BLOCKS OF QUOTIENTS OF MACKEY ALGEBRAS

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

MASTER OF SCIENCE

IN

MATHEMATICS

By

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August, 2015

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We certify that I have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

BLOCKS OF QUOTIENTS OF MACKEY ALGEBRAS

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August, 2015

We review a theorem by Boltje and Külshammer which states that under certain circumstances the endomorphism ring $\mathrm{End}_{\mathrm{RG}}(\mathrm{RX})$ has only one block. We study the double Burnside ring, the Burnside ring and the transformations between two bases of it, namely the transitive G-set basis and the primitive idempotent basis. We introduce algebras Λ, Λ^{def} and Υ which are quotient algebras of the inflation Mackey algebra, the deflation Mackey algebra and the ordinary Mackey algebra respectively. We examine the primitive idempotents of $Z(\Upsilon)$. We prove that the algebra Λ has a unique block and give an example where Λ^{def} has two blocks.

Keywords: blocks, double Burnside ring, inflation Mackey algebra, deflation Mackey algebra, ordinary Mackey algebra.

ÖZET

MACKEY CEBİRLERİNİN BÖLÜM CEBİRLERİNİN BLOKLARI

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Ağustos, 2015

Boltje ve Külshammer'ın bazı özel koşullar altında özyapı dönüşüm halkası $\mathrm{End}_{\mathrm{RG}}(\mathrm{RX})$'in yalnızca bir bloku olduğunu gösteren bir teoremini sunacağız. İkili Burnside halkasını ve Burnside halkasını çalışacağız ve iki bazı arasındaki dönüşümü göstereceğiz. Λ, Λ^{def} ve Υ şeklinde göstereceğimiz üç cebir tanımlayacağız. Bu cebirler şişirme Mackey cebiri, söndürme Mackey cebiri ve adi Mackey cebirinin bölüm cebirleridir. Ardından $Z(\Upsilon)$'un ilkel idempotentlerini inceleyeceğiz. Λ cebirinin sadece bir bloku olduğunu gösterdikten sonra, Λ^{def}'in iki blokunun olduğu bir örnek vereceğiz.

Anahtar sözcükler: blok, ikili Burnside halkası, şişirme Mackey cebiri, söndürme Mackey cebiri, adi Mackey cebiri .

Acknowledgement

First of all, I would like to thank my thesis supervisor, Assoc. Prof. Dr. Laurence J. Barker, for his expertise and knowledge that he generously shared, his valuable guidance, patience and encouragement. Also he was really kind to share his unpublished results. My work would not have been possible without his support. I would like to sincerely thank my father Ender Doğan and my mother Nuran Doğan for every kind of support they provided, they are the most beautiful people I have ever met.

Also I want to thank my dearest friends Bengi Ruken Yavuz, Merve Demirel and Zeliha Ural for providing information and sharing experiences. I know that without them my life would be surely dull.

Finally, I want to thank my dear husband Salman Ul Hassan Dar for his support and encouragement. His existence is the best gift I have ever received.

Contents

Chapter 1

Introduction

In this thesis we focus on finding the blocks of some specific algebras. Throughout the thesis, $\mathbb{K}$ will denote a field with characteristic zero and $\mathfrak{K}$ a finite set of finite groups that is closed under subquotients up to isomorphism. The Burnside ring $B(G)$ of a finite group G, introduced by Dress [1], is the Grothendieck group of the category of finite G-sets with multiplication coming from direct product.

In Chapter 2, we give definition of a block and a theorem of Boltje and Külshammer [2] which states that under certain circumstances $\mathrm{End}_{\mathrm{RG}}(\mathrm{RX})$ has only one block.

In Chapter 3, we give the definition of bisets and the double Burnside ring. Also we see that the Burnside ring has two bases, namely the transitive G-set basis and the primitive idempotent basis. In addition, we will give the transformation between these two bases which is found by Gluck [5] and Yoshida [6], independently.

We have the inflation Mackey category, $B^{\triangleright} = B^{\triangleright}_{\mathfrak{K}}$, generated by ordinary inductions, ordinary restrictions and inflations. We also consider the deflation Mackey category , $B^{\triangleleft} = B^{\triangleleft}_{\mathfrak{K}}$, similarly defined.

Let ${}^{\oplus}\mathbb{K}B = \bigoplus_{F,G\in\mathfrak{K}} \mathbb{K}B(F,G)$ and define ${}^{\oplus}\mathbb{K}B^{\triangleright}$ and ${}^{\oplus}\mathbb{K}B^{\triangleleft}$ similarly. The problem of classifying the blocks of ${}^{\oplus}\mathbb{K}B^{\triangleright}$ is equivalent to the problem of classifying the blocks of ${}^{\oplus}\mathbb{K}B^{\triangleleft}$, because these two are opposite algebras of each other up to isomorphism and therefore they have isomorphic centres. However this problem is hard. In this thesis we will consider the quotient algebras Λ and Λ^{def}, defined below, instead of those two algebras. Now let ${}^{\oslash}\mathbb{K}B = \bigoplus_{G\in\mathfrak{K}} \mathbb{K}B(G)$, which is a module of the quiver algebra ${}^{\oplus}\mathbb{K}B$. Let ρ: ${}^{\oplus}\mathbb{K}B \to End_{\mathbb{K}}({}^{\oslash}\mathbb{K}B)$ be the representation associated with this module . We introduce the algebras

$$\Lambda = \Lambda_{\mathfrak{K}} := \rho({}^{\oplus}\mathbb{K}B^{\triangleright})$$

$$\Lambda^{\mathrm{def}} = \Lambda^{\mathrm{def}}_{\mathfrak{K}} := \rho({}^{\oplus}\mathbb{K}B^{\triangleleft}).$$

In Chapter 4, we show that Λ has a unique block. Also, we will give an example where Λ^{def} has two blocks.

Chapter 2

Blocks of Endomorphism Rings

2.1 Blocks of Endomorphism Rings

Let R be a unital ring. Recall that an idempotent of R is an element e such that $e^2 = e$. Also we call an idempotent e primitive if it cannot be written as a sum of two orthogonal idempotents. In other words, e cannot be written as $e = e_1 + e_2$ such that $e_1 e_2 = 0 = e_2 e_1$ where e_1 and e_2 are nonzero idempotents. We define a block of R to be a primitive idempotent of $Z(R)$.

Remark 2.1. Let A be a finite dimensional algebra over a field. Then $1 = \sum_{i=1}^{n} e_i$ where the e_i's are the primitive idempotents of $Z(A)$. Then

$$A = \bigoplus_{i=1}^{n} Ae_i$$

as a direct sum of algebras.

Remark 2.2. Let Λ be a ring and let $1_\Lambda = e_1 + e_2 + \cdots + e_n$ be a decomposition of 1_Λ into primitive idempotents $e_1, \ldots, e_n$ of Λ. Then every central idempotent e of Λ is equal to the subsum $e = \sum_{i \in I} e_i$ where I denotes the set of all elements $i \in \{1, 2, \ldots n\}$ satisfying $e_i e = e_i$.

In fact, for an arbitrary i , ee_i and $(1_\Lambda - e)e_i$ are orthogonal idempotents whose sum is e_i. Since e_i is primitive, we get $ee_i = e_i$ or $ee_i = 0$. If we multiply both sides of the equation $1_\Lambda = e_1 + e_2 + \cdots + e_n$ with e, we get the desired expression for e.

The next lemma and theorem is from Boltje and Külshammer [2].

Lemma 2.3 (Boltje-Külshammer)**.** *Let G be a finite group, R be an integral domain and X be a transitive G-set. If no prime divisor of $|X|$ is invertible in R, then the RG- permutation module RX is indecomposable.*

Proof. We may assume that $|X| \neq 1$. Assume that $RX = M \bigoplus N$ is a direct sum decomposition into RG-submodules and assume that $M \neq \{0\}$. Then M and N are finitely generated free R-modules and they have a well defined R-rank. Let $x \in X$ and $H := stab_G(x)$ and let p be a prime divisor of $|X| = [G : H]$. Since $pR \neq R$, there exists a maximal ideal P of R such that $p \in P$. Then $F := R/P$ is a field of characteristic p. Let $\overline{F}$ denote an algebraic closure of F. Then $\overline{F}X \cong (\overline{F} \otimes_R M) \oplus (\overline{F} \otimes_R N)$ where $\overline{F}X$, $\overline{F} \otimes_R M$ and $\overline{F} \otimes_R N$ are relatively H-projective $\overline{F}G$-modules. By Green's Indecomposibility Theorem [3], the p-part $[G : H]_p = |X|_p$ of $|X|$ divides

$$dim_{\overline{F}}(\overline{F} \otimes_R M) = dim_F(F \otimes_R M) = rk_{R_p}(R_p \otimes_R M) = rk_R(M)$$

Since p is arbitrary, we conclude that $|X|$ divides $rk_R(M)$. But,

$$0 \neq rk_R(M) \leq rk_R(M) + rk_R(N) = rk_R(M \oplus N) = rk_R(RX) = |X|,$$

which implies $rk_R(M) = |X|$ and $rk_R(N) = 0$. Thus, $N = 0$ and $M = RX$. □

Theorem 2.4 (Boltje-Külshammer)**.** *Let G be a finite group, X be a finite G-set and R be an integral domain. Assume that, for every $x \in X$ and for every prime divisor p of $[G : stab_G(x)]$, one has $\{0\} \neq pR \neq R$. Then the ring $\mathrm{End_{RG}(RX)}$ has a unique block.*

Proof. Let K denote the field of fractions of R. We decompose X into G-orbits, $X = X_1 \bigsqcup \cdots \bigsqcup X_n$, and obtain decompositions

$$RX = RX_1 \oplus \ldots RX_n \quad and \quad KX = KX_1 \oplus \ldots KX_n \tag{1.1}$$

into RG-submodules and KG-submodules, respectively. We decompose KX_i, for each $i = 1, 2, \dots, n$, into indecomposable KG- submodules:

$$KX_i = V_i^1 \oplus \cdots \oplus V_i^{r_i}. \tag{1.2}$$

We may assume that $V_i^1 \cong K$, the trivial KG-module. In fact, the hypothesis on R and X implies that $|X_i| \neq 0$ in K. This implies that $\iota : K \longrightarrow KX_i$, $1 \longmapsto |X_i|^{-1} \sum_{x \in X_i} x$ and $\pi : KX_i \longrightarrow K$, $x \longmapsto 1$ are KG-module homomorphisms with $\pi \circ \iota = id_K$, so that K is isomorphic to a direct summand of KX_i. Let $e_i \in \mathrm{End}_{RG}(RX)$ denote the idempotent which is the projection onto the i-th component in the first decomposition in 1.1. Then e_i is primitive in $\mathrm{End}_{RG}(RX)$ by Lemma 2.3. We view $\mathrm{End}_{RG}(RX)$ as a subring of $\mathrm{End}_{KG}(KX)$ via the canonical embedding and decompose e_i in $\mathrm{End}_{KG}(KX)$ further into primitive idempotents corresponding to the decomposition in 1.2.

$$e_i = e_i^{(1)} + \cdots + e_i^{(r_i)}.$$

Altogether we have a primitive decomposition

$$1 = (e_1^{(1)} + e_1^{(2)} + \cdots + e_1^{(r_1)}) + \cdots + (e_n^{(1)} + e_n^{(2)} + \cdots + e_n^{(r_n)}) \tag{1.3}$$

in $\mathrm{End}_{KG}(KX)$. Now let e be a non-zero central idempotent of $\mathrm{End}_{RG}(RX)$. Since $1 = e_1 + \cdots + e_n$ is a primitive decomposition of 1 in $\mathrm{End}_{RG}(RX)$, we have $e = \sum_{i \in I} e_i$ for some $\emptyset \neq I \subseteq \{1, 2, \dots n\}$ by Remark 2.2. Since e is also a central idempotent of $\mathrm{End_{KG}(KX)}$, it is also a subsum of the decomposition in (1.3). Since $\emptyset \neq I \subseteq \{1, 2, \dots n\}$, there exists an element $i \in I$, and we have $e_i e = e_i$. This implies that $e_i^{(1)} e = e_i^{(1)}$. For every $j \in \{1, 2, \dots n\}$ there exists an isomorphism $\alpha : KX \longrightarrow KX$ such that $\alpha e_i^{(1)} \alpha^{-1} = e_j^{(1)}$. The equation $e_i^{(1)} e = e_i^{(1)}$ implies

$$e_j^{(1)} = \alpha e_i^{(1)} \alpha^{-1} = \alpha e_i^{(1)} e \alpha^{-1} = \alpha e_i^{(1)} \alpha^{-1} e = e_j^{(1)} e.$$

This implies that $e_j e \neq 0$ and Remark 2.2 implies that $j \in I$. Thus $I = \{1, 2, \dots n\}$ and $e = 1$. □

Chapter 3

Double Burnside Ring

In this chapter we give the theory of bisets which was initiated by Bouc [4] and we define the double Burnside ring. Also we exhibit two bases of the Burnside algebra and give the transformation between them, which was found by Gluck [5] and Yoshida [6] independently.

3.1 Bisets and the Double Burnside Ring

In this section, we explain how general notions of induction and restriction can be expressed using bisets.

Definition 3.1. Let G and H be groups. An (G, H)-biset U is a set with a left G-action and a right H-action such that these actions commute, i.e.,

$$\forall g \in G, \forall u \in U, \forall h \in H, \qquad (g.u).h = g.(u.h).$$

Definition 3.2. Let G and H be finite groups. The double Burnside ring $B(G, H)$ consists of the formal differences of isomorphism classes of finite (G, H)-bisets. The addition is defined to be disjoint union of (G, H)-bisets, and multiplication is as follows

Definition 3.3 (Product of two bisets)**.** Let G, H and K be groups. The product of (G,H)-biset U and (H,K)-biset V is defined as the set of H orbits of the cartesian product $U \times V$ where the action of H is defined by $(u,v).h := (u.h, h^{-1}.v)$. It is denoted by $U \times_H V$ and the H-orbit of (u,v) is denoted by $(u,_H v)$. The set $U \times_H V$ is a (G,K)-biset with the actions $g.(u,_H v).k = (g.u,_H v.k)$.

Definition 3.4. Let U be a (G,H)-biset. Then for $u \in U$ we define the orbit of u as the set of elements whose form is guh where $g \in G$ and $h \in H$.

So we can write U as a disjoint union of its orbits:

$$U = \bigsqcup_{u \in G \backslash U / H} GuH$$

where u runs through the representatives of (G,H) orbits.

Definition 3.5. Let U be (G,H)-biset. U is called transitive if it has only one orbit.

We can see every (G,H)-biset as a $(G \times H)$-set by defining the action as

$$(g,h).u := guh^{-1}.$$

When (G,H)-biset U has only one orbit, i.e., U is transitive, it is isomorphic to $[(G \times H)/L_u]$ where L_u is the stabilizer of any element u of U in $G \times H$, i.e.,

$$L_u = (G,H)_u = \{(g,h) \in G \times H \mid gu = uh, \quad u \in U\}.$$

The isomorphism is

$$(g,h)L_u \in [(G \times H)/L_u] \to guh^{-1} \in U.$$

Since every (G,H)-biset is a disjoint union of transitive (G,H)-bisets, the double Burnside ring is the free $\mathbb{Z}$ module whose generators are the set of isomorphism classes of transitive (G,H)-bisets, ie,

$$B(G,H) = \bigoplus_{L \leq_{G,H} G \times H} \mathbb{Z} \left[\frac{G \times H}{L} \right].$$

Given a group homomorphism $\alpha : H \leftarrow G$, we define transitive morphisms induction as an (H, G)-biset such that,

$$ {}_H\text{ind}_G^\alpha = [H \times G/\{(\alpha(g), g) : g \in G\}] $$

and restriction as a (G, H)-biset such that

$$ {}_G\text{res}_H^\alpha := [G \times H/\{(g, \alpha(g)) : g \in G\}] . $$

When α is injective, following [7], we call ${}_H\text{ind}_G^\alpha$ an ordinary induction and ${}_G\text{res}_H^\alpha$ an ordinary restriction. When α is surjective we write ${}_H\text{def}_G^\alpha = {}_H\text{ind}_G^\alpha$ which we call deflation and we write ${}_G\text{inf}_H^\alpha = {}_G\text{res}_H^\alpha$ which we call inflation. When α is an isomorphism we write ${}_H\text{iso}_G^\alpha = {}_H\text{ind}_G^\alpha = {}_G\text{res}_H^{\alpha^{-1}}$ and call it isogation. When α is an inclusion we omit the symbol α from the notation, just writing ${}_H\text{ind}_G$ and ${}_G\text{res}_H$.

Following the notation of Bouc [8], let

$$ k_1(L) := \{h \in H | (h, 1) \in L\} $$

$$ k_2(L) := \{g \in G | (1, g) \in L\} $$

$$ p_1(L) := \{h \in H | \exists g \in G, (h, g) \in L\} $$

$$ p_2(L) := \{g \in G | \exists h \in H, (h, g) \in L\} . $$

Definition 3.6. (Star Product) The star product $*$ of two subgroups $L \leq G \times H$ and $M \leq H \times K$ is defined as

$$ L * M = \{(g, k) : (g, h) \in L \text{ and } (h, k) \in M \text{ for some } h \in H\} . $$

Due to Bouc [8], we have a formula for the product of two bisets.

Theorem 3.7 (Mackey Product Formula, [8]). *Let G, H, K be finite groups and let $L \leq G \times H$ and $M \leq H \times K$. Then*

$$ \left[\frac{G \times H}{L}\right] \times_H \left[\frac{H \times K}{M}\right] = \sum_{h \in [p_2(L) \backslash H / p_1(M)]} \left[\frac{G \times K}{L *^{(h,1)} M}\right]. $$

Also again by Bouc [8], we know that every transitive (G,H)-biset can be written as the composition of the five elementary bisets defined above.

Theorem 3.8 ([8])**.** *Let H and G be groups and $L \leq H \times G$.*

$$\left[\frac{H \times G}{L}\right] =_H \mathrm{ind}_D \mathrm{inf}_{D/C} \mathrm{iso}^{\varphi}_{B/A} \mathrm{def}_B \mathrm{res}_G$$

where $D = p_1(L)$, $C = k_1(L)$, $B = p_2(L)$, $A = k_2(L)$ and $\varphi : B/A \to D/C$ is an isomorphism.

3.2 Two Bases of the Burnside Algebra

In this section, we will describe two bases of the Burnside algebra and the transformation between these two bases found by Gluck [5] and independently by Yoshida [6].

A finite G-set X is a finite set on which G acts associatively. A G-set X is transitive when there is only one G-orbit in X. In that case, let $x \in X$ and let H be the stabilizer of x. Then there is an isomorphism between $[X]$ and $[G/H]$ (the left cosets of H in G). The isomorphism is

$$gx \in X \to gH \in G/H \text{ for } g \in G.$$

Let H and K be subgroups of G. Call H and K as G-conjugate, denoted by $H =_G K$, if $gHg^{-1} = K$ for some $g \in G$. Also, if $gHg^{-1} \subseteq K$ for some $g \in G$, we write $H \leq_G K$, and say that H is subconjugate to K.

Given arbitrary G-sets X and Y , we form their disjoint union $X \amalg Y$ and cartesian product $X \times Y$, both of which are G-sets. The action of G on $X \times Y$ is defined by

$$g.(x,y) = (gx, gy) \text{ for } g \in G,\ x \in X \text{ and } y \in Y.$$

Definition 3.9. The Burnside ring of a finite group G, denoted by $B(G)$, is the abelian group generated by the isomorphism classes $[X]$ of finite G-sets X with addition $[X]+[Y]=[X \uplus Y]$, the disjoint union of the G-sets X and Y. We define the multiplication for G-sets X and Y by $[X][Y]=[X \times Y]$, the direct product, which makes $B(G)$ a unital commutative ring.

Every G-set can be written as a disjoint union of transitive G-sets. Therefore, $\{[G/H] : H \leq_G G\}$ is a basis for $B(G)$, ie, $B(G) = \bigoplus_{H \leq_G G} \mathbb{Z}\,[G/H]$.

Note that, as $\mathbb{Z}$-modules, we can identify $B(G,H) = B(G \times H)$, but the product in the previous section is different from the ring multiplication defined in this section.

We define the Burnside algebra over $\mathbb{C}$ to be

$$\mathbb{C}B(G) = \mathbb{C} \otimes_Z B(G) = \bigoplus_{H \leq_G G} \mathbb{C}\,[G/H]\,.$$

Let $\left\{\epsilon_I^G : I \leq_G G\right\}$ be the set of algebra maps $\mathbb{C}B(G) \to \mathbb{C}$ where $\epsilon_I^G\,[X] = |X^I|$ and $|X^I|$ is the number of elements fixed by I for a G-set X . The set of primitive idempotents of $\mathbb{C}B(G)$ can be written as $\left\{e_I^G : I \leq_G G\right\}$ where $\epsilon_I^G(e_{I'}^G) = \delta_{(I,I')}$. Here $\delta_{(I,I')}$ is 1 if $I =_G I'$ and 0 otherwise. The next well known theorem can be found in, for instance, Ayşe Yaman's thesis [10].

Theorem 3.10. *$\left\{e_I^G : I \leq_G G\right\}$ gives another basis for $\mathbb{C}B(G)$.*

The table of marks, which we now define, is the transformation matrix from coordinates with respect to the basis $\{[G/U] : U \leq_G G\}$ to coordinates with respect to the basis $\left\{e_I^G : I \leq_G G\right\}$. Detailed information about it can be found in Ayşe Yaman's thesis [10].

Definition 3.11 (the table of marks)**.** The matrix $M_G = (m_G(I,U))_{I,U \leq_G G}$ with rows and columns indexed by representatives of the conjugacy classes of the subgroups of G, is called the Table of Marks where

$$m_G(I,U) = \epsilon_I^G\,[G/U] = |\,\{gU \subseteq G : IgU = gU\}\,| = |\,\{g \in G : I \leq^g U\}\,|/|U|.$$

We write the inverse of the table of marks as $M_G^{-1} = (m_G^{-1}(U,I))_{I,U \leq_G G}$.

We will use the transformation between these two bases in the next chapter, the transformations are

$$[G/U] = \sum_{I \leq_G G} m_G(I,U) e_I^G \qquad \textit{and} \qquad e_I^G = \sum_{U \leq_G G} m_G^{-1}(U,I)\,[G/U].$$

Remark 3.12. Let x be in $\mathbb{C}B(G)$, then x can be written as,

$$x = \sum_{I \leq_G G} \epsilon_I^G(x) e_I^G.$$

Chapter 4

The Blocks of Λ and Λ^{def}

In this chapter we will introduce algebras Λ, Λ^{def} and Υ. We will show that Λ has a unique block after classifying the blocks of Υ. Also we will give an example where Λ^{def} has two blocks. Our account is influenced by Barker and draws some parts from his unpublished notes.

4.1 Fundementals

In the previous chapter we defined the double Burnside ring. In this section we introduce algebras Λ, Λ^{def} and Υ. Let $\mathbb{K}$ be a field with characteristic zero and $\mathfrak{K}$ be a finite set of finite groups that is closed under subquotients up to isomorphism, i.e., if $K \trianglelefteq H \leq G \in \mathfrak{K}$ then an isomorphic copy of H/K belongs to $\mathfrak{K}$.

Definition 4.1. $B_{\mathfrak{K}}$ is the full subcategory of the biset category such that $Obj(B_{\mathfrak{K}}) = \mathfrak{K}$ and the $\mathbb{Z}$-module of morphisms $F \leftarrow G$ in $B_{\mathfrak{K}}$ is $B(F, G) = B(F \times G)$ where the composition operation $B(F, G) \times B(G, H) \to B(F, H)$ given by taking G-orbits of direct products. This category is generated by ordinary restrictions, ordinary inductions, deflations, inflations and isogations by Theorem 3.8.

Definition 4.2. (the inflation Mackey category) $B^{\triangleright} = B^{\triangleright}_{\mathfrak{K}}$ is the subcategory of $B_{\mathfrak{K}}$ such that the morphisms are generated by inflations, ordinary inductions and ordinary restrictions. The category $B^{\triangleright}$ is called the **inflation Mackey category** for $\mathfrak{K}$. Here, the transitive morphisms $[(F \times G)/I]$ are such that $k_2(I) = 1$. By Theorems 3.7 and 3.8, we have, for some epimorphism $\tau_I : p_1(I) \to p_2(I)$,

$$[(F \times G)/I] = {}_F\mathrm{ind}_{p_1(I)}\mathrm{inf}^{\tau_I}_{p_2(I)}\mathrm{res}_G.$$

These transitive morphisms comprise a basis for $B^{\triangleright}(F, G)$.

Definition 4.3. (the deflation Mackey category) $B^{\triangleleft} = B^{\triangleleft}_{\mathfrak{K}}$ is the subcategory of $B_{\mathfrak{K}}$ such that the morphisms are generated by deflations,ordinary inductions and ordinary restrictions. The category $B^{\triangleleft}$ is called the **deflation Mackey category** for $\mathfrak{K}$. Here, the transitive morphisms $[(F \times G)/I]$ are such that $k_1(I) = 1$. We have, for some epimorphism $\tau_I : p_1(I) \leftarrow p_2(I)$,

$$[(F \times G)/I] = {}_F\mathrm{ind}_{p_1(I)}\mathrm{def}^{\tau_I}_{p_2(I)}\mathrm{res}_G.$$

These transitive morphisms comprise a basis for $B^{\triangleleft}(F, G)$.

Definition 4.4. (the ordinary Mackey category) $B^{\Delta} = B^{\Delta}_{\mathfrak{K}}$ is the subcategory of $B_{\mathfrak{K}}$ such that the morphisms are generated by ordinary inductions and ordinary restrictions. The category B^{Δ} is called the **ordinary Mackey category** for $\mathfrak{K}$. Here, transitive morphisms $[(F \times G)/I]$ are such that $k_1(I) = 1 = k_2(I)$. We have, for some isomorphism $\tau_I : p_1(I) \to p_2(I)$,

$$[(F \times G)/I] = {}_F\mathrm{ind}_{p_1(I)}\mathrm{iso}^{\tau_I}_{p_2(I)}\mathrm{res}_G.$$

These transitive morphisms comprise a basis for $B^{\Delta}(F, G)$.

Now we consider the problem of classifying the blocks of the category $\mathbb{K}B^{\triangleright}$ for given $\mathbb{K}$ and $\mathfrak{K}$, we mean, the blocks of the algebra

$${}^{\oplus}\mathbb{K}B^{\triangleright} = \bigoplus_{F,G \in \mathfrak{K}} \mathbb{K}B^{\triangleright}(F, G).$$

This is equivalent to the problem of classifying the blocks of ${}^{\oplus}\mathbb{K}B^{\triangleleft}$ because these two are opposite algebras of each other up to isomorphism and therefore they have isomorphic centres. However this problem is hard. In this thesis we will consider quotients of these instead of these two algebras. Now let

${}^{\oslash}\mathbb{K}B ={}^{\oslash} \mathbb{K}B_{\mathfrak{K}} = \bigoplus_{G\in\mathfrak{K}} \mathbb{K}B(G)$. We make ${}^{\oslash}\mathbb{K}B$ a ${}^{\oplus}\mathbb{K}B$-module, via the evident isomorphism ${}^{\oslash}\mathbb{K}B \cong \bigoplus_{G\in\mathfrak{K}} \mathbb{K}B(G,1)$.

Let ρ: ${}^{\oplus}\mathbb{K}B \to End_{\mathbb{K}}({}^{\oslash}\mathbb{K}B)$ be the representation associated with this module. We introduce the algebras

$$\Lambda = \Lambda_{\mathfrak{K}} := \rho({}^{\oplus}\mathbb{K}B^{\triangleright})$$

$$\Lambda^{\mathrm{def}} = \Lambda^{\mathrm{def}}_{\mathfrak{K}} := \rho({}^{\oplus}\mathbb{K}B^{\triangleleft}) \quad \textit{and}$$

$$\Upsilon = \Upsilon_{\mathfrak{K}} := \rho({}^{\oplus}\mathbb{K}B^{\Delta}).$$

4.2 The Blocks of Υ

In this section we will investigate the blocks of Υ which will be used in the next section to find the blocks of Λ.
For a finite-dimensional algebra A, the Jacobson radical $J(A)$ is the unique maximal nilpotent ideal of A, i.e., it is the unique maximal ideal such that there exists a natural number k satisfying $(J(A))^k = 0$. Also, it is well known that $J(A)$ is the unique minimal ideal such that $A/J(A)$ is semisimple.

Theorem 4.5. *We have* ${}^{\oplus}\mathbb{K}B^{\triangleright} = {}^{\oplus}\mathbb{K}B^{\Delta} \oplus J({}^{\oplus}\mathbb{K}B^{\triangleright})$ *and* ${}^{\oplus}\mathbb{K}B^{\triangleleft} = {}^{\oplus}\mathbb{K}B^{\Delta} \oplus J({}^{\oplus}\mathbb{K}B^{\triangleleft})$. *In particular* ${}^{\oplus}\mathbb{K}B^{\Delta}$ *is semisimple.*

Proof. Proof of this theorem can be found in the paper of Barker [9, Theorem 5.3]. □

Now we will give an alternative proof to a lemma which can be found in a paper of Boltje-Külshammer [2, Theorem 5.2].

Lemma 4.6. *Let A be a unital ring and suppose that $A = B \oplus N$ where B is a unital subring with $1_A = 1_B$ and N is a nilpotent ideal. Then every idempotent of $Z(A)$ belongs to $Z(B)$.*

Proof. Let a be an idempotent of $Z(A)$. Since a is an idempotent we have $a^2 = a$ and therefore $a = a^i$ for every positive integer i. We write $a = b + n$ for some $b \in B$ and $n \in N$. Again since a is an idempotent we have

$$b + n = (b + n)^2 = b^2 + nb + bn + n^2$$

$b^2 \in B$ since B is a subring and $nb + bn + n^2 \in N$ since N is an ideal. Since we have a direct sum, these give $b^2 = b$ and therefore $b^i = b$ for every positive integer i.

Since N is nilpotent $n^k = (a - b)^k = 0$ for some positive integer k. This with $a^i = a$ for $i = 1, 2, \ldots, k$ gives

$$\begin{aligned} 0 = (a-b)^k &= a^k + \binom{k}{k-1} a^{k-1}(-b) + \binom{k}{k-2} a^{k-2}(-b)^2 + \cdots + (-b)^k \\ &= a + \binom{k}{k-1} a(-b) + \binom{k}{k-2} a(-b)^2 + \cdots + (-b)^k. \end{aligned}$$

If we multiply both sides with a and reduce the powers of a again, we get

$$\begin{aligned} 0 = a(a-b)^k &= a^{k+1} + \binom{k}{k-1} a^k(-b) + \binom{k}{k-1} a^{k-2}(-b)^2 + \cdots + a(-b)^k \\ &= a + \binom{k}{k-1} a(-b) + \binom{k}{k-2} a(-b)^2 + \cdots + a(-b)^k. \end{aligned}$$

This gives us $a(-b)^k = (-b)^k$ which implies with the fact that $b^i = b$, $ab = b$.
We have $ab = b$ which gives $(b + n)b = b^2 + nb = b$. Since $b^2 = b$, we get $nb = 0$.
Now since $b + n = a$ is in $Z(A)$, we have

$$bn + n^2 = (b + n)n = an = na = n(b + n) = nb + n^2$$

which implies $bn = nb$.
Since $a = b + n$ is an idempotent, we have $(b+n)^2 = b+n$. Since $bn = nb = 0$ we have $b^2 + n^2 = b + n$ which gives $n^2 = n$. Therefore $n^i = n$ for all positive integer i. Therefore we have $n = n^k = 0$ which means $a = b + n = b$. □

Definition 4.7. Let G and H be groups. If U is an (H, G)-biset, then the opposite biset U^{op} is the (G, H)-biset equal to U as a set, with actions defined by

$$\forall g \in G, \forall u \in U, \forall h \in H, \; guh(in\ U^{op}) = h^{-1}ug^{-1}(in\ U).$$

Definition 4.8. If G and H are groups, and L is a subgroup of $H \times G$, then the opposite subgroup L° is the subgroup of $G \times H$ defined by

$$L^\circ = \{(g,h) \in G \times H \mid (h,g) \in L\}.$$

Corollary 4.9. *Every idempotent of $Z({}^{\oplus}\mathbb{K}B^{\triangleright})$ and $Z({}^{\oplus}\mathbb{K}B^{\triangleleft})$ belongs to $Z({}^{\oplus}\mathbb{K}B^{\Delta})$.*

Proof. By using Theorem 4.5, and Lemma 4.6 we get the result for ${}^{\oplus}\mathbb{K}B^{\triangleright}$. Now take an idempotent $e = \sum_{L \leq (G\times G)} \lambda_L \left[\frac{G \times G}{L}\right]$ from $Z({}^{\oplus}\mathbb{K}B^{\triangleleft})$ where $G \in \mathfrak{K}$ and $\lambda_L \in \mathbb{K}$. Define $\phi : {}^{\oplus}\mathbb{K}B^{\triangleleft} \to {}^{\oplus}\mathbb{K}B^{\triangleright}$, $x \to x^\circ$ to be the linear map such that $\left[\frac{G \times G}{L}\right] \to \left[\frac{G \times G}{L^\circ}\right]$. Then $\phi(e)\phi(e) = \phi(e^2) = \phi(e)$.

Also if $\lambda_L \neq 0$ then $k_1(L) = 1$, and $k_2(L^\circ) = 1$ which implies $\phi(e)$ is an idempotent which belongs to $Z({}^{\oplus}\mathbb{K}B^{\triangleright})$. Then by the first part $\phi(e)$ belongs to $Z({}^{\oplus}\mathbb{K}B^{\Delta})$. Therefore, $e = \phi(\phi(e))$ also belongs to $Z({}^{\oplus}\mathbb{K}B^{\Delta})$ because opposite of the ordinary Mackey category is itself. □

Corollary 4.10. *The algebra Υ is semisimple and every idempotent of $Z(\Lambda)$ and $Z(\Lambda^{\text{def}})$ belongs to $Z(\Upsilon)$.*

Proof. By Theorem 4.5, we have $\rho({}^{\oplus}\mathbb{K}B^{\triangleright}) = \rho({}^{\oplus}\mathbb{K}B^{\Delta}) + \rho(J({}^{\oplus}\mathbb{K}B^{\triangleright}))$. $\rho(J({}^{\oplus}\mathbb{K}B^{\triangleright}))$ is a nilpotent ideal by being the image of a nilpotent ideal under ρ. Since Υ is semisimple, the intersection of Υ with any nilpotent ideal must be zero. Therefore $\Upsilon \cap \rho(J({}^{\oplus}\mathbb{K}B^{\triangleright})) = 0$. The result now follows from Lemma 4.6. □

Lemma 4.11, 4.14 and 4.17 can be found in the paper by Yoshida [6].

Lemma 4.11. *Given finite groups $H \leq G \geq I$, then*

$${}_H\text{res}_G(e_I^G) = \sum_{I' \leq_H H : I' =_G I} e_{I'}^H.$$

Proof. Since ${}_H\mathrm{res}_G(e_I^G) \in \mathbb{C}B(H)$ by Remark 3.12 we have,

$$ {}_H\mathrm{res}_G(e_I^G) = \sum_{I' \leq_H H} \epsilon_{I'}^H({}_H\mathrm{res}_G(e_I^G))e_{I'}^H. $$

Note that for $J \leq H \leq G$ and for any G-set X, we have

$$ \epsilon_J^H({}_H\mathrm{res}_G[X]) = \epsilon_J^G[X]. $$

By using these, we get,

$$ {}_H\mathrm{res}_G(e_I^G) = \sum_{I' \leq_H H} \epsilon_{I'}^H({}_H\mathrm{res}_G(e_I^G))e_{I'}^H = \sum_{I' \leq_H H} \epsilon_{I'}^G(e_I^G))e_{I'}^H = \sum_{I' \leq_H H : I' =_G I} e_{I'}^H. $$

□

Lemma 4.12 (Mackey Formula, [8])**.** *Let H and K be subgroups of G. Then*

$$ {}_K\mathrm{res}_G\mathrm{ind}_H = \sum_{g \in [K \backslash G/H]} {}_K\mathrm{ind}_{K \cap {}^gH}\mathrm{con}^g_{K^g \cap H}\mathrm{res}_H $$

where $[K \backslash G/H]$ is a set of representatives of (K, H)-double cosets in G and $\mathrm{con}^{\mathrm{g}}$ *is the group isomorphism induced by conjugation by g.*

Proof. By Theorem 3.7, we have

$$ {}_K\mathrm{res}_G\mathrm{ind}_H = \left[\frac{K \times G}{\{(k,k) : k \in K\}}\right]\left[\frac{G \times H}{\{(h,h) : h \in H\}}\right] $$

$$ = \sum_{g \in [K \backslash G/H]} \left[\frac{K \times H}{\{(k,k) : k \in K\} *^{(g,1)} \{(h,h) : h \in H\}}\right] $$

$$ = \sum_{g \in [K \backslash G/H]} \left[\frac{K \times H}{\{({}^gh, h) : h \in K^g \cap H\}}\right] $$

$$ = \sum_{g \in [K \backslash G/H]} \left[\frac{K \times (K \cap {}^gH)}{\{({}^gh, {}^gh) : {}^gh \in K \cap {}^gH\}}\right]\left[\frac{(K \cap {}^gH) \times (K^g \cap H)}{\{({}^gh, h) : h \in K^g \cap H\}}\right]\left[\frac{(K^g \cap H) \times H}{\{(h,h) : h \in K^g \cap H\}}\right] $$

$$ = \sum_{g \in [K \backslash G/H]} {}_K\mathrm{ind}_{K \cap {}^gH}\mathrm{con}^g_{K^g \cap H}\mathrm{res}_H. $$

□

Lemma 4.13. *For $H \leq G$,*

$$ {}_G\mathrm{ind}_H(e_H^H) = |N_G(H) : H|e_H^G. $$

Proof. Take any $K \leq G$, we have

$$\epsilon_K^G({}_G\text{ind}_H(e_H^H)) = \epsilon_K^K({}_K\text{res}_G\text{ind}_H(e_H^H)).$$

By using Lemma 4.12,

$$= \epsilon_K^K(\sum_{KgH \subseteq G} {}_K\text{ind}_{K \cap {}^gH}\text{con}_{K^g \cap H}^g\text{res}_H(e_H^H)).$$

By the Lemma 4.11 we know that restriction of e_H^H to any proper subset of H is zero. Therefore $K^g \cap H = H$ and $K = K \cap {}^gH$, so $K^g = H$ and $K = {}^gH$.

$$= \sum_{KgH \subseteq G : K = {}^gH} \epsilon_K^K({}_{{}^gH}\text{con}_H^g(e_H^H)).$$

Here we have, $\epsilon_K^K({}_{{}^gH}\text{con}_H^g(e_H^H)) = \epsilon_{{}^gH}^{{}^gH}(e_{{}^gH}^{{}^gH}) = 1$. Therefore,

$$\epsilon_K^G({}_G\text{ind}_H(e_H^H)) = \begin{cases} |N_G(H) : H|, & \text{if } K =_G H \\ 0, & \text{otherwise} \end{cases}$$

Therefore we have;

$${}_G\text{ind}_H(e_H^H) = \sum_{K \leq_G G} \epsilon_K^G({}_G\text{ind}_H(e_H^H))e_K^G = \sum_{K \leq_G G : K =_G H} |N_G(H) : H|e_K^G = |N_G(H) : H|e_H^G.$$

□

Lemma 4.14. *Given finite groups $J \leq H \leq G$, then*

$${}_G\text{ind}_H(e_J^H) = |N_G(J) : N_H(J)|e_J^G.$$

Proof. Let $J \leq H$, we have

$$|N_G(J) : J|e_J^G = {}_G\text{ind}_J(e_J^J) = {}_G\text{ind}_H\text{ind}_J(e_J^J) = |N_H(J) : J|{}_G\text{ind}_H(e_J^H)$$

which implies;

$${}_G\text{ind}_H(e_J^H) = |N_G(J) : N_H(J)|e_J^G.$$

□

For $K \in \mathfrak{K}$, let S_K be the subspace of the ${}^{\oslash}\mathbb{K}B$ spanned by the elements e_I^F such that $K \cong I \leq F \in \mathfrak{K}$. As a direct sum of Υ-modules,

$${}^{\oslash}\mathbb{K}B = \bigoplus_{K \in_{\bigstar} \mathfrak{K}} S_K$$

where K runs over representatives of the isomorphism classes of groups in $\mathfrak{K}$.

Proposition 4.15. *For each $K \in \mathfrak{K}$, the Υ-submodule S_K of ${}^{\oslash}\mathbb{K}B$ is simple.*

Proof. Let S be a non zero Υ-submodule of S_K. Take a non zero element $s \in S$. We have $s = \sum_{G,J} a_J^G e_J^G$ where G runs over the groups in $\mathfrak{K}$ and J runs over the G-conjugacy classes of subgroups of G such that $J \cong K$. There exists at least one pair (G, J) such that $a_J^G \neq 0$. Let $\phi : G \leftarrow K$ be a group monomorphism such that $J = \phi(K)$. By Lemma 4.11, we have

$${}_K\mathrm{res}_G^{\phi}(e_J^G s)/(a_J^G) = e_K^K \in S.$$

Also let I and F be such that $K \cong I \leq F \in \mathfrak{K}$. Let $\psi : F \leftarrow G$ be a group monomorphism such that $I = \psi(K)$. By Lemma 4.14, we have

$${}_F\mathrm{ind}_K^{\psi}(e_K^K)/|N_F(I)| = e_I^F \in S.$$

□

For every $K \in \mathfrak{K}$, let d_K be the $\mathbb{K}$-endomorphism of ${}^{\oslash}\mathbb{K}B$ projecting to S_K and annihilating all the other simple modules, i.e., $d_K(e_I^G) = \lfloor K \cong I \rfloor e_I^G$.

Proposition 4.16. *The set $\{d_K : K \in_{\bigstar} \mathfrak{K}\}$ is the set of primitive idempotents of $Z(\Upsilon)$.*

Proof. Since Υ acts faithfully on ${}^{\oslash}\mathbb{K}B = \bigoplus_{K \in_{\bigstar} \mathfrak{K}} S_K$, we have

$$\Upsilon = \mathrm{End}_{\mathbb{K}}(S_1) \oplus \cdots \oplus \mathrm{End}_{\mathbb{K}}(\mathrm{S_K}).$$

Also, since Υ is semisimple, we have

$$\Upsilon = \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(\mathbb{K}).$$

Without loss of generality, one can say $\mathrm{Mat}_{n_i}(\mathbb{K}) \cong \mathrm{End}_{\mathbb{K}}(S_i)$. Now $Z(\Upsilon) = \bigoplus_{i=1}^{k} \mathbb{K} I_i$ where I_i is the identity matrix for $\mathrm{Mat}_{n_i}(\mathbb{K})$. So primitive idempotents are $(\cdots, 0, I_i, 0, \cdots)$ for $i = 1, \cdots, k$ which corresponds to d_k's.

□

4.3 The Blocks of Λ

We will show that Λ has a unique block by using the blocks of Υ which were found in the previous section.

Lemma 4.17. *Given finite groups $N \trianglelefteq G$ and $N \le H \le G$, then*

$$ {}_G\mathrm{inf}_{G/N}(e_{H/N}^{G/N}) = \sum_{I \le_G G : IN =_G H} e_I^G . $$

Proof. Regarding a G/N-set X as a G-set by inflation, we have $X^I = X^{IN/N}$, ie, $\epsilon_I^G(X) = \epsilon_{IN/N}^{G/N}[X]$. So we have,

$$ {}_G\mathrm{inf}_{G/N}(e_{H/N}^{G/N}) = \sum_{I \le_G G} \epsilon_I^G({}_G\mathrm{inf}_{G/N}(e_{H/N}^{G/N})) e_I^G = \sum_{I \le_G G} \epsilon_{IN/N}^{G/N}(e_{H/N}^{G/N}) e_I^G = \sum_{I \le_G G : IN =_G H} e_I^G . $$

□

Theorem 4.18. *The algebra Λ has a unique block.*

Proof. By Corollary 4.10 and Proposition 4.16, every idempotent of $Z(\Lambda)$ is a sum of idempotents having the form d_K where $K \in \mathfrak{K}$. Given an idempotent d of $Z(\Lambda)$ and $K \in \mathfrak{K}$, we have $dd_K = d_K$ or $dd_K = 0$. We define an equivalence relation $\equiv$ such that, given $K, K' \in \mathfrak{K}$, $K \equiv K'$ provided, for all idempotents d of $Z(\Lambda)$, we have $dd_K = d_K$ if and only if $dd_{K'} = d_{K'}$.

Let δ_K be the unique block of Λ such that $\delta_K d_K = d_K$. So we have $\delta_K = \sum_{K'} d_{K'}$ where K' runs over representatives of the isomorphism classes of groups in $\mathfrak{K}$ such that $K \equiv K'$. Now using Lemma 4.17,

$$\begin{aligned}({}_K\mathrm{inf}_1.\delta_{\mathrm{K}})(\mathrm{e}_1^1\mathrm{e}_{\mathrm{K}}^{\mathrm{K}}) = (\delta_{\mathrm{K}}.{}_{\mathrm{K}}\mathrm{inf}_1)(\mathrm{e}_1^1\mathrm{e}_{\mathrm{K}}^{\mathrm{K}}) &= \delta_K(e_K^K) + \sum_{H \lneqq_K K} \delta_K e_H^K e_K^K \\ &= (\delta_K d_K)(e_K^K) + \sum_{H \lneqq_K K} (\delta_K d_H) e_H^K e_K^K \\ &= d_K(e_K^K) = e_K^K \neq 0.\end{aligned}$$

Therefore $0 \neq \delta_K(e_1^1 e_K^K) = \delta_K.d_1(e_1^1 e_K^K)$. In particular, $\delta_K.d_1 \neq 0$. Therefore $K \equiv 1$ for arbitrary $K \in \mathfrak{K}$. The equivalence relation $\equiv$ has a unique equivalence class. □

4.4 The Case $\mathfrak{K} = \{1, C_2, V_4\}$

In this section we shall show that Λ has only one block and Λ^{def} has two blocks for $\mathfrak{K} = \{1, C_2, V_4\}$. Let $V_4 = \{1, a, b, c\}$, $A = \{1, a\}$, $B = \{1, b\}$ and $C = \{1, c\}$. For $\mathfrak{K} = \{1, C_2, V_4\}$, we have the basis $\{e_1^1, e_1^{C_2}, e_{C_2}^{C_2}, e_1^{V_4}, e_A^{V_4}, e_B^{V_4}, e_C^{V_4}, e_{V_4}^{V_4}\}$ for the Burnside algebra. We will write inductions, restrictions, isogations, inflations and deflations below. All basis elements goes to zero unless stated otherwise.

$${}_{\mathrm{C_2}}\mathrm{ind}_1 = \left\{ e_1^1 \to 2e_1^{C_2} \right. , \quad {}_{\mathrm{V_4}}\mathrm{ind}_1 = \left\{ e_1^1 \to 4e_1^{V_4} \right.$$

$${}_{V_4}\mathrm{ind}_{\mathrm{A}} = \begin{cases} e_1^{C_2} & \to 2e_1^{V_4} \\ e_{C_2}^{C_2} & \to 2e_A^{V_4} \end{cases}, \quad {}_{\mathrm{V_4}}\mathrm{ind}_{\mathrm{B}} = \begin{cases} e_1^{C_2} & \to 2e_1^{V_4} \\ e_{C_2}^{C_2} & \to 2e_B^{V_4} \end{cases}, \quad {}_{\mathrm{V_4}}\mathrm{ind}_{\mathrm{C}} = \begin{cases} e_1^{C_2} & \to 2e_1^{V_4} \\ e_{C_2}^{C_2} & \to 2e_C^{V_4} \end{cases}$$

$${}_1\mathrm{res}_{\mathrm{C_2}} = \left\{ e_1^{C_2} \to e_1^1 \right. , \quad {}_1\mathrm{res}_{\mathrm{V_4}} = \left\{ e_1^{V_4} \to e_1^1 \right.$$

$${}_A\mathrm{res}_{\mathrm{V_4}} = \begin{cases} e_1^{V_4} & \to e_1^{C_2} \\ e_A^{V_4} & \to e_{C_2}^{C_2} \end{cases}, \quad {}_{\mathrm{B}}\mathrm{res}_{\mathrm{V_4}} = \begin{cases} e_1^{V_4} & \to e_1^{C_2} \\ e_B^{V_4} & \to e_{C_2}^{C_2} \end{cases}, \quad {}_{\mathrm{C}}\mathrm{res}_{\mathrm{V_4}} = \begin{cases} e_1^{V_4} & \to e_1^{C_2} \\ e_C^{V_4} & \to e_{C_2}^{C_2} \end{cases}$$

$$
{}_{1}\mathrm{iso}_{1} = \begin{cases} e_1^1 & \to e_1^1 \end{cases}, \quad {}_{\mathrm{C_2}}\mathrm{iso}_{\mathrm{C_2}} = \begin{cases} e_1^{C_2} & \to e_1^{C_2} \\ e_{C_2}^{C_2} & \to e_{C_2}^{C_2} \end{cases}, \quad {}_{\mathrm{V_4}}\mathrm{iso}_{\mathrm{V_4}} = \begin{cases} e_1^{V_4} & \to e_1^{V_4} \\ e_A^{V_4} & \to e_A^{V_4} \\ e_B^{V_4} & \to e_B^{V_4} \\ e_C^{V_4} & \to e_C^{V_4} \\ e_{V_4}^{V_4} & \to e_{V_4}^{V_4} \end{cases}
$$

$$
{}_{C_2}\mathrm{inf}_{1} = \begin{cases} e_1^1 & \to e_1^{C_2} + e_{C_2}^{C_2} \end{cases}, \quad {}_{\mathrm{V_4}}\mathrm{inf}_{1} = \begin{cases} e_1^1 & \to e_1^{V_4} + e_A^{V_4} + e_B^{V_4} + e_C^{V_4} + e_{V_4}^{V_4} \end{cases}
$$

$$
{}_{V_4}\mathrm{inf}_{\mathrm{V_4/A}} = \begin{cases} e_1^{C_2} & \to e_1^{V_4} + e_A^{V_4} \\ e_{C_2}^{C_2} & \to e_B^{V_4} + e_C^{V_4} + e_{V_4}^{V_4} \end{cases}, \quad {}_{\mathrm{V_4}}\mathrm{inf}_{\mathrm{V_4/B}} = \begin{cases} e_1^{C_2} & \to e_1^{V_4} + e_B^{V_4} \\ e_{C_2}^{C_2} & \to e_A^{V_4} + e_C^{V_4} + e_{V_4}^{V_4} \end{cases}
$$

$$
{}_{V_4}\mathrm{inf}_{\mathrm{V_4/C}} = \begin{cases} e_1^{C_2} & \to e_1^{V_4} + e_C^{V_4} \\ e_{C_2}^{C_2} & \to e_A^{V_4} + e_B^{V_4} + e_{V_4}^{V_4} \end{cases}
$$

There is no short formula for deflations for the primitive idempotent basis. Therefore, we computed deflations by the transitive G-set basis and then passed to the primitive idempotent basis by using the transformation formula by Gluck [5]. For this, we need the table of marks and the inverse of it.

$M_G(I,U)$	1	A	B	C	V_4
1	4	2	2	2	1
A	0	2	0	0	1
B	0	0	2	0	1
C	0	0	0	2	1
V_4	0	0	0	0	1

$$ {}_1\mathrm{def}_{\mathrm{C_2}} = \begin{cases} e_1^{C_2} & \to 1/2e_1^1 \\ e_{C_2}^{C_2} & \to 1/2e_1^1 \end{cases}, \quad {}_1\mathrm{def}_{\mathrm{V_4}} = \begin{cases} e_1^{V_4} & \to 1/4e_1^1 \\ e_A^{V_4} & \to 1/4e_1^1 \\ e_B^{V_4} & \to 1/4e_1^1 \\ e_C^{v_4} & \to 1/4e_1^1 \end{cases} $$

$$ {}_{V_4/A}\mathrm{def}_{\mathrm{V_4}} = \begin{cases} e_1^{V_4} & \to 1/2e_1^{C_2} \\ e_A^{V_4} & \to 1/2e_1^{C_2} \\ e_B^{V_4} & \to 1/2e_{C_2}^{C_2} \\ e_C^{V_4} & \to 1/2e_{C_2}^{C_2} \end{cases}, {}_{\mathrm{V_4/B}}\,\mathrm{def}_{\mathrm{V_4}} = \begin{cases} e_1^{V_4} & \to 1/2e_1^{C_2} \\ e_A^{V_4} & \to 1/2e_{C_2}^{C_2} \\ e_B^{V_4} & \to 1/2e_1^{C_2} \\ e_C^{V_4} & \to 1/2e_{C_2}^{C_2} \end{cases} $$

$$ {}_{V_4/C}\mathrm{def}_{\mathrm{V_4}} = \begin{cases} e_1^{V_4} & \to 1/2e_1^{C_2} \\ e_A^{V_4} & \to 1/2e_{C_2}^{C_2} \\ e_B^{V_4} & \to 1/2e_{C_2}^{C_2} \\ e_C^{V_4} & \to 1/2e_1^{C_2} \end{cases} $$

Also we can give them in matrix form. Let $a_{i,j}$ represent the entry in the i'th row and j'th column of the corresponding matrix. All entries of the matrices below are zero unless stated otherwise.

$$ \begin{aligned} {}_{C_2}\mathrm{ind}_1 &= \left\{a_{(2,1)} = 2\right\}, & {}_{V_4}\mathrm{ind}_1 &= \left\{a_{(4,1)} = 4\right\}, \\ {}_{V_4}\mathrm{ind}_{\mathrm{A}} &= \left\{a_{(4,2)} = a_{(5,3)} = 2\right\}, & {}_{V_4}\mathrm{ind}_{\mathrm{B}} &= \left\{a_{(4,2)} = a_{(6,3)} = 2\right\}, \\ {}_{V_4}\mathrm{ind}_{\mathrm{C}} &= \left\{a_{(4,2)} = a_{(7,3)} = 2\right\}, & {}_1\mathrm{res}_{\mathrm{C_2}} &= \left\{a_{(1,2)} = 1\right\}, \\ {}_1\mathrm{res}_{\mathrm{V_4}} &= \left\{a_{(1,4)} = 1\right\}, & {}_A\mathrm{res}_{\mathrm{V_4}} &= \left\{a_{(2,4)} = a_{(3,5)} = 1\right\} \\ {}_B\mathrm{res}_{\mathrm{V_4}} &= \left\{a_{(2,4)} = a_{(3,6)} = 1\right\}, & {}_C\mathrm{res}_{\mathrm{V_4}} &= \left\{a_{(2,4)} = a_{(3,7)} = 1\right\} \\ {}_1\mathrm{iso}_1 &= \left\{a_{(1,1)} = 1\right\}, & {}_{C_2}\mathrm{iso}_{\mathrm{C_2}} &= \left\{a_{(2,2)} = a_{(3,3)} = 1\right\} \end{aligned} $$

$$ \text{and } {}_{V_4}\mathrm{iso}_{\mathrm{V_4}} = \left\{\mathrm{a}_{(4,4)} = \mathrm{a}_{(5,5)} = \mathrm{a}_{(6,6)} = \mathrm{a}_{(7,7)} = \mathrm{a}_{(8,8)} = 1\right\} $$

$$
\begin{aligned}
{}_{C_2}\mathrm{inf}_1 &= \{a_{(2,1)} = a_{(3,1)} = 1\} \\
{}_{V_4}\mathrm{inf}_1 &= \{a_{(4,1)} = a_{(5,1)} = a_{(6,1)} = a_{(7,1)} = a_{(8,1)} = 1\} \\
{}_{V_4}\mathrm{inf}_{\mathrm{V_4/A}} &= \{a_{(4,2)} = a_{(5,2)} = a_{(6,3)} = a_{(7,3)} = a_{(8,3)} = 1\} \\
{}_{V_4}\mathrm{inf}_{\mathrm{V_4/B}} &= \{a_{(4,2)} = a_{(5,3)} = a_{(6,2)} = a_{(7,3)} = a_{(8,3)} = 1\} \\
{}_{V_4}\mathrm{inf}_{\mathrm{V_4/C}} &= \{a_{(4,2)} = a_{(5,3)} = a_{(6,3)} = a_{(7,2)} = a_{(8,3)} = 1\} \\
{}_1\mathrm{def}_{\mathrm{C_2}} &= \{a_{(1,2)} = a_{(1,3)} = 1/2\} \\
{}_1\mathrm{def}_{\mathrm{V_4}} &= \{a_{(1,4)} = a_{(1,5)} = a_{(1,6)} = a_{(1,7)} = 1/4\} \\
{}_{V_4/A}\mathrm{def}_{\mathrm{V_4}} &= \{a_{(2,4)} = a_{(2,5)} = a_{(3,6)} = a_{(3,7)} = 1/2\} \\
{}_{V_4/B}\mathrm{def}_{\mathrm{V_4}} &= \{a_{(2,4)} = a_{(3,5)} = a_{(2,6)} = a_{(3,7)} = 1/2\} \\
{}_{V_4/C}\mathrm{def}_{\mathrm{V_4}} &= \{a_{(2,4)} = a_{(3,5)} = a_{(3,6)} = a_{(2,7)} = 1/2\}
\end{aligned}
$$

By Corollary 4.10, we know that every idempotent of $Z(\Lambda)$ belongs to $Z(\Upsilon)$. So every idempotent of $Z(\Lambda)$ is a sum of d_k's for $k = 1,2,3$. Here $d_1 = diag(1,1,0,1,0,0,0,0)$, $d_2 = diag(0,0,1,0,1,1,1,0)$, $d_3 = diag(0,0,0,0,0,0,0,1)$. We have to find which of those commute with inflations.

Now take an idempotent δ from Λ. Then δ has the form $\delta = ad_1 + bd_2 + cd_3$ where $a, b \in \{0,1\}$. Commutativity with ${}_{V_4}\mathrm{inf}_1$ gives us $a = b = c$. So only idempotents are 0 and 1. It has just one block.

However that is not the case for Λ^{def}.

Theorem 4.19. Λ^{def} *has two blocks for* $\mathfrak{K} = \{1, C_2, V_4\}$.

Proof. By Corollary 4.10, we know that every idempotent of $Z(\Lambda^{\mathrm{def}})$ belongs to $Z(\Upsilon)$. So every idempotent of $Z(\Lambda^{\mathrm{def}})$ is a sum of d_k as above. We have to find which of those commute with deflations.

Now take an idempotent δ which $\delta d_1 = d_1$. Then δ has the form $\delta = d_1 + ad_2 + bd_3$ where $a, b \in \{0,1\}$. Commutativity with ${}_{V_4/A}\mathrm{def}_{\mathrm{V_4}}$ gives us $a = 1$. So every idempotent that contains d_1 should also contain d_2. However no other deflation gives any further restriction. Therefore we have $\delta = d_1 + d_2$ or $\delta = d_1 + d_2 + d_3$.

Now take δ which $\delta d_3 = d_3$. So δ has the form $\delta = ad_1 + bd_2 + d_3$ where

$a, b \in \{0, 1\}$. If we check commutativity with deflations for this δ we get no restriction except that if it contains d_1, it has to contain d_2 too. Therefore we have another idempotent d_3.

Between $0, 1$ and these three idempotents, $d_1 + d_2$ and d_3 are primitive. $\square$

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