, and
- the set of conjugacy classes of the subgroups of $P \times Q$.

Here the correspondence is given by $[X] \leftrightarrow [L]$ if and only if the stabilizer of a point $x \in X$ is $P \times Q$-conjugate to L (Here $[X]$ denotes the isomorphism class of X and $[L]$ denotes the conjugacy class of L).

Recall that, for a finite group G, the **Burnside group** $B(G)$ is defined as the $\mathbb{Z}$-module spanned by the isomorphism classes of transitive G-sets. Similarly, the **double Burnside group** $B(P, Q)$ is defined as the $\mathbb{Z}$-module spanned by the isomorphism classes of transitive P-Q-bisets. By the bijective correspondence above, we can equivalently define $B(P, Q)$ as a $\mathbb{Z}$-module having a basis

$$\left\{ \left[\frac{P \times Q}{L} \right] \mid L \in \mathcal{L} \right\}$$

where $\mathcal{L}$ denotes the set of conjugacy classes of the subgroups of $P \times Q$. This is a group under disjoint union of bisets.

Let $p_1 : P \times Q \to P$ and $p_2 : P \times Q \to Q$ denote the canonical projections, for $L \leq P \times Q$, set

$$k_1(L) = \{u \in P \mid (u, 1) \in L\} \text{ and } k_2(L) = \{v \in Q \mid (1, v) \in L\}.$$

Then $k_i(L) \trianglelefteq p_i(L)$ for $i = 1, 2$.

A P-Q-biset is called **left-free** if the left P-action is free and **right-free** if the right Q-action is free and **bifree** if both of the actions on either sides are free.

We have the following lemma whose proof is clear from definitions.

Lemma 2.2.1. *A P-Q-biset X is left-free if and only if $k_1(\mathrm{stab}_{P\times Q}(x)) = 1$ for all $x \in X$, and X is right-free if and only if $k_2(\mathrm{stab}_{P\times Q}(x)) = 1$ for all $x \in X$. Thus X is bifree if and only if $k_1(\mathrm{stab}_{P\times Q}(x)) = k_2(\mathrm{stab}_{P\times Q}(x)) = 1$ for all $x \in X$.*

As a consequence of that lemma, we have a bijective correspondence between

- the set of isomorphism classes of bifree and transitive P-Q-bisets, and

- the set of conjugacy classes of the subgroups L of $P \times Q$ subject to the property that $k_1(L) = k_2(L) = 1$.

The **bifree double Burnside group** $B^{\triangle}(P, Q)$ is defined as the $\mathbb{Z}$-module spanned by the isomorphism classes of transitive bifree P-Q-bisets. By the bijective correspondence above we can equivalently define $B^{\triangle}(P, Q)$ as the $\mathbb{Z}$-module having the basis set

$$\left\{ \left[\frac{P \times Q}{L}\right] \mid L \in \mathcal{L},\ k_1(L) = k_2(L) = 1 \right\}$$

where $\mathcal{L}$ denotes the set of conjugacy classes of the subgroups of $P \times Q$. Observe that $B^{\triangle}(P, Q) \leq B(P, Q)$.

Let X be a P-Q-biset, Y be a Q-R-biset. Then we define the **Mackey product** $X \times_Q Y$ as the set of Q-orbits of the cartesian product $X \times Y$. Here Q acts via $v.(x, y) := (x.v^{-1}, v.y)$ and we write $(x,_Q y)$ to denote an arbitrary element in $X \times_Q Y$. The set $X \times_Q Y$ is a P-R-biset via

$$u.(x,_Q y).r := (u.x,_Q y.r).$$

This product induces bilinear maps

$$B(P, Q) \times B(Q, R) \to B(P, R) \text{ and } B^{\triangle}(P, Q) \times B^{\triangle}(Q, R) \to B^{\triangle}(P, R).$$

Observe that $B(P, P)$ and $B^{\triangle}(P, P)$ become rings under this product.

2.3 Characteristic bisets

There is a close relationship between saturated fusion systems defined on a p-group P and special type of bisets called characteristic bisets lying in the bifree Burnside ring $B^{\triangle}(P, P)$.

To introduce the theory of characteristic bisets, we need to fix some notation. For a subgroup $Q \leq P$, and a group homomorphism $\varphi : Q \to P$, let

$$P \times_{(Q,\varphi)} P = (P \times P)/\sim$$

where $(x\varphi(u), y) \sim (x, uy)$ for $x, y \in P$ and $u \in Q$. Let $< x, y >$ denote the equivalence class of (x, y) under $\sim$. We can view this set as a P-P-biset via

$$p < x, y >=< px, y > \text{ and } < x, y > p =< x, yp >$$

for $x, y, p \in P$. This set is free on the left and it is free on the right if φ is injective. Furthernore, there is a P-P-biset isomorphism

$$P \times_{(Q,\varphi)} P \simeq (P \times P)/\triangle(\varphi(Q), \varphi, Q)$$

where $\triangle(\varphi(Q), \varphi, Q) = \{(\varphi(v), v) \mid v \in Q\}$.

For a P-P-biset X, and a group homomorphism $\varphi : Q \to P$, let ${}_QX$ denote the Q-P-biset obtained from X by restricting the left P-action to Q and ${}_\varphi X$ denote the Q-P-biset obtained from X where the left Q-action is induced by φ. Broto-Levi-Oliver show every saturated fusion system defined on a p-group has a characteristic biset.

Theorem 2.3.1 ([11], Proposition 5.5)**.** *For any saturated fusion system $\mathcal{F}$ on a finite p-group P, there is a P-P-biset X with the following properties:*

(i) Each transitive subbiset of X is of the form $P \times_{(Q,\varphi)} P$ for some $Q \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$.

(ii) For each $Q \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$, ${}_QX$ and ${}_\varphi X$ are isomorphic as Q-P-bisets.

(iii) $\frac{|X|}{|P|} \not\equiv 0 \bmod p$.

A biset satisfying the three conditions in the theorem above is called **a characteristic biset corresponding to** $\mathcal{F}$. The properties above were formulated by Linckelmann and Webb in an unpublished work.

Ragnarsson-Stancu showed that given such a P-P-biset we can recover a saturated fusion system $\mathcal{F}$ (see [20]). In fact, it was shown that there is a bijection between the set of saturated fusion systems defined on a p-group P and the set of characteristic idempotents in $\mathbb{Z}_{(p)}B^{\triangle}(P, P)$.

2.4 Park groups

In [19], Park constructs a finite group such that a given saturated fusion system can be realized by that group. He uses the characteristic biset as a tool in the construction of that group.

Let G be a finite group and P be a p-subgroup of G. We denote by $\mathcal{F}_P(G)$ the fusion system on P whose morphism set is

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \{\varphi : Q \to R \mid \exists g \in G \text{ s.t. } \varphi(q) = gqg^{-1}\ \forall q \in Q\}$$

for every $Q, R \leq P$. It is known that when $P \in \mathrm{Syl}_p(G)$, the fusion system $\mathcal{F}_P(G)$ is saturated.

Theorem 2.4.1 ([19], Theorem 3)**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, X be a characteristic biset corresponding to $\mathcal{F}$. Let $Q \leq P$ and let $\varphi : Q \to P$ be an injective group homomorphism. Then the following are equivalent:*

(i) *φ is a morphism in $\mathcal{F}$.*

(ii) *The Q-P-bisets ${}_QX$ and ${}_\varphi X$ are isomorphic.*

(iii) *φ is a morphism in $\mathcal{F}_{\varrho(P)}(G)$, where $G = \mathrm{Aut}({}_1X)$, that is, the group of bijections preserving the right P-action and P is identified with a subgroup of G via*

$$P \xrightarrow{\varrho} \mathrm{Aut}({}_1X)$$
$$p \mapsto (x \mapsto px).$$

(iv) *The fixed point set $X^{\triangle(\varphi(Q),\varphi,Q)} \neq \emptyset$.*

We call the group G that makes $\mathcal{F} = \mathcal{F}_{\varrho(P)}(G)$, **a Park group of** $\mathcal{F}$. Since Park group depends both on the fusion system and the characteristic biset X corresponding to it, we will use the notation $\mathrm{Park}(\mathcal{F}, X)$ to denote this group.

Remark 2.4.2. In the theorem above, since ${}_1X$ is a right-free P-set, the automorphism group $\mathrm{Aut}({}_1X) \cong P \wr S_n$ for $n = \frac{|X|}{|P|}$, thus $G \cong P \wr S_n$.

Chapter 3

Scott Modules

In this section, we give the definition of a Scott module and quote results about basic properties of it. We use [18] as a reference for this chapter.

3.1 Relative trace maps

Let G be a finite group and k be a commutative ring with identity. For a kG-module M and $Q \leq H \leq G$, the **relative trace map** is the map

$$\mathrm{Tr}_Q^H : M^Q \to M^H$$

such that $\mathrm{Tr}_Q^H(m) = \sum_{h \in H/Q} hm$ for $m \in M^Q$. We set $M_Q^H = \mathrm{Tr}_Q^H(M^Q)$. The **Brauer quotient** is the quotient

$$M(H) = M^H / (\sum_{Q<H} M_Q^H)$$

and Br_H denotes the canonical homomorphism from M^H to $M(H)$. Conjugation by $g \in G$ induces a kG-module isomorphism between $M(H)$ and $M({}^gH)$. So if $M(H) \neq 0$, then $M(H)$ inherits a natural $kN_G(H)$-module-structure. Since H acts trivially on M^H, we can also view $M(H)$ as a $kN_G(H)/H$-module.

Let M and M' be kG-modules. Then, the set $\mathrm{Hom}_k(M, M')$ becomes a kG-module via

$$^g f(m) := gf(g^{-1}m).$$

Moreover, $\mathrm{Hom}_k(M, M')^H = \mathrm{Hom}_{kH}(M, M')$ for all $H \leq G$. Thus for $Q \leq H \leq G$, the relative trace map becomes

$$\mathrm{Tr}_Q^H : \mathrm{Hom}_{kQ}(M, M') \to \mathrm{Hom}_{kH}(M, M')$$

$\mathrm{Tr}_Q^H(f)(m) = \sum_{h \in H/Q} hf(h^{-1}m)$.

For a G-algebra A over k, relative trace maps, Brauer homomorphism and Brauer quotient are defined in a similar way.

3.2 Relative projectivity

From now on, we will study with a p-modular system. Let p be a prime number. A p**-modular system** is a triple $(K, \mathcal{O}, k)$ where $\mathcal{O}$ is a local principal ideal domain, K is the field of quotients of $\mathcal{O}$ and k is the quotient field $\mathcal{O}/J(\mathcal{O})$ such that the following hold:

- $\mathcal{O}$ is complete with respect to the natural topology induced by its unique maximal ideal $J(\mathcal{O})$.
- K has characteristic 0.
- k has characteristic p.

For a finite group G, in order to avoid complications arising from rationality considerations, we assume that $\mathcal{O}$ is such that K contains a primitive $|G|$th root of unity and k is algebraically closed. We assume also that all kG-modules that we are dealing with are finite dimensional.

For $H \leq G$, a kG-module M is called **relatively H-projective** if there is a kH-module W such that M is a direct summand of $\mathrm{Ind}_H^G W$ and write $M \mid \mathrm{Ind}_H^G W$.

Observe that, the definition of projectivity for kG-modules coincides with $\{1\}$-projectivity.

The relationship between relative projectivity and trace map is given by Higman as follows:

Theorem 3.2.1 (Higman, [15])**.** *Let M be a kG-module and $H \leq G$. Then the following are equivalent:*

(i) M *is relatively* H*-projective.*

(ii) $M \mid \mathrm{Ind}_H^G \mathrm{Res}_H^G M$.

(iii) The identity map on M *is in the image of* $\mathrm{Tr}_H^G : \mathrm{Hom}_{kH}(M, M) \to \mathrm{Hom}_{kG}(M, M)$.

As a corollary of Higman's theorem, we have the following remark.

Remark 3.2.2. If $(|G : H|, p) = 1$, then every kG-module M is relatively H-projective. In particular, M is relatively P-projective if $P \in \mathrm{Syl}_p(G)$.

Proof. Since $|G : H|$ is a unit in k, we have $\mathrm{id}_M = \mathrm{Tr}_H^G(|G : H|^{-1}\mathrm{id}_M)$. □

Theorem 3.2.3 ([18], Theorem 4.3.3)**.** *If M is an indecomposable kG-module, then there exists a p-subgroup of G determined up to G-conjugacy such that the following statements hold:*

(i) M *is relatively* P*-projective.*

(ii) If M *is relatively* H*-projective for some* $H \leq G$*, then* $P \leq_G H$.

Any p-subgroup P of G satisfying the two conditions in the theorem above is called **a vertex of** M.

3.3 Scott modules as p-permutation modules

A kG-module M is called **a p-permutation module** if for every p-subgroup P of G, there exists a k-basis of M which is stabilized by P.

Lemma 3.3.1 ([12], Theorem 3.1)**.** *Let M be a $p-$permutation kG-module, and let P be a p-subgroup of G. Then $M(P)$ is a p-permutation $kN_G(P)/P$-module.*

Theorem 3.3.2 ([12], Theorem 3.2)**.** *Let G be a finite group. Then the following statements are true.*

1. *The vertices of an indecomposable p-permutation kG-module M are the maximal p-subgroup P such that $M(P) \neq 0$.*
2. *An indecomposable p-permutation kG-module M has vertex P if and only if $M(P)$ is a nontrivial projective $kN_G(P)/P$-module.*
3. *The correspondence $M \to M(P)$ induces a bijection between the isomorphism classes of indecomposable p-permutation kG-modules with vertex P and the isomorphism classes of indecomposable projective $kN_G(P)/P$-modules.*

The definition of a Scott module is given by the following theorem.

Theorem 3.3.3 (Scott-Alperin)**.** *For a p-subgroup P of G, there exists an indecomposable p-permutation kG-module with vertex P denoted by $S_P(G,k)$, uniquely defined up to isomorphism by the following equivalent properties:*

(i) $k_G \mid \mathrm{soc}(S_P(G,k))$*(:= largest semisimple submodule of $S_P(G,k)$).*

(ii) $k_G \mid \mathrm{hd}(S_P(G,k))$*(:= largest semisimple quotient of $S_P(G,k)$).*

where k_G denotes the trivial kG-module. Moreover, $S_P(G,k)$ is isomorphic to its dual, and is a direct summand of $\mathrm{Ind}_H^G k$ if and only if P is G-conjugate to a Sylow p-subgroup of H.

As a corollary of the last two theorems we have the following.

Corollary 3.3.4. *$S_P(G,k)(P)$ is the projective cover of the trivial $kN_G(P)/P$-module.*

Chapter 4

On Mackey category corresponding to a fusion system

4.1 The category and its quiver algebra

In this chapter, we will present the Mackey category corresponding to a fusion system. We will introduce the quiver algebra coming out of this category and prove the semisimplicity of this algebra by finding the set of centrally primitive mutually orthogonal idempotents of the ghost algebra of the quiver algebra which is isomorphic to the quiver algebra.

Let $\mathcal{F}$ be an arbitrary fusion system defined on $\mathcal{P}$. We define **Mackey category corresponding to $\mathcal{F}$,** which is denoted by $\mathcal{M}_{\mathcal{F}}$, to be a category where

- $\text{Obj}(\mathcal{M}_{\mathcal{F}}) = \mathcal{P}$,

- Given $P, Q \in \mathcal{P}$, $\text{Hom}_{\mathcal{M}_{\mathcal{F}}}(P, Q) = B^{\triangle_{\mathcal{F}}}(P, Q)$. Here $B^{\triangle_{\mathcal{F}}}(P, Q)$ is the $\mathbb{Z}$-submodule of $B^{\triangle}(P, Q)$ having a basis consisting of elements of the form $[\frac{P\times Q}{\triangle(U,\varphi,V)}]$ where $V \leq Q$, $U \leq P$ and $\varphi : V \to U$ is an isomorphism in $\mathcal{F}$ and

$$\triangle(U, \varphi, V) = \{(\varphi(v), v) \mid v \in V\}.$$

• Composition of morphisms in $\mathcal{M}_{\mathcal{F}}$ is induced by Mackey product of bisets and composition of incompatible morphisms are defined to be zero.

We define the category $\mathbb{C}\mathcal{M}_{\mathcal{F}}$ where objects are the same as objects of $\mathcal{M}_{\mathcal{F}}$ and $\mathrm{Hom}_{\mathbb{C}\mathcal{M}_{\mathcal{F}}}(Q,P) = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{M}_{\mathcal{F}}(P,Q)$ and composition of morphisms are given by $\mathbb{C}$-linear extension of the composition of morphisms of $\mathcal{M}_{\mathcal{F}}$.

The **quiver ring** ${}^{\oplus}\mathcal{M}_{\mathcal{F}}$ is defined as

$$^{\oplus}\mathcal{M}_{\mathcal{F}} = \bigoplus_{P,Q \in \mathcal{P}} \mathcal{M}_{\mathcal{F}}(P,Q)$$

where multiplication is induced from the composition of morphisms in the category $\mathcal{M}_{\mathcal{F}}$. We will be concerned about the quiver algebra ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}} := \mathbb{C} \otimes_{\mathbb{Z}} {}^{\oplus}\mathcal{M}_{\mathcal{F}}$. The main aim of this chapter is to prove the following theorem by a different method than they use.

Theorem 4.1.1. (Boltje-Danz, [8]) *The algebra* ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$ *is semisimple.*

4.2 Parametrization of simple ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$-modules

In [14], Díaz and Park gives the parametrization of simple Mackey functors for fusion systems. They prove that there is a one-to-one correspondence between the set of simple Mackey functors over $\mathbb{C}$ defined for a fusion system $\mathcal{F}$ and the equivalence classes of seeds of $\mathcal{F}$ over $\mathbb{C}$.

A **seed of** $\mathcal{F}$ **over** $\mathbb{C}$ is defined to be a pair (K, χ) where $K \in \mathcal{P}$ and $\chi \in \mathrm{Irr}(\mathbb{C}\mathrm{Out}_{\mathcal{F}}(K))$, for $\mathrm{Out}_{\mathcal{F}}(K) = \mathrm{Aut}_{\mathcal{F}}(K)/\mathrm{Inn}(K)$. There is an equivalence relation defined on the set of all seeds, namely, two seeds (K, χ) and (K', χ') are **equivalent** provided there exists an isomorphism $\phi : K \to K'$ in $\mathcal{F}$ such that $\chi'(\phi\tau\phi^{-1}) = \chi(\tau)$ for all $\tau \in \mathrm{Out}_{\mathcal{F}}(K)$. Let Ω denote the set of equivalence classes of seeds of $\mathcal{F}$ over $\mathbb{C}$.

We define **a Mackey functor for a fusion system** $\mathcal{F}$ **over** $\mathbb{C}$ to be a $\mathbb{C}$-linear functor from the category $\mathbb{C}\mathcal{M}_{\mathcal{F}}$ to the category $\mathbb{C}$-Mod of $\mathbb{C}$-modules and refer to it as an $\mathcal{M}_{\mathcal{F}}$-functor.

Remark 4.2.1. $\mathcal{M}_{\mathcal{F}}$-functors can be regarded as modules of the quiver algebra ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$. The correspondence sends an $\mathcal{M}_{\mathcal{F}}$-functor F to the ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$-module $\oplus_{Q\in\mathcal{P}}F(Q)$, where the action of a P-Q-biset ${}_PX_Q$ on the summand $F(Q)$ is given by $F({}_PX_Q)$ and zero on the other summands. Conversely, for each $Q\in\mathcal{P}$ there is an idempotent Q-Q-biset ${}_QQ_Q$ and for a ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$-module A, F defines an $\mathcal{M}_{\mathcal{F}}$-functor for $F(Q) :=_Q Q_Q A$.

From Proposition 3.1 of [14] we deduce a correspondence between

- the set of simple $\mathcal{M}_{\mathcal{F}}$-functors, and
- the elements in Ω.

Therefore from Remark 4.2.1, we deduce that there is a bijective correspondence between

- the set of simple ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$-modules, and
- the elements in Ω.

4.3 The ghost algebra

For the construction of the ghost algebra, we follow Boltje-Danz's construction introduced in [8]. The only difference is that they introduce the algebra for more general categories, we are specializing to fusion systems.

For $P,Q\in\mathcal{P}$, and a fusion system on $\mathcal{P}$,

$$\triangle_{\mathcal{F}}(P,Q)=\{(U,\alpha,V)\mid U\le P,\ V\le Q,\ \alpha:V\to U\}$$

For each triple $(U,\alpha,V)\in\triangle_{\mathcal{F}}(P,Q)$, we introduce the elements $e_{P\times Q}(U,\alpha,V)$ where the group $P\times Q$ acts on them as

$$^{(x,y)}e_{P\times Q}(U,\alpha,V) := e_{P\times Q}({}^xU, c_y\alpha c_{x^{-1}}, {}^yV)$$

for $x\in P$ and $y\in Q$. Let us introduce the free $\mathbb{Z}$-module $\widehat{\mathcal{M}_{\mathcal{F}}}(P,Q)$ with a free

$\mathbb{Z}$-basis $\{e_{P\times Q}(U,\alpha,V) \mid (U,\alpha,V)\in \triangle_{\mathcal{F}}(P,Q)\}$, that is

$$\widehat{\mathcal{M}_{\mathcal{F}}}(P,Q) = \bigoplus_{(U,\alpha,V)\in\triangle_{\mathcal{F}}(P,Q)} \mathbb{Z}\ e_{P\times Q}(U,\alpha,V)$$

The direct sum ${}^{\oplus}\widehat{\mathcal{M}_{\mathcal{F}}} := \bigoplus_{P,Q\in\mathcal{P}} \widehat{\mathcal{M}_{\mathcal{F}}}(P,Q)$ happens to be a ring via the multiplication

$$e_{P\times Q}(U,\alpha,V)\, e_{Q'\times R}(V',\beta,W) = \begin{cases} \frac{|C_Q(V)|}{|Q|}\ e_{P\times R}(U,\alpha\beta,W), & \text{if } Q=Q' \text{ and } V=V' \\ 0, & \text{otherwise.} \end{cases}$$

The **ghost ring** is defined to be

$${}^{\oplus}\widetilde{\mathcal{M}_{\mathcal{F}}} := \bigoplus_{P,Q\in\mathcal{P}} \widehat{\mathcal{M}_{\mathcal{F}}}(P,Q)^{P\times Q}.$$

Setting

$$\tilde{e}_{P\times Q}(U,\alpha,V) := \sum_{(x,y)} {}^{(x,y)}e_{P\times Q}(U,\alpha,V)$$

where the sum runs through the set of equivalence classes of the stabilizers of the orbits of $e_{P\times Q}(U,\alpha,V)$, then $P\times Q$ fixes $\tilde{e}_{P\times Q}(U,\alpha,V)$ for each $(U,\alpha,V)\in \triangle_{\mathcal{F}}(P,Q)$. Hence, if we let

$$\widetilde{\mathcal{M}_{\mathcal{F}}}(P,Q) := \bigoplus_{(U,\alpha,V)\in_{P\times Q}\triangle_{\mathcal{F}}(P,Q)} \mathbb{Z}\ \tilde{e}_{P\times Q}(U,\alpha,V)$$

we can interpret the ghost ring as

$${}^{\oplus}\widetilde{\mathcal{M}_{\mathcal{F}}} = \bigoplus_{P,Q\in\mathcal{P}} \widetilde{\mathcal{M}_{\mathcal{F}}}(P,Q).$$

We will be working with the complex algebra ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}} := \mathbb{C}\otimes_{\mathbb{Z}}\ {}^{\oplus}\widetilde{\mathcal{M}_{\mathcal{F}}}$.

To relate the ghost algebra ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$ and the quiver algebra ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$, we define the mark map

$$\rho_{P,Q} : \mathbb{C}\mathcal{M}_{\mathcal{F}}(P,Q) \to \mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}(P,Q)$$

to be the linear map defined for a P-Q-biset X, then

$$\rho_{P,Q}[X] = \sum_{(U,\alpha,V)\in_{P\times Q}\triangle_{\mathcal{F}}(P,Q)} \frac{|X^{\triangle(U,\alpha,V)}|}{|C_P(U)|}\ \tilde{e}_{P\times Q}(U,\alpha,V).$$

Letting P and Q run over all objects of $\mathcal{P}$, we obtain a $\mathbb{C}$-linear map

$$\rho : {}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}} \to {}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}.$$

Theorem 4.3.1 ([8], Theorem 4.7)**.** *The map ρ is an isomorphism of $\mathbb{C}$-algebras.*

We aim to show semisimplicity of ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$. The method we will use here is to compute mutually orthogonal centrally primitive idempotents of the ghost algebra ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$ and then using mark isomorphism we can conclude semisimplicity.

4.4 Abelian Case

In this section, we concentrate on the fusion systems $\mathcal{F}$ on $\mathcal{P}$ where $\mathcal{P}$ consists of abelian groups. In this case, the basis elements of the ghost algebra satisfies $\tilde{e}_{P\times Q}(U,\alpha,V) = e_{P\times Q}(U,\alpha,V)$ for all $P,Q \in \mathcal{P}$ and all $(U,\alpha,V) \in \triangle_{\mathcal{F}}(P,Q)$. Hence, the multiplication of the basis elements is easier to deal with in this case than the non-abelian case.

Notation: Let $\mathcal{P}_K^{\mathcal{F}}$ be a subset of $\mathcal{P}\times\mathcal{P}$ defined by

$$\mathcal{P}_K^{\mathcal{F}} = \{(J,P) \mid J,P \in \mathcal{P},\ J =_{\mathcal{F}} K,\ P \geq J\}$$

where $J =_{\mathcal{F}} K$ denotes there is an J and K are $\mathcal{F}$-isomorphic. Note that if $K =_{\mathcal{F}} K'$, then $\mathcal{P}_K^{\mathcal{F}} = \mathcal{P}_{K'}^{\mathcal{F}}$. In the following theorem $K \in_{\mathcal{F}} \mathcal{P}$ is used to denote that K is running over $\mathcal{F}$-isomorphism classes of $\mathcal{P}$.

Theorem 4.4.1. *Let $\mathcal{P}$ be a set of abelian groups closed under taking subgroups and let $\mathcal{F}$ be a fusion system defined on $\mathcal{P}$ and*

$$i_K = \sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} e_{P\times P}(J,\mathrm{id},J).$$

The set $\{i_K \mid K \in_{\mathcal{F}} \mathcal{P}\}$ is the set of mutually orthogonal idempotents of the center $Z({}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}})$ and

$$e = \sum_{K\in_{\mathcal{F}}\mathcal{P}} i_K.$$

Proof. Since $K =_{\mathcal{F}} K'$ implies $\mathcal{P}_K^{\mathcal{F}} = \mathcal{P}_{K'}^{\mathcal{F}}$, we get $i_K = i_{K'}$. For the pair of groups K, K' with $K \neq_{\mathcal{F}} K'$, we have $i_K \neq i_{K'}$, because $\mathcal{P}_K^{\mathcal{F}} = \mathcal{P}_{K'}^{\mathcal{F}}$. For that pair of groups, we have $i_K.i_{K'} = 0$ since $\mathcal{P}_K^{\mathcal{F}} \cap \mathcal{P}_{K'}^{\mathcal{F}} = \emptyset$, so these elements are mutually orthogonal.

We claim that i_K is central for all $K \in_{\mathcal{F}} \mathcal{P}$. Let $m \in {}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$ be an arbitrary element. Hence, m can be uniquely written as

$$m = \sum_{\substack{P,Q \in \mathcal{P}, \\ (U,\alpha,V) \in \triangle_{\mathcal{F}}(P,Q)}} m_{P\times Q}(U,\alpha,V)\ e_{P\times Q}(U,\alpha,V)$$

where $m_{P\times Q}(U,\alpha,V)$ are the coefficients in $\mathbb{C}$. We have

$$i_K.m = \sum_{\substack{(J,P) \in \mathcal{P}_K^{\mathcal{F}},\ Q \in \mathcal{P}, \\ (J,\alpha,V) \in \triangle_{\mathcal{F}}(P,Q)}} m_{P\times Q}(J,\alpha,V)\ e_{P\times Q}(J,\alpha,V)$$

$$m.i_K = \sum_{\substack{P,Q \in \mathcal{P},\ (U,\alpha,J) \in \triangle_{\mathcal{F}}(P,Q), \\ J =_{\mathcal{F}} K}} m_{P\times Q}(U,\alpha,J)\ e_{P\times Q}(U,\alpha,J)$$

observing that the sets identified under the sum signs above are in fact coincide, we conclude that $i_K \in Z({}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}})$.

Now, we claim that i_K is idempotent for all $K \in \mathcal{P}$, because

$$\begin{aligned} i_K.i_K &= \left(\textstyle\sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} e_{P\times P}(J,\mathrm{id},J)\right) . \left(\textstyle\sum_{(J',P')\in\mathcal{P}_K^{\mathcal{F}}} e_{P'\times P'}(J',\mathrm{id},J')\right) \\ &= \textstyle\sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} e_{P\times P}(J,\mathrm{id},J) \\ &= i_K. \end{aligned}$$

We observe that the identity of the ghost algebra ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$ is

$$e = \sum_{J,P \in \mathcal{P},\ P \geq J} e_{P\times P}(J,\mathrm{id},J)$$

and therefore we have $\sum_{K \in_{\mathcal{F}} \mathcal{P}} i_K = e$. □

The idempotents given in the theorem above are not necessarily primitive as can be seen from the following lemma.

Lemma 4.4.2. *For any $K \in_{\mathcal{F}} \mathcal{P}$, the algebras ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}.i_K$ and $\mathbb{C}Out_{\mathcal{F}}(K)$ are Morita equivalent.*

Proof. We recall Theorem 9.9 of [23]. An algebra A and its subalgebra eAe are Morita equivalent if and only if e is an idempotent of A such that $AeA = A$. Set $A =^{\oplus} \mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}.i_K$ and $e = e_{K\times K}(K, \mathrm{id}, K)$, then we have $eAe \simeq \mathbb{C}\mathrm{Out}_{\mathcal{F}}(K)$ and $AeA = A$ as claimed. □

From this lemma, we get a bijection between $\mathrm{Irr}(^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}.i_K)$ and $\mathrm{Irr}(\mathbb{C}\mathrm{Out}_{\mathcal{F}}(K))$. Hence, it is not surprising to have the following theorem.

Theorem 4.4.3. *Let $\mathcal{P}$ be a set of abelian groups closed under taking subgroups and let $\mathcal{F}$ be a fusion system defined on $\mathcal{P}$,*

$$i_{K,\chi} = \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(K)|} \sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} \sum_{\beta\in\mathrm{Out}_{\mathcal{F}}(J)} \chi(\beta^{-1}) e_{P\times P}(J,\beta,J).$$

The set $\{i_{K,\chi} \mid (K,\chi)\in\Omega\}$ is the set of mutually orthogonal centrally primitive idempotents of $^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$ and

$$1 = \sum_{(K,\chi)\in\Omega} i_{K,\chi}.$$

Remark 4.4.4. In the innermost sum of the formula, χ is regarded as a $\mathbb{C}\mathrm{Out}_{\mathcal{F}}(J)$-character. Indeed, we do this by transporting the structure as follows: let $\phi : K \to J$ be an isomorphism in $\mathcal{F}$, then it induces an isomorphism $\bar{\phi} : \mathrm{Out}_{\mathcal{F}}(J) \to \mathrm{Out}_{\mathcal{F}}(K)$ where $\bar{\phi}(\beta) := \phi^{-1}\beta\phi$. Hence, we set

$$\chi(\beta) =: \chi(\bar{\phi}(\beta)).$$

Note that, this setting does not depend on our choice of the isomorphism ϕ, indeed if ϕ' is another $\mathcal{F}$-isomorphism from K to J, then $\bar{\phi}(\beta) =^{\phi^{-1}\phi'} \bar{\phi}'(\beta)$ where $\phi^{-1}\phi' \in \mathrm{Out}_{\mathcal{F}}(K)$.

Proof. If (K,χ) and (K',χ') lie in the same equivalence class of seeds, then we have $i_{K,\chi} = i_{K',\chi'}$, if they lie in different equivalence classes, then $i_{K,\chi} \neq i_{K',\chi'}$ by definition of the equivalence of seeds.

We claim that $i_{K,\chi}$ is central for all $(K,\chi)\in\Omega$. Let

$$m = \sum_{\substack{P,Q\in\mathcal{P},\\ (U,\alpha,V)\in\triangle_{\mathcal{F}}(P,Q)}} m_{P\times Q}(U,\alpha,V)\ e_{P\times Q}(U,\alpha,V)$$

be an arbitrary element of ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$, then

$$i_{K,\chi}.m = \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(K)|} \sum_{\substack{(J,P)\in\mathcal{P}_K^{\mathcal{F}},\\ Q\in\mathcal{P}}} \sum_{\substack{(J,\alpha,V)\in\triangle_{\mathcal{F}}(P,Q),\\ \beta\in\mathrm{Out}_{\mathcal{F}}(J)}} \chi(\beta^{-1}) m_{P\times Q}(J,\alpha,V)\ e_{P\times Q}(J,\beta\alpha,V)$$

$$m.i_{K,\chi} = \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(K)|} \sum_{\substack{P,Q\in\mathcal{P},\\ (U,\alpha,V)\in\triangle_{\mathcal{F}}(P,Q)}} \sum_{\substack{(V,Q)\in\mathcal{P}_K^{\mathcal{F}},\\ \beta\in\mathrm{Out}_{\mathcal{F}}(V)}} \chi(\beta^{-1}) m_{P\times Q}(U,\alpha,V)\, e_{P\times Q}(U,\alpha\beta,V).$$

The sums above give the same result because the sets which the sums run through coincide.

Now, we claim that $i_{K,\chi}$ is an idempotent element. Let

$$e_{\chi,J,P} := \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(J)|} \sum_{\beta\in\mathrm{Out}_{\mathcal{F}}(J)} \chi(\beta^{-1}) e_{P\times P}(J,\beta,J),$$

then

$$i_{K,\chi} i_{K,\chi} = \Big(\sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} e_{\chi,J,P}\Big)\Big(\sum_{(J',P')\in\mathcal{P}_K^{\mathcal{F}}} e_{\chi,J',P'}\Big)$$

and since $e_{\chi,J,P}$ are primitive idempotents of the group algebra $\mathbb{C}\mathrm{Out}_{\mathcal{F}}(J)$ we have $e^2_{\chi,J,P} = e_{\chi,J,P}$ and $e_{\chi,J,P}.e_{\chi,J',P'} = 0$ for $P \neq P'$ or $J \neq J'$. Therefore, idempotency is clear.

We have

$$\begin{aligned}
\sum_{\chi\in\mathrm{Irr}(\mathbb{C}\mathrm{Out}_{\mathcal{F}}(K))} i_{K,\chi} &= \sum_{\chi\in\mathrm{Irr}(\mathbb{C}\mathrm{Out}_{\mathcal{F}}(K))} \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(K)|} \sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} \sum_{\beta\in\mathrm{Out}_{\mathcal{F}}(J)} \chi(\beta^{-1}) e_{P\times P}(J,\beta,J) \\
&= \sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} \sum_{\beta\in\mathrm{Out}_{\mathcal{F}}(J)} \frac{e_{P\times P}(J,\beta,J)}{|\mathrm{Out}_{\mathcal{F}}(J)|} \sum_{\chi\in\mathrm{Irr}(\mathbb{C}\mathrm{Out}_{\mathcal{F}}(J))} \chi(1).\chi(\beta^{-1}) \\
&= \sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} e_{P\times P}(J,\mathrm{id},J) \\
&= i_K
\end{aligned}$$

passing from second line to the third line, we use the second orthogonality relation of the characters. Hence, we have $\sum_{(K,\chi)\in\Omega} i_{K,\chi} = \sum_{K\in_{\mathcal{F}}\mathcal{P}} i_K = 1$.

The primitiveness of $i_{K,\chi}$ comes from the classification of simple ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$-modules given in the Section 4.2.

□

4.5 Non-abelian Case

We continue with the case where $\mathcal{P}$ may contain some non-abelian groups. In this case the basis elements of the ghost algebra has a more complicated multiplication. To simplify it, we will change the basis of the ghost algebra as follows:

For $P, Q \in \mathcal{P}$ and $(U, \alpha, V) \in \triangle_{\mathcal{F}}(P, Q)$, we introduce

$$e'_{P\times Q}(U, \alpha, V) = \sqrt{\frac{|P|.|Q|}{|C_P(U)|.|C_Q(V)|}}\ e_{P\times Q}(U, \alpha, V)$$

so that have the following multiplication

$$e'_{P\times Q}(U, \alpha, V)\ e'_{Q'\times R}(V', \beta, W) = \begin{cases} e'_{P\times R}(U, \alpha\beta, W), & \text{if } Q = Q' \text{ and } V = V' \\ 0, & \text{otherwise.} \end{cases}$$

Similar to the previous construction, set

$$\tilde{e}'_{P\times Q}(U, \alpha, V) := \sum_{(x,y)\in P\times Q} e'_{P\times Q}(U, \alpha, V)$$

then $P \times Q$ fixes $\tilde{e}'_{P\times Q}(U, \alpha, V)$ for each $(U, \alpha, V) \in \triangle_{\mathcal{F}}(P, Q)$. Thus, the set $\{\tilde{e}'_{P\times Q}(U, \alpha, V) \mid (U, \alpha, V) \in_{P\times Q} \triangle_{\mathcal{F}}(P, Q)\}$ constitutes a basis for $\widetilde{\mathcal{M}_{\mathcal{F}}}(P, Q)$. Note that,

$$\tilde{e}'_{P\times Q}(U, \alpha, V)\ \tilde{e}'_{Q'\times R}(V', \alpha, W) = 0$$

when $Q \neq Q'$ or when $Q = Q'$ and V is not Q-conjugate to V'.

Lemma 4.5.1. *The identity element e of ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$ is*

$$e = \sum_{P\in\mathcal{P}, J\leq_P P} \frac{\tilde{e}'_{P\times P}(J, id, J)}{|N_P(J)|.|P|}.$$

Proof. e is a central element since it is symmetric. Let m be an arbitrary element of ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$, then m can be uniquely written as

$$m = \sum_{\substack{P,Q\in\mathcal{P},\\ (U,\alpha,V)\in_{P\times Q}\triangle_{\mathcal{F}}(P,Q)}} m_{P\times Q}(U, \alpha, V)\ \tilde{e}'_{P\times Q}(U, \alpha, V).$$

We have

$$\begin{aligned}
e.m &= \sum_{P,Q \in \mathcal{P}, J \leq_P P} \sum_{(J,\alpha,V) \in_{P\times Q} \triangle_{\mathcal{F}}(P,Q)} m_{P\times Q}(J,\alpha,V) \frac{\tilde{e'}_{P\times P}(J,\mathrm{id},J)\ .\ \tilde{e'}_{P\times Q}(J,\alpha,V)}{|N_P(J)|.|P|} \\
&= \sum_{\substack{P,Q\in\mathcal{P},\\ (J,\alpha,V)\in_{P\times Q}\triangle_{\mathcal{F}}(P,Q)}} m_{P\times Q}(J,\alpha,V)\ \tilde{e'}_{P\times Q}(J,\alpha,V) \\
&= m
\end{aligned}$$

because for $C := \tilde{e'}_{P\times P}(J,\mathrm{id},J)\ .\ \tilde{e'}_{P\times Q}(J,\alpha,V)$,

$$\begin{aligned}
C &= \sum_{(p_1,p_2)\in P\times P} e'_{P\times P}({}^{p_1}J, c_{p_1}c_{p_2^{-1}}, {}^{p_2}J)\ .\ \sum_{(p,q)\in P\times Q} e'_{P\times Q}({}^{p}J, c_p\alpha c_{q^{-1}}, {}^{q}V) \\
&= \sum_{(p_1,q)\in P\times Q} \sum_{p_2^{-1}p\in N_P(J)} e'_{P\times Q}({}^{p_1}J, c_{p_1}c_{p_2^{-1}}c_p\alpha c_{q^{-1}}, {}^{q}V) \\
&= |P| \sum_{g\in N_P(J)} \sum_{(p_1,q)\in P\times Q} e'_{P\times Q}({}^{p_1}J, c_{p_1}c_g\alpha c_{q^{-1}}, {}^{q}V) \\
&= |P| \sum_{g\in N_P(J)} \tilde{e'}_{P\times Q}(J, c_g\alpha, V) \\
&= |P|.|N_P(J)|\tilde{e'}_{P\times Q}(J, c_g\alpha, V) \\
&= |P|.|N_P(J)|\tilde{e'}_{P\times Q}(J, \alpha, V).
\end{aligned}$$

Hence, $e.m = m$. □

Next, we state a theorem which gives the set of mutually orthogonal idempotents. Similar to the previous case, we fix our notation as follows:

Notation: Let $\mathcal{P}_K^{\mathcal{F}}$ be a subset of $\mathcal{P}\times\mathcal{P}$ which is defined as

$$\mathcal{P}_K^{\mathcal{F}} = \{(J,P) \mid J,P\in\mathcal{P},\ J =_{\mathcal{F}} K,\ J \leq_P P\}.$$

Observe that this definition coincides with the definition given in the case for abelian groups; because when P is abelian, the set of P-conjugacy classes of subgroups of P is exactly the same as the set of subgroups of P.

Theorem 4.5.2. *Let $\mathcal{P}$ be a set of finite groups closed under taking subgroups and let $\mathcal{F}$ be a fusion system defined on $\mathcal{P}$,*

$$i_K = \sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} \frac{\tilde{e'}_{P\times P}(J,\mathrm{id},J)}{|N_P(J)|.|P|}.$$

The set $\{i_K \mid K \in_{\mathcal{F}} \mathcal{P}\}$ *is the set of mutually orthogonal idempotents of the center* $Z({}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}})$ *and*

$$1 = \sum_{K \in_{\mathcal{F}} \mathcal{P}} i_K.$$

Proof. Since $K =_{\mathcal{F}} K'$ implies $\mathcal{P}_K^{\mathcal{F}} = \mathcal{P}_{K'}^{\mathcal{F}}$, so we get $i_K = i_{K'}$. For the pair of groups K, K' with $K \neq_{\mathcal{F}} K'$, we have $i_K \neq i_{K'}$, similarly. Also, we have $i_K.i_{K'} = 0$ since $\mathcal{P}_K^{\mathcal{F}} \cap \mathcal{P}_{K'}^{\mathcal{F}} = \emptyset$, so that these elements are mutually orthogonal.

We claim that i_K is central for all $K \in_{\mathcal{F}} \mathcal{P}$. Let $m \in {}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$ be an arbitrary element. Hence, m can be uniquely written as

$$m = \sum_{\substack{P,Q \in \mathcal{P}, \\ (U,\alpha,V) \in_{P\times Q} \triangle_{\mathcal{F}}(P,Q)}} m_{P\times Q}(U,\alpha,V)\ \tilde{e'}_{P\times Q}(U,\alpha,V)$$

where $m_{P\times Q}(U,\alpha,V)$ are coefficients in $\mathbb{C}$. We have

$$i_K.m = \sum_{\substack{(J,P) \in \mathcal{P}_K^{\mathcal{F}},\ Q \in \mathcal{P}, \\ (J,\alpha,V) \in_{P\times Q} \triangle_{\mathcal{F}}(P,Q)}} m_{P\times Q}(J,\alpha,V)\ \tilde{e'}_{P\times Q}(J,\alpha,V) \text{ and } m.i_K = \sum_{\substack{P,Q \in \mathcal{P},\ (U,\alpha,J) \in_{P\times Q} \triangle_{\mathcal{F}}(P,Q), \\ J =_{\mathcal{F}} K}} m_{P\times Q}(U,\alpha,J)\ \tilde{e'}_{P\times Q}(U,\alpha,J).$$

Observing that the sets identified under the sum signs above coincide, we conclude that $i_K \in Z({}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}})$. Now, we claim that i_K is an idempotent for all $K \in \mathcal{P}$, because

$$\begin{aligned}
i_K.i_K &= \left(\sum_{(J,P) \in \mathcal{P}_K^{\mathcal{F}}} \frac{\tilde{e'}_{P\times P}(J,\mathrm{id},J)}{|N_P(J)|.|P|} \right) \cdot \left(\sum_{(J',P') \in \mathcal{P}_K^{\mathcal{F}}} \frac{\tilde{e'}_{P'\times P'}(J',\mathrm{id},J')}{|N_{P'}(J')|.|P'|} \right) \\
&= \sum_{(J,P) \in \mathcal{P}_K^{\mathcal{F}}} \sum_{\substack{(p_1,p_2),(p_3,p_4) \in P\times P \\ p_2 p_3^{-1} \in N_P(J)}} \frac{e'_{P\times P}({}^{p_1}J, c_{p_1} c_{p_2^{-1}} c_{p_3} c_{p_4^{-1}}, {}^{p_4}J)}{|N_P(J)|^2.|P|^2} \\
&= \sum_{(J,P) \in \mathcal{P}_K^{\mathcal{F}}} \frac{1}{|P|} \sum_{\substack{(p_1,p_4) \in P\times P \\ g \in N_P(J)}} \frac{e'_{P\times P}({}^{p_1}J, c_{p_1} c_g c_{p_4^{-1}}, {}^{p_4}J)}{|N_P(J)|^2} \\
&= \sum_{(J,P) \in \mathcal{P}_K^{\mathcal{F}}} \frac{1}{|P|} \sum_{g \in N_P(J)} \frac{\tilde{e'}_{P\times P}(J, c_g, J)}{|N_P(J)|^2} \\
&= \sum_{(J,P) \in \mathcal{P}_K^{\mathcal{F}}} \frac{\tilde{e'}_{P\times P}(J,\mathrm{id},J)}{|N_P(J)|.|P|} \\
&= i_K
\end{aligned}$$

because of the multiplication rule introduced.

It is clear that $1 = \sum_{K \in_{\mathcal{F}} \mathcal{P}} i_K$. $\square$

Lemma 4.5.3. *For any $K \in_{\mathcal{F}} \mathcal{P}$, the algebras ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}.i_K$ and $\mathbb{C}Out_{\mathcal{F}}(K)$ are Morita equivalent.*

Proof. We recall again [23] Theorem 9.9. An algebra A and its subalgebra eAe are Morita equivalent if and only if e is an idempotent of A such that $AeA = A$. Set $A =^{\oplus} \mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}.i_K$ and $e = \tilde{e'}_{K\times K}(K, \mathrm{id}, K)$, then we have $eAe \cong \mathbb{C}\mathrm{Out}_{\mathcal{F}}(K)$ and $AeA = A$ and the result follows. $\square$

Now, we can state non-abelian version of Theorem 4.4.3.

Theorem 4.5.4. *Let $\mathcal{P}$ be a set of finite groups closed under taking subgroups and let $\mathcal{F}$ be a fusion system defined on $\mathcal{P}$, and*

$$i_{K,\chi} = \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(K)|} \sum_{(J,P)\in \mathcal{P}_K^{\mathcal{F}}} \sum_{\beta \in \mathrm{Out}_{\mathcal{F}}(J)} \chi(\beta^{-1}) \frac{\tilde{e'}_{P\times P}(J, \beta, J)}{|N_P(J)|.|P|}.$$

The set $\{i_{K,\chi} \mid (K,\chi) \in \Omega\}$ is the set of mutually orthogonal centrally primitive idempotents of ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$ and

$$1 = \sum_{(K,\chi)\in\Omega} i_{K,\chi}.$$

Proof. If (K,χ) and (K',χ') lie in the same equivalence class, then we have $i_{K,\chi} = i_{K',\chi'}$, if they lie in different equivalence classes, then $i_{K,\chi} \neq i_{K',\chi'}$ by definition of the equivalence of seeds.

We claim that $i_{K,\chi}$ is central for all $(K,\chi) \in \Omega$. Let

$$m = \sum_{\substack{P,Q\in\mathcal{P}, \\ (U,\alpha,V)\in {}_{P\times Q}\triangle_{\mathcal{F}}(P,Q)}} m_{P\times Q}(U,\alpha,V)\ \tilde{e'}_{P\times Q}(U,\alpha,V)$$

be an arbitrary element of ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}$, then

$$i_{K,\chi}.m = \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(K)|} \sum_{\substack{(J,P)\in\mathcal{P}_K^{\mathcal{F}}, \\ Q\in\mathcal{P}}} \sum_{\substack{(J,\alpha,V)\in {}_{P\times Q}\triangle_{\mathcal{F}}(P,Q), \\ \beta\in\mathrm{Out}_{\mathcal{F}}(J)}} \frac{\chi(\beta^{-1}) m_{P\times Q}(J,\alpha,V)}{|N_P(J)|.|P|} A_{P,Q,J,V,\beta,\alpha}$$

where $A_{P,Q,J,V,\beta,\alpha} = \tilde{e'}_{P\times P}(J,\beta,J)\ .\ \tilde{e'}_{P\times Q}(J,\alpha,V)$. In fact, it is

$$\begin{aligned} A_{P,Q,J,V,\beta,\alpha} &= \left(\sum_{(p_1,p_2)\in P\times P} e'_{P\times P}({}^{p_1}J, c_{p_1}\beta c_{p_2^{-1}}, {}^{p_2}J)\right)\ .\ \left(\sum_{(p,q)\in P\times Q} e'_{P\times Q}({}^{p}J, c_p\alpha c_{q^{-1}}, {}^{q}V)\right) \\ &= \sum_{\substack{(p_1,p_2)\in P\times P \\ (p,q)\in P\times Q, pp_2^{-1}\in N_P(J)}} e'_{P\times Q}({}^{p_1}J, c_{p_1}\beta c_{p_2^{-1}} c_p\alpha c_{q^{-1}}, {}^{q}V) \\ &= |P| \sum_{\substack{(p_1,q)\in P\times Q, \\ g\in N_P(J)}} e'_{P\times Q}({}^{p_1}J, c_{p_1}\beta c_g\alpha c_{q^{-1}}, {}^{q}V) \\ &= |P| \sum_{g\in N_P(J)} \tilde{e'}_{P\times Q}(J, \beta c_g\alpha, V) \\ &= |P|.|N_P(J)|\tilde{e'}_{P\times Q}(J,\beta\alpha,V), \end{aligned}$$

here, when the last line is due to the fact that $\beta c_g = \beta$ for all $\beta \in \mathrm{Out}_{\mathcal{F}}(J)$ and $g \in N_P(J)$. If we multiply with vice versa, we get

$$m.i_{K,\chi} = \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(K)|} \sum_{\substack{P,Q\in\mathcal{P}, \\ (U,\alpha,V)\in_{P\times Q}\triangle_{\mathcal{F}}(P,Q)}} \sum_{\substack{(V,Q)\in\mathcal{P}_K^{\mathcal{F}}, \\ \beta\in\mathrm{Out}_{\mathcal{F}}(V)}} \frac{\chi(\beta^{-1}) m_{P\times Q}(U,\alpha,V)}{|N_Q(V)|.|Q|} B_{P,Q,U,V,\beta,\alpha}$$

where $B_{P,Q,U,V,\beta,\alpha} = \tilde{e'}_{P\times Q}(U,\alpha,V)\ .\ \tilde{e'}_{Q\times Q}(V,\beta,V)$. Similarly we have

$$B_{P,Q,U,V,\beta,\alpha} = |Q|.|N_Q(V)|\tilde{e'}_{P\times Q}(U,\alpha\beta,V).$$

Letting $U = J$, we conclude that the coefficients of $\tilde{e'}_{P\times Q}(J,\_,V)$ are all equal. Therefore, we conclude that $i_{K,\chi}$ is central for all $(K,\chi)\in\Omega$.

Now, we claim that $i_{K,\chi}$ is an idempotent element. Similar to the abelian case let

$$e_{\chi,J,P} := \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(J)|} \sum_{\beta\in\mathrm{Out}_{\mathcal{F}}(J)} \chi(\beta^{-1}) \frac{\tilde{e'}_{P\times P}(J,\beta,J)}{|N_P(J)|.|P|},$$

then

$$i_{K,\chi} i_{K,\chi} = (\sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} e_{\chi,J,P})(\sum_{(J',P')\in\mathcal{P}_K^{\mathcal{F}}} e_{\chi,J',P'})$$

and since $e_{\chi,J,P}$ are primitive idempotents of the group algebra $\mathbb{C}\mathrm{Out}_{\mathcal{F}}(J)$ we have $e^2_{\chi,J,P} = e_{\chi,J,P}$ and $e_{\chi,J,P}.e_{\chi,J',P'} = 0$ for $P \neq P'$ or $J \neq J'$. Therefore, idempotency is clear.

We have

$$\begin{aligned}
\sum_{\chi\in\mathrm{Irr}(\mathbb{C}\mathrm{Out}_{\mathcal{F}}(K))} i_{K,\chi} &= \sum_{\chi\in\mathrm{Irr}(\mathbb{C}\mathrm{Out}_{\mathcal{F}}(K))} \frac{\chi(1)}{|\mathrm{Out}_{\mathcal{F}}(K)|} \sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} \sum_{\beta\in\mathrm{Out}_{\mathcal{F}}(J)} \chi(\beta^{-1}) \frac{\tilde{e'}_{P\times P}(J,\beta,J)}{|N_P(J)|.|P|} \\
&= \sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} \sum_{\beta\in\mathrm{Out}_{\mathcal{F}}(J)} \frac{\tilde{e'}_{P\times P}(J,\beta,J)}{|\mathrm{Out}_{\mathcal{F}}(J)|.|N_P(J)|.|P|} \left(\sum_{\chi\in\mathrm{Irr}(\mathbb{C}\mathrm{Out}_{\mathcal{F}}(J))} \chi(1).\chi(\beta^{-1}) \right) \\
&= \sum_{(J,P)\in\mathcal{P}_K^{\mathcal{F}}} \frac{\tilde{e'}_{P\times P}(J,\mathrm{id},J)}{|N_P(J)|.|P|} \\
&= i_K
\end{aligned}$$

passing from second line to the third line, we use the second orthogonality relation of the characters. Hence, we have $\sum_{(K,\chi)\in\Omega} i_{K,\chi} = \sum_{K\in_{\mathcal{F}}\mathcal{P}} i_K = 1$.

The primitiveness of $i_{K,\chi}$ comes from the classification of simple ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$-modules given in the Section 4.2.

□

4.6 Proof of Theorem 4.1.1

Both for the abelian and non-abelian cases, we have the semisimplicity of the ghost algebra.

Proof. From Theorem 4.5.4, we have ${}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}} = \oplus_{(K,\chi)\in\Omega}\, {}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}.i_{K,\chi}$. Moreover, since we have

$$\bigoplus_{\chi\in\mathrm{Irr}(\mathbb{C}\mathrm{Out}_{\mathcal{F}}(K))} {}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}.i_{K,\chi} = {}^{\oplus}\mathbb{C}\widetilde{\mathcal{M}_{\mathcal{F}}}.i_K$$

which is semisimple by Lemma 4.5.3. The result follows.

□

The mark map induces an isomorphism between the ghost algebra and the quiver algebra. Since isomorphism of algebras preserves semisimplicity, we deduce semisimplicity of ${}^{\oplus}\mathbb{C}\mathcal{M}_{\mathcal{F}}$ hence prove Theorem 4.1.1.

Chapter 5

On fusion systems defined on p-permutation algebras

Let p be a prime number, G a finite group, and k an algebraically closed field of characteristic p. As we mention in the introduction, we are interested in the following question:

Given a saturated fusion system $\mathcal{F}$ on a finite p-group P, does there exist a finite group G, a p-permutation G-algebra A and a primitive idempotent b of A^G such that $\mathcal{F} = \mathcal{F}_{(P,e_P)}(A, b, G)$ for some maximal (A, b, G)-Brauer pair (P, e_P)?

We have the following conjecture:

Conjecture. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, X be a characteristic biset for $\mathcal{F}$, $G = \mathrm{Park}(\mathcal{F}, X)$ and $S_P(G, k)$ be the Scott kG-module with vertex P. Then for $A = \mathrm{End}_k(S_P(G, k))$ we have*

$$\mathcal{F} = \mathcal{F}_{(P,1_{A(P)})}(A, 1_A, G).$$

For some particular saturated fusion systems, we prove this conjecture. In fact, the question is reduced to finding Brauer indecomposable p-permutation modules by the work of Kessar-Kunugi-Mitsuhashi. We show in Theorems 5.3.2 and 5.3.6 that the corresponding Scott modules become Brauer indecomposable, hence they provide examples that support the conjecture.

5.1 A sufficient condition for saturation

In this section, we define Brauer pairs and fusion systems for primitive idempotents of G-fixed subalgebras of p-permutation G-algebras. We shall state a sufficient condition for such a fusion system to be saturated.

A G-algebra is called **p-permutation G-algebra** if for any p-subgroup Q of G, it has a basis which is Q-stable. For a p-permutation G-algebra A, a primitive idempotent b of A^G, we define an **(A, b, G)-Brauer pair** to be a pair (Q, f) such that Q is a p-subgroup of G such that $A(Q) \neq 0$, f is a block (centrally primitive idempotent) of $A(Q)$ where $\mathrm{Br}_Q(b) \neq 0$ and $\mathrm{Br}_Q(b)f \neq 0$. We call (A, b, G) a **saturated triple** if b is a central idempotent of A, and for each (A, b, G)-Brauer pair (Q, f), the idempotent f is primitive in $A(Q)^{C_G(Q,f)}$. (Here $C_G(Q, f)$ denotes the subgroup of $C_G(Q)$ which stabilizes f).

Broué and Puig defined, in [13], the notion of inclusion on Brauer pairs as follows. Let (Q, f) and (P, e) be (A, b, G)-Brauer pairs, then $(Q, f) \leq (P, e)$ if $Q \leq P$ and whenever i is a primitive idempotent of A^P such that $\mathrm{Br}_P(i)e \neq 0$, then $\mathrm{Br}_Q(i)f \neq 0$. For an element $x \in G$, the conjugate of (P, e) by x is the (A, b, G)-Brauer pair ${}^x(P, e) := ({}^xP,\ {}^xe)$.

The following theorem gives fundamental properties about the inclusion of (A, b, G)-Brauer pairs.

Theorem 5.1.1 ([13], Theorem 1.8)**.** *Let (P, e) be an (A, b, G)-Brauer pair and let $Q \leq P$.*

(i) There exists a unique block f of $A(Q)$ such that (Q, f) is an (A, b, G)-Brauer pair and $(Q, f) \leq (P, e)$.

(ii) The set of (A, b, G)-Brauer pairs is a G-poset under the action of G defined above.

The properties of maximal (A, b, G)-Brauer pairs are given in the following theorem.

Theorem 5.1.2 ([13], Theorem 1.14)**.** *Let A be a p-permutation G-algebra and b be a primitive idempotent of A^G. Then,*

(i) The group G acts transitively on the set of maximal (A, b, G)-Brauer pairs.

(ii) Let (P, e) be an (A, b, G)-Brauer pair. The following are equivalent.

(a) (P, e) is a maximal Brauer pair.

(b) $\mathrm{Br}_P(b) \neq 0$ and P is maximal amongst p-subgroups Q of G with the property that $\mathrm{Br}_Q(b) \neq 0$.

(c) $b \in \mathrm{Tr}_P^G(A^P)$ and P is minimal amongst subgroups H of G such that $b \in \mathrm{Tr}_H^G(A^H)$.

If Q, R are subgroups of G and $g \in G$ is such that ${}^gQ \leq R$, then $c_g : Q \to R$ denotes the conjugation map which sends an element q of Q to the element ${}^gq = gqg^{-1}$ of R.

Now, let (P, e_P) be a maximal (A, b, G)-Brauer pair. For each subgroup Q of P, let (Q, e_Q) be the unique (A, b, G)-Brauer pair such that $(Q, e_Q) \leq (P, e_P)$. The category $\mathcal{F}_{(P,e_P)}(A, b, G)$ is the category whose objects are the subgroups of P, whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{F}_{(P,e_P)}(A,b,G)}(Q, R) := \{c_g : Q \to R \mid g \in G,\ {}^g(Q, e_Q) \leq (R, e_R)\}$$

for $Q, R \leq P$ and where composition of morphisms is the usual composition of functions. This category is in fact a fusion system as the following theorem implies.

Theorem 5.1.3. *Let A be a p-permutation G-algebra and b be a primitive idempotent of A^G and (P, e_P) a maximal Brauer pair. Then $\mathcal{F} := \mathcal{F}_{(P,e_P)}(A, b, G)$ satisfies the following.*

(i) $\mathrm{Hom}_P(Q,R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q,R) \subseteq \mathrm{Inj}(Q,R)$ *for all* $Q, R \leq P$ *where* $\mathrm{Hom}_P(Q,R)$ *denotes the set of all group homomorphisms from* Q *to* R *which are induced by conjugation by some element of* P*.*

(ii) For any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$*, the induced isomorphism* $Q \simeq \phi(Q)$ *and its inverse are morphisms in* $\mathcal{F}$*.*

The fusion system $\mathcal{F}_{(P,e)}(A,b,G)$ is not always saturated. The following theorem gives a sufficient condition for saturation.

Theorem 5.1.4 ([16], Theorem 1.6)**.** *Let* A *be a* p*-permutation* G*-algebra and* b *be a primitive idempotent of* A^G *and* (P, e_P) *a maximal Brauer pair. Suppose that* (A,b,G) *is a saturated triple, then for any maximal* (A,b,G)*-Brauer pair* (P,e)*,* $\mathcal{F}_{(P,e)}(A,b,G)$ *is a saturated fusion system.*

Hence, the theorem implies that in order to have a saturated fusion system, we should have a saturated triple.

5.2 Relation to Brauer indecomposability

We give a criterion for a particular triple to be saturated following the work of Kessar-Kunugi-Mitsuhashi in [16].

For a finite dimensional kG-module M and a p-subgroup Q of G, the Brauer quotient $M(Q)$ with respect to Q, is naturally a $kN_G(Q)/Q$-module (see Section 3.1), hence by restriction is a $kC_G(Q)/Q$-module. We say that M is **Brauer indecomposable** if for any p-subgroup Q of G, $M(Q)$ is indecomposable or zero as a $kQC_G(Q)/Q$-module.

Now, let M be an indecomposable p-permutation kG-module with vertex P and set $A = \mathrm{End}_k(M)$. Then A is a G-algebra via

$$G \times A \to A$$

$$(g,\phi) \mapsto {}^g\phi$$

where ${}^{g}\phi(m) := g\phi(g^{-1}m)$ for $m \in M$. Since M is a p-permutation module, A is a p-permutation algebra and since M is indecomposable, $1_A = \mathrm{id}_M$ is primitive. Thus, we can introduce $(A, 1_A, G)$-Brauer pairs in this setting. The following theorem gives a necessary and sufficient condition for the triple $(A, 1_A, G)$ to be saturated.

Theorem 5.2.1 ([16], Proposition 4.1). *With the notation above, the $(A, 1_A, G)$-Brauer pairs are the pairs $(Q, 1_{A(Q)})$ such that $M(Q) \neq 0$ and $(P, 1_{A(P)})$ is a maximal $(A, 1_A, G)$-Brauer pair. Further,*

(i) $\mathcal{F}_{(P,1_{A(P)})}(A, 1_A, G) = \mathcal{F}_P(G)$.

(ii) The triple $(A, 1_A, G)$ is saturated if and only if M is Brauer indecomposable.

Here, the fusion system $\mathcal{F}_P(G)$ is the category whose objects are the subgroups of P and whose morphism set from Q to R is $\mathrm{Hom}_G(Q, R)$. This theorem suggests us to find Brauer indecomposable p-permutation modules in order for $(A, 1_A, G)$ to be a saturated triple.

5.3 Brauer indecomposability of Scott modules for some Park groups

Kessar-Kunugi-Mitsuhashi showed for the special case when $M = S_P(G, k)$, the triple $(A, 1_A, G)$ is saturated for $A = \mathrm{End}_k(M)$ for the case when P is an abelian p-group as in the following:

Theorem 5.3.1 ([16], Theorem 1.2). *Let P be abelian p-subgroup of a finite group G. If $\mathcal{F}_P(G)$ is saturated then $S_P(G, k)$ is Brauer indecomposable and hence $(A, 1_A, G)$ is a saturated triple for $A = \mathrm{End}_k(S_P(G, k))$.*

We extend this result to some different fusion systems $\mathcal{F}$ defined on P where P is not necessarily abelian. Our first theorem is the following:

Theorem 5.3.2. *Let P be a finite p-group. For $n \in \mathbb{Z}^+$, let $G = P \wr S_n$ and ι be the diagonal embedding of P into G. The kG-module $S_{\iota(P)}(G,k)$ is Brauer indecomposable.*

Remark 5.3.3. In this theorem, since S_n acts trivially on $\iota(P)$, $\mathcal{F}_{\iota(P)}(G) = \mathcal{F}_{\iota(P)}(\iota(P))$. Hence, the fusion system $\mathcal{F}_{\iota(P)}(G)$ is saturated. Here, the group G is not a Park group, but is closely related to Park group, because of this we will call it as Park type group.

We use couple of lemmas in order to prove the theorem.

Lemma 5.3.4. *Let G be a finite p-group and $P \leq G$. If $\mathcal{F}_P(G)$ is saturated, then $S_P(G,k)$ is Brauer indecomposable.*

Proof. Let Q be a fully $\mathcal{F}$-normalized subgroup of P, then by Theorem 5.2 of [21] we have $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_G(Q))$. Thus, we have $\mathrm{Aut}_P(Q) = \mathrm{Aut}_G(Q)$ and $N_G(Q) = N_P(Q)C_G(Q)$ since G is a p-group. Consequently, by Alperin's Fusion Theorem (see Theorem A.10 in [11] for example), we have $\mathcal{F}_P(G) = \mathcal{F}_P(P)$.

Since G is a p-group, $\mathrm{Ind}_P^G k$ is an indecomposable kG-module by Green's Indecomposability Theorem, so $M := S_P(G,k) = \mathrm{Ind}_P^G k$. By the Mackey formula,

$$\mathrm{Res}^G_{N_G(Q)} M = \bigoplus_{g \in N_G(Q) \backslash G / P} \mathrm{Ind}^{N_G(Q)}_{N_G(Q) \cap {}^g P} k.$$

Taking Brauer quotient gives,

$$M(Q) = \bigoplus_{g \in N_G(Q) \backslash G/P,\ Q \leq {}^g P} \mathrm{Ind}^{N_G(Q)}_{N_{{}^g P}(Q)} k.$$

We claim that there is only one coset in the direct sum above. Indeed, if $g \in G$ is such that $Q \leq^g P$, then $c_{g^{-1}} : Q \to {}^{g^{-1}}Q$ is in $\mathcal{F}_P(G)$, so is in $\mathcal{F}_P(P)$. Thus $g \in PC_G(Q)$, which establishes our claim. Therefore, $M(Q) = \mathrm{Ind}^{N_G(Q)}_{N_P(Q)} k$ is an indecomposable $kN_G(Q)$-module by Green's Indecomposability Theorem. As a $kQC_G(Q)$-module,

$$\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} M(Q) = \bigoplus_{g \in QC_G(Q) \backslash N_G(Q) / N_P(Q)} \mathrm{Ind}^{QC_G(Q)}_{QC_G(Q) \cap {}^g N_P(Q)} k$$

by the Mackey formula. Since we have $N_G(Q) = N_P(Q)C_G(Q)$, there exists only one coset. Therefore, $M(Q) = \mathrm{Ind}_{QC_P(Q)}^{QC_G(Q)}k$ is an indecomposable $kQC_G(Q)$-module again by Green's Indecomposability Theorem and hence an indecomposable $kQC_G(Q)/Q$-module. Moreover, since any subgroup is $\mathcal{F}$-conjugate (hence G-conjugate) to a fully $\mathcal{F}$-normalized subgroup, the result holds for all subgroups of P. □

Lemma 5.3.5. *We have*

$$S_{\iota(P)}(G,k) = \mathrm{Ind}_{\iota(P)\rtimes S_n}^{G}k \otimes_k \mathrm{Inf}_{S_n}^{G}\mathrm{PIM}(S_n)$$

where $\mathrm{PIM}(S_n)$ *is the projective cover of the trivial* kS_n*-module. Here,* G *acts diagonally on the tensor product.*

Proof. Set $T = \mathrm{Ind}_{\iota(P)\rtimes S_n}^{G}k$ and $U = \mathrm{Inf}_{S_n}^{G}\mathrm{PIM}(S_n)$. It is enough to show that $T \otimes_k U$ is a an indecomposable p-permutation kG-module whose vertex is $\iota(P)$ and whose socle contains the trivial module.

Let $D := P \times \ldots \times P$, we have

$$\mathrm{Res}_D^G\mathrm{Ind}_{\iota(P)\rtimes S_n}^{G}k = \mathrm{Ind}_{D\cap(\iota(P)\rtimes S_n)}^{D}k = \mathrm{Ind}_{\iota(P)}^{D}k \tag{5.1}$$

by the Mackey formula and $G = D \rtimes S_n$. Since D is a p-group, the module on the right is an indecomposable kD-module by Green's Indecomposability Theorem. Therefore, from Proposition 2.1 of [17], we deduce that $T\otimes_k U$ is an indecomposable kG-module. We note also that, both T and U are p-permutation modules. Hence, $T \otimes_k U$ is also a p-permutation module.

By Theorem 3.2.1 T is $\iota(P)\rtimes S_n$-projective and U is D-projective since D acts trivially on U. Hence, $T\otimes_k U$ is both $\iota(P)\rtimes S_n$ and D-projective (see [18], Chapter 4, Lemma 2.1 (iii)). Hence, a vertex of $T\otimes_k U$ lies inside $(\iota(P)\rtimes S_n)\cap D = \iota(P)$. On the other hand,

$$T \otimes_k U(\iota(P)) \simeq T(\iota(P)) \otimes_k U \neq 0.$$

So, $\iota(P)$ is contained in a vertex of $T \otimes_k U$. Therefore, $T \otimes_k U$ has vertex $\iota(P)$.

Finally, since both socle(T) and socle(U) contains k as a kG-submodule, the socle of the product $T \otimes_k U$ contains k as a kG-submodule. □

Proof of Theorem 5.3.2. By the previous lemma, it remains to show that for $T \otimes_k U(\iota(Q))$ is $k[\iota(Q)C_G(\iota(Q))]$ -indecomposable for all $Q \leq P$. We have

$$T \otimes_k U(\iota(Q)) \simeq T(\iota(Q)) \otimes_k \mathrm{Inf}_{S_n}^{\iota(Q)C_G(\iota(Q))}\mathrm{PIM}(S_n)$$

as $\iota(Q)C_G(\iota(Q))$- modules since $\iota(Q)$ acts trivially on $\mathrm{PIM}(S_n)$.

From the identity 5.1, we get $\mathrm{Res}_D^G T = S_{\iota(P)}(D,k)$ because D is a p-group. Hence, Lemma 5.3.4 implies that $T(\iota(Q))$ is $k[\iota(Q)C_D(\iota(Q))]$-indecomposable. Therefore, by Proposition 2.1 of [17], we conclude that $T(\iota(Q)) \otimes_k \mathrm{Inf}_{S_n}^{\iota(Q)C_G(\iota(Q))}\mathrm{PIM}(S_n)$ is $k[\iota(Q)C_G(\iota(Q))]$-indecomposable.

□

Our second result is the following theorem.

Theorem 5.3.6. *Let P be a finite p-group, $E \leq Aut(P)$, and $n = |E|$ such that $(n,p) = 1$. For $\varrho(P) = \{(e_1(p), \dots, e_n(p); id) \mid p \in P\} \leq G := P \wr S_n$ where $e_i \in E$ for $i = 1, \dots, n$. The kG-module $S_{\varrho(P)}(G,k)$ is Brauer indecomposable.*

Remark 5.3.7. Since P is a Sylow p-subgroup of $P \rtimes E$, the fusion system $\mathcal{F}_P(P \rtimes E)$ is saturated and $P \rtimes E$ is a characteristic biset corresponding to $\mathcal{F}_P(P \rtimes E)$. We observe also that the subgroup $\varrho(P)$ is Park's embedding. Hence by Theorem 2.4.1, for $G = \mathrm{Park}(\mathcal{F}_P(P \rtimes E), P \rtimes E)$, we have $\mathcal{F}_{\varrho(P)}(G) = \mathcal{F}_P(P \rtimes E)$, thus the fusion system $\mathcal{F}_{\varrho(P)}(G)$ is saturated.

Proof. Let $H = P \rtimes E$, $D = P \times \ldots \times P$. Since H acts on itself by left multiplication, the embedding ϱ can be extended to H, so that $\varrho(H) = \varrho(P) \rtimes \varrho(E) \leq G$, where $\varrho(E) \cap D = 1$. Hence, $\mathcal{F}_{\varrho(P)}(\varrho(H)) = \mathcal{F}_{\varrho(P)}(G)$. Moreover, since E acts faithfully on P, is $C_G(\varrho(P)) = \{(p_1, \dots, p_n; \mathrm{id}) \mid p_i \in Z(P)\}$. Together with this and the relation $\mathrm{Aut}_G(\varrho(P)) \simeq \mathrm{Aut}_H(P)$, we get $|N_G(\varrho(P))| = n|Z(P)|^{n-1}|P|$.

We claim that $S_{\varrho(P)}(G,k) = \mathrm{Ind}_{\varrho(H)}^G k$. Since $\varrho(P) \in \mathrm{Syl}_p(\varrho(H))$, we get $S_{\varrho(P)}(G,k) = S_{\varrho(H)}(G,k)$ ([18], Chapter 4, Corollary 8.5). Thus, we deduce that $S_{\varrho(P)}(G,k) \mid \mathrm{Ind}_{\varrho(H)}^G k$. Now, suppose

$$\mathrm{Ind}_{\varrho(H)}^G k = S_{\varrho(P)}(G,k) \oplus Y$$

for some kG-module Y. By Corollary 3.3.4, $S_{\varrho(P)}(G,k)(\varrho(P))$ is the projective cover of the trivial $kN_G(\varrho(P))/\varrho(P)$- module. Thus $|\frac{N_G(\varrho(P))}{\varrho(P)}|_p = |Z(P)|^{n-1}$ divides the dimension of $S_{\varrho(P)}(G,k)(\varrho(P))$. We have

$$\dim \mathrm{Ind}_{\varrho(H)}^G k(\varrho(P)) = |\{g\varrho(H) \mid g \in G,\ {}^g\varrho(P) \le \varrho(H)\}|.$$

The condition ${}^g\varrho(P) \le \varrho(H)$ implies that ${}^g\varrho(P) \le D \cap \varrho(H) = \varrho(P)$, thus

$$\dim \mathrm{Ind}_{\varrho(H)}^G k(\varrho(P)) = \left|\frac{N_G(\varrho(P))}{\varrho(H)}\right| = |Z(P)|^{n-1}$$

which gives $Y(\varrho(P)) = 0$. On the other hand, since $Y \mid \mathrm{Ind}_{\varrho(H)}^G k$, by the Mackey formula $\mathrm{Res}_D^G Y \mid \mathrm{Res}_D^G \mathrm{Ind}_{\varrho(H)}^G k = \oplus_{g\in D\backslash G/H} \mathrm{Ind}_{D\cap^g\varrho(H)}^D k$. Thus $D\cap^g \varrho(H) =^g \varrho(P)$ forces $Y({}^g\varrho(P)) \ne 0$. This is a contradiction. Therefore, $Y = 0$ and the claim is established.

It remains to show that $S_{\varrho(P)}(G,k)$ is Brauer indecomposable. For $M := S_{\varrho(P)}(G,k) = \mathrm{Ind}_{\varrho(H)}^G k$, let us first find what $M(\varrho(Q))$ is as a $kN_G(\varrho(Q))$-module. We have

$$\dim M(\varrho(Q)) = |\{g\varrho(H) \mid g \in G,\ {}^g\varrho(Q) \le \varrho(H)\}|$$

and since any conjugate of $\varrho(Q)$ lies in D, the set above counts, in fact, the cosets for which ${}^g\varrho(Q) \le \varrho(H)\cap D = \varrho(P)$. Or, equivalently it counts the elements $g \in G$ which induces a conjugation map $c_g : \varrho(Q) \to^g \varrho(Q)$ in $\mathcal{F}_{\varrho(P)}(G) = \mathcal{F}_{\varrho(P)}(\varrho(H))$, this forces g to be in $\varrho(H)C_G(\varrho(Q))$. Hence

$$\dim M(\varrho(Q)) = \frac{|\varrho(H)C_G(\varrho(Q))|}{|\varrho(H)|} = \frac{|C_G(\varrho(Q))|}{|C_{\varrho(H)}(\varrho(Q))|}. \tag{5.2}$$

On the other hand, by Mackey formula,

$$\mathrm{Res}_{N_G(\varrho(Q))}^G M = \bigoplus_{g\in N_G(\varrho(Q))\backslash G/\varrho(H)} \mathrm{Ind}_{N_G(\varrho(Q))\cap^g\varrho(H)}^{N_G(\varrho(Q))} k.$$

Besides, for all $Q \le P$, we have $\mathrm{Aut}_G(\varrho(Q)) = \mathrm{Aut}_{\varrho(H)}(\varrho(Q))$, thus

$$\frac{|N_G(\varrho(Q))|}{|C_G(\varrho(Q))|} = \frac{|N_{\varrho(H)}(\varrho(Q))|}{|C_{\varrho(H)}(\varrho(Q))|}. \tag{5.3}$$

Hence by Equations 5.2 and 5.3, we conclude that

$$M(\varrho(Q)) = \mathrm{Ind}_{N_{\varrho(H)}(\varrho(Q))}^{N_G(\varrho(Q))} k.$$

When viewed as $k\varrho(Q)C_G(\varrho(Q))$-module, we claim that $M(\varrho(Q))$ is in fact a $k\varrho(Q)C_G(\varrho(Q))$-Scott module with vertex $\varrho(Q)C_{\varrho(P)}(\varrho(Q))$ and this will automatically give the indecomposability of $M(\varrho(Q))$ as a $k\varrho(Q)C_G(\varrho(Q))$-module. The restricted module is

$$\mathrm{Res}^{N_G(\varrho(Q))}_{\varrho(Q)C_G(\varrho(Q))} M(\varrho(Q)) = \mathrm{Ind}^{\varrho(Q)C_G(\varrho(Q))}_{\varrho(Q)C_G(\varrho(Q))\cap N_{\varrho(H)}(\varrho(Q))} k = \mathrm{Ind}^{\varrho(Q)C_G(\varrho(Q))}_{\varrho(Q)C_{\varrho(H)}(\varrho(Q))} k$$

by the Mackey formula and by $\varrho(Q)C_G(\varrho(Q))N_{\varrho(H)}(\varrho(Q)) = N_G(\varrho(Q))$.

Set $A = \varrho(Q)C_{\varrho(P)}(\varrho(Q))$ and $B = \varrho(Q)C_G(\varrho(Q))$ and let $S := S_A(B,k)$. Since $A \in \mathrm{Syl}_p(\varrho(Q)C_{\varrho(H)}(\varrho(Q)))$ and by the equation above, S is a direct summand of $M(\varrho(Q))$ ([18], Chapter 4, Corollary 8.5). Let

$$M(\varrho(Q)) = S \oplus X$$

for some $k\varrho(Q)C_G(\varrho(Q))$- module X. We will show that $X = 0$. Observe that $N_B(A) = \varrho(Q)(N_G(\varrho(P)) \cap C_G(\varrho(Q)))$, thus

$$\frac{|N_B(A)|}{|A|} = \frac{|N_G(\varrho(P)) \cap C_G(\varrho(Q))|}{|C_{\varrho(P)}(\varrho(Q)|}.$$

Since $S(A)$ is the projective cover of the trivial $N_B(A)/A$- module, $\left|N_B(A)/A\right|_p$ divides the dimension of $S(A)$, hence $\frac{|N_G(\varrho(P))\cap C_G(\varrho(Q))|_p}{|C_{\varrho(P)}(\varrho(Q)|}$ divides the dimension of $S(A)$. On the other hand,

$$\begin{aligned}\dim M(\varrho(Q))(A) &= |\{\ g\ \varrho(Q)C_{\varrho(H)}(\varrho(Q)) \mid g \in B,\ {}^gA \leq \varrho(Q)C_{\varrho(H)}(\varrho(Q))\ \}| \\ &= \frac{|\varrho(Q)(N_G(\varrho(P)) \cap C_G(\varrho(Q))|}{|\varrho(Q)C_{\varrho(H)}(\varrho(Q))|} \\ &= \frac{|(N_G(\varrho(P)) \cap C_G(\varrho(Q))|}{|C_{\varrho(H)}(\varrho(Q))|}\end{aligned}$$

where the first equality comes from the fact that $\varrho(P) \trianglelefteq \varrho(H)$. We claim that the two numbers $\dfrac{|N_G(\varrho(P)) \cap C_G(\varrho(Q))|_p}{|C_{\varrho(P)}(\varrho(Q)|}$ and $\dfrac{|(N_G(\varrho(P)) \cap C_G(\varrho(Q))|}{|C_{\varrho(H)}(\varrho(Q))|}$ are equal and this will in turn imply $S(A) = M(\varrho(Q))(A)$. Let $g \in N_G(\varrho(P)) \cap C_G(\varrho(Q))$, then $c_g : \varrho(P) \to \varrho(P)$ is in $\mathcal{F}_{\varrho(P)}(G) = \mathcal{F}_{\varrho(P)}(\varrho(H))$, thus $g \in \varrho(H)C_G(\varrho(P))$ and so $g \in C_{\varrho(H)C_G(\varrho(P))}(\varrho(Q))$. Conversely, let $g \in C_{\varrho(H)C_G(\varrho(P))}(\varrho(Q))$, then since $\varrho(H) \leq N_G(\varrho(P))$, $g \in N_G(\varrho(P))$. Hence, $N_G(\varrho(P)) \cap C_G(\varrho(Q)) = C_{\varrho(H)C_G(\varrho(P))}(\varrho(Q))$. This yields

$$|N_G(\varrho(P)) \cap C_G(\varrho(Q))|_{p'} = |C_{\varrho(H)C_G(\varrho(P))}(\varrho(Q))|_{p'} = \frac{|C_{\varrho(H)|}(\varrho(Q))|}{|C_{\varrho(P)|}(\varrho(Q))|}$$

since $C_G(\varrho(P)) \leq D$ and D is a p-group. This establishes the claim and that $X(A) = 0$.

Let $D' := \varrho(Q)C_D(\varrho(Q))$. Since $X \mid \mathrm{Ind}^B_{\varrho(Q)C_{\varrho(H)}(\varrho(Q))}k$,

$$\mathrm{Res}^B_{D'}X \mid \bigoplus_{g\in D'\backslash B/\varrho(Q)C_{\varrho(H)}(\varrho(Q))} \mathrm{Ind}^{D'}_{D'\cap^g(\varrho(Q)C_{\varrho(H)}(\varrho(Q)))}k.$$

Moreover,

$$D' \cap^g (\varrho(Q)C_{\varrho(H)}(\varrho(Q))) = \varrho(Q)C_{D\cap^g\varrho(H)}(\varrho(Q)) = \varrho(Q)C_{^g\varrho(P)}(\varrho(Q)) =^g A$$

since ${}^gD = D$ for all $g \in B$. Thus, $X({}^gA) \neq 0$, which contradicts with the result in the previous paragraph. Therefore, we conclude that $X = 0$ and

$$M(\varrho(Q)) = S_{\varrho(Q)C_{\varrho(P)}(\varrho(Q))}(\varrho(Q)C_G(\varrho(Q)), k)$$

is an indecomposable $k\varrho(Q)C_G(\varrho(Q))$-module. □

Chapter 6

On real representation spheres and real monomial Burnside ring

This chapter contains the presentation of the paper [5]. We introduce a restriction morphism, called the Boltje morphism, from a given ordinary representation functor to a given monomial Burnside functor. In the case of a sufficiently large fibre group, this is Robert Boltje's splitting of the linearization morphism. By considering a monomial Lefschetz invariant associated with real representation spheres, we show that, in the case of the real representation ring and the fibre group $\{\pm 1\}$, the image of a modulo 2 reduction of the Boltje morphism is contained in a group of units associated with the idempotents of the 2-local Burnside ring. We deduce a relation on the dimensions of the subgroup-fixed subspaces of a real representation.

6.1 Results

We shall be making a study of some restriction morphisms which, at one extreme, express Boltje's canonical induction formula [7] while, at the other extreme, they generalize a construction initiated by tom Dieck [25, 5.5.9], namely, the tom Dieck morphism associated with spheres of real representations. A connection between canonical induction and the tom Dieck morphism has appeared before, in Symonds [22], where the integrality property of Boltje's restriction morphism was proved by using the natural fibration of complex projective space as a monomial analogue of the sphere.

Generally, our concern will be with finite-dimensional representations of a finite group G over a field $\mathbb{K}$ of characteristic zero. A little more specifically, our concern will be with the old idea of trying to synthesize information about $\mathbb{K}G$-modules from information about certain 1-dimensional $\mathbb{K}I$-modules where I runs over some or all of the subgroups of G. Throughout, we let C be a torsion subgroup of the unit group $\mathbb{K}^\times = \mathbb{K} - \{0\}$. The 1-dimensional $\mathbb{K}I$-modules to which we shall be paying especial attention will be those upon which each element of I acts as multiplication by an element of C. Some of the results below are specific to the case where $\mathbb{K} = \mathbb{R}$ and $C = \{\pm 1\}$, and some of them are also specific to the case where G is a 2-group.

Fixing C, we write $O_C(G)$, or just $O(G)$, to denote the smallest normal subgroup of G such that the quotient group $G/O(G)$ is abelian and every element of $G/O(G)$ has the same order as some element of C. In other words, $O(G)$ is intersection of the kernels of the group homomorphisms $G \to C$.

Consider a $\mathbb{K}G$-module M, finite-dimensional as we deem all $\mathbb{K}G$-modules to be. Given a subgroup $I \leq G$, then the $O(I)$-fixed subspace $M^{O(I)}$ of M is the sum of those 1-dimensional $\mathbb{K}I$-submodules of M that are inflated from $I/O(I)$. For elements $c \in C$ and $i \in I$, we write $M_c^{I,i}$ to denote the c-eigenspace of the action of i on $M^{O(I)}$. By Maschke's Theorem,

$$M^{O(I)} = \bigoplus_{c \in C} M_c^{I,i} , \qquad \dim(M^{O(I)}) = \sum_{c \in C} \dim(M_c^{I,i}) .$$

We shall introduce a restriction morphism, denoted $\dim^c$, whereby the isomorphism class $[M]$ of M is associated with the function

$$(I, i) \mapsto \dim(M_c^{I,i}) .$$

We shall define the *Boltje morphism* to be the restriction morphism

$$\mathrm{bol}^{\mathbb{K},C} = \sum_{c \in C} c \, \dim^c .$$

This morphism is usually considered only in the case where C is sufficiently large in the sense that each element of G has the same order as some element of C. In that case, the field $\mathbb{K}$ splits for G, the Boltje morphism is a splitting for linearization and we have a canonical induction formula. At the other extreme though, when $C = \{1\}$, the monomial dimension morphism $\dim^1$ is closely related to the tom Dieck morphism die(), both of those morphisms associating the isomorphism class $[M]$ with the function

$$I \mapsto \dim_{\mathbb{R}}(M^I)\,.$$

The vague comments that we have just made are intended merely to convey an impression of the constructions. In Section 2, we shall give details and, in particular, we shall be elucidating those two extremal cases.

For the rest of this introductory section, let us confine our discussion to the case where we have the most to say, the case $\mathbb{K} = \mathbb{R}$. Here, the only possibilities for C are $C = \{1\}$ and $C = \{\pm 1\}$. We shall be examining the modulo 2 reductions of the morphisms $\dim^c$ and $\mathrm{bol}^{\mathbb{R},C}$. We shall be making use of the following topological construction. Given an $\mathbb{R}G$-module M, we let $S(M)$ denote the unit sphere of M with respect to any G-invariant inner product on M. Up to homotopy, $S(M)$ can be regarded as the homotopy G-sphere obtained from M by removing the zero vector.

Let us make some brief comments concerning the case $C = \{1\}$. The reduced tom Dieck morphism $\overline{\mathrm{die}}$ is so-called because it can be regarded as a modulo 2 reduction of the tom Dieck morphism die(). Via $\overline{\mathrm{die}}$, the isomorphism class $[M]$ is associated with the function

$$I \mapsto \mathrm{par}(\dim(M^I))$$

where $\mathrm{par}(n) = (-1)^n$ for $n \in \mathbb{Z}$. We can view $\overline{\mathrm{die}}$ as a morphism of biset functors

$$\overline{\mathrm{die}} \;:\; A_{\mathbb{R}} \to \beta^{\times}$$

where the coordinate module $A_{\mathbb{R}}(G)$ is the real representation ring of G and the coordinate module $\beta^{\times}(G)$ is the unit group of the ghost ring $\beta(G)$ associated with the Burnside ring $B(G)$ of G. But we shall be changing the codomain. A result of tom Dieck asserts that the image of the coordinate map $\overline{\mathrm{die}}_G : A_{\mathbb{R}}(G) \to \beta^{\times}(G)$

is contained in the unit group $B^\times(G)$ of $B(G)$. His proof makes use of the fact that the function $I \mapsto \mathrm{par}(\dim(M^I))$ is determined by the Lefchetz invariant of $S(M)$. Hence, we can regard the reduced tom Dieck morphism as a morphism of biset functors

$$\overline{\mathrm{die}} : A_{\mathbb{R}} \to B^\times .$$

The main substance of this chapter concerns the case $C = \{\pm 1\}$, still with $\mathbb{K} = \mathbb{R}$. We now replace the ordinary Burnside ring $B(G)$ with the *real Burnside ring* $B_{\mathbb{R}}(G) = B_{\{\pm 1\}}(G)$, we mean to say, the monomial Burnside ring with fibre group $\{\pm 1\}$. For the rest of this section, we assume that $C = \{\pm 1\}$. Thus, the group $O(G) = O_C(G)$ is the smallest normal subgroup of G such that $G/O(G)$ is an elementary abelian 2-group. We write $O^2(G)$ to denote the smallest normal subgroup of G such that $G/O^2(G)$ is a 2-group.

In a moment, we shall define a restriction morphism $\overline{\mathrm{bol}}$, called the **reduced Boltje morphism**, whereby $[M]$ is associated with the function

$$I \mapsto \mathrm{par}(\dim(M^{O(I)})) .$$

Some more notation is needed. Recall that the algebra maps $\mathbb{Q}B(G) \to \mathbb{Q}$ are the maps $\epsilon_I^G : \mathbb{Q}B(G) \to \mathbb{Q}$, indexed by representatives I of the conjugacy classes of subgroups of G, where $\epsilon_I^G[\Omega] = |\Omega^I|$, the notation indicating that the isomorphism class $[\Omega]$ of a G-set Ω is sent to the number of I-fixed elements of Ω. Also recall that any element x of $\mathbb{Q}B(G)$ has coordinate decomposition

$$x = \sum_I \epsilon_I^G(x)\, e_I^G$$

where each e_I^G is the unique primitive idempotent of $\mathbb{Q}B(G)$ such that $\epsilon_I^G(e_I^G) \neq 0$. The ghost ring $\beta(G)$ is defined to be the set consisting of those elements x such that each $\epsilon_I^G(x) \in \mathbb{Z}$. Evidently, the unit group $\beta^\times(G)$ of $\beta(G)$ consists of those elements x such that each $\epsilon_I^G(x) \in \{\pm 1\}$. In particular, $\beta^\times(G)$ is an elementary abelian 2-group, and it can be regarded as a vector space over the field of order 2. Our notation follows [4, Section 3], where fuller details of these well-known constructions are given. We define $\overline{\mathrm{bol}}_G : A_{\mathbb{R}}(G) \to \beta^\times(G)$ to be the $\mathbb{Q}$-linear map such that

$$\overline{\mathrm{bol}}_G[M] = \sum_I \mathrm{par}(\dim(M^{O(I)}))\, e_I^G .$$

Evidently, we can view $\overline{\text{bol}}$ as a morphism of restriction functors $A_{\mathbb{R}} \to \beta^{\times}$. Extending to the ring $\mathbb{Z}_{(2)}$ of 2-local integers, we can view $\overline{\text{bol}}$ as a morphism of restriction functors $\mathbb{Z}_{(2)} A_{\mathbb{R}} \to \beta^{\times}$.

Let $\beta^{\times}_{(2)}$ denote the restriction subfunctor of $\beta^{\times}$ such that $\beta^{\times}_{(2)}(G)$ consists of those units in $\beta^{\times}(G)$ which can be written in the form $1 - 2y$, where y is an idempotent of $\mathbb{Z}_{(2)} B(G)$. In analogy with the above result of tom Dieck, we shall prove the following result in Section 6.3.

Theorem 6.1.1. *The image of the map* $\overline{\text{bol}}_G : \mathbb{Z}_{(2)} A_{\mathbb{R}}(G) \to \beta^{\times}(G)$ *is contained in* $\beta^{\times}_{(2)}(G)$*. Hence,* $\overline{\text{bol}}$ *can be regarded as a restriction morphism* $\overline{\text{bol}} : \mathbb{Z}_{(2)} A_{\mathbb{R}} \to \beta^{\times}_{(2)}$.

In Section 6.4, using Theorem 6.1.1 together with a characterization of idempotents due to Dress, we shall obtain the following result. We write $\equiv_2$ to denote congruence modulo 2.

Theorem 6.1.2. *Given an* $\mathbb{R}G$*-module* M*, then* $\dim(M^{O(I)}) \equiv_2 \dim(M^{O^2(I)})$ *for all* $I \leq G$.

Specializing to the case of a finite 2-group, and using a theorem of Tornehave, we shall deduce the next result, which expresses a constraint on the units of the Burnside ring of a finite 2-group. We shall also give a more direct alternative proof, using the same theorem of Tornehave and also using an extension in [3] of Bouc's theory [9, Chapter 9] of genetic sections.

Theorem 6.1.3. *Suppose that* G *is a* 2*-group. Then, for all* $I \leq G$ *and all units* $x \in B^{\times}(G)$*, we have* $\epsilon^G_{O(I)}(x) = \epsilon^G_1(x)$.

6.2 Boltje morphisms

For an arbitrary field $\mathbb{K}$ with characteristic zero, an arbitrary torsion subgroup C of the unit group $\mathbb{K}^{\times}$ and an arbitrary element $c \in C$, we shall define a restriction morphism $\dim^c$, called the *monomial dimension morphism* for eigenvalue c, and we shall define a restriction morphism $\mathrm{bol}^{C,\mathbb{K}}$, called the *Boltje morphism* for C and $\mathbb{K}$. In this section, we shall explain how, in one extremal case, $\mathrm{bol}^{C,\mathbb{K}}$ is associated with canonical induction while, in another extremal case, $\mathrm{bol}^{C,\mathbb{K}}$ is associated with dimension functions on real representation spheres.

We shall be considering three kinds of group functors, namely, restriction functors, Mackey functors, biset functors. All of our group functors are understood to be defined on the class of all finite groups, except when we confine attention to the class of all finite 2-groups. For any group functor L, we write $L(G)$ for the coordinate module at G. For any morphism of group functors $\theta : L \to L'$, we write $\theta_G : L(G) \to L'(G)$ for the coordinate map at G. Any group isomorphism $G \to G'$, gives rise to an isogation map (sometimes awkwardly called an isomorphism map) $L(G) \to L(G')$, which is to be interpreted as transport of structure. Restriction functors are equipped with isogation maps and restriction maps. Mackey functors are further equipped with induction maps, biset functors are yet further equipped with inflation and deflation maps. A good starting-point for a study of these briefly indicated notions is Bouc [9].

Recall that the representation ring of the group algebra $\mathbb{K}G$ coincides with the character ring of $\mathbb{K}G$. Denoted $A_{\mathbb{K}}(G)$, it is a free $\mathbb{Z}$-module with basis $\mathrm{Irr}(\mathbb{K}G)$, the set of isomorphism classes of simple $\mathbb{K}G$-modules, which we identify with the set of irreducible $\mathbb{K}G$-characters. The sum and product on $A_{\mathbb{K}}(G)$ are given by direct sum and tensor product. We can understand $A_{\mathbb{K}}$ to be a biset functor for the class of all finite groups, equipped with isogation, restriction, induction, inflation, deflation maps. Actually, the inflation and deflation maps will be of no concern to us in this chapter, and we can just as well regard $A_{\mathbb{K}}(G)$ as a Mackey functor, equipped only with isogation, restriction and induction maps.

The monomial Burnside ring of G with fibre group C, denoted $B_C(G)$, is defined similarly, but with C-fibred G-sets in place of $\mathbb{K}G$-modules. Recall that a **C-fibred G-set** is a permutation set Ω for the group $CG = C \times G$ such that C acts freely and the number of C-orbits is finite. A C-orbit of Ω is called a **fibre** of Ω. It is well-known that B_C can be regarded as a biset functor. For our purposes, we can just as well regard it as a Mackey functor.

Let us briefly indicate two coordinate decompositions that were reviewed in more detail in [2, Equations 1, 2]. Defining a C**-subcharacter** of G to be a pair (U, μ) where $U \leq G$ and $\mu \in \mathrm{Hom}(U, C)$, then we have a coordinate decomposition

$$B_C(G) = \bigoplus_{(U,\mu)} \mathbb{Z}\, d^G_{U,\mu}$$

where (U, μ) runs over representatives of the G-conjugacy classes of C-subcharacters, and $d^G_{U,\mu}$ is the isomorphism class of a transitive C-fibred G-set such that U is the stabilizer of a fibre and U acts via μ on that fibre. The other coordinate decomposition concerns the algebra $\mathbb{K}B_C(G) = \mathbb{K} \otimes B_C(G)$. We define a C**-subelement** of G to be a pair $(I, iO_C(I))$, where $i \in I \leq G$. As an abuse of notation, we write (I, i) instead of $(I, iO_C(I))$. For each C-subelement (I, i), let $\epsilon^G_{I,i}$ be the algebra map $\mathbb{K}B_C(G) \to \mathbb{K}$ associated with (I, i). Recall that, given a C-fibred G-set Ω, then $\epsilon^G_{I,i}[\Omega] = \sum_\omega \phi_\omega$, where ω runs over the fibres stabilized by I and i acts on ω as multiplication by ϕ_ω. Let $e^G_{I,i}$ be the unique primitive idempotent of $\mathbb{K}B_C(G)$ such that $\epsilon^G_{I,i}(e^G_{I,i}) = 1$. Note that we have G-conjugacy $(I, i) =_G (J, j)$ if and only if $\epsilon^G_{I,i} = \epsilon^G_{J,j}$, which is equivalent to the condition $e^G_{I,i} = e^G_{J,j}$. We have

$$\mathbb{K}B_C(G) = \bigoplus_{(I,i)} \mathbb{K}\, e^G_{I,i}$$

where (I, i) runs over representatives of the G-conjugacy classes of C-subelements. Thus, given an element $x \in \mathbb{K}B_C(G)$, then

$$x = \sum_{(I,i)} \epsilon^G_{I,i}(x)\, e^G_{I,i} \, .$$

Recall that there is an embedding $B(G) \hookrightarrow B_C(G)$ such that $[\mho] \mapsto [C\mho]$, where each element ω of a given G-set $\mho$ corresponds to a fibre $\{c\omega : c \in C\}$ of the C-fibred G-set $C\mho = C \times \mho$. The embedding is characterized by an easy remark [2, 7.2], which says that, given $x \in B_C(G)$, then $x \in B(G)$ if and only if $\epsilon^G_{I,i}(x) = \epsilon^G_{I,i'}(x)$ for all $i, i' \in I$, in which case, $\epsilon^G_I(x) = \epsilon^G_{I,i}(x)$ for all $i \in I$. We shall be needing the following remark in the next section.

Remark 6.2.1. Let R be a unital subring of $\mathbb{K}$. Then $\mathbb{K}B(G) \cap RB_C(G) = RB(G)$.

Proof. Let $\pi_C : B_C(G) \to B(G)$ be the projection such that $[\Omega] \mapsto [C\backslash\Omega]$, where $C\backslash\Omega$ denotes the G-set of fibres of a given C-fibred G-set Ω. Extending linearly, we obtain projections $\pi_C : RB_C(G) \to RB(G)$ and $\pi_C : \mathbb{K}B_C(G) \to \mathbb{K}B(G)$. Given $x \in \mathbb{K}B(G) \cap RB_C(G)$, then $x = \pi_C(x) \in RB(G)$. So $\mathbb{K}B(G) \cap RB_C(G) \subseteq RB(G)$. The reverse inclusion is obvious. □

We mention that the projection $\pi_C : \mathbb{K}B_C(G) \to \mathbb{K}B(G)$ is an algebra map and, since $\epsilon_I^G[C\backslash\Omega] = \epsilon_{I,1}^G[\Omega]$, we have $\pi_C(e_{I,i}^G) = e_I^G$ if $i \in O(I)$ while $\pi_C(e_{I,i}^G) = 0$ otherwise.

We shall also be making use of the primitive idempotents of $\mathbb{K}A_{\mathbb{K}}(G)$. Regarding $\mathbb{K}A_{\mathbb{K}}(G)$ as the $\mathbb{K}$-vector space of G-invariant functions $G \to \mathbb{K}$, then the algebra maps $\mathbb{K}A_{\mathbb{K}}(G) \to \mathbb{K}$ are the maps ϵ_g^G, indexed by representatives g of the conjugacy classes of G, where $\epsilon_g^G(\chi) = \chi(g)$ for $\chi \in \mathbb{K}A_{\mathbb{K}}(G)$. Letting e_g^G be the primitive idempotent such that $\epsilon_g^G(e_g^G) = 1$, then

$$\chi = \sum_g \epsilon_g^G(\chi)\, e_g^G = \sum_g \chi(g)\, e_g^G$$

where g runs over representatives of the conjugacy classes of G. The linearization morphism

$$\mathrm{lin}^{C,\mathbb{K}} : \mathbb{K}B_C \to \mathbb{K}A_{\mathbb{K}}$$

has coordinate morphisms $\mathrm{lin}_G^{C,\mathbb{K}} : \mathbb{K}B_C(G) \to \mathbb{K}A_{\mathbb{K}}(G)$ such that

$$\mathrm{lin}_G^{C,\mathbb{K}}[d_{U,\mu}^G] = \mathrm{ind}_{G,U}(\mu)\,.$$

Letting Ω be a C-fibred G-set, and letting $\mathbb{K}\Omega = \mathbb{K} \otimes_C \Omega$ be the evident extension of Ω to a $\mathbb{K}G$-module, then $\mathrm{lin}_G^{C,\mathbb{K}}[\Omega] = [\mathbb{K}\Omega]$.

Remark 6.2.2. Given a primitive idempotent $e_{I,i}^G$ of $\mathbb{K}B_C(G)$, then $\mathrm{lin}_G^{C,\mathbb{K}}(e_{I,i}^G) \neq 0$ if and only if I is cyclic with generator i, in which case $\mathrm{lin}_G^{C,\mathbb{K}}(e_{I,i}^G) = e_i^G$.

Proof. It suffices to show that $\epsilon_{\langle i\rangle,i}^G[\Omega] = \epsilon_i^G[\mathbb{K}\Omega]$. Letting x run over representatives of the fibres of Ω, then x runs over the elements of a basis for the $\mathbb{K}G$-module $\mathbb{K}\Omega$. With respect to that basis, the action of i on $\mathbb{K}\Omega$ is represented by a matrix which has exactly one entry in each row and likewise for each column. The two sides of the required equation are plainly both equal to the trace of that matrix. □

Given $c \in C$, we define a $\mathbb{K}$-linear map

$$\dim_G^c \,:\, \mathbb{K}A_{\mathbb{K}}(G) \to \mathbb{K}B_C(G)$$

such that $\epsilon_{I,i}^G(\dim_G^c[M]) = \dim(M_c^{I,i})$ for a $\mathbb{K}G$-module M. In other words,

$$\dim_G^c[M] = \sum_{(I,i)} \dim(M_c^{I,i})\, e_{I,i}^G \,.$$

Since $\epsilon_{I,i}^H(\mathrm{res}_{H,G}(x)) = \epsilon_{I,i}^G(x)$ for all intermediate subgroups $I \leq H \leq G$, the maps $\dim_G^c$ commute with restriction. Plainly, the maps $\dim_G^c$ also commute with isogation. Thus, the maps $\dim_G^c$ combine to form a restriction morphism

$$\dim^c \,:\, \mathbb{K}A_{\mathbb{K}} \to \mathbb{K}B_C \,.$$

Let us define the **Boltje morphism** to be the restriction morphism

$$\mathrm{bol}^{C,\mathbb{K}} = \sum_{c \in C} c \dim^c \,:\, \mathbb{K}A_{\mathbb{K}} \to \mathbb{K}B_C \,.$$

The sum makes sense because, for each G, the sum $\mathrm{bol}_G^{C,\mathbb{K}} = \sum_{c \in C} c \dim_G^c$ is finite, indeed, $\dim_G^c$ vanishes for all c whose order does not divide $|G|$. When C is sufficiently large, the Boltje morphism is a splitting for linearization. We mean to say, if every element of G has the same order as an element of C, then

$$\mathrm{lin}_G^{C,\mathbb{K}} \circ \mathrm{bol}_G^{C,\mathbb{K}} = \mathrm{id}_{\mathbb{K}A_{\mathbb{K}}(G)} \,.$$

To see this, first note that, for arbitrary C and $\mathbb{K}$, we have

$$\mathrm{bol}_G^{C,\mathbb{K}}[M] = \sum_{(I,i)} \chi_I(i) e_{I,i}^G$$

where χ_I is the $\mathbb{K}I$-character of the $\mathbb{K}I$-module $M^{O(I)}$. Using Remark 6.2.2,

$$\mathrm{lin}_G^{C,\mathbb{K}}(\mathrm{bol}_G^{C,\mathbb{K}}[M]) = \sum_{(I,i)} \chi_I(i) \,\mathrm{lin}_G^{C,\mathbb{K}}(e_{I,i}) = \sum_i \chi(i)\, e_i$$

where χ is the $\mathbb{K}G$-character of M and, in the final sum, i runs over representatives of those conjugacy classes of elements of G such that the order of i divides $|G|$. When C is sufficiently large in the sense specified above, i runs over representatives of all the conjugacy classes, and $\sum_i \chi(i)\, e_i = [M]$, as required.

Let us confirm that the assertion we have just made is just a reformulation of the splitting result in Boltje [7]. Suppose, again, that C is sufficiently large. Then, in particular, $\mathbb{K}$ is a splitting field for G. We must now resolve two different notations. Where we write $B_C(G)$ and $A_{\mathbb{K}}(G)$ and $\mathrm{lin}_G^{C,\mathbb{K}}$ and $d_{U,\mu}^G$, Boltje [7] writes $R_+(G)$ and $R(G)$ and b_G and $\overline{(U,\mu)}^G$, respectively. Note that, because of the hypothesis on C, the scenario is essentially independent of C and $\mathbb{K}$. In [7, 2.1], he shows that there exists a unique restriction morphism $a : A_{\mathbb{K}} \to B_C$ such that $a_G(\phi) = d_{G,\phi}^G$ for all $\phi \in \mathrm{Hom}(G, C)$. Since

$$\epsilon_{I,i}^G(\mathrm{bol}_G^{C,\mathbb{K}}(\phi)) = \phi(i) = \epsilon_{I,i}^G(d_{G,\phi}^G) = \epsilon_{I,i}^G(a_G(\phi))$$

we have $\mathrm{bol}_G^{C,\mathbb{K}} = a_G$ and $\mathrm{bol}^{C,\mathbb{K}} = a$. But the splitting property that we have been discussing is just a preliminary to a deeper result about integrality. Having resolved the two different notations, we can now interpret Boltje [7, 2.1(b)] as the following theorem, which expresses the integrality property too.

Theorem 6.2.3. (Boltje) *Suppose that every element of G has the same order as an element of C. Then the restriction morphism $\mathrm{bol}^{C,\mathbb{K}} : \mathbb{K}A_{\mathbb{K}} \to \mathbb{K}B_C$ is the $\mathbb{K}$-linear extension of the unique restriction morphism $\mathrm{bol}^{C,\mathbb{K}} : A_{\mathbb{K}} \to B_C$ such that $\mathrm{lin}^{C,\mathbb{K}} \circ \mathrm{bol}^{C,\mathbb{K}} = id$.*

When the hypothesis on C is relaxed, the splitting property and the integrality property in the conclusion of the theorem can fail. Nevertheless, as we shall see in the next section, the Boltje morphism $\mathrm{bol}^{C,\mathbb{K}}$ does appear to be of interest even in the two smallest cases, where $C = \{1\}$ or $C = \{\pm 1\}$. Let us comment on a connection between the tom Dieck morphism die() and the Boltje morphism in the case $C = \{1\}$. Our notation die() is taken from a presentation in [4, 4.1] of a result of Bouc–Yalçın [10, page 828]. Letting B^* denote the dual of the Burnside functor B, then the tom Dieck morphism $\mathrm{die} : A_{\mathbb{K}} \to B^*$ is given by

$$\mathrm{die}_G[M] = \sum_I \dim(M^I)\, \delta_I^G$$

where I runs over representatives of the G-conjugacy classes of subgroups of G, and the elements δ_I^G comprise a $\mathbb{Z}$-basis for $B^*(G)$ that is dual to the $\mathbb{Z}$-basis of $B(G)$ consisting of the isomorphism classes of transitive G-sets $d_I^G = [G/I]$. On the other hand, the morphism $\mathrm{bol}^{\{1\},\mathbb{K}} = \dim^1 : A_{\mathbb{K}} \to B$ is given by

$$\mathrm{bol}_G^{\{1\},\mathbb{K}}[M] = \dim_G^1[M] = \sum_I \dim(M^I)\, e_I^G \,.$$

Thus, although die() and $\mathrm{bol}^{\{1\},\mathbb{K}}$ have different codomains, their defining formulas are similar. A closer connection will transpire, however, when we pass to the reduced versions of those two morphisms in the special case $\mathbb{K} = \mathbb{R}$.

6.3 The reduced Boltje morphism

Still allowing the finite group G to be arbitrary, we now confine our attention to the case $\mathbb{K} = \mathbb{R}$. The only torsion units of $\mathbb{R}$ are 1 and -1, so the only possibilities for C are $C = \{1\}$ and $C = \{\pm 1\}$. We shall be discussing modulo 2 reductions of the tom Dieck morphism die() and the Boltje morphisms $\mathrm{bol}^{\{1\},\mathbb{R}}$ and $\mathrm{bol}^{\{\pm 1\},\mathbb{R}}$, realizing the reductions as morphisms by understanding their images to be contained in the unit groups $B^\times(G)$ and $\beta^\times(G)$, respectively. Although those unit groups are abelian, it will be convenient to write their group operations multiplicatively.

In preparation for a study of the case $C = \{\pm 1\}$, we first review the case $C = \{1\}$, drawing material from [4] and Bouc–Yalçın [10]. The parity function $\mathrm{par} : n \mapsto (-1)^n$ is, of course, modulo 2 reduction of rational integers written multiplicatively (with the codomain C_2, the cyclic group with order 2, taken to be $\{\pm 1\}$ instead of $\mathbb{Z}/2\mathbb{Z}$). Thus, fixing an $\mathbb{R}G$-module M, and letting I run over representatives of the conjugacy classes of subgroups of G, the function $\overline{\mathrm{die}} : I \mapsto \mathrm{par}(\dim(M^I))$ is the modulo 2 reduction of the function $\mathrm{die} : I \mapsto \dim(M^I)$. In Section 2, we realized die() as a morphism with codomain B^*. But we shall be realizing $\overline{\mathrm{die}}$ as a morphism with codomain $B^\times$. Let us explain the relationship between those two codomains. Recall that the **ghost ring** associated with $B(G)$ is defined to be the $\mathbb{Z}$-span of the primitive idempotents $\beta(G) = \bigoplus_I \mathbb{Z}\, e_I^G$. We have $B(G) \leq \beta(G) < \mathbb{Q}B(G)$, and an element $x \in \mathbb{Q}B(G)$ belongs to $\beta(G)$ if and only if $\epsilon_I^G(x) \in \mathbb{Z}$ for each $I \leq G$. We also have an inclusion of unit groups $B^\times(G) \leq \beta^\times(G)$, and $x \in \beta^\times(G)$ if and only if each $\epsilon_I^G(x) \in \{\pm 1\}$. We shall be making use of Yoshida's characterization [28, 6.5] of $B^\times(G)$ as a subgroup of $\beta^\times(G)$.

Theorem 6.3.1. (Yoshida's Criterion) *Given an element $x \in \beta^\times(G)$, then $x \in B^\times(G)$ if and only if, for all $I \leq G$, the function $N_G(I)/I \ni gI \mapsto \epsilon_{\langle I,g\rangle}^G(x)/\epsilon_I^G(x) \in \{\pm 1\}$ is a group homomorphism.*

As discussed in [4, Section 10], the modulo 2 reduction of the biset functor B^* can be identified with the biset functor $\beta^\times$, and the modulo 2 reduction of the morphism of biset functors die() from $A_\mathbb{R}$ to B^* can be identified with the morphism of biset functors $\overline{\text{die}}$ from $A_\mathbb{R}$ to $\beta^\times$ given by

$$\overline{\text{die}}_G[M] = \sum_I \text{par}(\dim(M^I))\, e_I^G \,.$$

A well-known result of tom Dieck asserts that the image $\overline{\text{die}}_G(A_\mathbb{R}(G))$ is contained in $B^\times(G)$. Since $B^\times$ is a biset subfunctor of $\beta^\times$, we can regard $\overline{\text{die}}$ as a morphism of biset functors

$$\overline{\text{die}} \,:\, A_\mathbb{R} \to B^\times \,.$$

We call $\overline{\text{die}}$ the **reduced tom Dieck morphism**. (In [4], the tom Dieck morphism die() was called the "lifted tom Dieck morphism" for the sake of clear contradistinction.)

Below, our strategy for proving Theorem 6.1.1 will be to treat it as a monomial analogue of tom Dieck's inclusion $\overline{\text{die}}(A_\mathbb{R}) \leq B^\times$. Just as an interesting aside, let us show how Yoshida's Criterion yields a quick direct proof of tom Dieck's inclusion. Consider an $\mathbb{R}G$-module M and an element $g \in G$. Let $m_+(g)$ and $m_-(g)$ be the multiplicities of 1 and -1, respectively, as eigenvalues of the action of g on M. Let $m(g)$ be the sum of the multiplicities of the non-real eigenvalues. Then $\dim(M) = m_+(g) + m_-(g) + m(g)$. Since the non-real eigenvalues occur in complex conjugate pairs, $m(g)$ is even and the determinant of the action of g is

$$\det(g : M) = \text{par}(m_-(g)) = \text{par}(m_+(g) - \dim(M)) = \frac{\text{par}(\dim(M^{\langle g\rangle}))}{\text{par}(\dim(M))} \,.$$

Let $x = \overline{\text{die}}_G[M]$. Consider a subgroup $I \leq G$ and an element $gI \in N_G(I)/I$. Replacing the $\mathbb{R}G$-module M with the $\mathbb{R}N_G(I)/I$-module M^I, we have

$$\det(gI : M^I) = \frac{\text{par}(\dim(M^{\langle I,g \rangle}))}{\text{par}(\dim(M^I))} = \frac{\epsilon^G_{\langle I,g \rangle}(x)}{\epsilon^G_I(x)} \, .$$

By the multiplicative property of determinants, x satisfies the criterion in Theorem 3.1, hence $x \in B^\times(G)$. The direct proof of the inclusion $\overline{\text{die}}(A_{\mathbb{R}}) \leq B^\times$ is complete.

However, lacking an analogue of Theorem 6.3.1 for the case $C = \{\pm 1\}$, we shall be unable to adapt the argument that we have just given. Tom Dieck's original proof of the inclusion $\overline{\text{die}}(A_{\mathbb{R}}) \leq B^\times$ is well-known, but let us briefly present it. Let K be an admissible G-equivariant triangulation of the G-sphere $S(M)$. Thus, K is a G-simplicial complex, admissible in the sense that the stabilizer of any simplex fixes the simplex, and the geometric realization of K is G-homeomorphic to $S(M)$. Recall that the Lefschetz invariant of $S(M)$ is

$$\Lambda_G(S(M)) = \sum_{\sigma \in_G K} \text{par}(\ell(\sigma)) \, [\text{Orb}_G(\sigma)]$$

as an element of $B(G)$, summed over representatives σ of the G-orbits of simplexes in K, where $\text{Orb}_G(\sigma)$ denotes the G-orbit of σ as a transitive G-set and $\ell(\sigma)$ denotes the dimension of σ. Here, we are not including any (-1)-simplex. For $I \leq G$, the subcomplex K^I consisting of the I-fixed simplexes is a triangulation of the I-fixed sphere $S(M)^I = S(M^I)$. Summing over all the simplexes σ in K^I, we have

$$\epsilon^G_I(\Lambda_G(S(M))) = \sum_{\sigma \in K^I} \text{par}(\ell(\sigma)) = \chi(S(M)^I) = 1 - \text{par}(\dim(M^I)) = \epsilon^G_I(1 - \overline{\text{die}}_G[M])$$

where χ denotes the Euler characteristic, equal to 2 or 0 for even-dimensional or odd-dimensional spheres, respectively. Therefore $\overline{\text{die}}_G[M] = 1 - \Lambda_G(S(M))$ and, perforce, $\overline{\text{die}}_G[M] \in B(G)$. But $\overline{\text{die}}_G[M] \in \beta^\times(G)$, hence $\overline{\text{die}}_G[M] \in B^\times(G)$. We have again established the inclusion $\overline{\text{die}}(A_{\mathbb{R}}) \leq B^\times$.

For the rest of this section, we put $C = \{\pm 1\}$. Thus, given a subgroup $I \leq G$, then $I/O(I)$ is the largest quotient group of I such that $I/O(I)$ is an elementary abelian 2-group. We shall prove Theorem 6.1.1 by adapting the above topological proof of the inclusion $\overline{\mathrm{die}}(A_{\mathbb{R}}) \leq B^{\times}$.

Let M be an $\mathbb{R}G$-module. Allowing C to act multiplicatively on M and on $S(M)$, let K be an admissible CG-equivariant triangulation of $S(M)$. Thus, the hypothesis on K is stronger than before, the extra condition being that, when we identify the vertices of K with their corresponding points of $S(M)$, the vertices occur in pairs, z and $-z$. More generally, identifying the simplexes of K with their corresponding subsets of $S(M)$, the simplexes occur in pairs, σ and $-\sigma$, the points of any simplex being the negations of the points of its paired partner. As an element of $B_C(G)$, we define the C**-monomial Lefschetz invariant** of M to be

$$\Lambda_{CG}(M) = \sum_{\sigma} \mathrm{par}(\ell(\sigma))\,[\mathrm{Orb}_{CG}(\sigma)]$$

where σ now runs over representatives of the CG-orbits of simplexes in K, and $[\mathrm{Orb}_{CG}(\sigma)]$ denotes the isomorphism class of the CG-orbit $\mathrm{Orb}_{CG}(\sigma)$ as a C-fibred G-set. A similar monomial Lefchetz invariant, in the context of a sufficiently large fibre group, was considered by Symonds in [22, Section 2]. To see that $\Lambda_{CG}(M)$ is an invariant of the CG-homotopy class of $S(M)$, observe that, regarding M as a CG-module and regarding $S(M)$ as a CG-space, then $\Lambda_{CG}(M)$ is determined by the usual Lefschetz invariant $\Lambda_{CG}(S(M)) \in B(CG)$, which is given by the same formula, but with $[\mathrm{Orb}_{CG}(\sigma)]$ reinterpreted as the isomorphism class of $\mathrm{Orb}_{CG}(\sigma)$ as a transitive CG-set.

Theorem 6.3.2. *Still assuming that $C = \{\pm 1\}$ and that M is an $\mathbb{R}G$-module then, for any C-subelement (I, i) of G, we have*

$$\epsilon^G_{I,i}(\Lambda_{CG}(M)) = \sum_{\psi \in \mathrm{Irr}_M(\mathbb{R}I)} \psi(i)$$

where $Irr_M(\mathbb{R}I)$ denotes the subset of $Irr(\mathbb{R}I)$ consisting of those irreducible $\mathbb{R}I$-characters that have odd multiplicity in the $\mathbb{R}I$-module $M^{O(I)}$. In particular, $\epsilon^G_{I,i}(\Lambda_{CG}(M)) \equiv_2 \dim_{\mathbb{R}}(M^{O(I)})$.

Proof. We have $\dim_{\mathbb{R}}(M^{O(I)}) = \sum_{\psi} m_{\psi}$ where, for the moment, ψ runs over all the irreducible $\mathbb{R}I$-characters and m_{ψ} is the multiplicity of ψ in the $\mathbb{R}I$-character of $M^{O(I)}$. If $m_\psi \neq 0$ then ψ is the inflation of an irreducible $\mathbb{R}I/O(I)$-character and, in particular, $\psi(i) = \pm 1$. Therefore, $\dim_{\mathbb{R}}(M^{O(I)}) \equiv_2 \sum_\psi \psi(i)$, where ψ now runs over those irreducible $\mathbb{R}\overline{I}$-characters such that m_ψ is odd. So the rider will follow from the main equality.

Put $\Lambda = \Lambda_{CG}(M)$. Since $\epsilon^G_{I,i}(\Lambda) = \epsilon^I_{I,i}(\mathrm{res}_{I,G}(\Lambda)) = \epsilon^I_{I,i}(\Lambda_{CI}(\mathrm{res}_{I,G}(M)))$, we can replace M with $\mathrm{res}_{I,G}(M)$. In other words, we may assume that $I = G$. Let K be an admissible CG-equivariant triangulation of the sphere $S(M)$. We have

$$\epsilon^G_{G,i}(\Lambda) = \sum_\sigma \mathrm{par}(\ell(\sigma))\, \epsilon^G_{G,i}[\mathrm{Orb}_{CG}(\sigma)]$$

where σ runs over representatives of the CG-orbits of simplexes of K. By the definition of $\epsilon^G_{G,i}$, contributions to the sum come from only those representatives σ such that the fibre $\{\sigma, -\sigma\}$ is stabilized by G, in other words, $\{\sigma, -\sigma\} = \mathrm{Orb}_{CG}(\sigma)$. Let A be the set of simplexes ρ of K whose fibre is stablized by G. Let $\overline{G} = G/O(G)$, and regard the irreducible $\mathbb{R}\overline{G}$-characters as irreducible $\mathbb{R}G$-characters by inflation. For all $\rho \in A$, we have

$$\epsilon^G_{G,i}[\mathrm{Orb}_{CG}(\rho)] = \epsilon^G_{G,i}[\{\rho, -\rho\}] = \psi_\rho(i)$$

where ψ_ρ is the irreducible $\mathbb{R}\overline{G}$-character such that $i\rho = \psi_\rho(i)\rho$. Since each CG-orbit in A owns exactly two simplexes,

$$2\epsilon^G_{G,i}(\Lambda) = \sum_{\rho \in A} \psi_\rho(i)\, \mathrm{par}(\ell(\rho)) \; .$$

Defining $A_\psi = \{\rho \in A \, : \, \psi_\rho = \psi\}$, we have a disjoint union $A = \bigcup_\psi A_\psi$ where ψ runs over the irreducible $\mathbb{R}\overline{G}$-characters. So

$$2\epsilon^G_{G,i}(\Lambda) = \sum_{\psi \in \mathrm{Irr}(\mathbb{R}\overline{G})} \psi(i) \sum_{\rho \in A_\psi} \mathrm{par}(\ell(\rho)) \; .$$

Meanwhile, we have a direct sum of $\mathbb{R}\overline{G}$-modules $M^{O(G)} = \bigoplus_\psi M_\psi$, where M_ψ is the sum of the $\mathbb{R}\overline{G}$-modules with character ψ. We claim that A_ψ is a triangulation of $S(M_\psi)$. To demonstrate the claim, we shall make use of the admissibility of K as a CG-complex. We have $M_\psi = M^{G_\psi}$, where G_ψ be the index 2 subgroup of CG such that if $\psi(i) = 1$ then $i \in G_\psi \not\ni -i$, otherwise

$i \notin G_\psi \ni -i$. But A_ψ is precisely the set of simplexes in K that are fixed by G_ψ. By the admissibility of K as a CG-complex, A_ψ is a triangulation of $S(M^{G_\psi})$. The claim is established. Therefore

$$\sum_{\rho \in A_\psi} \mathrm{par}(\ell(\rho)) = \chi(S(M_\psi)) = 1 - \mathrm{par}(\dim_{\mathbb{R}}(M_\psi)) \ .$$

We have shown that $\epsilon_G^{G,i}(\Lambda) = \sum_{\psi \in \mathrm{Irr}_M(\mathbb{R}G)} \psi(i)$, as required. □

We need to introduce a suitable ghost ring. As a subring of $\mathbb{Q}B_{\mathbb{R}}(G)$, we define

$$\beta_{\mathbb{R}}(G) = \bigoplus_{(I,i)} \mathbb{Z}\, e_{I,i}^G$$

where, as usual, (I, i) runs over representatives of the G-conjugacy classes of C-subelements of G. To distinguish $\beta_{\mathbb{R}}(G)$ from other ghost rings that are sometimes considered in other contexts, let us call $\beta_{\mathbb{R}}(G)$ the **full ghost ring** associated with $B_{\mathbb{R}}(G)$. We have $B_{\mathbb{R}}(G) \leq \beta_{\mathbb{R}}(G) < \mathbb{Q}B_{\mathbb{R}}(G)$, and an element $x \in \mathbb{Q}B_{\mathbb{R}}(G)$ belongs to $\beta_{\mathbb{R}}(G)$ if and only if each $\epsilon_{I,i}^G(x) \in \mathbb{Z}$. Let us mention that $\beta_{\mathbb{R}}(G)$ can be characterized in various other ways: as the $\mathbb{Z}$-span of the primitive idempotents of $\mathbb{Q}B_{\mathbb{R}}(G)$; as the integral closure of $B_{\mathbb{R}}(G)$ in $\mathbb{Q}B_{\mathbb{R}}(G)$; as the unique maximal subring of $\mathbb{Q}B_{\mathbb{R}}(G)$ that is finitely generated as a $\mathbb{Z}$-module.

Since $\epsilon_{I,i}^H(\mathrm{res}_{H,G}(x)) = \epsilon_{I,i}^G(x)$ for all $I \leq H \leq G$, the rings $\beta_{\mathbb{R}}(G)$ combine to form a restriction functor $\beta_{\mathbb{R}}$. Let us mention that, by [2, 5.4, 5.5], $\beta_{\mathbb{R}}$ commutes with induction as well as restriction and isogation, so we can regard $\beta_{\mathbb{R}}$ as a Mackey functor defined on the class of all finite groups. In fact, some unpublished results of Boltje and Olcay Coşkun imply that $\beta_{\mathbb{R}}$ is a biset functor. Let $\beta_{\mathbb{R}}^\times(G)$ denote the unit group of $\beta_{\mathbb{R}}(G)$. We have $B_{\mathbb{R}}^\times(G) \leq \beta_{\mathbb{R}}^\times(G)$, and $x \in \beta_{\mathbb{R}}^\times(G)$ if and only if each $\epsilon_{I,i}^G(x) \in C$. For the same reason as before, $\beta_{\mathbb{R}}^\times$ is a restriction functor. Actually, part of [2, 9.6] says that $\beta_{\mathbb{R}}^\times$ is a Mackey functor.

Lemma 6.3.3. *Let x be an element of $\mathbb{Z}_{(2)}B_{\mathbb{R}}(G)$ such that $\epsilon_{I,i}^G(x) \equiv_2 \epsilon_{I,j}^G(x)$ for all $I \leq G$ and $i, j \in I$. Write $\lim(x)$ to denote the idempotent of $\beta(G)$ such that $\epsilon_I^G(\lim(x)) \equiv_2 \epsilon_{I,i}^G(x)$. Then $\lim(x) \in \mathbb{Z}_{(2)}B(G)$.*

Proof. For any sufficiently large positive integer m, we have $2^m \mathbb{Z}_{(2)} \beta_{\mathbb{R}}(G) \subseteq \mathbb{Z}_{(2)} B_{\mathbb{R}}(G)$. Choose and fix such m. Let z be the element of $\mathbb{Z}_{(2)} \beta_{\mathbb{R}}(G)$ such that $\lim(x) = x + 2z$. Then

$$\lim(x) = \lim(x)^{2^n} = x^{2^n} + \sum_{j=1}^{2^n} \binom{2^n}{j} 2^j z^j x^{2^n - j}$$

for all positive integers n. When n is sufficiently large, 2^m divides all the binomial coefficients indexed by integers j in the range $1 \leq j \leq m - 1$. Choose and fix such n. Then $\lim(x) - x^{2^n}$ belongs to the subset $2^m \mathbb{Z}_{(2)} \beta_{\mathbb{R}}(G)$ of $\mathbb{Z}_{(2)} B_{\mathbb{R}}(G)$. Therefore $\lim(x) \in \mathbb{Z}_{(2)} B_{\mathbb{R}}(G)$. But $\lim(x)$ also belongs to $\mathbb{R}B(G)$, and the required conclusion now follows from Remark 6.2.1. $\square$

The rationale for the notation $\lim(x)$ is that, under the 2-adic topology, $\lim(x) = \lim_n x^{2^n}$.

We now turn to the reduced Boltje morphism $\overline{\text{bol}}$, which we defined in Section 1. Note that $\overline{\text{bol}}$ can be regarded as the modulo 2 reduction of $\text{bol}^{\{\pm 1\}, \mathbb{R}}$ because

$$\epsilon^G_{I,i}(\text{bol}^{\{\pm 1\}, \mathbb{R}}_G[M]) = \chi_I(i) \equiv_2 \dim(M^{O(I)})$$

where χ_I is the $\mathbb{R}I$-character of $M^{O(I)}$.

Theorem 6.3.4. *Still putting* $C = \{\pm 1\}$ *and letting* M *be an* $\mathbb{R}G$*-module, then*

$$\overline{\text{bol}}_G[M] = 1 - 2 \lim(\Lambda_{CG}(M)) \; .$$

Furthermore, $\lim(\Lambda_{CG}(M)) \in \mathbb{Z}_{(2)} B(G)$ *and* $\overline{\text{bol}}_G[M] \in \beta^{\times}_{(2)}(G)$.

Proof. By Theorem 6.3.2, $\epsilon^G_{I,i}(\Lambda_{CG}(M)) \equiv_2 \dim_{\mathbb{R}}(M^{O(I)})$ for any C-subelement (I, i). So the expression $\lim(\Lambda_{CG}(M))$ makes sense and the asserted equality holds. The rider follows from Lemma 6.3.3. $\square$

The proof of Theorem 6.1.1 is complete. As an aside, it is worth recording the following description of $\overline{\text{die}}_G[M]$ in terms of monomial Lefschetz invariants of M and $M \oplus \mathbb{R}$, where $\mathbb{R}$ denotes the trivial $\mathbb{R}G$-module.

Corollary 6.3.5. *Still putting $C = \{\pm 1\}$ and letting M be an $\mathbb{R}G$-module, then*

$$\overline{\mathrm{die}}_G[M] = \Lambda_{CG}(M \oplus \mathbb{R}) - \Lambda_{CG}(M) \ .$$

Proof. Let $\Lambda = \Lambda_{CG}(M)$ and $\Gamma = \Lambda_{CG}(M \oplus \mathbb{R})$. In the notation of Theorem 6.3.2,

$$\epsilon^G_{I,i}(\Gamma - \Lambda) = \begin{cases} 1 & \text{if the trivial } \mathbb{R}I\text{-module has odd multiplicity in } (M \oplus \mathbb{R})^{O(I)}, \\ -1 & \text{if the trivial } \mathbb{R}I\text{-module has odd multiplicity in } M^{O(I)}, \end{cases}$$

$$= \begin{cases} 1 & \text{if the trivial } \mathbb{R}I\text{-module has odd multiplicity in } M \oplus \mathbb{R}, \\ -1 & \text{if the trivial } \mathbb{R}I\text{-module has odd multiplicity in } M, \end{cases}$$

$$= \mathrm{par}(\dim_{\mathbb{R}}(M^I)) = \epsilon^G_I(\overline{\mathrm{die}}[M]) \ .$$

Since this is independent of i, we have $\Gamma - \Lambda \in B(G)$ and $\epsilon^G_I(\Gamma - \Lambda) = \epsilon^G_I(\overline{\mathrm{die}}[M])$.
□

6.4 Dimensions of subspaces fixed by subgroups

We shall prove Theorem 6.1.2, we shall show that Theorem 6.1.2 implies Theorem 6.1.3 and we shall also give a more direct proof of Theorem 6.1.3.

Let us begin with a direct proof of a special case of Theorem 6.1.2.

Theorem 6.4.1. *If G is a 2-group, then $\dim(M^{O(I)}) \equiv_2 \dim(M)$ for any $\mathbb{R}G$-module M and any subgroup $I \leq G$.*

Proof. First assume that G has a cyclic subgroup A such that $|G : A| \leq 2$. Letting $x = \overline{\mathrm{die}}_G[M]$, then $\epsilon^G_I(x) = \mathrm{par}(\dim(M^I))$, and we are to show that $\epsilon^G_{O(I)}(x) = \epsilon^G_1(x)$. Our assumption implies that one of the following holds: G is trivial; $O(I) = A < G$ and G is cyclic; $O(I) < A$. By dealing with each of those three possibilities separately, it is easy to see that $O(I)$ is cyclic with generator t^2 for some $t \in G$. A special case of Theorem 3.1 asserts that the function $G \ni g \mapsto \epsilon^G_{\langle g \rangle}(x)/\epsilon^G_1(x) \in \{\pm 1\}$ is a group homomorphism. Therefore

$\epsilon^G_{O(I)}(x)/\epsilon^G_1(x) = (\epsilon^G_{\langle t\rangle}(x)/\epsilon^G_1(x))^2 = 1$. The assertion is now established in the special case of the assumption.

For the general case, we shall argue by induction on $|G|$. We may assume that M is simple. Let us recall some material from [3], restating only those conclusions that we need, and only in the special cases that we need. A finite 2-group is called a **Roquette 2-group** provided every normal abelian subgroup is cyclic. A well-known result of Peter Roquette asserts that those 2-groups are precisely as follows: the cyclic 2-groups, the generalized quaternion 2-groups with order at least 8, the dihedral 2-groups with order at least 16, the semidihedral 2-groups with order at least 16. Part of the Genotype Theorem [3, 1.1] says that the simple $\mathbb{R}G$-module M can be written as an induced module $M = \mathrm{Ind}_{G,H}(S)$, where S is a simple $\mathbb{R}H$-module and $H/\mathrm{Ker}(S)$ is a Roquette 2-group.

If M is not absolutely simple, then the $\mathbb{C}G$-module $\mathbb{C} \otimes_{\mathbb{R}} M$ is the sum of two conjugate simple $\mathbb{C}G$-modules, hence each $\dim(M^{O(I)})$ is even and the required conclusion is trivial. So we may assume that M is absolutely simple. Then S must be absolutely simple too.

Suppose that $H = G$. If M is not faithful, then the required conclusion follows from the inductive hypothesis. If M is faithful, then G is a Roquette 2-group. By Roquette's classification, every Roquette 2-group has a cyclic subgroup with index at most 2, and we have already dealt with that case.

So we may assume that $H < G$. Let J be a maximal subgroup of G containing H and let $T = \mathrm{Ind}_{J,H}(S)$. The $\mathbb{R}J$-module T is absolutely simple because $M = \mathrm{Ind}_{G,J}(T)$. Let $x \in G - J$.

Suppose that $\dim(T) = 1$. Then the kernel $N = \mathrm{Ker}(T)$ has index at most 2 in J, so the kernel $N \cap {}^xN = \mathrm{Ker}(M)$ has index at most 2 in N and at most 8 in G. Moreover, if $\mathrm{Ker}(M) \neq N$ then $G/\mathrm{Ker}(M)$ is non-abelian. Replacing G with $G/\mathrm{Ker}(M)$, we reduce to the case where either $|G| = 2$ or else $|G| = 4$ or else G is non-abelian and $|G| = 8$. Any such G has a cyclic subgroup with index at most 2 and, again, the argument is complete in this case.

So we may assume that $\dim(T) \geq 2$. We shall deduce that $\dim(M^{O(I)})$ is even for all $I \leq G$. Identifying T with the subspace $1 \otimes T$ of M, we have $M = T \oplus xT$ as a direct sum of two simple $\mathbb{R}J$-modules. Noting that $O(I) \leq O(G) \leq J$, we have

$$M^{O(I)} = T^{O(I)} \oplus (xT)^{O(I)}$$

as a direct sum of real vector spaces. We are to show that

$$\dim(T^{O(I)}) \equiv_2 \dim((xT)^{O(I)}) \, .$$

If $I \leq J$, then $\dim(T^{O(I)}) \equiv_2 \dim(T) = \dim(xT) \equiv_2 \dim((xT)^{O(I)})$ because, by the inductive hypothesis, the assertion holds for J. Finally, suppose that $I \not\leq J$, in other words, $IJ = G$. The conjugation action of x^{-1} on J induces a transport of structure whereby $O(I)$ is sent to $x^{-1}O(I)x$ and the isomorphism class of xT is sent to the isomorphism class of T. Therefore $\dim((xT)^{O(I)}) = \dim(T^{x^{-1}O(I)x})$. But the element $x \in G - J$ was chosen arbitrarily and, since $IJ = G$, we may insist that $x \in I$, whereupon $x^{-1}O(I)x = O(I)$ and $\dim((xT)^{O(I)}) = \dim(T^{O(I)})$. □

We shall be needing the following result of Tornehave [26]. Another proof of it was given by Yalçın [27, 1.1].

Theorem 6.4.2. (Tornehave) *Supposing that G is a 2-group, then the reduced tom Dieck map $\overline{\text{die}}_G : A_{\mathbb{R}}(G) \to B^{\times}(G)$ is surjective.*

In view of Theorem 6.4.2, we see that Theorem 6.1.3 is equivalent to Theorem 6.4.1. Our direct proof of Theorem 6.1.3 is complete.

We mention another way of expressing Theorem 6.1.3. Let $\overline{\text{sgn}} : B^{\times} \to \beta_{(2)}$ be the unique restriction morphism such that, for any finite group G, the coordinate map $\overline{\text{sgn}}_G$ has image $\overline{\text{sgn}}_G(B^{\times}) = \{\pm 1_{B(G)}\}$. Thus, $\epsilon_I^G(\overline{\text{sgn}}(x)) = \epsilon_1^G(x)$ for all $I \leq G$ and $x \in B^{\times}(G)$. Plainly, Theorem 1.3 can be expressed as follows.

Theorem 6.4.3. *As restriction functors for the class of finite 2-groups, $\overline{\text{bol}} = \overline{\text{sgn}} \circ \overline{\text{die}}$.*

We now turn towards the task of proving Theorem 6.1.2. The following theorem of Andreas Dress can be found in, for instance, Benson [6, 5.4.8]. Let p be a prime. We write $\mathbb{Z}_{(p)}$ for the ring of p-local integers. We write $O^p(G)$ for the largest normal subgroup of G such that $G/O^p(G)$ is a p-group. Recall that G is said to be p-**perfect** provided $G = O^p(G)$.

Theorem 6.4.4. (Dress) *Given a prime p and an idempotent $y \in \mathbb{Q}B(G)$, then $y \in \mathbb{Z}_{(p)}B(G)$ if and only if $\epsilon_I^G(y) = \epsilon_{O^p(I)}^G(y)$ for all $I \leq G$. In particular, the condition $\epsilon_H^G(y) = 1$ characterizes a bijective correspondence between the primitive idempotents y of $\mathbb{Z}_{(p)}B(G)$ and the conjugacy classes of p-perfect subgroups H of G.*

The next corollary is worth mentioning, although it yields no constraints on the units of $B(G)$ and it will not be used below.

Corollary 6.4.5. *Given $x \in \mathbb{Z}_{(2)}B(G)$, then $\epsilon_I^G(x) \equiv_2 \epsilon_{O^2(I)}^G(x)$ for all $I \leq G$.*

Proof. The hypothesis on x implies that $\epsilon_{I,i}^G(x) = \epsilon_{I,j}^G(x)$ for all $I \leq G$ and all $i, j \in I$. By Lemma 3.3 and Theorem 4.3, $\epsilon_I^G(x) \equiv_2 \epsilon_I^G(\lim(x)) = \epsilon_{O^2(I)}^G(\lim(x)) \equiv_2 \epsilon_{O^2(I)}^G(x)$. □

Putting $C = \{\pm 1\}$ and letting M be an $\mathbb{R}G$-module, Theorems 6.3.4 and 6.4.3 yield

$$\dim(M^{O(I)}) \equiv_2 \epsilon_I^G(\Lambda_{CG}(M)) = \epsilon_{O^2(I)}^G(\Lambda_{CG}(M)) \equiv_2 \dim(M^{O(O^2(I))}) = \dim(M^{O^2(I)}) \ .$$

The proof of Theorem 6.1.2 is complete.

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