ON SOME OF THE SIMPLE COMPOSITION FACTORS OF THE BISET FUNCTOR OF P-PERMUTATION MODULES

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ON SOME OF THE SIMPLE COMPOSITION FACTORS OF THE BISET FUNCTOR OF *P*-PERMUTATION MODULES

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Let k be an algebraically closed field of characteristic p, which is a prime, and $\mathbb{C}$ denote the field of complex numbers. Given a finite group G, letting $pp_k(G)$ denote the Grothendieck group of p-permutation kG-modules, we consider the biset functor of p-permutation modules, $\mathbb{C}pp_k$, by tensoring with $\mathbb{C}$. By a theorem of Serge Bouc, it is known that the simple biset functors $S_{H,V}$ are parametrized by pairs (H, V) where H is a finite group, and V is a simple $\mathbb{C}\mathrm{Out}(H)$-module. At present, the full classification of the simple biset functors apparent in $\mathbb{C}pp_k$ is not known. In this thesis, we find new simple functors $S_{H,V}$ apparent in $\mathbb{C}pp_k$ where H is a specific type of p-hypo-elementary B-group. The technique for this result makes use of Maxime Ducellier's notion of a p-permutation functor and his use of D-pairs to classify the simple factors of the p-permutation functor of p-permutation modules $\mathbb{C}pp_k^{p-perm.}$.

Keywords: biset functors, p-permutation modules, simple composition factors.

ÖZET

P-PERMÜTASYON İKİLİ İZLEÇLERİNİN BAZI BASİT KOMPOSİZYON FAKTÖRLERİ

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k karakteristiği asal sayı p olan, cebirsel olarak kapalı bir cisim ve $\mathbb{C}$ karmaşık sayıların cismi olsun. Verilen sonlu bir grup G için, $pp_k(G)$, p-permütasyon kG-modüllerinin Grothendieck grubunu simgeler ve $\mathbb{C}$ ile tensor çarpımını alarak ikili küme izleci olan $\mathbb{C}pp_k$'yı tanımlarız. Serge Bouc'un bir teoremi tarafından bilindiği üzere, basit ikili küme izleçleri olan $S_{H,V}$'ler, (H, V) çiftleriyle tanımlanır, öyle ki, burada H sonlu bir grup ve V basit bir $\mathbb{C}\text{Out}(H)$-modülüdür. Şu an için, $\mathbb{C}pp_k$'da görülen basit ikili izleçler olan $S_{H,V}$'lerin tüm sınıflandırılması bilinmemektedir. Bu tezde, $\mathbb{C}pp_k$'da görülen yeni basit izleçler olan $S_{H,V}$'leri buluyoruz, öyle ki burada H belirli bir p-hipoelementer B-grup'tur. Bu sonuç için kullanılan teknik Maxime Ducellier'in tanımı olan p-permütasyon izlecine ve p-permütasyon modüllerinden oluşan p-permütasyon izleci olan $\mathbb{C}pp_k^{p-perm.}$'un basit faktörlerini sınıflandırmak için kullandığı D-ikililerine dayanmaktadır.

Anahtar sözcükler: ikili küme izleçleri, p-permütasyon modülleri, basit komposizyon faktörleri.

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Contents

Chapter 1

Introduction

Let $\mathbb{C}$ be the field of complex numbers. The biset category $\mathbb{C}\mathcal{C}$ is defined as follows:

(i) The objects are finite groups,

(ii) $\mathrm{Hom}_{\mathbb{C}\mathcal{C}}(G, H) = \mathbb{C} \otimes_{\mathbb{Z}} B(H, G)$ where $B(H, G)$ is the Grothendieck group of isomorphism classes of finite $(H \times G)$-sets.

(iii) The composition is defined to be $\mathbb{C}$-linear extension of the composition $[V] \circ [U] = [V \times_H U]$ given $[U] \in B(K, H)$ and $[V] \in B(H, G)$, where $V \times_H U$ denotes the set of H-orbits of $V \times U$.

Then, a biset functor defined on $\mathbb{C}\mathcal{C}$ is a $\mathbb{C}$-linear functor from $\mathbb{C}\mathcal{C}$ to $\mathbb{C}$-Mod, which is the category of finite dimensional vector spaces over $\mathbb{C}$.

By the work of Serge Bouc [1] , we know that the simple biset functors $S_{H,V}$ are associated to pairs (H, V) where H is a finite group and V is a simple $\mathbb{C}\mathrm{Out}(H)$-module up to some equivalence. A simple biset functor $S_{H,V}$ is said to be a composition factor of a biset functor F if there exists subfunctors $F_1 \subseteq F_2 \subseteq F$ with $F_2/F_1 \cong S_{H,V}$. Classifying the simple composition factor structure of a biset functor is the main concern, and some of such classifications can be found in the literature [1].

Let k be an algebraically closed field of characteristic p, where p is a prime. We define a p-permutation module to be a direct summand of a permutation module. Then, $\mathbb{C}pp_k$ is a biset functor assigning a finite group G to $\mathbb{C}pp_k(G)$, the Grothendieck group of isomorphism classes of finite p-permutation kG-modules. The classification of simple composition factors of the biset functor of p-permutation modules $\mathbb{C}pp_k$ is not known at the time of writing. However, Mélanie Baumann has found some of the simple composition factors of $\mathbb{C}pp_k$ in [2], [3] and [4].

Recall that a finite group H is called p-hypo-elementary if H has a normal p-subgroup P such that $H/P \cong C_l$ where C_l is a cyclic group of order l and $(p, l) = 1$. By the Schur-Zassenhaus theorem, this extension splits so we have a semidirect product $H = P \rtimes C_l$. On the other hand, we call a group H a B-group if for any non-trivial $N \trianglelefteq H$, the deflation number introduced by Bouc is zero, i.e., $m_{H,N} = \frac{1}{|H|} \sum\limits_{XN=H} |X|\mu(X, H) = 0$ where μ is the Möbius function of the poset of subgroups of H.

Baumann has found that for any p-hypo-elementary B-group $H = P \rtimes C_l$, the simple biset functor $S_{H,\mathbb{C}}$ is necessarily a composition factor of $\mathbb{C}pp_k$. In [3], it was conjectured that for such type of a group H, the simple biset functor $S_{H,V}$ is apparent in $\mathbb{C}pp_k$ if and only if V is the trivial $\mathbb{C}\mathrm{Out}(H)$-module $\mathbb{C}$ with multiplicity $\Phi(l)$. In this thesis, we refute that conjecture and discuss the appearance of other simple $\mathbb{C}\mathrm{Out}(H)$-modules.

The method we use is due to Maxime Ducellier who, in [5], introduced the p-permutation category $\mathbb{C}\mathcal{C}^{pp_k}$ as follows:

(i) The objects are finite groups,

(ii) $\mathrm{Hom}_{\mathbb{C}\mathcal{C}^{pp_k}}(G, H) = \mathbb{C} \otimes_{\mathbb{Z}} pp_k(H, G)$ where $pp_k(H, G) = pp_k(H \times G)$ is the Grothendieck group of p-permutation $(H \times G)$-modules,

(iii) The composition is defined to be $\mathbb{C}$-linear extension of the composition $[X] \circ [Y] = [X \otimes_{kH} Y]$, given $[X] \in pp_k(G, H)$ and $[Y] \in pp_k(H, K)$, where $\otimes_{kH}$ is the tensor product over the group algebra kH.

Then, a p-permutation functor is a $\mathbb{C}$-linear functor defined from $\mathbb{C}\mathcal{C}^{pp_k}$ to $\mathbb{C}$-Mod.

In [5], Ducellier examined the p-permutation functor of p-permutation modules denoted by $\mathbb{C}pp_k^{p-perm.}$, and found that the simple p-permutation composition factors of $\mathbb{C}pp_k^{p-perm.}$ are indexed by p-hypo-elementary B-groups.

At this point, we mention that the structure of the biset functor $\mathbb{C}pp_k$ and the p-permutation functor $\mathbb{C}pp_k^{p-perm.}$ are different. We can interpret $\mathbb{C}pp_k$ as a module $\bigoplus_{G\in Obj(\mathbb{C}\mathcal{C})} \mathbb{C}pp_k(G)$ of the quiver algebra $\bigoplus_{H,K\in Obj(\mathbb{C}\mathcal{C})} \mathbb{C}B(H,K)$; whereas, $\mathbb{C}pp_k^{p-perm.}$ can be thought as a module $\bigoplus_{G\in Obj(\mathbb{C}\mathcal{C}^{pp_k})} \mathbb{C}pp_k(G)$ of the quiver algebra $\bigoplus_{H,K\in Obj(\mathbb{C}\mathcal{C}^{pp_k})} \mathbb{C}pp_k(H,K)$. The latter interpretation produces some extra maps that are generally called as diagonal maps

$$\delta : \mathbb{C}pp_k(H,G) \times \mathbb{C}pp_k(G,1) \to \mathbb{C}pp_k(H,1).$$

These maps are the basic reasons of the fact that the simple composition factor structure of $\mathbb{C}pp_k^{p-perm.}$ is far coarser than that of $\mathbb{C}pp_k$.

The method we use is to restrict the simple p-permutation factors of $\mathbb{C}pp_k^{p-perm.}$ to obtain new classification of simple composition biset factors of $\mathbb{C}pp_k$ which are indexed by a specific genre of p-hypo-elementary B-groups.

In detail, given a p-hypo-elementary B-group $H = P \rtimes C_l$ with $C_l = \langle s \rangle$, in Chapter 4, we shall see that a simple p-permutation functor $S_{H,W_{P,s}}^{p-perm.}$ is a composition factor of $\mathbb{C}pp_k^{p-perm.}$. We consider the restriction of $S_{H,W_{P,s}}^{p-perm.}$ to biset functors and obtain the following theorem:

Theorem 5.1.1. Suppose that $H = P \rtimes C_l$ be a p-hypo-elementary B-group such that every non-trivial $\mathbb{F}_p C_l$-module is apparent in P. Then, for every $\varphi \in \mathrm{Out}(C_l)$, the simple biset functor $S_{H,\mathbb{C}_\varphi}$ is apparent as a composition factor of the biset functor $\mathbb{C}pp_k$ where $\mathbb{C}_\varphi$ is the inflation of the vector space $\mathbb{C}$ on which the group $\mathrm{Out}(C_l)$ acts by φ.

We shall also provide a much detailed outline of the thesis:

In Chapter 2, we recall some background information on biset functors as well as crucial examples of biset functors such as the Burnside functor $\mathbb{C}B$, the biset functors

$\mathbb{C}R_k$, $\mathbb{C}R_{\mathbb{C}}$, the biset functor of p-permutation modules $\mathbb{C}pp_k$, and the monomial Burnside functor $\mathbb{C}B_{k^\times}$. We study primitive idempotent basis of $\mathbb{C}pp_k(G)$, and induction, restriction, isogation, inflation and deflation formulas which can be found in [5] and [6]. In this chapter, we shall also provide an alternative way to compute deflation formula for the primitive idempotents of $\mathbb{C}pp_k(G)$ by using the linearization map between $\mathbb{C}B_{k^\times}$ and $\mathbb{C}pp_k$.

In Chapter 3, we review some of the known simple composition factors of $\mathbb{C}pp_k$ found by Baumann in [2], and [3], [4]. We are particularly interested in the special type of groups named p-hypo-elementary B-groups. This chapter involves the proof of the classification of p-hypo-elementary B-groups by Baumann. The final part of this chapter is devoted to provide a counter-example to Conjecture 3.4.1 which claims that for a p-hypo-elementary B-group H, the simple biset functor $S_{H,V}$ is apparent in $\mathbb{C}pp_k$ if and only if V is the trivial $\mathbb{C}\mathrm{Out}(H)$-module $\mathbb{C}$. To do so, we show that for the alternating group A_4 which is a 2-hypo-elementary B-group, the simple biset functors $S_{A_4,\mathbb{C}}$ and $S_{A_4,\mathbb{C}_{-1}}$ are composition factors of $\mathbb{C}pp_k$ with each multiplicity 1 where k has characteristic 2. Along the way, by using the method of Baumann in [3], we compute the composition factors associated to small ordered groups C_1, C_2, C_3, V_4 when p is 2.

In Chapter 4, we study the p-permutation functors which are introduced by Ducellier in [5]. We review the notion of D-pairs as well as the simple composition factor structure of p-permutation functor of p-permutation modules $\mathbb{C}pp_k^{p-perm.}$. We shall provide a proof for the classification theorem of D-pairs by Ducellier which will show us that they are in a bijective correspondence with p-hypo-elementary B-groups.

In Chapter 5, we obtain a relaxation of Baumann's sufficient condition for the appearence of simple biset functors indexed by a special genre of p-hypo-elementary B-groups in $\mathbb{C}pp_k$. More precisely, we show that for a p-hypo-elementary B-group H such that every non-trivial $\mathbb{F}_pC_l$-module is apparent in P, for every $\varphi \in \mathrm{Out}(C_l)$, the simple biset functor $S_{H,\mathbb{C}_\varphi}$ is apparent as a composition factor of the biset functor $\mathbb{C}pp_k$ where $\mathbb{C}_\varphi$ is the inflation of $\mathbb{C}$ on which the group $\mathrm{Out}(C_l)$ acts by φ. In the final part, we shall see that this result implies that the simple p-permutation factor $S^{p-perm.}_{H,W_{P,s}}$ partially decomposes into simple biset factors $S_{H,\varphi}$ for every $\varphi \in \mathrm{Out}(C_l)$, where $C_l = \langle s \rangle$.

Chapter 2

Biset functors

2.1 Biset functors

In this section, we want to define biset functors and examine the simple composition structure of some crucial biset functors. We shall start with some review on background material which can be found in Bouc [1]. Recall that a left G-set X is a set with a left G-action satisfying:

(i) If $g, h \in G$ and $x \in X$, then $g \cdot (h \cdot x) = (gh) \cdot x$.

(ii) If $x \in X$, and 1_G is the identity element of G, then $1_G \cdot x = x$.

A finite G-set X is called *transitive* if X has a single G-orbit. Any transitive G-set has form G/H for some subgroup H of G. Note that G/H is isomorphic to G/K if and only if H and K are G-conjugate. Moreover, any G-set X can be expressed as a direct sum of transitive G-sets, i.e., where $[G/X]$ is a set of representatives of the G-orbits in X,

$$X \cong \bigsqcup_{x \in [G/X]} G/G_x$$

We denote the isomorphism class of finite G-set X by $[X]$.

Definition 2.1.1 (Burnside Group). *Let G be a finite group. The Burnside group $B(G)$ of G is the Grothendieck group of the category of G-sets. In other words, it is the quotient of free abelian group on the set of isomorphism classes of finite G-sets, by the subgroup generated by the elements of the form $[X \sqcup Y] - [X] - [Y]$ where X and Y are finite G-sets.*

The Burnside group $B(G)$ has also a natural ring structure which is given by Cartesian product of given G-sets, i.e., $[X_1] \cdot [Y_1] = [X_1 \times Y_1]$ where the identity element of $B(G)$ corresponds one point set with trivial action and the zero element is the empty set. One can note that $\{[G/H] : H \leqslant_G G\}$ forms a basis for $B(G)$ called the transitive basis. The following formula provides an explanation to multiplication structure of this basis elements.

Lemma 2.1.2 (Mackey Product Formula for Burnside Groups). *Let G be a finite group, H and K be subgroups of G. Then,*

$$[G/H] \cdot [G/K] = \sum_{HgK \leqslant_G G} [G/(H \cap {}^gK)] = \sum_{HgK \leqslant_G G} [G/(H^g \cap K)].$$

Definition 2.1.3. *For finite groups G and H, an $(H \times G^{op})$-set U is called (H, G)-biset. U can be thought as a both left H-set and a G-set in which actions of H and G commute, that is,*

$$\forall h \in H,\ \forall u \in U,\ \forall g \in G,\ (h \cdot u) \cdot g = h \cdot (u \cdot g).$$

Let $H\backslash U/G$ denote the double coset of U. We call a (H, G)-biset U transitive if $H\backslash U/G$ has a cardinality 1.

Lemma 2.1.4 ([1], p19). *Let G and H be groups.*

(1) If L is a subgroup of $H \times G$, then the set $(H \times G)/L$ is a transitive (H, G)-biset for the actions defined by

$$\forall h \in H,\ \forall (b, a)L \in (H \times G)/L,\ \forall g \in G,\ h \cdot (b, a)L \cdot g = (hb, g^{-1}a)L.$$

(2) If U is an (H, G)-biset, choose a set $[H\backslash U/G]$ of representatives of (H, G)-orbits on U. Then, there is an isomorphism of (H, G)-bisets

$$U \cong \bigsqcup_{u \in [H\backslash U/G]} (H \times G)/L_u$$

where $L_u = (H,G)_u = \{(h,g) \in H \times G \mid h \cdot u = u \cdot g\}$.

Definition 2.1.5. *Let G, H and K be finite groups. For (H,G)-biset U and (K,H)-biset V, we define the composition of V and U, namely $V \times_H U$ as the set of H-orbits of the cartesian product $V \times U$, where the H-action is defined as follows: for $(v,u) \in V \times U$ and for each $h \in H$,*

$$(v,u) \cdot h = (v \cdot h, h^{-1} \cdot u).$$

We denote the H-orbit of an element $(v,u) \in V \times U$ by $(v,_H u)$. Moreover, the set $V \times_H U$ is an (K,G)-biset for the action defined by:

$$k \cdot (v,_H u) \cdot g = (k \cdot v,_H u \cdot g)$$

for each $k \in K$, $(v,_H u) \in V \times_H U$, $g \in G$.

Now we will define five elementary bisets: Let G be a finite set.

(1) G can be thought as (G,G)-biset where the G-action is the usual group multiplication. We denote this biset by Id_G.

(2) Let $H \leqslant G$. Then G can be thought as (H,G)-biset, denoted by Res^G_H.

(3) Let $H \leqslant G$. Then G can be thought as (G,H)-biset, denoted by Ind^G_H.

(4) Let $N \trianglelefteq G$ and $H = G/N$. Then H can be thought as an (G,H)-biset, denoted by Inf^G_H for the right action of H by multiplication, and the left action of G by projection to H, then left multiplication in H.

(5) Let $N \trianglelefteq G$ and $H = G/N$. H can be thought as an (H,G)-biset, denoted by Def^G_H for the left action of H by multiplication, and the right action of G by projection to H, and then right multiplication in H.

(6) Let $f : G \to H$ be a group isomorphism, then H can be thought as (H,G)-biset, denoted by $\mathrm{Iso}^G_H(f)$ for the left action of H by multiplication and right action of G is taken by image of f.

Recall that for a finite group G, a section (T,S) of G is defined by subgroups of G, T and S such that $S \trianglelefteq T$.

Lemma 2.1.6. *([1], Goursat's Lemma) Let G and H be groups.*

(1) If (D,C) is a section of H and (B,A) is a section of G such that there exists a group isomorphism $f : B/A \longrightarrow D/C$, then

$$L_{(D,C),f,(B,A)} = \{(h,g) \in H \times G | h \in D, g \in B, hC = f(gA)\}$$

is a subgroup of $H \times G$.

(2) Conversely, if L is a subgroup of $H \times G$, then there exists a unique section (D,C) of H, a unique section (B,A) of G, and a unique group isomorphism $f : B/A \longrightarrow D/C$, such that $L = L_{(D,C),f,(B,A)}$.

Lemma 2.1.7. *([1], Butterfly Factorization) Let G and H be groups. If L is a subgroup of $H \times G$, let (D,C) and (B,A) be the sections of H and G respectively, and f be the group isomorphism $B/A \longrightarrow D/C$ such that $L = L_{(D,C),f,(B,A)}$. Then there is an isomorphism of (H,G)-bisets*

$$(H \times G)/L \simeq {}_H Ind_D Inf_{D/C} Iso(f)_{B/A} Def_B Res_G$$

Definition 2.1.8. *Let G and H be finite groups. The biset Burnside group $B(H,G)$ is the Burnside group $B(H \times G^{op})$, i.e., the Grothendieck group of the isomorphism classes of finite (H,G)-bisets for the disjoint union.*

Remark 2.1.9. *Let G, H and K be finite groups. There is a unique bilinear map $\times_H : B(K,H) \times B(H,G) \to B(K,G)$ such that $[V] \times_H [U] = [V \times_H U]$, whenever U is a finite (H,G)-biset and V is a finite (K,H)-biset.*

Remark 2.1.10. *Any element $[X] \in B(H,G)$ can be written as a linear combination of isomorphism classes of transitive (H,G)-bisets, namely,*

$$[X] = \sum_{L \leq_{H \times G} H \times G} \lambda_L(X)[(H \times G)/L].$$

By Butterfly Factorization, we can say that elements of $B(H,G)$ are generated by induction, inflation, isogation, deflation and restriction maps.

Definition 2.1.11. *([1], p41) The biset category $\mathcal{C}$ is defined as follows:*

(i) The objects of $\mathcal{C}$ are finite groups,

(ii) For $G, H \in Obj(\mathcal{C})$, then $Hom_{\mathcal{C}}(H, G) = B(H, G)$,

(iii) The composition is given by $[V] \circ [U] = [V \times_H U]$ for $[V] \in Hom_{\mathcal{C}}(H, K)$, $[U] \in Hom_{\mathcal{C}}(G, H)$,

(iv) For any finite group G, the identity morphism of G in $\mathcal{C}$ is $[Id_G]$.

Definition 2.1.12. *Let $\mathbb{C}$ be the field of complex numbers. Then we define the biset category $\mathbb{C}\mathcal{C}$ as follows:*

(i) The objects of $\mathbb{C}\mathcal{C}$ are finite groups,

(ii) For $G, H \in Obj(\mathcal{C})$, then $Hom_{\mathbb{C}\mathcal{C}}(H, G) = \mathbb{C} \otimes_{\mathbb{Z}} B(H, G)$,

(iii) The composition of morphisms in $\mathbb{C}\mathcal{C}$ is the $\mathbb{C}$-linear extension of the composition in $\mathbb{C}\mathcal{C}$,

(iv) For any finite group G, the identity morphism of G in $\mathbb{C}\mathcal{C}$ is $\mathbb{C} \otimes_{\mathbb{Z}} [Id_G]$.

Remark 2.1.13. *The biset category $\mathbb{C}\mathcal{C}$ is $\mathbb{C}$- linear category which means that the set of its morphisms are $\mathbb{C}$-modules, and the composition is $\mathbb{C}$-bilinear.*

Definition 2.1.14. *Let $\mathcal{D}$ be a $\mathbb{C}$-linear subcategory of $\mathbb{C}\mathcal{C}$. A biset functor on $\mathcal{D}$ is a $\mathbb{C}$-linear functor from $\mathcal{D}$ to $\mathbb{C} - Mod$. Moreover, biset functors on the subcategory $\mathcal{D}$ form a category denoted by $\mathcal{F}_{\mathcal{D},\mathbb{C}}$ where the homomorphism sets are natural transformations of functors and compositions are composition of natural transformations.*

In [1], Bouc provided a classification for simple objects of this category, namely, $\mathcal{F}_{\mathcal{D},\mathbb{C}}$. We shall review some basic definitions and results for this purpose.

If F is an object of $\mathcal{F}_{\mathcal{D},\mathbb{C}}$, then we define a minimal group for F to be an object H of $\mathcal{D}$ such that $F(H) \neq \{0\}$ and for every object K of $\mathcal{D}$ with $|K| < |H|$. The set of minimal objects for F is denoted by $Min(F)$.

Definition 2.1.15. *A full-subcategory $\mathcal{D}$ of $\mathbb{C}\mathcal{C}$ is called replete if its object set is closed under taking subquotients that is any group is isomorphic to a subquotient of an element of $\mathcal{D}$ is in $\mathcal{D}$.*

Definition 2.1.16. *A simple biset functor on $\mathcal{D}$ is a simple object of $\mathcal{F}_{\mathcal{D},\mathbb{C}}$ which is a non-zero functor F whose only subfunctors are itself and the zero functor.*

Proposition 2.1.17. *Suppose that $\mathcal{D}$ is a replete subcategory of $\mathbb{C}\mathcal{C}$ and let $\mathcal{E}$ be a full-subcategory of $\mathcal{D}$. If F is simple object of $\mathcal{F}_{\mathcal{D},\mathbb{C}}$ and $Res^{\mathcal{D}}_{\mathcal{E}}F \neq 0$, then $Res^{\mathcal{D}}_{\mathcal{E}}F$ is a simple object of $\mathcal{F}_{\mathcal{E},\mathbb{C}}$.*

The following results are due to Bouc and can be found in [1].

Definition 2.1.18. *Let G be an object of $\mathcal{D}$ and V be an $End_{\mathcal{D}}(G)-$ module. We define the biset functor $L_{G,V}$ as follows:*

(i) For every object H of $\mathcal{D}$, we set

$$L_{G,V}(H) = Hom_{\mathcal{D}}(G,H) \otimes_{End_{\mathcal{D}}(G)} V = \mathbb{C}B(H,G) \otimes_{\mathbb{C}B(G,G)} V.$$

(ii) For every $\phi : H \to K$ in $\mathcal{D}, L_{G,V}(\phi) : L_{G,V}(H) \to L_{G,V}(K)$ is defined by $\vartheta \otimes v \mapsto (\phi \circ \vartheta) \otimes v$.

It is clear from the definition that $L_{G,V}(G) \cong V$.

Proposition 2.1.19 ([1], Corollary 4.2.4, p58)**.** *Let G be an object of $\mathcal{D}$ and V be a simple $End_{\mathcal{D}}(G)$-module. Then the biset functor $L_{G,V}$ has a unique proper maximal subfunctor denoted by $J_{G,V}$ and the quotient $S_{G,V} = L_{G,V}/J_{G,V}$ is a simple object of $\mathcal{F}_{\mathcal{D},\mathbb{C}}$ such that $S_{G,V}(G) \cong V$.*

Now, we let $\mathcal{D}$ be a subcategory of the biset category $\mathbb{C}\mathcal{C}$ which contains group isomorphisms.

Definition 2.1.20. *A pair (G,V), where G is an object of $\mathcal{D}$ and V is a simple $\mathbb{C}Out(G)$-module, is called a seed of $\mathcal{D}$. We call two pairs of $\mathcal{D}$ (G,V) and (G',V') as isomorphic if there exists a group isomorphism $\phi : G \to G'$ and an $\mathbb{C}$-module isomorphism $\psi : V \to V'$ such that*

$$\forall v \in V,\ \ \forall a \in Out(G),\ \ \psi(a \cdot v) = (\phi a \phi^{-1}) \cdot \psi(v).$$

Lemma 2.1.21 ([1], Lemma 4.3.9, p61). *Let G be a finite group and V be a simple $\mathbb{C}Out(G)$-module. If H is a finite group such that $S_{G,V}(H) \neq \{0\}$, then G is isomorphic to a subquotient of H.*

Theorem 2.1.22 ([1], p62). *Let $\mathcal{D}$ be an admissible subcategory of $\mathbb{C}\mathcal{C}$. There is a one to one correspondence between the set of isomorphism classes of simple objects of $\mathcal{F}_{\mathcal{D},\mathbb{C}}$ and the set of isomorphism classes of seeds of $\mathcal{D}$, sending the class of the simple functor S to the isomorphism class of a pair $(G, S(G))$, where G is any minimal group for S. The inverse correspondence maps the class of the seed (G, V) to the class of the functor $S_{G,V}$.*

Definition 2.1.23. *Let F be a biset functor on $\mathcal{D}$. We call a simple functor S as a composition factor of the biset functor F if there exists subfunctors $F^{'} \subseteq F^{''} \subseteq F$ such that $F^{''}/F^{'} \cong S$.*

We now take a full-subcategory $\mathcal{E}$ of $\mathcal{D}$ and a biset functor F on $\mathcal{D}$. We can also consider F to be a biset functor on the subcategory $\mathcal{E}$ which we denote by $\mathrm{Res}^{\mathcal{D}}_{\mathcal{E}}\mathrm{F}$. The following result is called finite reduction principle for biset functors and we will make use of this result for our main theorem by considering a specific full-subcategory.

Proposition 2.1.24 (Finite Reduction Principle For Biset Functors). *Let G be an object of $\mathcal{D}$ and V be a simple $\mathbb{C}Out(G) -$ module. If $S_{G,V}$ is a composition factor of $Res^{\mathcal{D}}_{\mathcal{E}}$ on $\mathcal{E}$, then $S_{G,V}$ is also a composition factor of F on $\mathcal{D}$.*

Definition 2.1.25. *We define $\mathcal{F}_n$ to be a full-subcategory of the biset category $\mathcal{D}$ such that the objects are all finite groups whose orders are less than or equal to n, where n is a positive integer.*

2.2 Examples of biset functors

2.2.1 The Burnside functor $\mathbb{C}B$

For this part, we always assume that $\mathcal{D}$ is a replete subcategory of the biset category $\mathbb{C}\mathcal{C}$ where $\mathbb{C}$ is the field of complex numbers.

Let G be a finite group then we have defined $B(G)$ as the Burnside ring of G. Moreover, for any finite (H,G)-biset U, we can define the following map:

$$B([U]) : B(G) \to B(H) \quad by \quad [V] \mapsto [U \times_G V],$$

for every finite G-set V.

We can extend this map $\mathbb{C}$-linearly to the map $\mathbb{C}B([U]) : \mathbb{C}B(G) \to \mathbb{C}B(H)$ where $\mathbb{C}B(G) = \mathbb{C} \otimes_{\mathbb{Z}} B(G)$. This defines a biset functor, the Burnside functor.

The Burnside ring $B(G)$ has another basis called the primitive basis: $\{e_H^G : H \leq_G G\}$ with a nicer multiplication compared to transitive basis:

$$e_K^G \cdot e_H^G = \begin{cases} e_K^G & if \quad K =_G H \\ 0 & otherwise \end{cases}$$

These two different bases of $B(G)$ are related by the following inversion formula proven by Gluck and Yoshida separately:

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{K \leqslant H} |K| \mu(K,H)[G/K]$$

where μ is the Möbius function of the poset of subgroups of G.

Since every biset functor on $\mathcal{D}$ can be thought as a module of the quiver algebra $\bigoplus_{\forall H,G \in Obj(\mathcal{D})} B(H,G)$, and since we know that by the Butterfly factorization lemma, every element of biset Burnside ring $B(H,G)$ is generated by finite elementary maps, induction, inflation, isogation, deflation and restriction, it is meaningful to study the effects of these maps on the primitive basis of the Burnside ring which has just shown to possess a biset functor structure.

Theorem 2.2.1 ([1], Theorem 5.2.4., p77)**.** *Let G be a finite group.*

1. *Let H and K be subgroups of G. Then,*

$$Res_K^G(e_H^G) = \sum_{x \in [N_G(H) \backslash T_G(H,K)/K]} e_{H^x}^K$$

where x runs through a set of representatives of $(N_G(H), K)$- orbits on the set $T_G(H,K) = \{g \in G | H^g \subseteq K\}$.

2. Let $K \leq H$ be subgroups of G. Then,

$$Ind_H^G e_K^H = \frac{|N_G(K)|}{|N_H(K)|} e_K^G.$$

3. Let $N \trianglelefteq G$. Then, for any subgroup H of G containing N,

$$Inf_{G/N}^G e_{H/N}^{G/N} = \sum_{KN =_G H, K =_G G} e_K^G.$$

4. Let $N \trianglelefteq G$. Then,

$$Def_{G/N}^G e_G^G = m_{G,N} e_{G/N}^{G/N},$$

where $m_{G,N} = \frac{1}{|G|} \sum_{XN=G} |X| \mu(X, G)$.

5. If $\phi : G \to G'$ is a group isomorphism, and $H \leq G$, then

$$Iso(\phi)(e_H^G) = e_{\phi(H)}^{G'}.$$

In this part, we shall review Bouc's result on the classification of the simple composition factors for particularly our case that is when the ground field is $\mathbb{C}$ the complex field. More general cases can be found in [1].

For this purpose, let us define a specific subfunctor of $\mathbb{C}B$ on a replete subcategory $\mathcal{D}$ of the biset category $\mathbb{C}\mathcal{C}$. Suppose we are given an object G of the category $\mathcal{D}$, we denote e_G to be the subfunctor of $\mathbb{C}B$ generated by the primitive idempotent $e_G^G \in \mathbb{C}B(G)$. To be more precise, for any object $H \in Obj(\mathcal{D})$, $e_G(H) = Hom_{\mathcal{D}}(G, H)(e_G^G)$.

Moreover, a finite group G in $\mathcal{D}$ is called B-group if for every non-trivial normal subgroup N of G, we have $m_{G,N} = 0$. We denote the class of B-groups in $\mathcal{D}$ by $B - gr(\mathcal{D})$ and we denote the set of representatives of isomorphism classes of these B-groups by $[B - gr(\mathcal{D})]$.

Bouc showed that for every finite group G in $\mathcal{D}$, we can define a group denoted by $\beta(G)$ to be a quotient G/N for some normal subgroup N of G such that $m_{G,N} \neq 0$ and G/N is a B-group. He showed that $\beta(G)$ is well-defined up to group isomorphism; however, the normal subgroup N is not unique in general.

Proposition 2.2.2 ([1], p89). *1. Let G be a B-group over $\mathbb{C}$. Then the subfunctor e_G of $\mathbb{C}B$ has a unique maximal subfunctor, equal to*

$$j_G = \sum_{H \in [B-gr_{\mathbb{C}}(\mathcal{D})], H \gg G, H \ncong G} e_H,$$

and the quotient functor e_G/j_G is isomorphic to the simple functor $S_{G,\mathbb{C}}$.

2. If $F \subseteq F^{'}$ are subfunctors of $\mathbb{C}B$ such that $F^{'}/F$ is simple, then there exists a unique $G \in [B - gr_{\mathbb{C}}(\mathcal{D})]$ such that $e_G \subseteq F^{'}$ and $e_G \nsubseteq F$. In particular, $e_G + F = F^{'}$, $e_G \cap F = j_G$, and $F^{'}/F \cong S_{G,\mathbb{C}}$.

This proposition shows us that the composition factors of $\mathbb{C}B$ on $\mathcal{D}$ are exactly the simple functors $S_{G,\mathbb{C}}$.

Remark 2.2.3. *Let p be a prime number. In characteristic 0, it is known due to Bouc that a p-group G is a B-group if and only if G is trivial or isomorphic to $C_p \times C_p$. Therefore, if we consider a full-subcategory $\mathcal{C}_p$ of the biset category $\mathbb{C}\mathcal{C}$ whose objects are $p-$groups, then the simple composition factors of $\mathbb{C}B$ on $\mathcal{C}_p$ are $S_{C_1,\mathbb{C}}$ and $S_{C_p \times C_p,\mathbb{C}}$ with multiplicity 1.*

2.2.2 The biset functors $\mathbb{C}R_{\mathbb{C}}$ and $\mathbb{C}R_k$

Let $\mathbb{C}$ be the field of complex numbers.

Definition 2.2.4. *([1], Chapter7) Let G be a finite group, $R_{\mathbb{C}}(G)$ is defined to be the Grothendieck group of the category of finite dimensional $\mathbb{C}G$-modules. For any finite $(H,G)-$ biset U, we define $R_{\mathbb{C}}([U]) : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(H)$ by*

$$R_{\mathbb{C}}([U])([E]) = [\mathbb{C}U \otimes_{\mathbb{C}G} E],$$

where $[E] \in R_{\mathbb{C}}(G)$ denotes the isomorphism class of a finite dimensional $\mathbb{C}G$-module E, and $\mathbb{C}U$ is the $(\mathbb{C}H, \mathbb{C}G)$-permutation bimodule associated to U. We can extend this map $\mathbb{C}$-linearly. This construction provides $\mathbb{C}R_{\mathbb{C}}$ with biset functor structure.

Definition 2.2.5. *([1], Definition 7.3.1.) A character $\xi : (\mathbb{Z}/m\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$ is called primitive if it cannot be factored through any quotient $(\mathbb{Z}/n\mathbb{Z})^{\times}$ of $(\mathbb{Z}/m\mathbb{Z})^{\times}$, where n is a proper divisor of m.*

The simple composition factors of the biset functor $\mathbb{C}R_{\mathbb{C}}$ are classified by Bouc as follows:

Proposition 2.2.6. *([1], Corollary 7.3.5, p133) $\mathbb{C}R_{\mathbb{C}}$ is semisimple, and*

$$\mathbb{C}R_{\mathbb{C}} \cong \bigoplus_{(m,\xi)} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi},$$

where (m,ξ) runs through the set of pairs consisting of a positive integer m and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$.

Definition 2.2.7. *Let k be an algebraically closed field of characteristic p, prime. Let $\mathcal{C}_{p'}$ denote the full-subcategory of the biset category $\mathbb{C}\mathcal{C}$ whose objects are formed by finite $p^{'}$-groups. For a finite $p^{'}$-group G, we can define $R_k(G)$ to be the Grothendieck group of the category of finite dimensional kG-modules. In the same way, for every (H,G)-biset U, we can define $R_k([U]) : R_k(G) \to R_k(H)$ by*

$$R_k([U])([E]) = [kU \otimes_{kG} E],$$

for every kG-module E. Then, we can extend it $\mathbb{C}$-linearly. This tells us that the biset functor $\mathbb{C}R_k$ has a biset functor structure on the category $\mathcal{C}_{p'}$.

Remark 2.2.8. *We have, for every finite group G whose order is coprime to p, $\mathbb{C}R_k(G) \cong \mathbb{C}R_{\mathbb{C}}(G)$. Therefore, on the category $\mathcal{C}_{p'}$, $\mathbb{C}R_k$ is isomorphic to $\mathbb{C}R_{\mathbb{C}}$. For this reason, we can say that*

$$\mathbb{C}R_k \cong \bigoplus_{(m,\xi)} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi},$$

where (m,ξ) runs through the set of pairs consisting of a positive integer m coprime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$.

2.2.3 The biset functor of p-permutation modules $\mathbb{C}pp_k$

Let $\mathbb{C}$ be the field of complex numbers and k be an algebraically closed field of characteristic p where p is prime.

We shall start with some basic definitions which can be found in [7] and [8].

Definition 2.2.9. *Let M be an indecomposable kG-module. A minimal subgroup Q of G for which M is a direct summand of $Ind_Q^G Res_Q^G(M)$ is called* a vertex *of M and is defined up to G-conjugacy. It is known that for such a field k, the vertex of every indecomposable kG-module is a p-group.*

Definition 2.2.10. A source *of M is an indecomposable kQ-module M_0, where Q is a vertex of M, such that M is a direct summand of $Ind_Q^G(M_0)$.*

Definition 2.2.11. *We call a kG-module M by* a trivial source module *if each indecomposable summand of M has the trivial module k as its source.*

Definition 2.2.12. *An kG-module N is called* a permutation module *if there exists a G-set X with $N = kX$, that is to say, N has a G-stable k-basis.*

Note that we may decompose X as a disjoint union of G-orbits which gives us a direct sum decomposition of kX as an kG-module. If we let X to be a transitive G-set, then we have $kX \cong \mathrm{Ind}_H^G(k)$ where H is a stabilizer of some element x of X, and k is the trivial kH-module.

Hence, we can think any arbitrary permutation kG-module as a direct sum of modules of the form $\mathrm{Ind}_H^G(k)$ for some subgroups $H \leq G$. Note that $\mathrm{Ind}_H^G(k)$ is a permutation kG-module with G-basis $\{g \otimes 1_k \mid g \in [G/H]\}$. Moreover, if kX is a permutation kH-module on X, then $\mathrm{Ind}_H^G(kX)$ is a permutation kG-module with G-basis given $\{g \otimes x \mid g \in [G/H],\ x \in X\}$. Therefore, induction preserves permutation modules, and it is clear that restriction and conjugation, too.

Definition 2.2.13. *An kG-module M is called* a p-permutation kG-module *if $Res_Q^G(M)$ is a permutation kQ-module for every p-subgroup Q of G.*

Suppose that P is a Sylow p-subgroup of G. Since we have $\mathrm{Res}_{{}^gP}^G(M) = {}^g(\mathrm{Res}_P^G(M))$ and restriction and conjugation preserves permutation modules, we only need to have $\mathrm{Res}_P^G(M)$ to be a permutation module to conclude that M is a p-permutation module. It means that an kG-module M is a p-permutation kG-module if $\mathrm{Res}_P^G(M)$ is a permutation kP-module where P is a Sylow p-subgroup of G.

Clearly, p-permutation modules are preserved by direct sums, tensor products, restriction and conjugation. We shall use another definition of p-permutation modules to show that induction also preserves p-permutation modules as well.

Remark 2.2.14. *The following conditions are equivalent:*

(i) M is a p-permutation kG-module,

(ii) M is a trivial source kG-module.

Proof. $(\Rightarrow)$: Suppose that M is an indecomposable p-permutation kG-module. Let P be a p-subgroup of G which is a vertex of M. Then, we know that M is a direct summand of $\text{Ind}_P^G\text{Res}_P^G(M)$. By definition, we have $\text{Res}_P^G(M)$ a permutation kP-module.

Thus, $\text{Res}_P^G(M)$ is a direct sum of modules of the form $\text{Ind}_Q^P(k)$ where $Q \leq P$.

Thus, there exists $Q \leq P$ such that M is a summand of $\text{Ind}_Q^G(k)$. But since P is a vertex of M, we have $Q \cong P$. Thus, M is a direct summand of $\text{Ind}_P^G(k)$, that is to say, M has a trivial source.

$(\Leftarrow)$: For this part, we shall show that any summand of $\text{Ind}_H^G(k)$ is a p-permutation module. Then, it would imply that any module with trivial source is a p-permutation module.

Let M' be an indecomposable trivial source kG-module with vertex Q, which is known to be a p-group, then M' is isomorphic to a direct summand of $\text{Ind}_Q^G(k)$, which is a permutation kG-module.

Now, we wish to show that M' is a p-permutation kG-module. Let us denote $M = \text{Ind}_Q^G(k)$. Clearly, M is a p-permutation kG-module. Now, let P be a p-subgroup of G. Then, $\text{Res}_P^G(M')$ is a summand of $\text{Res}_P^G(M)$. By using the definition of being p-permutation module, we know that $\text{Res}_P^G(M)$ is a permutation kP-module. Thus, $\text{Res}_P^G(M) \cong \bigoplus_i \text{Ind}_{Q_i}^P(k)$ for some subgroups $Q_i \leq P$.

<u>Claim:</u> $\text{Ind}_{Q_i}^P(k)$ is indecomposable.

<u>Proof:</u> Since we have the isomorphism $\text{Hom}_{kP}(\text{Ind}_{Q_i}^P(k), k) \cong \text{Hom}_{kQ_i}(k, \text{Res}_{Q_i}^P(k)) \cong \text{Hom}_{kQ_i}(k, k) \cong k$, and the fact that only simple kP-module up to isomorphism is k, if we suppose $\text{Ind}_{Q_i}^P(k) = M_1 \oplus M_2$, we have non-zero and linearly independent maps, $f_i : M_i \to k$, for $i = 1, 2$, which extends $f_i : \text{Ind}_{Q_i}^P(k) \to k$, contradicting the fact that $\text{Hom}_{kP}(\text{Ind}_{Q_i}^P(k), k) \cong k$.

Then, we have $\mathrm{Res}^G_P(M') \cong \mathrm{Ind}^P_{Q_i}(k)$ which is a permutation module, i.e., M' is a p-permutation kG-module. □

Definition 2.2.15 ([6], Definition 2.6)**.** *Let G be a finite group. The p-permutation ring denoted by $pp_k(G)$ is the Grothendieck group of the isomorphism classes of p-permutation kG-modules, with the relation $[M]+[N] = [M \oplus N]$, and the ring structure is induced by the tensor product of modules over k. The identity element of $pp_k(G)$ is the class of the trivial kG-module k.*

Definition 2.2.16 (The biset functor of p-permutation modules $\mathbb{C}pp_k$)**.** *For every (H,G)-biset U, we define*

$$pp_k([U]) : pp_k(G) \to pp_k(H) \quad by$$
$$[M] \mapsto [kU \otimes_{kG} M]$$

for every p-permutation kG-module M. Similarly to previous examples, we extend this map $\mathbb{C}$-linearly, $\mathbb{C}pp_k([U]) : \mathbb{C}pp_k(G) \to \mathbb{C}pp_k(H)$ where $\mathbb{C}pp_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} pp_k(G)$. Moreover, we define $\mathbb{C}pp_k(u)$ for every $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H,G)$ where $u = \sum_{i=1}^{n} \lambda_i [U_i]$ by $\mathbb{C}pp_k(u) = \sum_{i=1}^{n} \lambda_i \mathbb{C}pp_k([U_i])$ which defines the biset functor structure of $\mathbb{C}pp_k$.

Now, we shall provide some further remarks on two bases of $\mathbb{C}pp_k$ which can be found in [6].

We can think $pp_k(G)$ as the free abelian group of the set of isomorphism classes of indecomposable p-permutation kG-modules.

Recall that for a given kG-module M, and a p-subgroup P of G, *the relative trace map* is the map $tr^P_Q : M^Q \to M^P$ given by $tr^P_Q(m) = \sum_{x \in [P/Q]} x \cdot m$ with $Q \leq P$.

Furthermore, we define *the Brauer quotient* of M at P to be the k-vector space $M[P] = M^P / \sum_{Q<P} tr^P_Q M^Q$ which has a natural $kN_G(P)/P$-module structure and for any finite group H which is not a p-group, $M[H]$ is zero by using the fact that the map tr^H_P is onto where P is a Sylow p-subgroup of H.

Now, we shall refer the following theorem which can be found in [6] and [9]:

Theorem 2.2.17 ([9], Theorem 3.2)**.**

1. *The vertices of an indecomposable p-permutation kG-module M are the maximal p-subgroups P of G such that* $M[P] \neq 0$*.*

2. *An indecomposable p-permutation kG-module has vertex P if and only if* $M[P]$ *is a non-zero projective* $kN_G(P)/P$*-module.*

3. *The correspondence* $M \mapsto M[P]$ *induces a bijection between the isomorphism classes of indecomposable p-permutation kG-modules with vertex P and the isomorphism classes of indecomposable projective* $kN_G(P)/P$*-modules.*

Now, we let $\mathcal{P}_{G,p}$ be the set of pairs (P, E) such that P is a p-subgroup of G, and E is an indecomposable projective $kN_G(P)/P$-module. We have a G-action on $\mathcal{P}_{G,p}$ by conjugation and the set of G-orbits are denoted by $[\mathcal{P}_{G,p}]$. Given (P, E), and by using Theorem 2.2.17, we let $M_{P,E}$ denote the indecomposable p-permutation kG-module such that $M_{P,E}[P] \cong E$. Then, we have the following result:

Corollary 2.2.18 ([6], Corollary 2.9)**.** *The isomorphism classes of* $M_{P,E}$ *form a* $\mathbb{Z}$*-basis of* $pp_k(G)$ *where* $(P, E) \in [\mathcal{P}_{G,p}]$*.*

Now, we move to explanation of the primitive basis of $\mathbb{C}pp_k$ which is found by Bouc and Thévenaz in [6].

Firstly, let $\mathcal{Q}_{G,p}$ denote the set of pairs (P, s) where P is a p-group of G, and s is a $p^{'}$-element of $N_G(P)/P$, and G acts on $\mathcal{Q}_{G,p}$ by conjugation and we denote the set of G-orbits by $[\mathcal{Q}_{G,p}]$.

Now, we are ready to define the species for $pp_k(G)$:

Given $(P, s) \in \mathcal{Q}_{G,p}$, we define $\tau^G_{P,s}$ to be the additive map from $pp_k(G)$ to $\mathbb{C}$ given by assigning the class of a p-permutation kG-module M to the value at s of the Brauer character of the $N_G(P)/P$-module $M[P]$.

Proposition 2.2.19 ([6], Proposition 2.18)**.**

1. *The map* $\tau^G_{P,s}$ *is a ring homomorphism* $pp_k(G) \to \mathbb{C}$ *and extends a* $\mathbb{C}-$ *algebra homomorphism* $\tau^G_{P,s} : \mathbb{C}\otimes_{\mathbb{Z}} pp_k(G) \to \mathbb{C}$*,*

2. *The set* $\{\tau^G_{P,s} | (P,s) \in [\mathcal{Q}_{G,p}]\}$ *is the set of all distinct species from* $\mathbb{C}\otimes_{\mathbb{Z}} pp_k(G)$ *to* $\mathbb{C}$. *Then, we have the following* $\mathbb{C}$*-algebra isomorphism*

$$\prod_{(P,s)\in[\mathcal{Q}_{G,p}]} \tau^G_{P,s} : \mathbb{C}\otimes_{\mathbb{Z}} pp_k(G) \to \prod_{(P,s)\in[\mathcal{Q}_{G,p}]} \mathbb{C}.$$

Corollary 2.2.20 ([6], Corollary 2.19)**.** *The* $\mathbb{C}$*-algebra* $\mathbb{C}\otimes_{\mathbb{Z}} pp_k(G)$ *is a semisimple commutative* $\mathbb{C}$*-algebra and its primitive idempotents are* $F^G_{P,s}$ *indexed by* $(P,s) \in [\mathcal{Q}_{G,p}]$ *such that*

$$\forall\ (R,u) \in \mathcal{Q}_{G,p}\ , \tau^G_{R,u}(F^G_{P,s}) = \begin{cases} 1 & if \quad (R,u) =_G (P,s), \\ 0 & otherwise. \end{cases}$$

Remark 2.2.21. *Letting* p *to be a prime which is a characteristic of the field* k*, we have* $dim_{\mathbb{C}}\mathbb{C}pp_k(G) = \sum_P l_p(N_G(P)/P)$ *where* P *runs through* G*-conjugacy classes of all* p*-subgroups, and* $l_p(N_G(P)/P)$ *denotes the number of* p'*-elements of* $N_G(P)/P$.

We have the following formulas:

Proposition 2.2.22 ([6], 3.1 Proposition)**.** *Suppose that* H *is a subgroup of* G*, and let* $F^G_{P,s}$ *be a primitive idempotent of* $\mathbb{C}pp_k(G)$*. Then,*

$$Res^G_H F^G_{P,s} = \sum_{(Q,t)} F^H_{Q,t},$$

where (Q,t) *runs through a set of representatives of* H*-conjugacy classes of* G*-conjugates of the pair* (P,s) *contained in* H.

Proposition 2.2.23 ([6], 3.2 Proposition)**.** *Suppose* H *is a subgroup of* G *and and let* $F^H_{Q,t}$ *be a primitive idempotent of* $\mathbb{C}pp_k(G)$*. Then,*

$$Ind^G_H F^H_{Q,t} = |N_G(Q,t) : N_H(Q,t)| F^G_{P,s},$$

where $N_G(Q,t)$ *is the set of elements* g *in* $N_G(P)$ *such that* $gsg^{-1} = s$.

Proposition 2.2.24. *Let* $(P,s) \in \mathcal{Q}_{G,p}$ *and let* $\phi : G \to G'$ *be a group isomorphism. Then,*

$$Iso(\phi) F^G_{P,s} = F^{G'}_{\phi(P),\phi(s)}.$$

The inflation and deflation formulas for primitive idempotents of $\mathbb{C}pp_k$ are found by Ducellier.

Proposition 2.2.25 ([5], p44)**.** *Let N be a normal subgroup of G. Then, we have*

$$Inf_{G/N}^{G} F_{P,s}^{G/N} = \sum_{(Q,t)\in I} F_{Q,t}^{G}$$

with $I := \{(Q,t) \in [\mathcal{Q}_{G,p}] | \exists \bar{g} = G/N, QN/N = {}^{\bar{g}}P, \bar{t} = {}^{g}s\}$. where $\bar{t}$ is the projection of t onto $\bar{N}_{G/N}(QN/N)$.

Proposition 2.2.26 ([5], Lemma 3.1.4, p45)**.** *Let G be a finite group and $(P,s) \in [\mathcal{Q}_{G,p}]$, and N be a normal subgroup of G. Then,*

$$Def_{G/N}^{G} F_{P,s}^{G} = m_{P,s,N} F_{Q,t}^{G/N},$$

where Q is a p-subgroup of G/N and t is a p'-element of $\bar{N}_{G/N}(Q)$.

For a specific case, Ducellier computed the deflation numbers $m_{P,s,N}$ more precisely, as follows:

Corollary 2.2.27 ([5], Corollary 3.1.9, p52)**.** *Let G be a semidirect product of p-group P and p'-element s acting on P that is to say $G = P \rtimes \langle s \rangle$. and let N be a normal subgroup of G, then we have*

$$m_{P,s,N} = \frac{|s|}{|N \cap \langle s \rangle||C_G(s)|} \sum_{\substack{Q \leqslant P \\ Q^s = Q \\ \langle Rs \rangle N = G}} |C_Q(s)| \mu((Q,P)^s),$$

where $\mu((Q,P)^s)$ is the Möbius function defined on the poset of subgroups of G normalized by s.

At this point, we shall provide further reminders about the Möbius function of posets which can be found [10] and [11].

Remark 2.2.28. *Let $(X, \leq)$ be a poset, and denote the set of chains $x_0 < x_1 < ... < x_n$ of cardinality $n+1$ of elements of X by $Sd_n(X)$. Now, the chain complex $C_*(X,\mathbb{Z})$ is formed by the module $C_n(X,\mathbb{Z})$ which is the free $\mathbb{Z}$-module with basis $Sd_n(X)$ and the differentials $d_n : C_n(X,\mathbb{Z}) \to C_{n-1}(X,\mathbb{Z})$ given by*

$$d_n(x_0, x_1, ..., x_n) = \sum_{i=0}^{n} (-1)^n (x_o, x_1, ..., \hat{x}_i, ..., x_n),$$

where $(x_0, x_1, ..., \hat{x_i}, ..., x_n)$ *denotes the chain* $(x_0, ..., x_n) - \{x_i\}$*. Moreover, one can also consider the augmented chain complex of* $C_*(X, \mathbb{Z})$, $\widetilde{C_*}(X, \mathbb{Z})$ *which is defined by setting* $\widetilde{C}_{-1}(X, \mathbb{Z}) = \mathbb{Z}$ *and* $\widetilde{C}_n(X, \mathbb{Z}) = C_n(X, \mathbb{Z})$ *and* $d_n = \tilde{d}_n$ *for* $n \geq 0$*, and the augmentation map* $\epsilon : \widetilde{C}_0(X, \mathbb{Z}) \to \widetilde{C}_{-1}(X, \mathbb{Z})$ *sending* $x_0 \mapsto 1$*. Recall that the homology group of these chain complexes* $H_n(X, \mathbb{Z}) = \frac{Ker d_n}{Im d_{n+1}}$ *and* $\widetilde{H}_n(X, \mathbb{Z}) = \frac{Ker \tilde{d}_n}{Im \tilde{d}_{n+1}}$.

If we are given two posets X *and* Y*, then a map of posets* $f : X \to Y$ *is defined to be a map such that whenever* $x \leq x^{'}$ *in* X*, we have* $f(x) \leq f(x^{'})$*. Given such a poset map, there is an induced map of chain complexes* $C_*(f, \mathbb{Z})$*:* $C_*(X, \mathbb{Z}) \to C_*(Y, \mathbb{Z})$ *such that*

$$C_n(f, \mathbb{Z})(x_0, x_1, ..., x_n) = \begin{cases} (f(x_0), f(x_1), ..., f(x_n)) & if \quad f(x_0) < f(x_1) < ... < f(x_n), \\ 0 & otherwise \end{cases}$$

Similarly, we can define the induced map of reduced chain complexes $\widetilde{C_*}(f, \mathbb{Z})$: $\widetilde{C_*}(X, \mathbb{Z}) \to \widetilde{C_*}(Y, \mathbb{Z})$ *by* $\widetilde{C_n}(f, \mathbb{Z}) = C_n(f, \mathbb{Z})$ *for* $n \geq 0$ *and* $\widetilde{C_{-1}}(f, \mathbb{Z}) = Id_{\mathbb{Z}}$.

Now, we define the Euler-Poincaré characteristic $\chi(X)$ *of a finite poset* X *to be*

$$\chi(X) = \sum_{n \geq 0} (-1)^n rank_{\mathbb{Z}} C_n(X, \mathbb{Z}).$$

Similarly,the reduced Euler-Poincaré characteristic $\tilde{\chi}(X)$ *of a finite poset* X *is defined by*

$$\tilde{\chi}(X) = \sum_{n \geq -1} (-1)^n rank_{\mathbb{Z}} \widetilde{C_n}(X, \mathbb{Z}).$$

Recall that the Möbius function μ *is the unique function from* $X \times X$ *to* $\mathbb{Z}$ *satisfying* $\mu(x, y) = 0$ *unless* $x \leq y$ *and the recursion formula*

$$\sum_{\substack{y \in X \\ x \leq y \leq z}} \mu(y, z) = \delta(x, z) = \begin{cases} 1 & if \quad x = z, \\ 0 & otherwise \end{cases}$$

We have a correspondence between the reduced Euler-Poincaré characteristic and the Möbius function ([11], Proposition 3.8.5., p121) as follows: if μ_X *is the Möbius function on the poset* X*, and* $x, y \in X$*, then we have* $\mu_X(x, y) = \tilde{\chi}((x, y)_X)$ *where* $(x, y)_X := \{z \in X \mid x < z < y\}$.

2.2.4 The monomial Burnside functor $\mathbb{C}B_{k^\times}$

Now, we are going to provide an alternative formula for deflation of these primitive idempotents. For this part, we need to briefly review Monomial Burnside ring which has a structure of a biset functor. The following definitions and formulas can be found in [12] and [13], Section 1.4 and 2.7.

Definition 2.2.29 (Monomial Burnside Ring $B_{k^\times}(G)$)**.** *Let $\mathbb{C}$ be the algebraically closed field of characteristic 0, and k be an algebraically closed field of characteristic p, and $k^\times$ denote the unit group of the field k, and suppose that G is a finite group. Let us denote the set of $k^\times$-subcharacters of G by*

$$\mathcal{C}(G) := \{(U, \mu) : U \leq G;\ \ \mu : U \to k^\times\},$$

which is a G-poset and a G-set under conjugation. Then, we define $B_{k^\times}(G)$ to be the free abelian group on the G-conjugacy classes of $(U, \mu)_G$ of elements in $\mathcal{C}(G)$. By taking the tensor product over $\mathbb{C}$, we define $\mathbb{C}B_{k^\times}(G)$.

Moreover, there is also a primitive basis of this ring. For this, let us denote the set of $k^\times$-subelements by

$$el(k^\times, G) := \{(H, hO(H)) : H \leq G, hO(H) \in H/O(H)\},$$

where $O(H)$ corresponds to the minimal normal subgroup of H such that $H/O(H)$ is an abelian p'-group. The element of this set, $(H, hO(H))$ will be denoted as (H, h). Then, we have

$$\mathbb{C}B_{k^\times}(G) \cong \bigoplus_{(H,h)\in_G el(k^\times,G)} \mathbb{C}e^G_{H,h}.$$

The Monomial Burnside ring has also a biset functor structure.

Remark 2.2.30 ([12], Lemma 7.4)**.** *For $H \leq G$, the primitive idempotent $e^G_H \in \mathbb{C}B(G)$ decomposes as a sum of primitive idempotents of $\mathbb{C}B_{k^\times}(G)$ as follows:*

$$e^G_H = \sum_{(I,i)\in_G \mathcal{I}} e^G_{I,i}$$

where $\mathcal{I}$ is the set of $k^\times$-subelements of G such that $I =_G H$.

The following inflation and deflation formulas for Monomial Burnside Ring can be found in [14]:

Proposition 2.2.31. *Let G be a finite group and N be a normal subgroup of G. Then,*

$$Inf_{G/N}^{G}(e_{K/N,kN}^{G/N}) = \sum_{(I,i)\in_G el(k^{\times},G):(IN/N,iN)=_{G/N}(K/N,kN)} e_{I,i}^{G}.$$

Proposition 2.2.32. *Let G and N be as above. Then,*

$$Def_{G/N}^{G}(e_{I,i}^{G}) = \beta_G(I/(I\cap N),I,i)e_{IN/N,iN}^{G/N}$$

where

$$\beta_G(I/(I\cap N),I,i) = \frac{|N_{G/N}(IN/N,iN):IN/N|}{|N_G(I,i):I|}\beta^{k^{\times}}(I/(I\cap N),I,i)$$

and

$$\beta^{k^{\times}}(I/(I\cap N),I,i) = \frac{1}{|O(I)(I\cap N)|}\sum_{U\leq I:U(I\cap N)=I}|U\cap iO(I)|\mu(U,I)$$

with $O(I)$ is defined as earlier.

There is a surjective map which can be found in [13], Section 4.3 and 4.7, and [15] Section 1.5, from $B_{k^{\times}}(G)$ to $pp_k(G)$. It provides us with an alternative formula for deflation of primitive idempotents of $\mathbb{C}pp_k(G)$.

Remark 2.2.33. *There is a surjective biset functor morphism called linearization map from the Monomial Burnside functor $\mathbb{C}B_{k^{\times}}$ to the biset functor of p-permutation modules $\mathbb{C}pp_k$ defined as follows: for a finite group G,*

$$\boldsymbol{lin}_G : \mathbb{C}B_{k^{\times}}(G) \to \mathbb{C}pp_k(G)$$

sending

$$e_{H,h}^{G} \mapsto \begin{cases} F_{P,h}^{G} & if \quad H=\langle P,h\rangle, \\ 0 & otherwise. \end{cases}$$

We have the following commutative diagram for deflations:

$$\begin{array}{ccc} e_{\langle P,h\rangle,h}^{G} & \xmapsto{Def_{G/N}^{G}} & \beta_G(\langle P,h\rangle/(\langle P,h\rangle\cap N),\langle P,h\rangle,h)e_{\langle P,h\rangle N/N,hN}^{G/N} \\ {\scriptstyle lin_G}\downarrow & & \downarrow{\scriptstyle lin_{G/N}} \\ F_{P,h}^{G} & \xmapsto[Def_{G/N}^{G}]{} & \beta_G(\langle P,h\rangle/(\langle P,h\rangle\cap N),\langle P,h\rangle,h)F_{PN/N,hN}^{G/N} \end{array}$$

Chapter 3

Work of Baumann on the simple composition factors of $\mathbb{C}pp_k$

Throughout this chapter, we suppose that $\mathbb{C}$ is the algebraically closed field of characteristic 0 and k is an algebraically closed field of prime characteristic p.

We have already mentioned that the simple objects of the category formed by biset functors are parametrized by pairs (G, V) and denoted by $S_{G,V}$ where G is a finite group and V is a simple $\mathbb{C}\mathrm{Out}(G)-$module. In the previous chapter, we saw the full classification for which pairs (G, V), the associated simple biset functor $S_{G,V}$ appears as a simple composition factor for the biset functors $\mathbb{C}B$ and $\mathbb{C}R_{\mathbb{C}}$ and $\mathbb{C}R_k$ on some restriction full subcategory of biset category. However, for the biset functor of $p-$permutation modules, $\mathbb{C}pp_k$, the classification of these pairs (G, V) is not completely known. By the work of Baumann [3], we have some partial information about for which pairs (G, V), the simple composition factors $S_{G,V}$'s are apparent in $\mathbb{C}pp_k$.

In this chapter, we shall review the results on some of the simple composition factors of $\mathbb{C}pp_k$ obtained by Baumann, defining a special type of group called p-hypo-elementary B-group whose classification are provided by Baumann. Secondly, we review the method that she introduced to find simple composition factors associated to groups with small order and find the full list of simple composition factors indexed by C_1, C_2, C_3, V_4 when $p = 2$.

Then, we will use these results to obtain the following theorem:

Theorem 3.0.34 (The alternating group $A_4 = V_4 \rtimes C_3$)**.** *If k is an algebraically closed field of characteristic $p = 2$, then both $S_{A_4,\mathbb{C}}$ and $S_{A_4,\mathbb{C}_{-1}}$ are the only simple composition factors of $\mathbb{C}pp_k$ associated to A_4 and their multiplicity is* 1*.*

Since A_4 is a p-hypo-elementary B-group for $p = 2$, this theorem will help us to disprove the following conjecture due to Baumann:

Conjecture 3.0.35. *([3], Conjecture 4.24, p59) Let k be an algebraically closed field of characteristic p and $\mathbb{C}$ be the algebraically closed field of characteristic* 0*. Suppose $H = P \rtimes C_l$ is a p-hypo-elementary B-group. Then, $S_{H,V}$ is a simple composition factor of $\mathbb{C}pp_k$ if and only if V is the trivial $\mathbb{C}Out(H)-$module, i.e. $S_{H,\mathbb{C}}$. Moreover, the multiplicity of $S_{H,\mathbb{C}}$ as a simple composition factor of $\mathbb{C}pp_k$ is $\Phi(l)$.*

3.1 Some of the simple composition factors of $\mathbb{C}pp_k$

We start this chapter with the review of the following findings by Baumann:

Theorem 3.1.1. *The simple functors $S_{C_m,\mathbb{C}_\xi}$ where m is a positive integer coprime to p, and ξ is a primitive character $(\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ are composition factors of $\mathbb{C}pp_k$.*

Proof. We first let $\mathcal{C}_{p'}$ to be a full-subcategory of the biset category $\mathbb{C}\mathcal{C}$ whose objects are finite groups of order coprime to p.

Claim: For any finite group G in $Obj(\mathcal{C}_{p'})$, we have $pp_k(G) \cong R_k(G)$.

Let M be an indecomposable $kG-$module. Moreover, we know that the vertices of M, which are all conjugate, must be a $p-$subgroup of G. However, G has order coprime to p, so the vertex of M must be trivial.

Now, the only indecomposable $k1-$module is k implies that M must be an indecomposable direct summand of $\mathrm{Ind}_1^G k = kG$. In particular, being an indecomposable direct summand of permutation module, M is a $p-$permutation module. That is to say, every $kG-$module is a $p-$permutation $kG-$module. Thus, it implies that $\mathbb{C}pp_k(G) = \mathbb{C}R_k(G)$ for each $G \in Obj(\mathcal{C}_{p'})$. Moreover, by definition, we already know that $\mathbb{C}pp_k(u) = \mathbb{C}R_k(u)$ for each $u \in \mathbb{C}B(H,G)$ with $H, G \in Obj(\mathcal{C}_{p'})$. Consequently, we have $\mathbb{C}pp_k^{\mathcal{C}_{p'}} = \mathbb{C}R_k^{\mathcal{C}_{p'}}$.

Moreover, recall by Remark 2.2.8 in Chapter 2, that the simple composition factors of $\mathbb{C}R_k$ on $\mathcal{C}_{p'}$ are precisely $S_{C_m,\mathbb{C}_\xi}$ where m is a positive integer coprime to p and ξ is a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$. Now, by the equality above, we conclude that these simple functors are also composition factors of $\mathbb{C}pp_k^{\mathcal{C}_{p'}}$. Now, by finite reduction principle for biset functors, we find that these simple functors are composition factors of $\mathbb{C}pp_k$ on $\mathbb{C}\mathcal{C}$ as required. □

Theorem 3.1.2. *The simple functors $S_{C_p\times C_p,\mathbb{C}}$ and $S_{1,\mathbb{C}}$ are composition factors of $\mathbb{C}pp_k$ where k is an algebraically closed field of characteristic p, prime.*

Proof. For this part, we work on full-subcategory defined on family of groups closed under taking quotients, namely, $\mathcal{F} = \{C_1, C_p, C_p \times C_p\}$. Firstly, recalling the correspondence between primitive idempotent bases of $\mathbb{C}B(G)$ and $\mathbb{C}B_{k^\times}(G)$ for any finite group G, we have $e_{C_1}^{C_1} = e_{C_1,1}^{C_1}$, $e_{C_1}^{C_p} = e_{C_p,1}^{C_p}$, and $e_{C_1}^{C_p\times C_p} = e_{C_1,1}^{C_p\times C_p}$, $e_{C_p}^{C_p\times C_p} = e_{C_p,1}^{C_p\times C_p}$, $e_{C_p\times C_p}^{C_p\times C_p} = e_{C_p\times C_p,1}^{C_p\times C_p}$, that is to say $\mathbb{C}B_{k^\times}^{\mathcal{F}} \cong \mathbb{C}B^{\mathcal{F}}$. Now, it is clear by use of linearization map between $\mathbb{C}B_{k^\times}$ and $\mathbb{C}pp_k$ that $\mathbb{C}B_{k^\times}^{\mathcal{F}} \cong \mathbb{C}pp_k^{\mathcal{F}}$. Thus, we showed that $\mathbb{C}B$ and $\mathbb{C}pp_k$ are isomorphic on $\mathcal{F}$. Moreover, we know that simple composition factors of $\mathbb{C}B$ are indexed by B-groups. However, the only B-groups which are also p-groups are C_1 and $C_p \times C_p$. Hence, on $\mathcal{F}$, the simple composition factors of $\mathbb{C}B$ are precisely $S_{C_1,\mathbb{C}}$ and $S_{C_p\times C_p,\mathbb{C}}$. Due to isomorphism, we obtain that these are also simple composition factors of $\mathbb{C}pp_k^{\mathcal{F}}$. Now, by finite reduction principle for biset functors, we conclude the desired result. □

The next result is again due to Baumann which was found by restricting the biset category $\mathbb{C}\mathcal{C}$ to full-subcategory $\mathcal{C}_{p\times p'}$ whose objects are all abelian groups. On this subcategory, she found the composition factors of $\mathbb{C}pp_k$ to be precisely the simple functors $S_{C_m,\mathbb{C}_\xi}$ and $S_{C_p\times C_p\times C_m,\mathbb{C}_\xi}$ where (m,ξ) runs through the set of positive integers m coprime to p, and primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ with multiplicity 1. Now, by the finite reduction principle for biset functors, we obtain the following theorem:

Theorem 3.1.3 ([4], Corollary 44). *The simple functors $S_{C_m,\mathbb{C}_\xi}$ and $S_{C_p\times C_p\times C_m,\mathbb{C}_\xi}$ where (m,ξ) runs through the set of positive integers m coprime to p, and primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ with multiplicity* 1*.*

3.2 p-Hypo-elementary B-groups and some simple composition factors of $\mathbb{C}pp_k$ indexed by them

Now, we are going to study some of the simple composition factors of the biset functor of $p-$permutation modules which are indexed by a special type of groups, named p-hypo-elementary B-group H. To do so, we firstly start with the definition of this special group, as follows:

Definition 3.2.1 (p-hypo-elementary group). *Let p be a prime number. A group H is said to be p-hypo-elementary if the quotient $H/O_p(H)$ is cyclic where $O_p(H)$ denotes the largest normal p-subgroup of H. This means that H has a normal p-subgroup such that the quotient is a cyclic $p^{'}$-group.*

In this thesis, we are particularly interested in finite groups which are both p-hypo-elementary and B-group which we defined in Chapter 2. Note that one of the examples of p-hypo-elementary B-group is the alternating group A_4 when $p = 2$, since $O_p(A_4) = V_4$ and $A_4/V_4 \cong C_3$ and $m_{A_4,V_4} = m_{A_4,A_4} = 0$.

Baumann has found a partial result about the appearance of simple composition factors $S_{H,V}$ of $\mathbb{C}pp_k$ where H is a finite p-hypo-elementary B-group as follows:

Theorem 3.2.2 ([2], Theorem 30). *The simple functors $S_{H,\mathbb{C}}$ with H is a finite p-hypo-elementary B-group are composition factors of $\mathbb{C}pp_k$. However, the multiplicity of $S_{H,\mathbb{C}}$ as a composition factor of $\mathbb{C}pp_k$ is not known.*

She also found the following result:

Theorem 3.2.3 ([3], Theorem 4.15., p48). *Let k be an algebraically closed field of characteristic p, prime. Let $G = C_p \rtimes C_l$ where $l > 1$ and $(l, p) = 1$, and the action of*

C_l on C_p is faithful. Then, the simple functor $S_{C_p \rtimes C_l, V}$ is a simple composition factor of $\mathbb{C}pp_k$ if and only if V is the trivial $\mathbb{C}Out(C_p \rtimes C_l)$-module $\mathbb{C}$ i.e. $S_{C_p \rtimes C_l, \mathbb{C}}$. Moreover, the multiplicity of $S_{C_p \rtimes C_l, \mathbb{C}}$ as a composition factor of $\mathbb{C}pp_k$ is equal to $\Phi(l)$.

It should be noted at this point that any p-hypo-elementary group has form $H = O_p(H) \rtimes C_l$ with $(l, p) = 1$ which follows from Schur-Zassenhaus Theorem.

3.3 The classification of p-hypo-elementary B-groups

Now, we shall state the classification of p-hypo-elementary B-groups which are completed by Baumann:

Theorem 3.3.1 ([2], Theorem 43)**.** *$G \cong P \rtimes C_n$ is a p-hypo-elementary B-group if and only if*

(i) P is elementary abelian,

(ii) The action of C_n on P is faithful,

(iii) In a decomposition of P as a direct sum of simple $\mathbb{F}_p C_n$-modules, every simple $\mathbb{F}_p C_n$-module appears at most one time, except the trivial module which may appear 0 or 2 times.

Proof. Let us suppose that $G = P \rtimes C_n$ is a B-group.

STEP 1: P is elementary abelian group.

Proof. We start with the following claim:

Claim: $\Phi(P) \subseteq \Phi(G)$.

Proof: Let M be a maximal subgroup of G. It is enough to show that $\Phi(P) \subseteq M$. Since P is the unique Sylow p-subgroup of G, we have $R = M \cap P$ is a normal Sylow p-subgroup of M. Now, if $R = P$, then it would clearly imply that $\Phi(P) \subseteq P \subseteq M$.

If $R \neq P$, then we consider the subgroups $\Phi(P)R$ and $\Phi(P)M$, which are well-defined because of the fact that $\Phi(P) \trianglelefteq G$ by noting $\Phi(P)$ is a characteristic group of P and $P \trianglelefteq G$. Now, since M is a maximal subgroup of G, we have either $\Phi(P)M = M$ or $\Phi(P)M = G$. But $(\Phi(P)M) \cap P = \Phi(P)R \neq P$, because of our assumption $R \neq P$. Therefore, it implies that we must have $\Phi(P)M = M$. Then, $\Phi(P) \subseteq M$, that is to say, $\Phi(P) \subseteq \Phi(G)$.

Now, suppose that $\Phi(P) \neq 1$. Then, since G is a B-group and $1 \neq \Phi(P) \trianglelefteq G$, we must have

$$m_{G,\Phi(P)} = \frac{1}{|G|} \sum_{X\Phi(P)=G} |X|\mu(X,G) = 0.$$

But note that since $X\Phi(P) = G$ and $\Phi(P) \subseteq \Phi(G)$, we have $X\Phi(G) = G$. But then by the property of Frattini subgroup, we have $X = G$. Then,

$$m_{G,\Phi(P)} = \frac{1}{|G|}|G|\mu(G,G) = 1 \neq 0,$$

which is a contradiction. Thus, $\Phi(P) = 1$. Since P is a p-group, it is possible if and only if P is elementary abelian, as required. □

STEP 2: C_n acts trivially on P.

Proof. For this part, we shall start with the result on calculation of deflation numbers:

Proposition 3.3.2 ([1], Proposition 5.6.4.)**.** *If N is a minimal normal abelian subgroup of G, then*

$$m_{G,N} = 1 - \frac{|K_G(N)|}{|N|},$$

where $K_G(N)$ is the set of complements of N in G.

Moreover, if G is solvable, then G is a B-group if and only if $|K_G(N)| = |N|$ for all minimal normal subgroups N of G.

Claim: Any p-hypo-elementary B-group $G \cong P \rtimes C_n$ is solvable.

Proof: Recall that if $B \trianglelefteq A$, and both B and A/B are solvable groups then A is solvable (cf. [16], Proposition 3.25, p188). Now, since P is a finite p-group, it is necessarily solvable by using the fact above, and the fact that $1 \neq Z(P) \trianglelefteq P$ and induction. Then, again by the same fact, since $P \trianglelefteq G$, and both P and $C_n \cong G/P$ are solvable, we have G is solvable, as claimed.

Therefore, we can make use of the latter part of Proposition 3.3.2, i.e., $G = P \rtimes C_n$ is a B-group if and only if $|K_G(N)| = |N|$, for all minimal normal subgroups N of G.

Now, $n = 1$ case is trivial so we suppose $n \geq 2$. Let us denote the action of C_n on P by $\varphi : H \to \mathrm{Aut}(P)$ with $Ker\varphi = C_d$.

Assume for a contradiction that the action is not faithful, i.e., $C_d \neq 1$. Then, we have $C_d \trianglelefteq G = P \rtimes C_n$. Now, there exists a minimal normal subgroup $1 \neq N$ in C_d. We shall observe that there can be at most one complement of N in G. Suppose that C is a complement of N in G, then since C contains the normal Sylow p-subgroup P of G, it has a form $C = P \rtimes K$ where $K \leq C_n$. Note that since C is a complement of N in C_n, we have at most one possibility for K. Therefore, $|K_G(N)| \leq 1$. However, since G is a B-group which is solvable, we must have $|N| = |K_G(N)|$, a contradiction since $N \neq 1$. Therefore, the kernel of the action is trivial as required. □

STEP 3: Condition(iii) is satisfied.

Proof. We shall start with an observation:

Claim: Any minimal normal subgroup N of G is always contained in P.

Proof: Let $1 \neq N$ be a minimal normal subgroup of G. We may suppose that $N \cap P = 1$, because otherwise, by the minimality of N, we would have $N \leqslant P$. Then, note that since $N, P \trianglelefteq G$ and $N \cap P = 1$, we must have $[N, P] = 1$. Thus, $N \leq C_G(P)$, i.e., $P \leq C_G(P) \leq PC_n$. But then, $C_G(P) = P(C_G(P) \cap C_n) = PC_{C_n}(P) = P$ since $C_{C_n}(P) = 1$ by the last part of Step 2. Hence, we obtain $C_G(P) = P$. However, we also found that $N \leq C_G(P) = P$. Therefore, $N = 1$, a contradiction. Therefore, we must have $N \subseteq P$.

Now, we need to describe complements of N:

Claim: Let $N \trianglelefteq G$ such that $N \leq P$. Then, every complement of N is of the form $C \rtimes Q$ where $C \trianglelefteq G$ which is a complement of N in P and Q is a subgroup of G conjugate to C_n.

Proof: Let X be a complement of N in G. Then, we define $C = X \cap P$, which is a normal Sylow p-subgroup of X. Moreover, since P is abelian, we also have $C \trianglelefteq P$, therefore, $C \trianglelefteq G$.

Now, by Schur-Zassenhaus Theorem, since the order of C and X/C is coprime, there exists a subgroup $Q \cong X/C$ of X such that $X = C \rtimes Q$. Since $N \leq P$, we must have $|Q| = n$. But since N is solvable, by second part of Schur-Zassenhaus theorem, every subgroup of order n is conjugate in G. Thus, we have $Q =_G C_n$.

On the other hand, suppose that $X = C \rtimes Q$ such that $C \trianglelefteq G$ which is a complement of N in P, and $Q \leq G$ such that $|Q| = n$. Then, we have $N \cap X = N \cap C = 1$ and $NX = N(C \rtimes Q) = NC \rtimes Q = P \rtimes Q = G$, i.e., X is a complement of N in G, completing the proof of the claim.

Now, by Step 1, P is elementary abelian group, so it can be thought as an $\mathbb{F}_p$-vector space, and since C_n acts on P, P has an $\mathbb{F}_p C_n$-module structure. We shall note that since $(p, n) = 1$, every $\mathbb{F}_p C_n$-module is semisimple, so is P.

Then, let $P = \bigoplus_{i=1}^{t} P_i$ be the homogeneous componentwise decomposition of $\mathbb{F}_p C_n$-module P, where each P_i corresponds to $P_i \cong \bigoplus_{j=1}^{m_i} S_i$ with a simple $\mathbb{F}_p C_n$-module S_i.

At this point, we suppose S_1 to be the trivial $\mathbb{F}_p C_n$-module and if it does not appear in the decomposition of P, we add it.

Moreover, note that $\mathbb{F}_p C_n \cong \mathbb{F}_p[x]/(x^n - 1) = \prod_i \mathbb{F}_p[x]/m_i(x)$, each S_i corresponding to $\mathbb{F}_p[x]/m_i(x)$ where $m_i(x)$ is irreducible polynomial over $\mathbb{F}_p$, so we have S_i is a field over $\mathbb{F}_p$ i.e. for some $s_i \in \mathbb{N}$, $S_i \cong \mathbb{F}_{p^{s_i}}$.

Now, let N be a minimal normal subgroup of G. Then, we know that $|K_G(N)| = |N|$.

Moreover, we found that $N \leq P$ and since N is a normal subgroup of G, we can think N as a $\mathbb{F}_p C_n$-submodule of P. Note that by minimality of N, we must have $N \cong S_l$ for some $1 \leq l \leq t$.

By our claim above, we know the description of complements of such a normal subgroup N. In fact, they are all in the form $C \rtimes Q$ where $C \trianglelefteq G$, which is a complement of N in P, and Q is a cyclic subgroup of G of order n.

Determination of the number of possibilities for C:

Note that C can be thought as an $\mathbb{F}_p C_n$-submodule of P. Thus, by using the uniqueness of the homogeneous components, a complement C of N in P is of the form

$$H_l \oplus \bigoplus_{\substack{i=1 \\ i \neq l}}^{t} P_i,$$

where H_l is a complement of N in P_l.

Now, recall that we obtained $N \cong S_l = \mathbb{F}_{p^{s_l}}$, therefore, having $P_l \cong \bigoplus_{j=1}^{m_l} S_l$, we can think P_l as a vector space over $\mathbb{F}_{p^{s_l}}$. Therefore, in order to find the number of complements, it is sufficient to count the number of complements as $\mathbb{F}_{p^{s_l}}$-vector spaces.

Note that the number of complements of $N = \mathbb{F}_{p^{s_l}}$ of $P_l \cong \bigoplus_{j=1}^{m_l} S_l$ is equal to the difference between the number of hyperplanes of P_l and the number of hyperplanes of P_l that contain N. Note that the latter one is equal to the number of hyperplanes of P_l/N.

Thus, the number of complements of N in P_l is

$$\begin{bmatrix} m_l \\ 1 \end{bmatrix}_{p^{s_l}} - \begin{bmatrix} m_l - 1 \\ 1 \end{bmatrix}_{p^{s_l}} = \frac{(1-(p^{s_l})^{m_l})}{(1-p^{s_l})} - \frac{(1-(p^{s_l})^{m_l-1})}{(1-p^{s_l})} = p^{s_l(m_l-1)}.$$

Therefore, we obtain $p^{s_l(m_l-1)}$ many complements of N in P.

Determination of the number of possibilities for Q:

Since we have $NC = P$, we must find the number of complements of P in $P \rtimes C_n$ divided by the number of complements of C in $C \rtimes Q$.

Claim: The number of complements of P in $G = P \rtimes C_n$ is equal to p^{m-m_1}.

Proof: For this, we have to find the number of subgroups which are conjugate to C_n in G. At this point, we again refer to Schur-Zassenhaus theorem, which states that P acts transitively on the set of conjugates of C_n in G. Therefore, the number of such groups is equal to $\frac{|P|}{|C_P(C_n)|}$. We claim that $C_P(C_n) = P_1$, which is the homogeneous component of P associated to the trivial $\mathbb{F}_p C_n$-module S_1, i.e., $P_1 \cong \bigoplus_{j=1}^{m_1} S_1$. It is clear that $P_1 \leq C_P(C_n)$. Conversely, suppose that $p \in C_P(C_n)$, then we have for every $h \in C_n$, $(p^h p^{-1}, h) \in C_n$, so ${}^h p^{-1} = p^{-1}$, i.e., $p \in P_1$, as required.

But then, we have the number of complements of P in $G = P \rtimes C_n$ is equal to $\frac{|P|}{|P_1|} = \frac{p^m}{p^{m_1}} = p^{m-m_1}$.

<u>The calculation of the complements of C in $C \rtimes Q$:</u>

Suppose $N \cong S_1$, then $C \cong H_1 \oplus \bigoplus_{i \neq 1}^{t} P_i$ where H_1 is the complement of N in P_1. Now, we have $|C_C(Q)| = H_1$. Then, the number of complements of C in $C \rtimes Q$ is equal to $\frac{|C|}{|H_1|} = \frac{p^{m-s_1}}{p^{m_1-1}} = p^{m-m_1}$ noting that $s_1 = 1$ because $S_1 \cong \mathbb{F}_p$.

Suppose $N \cong S_l$ where S_l is non-trivial module. Then, we have $|C_C(Q)| = P_1$, and so the number of complements of C in $C \rtimes Q$ is equal to $\frac{p^{m-s_l}}{p^{m_1}} = p^{m-s_l-m_1}$.

Now, we have the number of possibilities for Q is equal to

$$= \begin{cases} \frac{p^{m-m_1}}{p^{m-m_1}} = 1 & if \quad l = 1 \\ \frac{p^{m-m_1}}{p^{m-s_l-m_1}} = p^{s_l} & if \quad l \neq 1 \end{cases}$$

Then, we obtain

$$|K_G(N)| = \begin{cases} p^{s_l(m_1-1)} = p^{(m_1-1)} & if \quad l = 1 \\ \frac{p^{s_l(m_l-1)}}{p^{s_l}} = p^{s_l m_l} & if \quad l \neq 1 \end{cases}$$

But since G is a B-group, we must have $|K_G(N)| = |N| = p^{s_l}$. Hence, if $l = 1$, then $p^{m_1-1} = p$ implies $m_1 = 2$ noting that S_0 may not be apparent in P at all, i.e., $m_1 = 0$. If $l \neq 1$, then $p^{s_l m_l} = p^{s_l}$ which is true if and only if $m_l = 1$ noting that $m_l = 0$ as well, if S_l is not apparent in P.

Therefore, we have shown that the condition(iii) is satisfied. □

Converse of this theorem follows easily by following the calculation for $|K_G(N)|$, and using the fact that each S_l apparent in P can be taken as a minimal normal subgroup of G.

□

Remark 3.3.3. *Note that for our example $A_4 = V_4 \rtimes C_3$ when $p = 2$, we have V_4 is elementary abelian, $C_{C_3}(V_4) = 1$ that is to say the action of C_3 on V_4 is faithful, and as an $\mathbb{F}_2C_3$-module, $V_4 \cong S_2$ where S_2 is the 2−dimensional simple $\mathbb{F}_2C_3$-module appearing once and the trivial $\mathbb{F}_2C_3$−module S_1 appears zero times.*

3.4 The conjecture of Baumann on the appearence of simple composition factors of $\mathbb{C}pp_k$ indexed by p-hypo-elementary B-groups

We have reviewed notion of B-groups and the fact that the simple composition factors of $\mathbb{C}B$ are precisely $S_{G,\mathbb{C}}$ where G is a finite B-group. Moreover, we studied that Baumann found out that for a p-hypo-elementary B-group H, $S_{H,\mathbb{C}}$ is a simple composition factor of $\mathbb{C}pp_k$. Combining these observations, and Theorem 3.2.3, it is reasonable to expect that for a p-hypo-elementary B-group H, the only composition factors of $\mathbb{C}pp_k$ are in the form of $S_{H,\mathbb{C}}$. Baumann has a conjecture on this as follows:

Conjecture 3.4.1 ([3], Conjecture 4.24, p59)**.** *Let k be an algebraically closed field of characteristic p and $\mathbb{C}$ be the algebraically closed field of characteristic 0. Suppose $H = P \rtimes C_l$ is a p-hypo-elementary B-group. Then, $S_{H,V}$ is a simple composition factor of $\mathbb{C}pp_k$ if and only if V is the trivial $\mathbb{C}Out(H)$−module, i.e. $S_{H,\mathbb{C}}$. Moreover, the multiplicity of $S_{H,\mathbb{C}}$ as a simple composition factor of $\mathbb{C}pp_k$ is $\Phi(l)$.*

In this last part of this chapter, we want to disprove this conjecture by considering the alternating group A_4 which is a p-hypo-elementary B-group for $p = 2$. However, firstly, we need the following remarks and methods which can be found in [3], Chapter 4, Section 4.3. We are going to apply her method to find all simple composition factors of $\mathbb{C}pp_k$ indexed by finite groups C_1, C_2, C_3, V_4 and associated simple modules when $p = 2$. However, as we will see, this method will not provide us with finding the explicit simple composition factors indexed by A_4 when $p = 2$. Therefore, we will provide an alternative method to find some simple composition factors indexed by A_4 when $p = 2$. This method will be generalized to some p-hypo-elementary B-groups when characteristic of k is p to find new simple composition factors of $\mathbb{C}pp_k$.

Remark 3.4.2. *We have the following equalities, for any finite group G:*

$$dim_{\mathbb{C}}\mathbb{C}pp_k(G) = \sum_{(H,V)\in CF(G)} m_{H,V} dim_{\mathbb{C}} S_{H,V}(G),$$

where $CF(G)$ is the set of pairs (H, V) where H is a subquotient of G and V is a simple $\mathbb{C}Out(H)-$module, and $m_{H,V}$ is the multiplicity of $S_{H,V}$ as a composition factor in $\mathbb{C}pp_k$. (by Bouc, we know that if $dim_{\mathbb{C}} S_{H,V}(G) \neq 0$ then H has to be a subquotient of G.) On the other hand, we have

$$dim_{\mathbb{C}}\mathbb{C}pp_k(G) = \sum_{P} l_p(N_G(P)/P),$$

where P runs throught the set of $p-$subgroups of G up to conjugacy, and $l_p(N_G(P)/P)$ denotes the number of conjugacy classes of $p^{'}-$elements in $N_G(P)/P$.

The following remarks will help us to compute the dimensions of simple functors $S_{G,V}$ evaluated at some H over $\mathbb{C}$.

Remark 3.4.3 ([1], Theorem 5.5.4, p91)**.** *Let G be a $B-group$, then $dim_{\mathbb{C}} S_{G,\mathbb{C}}(H)$ is equal to the number of conjugacy classes of subgroups K of H such that $\beta(K) \cong G$. In particular, $dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(H)$ is equal to the number of conjugacy classes of cyclic subgroups of G.*

Remark 3.4.4 ([1], Corollary 7.4.3, p134)**.** *Let H be a finite group, then $dim_{\mathbb{C}} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi}(H)$ is equal to the number of conjugacy classes of cyclic groups K of H, of order multiple of m, for which the natural image of $N_H(K)$ in $(\mathbb{Z}/m\mathbb{Z})^\times$ is contained in kernel of ξ.*

Let us use Baumann's method to find simple composition factors $S_{G,V}$ of $\mathbb{C}pp_k$ where G is a group with small order and V is a simple $\mathbb{C}Out(G)-$module to compute those that are indexed by groups $G = C_1, C_2, C_3, V_4$ in characteristic $p = 2$. The results that we obtain will be used when we will try to find composition factors associated to the alternating group A_4 for $p = 2$ due to the fact that if $dim_{\mathbb{C}} S_{G,V}(A_4) \neq 0$ then G has to be a subquotient of A_4, namely the candidates for G are C_1, C_2, C_3, V_4 and A_4 itself.

Let us recall the result due to Bouc in this chapter once again, $dim_{\mathbb{C}} S_{G,V}(H) \neq \{0\}$, then G must be a subquotient of H.

Proposition 3.4.5. *Suppose that k is an algebraically closed field of characteristic 2. The simple biset functor $S_{C_1,V}$ is a composition factor of $\mathbb{C}pp_k$ if and only if $V = \mathbb{C}$ and the multiplicity of $S_{C_1,\mathbb{C}}$ as a composition factor of $\mathbb{C}pp_k$ is 1.*

Proof. We have

$$1 = dim_{\mathbb{C}} \mathbb{C}pp_k(C_1) = \sum_{(H,V)\in CF(C_1)} m_{H,V} dim_{\mathbb{C}} S_{H,V}(C_1),$$

where $CF(C_1)$ is the set of pairs (H, V) where H is a subquotient of C_1 and V is a simple $\mathbb{C}Out(C_1)-$module. Now, clearly, $H = C_1$ and $V = \mathbb{C}$ the trivial module. That is to say, we have

$$1 = dim_{\mathbb{C}} \mathbb{C}pp_k(C_1) = m_{C_1,\mathbb{C}} dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(C_1),$$

where $m_{C_1,\mathbb{C}}$ is the multiplicity of $S_{C_1,\mathbb{C}}$ as a composition factor of $\mathbb{C}pp_k$. Due to Remark 3.4.3, we know that $dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(C_1) = 1$, which implies that the multiplicity of $S_{C_1,\mathbb{C}}$ is 1. Now, by finite reduction principle for biset functors, we conclude that $S_{C_1,\mathbb{C}}$ is a composition factor of $\mathbb{C}pp_k$ on $\mathbb{C}\mathcal{C}$ with multiplicity 1. □

Proposition 3.4.6. *Suppose that k is an algebraically closed field of characteristic 2. The simple biset functor $S_{C_2,V}$ never appears as a composition factor of $\mathbb{C}pp_k$ for any simple $\mathbb{C}Out(C_2)$-module V.*

Proof. We have

$$2 = dim_{\mathbb{C}} \mathbb{C}pp_k(C_2) = \sum_{(H,V)\in CF(C_2)} m_{H,V} dim_{\mathbb{C}} S_{H,V}(C_2),$$

where $CF(C_2)$ is the set of pairs (H, V) where H is a subquotient of C_2 and V is a simple $\mathbb{C}Out(H)-$module. Hence, candidates for H are C_1 and C_2. We already found that $S_{C_1,\mathbb{C}}$ is apparent with multiplicity 1 in $\mathbb{C}pp_k$. Moreover, we have $dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(C_2) = 2$ due to Remark 3.4.3.

We have the following picture:

p	2
$dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(C_2)$	2
.....	.....
$dim_{\mathbb{C}} \mathbb{C}pp_k(C_2)$	2

Note that if there was another pair (C_2, V) in $CF(C_2)$ such that $S_{C_2,V}$ is apparent as a composition factor of $\mathbb{C}pp_k$, then since $dim_{\mathbb{C}} S_{C_2,V}(C_2) = dim_{\mathbb{C}} V \neq 0$, we would have

$$2 = dim_{\mathbb{C}} \mathbb{C}pp_k(C_2) < \sum_{(H,V) \in CF(C_2)} m_{H,V} dim_{\mathbb{C}} S_{H,V}(C_2),$$

which is a contradiction. Hence, there is no simple composition factor associated with C_2 of $\mathbb{C}pp_k$ when $p = 2$. □

Proposition 3.4.7. *Suppose that k is an algebraically closed field of characteristic 2. The simple biset functor $S_{C_3,V}$ is a simple composition factor of $\mathbb{C}pp_k$ if and only if $V = \mathbb{C}_{\xi_2}$ where ξ_2 denotes the primitive character $\xi_2 : (\mathbb{Z}/3\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$. Moreover, the multiplicity of $S_{C_3,\mathbb{C}_{\xi_2}}$ is 1.*

Proof. When p=2, we have

$$3 = dim_{\mathbb{C}} \mathbb{C}pp_k(C_3) = \sum_{(H,V) \in CF(C_3)} m_{H,V} dim_{\mathbb{C}} S_{H,V}(C_3),$$

and the candidates for H is C_1 and C_3.

By Theorem 3.1.3, we know that $S_{C_3,\mathbb{C}_{\xi_2}}$ is a simple composition factor of $\mathbb{C}pp_k$ with multiplicity 1 and we already saw that $S_{C_1,\mathbb{C}}$ is apparent with multiplicity 1 as well. Moreover, by Remark 3.4.3 and Remark 3.4.4, we have $dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(C_3) = 2$ and $dim_{\mathbb{C}} S_{C_3,\mathbb{C}_{\xi_2}}(C_3) = 1$. Then, we have the following picture:

p	2
$dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(C_3)$	2
$dim_{\mathbb{C}} S_{C_3,\mathbb{C}_{\xi_2}}(C_3)$	1
....	....
$dim_{\mathbb{C}} \mathbb{C}pp_k(C_3)$	3

Thus, by similar argument above, we find that there cannot be any other composition factor of $\mathbb{C}pp_k$ indexed by C_3 except $S_{C_3,\mathbb{C}_{\xi_2}}$ which has a multiplicity 1. □

Proposition 3.4.8. *Suppose that k is an algebraically closed field of characteristic 2. The simple biset functor $S_{V_4,V}$ is a simple composition factor of $\mathbb{C}pp_k$ if and only if V is the trivial $\mathbb{C}Out(V_4)$−module. Moreover, the multiplicity of $S_{V_4,\mathbb{C}}$ is 1.*

Proof. Firstly, let us start with determination of $dim_{\mathbb{C}} \mathbb{C}pp_k(V_4)$, noting that we are taking the $p-subgroups$ of V_4 up to V_4-conjugacy, we have:

$char(k)$	P	$N_{V_4}(P)$	$N_{V_4}(P)/P$	$l_2(N_{V_4}(P)/P)$
	C_1	V_4	V_4	1
$p=2$	C_2^1	V_4	C_2	1
	C_2^2	V_4	C_2	1
	C_2^3	V_4	C_2	1
	V_4	V_4	C_1	1

Thus, we have $dim_{\mathbb{C}} \mathbb{C}pp_k(V_4) = 5$ when $p = 2$.

So, we have

$$5 = dim_{\mathbb{C}} \mathbb{C}pp_k(V_4) = \sum_{(H,V)\in CF(V_4)} m_{H,V} dim_{\mathbb{C}} S_{H,V}(V_4),$$

where $CF(V_4)$ is the set of pairs (H, V) where H is a subquotient of V_4 and V is a simple $\mathbb{C}$Out(V_4)−module. Hence, the candidates of H are C_1, C_2, V_4.

We already found that $S_{C_1,\mathbb{C}}$ is a composition factor of $\mathbb{C}pp_k$ with multiplicity 1, and $S_{C_2,V}$ never appears for any simple $\mathbb{C}$Out(C_2)-module V.

Moreover, we already know that since $V_4 = C_2 \times C_2$, V_4 is a B-group. Therefore, by Theorem 3.2.2, $S_{V_4,\mathbb{C}}$ is a composition factor of $\mathbb{C}pp_k$, yet we do not know the multiplicity.

Now, we have $dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(V_4) = 4$ by Remark 3.4.3 and $dim_{\mathbb{C}} S_{V_4,\mathbb{C}}(V_4) = dim_{\mathbb{C}}(\mathbb{C}) = 1$. The picture is as follows:

p	2
$dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(V_4)$	4
$dim_{\mathbb{C}} S_{V_4,\mathbb{C}}(V_4)$	1
....	....
$dim_{\mathbb{C}} \mathbb{C}pp_k(V_4)$	5

By similar argument as above, we found that $S_{V_4,V}$ is a composition factor of $\mathbb{C}pp_k$ if and only if V is the trivial $\mathbb{C}\mathrm{Out}(V_4)$-module. Moreover, the multiplicity of $S_{V_4,\mathbb{C}}$ is 1 as a composition factor of $\mathbb{C}pp_k$. □

Now, we are ready to construct our example to disprove Baumann's conjecture:

Theorem 3.4.9 (The alternating group $A_4 = V_4 \rtimes C_3$)**.** *If k is an algebraically closed field of characteristic $p = 2$, then both $S_{A_4,\mathbb{C}}$ and $S_{A_4,\mathbb{C}_{-1}}$ are the only simple composition factors associated to A_4 and their multiplicity is* 1*.*

Proof. Firstly, note that we have A_4-conjugacy classes: $[1]_{A_4} = \{1\}$, $[(123)]_{A_4} = \{(123), (142), (134), (243)\}$ and $[(132)]_{A_4} = \{(132), (124), (143), (234)\}$ and $[(12)(34)]_{A_4} = \{(12)(34), (13)(24), (14)(23)\}$.

We have the following results for $H = A_4$,

$char(k)$	P	$N_{A_4}(P)$	$N_{A_4}(P)/P$	$l_2(N_{A_4}(P)/P)$
	C_1	A_4	A_4	3
$p = 2$	C_2	V_4	C_2	1
	V_4	A_4	C_3	3

Thus, $dim_{\mathbb{C}}\mathbb{C}pp_k(A_4) = 7$ when $p = 2$. Hence, we have

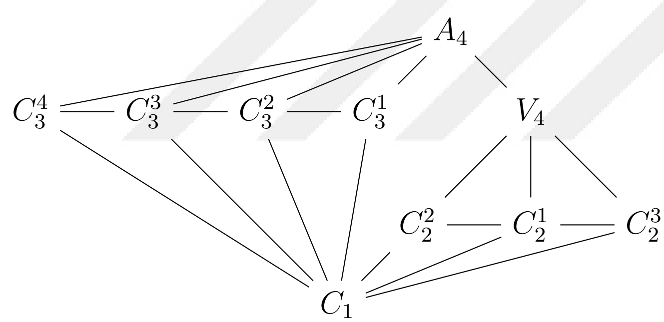
$$7 = dim_{\mathbb{C}}\mathbb{C}pp_k(A_4) = \sum_{(H,V)\in CF(A_4)} m_{H,V} dim_{\mathbb{C}} S_{H,V}(A_4),$$

where $CF(A_4)$ is the set of pairs (H, V) where H is a subquotient of A_4 and V is a simple $\mathbb{C}Out(H)-$module. The candidates for H are C_1, C_2, C_3, V_4, A_4.

By remarks above, for $H = C_1, C_2, C_3, V_4$, we know that we only have the simple composition factors : $S_{C_1,\mathbb{C}}$ with the multiplicity 1, $S_{C_3,\mathbb{C}_{\xi_2}}$ with the multiplicity 1, and $S_{V_4,\mathbb{C}}$ with the multiplicity 1.

We need to compute dimensions of these simple biset functors evaluated at A_4.

Let us recall the subgroup lattice of A_4:



By remark 3.4.3, we have $dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(A_4) = 3$ which is the number of conjugacy classes of cyclic subgroups of A_4, namely, C_1, C_2, C_3 since all C_2 groups are A_4-conjugates and so are C_3 groups of A_4. Moreover, $dim_{\mathbb{C}} S_{C_3,\mathbb{C}_{\xi_2}}(A_4) = 1$, and $dim_{\mathbb{C}} S_{V_4,\mathbb{C}}(A_4) = 1$ because $\beta(C_1) = C_1, \beta(C_2) = C_1, \beta(C_3) = C_1, \beta(A_4) = A_4$, there are only one subgroup K of A_4 such that $\beta(K) \cong V_4$, and this is only $K = V_4$ itself. Moreover, since A_4 is a p-hypo-elementary B-group, by Theorem 3.2.2, we know that $S_{A_4,\mathbb{C}}$ is apparent as a composition factor of $\mathbb{C}pp_k$ yet we do not know the multiplicity. Thus, we have the following result:

p	2
$dim_{\mathbb{C}} S_{C_1,\mathbb{C}}(A_4)$	3
$dim_{\mathbb{C}} S_{C_3,\mathbb{C}_{\xi_2}}(A_4)$	1
$dim_{\mathbb{C}} S_{V_4,\mathbb{C}}(A_4)$	1
$dim_{\mathbb{C}} S_{A_4,\mathbb{C}}(A_4)$	1
....	....
$dim_{\mathbb{C}} \mathbb{C}pp_k(A_4)$	7

This implies that we must have simple composition factors indexed by A_4 and a simple $\mathbb{C}\mathrm{Out}(A_4)$-module V. We have $\mathrm{Out}(A_4) \cong C_2$. This V can be either $\mathbb{C}$ in which case we would say $m_{A_4,\mathbb{C}} = 2$, or $V = \mathbb{C}_{-1}$ in which case we would conclude that $S_{A_4,\mathbb{C}}$ and $S_{A_4,\mathbb{C}_{-1}}$ are both the only composition factors of $\mathbb{C}pp_k$ with each of multiplicity 1.

For this part, Let $\mathcal{F}_{|A_4|}$ be a full-subcategory of the biset category $\mathbb{C}\mathcal{C}$ whose objects are finite groups of order less than or equal to the order of alternating group, $|A_4|$.

Moreover, we know that

$$[\mathcal{Q}_{A_4,p=2}] = \{(C_1,1),(C_1,(123)),(C_1,(132)),(C_2,1),(V_4,1),(V_4,(123)),(V_4,(132))\}.$$

Claim: $M_1 = span_{\mathbb{C}} < F^{A_4}_{V_4,(123)} + F^{A_4}_{V_4,(132)} >$ and $M_2 = span_{\mathbb{C}} < F^{A_4}_{V_4,(123)} - F^{A_4}_{V_4,(132)} >$ are both simple biset functors on the subcategory $\mathcal{F}_{|A_4|}$.

Proof. Since every biset functor can be thought as a module of the quiver algebra $\bigoplus_{\forall H,G\in\mathcal{F}_{|A_4|}} B(H,G)$, and due to the fact that elements of biset Burnside rings are generated by five elementary maps, namely, induction, inflation, isogation, deflation and restriction, we only need to check that M_1 and M_2 are closed under the action of those maps. Since both M_1 and M_2 are formed by the primitive idempotents $F^{A_4}_{V_4,(123)}$ and $F^{A_4}_{V_4,(132)}$, it suffices to consider the effects of these maps on the primitive idempotents $F^{A_4}_{V_4,(123)}$ and $F^{A_4}_{V_4,(132)}$:

(i) There is no induction nor inflation of the primitive idempotents $F^{A_4}_{V_4,(123)}$ and $F^{A_4}_{V_4,(132)}$ on the category $\mathcal{F}_{|A_4|}$ because there is no group in this category where A_4 is its subgroup or its subquotient.

(ii) Due to restriction formula, for any proper subgroup K of A_4, $\mathrm{Res}^{A_4}_{K} F^{A_4}_{V_4,(123)} = 0$ and $\mathrm{Res}^{A_4}_{K} F^{A_4}_{V_4,(132)} = 0$.

(iii) For deflations, we have non-trivial normal subgroups of A_4 are V_4 and A_4. In this chapter, we shall compute deflation via the linearization map. Note that in Chapter 4, we will show that these deflations are zero by using Ducellier's deflation formula.

Claim: $\mathrm{Def}^{A_4}_{A_4/A_4} F^{A_4}_{V_4,(123)} = 0$ and $\mathrm{Def}^{A_4}_{A_4/A_4} F^{A_4}_{V_4,(132)} = 0$.

Proof. Firstly, we note that $lin_{A_4}(e^{A_4}_{A_4,(123)}) = F^{A_4}_{V_4,(123)}$, and we have, recalling the deflation formula for the primitive idempotent basis of $\mathbb{C}B_{k^\times}(A_4)$,

$$\mathrm{Def}^{A_4}_{A_4/A_4} F^{A_4}_{V_4,(123)} = \lambda e^{C_1}_{C_1,1},$$

where $\lambda = \frac{|N_{A_4/A_4}(A_4A_4/A_4,(123)A_4):A_4A_4/A_4|}{|N_{A_4}(A_4,(123)):A_4|}\beta^{k^\times}(A_4/(A_4 \cap A_4), A_4, (123))$, and,

$$\beta^{k^\times}(A_4/(A_4 \cap A_4), A_4, (123)) = \frac{1}{|O(A_4)(A_4 \cap A_4)|} \sum_{\substack{U \leqslant A_4: \\ U(A_4 \cap A_4) = A_4}} |U \cap (123)O(A_4)| \mu(U, A_4).$$

We can see that U runs through all subgroups of A_4, that is,

$$U = \{C_1, C_2^1, C_2^2, C_2^3, C_3^1, C_3^2, C_3^3, C_3^4, V_4, A_4\},$$

and we have $O(A_4) = V_4$, and the coset $(123)V_4 = \{(123), (134), (243), (142)\}$, then we have $|C_1 \cap (123)V_4| = 0$, $|C_2 \cap (123)V_4| = 0$ and $|V_4 \cap (123)V_4| = 0$, $|C_3 \cap (123)V_4| = 1$ and $|A_4 \cap (123)V_4| = 4$. Thus, we have

$$\beta^{k^\times}(A_4/(A_4 \cap A_4), A_4, (123)) = \frac{1}{|O(A_4)(A_4 \cap A_4)|}(4 \cdot \mu(C_3, A_4) + 4 \cdot \mu(A_4, A_4)).$$

Now, due to properties of Möbius function, we know that $\sum_{H \leqslant F \leqslant K} \mu(H, K) = 0$ for $H < K$ and $\mu(G, G) = 1$ for any finite group G. Thus, letting $H = C_3$ and $K = A_4$, we know that $\mu(C_3, C_3) + \mu(C_3, A_4) = 0$. Hence, $\mu(C_3, A_4) = -1$. Moreover, also $\mu(A_4, A_4) = 1$. It shows that $\beta^{k^\times}(A_4/(A_4 \cap A_4), A_4, (123)) = 0$. Now, by the linearization map, we obtain the required result.

Visually, we have the following result:

$$\begin{array}{ccc}
e^{A_4}_{A_4,(123)} & \xrightarrow{\mathrm{Def}^{A_4}_{A_4/A_4}} & 0 \\
{\scriptstyle lin_{A_4}} \downarrow & & \downarrow {\scriptstyle lin_{A_4/A_4}} \\
F^{A_4}_{V_4,(123)} & \xrightarrow[\mathrm{Def}^{A_4}_{A_4/A_4}]{} & 0
\end{array}$$

Similar argument shows that $\mathrm{Def}^{A_4}_{A_4/A_4} F^{A_4}_{V_4,(132)} = 0$ as well.

□

Claim: $\mathrm{Def}^{A_4}_{A_4/V_4} F^{A_4}_{V_4,(123)} = 0$ and $\mathrm{Def}^{A_4}_{A_4/V_4} F^{A_4}_{V_4,(132)} = 0$.

Proof. As above, we have $lin_{A_4}(e^{A_4}_{A_4,(123)}) = F^{A_4}_{V_4,(123)}$, and we have

$$\mathrm{Def}^{A_4}_{A_4/V_4} F^{A_4}_{V_4,(123)} = \lambda e^{C_2}_{C_2,1},$$

where

$$\lambda = \frac{|N_{A_4/V_4}(A_4V_4/V_4,(123)V_4) : A_4V_4/V_4|}{|N_{A_4}(A_4,(123)) : A_4|}\beta^{k^\times}(A_4/(A_4 \cap V_4), A_4, (123)),$$

and,

$$\beta^{k^\times}(A_4/(A_4 \cap V_4), A_4, (123)) = \frac{1}{|O(A_4)(A_4 \cap V_4)|}\sum_{\substack{U \leqslant A_4: \\ U(A_4 \cap V_4) = A_4}} |U \cap (123)O(A_4)|\mu(U, A_4).$$

Thus, U runs through the subgroups $\{C_3^1, C_3^2, C_3^3, C_3^4, A_4\}$. Similarly to above calculations, we have

$$\beta^{k^\times}(A_4/(A_4 \cap V_4), A_4, (123)) = \frac{1}{|O(A_4)(A_4 \cap V_4)|}(4 \cdot \mu(C_3, A_4) + 4 \cdot \mu(A_4, A_4)) = 0.$$

Now, the linearization map, we get the required result.

Similarly, we can conclude that $\mathrm{Def}^{A_4}_{A_4/V_4} F^{A_4}_{V_4,(132)} = 0$.

□

(iv) Finally, we have to show that for any $\Psi \in \mathrm{Aut}(A_4)$, we have $\mathrm{Iso}(\Psi)M_1 \subseteq M_1$ and $\mathrm{Iso}(\Psi)M_2 \subseteq M_2$.

However, we have $\mathrm{Aut}(A_4) \cong S_4$ that is to say each automorphism comes from a conjugation by an element of S_4, and clearly $\mathrm{Inn}(A_4) \cong A_4$. Thus, $\mathrm{Out}(A_4) \cong$

C_2, i.e. the non-identity automorphism comes from a conjugation by an odd permutation in S_4.

Since the pairs $(V_4, (123))$ and $(V_4, (132))$ are taken up to $G-$conjugacy, instead of all automorphisms in $\mathrm{Aut}(A_4)$, we can just take Ψ to be in $\mathrm{Out}(A_4)$. We may take $\Psi = c_{(12)}$, conjugation by the element (12). Moreover, note that V_4 is a characteristic group of A_4.

Then, we have

$$\mathrm{Iso}(c_{(12)})F^{A_4}_{V_4,(123)} = F^{A_4}_{V_4,(132)} \quad \textit{and} \quad \mathrm{Iso}(c_{(12)})F^{A_4}_{V_4,(132)} = F^{A_4}_{V_4,(123)}.$$

Therefore, we have $\mathrm{Iso}(\Psi)M_1 = M_1$ and $\mathrm{Iso}(\Psi)M_2 = M_2$.

Thus, we know that both M_1 and M_2 are biset functors on $\mathcal{F}_{|A_4|}$. It is clear that both of them are simple.

Note that, for any $K < A_4$, $M_1(K) = 0$ and $M_2(K) = 0$, thus A_4 is the minimal group for both M_1, M_2, and $M_1(A_4) \cong \mathbb{C}$ and $M_2(A_4) \cong \mathbb{C}_{-1}$. That is to say, $M_1 = S_{A_4,\mathbb{C}}$ and $M_2 = S_{A_4,\mathbb{C}_{-1}}$ on $\mathcal{F}_{|A_4|}$.

□

Now, since $S_{A_4,\mathbb{C}}$ and $S_{A_4,\mathbb{C}_{-1}}$ are simple composition factors of $\mathbb{C}pp_k$ on $\mathcal{F}_{|A_4|}$, it follows from the finite reduction principle for biset functors that they are composition factors for $\mathbb{C}pp_k$ on the biset category $\mathbb{C}\mathcal{C}$.

□

Remark 3.4.10. *Noting that A_4 is a p-hypo-elementary B-group for $p = 2$, and by Theorem 3.4.9, we know that the simple biset functors $S_{A_4,\mathbb{C}}$ and $S_{A_4,\mathbb{C}_{-1}}$ are composition factors of $\mathbb{C}pp_k$, so Conjecture 3.4.1 does not hold. We shall see in the last chapter that we can generalize the idea in the construction of A_4 example to some p-hypo-elementary B-groups so that we aim to find new composition factors of $\mathbb{C}pp_k$ indexed by those p-hypo-elementary B-groups.*

Chapter 4

p-Permutation functors and D-pairs

Throughout this chapter, we let $\mathbb{C}$ be the algebraically closed field of characteristic 0 and k be an algebraically closed field of characteristic p, prime.

In chapter 2 and 3, we studied the biset functor of p-permutation modules denoted by $\mathbb{C}pp_k$ which is defined to be a $\mathbb{C}$-linear functor from the biset category $\mathbb{C}\mathcal{C}$ to the category of finitely generated $\mathbb{C}$-vector spaces $\mathbb{C}$-Mod. In this chapter, we shall study the notion of p-permutation functors introduced by Maxime Ducellier. He studied the p-permutation functor of p-permutation modules which we shall denote by $\mathbb{C}pp_k{}^{p-perm.}$, and the simple p-permutation factors of $\mathbb{C}pp_k{}^{p-perm.}$. Along the way, we shall review the notion of D-pairs introduced and classified by Ducellier. We shall see that the classification of D-pairs are precisely the same as the classification of p-hypo-elementary B-groups.

4.1 The p-permutation category $\mathbb{C}\mathcal{C}^{pp_k}$

We begin with the following definitions which can be found in [5].

Definition 4.1.1. *Any (kG, kH)-bimodule M can be seen as a $k(G \times H)$-module with the action defined as*

$$g \cdot m \cdot h = (g, h^{-1}) \cdot m.$$

M is called a p-permutation (kG, kH)-bimodule if it is a p-permutation $k(G \times H)$-module. We denote the Grothendieck group of p-permutation (kG, kH)-bimodules by $pp_k(G, H)$ which is isomorphic to $pp_k(G \times H)$. Moreover, we can extend the coefficients to $\mathbb{C}$ in the usual sense: $\mathbb{C}pp_k(G, H) := \mathbb{C} \bigotimes_{\mathbb{Z}} pp_k(G, H)$.

Definition 4.1.2 (Composition of two p-permutation bimodules)**.** *Let G, H and K be finite groups, and X be a (kG, kH)-bimodule and Y be a (kH, kK)-bimodule. Then, we define the tensor product*

$$X \circ Y := X \otimes_{kH} Y := X \otimes Y / \sim,$$

where $\sim$ is defined by

$$xh \otimes y \sim x \otimes hy, \forall x \in X, y \in Y, h \in H.$$

$X \otimes_{kH} Y$ has a (kG, kK)-bimodule structure with the action defined by

$$g \cdot (x \otimes y) \cdot k := (g \cdot x \otimes y \cdot k),$$

for every $g \in G$, $k \in K$, and $x \otimes y \in X \otimes_{kH} Y$.

Letting X to be a (kG, kH)-bimodule and Y to be a (kH, kK)-bimodule, we naturally have the following bilinear map from $pp_k(G, H) \times pp_k(H, K) \to pp_k(G, K)$ given by linearly extending the following map

$$([X], [Y]) \mapsto [X] \circ [Y] = [X \otimes_{kH} Y].$$

Now, we shall define the p-permutation category and p-permutation functors introduced by Ducellier:

Definition 4.1.3 (Ducellier2015)**.** *[the p-permutation category] The category $\mathbb{C}\mathcal{C}^{pp_k}$ is defined as follows:*

(i) The objects are finite groups,

(ii) For $G, H \in Obj(\mathbb{C}\mathcal{C}^{pp_k})$, $Hom_{\mathbb{C}\mathcal{C}^{pp_k}}(G, H) = \mathbb{C}pp_k(H, G)$,

(iii) For $[X] \in Hom_{\mathbb{C}\mathcal{C}^{pp_k}}(G, H)$ and $[Y] \in Hom_{\mathbb{C}\mathcal{C}^{pp_k}}(H, K)$, $[X] \circ [Y] = [X \otimes_{kH} Y]$ where $X \otimes_{kH} Y$ is defined as above.

Note that this bilinear map gives $pp_k(G, G)$ a ring structure. Thus, we conclude that $\mathbb{C}pp_k(G, G)$ is a $\mathbb{C}$-algebra.

4.2 Remarks on p-permutation functors

Definition 4.2.1 (p-permutation functor)**.** *A $\mathbb{C}$-linear functor defined from $\mathbb{C}\mathcal{C}^{pp_k}$ to $\mathbb{C}$-Mod, the category of finite dimensional $\mathbb{C}$-vector spaces, is called a p-permutation functor.*

Definition 4.2.2 (the p-permutation functor of p-permutation modules)**.** *The p-permutation functor $\mathbb{C}pp_k{}^{p-perm.}$ is defined as follows:*

(i) for any $G \in Obj(\mathbb{C}\mathcal{C}^{pp_k})$, $\mathbb{C}pp_k{}^{p-perm.}(G) := \mathbb{C}pp_k(G,1) = \mathbb{C}pp_k(G)$,

(ii) for $[U] \in Hom_{\mathbb{C}\mathcal{C}^{pp_k}}(G,H)$, we define

$$\mathbb{C}pp_k{}^{p-perm.}([U]) : \mathbb{C}pp_k{}^{p-perm.}(G) \to \mathbb{C}pp_k{}^{p-perm.}(H),$$

given by $[V] \mapsto \mathbb{C}pp_k{}^{p-perm.}([U])([V]) := [U] \circ [V] = [U \otimes_{kG} V]$.

We shall review the definition of simple p-permutation functors by Ducellier.

Definition 4.2.3. *(Minimal group) Let F be a p-permutation functor. A group H is said to be minimal for F if $F(H) \neq 0$ and for each group K such that $|K| < |H|$, $F(K) = 0$. The class of minimal groups for F is denoted by $Min(F)$.*

Definition 4.2.4. *Let G be a finite group, we define the essential algebra as follows:*

$$\overline{\mathbb{C}pp_k^{p-perm.}}(G,G) := \mathbb{C}pp_k(G,G)/I,$$

where I is two sided ideal of $\mathbb{C}pp_k(G,G)$ such that

$$I = \sum_{|H|<|G|} \mathbb{C}pp_k(G,H) \circ \mathbb{C}pp_k(H,G).$$

Definition 4.2.5. *(Simple p-permutation functors $S_{G,V}^{p-perm.}$) Let G be a finite group, and V be a simple $\overline{\mathbb{C}pp_k^{p-perm.}}(G,G)$-module which can be seen as a $\mathbb{C}pp_k^{p-perm.}(G,G)$-module by inflation, then we define the functor*

$$S_{G,V}^{p-perm.} := (Hom_{\mathbb{C}\mathcal{C}^{pp_k}}(G,H) \bigotimes_{\mathbb{C}pp_k(G,G)} V)/\mathcal{R},$$

where $\mathcal{R} := \{\sum \varphi_i \otimes v_i | \forall \psi \in \mathbb{C}pp_k(G,H), \sum(\psi\varphi_i) \cdot v_i = 0\}$.

Theorem 4.2.6 ([5], Theorem 2.16). *Let G be a group, and V be a simple module for $\overline{\mathbb{C}pp_k^{p-perm.}}(G,G)$ then $S_{G,V}^{p-perm.}$ is simple. On the other hand, let S be a simple p-permutation functor, then there exists a group G, and a simple $\overline{\mathbb{C}pp_k^{p-perm.}}(G,G)$-module V such that $S \simeq S_{G,V}^{p-perm.}$.*

4.3 Definitions of a pair (P,s) and D-pair

Ducellier has provided classification of simple composition factors of the p-permutation functor $\mathbb{C}pp_k{}^{p-perm.}$; however, we need to define some further notions:

Definition 4.3.1 (Pair). *A pair (P,s) is defined by a p-group P and a generator s of a cyclic $p^{'}$-group acting on P. We denote the semidirect product $P \rtimes \langle s\rangle$ by $\langle Ps\rangle$. If we let G to be a finite group, and P to be a p-subgroup of G with $s \in (N_G(P))_{p'}$, then the pair (P,s) is identified with an action of s on P induced by conjugation by s. Moroever, a pair (P,s) is contained in G if $\langle Ps\rangle \leq G$.*

Definition 4.3.2 (Isomorphic pairs). *Let (P,s) and (Q,t) be two pairs. A pair (P,s) is isomorphic to the pair (Q,t) if there exists $q \in Q$, a group isomorphism $\varphi : P \to Q$ and $\Psi : \langle s\rangle \to \langle {}^q t\rangle$ such that*

$$\Psi(s) = {}^q t \quad \textit{and} \quad \varphi(s \cdot u) = \Psi(s) \cdot \varphi(u), \forall u \in P.$$

We denote isomorphic pairs by $(P,s) \simeq (Q,t)$.

Proposition 4.3.3 ([5], Proposition 2.3.3., p15). *The followings are equivalent:*

1. *$(P,s) \simeq (Q,t)$,*
2. *There exists a group isomorphism $f : \langle Ps\rangle \to \langle Qt\rangle$ such that $f(s)$ is conjugate to t.*

Definition 4.3.4 (Quotient of a pair). *Let (P,s) and (Q,t) be two pairs. We say a pair (Q,t) is a quotient of a pair (P,s) if there exists $K \trianglelefteq \langle Ps\rangle$ such that $(Q,t) = (PK/K, sK)$, and we denote it by $(P,s) >> (Q,t)$.*

Definition 4.3.5 (D-pair). *A pair (P,s) is said to be **D-pair** if for every non-trivial normal subgroup N of G, we have $Def^{\langle Ps\rangle}_{\langle Ps\rangle/N} F^{\langle Ps\rangle}_{P,s} = 0$ that is to say $m_{P,s,N} = 0$.*

4.4 The simple composition factors of the p-permutation functor of p-permutation modules $\mathbb{C}pp_k{}^{p-perm.}$

Although we have no full classification of simple composition factors of the biset functor of p-permutation modules $\mathbb{C}pp_k$, thanks to Ducellier, we have information about all of the simple composition factors of p-permutation functor of p-permutation modules $\mathbb{C}pp_k{}^{p-perm.}$. Similar argument to Bouc's classification of simple composition factors of the biset Burnside functor who showed that they are precisely $S_{H,\mathbb{C}}$ where H is a B-group, Ducellier has shown that the simple composition factors of the p-permutation functor $\mathbb{C}pp_k{}^{p-perm.}$ are indexed by D-pairs as follows:

Here, we denote $e_{P,s}^{p-perm.}$ as the subfunctor of $\mathbb{C}pp_k{}^{p-perm.}$ generated by the primitive idempotent $F_{P,s}^{\langle Ps\rangle}$. We denote the representatives of isomorphism classes of D-pairs by $[D\text{-pairs}]$.

Theorem 4.4.1 ([5], Proposition 5.2.1)**.**

1. *Let (P,s) be a D-pair. Then, the subfunctor $e_{P,s}^{p-perm.}$ has a unique maximal subfunctor*
$$j_{P,s}^{p-perm.} = \sum_{\substack{(Q,t)\in[D\text{-}pair]\\ (Q,t)>>(P,s)\\ (Q,t)\not\simeq(P,s)}} e_{Q,t}^{p-perm.}.$$
The quotient functor $e_{P,s}^{p-perm.}/j_{P,s}^{p-perm.}$ is isomorphic to $S_{\langle Ps\rangle,W_{P,s}}^{p-perm.}$ where
$$W_{P,s} = \bigoplus_{\substack{(Q,t)\simeq(P,s)\\ \langle Qt\rangle=\langle Ps\rangle}} \mathbb{C}F_{Q,t}^{\langle Ps\rangle}.$$

2. *If $F \leq F'$ are subfunctors of $\mathbb{C}pp_k{}^{p-perm.}$ such that F'/F is simple, then there exists a unique D-pair $(P,s)\in[D\text{-}pair]$ such that $e_{P,s}^{p-perm.}\leq F'$, and $e_{P,s}^{p-perm.}\not\leq F$. In particular, $e_{P,s}^{p-perm.}+F=F'$, $e_{P,s}^{p-perm.}\cap F = j_{P,s}^{p-perm.}$ and $F'/F\simeq S_{\langle Ps\rangle,W_{P,s}}^{p-perm.}$.*

Theorem 4.4.1 tells us that the simple composition factors of the p-permutation functor of p-permutation modules $\mathbb{C}pp_k{}^{p-perm.}$ are precisely the simple p-permutation functors $S^{p-perm.}_{\langle Ps\rangle,W_{P,s}}$ where (P,s) is a D-pair.

Ducellier has also found the dimensions of these simple p-permutation functors evaluated at some finite groups:

Theorem 4.4.2 ([5], Theorem 5.2.4, p105)**.** *Let (P,s) be a D-pair. Then, $dim_{\mathbb{C}} S^{p-perm.}_{\langle Ps\rangle,W_{P,s}}(H)$ is equal to the number of conjugacy classes of pairs (Q,t) contained in H such that $(\tilde{Q},\tilde{t}) \simeq (P,s)$ where $(\tilde{Q},\tilde{t})$ denotes some quotient of the pair (Q,t).*

Corollary 4.4.3 ([5], Corollary 5.2.5, p106)**.** *Let H be a group. Then, $dim_{\mathbb{C}} S^{p-perm.}_{C_1,\mathbb{C}}(H)$ is equal to the number of H-conjugacy classes of pairs (Q,t) contained in H such that Q is cyclic and $t \in C_H(Q)$.*

4.5 The classification theorem of D-pairs

Now, we shall review the classification theorem of D-pairs thanks to Ducellier:

Theorem 4.5.1 ([5])**.** *A pair (P,s) is a D-pair if and only if*

(i) P is elementary abelian,

(ii) $C_{\langle s\rangle}(P) = 1$,

(iii) Each isotypic component of $\mathbb{F}_p(\langle s\rangle)$- module P is of multiplicity at most 1 if it corresponds to a non-trivial simple module, of multiplicity 0 or 2 if it corresponds to the trivial module.

Proof. Firstly, we suppose that we are given a D-pair (P,s). Then, by definition, we know that for every non-trivial $N \trianglelefteq G = \langle Ps\rangle$,

$$m_{P,s,N} = \frac{|s|}{|N \cap \langle s\rangle||C_G(s)|} \sum_{\substack{Q\leq P \\ Q^s=Q \\ \langle Qs\rangle N=\langle Ps\rangle}} |C_Q(s)|\mu((Q,P)^s) = 0. \tag{5.1}$$

STEP 1: ***P* is elementary abelian.** Assume for a contradiction that P is not elementary abelian group. Recall that the Frattini subgroup $\Phi(P)$ of a p-group P is trivial if and only if P is elementary abelian. Now, we shall start with the following result:

Proposition 4.5.2. *Let $Q \leq P \leq G$ and $s \in N_G(P)$ such that $Q^s = Q$, then if $\Phi(P) \nleq Q$, then we have $\mu((Q,P)^s) = 0$.*

Proof. By definition, we have $\mu((Q,P)^s) = \widetilde{\chi}((Q,P)^s)$ where $(Q,P)^s := \{K | Q < K < P \text{ s.t. } K^s = K\}$.

We define the map $\varphi : (Q,P)^s \to (Q,P)^s$ by sending $H \mapsto H\Phi(P)$.

__φ **is well-defined:**__ We have $Q < H < P$ and $H^s = H$. Then, since $\Phi(P)^s = \Phi(P)$, we have $(H\Phi(P))^s = H\Phi(P)$. Moreover, since $\Phi(P) \nleq Q$, we have $Q < H\Phi(P)$.

Claim: $H\Phi(P) < P$.

Proof of the claim: Suppose that $H\Phi(P) = P$. Then, we have $P = \langle h_i, \Phi(P) | h_i \in H \rangle$. Now, we know that $H < P$, since every proper subgroup of a finite group is contained in a maximal subgroup, we have that there exists a maximal subgroup M of P such that $H \leq M$. Moreover, we obtain $\Phi(P) \leq M$ because the Frattini subgroup is defined to be the intersection of all maximal subgroups. However, then it would imply that $P = \langle H, \Phi(P) \rangle \leq M$ which is a contradiction since M is maximal. Thus, we have $H\Phi(P) < P$, as claimed. Thus, the map φ is well-defined.

Now, we consider the poset $Y = Im\varphi$ and we let $X = (Q,P)^s$.

For this part, we shall firstly recall the following two well-known results which can be found in [10]:

Lemma 4.5.3. *Let f and g be two poset maps from X to Y such that f and g are comparable. Then, the induced maps of chain complexes of $C_*(f,\mathbb{Z})$ and $C_*(g,\mathbb{Z})$ are homotopic as well as the maps $\widetilde{C_*}(f,\mathbb{Z})$ and $\widetilde{C_*}(g,\mathbb{Z})$.*

Proof. We may assume that $f(x) \leq g(x)$ for every $x \in X$. We define the following map,

$$s_n : C_n(X, \mathbb{Z}) \to C_{n+1}(Y, \mathbb{Z})$$
$$(x_0, x_1, ..., x_n) \mapsto \sum_{i=0}^{n} (-1)^i (f(x_0), ..., f(x_i), g(x_i), ..., g(x_n)),$$

and we replace the term $(f(x_0), ..., f(x_i), g(x_i), ..., g(x_n))$ by zero if it is not strictly increasing.

Then, one can see that we have $d^Y_{n+1} \circ s_n + s_{n-1} \circ d^X_n = C_n(g, \mathbb{Z}) - C_n(f, \mathbb{Z})$, that is to say, $C_*(f, \mathbb{Z}) \simeq C_*(g, \mathbb{Z})$. The second part is the same, where we take $s_{-1} = 0$. □

Claim: $C_*(X, \mathbb{Z})$ and $C_*(Y, \mathbb{Z})$ are homotopic.

Proof: To show that these two induced chain complexes are homotopic, we shall only find two chain maps

$$\alpha_* : C_*(X, \mathbb{Z}) \to C_*(Y, \mathbb{Z}) \quad and \quad \beta_* : C_*(Y, \mathbb{Z}) \to C_*(X, \mathbb{Z})$$

such that $\alpha_* \circ \beta_* \simeq id_{C_*(Y,\mathbb{Z})}$ and $\beta_* \circ \alpha_* \simeq id_{C_*(X,\mathbb{Z})}$, where $\simeq$ refers to homotopy equivalence.

Recall that we have two poset maps $\varphi : X \to Y$ and the inclusion $inc_Y : Y \to X$ as stated above. Now, since we have for every element $y \in Y = Im\varphi$, there exists $H \in X$ such that $y = H\Phi(P)$, we obtain $\varphi \circ inc_Y(y) = \varphi(y) = y\Phi(P) = H\Phi(P)\Phi(P) = H\Phi(P) = y$. Thus, the map $\varphi \circ inc_Y = id_Y$. It is clear that $C_*(\varphi, \mathbb{Z}) \circ C_*(inc_Y, \mathbb{Z}) = C_*(\varphi \circ inc_Y, \mathbb{Z})$. Now, by Lemma 4.5.3, it is straightforward that the induced chain maps $C_*(\varphi \circ inc_Y, \mathbb{Z}) \simeq id_{C_*(Y,\mathbb{Z})}$.

Conversely, note that for any $H \in X = (Q, P)^s$, we have $id_X(H) = H \leq inc_Y \circ \varphi(H) = H\Phi(P)$ because of $\Phi(P) \nleq Q$ and by the choice of H i.e. $id_X \leq inc_Y \circ \varphi$ are comparable poset maps. It follows once again from Lemma 4.5.3 that the induced chain maps $C_*(inc_Y, \mathbb{Z}) \circ C_*(\varphi, \mathbb{Z}) = C_*(inc_Y \circ \varphi, \mathbb{Z}) \simeq id_{C_*(X,\mathbb{Z})}$. Thus, we conclude that $C_*(X, \mathbb{Z}) \simeq C_*(Y, \mathbb{Z})$.

The next claim is an application of the fact that whenever a poset owns smallest or maximal element, then it is contractible that is to say it is homotopy equivalent to a poset formed by a singleton.

Claim: $C_*(Y,\mathbb{Z})$ and $C_*(\{Q\Phi(P)\},\mathbb{Z})$ are homotopic.

Proof:

Recall that Y=Imφ where $\varphi:(Q,P)^s\to(Q,P)^s$ by $H\mapsto H\Phi(P)$.

Note that since $\Phi(P)\nleq Q$, we have $Q\Phi(P)\in(Q,P)^s$ so $\varphi(Q\Phi(P))=Q\Phi(P)\in$Y, noting that $Q\Phi(P)$ is the smallest element of Y.

Define poset maps $\alpha:Y\to\{Q\Phi(P)\}$ given by $H\Phi(P)\mapsto Q\Phi(P)$ and $\beta:\{Q\Phi(P)\}\to Y$ given by $Q\Phi(P)\mapsto Q\Phi(P)$.

Now, it is clear that $\alpha\circ\beta=id_{Q\Phi(P)}$. Thus, by Lemma 4.5.3, we obtain that $C_*(\alpha,\mathbb{Z})\circ C_*(\beta,\mathbb{Z})=C_*(\alpha\circ\beta,\mathbb{Z})\simeq id_{C_*(\{Q\Phi(P)\},\mathbb{Z})}$.

On the other hand, for every element $H\Phi(P)\in Y$, we have $\beta\circ\alpha(H\Phi(P))=\beta(Q\Phi(P))=Q\Phi(P)\leq id_Y(H\Phi(P))$ since $Q\Phi(P)$ is the smallest element of Y. Thus, by Lemma 4.5.3, we have $C_*(\beta,\mathbb{Z})\circ C_*(\alpha,\mathbb{Z})=C_*(\beta\circ\alpha,\mathbb{Z})\simeq id_{C_*(\{Q\Phi(P)\},\mathbb{Z})}$.

Thus, we obtain that $C_*(Y,\mathbb{Z})\simeq C_*(\{Q\Phi(P)\},\mathbb{Z})$. Then, we have $H_n(Y,\mathbb{Z})\cong H_n(\{Q\Phi(P)\},\mathbb{Z})$.

But, we have $H_n(\{Q\Phi(P)\},\mathbb{Z})=0$ for every $n\neq0$, and $H_0(\{Q\Phi(P)\},\mathbb{Z})=\mathbb{Z}$. Thus, $\widetilde{\chi}(Y)=0$ implying $\widetilde{\chi}(X)=\mu(X)=\mu((Q,P)^s)=0$, as claimed. □

Now, we turn back to our deflation numbers:

Since for every $Q\leq P, Q^s=Q$, whenever $\Phi(P)\nleq Q$, $\mu((Q,P)^s)$=0, we have

$$m_{P,s,N}=0+\frac{||s|}{|N\cap\langle s\rangle||C_G(s)|}\sum_{\substack{\Phi(P)\leq Q\leq P\\ Q^s=Q\\ \langle Qs\rangle N=\langle Ps\rangle}}|C_Q(s)|\mu((Q,P)^s).$$

Now, we make a small alteration:

Claim: Given $N\trianglelefteq\langle Ps\rangle$, $Q^s=Q$, then $\langle Qs\rangle N=\langle Ps\rangle$ if and only if $QN=PN$ if and only if $Q(N\cap P)=P$.

Proof: The second assertion is clear by using the fact that $Q \leq P \leq QN = PN$. Thus, we shall only provide the proof of the first if and only if.

($\Rightarrow$): Suppose that $\langle Qs \rangle N = \langle Ps \rangle$. Then, for any $(p, 1) \in \langle Ps \rangle$, there exists $n \in N$ and $(q, s^i) \in \langle Qs \rangle$ such that $(q, s^i) \cdot n = (p, 1)$. We also have $n = (n', s^j)$ since $N \trianglelefteq \langle Ps \rangle$, for some $n' \in P$ and $s^j \in \langle s \rangle$. Then, we have $(qs^i n' s^{-i}, s^{i+j}) = (p, 1)$ that is to say $qs^i n' s^{-i} = p \in P$, note that $n'' = s^i n' s^{-i} \in P$ since $P \trianglelefteq \langle Ps \rangle$. Moreover, since N is also normal subgroup of $\langle Ps \rangle$, we have $(1, s^j) \cdot (n', s^j) \cdot (1, s^{-j}) = (s^j n' s^{-j}, 1) \in N$. Thus, we have for any $p \in P$, $p = qn''$ where $n'' \in N \cap P$ i.e. $P \subseteq Q(N \cap P)$. The converse inclusion is clear. Thus, we obtain $P = Q(N \cap P)$.

($\Leftarrow$): Suppose that $Q(N \cap P) = P$. Clearly, $\langle Qs \rangle N \leq \langle Ps \rangle$. Let $(p, s^i) \in P \rtimes \langle s \rangle$. Since $Q(N \cap P) = P$, there exists $q \in Q$, and $n'' \in N \cap P$ such that $p = qn''$. Moreover, since $N \cap P \trianglelefteq \langle Ps \rangle$, there exist $n' \in N \cap P$ such that $n'' = s^i n' s^{-i}$. Thus, $(p, s^i) = (qn'', s^i) = (q, s^i) \cdot (n', 1) \in \langle Qs \rangle N$. Then, $\langle Ps \rangle \leq \langle Qs \rangle N$, as claimed.

Claim: Let $\Phi(P) \leq Q \leq P$ with $Q^s = Q$ and $Q(N \cap P) = P$, then we have $\mu((Q, P)^s) = \mu((Q/\Phi(P), P/\Phi(P))^{s\Phi(P)})$.

Proof: For this part, we must only show that $(Q, P)^s \simeq (Q/\Phi(P), P/\Phi(P))^{s\Phi(P)}$. Let $H \in (Q, P)^s$. Now, since $Q < H < P$ and $\Phi(P) \leq Q$, we have $\Phi(P) \leq H$. Moreover, clearly, since H is stabilized by s, $(H/\Phi(P))^{s\Phi(P)} = (H/\Phi(P))$. Thus, $H/\Phi(P) \in (Q/\Phi(P), P/\Phi(P))^{s\Phi(P)}$.

Conversely, given $X \in (Q/\Phi(P), P/\Phi(P))^{s\Phi(P)}$, we claim that $X\Phi(P) \in (Q, P)^s$. It is clear that we have $Q < X\Phi(P)$. Moreover, if $X\Phi(P) = P$, then $X = P$ by the definition of Frattini subgroup, which is a contradiction by the choice of X. Thus, we have $X\Phi(P) < P$. Now, clearly, we have $(X\Phi(P))^s \subseteq X\Phi(P)$. For the converse inclusion, we have $X^{s\Phi(P)} = X$. Thus, $X^{s\Phi(P)}\Phi(P) = X\Phi(P)$ and by using the fact that $\Phi(P) \trianglelefteq \langle Ps \rangle$,

$$s\Phi(P)Xs^{-1}\Phi(P)\Phi(P) = s\Phi(P)X\Phi(P)s^{-1} = sX\Phi(P)s^{-1} = X\Phi(P).$$

Thus, we reduced the deflation number into

$$m_{P,s,N} = \frac{||s|}{|N \cap \langle s\rangle||C_G(s)|} \sum_{\substack{Q/\Phi(P) \leq P/\Phi(P) \\ (Q/\Phi(P))^{s\Phi(P)} = Q/\Phi(P) \\ QN/\Phi(P) = PN/\Phi(P)}} |C_Q(s)|\mu((Q/\Phi(P), P/\Phi(P))^{s\Phi(P)}).$$

Now, we will refer to the following theorem:

Theorem 4.5.4 ([17], Theorem 5.3.15, page 188)**.** *Let A be a p'-group of automorphism of a p-group Q, and let N be an normal subgroup of Q which remains invariant under A. Then, $C_{Q/N}(A)$ is the image of $C_Q(A)$ in Q/N.*

Now, letting $A = \langle s\rangle$, and since $N = \Phi(P) \trianglelefteq Q$, we have a surjective map by the theorem above, $\pi : C_Q(s) \twoheadrightarrow C_{Q/\Phi(P)}(s)$ given by $x \mapsto x\Phi(P)$. Moreover, we have the inclusion $i : C_Q(s) \cap \Phi(P) = C_{\Phi(P)}(s) \hookrightarrow C_Q(s)$, and note that we have $\pi \circ i = 1$. Thus, we obtained the following short exact sequence:

$$1 \longrightarrow C_{\Phi(P)}(s) \overset{i}{\hookrightarrow} C_Q(s) \overset{\pi}{\twoheadrightarrow} C_{Q/\Phi(P)}(s) \longrightarrow 1.$$

Then, we have $|C_Q(s)| = |C_{\Phi(P)}(s)||C_{Q/\Phi(P)}(s)|$. Therefore,

$$m_{P,s,N} = \frac{|s||C_{\Phi(P)}(s)|}{|N \cap \langle s\rangle||C_G(s)|} \sum_{\substack{Q/\Phi(P) \leq P/\Phi(P) \\ (Q/\Phi(P))^{s\Phi(P)} = Q/\Phi(P) \\ QN/\Phi(P) = PN/\Phi(P)}} |C_{Q/\Phi(P)}(s)|\mu((Q/\Phi(P), P/\Phi(P))^{s\Phi(P)}).$$

We denote

$$\sigma_{P,s,N} = \sum_{\substack{Q/\Phi(P) \leq P/\Phi(P) \\ (Q/\Phi(P))^{s\Phi(P)} = Q/\Phi(P) \\ QN/\Phi(P) = PN/\Phi(P)}} |C_{Q/\Phi(P)}(s)|\mu((Q/\Phi(P), P/\Phi(P))^{s\Phi(P)}).$$

Now, we assumed that P is not an elementary abelian group, i.e., $\Phi(P) \neq 1$. Since (P, s) is a D-pair, we have for every $1 \neq N \trianglelefteq \langle Ps\rangle$, $m_{P,s,N} = 0$, which implies that $\sigma_{P,s,N} = 0$. Now, letting $N = \Phi(P)$, we have $\sigma_{P,s,\Phi(P)} = 0$. But note that since $\sigma_{P/\Phi(P),s\Phi(P),1} = \sigma_{P,s,\Phi(P)}$, we would have $\sigma_{P/\Phi(P),s\Phi(P),1} = 0$ which cannot be true because $m_{P/\Phi(P),s\Phi(P),1} = 1$. Therefore, we must have $\Phi(P) = 1$ i.e. P must be elementary abelian, as required. Thus, we have

$$\sigma_{P,s,N} = \sum_{\substack{Q \leq P \\ Q^s = Q \\ QN = PN}} |C_Q(s)| \mu((Q, P)^s).$$

STEP 2: CALCULATION OF $\sigma_{P,s,N}$

Note that since every elementary abelian p-group P can be thought as an $\mathbb{F}_p$-vector space, and since $P^s = P$, we can think P as an $\mathbb{F}_p\langle s\rangle$-module. Moreover, any subgroup $Q \leq P$ with $Q^s = Q$ can also be thought as an $\mathbb{F}_p\langle s\rangle$-submodule of P. Since $p \nmid |s|$, every $\mathbb{F}_p\langle s\rangle$-module is semisimple.

Now, we shall review the following fact:

Lemma 4.5.5. *For any $H \leq P$ with $H^s = H$, P and s as above, if $P = P_1 \oplus ... \oplus P_t$ where P_i denotes the homogeneous components of the $\mathbb{F}_p\langle s\rangle$-module P, then $H = (H \cap P_1) \oplus ... \oplus (H \cap P_t)$ and $H \cap P_i$'s are homogeneous components of the submodule H.*

Proof of the lemma: Suppose that $P = \bigoplus S_i$ where S_i is simple $\mathbb{F}_p\langle s\rangle$-module. Clearly, H is an $\mathbb{F}_p\langle s\rangle$-submodule of P. By semisimplicity, H can be written as a direct sum of homogeneous components, H_i, each corresponding to non-isomorphic simple modules S_i. Clearly, $H_i \subseteq P_i$ and $H \cap P_i$ is a sum of S_i by Jordan-Hölder Theorem, so $H \cap P_i \subseteq H_i$. But then, we have $H_i = H \cap P_i$, as claimed.

Now, we shall reduce the calculation of $\sigma_{P,s,N}$ into much smaller pieces, this result holds for every pair (P, s) where P is elementary abelian.

Claim: Let (P, s) be a pair, with P elementary abelian. Let $G = \langle Ps\rangle$, and $N \trianglelefteq \langle Ps\rangle$. Then, we have

$$\sigma_{P,s,N} = \prod_i \sigma_{P_i,s,N_i},$$

where $P = P_1 \oplus ... \oplus P_t$, P_i's are homogeneous components of $\mathbb{F}_p\langle s\rangle$-module P, and $N \cap P = (N_1) \oplus ... \oplus (N_t)$ with $N_i = (N \cap P_i)$'are homogeneous components of $\mathbb{F}_p\langle s\rangle$-module $N \cap P$.

Proof of the claim: Since both N and P are normal in $\langle Ps\rangle$, we have $(N\cap P)^s = N\cap P$, so $N\cap P$ can be thought as an $\mathbb{F}_p\langle s\rangle$-submodule of P. Now, by lemma above, we obtain the corresponding homogeneous components $N_i = (N\cap P_i)$. Now, we are asked to prove the following equality:

$$\sum_{\substack{Q\le P\\ Q^s=Q\\ Q(N\cap P)=P}} |C_Q(s)|\mu((Q,P)^s) = \prod_{i=1}^{t} \sum_{\substack{Q_i\le P_i\\ Q_i^s=Q_i\\ Q_i(N_i\cap P_i)=P_i}} |C_{Q_i}(s)|\mu((Q_i,P_i)^s).$$

It is clear that for a given $Q\le P$, as on the left hand side, we can obtain the corresponding Q_i's by letting $Q_i = Q\cap P_i$. Conversely, given such Q_i's, we can clearly obtain $Q = Q_1\times ...\times Q_t$ which satisfies $Q^s = Q$ and $Q(N\cap P) = P$, and clearly $|C_Q(s)| = |C_{Q_1}(s)|...|C_{Q_t}(s)|$, by making use of the fact that $P_i\cap P_j = 1$ whenever $i\neq j$.

Moreover, by the product rule, we have

$$\mu((Q,P)^s) = \mu((Q_1\times ...\times Q_t, P_1\times ...\times P_t)^s) = \mu((Q_1,P_1)^s)...\mu((Q_t,P_t)^s).$$

Thus, we have the desired equality.

Now, it requires us to calculate σ_{P_i,s,N_i}.

Proposition 4.5.6. *Let (P,s) be a pair with P elementary abelian and $N\trianglelefteq\langle Ps\rangle$. We have $\sigma_{P,s,N} = \prod_i \sigma_{P_i,s,N_i}$ where $P = P_1\oplus ...\oplus P_t$ and $N\cap P = N_1\oplus ...\oplus N_t$ with $P_i = \bigoplus_{j=1}^{n_i} S_i$ and $N_i = \bigoplus_{j=1}^{a_i} S_i$ with $S_i\cong \mathbb{F}_{p^{s_i}}$. Then, setting $q_i = p^{s_i}$, we have,*

$$\sigma_{P_i,s,N_i} = \begin{cases} |C_{P_i}(s)| & if \quad a_i = 0,\\ p^{n_i}(1-p^{n_i-2})...(1-p^{n_i-a_i-1}) & if \quad a_i\neq 0 \text{ and } S_i \text{ is trivial},\\ (1-q_i^{n_i-1})...(1-q_i^{n_i-a_i}) & if \quad a_i\neq 0 \text{ and } S_i \text{ is non-trivial}.\end{cases}$$

Proof of the proposition: Recall that $\sigma_{P_i,s,N_i} = \sum_{\substack{Q_i\le P_i\\ Q_i^s=Q_i\\ Q_iN_i=P_i}} |C_{Q_i}(s)|\mu((Q_i,P_i)^s)$.

Firstly, we define an $\mathbb{F}_p$-homomorphism $\Psi:\mathbb{F}_p\langle s\rangle\to End_{\mathbb{F}_p\langle s\rangle}(S_i)$ by $a\mapsto\varphi_a : S_i\to S_i$ which sends $x\mapsto a\cdot x$ for every $x\in S_i$. The well-definedness of Ψ is clear.

Now, note that since S_i is a simple $\mathbb{F}_p\langle s\rangle$-module, by Schur's lemma, $End_{\mathbb{F}_p\langle s\rangle}(S_i)$ is a division ring. Moreover, since S_i is finite, $End_{\mathbb{F}_p\langle s\rangle}(S_i)$ is finite, so $End_{\mathbb{F}_p\langle s\rangle}(S_i)$ is a field.

Moreover, since S_i is a simple $\mathbb{F}_p\langle s\rangle$-module, it is cyclic i.e. $S_i = (x) := \{bx | b \in \mathbb{F}_p\langle s\rangle\}$.

Claim: $\mathrm{Im}(\Psi)$ is a field.

Proof: Clearly, $\mathrm{Im}(\Psi)$ is a ring with multiplication $\varphi_a \cdot \varphi_b = \varphi_{a\cdot b}$, and by finiteness of $End_{\mathbb{F}_p\langle s\rangle}(S_i)$, we know that $\mathrm{Im}(\Psi)$ is finite. Thus, it is only required to show that $\mathrm{Im}(\Psi)$ is a division ring. Thus, let $\varphi \in \mathrm{Im}(\Psi)$, i.e. there exists a$\in \mathbb{F}_p\langle s\rangle$ such that $\varphi : S_i \to S_i$ given by $x \mapsto a \cdot x$ where x is the generator of S_i as defined above. Now, let $a \cdot x = x^{'}$, by simplicity of S_i, there exists $b \in \mathbb{F}_p\langle s\rangle$ such that $x = b \cdot x^{'}$. But then, $ax = x^{'}$ implies $ba \cdot x = bx^{'} = x$, and by commutativity, $ba \cdot x = ab \cdot x = x$.

Let $\Upsilon : S_i \to S_i$ be such that $x \mapsto bx$. Then, we have $(\varphi \cdot \Upsilon)(x) = ab \cdot x = x$ and conversely, $(\Upsilon \cdot \varphi)(x) = ba \cdot x = x$. Thus, $\Upsilon \cdot \varphi = \varphi \cdot \Upsilon = id_{S_i}$. Since, every element is invertible, and by finiteness, we obtain that $\mathrm{Im}(\Psi)$ is a field.

Note that the simple $\mathbb{F}_p\langle s\rangle$-module S_i can be thought as an $\mathrm{Im}(\Psi)$-vector space by the action $\varphi_a \cdot x = \varphi_a(x) = a \cdot x$.

Claim: $\dim_{Im(\Psi)} S_i = 1$.

Proof: We have $S_i = (x) := \{bx | b \in \mathbb{F}_p\langle s\rangle\}$. Then, we shall see that $\{x\}$ forms an $\mathrm{Im}(\Psi)$-basis of S_i. Let y be any element in S_i, then by simplicity of S_i, there exists $b \in \mathbb{F}_p\langle s\rangle$ such that $y = bx = \varphi_b \cdot x = \varphi_b(x)$, as required. Thus, $\dim_{Im(\Psi)} S_i = 1$, and $\mathrm{Im}(\Psi) \cong S_i$.

Now, denoting $ord(s) = l$, since $\mathbb{F}_p\langle s\rangle \cong \mathbb{F}_p[x]/(x^l - 1) = \prod_i \mathbb{F}_p[x]/m_i(x)$, each S_i corresponding to $\mathbb{F}_p[x]/m_i(x)$ where $m_i(x)$ is irreducible polynomial over $\mathbb{F}_p$, so we have S_i is a field over $\mathbb{F}_p$ i.e. for some $s_i \in \mathbb{N}$, $S_i \cong \mathbb{F}_{p^{s_i}}$.

The next claim will help us to express Möbius function of poset of subgroups that are stabilized by s in terms of poset of subspaces of a particular field.

Claim: Let $Q_i \leq P_i$ such that $Q_i^s = Q_i$. Then, Q_i is an $\mathrm{Im}(\Psi)$-vector subspace of P_i. Conversely, given $\mathrm{Im}(\Psi)$-vector subspace of P_i, then $Q_i \leq P_i$ and $Q_i^s = Q_i$.

Proof: Recall that we have $P_i = \bigoplus_{j=1}^{n_i} S_i$ as an $\mathbb{F}_p\langle s\rangle$-module.

$(\Rightarrow)$: By Lemma 4.5.5, we know that $Q_i = \bigoplus_{j=1}^{m_i} S_i$, $m_i \leq n_i$. We showed that S_i is an $\mathrm{Im}(\Psi)$-vector space, so it follows that Q_i, too.

$(\Leftarrow)$: Since P_i is a direct sum of S_i's, P_i is $\mathrm{Im}(\Psi)$-vector subspace. Now, let Q_i be an $\mathrm{Im}(\Psi)$-vector subspace of P_i. Thus, $Q_i \leq P_i$. Then, by the same argument as above, by Lemma 4.5.5, we have $Q_i = \bigoplus_{j=1}^{m_i} S_i$, $m_i \leq n_i$. But since each S_i is an $\mathbb{F}_p\langle s\rangle$-module, we have $Q_i^s = Q_i$ as required.

Now, we will calculate $\sigma_{P_i,s,N_i} = \sum_{\substack{Q_i \leq P_i \\ Q_i^s = Q_i \\ Q_i N_i = P_i}} |C_{Q_i}(s)|\mu((Q_i, P_i)^s)$ by induction on $\dim_{\mathrm{Im}(\Psi)}(N_i) = a_i$, where $N_i = \bigoplus_{j=1}^{a_i} S_i$.

Firstly, suppose that $a_i = 0$. Then, we have $N_i = 1$. Thus, $\sigma_{P_i,s,1} = |C_{P_i}(s)| \cdot \mu((P_i, P_i)^s) = |C_{P_i}(s)|$.

Now, suppose that $a_i \neq 0$. Let N_0 be a 1-dimensional $\mathrm{Im}(\Psi)$-vector subspace of N_i. Thus, we have

$$\sigma_{P_i,s,N_i} = \sum_{\substack{Q_i \leq P_i \\ Q_i^s = Q_i \\ Q_i N_i = P_i \\ Q_i \geqslant N_0}} |C_{Q_i}(s)|\mu((Q, P)^s) + \sum_{\substack{Q_i \leq P_i \\ Q_i^s = Q_i \\ Q_i N_i = P_i \\ Q_i \ngeqslant N_0}} |C_{Q_i}(s)|\mu((Q_i, P_i)^s).$$

Recalling our claim above that, Q_i's above are $\mathrm{Im}(\Psi)$-vector subspace of P_i.

Case(i): Suppose that s acts non-trivially on P_i.

Claim: In such a case, $C_{P_i}(s) = 1$.

Proof: We have $C_{P_i}(s) \leq P_i$ and by definition $(C_{P_i}(s))^s = C_{P_i}(s)$. Thus, by

Lemma[4.0.72], we have $C_{P_i}(s) = \bigoplus_{j=1}^{k_i} S_i$ with $k_i \leq n_i$ where $P_i = \bigoplus_{j=1}^{n_i} S_i$. Suppose that $C_{P_i}(s) \neq 1$, say $1 \neq p_i \in C_{P_i}(s)$. Then, we have $(p_i, 1) \cdot (1, s) \cdot (p_i^{-1}, 1) = (1, s)$, i.e. $sp_i^{-1}s^{-1} = p_i^{-1}$. But all S_i's are simple $\mathbb{F}_p\langle s \rangle$-modules, so having a fixed vector $1 \neq p_i^{-1}$ implies that S_i corresponds to the trivial $\mathbb{F}_p\langle s \rangle$-module. But, by our assumption, s acts non-trivially on P_i, a contradiction. Thus, we have $C_{P_i}(s) = 1$.

Thus, for every $Q_i \leq P_i$ with $Q_i^s = Q_i$, we have $C_{Q_i}(s) \subseteq C_{P_i}(s) = 1$.

We have,

$$\sigma_{P_i,s,N_i} = \sum_{\substack{Q_i \leq P_i \\ Q_i^s = Q_i \\ Q_i N_i = P_i \\ Q_i \geq N_0}} \mu((Q_i, P_i)^s) + \sum_{\substack{Q_i \leq P_i \\ Q_i^s = Q_i \\ Q_i N_i = P_i \\ Q_i \ngeq N_0}} \mu((Q_i, P_i)^s),$$

The first component is equal to $\sigma_{P_i/N_0, sN_0, N/N_0}$ and we name the second component by A.

Note that we can rewrite $A = \sum_{\substack{Q_i \leq P_i \\ Q_i^s = Q_i \\ Q_i N_i = P_i \\ Q_i \ngeq N_0}} \mu((Q_i, P_i)^s) = \sum_{\substack{R_i \leq P_i \\ R_i^s = R_i \\ R_i N_i = P_i \\ R_i \geq N_0}} \sum_{\substack{Q_i \leq R_i \\ Q_i^s = Q_i \\ Q_i N_0 = R_i \\ Q_i \ngeq N_0}} \mu((Q_i, P_i)^s),$

because given $Q_i \leq R_i, Q_i^s = Q_i, Q_i N_0 = R_i, Q_i \ngeq N_0$, we have $Q_i N_0 N_i = R_i N_i = P_i$ so $Q_i N_i = P_i$, conversely, given $Q_i \leq P_i, Q_i^s = Q_i, Q_i N_i = P_i, Q_i \ngeq N_0$, we can define $R_i = Q_i N_0 \leq P_i$, then clearly, $R_i^s = R_i$ and $R_i N_i = Q_i N_0 N_i = Q_i N_i P_i$.

Now, given R_i as above, and by our observation above, we know that R_i is an $\mathrm{Im}(\Psi)$-vector subspace of P_i, and given $Q_i \leq R_i, Q_i^s = Q_i, Q_i N_0 = R_i, Q_i \ngeq N_0$, Q_i is an $\mathrm{Im}(\Psi)$-vector subspace of R_i that does not contain N_0, but since $Q_i N_0 = R_i$, we have that Q_i is a hyperplane of R_i.

Now, letting $\dim_{Im(\Psi)} R_i = r$, with $|\mathrm{Im}(\Psi)| = |\mathbb{F}_{p^{s_i}}| = |S_i|$, and letting $q_i = p^{s_i}$ we have the number of hyperplanes of R_i which does not contain N_0 is equal to the difference between the number of hyperplanes of R_i and the number of hyperplanes of R_i/N_0, that is to say, $\begin{bmatrix} r \\ 1 \end{bmatrix}_{q_i} - \begin{bmatrix} r-1 \\ 1 \end{bmatrix}_{q_i} = \frac{(1-q_i^r)}{(1-q_i)} - \frac{(1-q_i^{(r-1)})}{(1-q_i)} = q_i^{r-1}$.

Thus, noting that $\dim_{\mathrm{Im}(\Psi)} Q_i = r-1$, we have

$$A = \sum_{\substack{R_i \leq P_i \\ R_i^s = R_i \\ R_i N_i = P_i \\ R_i \geq N_0}} q_i^{r-1} \mu((Q_i, P_i)^s).$$

Now, note that $\mu((Q_i, P_i)^s)$ is precisely the Möbius function associated to $\mathrm{Im}(\Psi)$-vector spaces between Q_i and P_i by our observation above.

We need the following well-known results before continuing our calculation:

Lemma 4.5.7. *Let $\mathbb{F}_q$ be a finite field with q elements and W be an $\mathbb{F}_q$-vector space with dimension n. Then, we have the following equality for the Möbius function of poset of $\mathbb{F}_q$-vector subspaces:*

$$\mu(\{0\}, W) = (-1)^n q^{n/2}.$$

Proof: We prove the result by induction on the dimension of W.

If n=0, then the result is trivial. Suppose now that the equality holds for any vector spaces of dimension less than or equal to n-1.

Let V_0 be 1-dimensional subspace of W. Now, the Crapo's complementation theorem says that,

$$\mu(\{0\}, W) = \sum_{W_0 \in I} \mu(0, W_0)\mu(W_0, W),$$

where I is the set of hyperplanes of W that do not contain V_0. We know that the cardinality of I is equal to the difference of the number of hyperplanes of W and the number of hyperplanes of W/V$_0$. Thus, we have $|I| = \begin{bmatrix} n \\ 1 \end{bmatrix}_q - \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q = \frac{(1-q^n)}{(1-q)} - \frac{(1-q^{n-1})}{(1-q)} = q^{n-1}$.

Thus, we have,

$$\mu(\{0\}, W) = \sum_{W_0 \in I} \mu(\{0\}, W_0)\mu(W_0, W) = q^{n-1}(-1)^{n-1} q^{\binom{n-1}{2}}(-1)^1 = (-1)^n q^{\binom{n}{2}},$$

proving the lemma.

Now, we have by Lemma 4.5.7, and our observation, and noting that $\dim_{\mathrm{Im}(\Psi)} P_i/Q_i = n_i - (r-1) = n_i - r + 1$,

$$\mu((Q_i, P_i)^s) = \mu((0, P_i/Q_i)^s) = (-1)^{n_i-r+1} q_i^{\binom{n_i-r+1}{2}}.$$

Thus,

$$A = \sum_{\substack{R_i \le P_i \\ R_i^s = R_i \\ R_i N_i = R_i \\ R_i \ge N_0}} q_i^{r-1} \mu((Q_i, P_i)^s) = \sum_{\substack{R_i \le P_i \\ R_i^s = R_i \\ R_i N_i = P_i \\ R_i \ge N_0}} q_i^{r-1} (-1)^{n_i-r+1} q_i^{\binom{n_i-r+1}{2}}.$$

Noting that

$$r - 1 + \binom{n_i - r + 1}{2} = r - 1 + \binom{n_i - r}{2} + \binom{n_i - r}{1} = n_i - 1 + \binom{n_i - r}{2},$$

we have,

$$A = \sum_{\substack{R_i \le P_i \\ R_i^s = R_i \\ R_i N_i = P_i \\ R_i \ge N_0}} (-q_i)^{n_i-1} (-1)^{n_i-r} q_i^{\binom{n_i-r}{2}} = -q_i^{n_i-1} \sum_{\substack{R_i \le P_i \\ R_i^s = R_i \\ R_i N_i = P_i \\ R_i \ge N_0}} (-1)^{n_i-r} q_i^{\binom{n_i-r}{2}}$$

and now by applying of Lemma 4.5.7 for $\mu((0, P_i/R_i)^s) = \mu((R_i, P_i)^s)$,

$$A = -q_i^{n_i-1} \sum_{\substack{R_i \le P_i \\ R_i^s = R_i \\ R_i N_i = P_i \\ R_i \ge N_0}} \mu((R_i, P_i)^s) = -q_i^{n_i-1} \sigma_{P_i/N_0, sN_0, N_i/N_0}.$$

Then, $\sigma_{P_i,s,N_i} = \sigma_{P_i/N_0, sN_0, N_i/N_0} + A = (1 - q_i^{n_i-1}) \sigma_{P_i/N_0, sN_0, N_i/N_0}$.

Now, we have $\dim_{\mathrm{Im}(\Psi)} P_i/N_0 = n_i - 1$ and $\dim_{\mathrm{Im}(\Psi)} N_i/N_0 = a_i - 1$ where $\mathrm{Im}(\Psi) \cong \mathbb{F}_{q_i} \cong \mathbb{F}_{p^{s_i}} \cong S_i$. Thus, by induction hypothesis, we have

$$\sigma_{P_i/N_0, sN_0, N_i/N_0} = (1 - q_i^{(n_i-1)-1}) ... (1 - q_i^{(n_i-1)-(a_i-1)}).$$

But then, we obtain,

$$\sigma_{P_i,s,N_i} = (1 - q_i^{n_i-1})(1 - q_i^{n_i-2}) ... (1 - q_i^{n_i-a_i}),$$

as claimed.

Case(ii): Suppose that s acts trivially on $\mathrm{P}_i \cong \bigoplus_{j=1}^{n_i} S_i$ i.e. $\mathrm{S}_i \cong \mathbb{F}_p$ is the trivial $\mathbb{F}_p\langle s\rangle$-module.

Clearly, for every $Q_i \leq P_i$ with $Q_i^s = Q_i$, we have $C_{Q_i}(s) = Q_i$ and $\mu((Q,P)^s) = \mu(Q,P)$.

Now, we have, letting N_0 be 1-dimensional $\mathrm{Im}(\Psi) \cong \mathbb{F}_p$-vector subspace of N_i,

$$\begin{aligned}\sigma_{P_i,s,N_i} &= \sum_{\substack{Q_i \leq P_i \\ Q_i^s = Q_i \\ Q_i N_i = P_i \\ Q_i \geq N_0}} |Q_i|\mu(Q_i,P_i) + \sum_{\substack{Q_i \leq P_i \\ Q_i^s = Q_i \\ Q_i N_i = P_i \\ Q_i \ngeq N_0}} |Q_i|\mu(Q_i,P_i) \\ &= |N_0|\sigma_{P_i/N_0, sN_0, N_i/N_0} + \mathrm{B} \\ &= p\sigma_{P_i/N_0, sN_0, N_i/N_0} + \mathrm{B}\end{aligned}$$

since $dim_{\mathrm{Im}(\Psi)} N_0 = 1$.

We shall calculate $\mathrm{B} = \sum_{\substack{Q_i \leq P_i \\ Q_i^s = Q_i \\ Q_i N_i = P_i \\ Q_i \ngeq N_0}} |Q_i|\mu(Q_i,P_i)$.

By similar argument as above, we rewrite B again,

$$B = \sum_{\substack{R_i \leq P_i \\ R_i \geq N_0 \\ R_i N_i = P_i}} \sum_{\substack{Q_i \leq R_i \\ Q_i N_0 = R_i \\ Q_i \ngeq N_0}} |Q_i|\mu(Q_i,P_i).$$

Letting $\mathrm{dim}_{\mathrm{Im}(\Psi)} R_i = dim_{\mathbb{F}_p} R_i = r$ so $|R_i| = p^r$, and then $|Q_i| = p^{r-1}$, we obtain,

$$\begin{aligned}B &= \sum_{\substack{R_i \leq P_i \\ R_i \geq N_0 \\ R_i N_i = P_i}} \sum_{\substack{Q_i \leq R_i \\ Q_i N_0 = R_i \\ Q_i \ngeq N_0}} p^{r-1}\mu(Q_i,P_i) \\ &= \sum_{\substack{R_i \leq P_i \\ R_i \geq N_0 \\ R_i N_i = P_i}} p^{r-1} \sum_{\substack{Q_i \leq R_i \\ Q_i N_0 = R_i \\ Q_i \ngeq N_0}} \mu(Q_i,P_i)\end{aligned}$$

Now, the second sum above, namely, $\sum_{\substack{Q_i \leq R_i \\ Q_i N_0 = R_i \\ Q_i \ngeq N_0}} \mu(Q_i,P_i) = p^{r-1}\mu(Q_i,P_i)$ where $p^{r-1} =$

$\begin{bmatrix} r \\ 1 \end{bmatrix}_p - \begin{bmatrix} r-1 \\ 1 \end{bmatrix}_p$ is the number of hyperplanes of R_i not containing N_0.

Moreover, by Lemma4.5.5, since $\dim_{\mathbb{F}_p} P_i/Q_i = n_i - (r-1) = n_i - r + 1$, $\mu(Q_i, P_i) = \mu(0, P_i/Q_i) = (-1)^{n_i - r + 1} p^{\binom{n_i - r + 1}{2}}$.

Hence, we obtain,

$$\begin{aligned} B &= \sum_{\substack{R_i \leq P_i \\ R_i \geq N_0 \\ R_i N_i = P_i}} p^{r-1} p^{r-1} \mu(Q_i, P_i) \\ &= \sum_{\substack{R_i \leq P_i \\ R_i \geq N_0 \\ R_i N_i = P_i}} p^{2r-2} (-1)^{n_i - r + 1} p^{\binom{n_i - r + 1}{2}} \end{aligned}$$

Noting that

$$\binom{n_i - r + 1}{2} = \binom{n_i - r}{2} + \binom{n_i - r}{1} = \binom{n_i - 1}{2} + n_i - r,$$

we have,

$$\begin{aligned} B &= (-1) p^{n_i - 1} \sum_{\substack{R_i \leq P_i \\ R_i \geq N_0 \\ R_i N_i = P_i}} (-1)^{n_i - r} p^{\binom{n_i - r}{2}} p^{r-1} \\ &= (-p^{n_i - 1}) \sum_{\substack{R_i \leq P_i \\ R_i \geq N_0 \\ R_i N_i = P_i}} \mu(R_i, P_i) p^{r-1} \\ &= (-p^{n_i - 1}) \sigma_{P_i/N_0, sN_0, N_i/N_0}. \end{aligned}$$

Now, we have

$$\sigma_{P_i, s, N_i} = (p - p^{n_i - 1}) \sigma_{P_i/N_0, sN_0, N_i/N_0} = p(1 - p^{n_i - 2}) \sigma_{P_i/N_0, sN_0, N_i/N_0}.$$

Thus, by induction, we have

$$\sigma_{P_i/N_0, sN_0, N_i/N_0} = (p^{n_i - 1})(1 - p^{n_i - 3}) ... (1 - p^{n_i - a_i - 1}).$$

Then, we obtain

$$\begin{aligned} \sigma_{P_i, s, N_i} &= p(1 - p^{n_i - 2})(p^{n_i - 1})(1 - p^{n_i - 3}) ... (1 - p^{n_i - a_i - 1}) \\ &= p^{n_i}(1 - p^{n_i - 2}) ... (1 - p^{n_i - a_i - 1}), \end{aligned}$$

as required.

STEP 3: Conditions (ii) and (iii)

Claim: For a D-pair (P,s), we have $C_{\langle s\rangle}(P) = 1$.

Proof: Note that $N = C_{\langle s\rangle}(P) \trianglelefteq \langle Ps\rangle$, and $N \cap P = 1$ which implies that for every i, $N_i = N \cap P_i = 1$ i.e. $a_i = 0$. But then, for every i, $\sigma_{P_i,s,N_i} = |C_{P_i}(s)| \neq 0$ which means that $m_{P,s,C_{\langle s\rangle}(P)} \neq 0$ which is a contradiction unless $C_{\langle s\rangle}(P) = 1$.

Claim: In the homogeneous componentwise decomposition of $\mathbb{F}_p\langle s\rangle$-module P, namely, P= $P_1 \oplus ... \oplus P_t$, $P_i = \bigoplus_{j=1}^{n_i} S_i$ where S_i is a simple $\mathbb{F}_p\langle s\rangle$-module, we have $n_i \in \{0, 2\}$ if S_i corresponds to the trivial $\mathbb{F}_p\langle s\rangle$-module, and $n_i = \{0, 1\}$ if S_i corresponds to a non-trivial $\mathbb{F}_p\langle s\rangle$-module.

Proof: Suppose that P_i is apparent in the homogeneous componentwise decomposition of $\mathbb{F}_p\langle s\rangle$- module P, with $P_i = \bigoplus_{j=1}^{n_i} S_i$ where S_i is a simple $\mathbb{F}_p\langle s\rangle$-module. Thus, S_i is an $\mathbb{F}_p$-vector subspace of P_i which is stable under the action of s, i.e. $S_i \leq P_i \leq P$ and $S_i^s = S_i$. Then, we can take $N = S_i \trianglelefteq \langle Ps\rangle$, which implies that $a_k = 0$ unless $k = i$, and $a_i = dim_{\mathrm{Im}(\Psi)} S_i = 1$ i.e. $N_i = S_i$ and $N_k = 1$.

Thus, we have $\sigma_{P_k,s,N_k} = |C_{P_k}(s)|$ whenever $k \neq i$, which means that since $\sigma_{P,s,N} = 0$, we have necessarily $\sigma_{P_i,s,N_i} = 0$. By using Proposition 4.5.6, we obtained that

$$0 = \sigma_{P_i,s,S_i} = \begin{cases} p^{n_i}(1 - p^{n_i-2}) & if \quad S_i \text{ corresponds to the trivial } \mathbb{F}_p\langle s\rangle\text{-module,} \\ (1 - q_i^{n_i-1}) & if \quad S_i \text{ corresponds to non-trivial } \mathbb{F}_p\langle s\rangle\text{-module.} \end{cases}$$

Therefore, $n_i = 2$ if S_i is trivial, and $n_i = 1$ if S_i is non-trivial. Noting that they may not appear at all as well.

For the converse part, that is, letting P to be elementary abelian, $C_{\langle s\rangle}(P) = 1$ and $n_i \in \{0, 2\}$ if S_i is trivial, and $n_i \in \{0, 1\}$ if S_i is non-trivial, the result directly follows from Proposition 4.5.6. □

Chapter 5

On some of the new simple composition factors of $\mathbb{C}pp_k$

5.1 The decomposition of simple p-permutation factors of $\mathbb{C}pp_k{}^{p-perm.}$ to biset functors in a special case

Throughout this chapter, we suppose k is an algebraically closed field of characteristic p, where p is prime, and $\mathbb{C}$ denotes the algebraically closed field of characteristic 0.

In this chapter, for a restricted type of p-hypo-elementary B-group $H = P \rtimes \langle s \rangle$, we extract information from the simple p-permutation factor $S^{p-perm.}_{\langle Ps \rangle, W_{P,s}}$ of the p-permutation functor $\mathbb{C}pp_k^{p-perm.}$ to obtain new simple composition factors of the biset functor $\mathbb{C}pp_k$.

The main result of this extraction, which we prove in this chapter, is the following theorem:

Suppose that $H = P \rtimes \langle s \rangle$ be a p-hypo-elementary B-group such that every non-trivial $\mathbb{F}_p\langle s \rangle$-module is apparent in the P. Then, for every $\varphi \in \mathrm{Out}(\langle s \rangle)$, the simple biset functor $S_{H,\mathbb{C}_\varphi}$ is apparent as a composition factor of the biset functor $\mathbb{C}pp_k$ where $\mathbb{C}_\varphi$ is the inflation of the vector space $\mathbb{C}$ on which the group $\mathrm{Out}(\langle s\rangle)$ acts by φ.

Along the way, we will observe that for such a group $H = P \rtimes \langle s\rangle$, we have $\dim_{\mathbb{C}} S^{p-perm.}_{\langle Ps\rangle, W_{P,s}}(H) = \phi(|s|)$. This observation supports the conjecture that for such a group H, $S_{H,V}$ is apparent as a simple composition factor of $\mathbb{C}pp_k$ if and only if V is the inflated module $\mathbb{C}_\varphi$ for some $\varphi \in \mathrm{Out}(\langle s\rangle)$.

We start with the following set up:

Let $H = P \rtimes \langle s\rangle$ be a p-hypo-elementary B-group such that in the homogeneous componentwise decomposition of $\mathbb{F}_p\langle s\rangle$-module P, every non-trivial simple $\mathbb{F}_p\langle s\rangle$-module S_i is apparent. Thus, we suppose the multiplicity n_0 of the trivial $\mathbb{F}_p\langle s\rangle$-module S_0 in P to be 0 or 2 and the multiplicity n_i of every non-trivial $\mathbb{F}_p\langle s\rangle$-module S_i in P to be necessarily 1.

We shall first show that, with this set up, there exists a surjective group homomorphism $f : \mathrm{Aut}(H) \twoheadrightarrow \mathrm{Aut}(\langle s\rangle)$. Let us denote $ord(s) = l$.

Let us define $f : \mathrm{Aut}(H) \twoheadrightarrow \mathrm{Aut}(\langle s\rangle)$ by sending $\Psi \mapsto \delta$ where $\Psi(s) =_H \delta(s)$.

f is well-defined:

Since $H = P \rtimes \langle s\rangle$ be a p-hypo-elementary B-group, by the classification of Baumann, we have $C_{\langle s\rangle}(P) = 1$. Thus, for every $1 \leq j \leq l-1$, $C_{\langle s^j\rangle}(P) = 1$.

Moreover, we have $C_H(s^j) = \langle s\rangle$ because if we suppose otherwise, then, there exists $(p, s^k) \in H$ with $p \neq 1$ such that $(p, s^k)\cdot(1, s^j)\cdot(s^{-k}p^{-1}s^k, s^{-k}) = (1, s^j)$, and so $(ps^jp^{-1}s^{-j}, s^j) = (1, s^j)$ implying that $s^jp^{-1}s^{-j} = p^{-1}$, which is possible if and only if $p = 1$ since $C_{\langle s^j\rangle}(P) = 1$, a contradiction.

Therefore, we obtain that $|[s^j]_H| = |H|/|\langle s\rangle| = |P|$. Clearly, $(1, s^i) \neq_H (1, s^j)$ whenever $i \neq j$. Thus, $[s^j]_H := \{(p, s^j) \mid \forall p \in P\}$.

Then, if we say $\Psi(s) = (p, s^j) \in H$, then $\Psi(s) =_H (1, s^j)$. Furthermore, since $\mathrm{ord}(\Psi(s)) = \mathrm{ord}(s)$, we have $\mathrm{ord}(s^j) = \mathrm{ord}(s) = l$. Thus, there exists $\delta \in \mathrm{Aut}(\langle s\rangle)$

such that $\delta(s) = s^j$.

<u>f is a group homomorphism:</u>

Let $\Psi, \Psi' \in \mathrm{Aut}(H)$. Suppose that $\Psi(s) = (p, s^k)$ and $\Psi'(s) = (p', s^{k'})$.

Claim: $f(\Psi \circ \Psi') = f(\Psi) \circ f(\Psi')$ and $f(id_{\mathrm{Aut}(H)}) = id_{\mathrm{Aut}(\langle s \rangle)}$.

Proof: The second part is obvious. For the first part, suppose that $f(\Psi) = \delta_k$ and $f(\Psi') = \delta_{k'}$ where $\delta_k, \delta_{k'} \in \mathrm{Aut}(\langle s \rangle)$ such that $\delta_k(s) = s^k$ and $\delta_{k'}(s) = s^{k'}$. We need to obtain that $\Psi \circ \Psi'(s) =_H s^{kk'}$.

Now, we have $\Psi \circ \Psi'(s) = \Psi(p', s^{k'}) = \Psi((p', 1) \cdot (1, s^{k'})) = \Psi(p', 1) \cdot \Psi(1, s^{k'}) = (p'', 1) \cdot \Psi(1, s^{k'})$ since P is a characteristic group of H which implies that we have $\Psi(p', 1) = (p'', 1)$ for some $p'' \in P$. Then, $(p'', 1) \cdot (1, s^{kk'}) = (p'', s^{kk'}) =_H (1, s^{kk'})$, that is to say, $\Psi \circ \Psi'(s) =_H (1, s^{kk'})$. Therefore, $f(\Psi \circ \Psi') = \delta_{kk'}$. On the other hand, $f(\Psi) \circ f(\Psi') = \delta_k \circ \delta_{k'} = \delta_{kk'}$, proving the claim.

<u>f is onto:</u>

Firstly, note that since $(p, l) = 1$, the polynomial $x^l - 1$ has no repeating roots. Now, considering the ring isomorphism $\mathbb{F}_p\langle s \rangle \cong \mathbb{F}_p[x]/(x^l - 1) \cong \prod_j \mathbb{F}_p[x]/m_j(x)$ where each $m_j(x)$ is distinct irreducible polynomial over $\mathbb{F}_p$, corresponding simple $\mathbb{F}_p\langle s \rangle$-module S_j, we obtain that $\mathbb{F}_p\langle s \rangle \cong \bigoplus_j S_j$.

Secondly, since $H = P \rtimes \langle s \rangle$ is a p-hypo-elementary B-group where all non-trivial $\mathbb{F}_p\langle s \rangle$-modules are apparent, by the classification of Baumann, we have two cases. Namely, as an $\mathbb{F}_p\langle s \rangle$-module, $P \cong S_0 \oplus \mathbb{F}_p\langle s \rangle$ or $P \oplus S_0 \cong \mathbb{F}_p\langle s \rangle$, where S_0 denotes the trivial $\mathbb{F}_p\langle s \rangle$-module.

Now, suppose that $\delta \in \mathrm{Aut}(\langle s \rangle)$ is given, i.e., $\delta(s) = s^i$ for some $i \in (\mathbb{Z}/l\mathbb{Z})^\times$. We need to construct $\Psi \in \mathrm{Aut}(\langle s \rangle)$ such that $f(\Psi) = \delta$.

We extend δ by defining $\widetilde{\delta} : \mathbb{F}_p\langle s \rangle \to \mathbb{F}_p\langle s \rangle$ by $\sum_{j=1}^{l-1} a_j s^j \mapsto \sum_{j=1}^{l-1} a_j \delta(s^j)$. Clearly, $\widetilde{\delta}$ is an $\mathbb{F}_p$-algebra automorphism of $\mathbb{F}_p\langle s \rangle$.

Case (i): If as an $\mathbb{F}_p\langle s\rangle$-module, $P \cong S_0 \oplus \mathbb{F}_p\langle s\rangle$.

We extend the $\mathbb{F}_p$-automorphism of $\mathbb{F}_p\langle s\rangle$ $\tilde{\delta}$ to an $\mathbb{F}_p$-homomorphism

$$\tilde{\tilde{\delta}} : S_0 \oplus \mathbb{F}_p\langle s\rangle \to S_0 \oplus \mathbb{F}_p\langle s\rangle \text{ by letting } \tilde{\tilde{\delta}}|_{S_0} = id_{S_0} \text{ and } \tilde{\tilde{\delta}}|_{\mathbb{F}_p\langle s\rangle} = \tilde{\delta}.$$

Then, for every $x \in \langle s\rangle$, $y \in P$, we have $y_o \in S_0$ and $y_1 \in \mathbb{F}_p\langle s\rangle$ such that

$$\begin{aligned} {}^{\tilde{\tilde{\delta}}(x)}\tilde{\tilde{\delta}}(y) &= {}^{\tilde{\tilde{\delta}}(x)}\,\tilde{\tilde{\delta}}(y_0 + y_1) \\ &= {}^{\tilde{\tilde{\delta}}(x)}\,y_0 + {}^{\tilde{\tilde{\delta}}(x)}\,\tilde{\tilde{\delta}}(y_1) \\ &= y_0 + \tilde{\tilde{\delta}}(x)\tilde{\tilde{\delta}}(y_1)\tilde{\tilde{\delta}}(x^{-1}) \\ &= y_0 + \tilde{\tilde{\delta}}({}^{x}y_1), \end{aligned}$$

since $\tilde{\delta}$ is an $\mathbb{F}_p$-algebra homomorphism of $\mathbb{F}_p\langle s\rangle$. But then, we have

$$\begin{aligned} &= y_0 + \tilde{\tilde{\delta}}({}^{x}y_1) \\ &= \tilde{\tilde{\delta}}(y_0) + \tilde{\tilde{\delta}}({}^{x}y_1) \\ &= \tilde{\tilde{\delta}}(y_0 +{}^{x} y_1) \\ &= \tilde{\tilde{\delta}}({}^{x}(y_0 + y_1)) = \tilde{\tilde{\delta}}({}^{x}y). \end{aligned}$$

Now, we define $\Psi : H \to H$ by $(y, x) \mapsto (\tilde{\tilde{\delta}}(y), \tilde{\tilde{\delta}}(x))$.

Ψ is a group homomorphism:

Let $x_1, x_2 \in \langle s\rangle$, and $y_1, y_2 \in P$. Then,

$$\begin{aligned} \Psi(y_1, x_1)\Psi(y_2, x_2) &= (\tilde{\tilde{\delta}}(y_1), \tilde{\tilde{\delta}}(x_1))(\tilde{\tilde{\delta}}(y_2), \tilde{\tilde{\delta}}(x_2)) \\ &= (\tilde{\tilde{\delta}}(y_1)^{\tilde{\tilde{\delta}}(x_1)}\tilde{\tilde{\delta}}(y_2), \tilde{\tilde{\delta}}(x_1)\tilde{\tilde{\delta}}(x_2)) \\ &= (\tilde{\tilde{\delta}}({y_1}^{x_1}y_2), \tilde{\tilde{\delta}}(x_1x_2)) \\ &= \Psi({y_1}^{x_1}y_2, x_1x_2) \\ &= \Psi((y_1, x_1) \cdot (y_2, x_2)). \end{aligned}$$

Moreover, it is clear that Ψ is one to one and onto. Therefore, we have $\Psi \in \mathrm{Aut}(H)$.

Case (ii): If as an $\mathbb{F}_p\langle s\rangle$-module, $P \oplus S_0 \cong \mathbb{F}_p\langle s\rangle$.

Similarly to the argument above, we consider the $\mathbb{F}_p$-algebra automorphism $\tilde{\delta}$: $\mathbb{F}_p\langle s\rangle \to \mathbb{F}_p\langle s\rangle$ by sending the element $\sum_j a_j s^j \mapsto \sum_j a_j \delta(s^j)$. Clearly, this maps sends S_0 to itself. Therefore, given $u \in P$, we have $\tilde{\delta}(u) \in P$. Now, for every $x \in \langle s\rangle$,

$$^{\tilde{\delta}(x)}\tilde{\delta}(u) = \tilde{\delta}(x)\tilde{\delta}(u)\tilde{\delta}(x^{-1}) = \tilde{\delta}(xux^{-1}) = \tilde{\delta}(^{x}u).$$

Then, again, we define $\Psi : H \to H$ by $(u, x) \mapsto (\tilde{\delta}(u), \tilde{\delta}(x))$. Same as above, it is a group isomorphism.

Thus, we showed that $f : \mathrm{Aut}(H) \twoheadrightarrow \mathrm{Aut}(\langle s\rangle)$ is a surjective group homomorphism sending $\Psi \mapsto \delta$ with $\Psi(s) =_H \delta(s)$. Now, $\mathrm{Ker}f := \{\Psi \in \mathrm{Aut}(H) : \Psi(s) =_H s\}$. Now, it is clear that $Inn(H) \trianglelefteq \mathrm{Ker}f$. Note that denoting $\mathbb{S} := \{\gamma \in \mathrm{Out}(H) : \gamma(s) = s\}$, we obtain

$$\mathrm{Out}(H)/\mathbb{S} \cong \mathrm{Aut}(H)/\mathrm{Ker}f \cong \mathrm{Aut}(\langle s\rangle) = \mathrm{Out}(\langle s\rangle).$$

Recall given $N \trianglelefteq G$, and an $\mathbb{C}G/N$-module V, we define the inflated module from $\mathbb{C}G/N$ to $\mathbb{C}G$ of V as $\mathbb{C}G/N \otimes_{\mathbb{C}G/N} V$. It is clear that V is a simple $\mathbb{C}G/N$-module if and only if the inflated module is a simple $\mathbb{C}G$-module.

Now, for any $\varphi \in \mathrm{Out}(\langle s\rangle) = (\mathbb{Z}/l\mathbb{Z})^{\times}$, we consider the vector space $\mathbb{C}$ on which the group $\mathrm{Out}(\langle s\rangle)$ acts via φ. With the setting above, we denote the associated inflated simple $\mathbb{C}\mathrm{Out}(H)$-module by $\mathbb{C}_\varphi$.

Now, we move to the our main observation:

Theorem 5.1.1. *Suppose that $H = P \rtimes \langle s\rangle$ be a p-hypo-elementary B-group such that every non-trivial $\mathbb{F}_p\langle s\rangle$-module is apparent in P. Then, for every $\varphi \in Out(\langle s\rangle)$, the simple biset functor $S_{H,\mathbb{C}_\varphi}$ is apparent as a composition factor of the biset functor $\mathbb{C}pp_k$ where $\mathbb{C}_\varphi$ is the inflation of the vector space $\mathbb{C}$ on which the group $Out(\langle s\rangle)$ acts by φ.*

Proof. Let $\mathcal{F}_{|H|}$ be the full-subcategory of $\mathbb{C}\mathcal{C}$ whose objects are all finite groups, up to isomorphism, of order less than or equal to $|H|$.

Now, let $\varphi \in \mathrm{Out}(\langle s\rangle)$ and suppose $\mathbb{C}_\varphi$ be the associated inflated $\mathbb{C}\mathrm{Out}(H)$-module as shown above.

Moreover, we have $[s^j]_H := \{(p, s^j) | \forall p \in P\}$ for every $1 \leq j \leq l-1$ as it is shown above for such a group H. Hence, $s^jP \neq s^kP$ whenever $j \neq k$. Since $P \trianglelefteq H$, for every $1 \leq j \leq l-1, s^j \in (N_H(P))_{p'}$. For this reason, we obtain that $(P, s^j) \in [Q_{H,p}]$ for every $1 \leq j \leq l-1$.

We define $M_\varphi := span_{\mathbb{C}}\langle \sum\limits_{\forall i \in (\mathbb{Z}/l\mathbb{Z})^\times} \varphi(i) F^H_{P,s^i}\rangle$.

<u>M_φ is a biset functor:</u>

Since every biset functor on $\mathcal{F}_{|H|}$ can be thought as a module of the quiver algebra $\bigoplus\limits_{F,G \in Obj(\mathcal{F}_{|H|})} B(F,G)$, it is adequate to show that M_φ is closed under the action of five elementary maps, induction, inflation, isogation, deflation and restriction.

By the definition of object set of $\mathcal{F}_{|H|}$, there is no induction or inflation of F^H_{P,s^i} for any $i \in (\mathbb{Z}/l\mathbb{Z})^\times$.

By the formula of restriction, whenever $K \lneqq H$, and $i \in (\mathbb{Z}/l\mathbb{Z})^\times$, $\mathrm{Res}^H_K(F^H_{P,s^i}) = 0$.

Note that since we obtained that the classification of p-hypo-elementary B-groups and D-pairs imply one another, and for any $i \in (\mathbb{Z}/l\mathbb{Z})^\times$, $H = P \rtimes \langle s^i\rangle$ is a p-hypo-elementary B-group, every such (P, s^i) is a D-pair. Therefore, for any $1 \neq N \trianglelefteq H$, $\mathrm{Def}^H_{H/N}(M_\varphi) = span_{\mathbb{C}}\langle \sum\limits_{\forall i \in (\mathbb{Z}/l\mathbb{Z})^\times} \varphi(i)\mathrm{Def}^H_{H/N} F^H_{P,s^i}\rangle = 0$.

For isogations, let $\Psi \in \mathrm{Aut}(H)$. We already observed that for given $i \in (\mathbb{Z}/l\mathbb{Z})^\times$, $\Psi(s^i) =_H (1, s^j)$ for some $j \in (\mathbb{Z}/l\mathbb{Z})^\times$. Since we tag the primitive idempotents up to H-conjugacy, we can suppose that $\Psi(s^i) = (1, s^j)$ for some $j \in (\mathbb{Z}/l\mathbb{Z})^\times$. Suppose that it corresponds to $\Psi(s) = s^k$ for some $k \in (\mathbb{Z}/l\mathbb{Z})^\times$.

Now, we have

$$\begin{aligned}
\mathrm{Iso}(\Psi)M_\varphi &= span_{\mathbb{C}}\langle \sum_{\forall i \in (\mathbb{Z}/l\mathbb{Z})^\times} \varphi(i) F^H_{P,s^{ik}} \rangle \\
&= span_{\mathbb{C}}\langle \sum_{\forall j \in (\mathbb{Z}/l\mathbb{Z})^\times} \varphi(k^{-1}j) F^H_{P,s^{j}} \rangle \\
&= span_{\mathbb{C}}\langle \sum_{\forall j \in (\mathbb{Z}/l\mathbb{Z})^\times} \varphi(k^{-1})\varphi(j) F^H_{P,s^{j}} \rangle \\
&= span_{\mathbb{C}}\langle \sum_{\forall j \in (\mathbb{Z}/l\mathbb{Z})^\times} \varphi(j) F^H_{P,s^{j}} \rangle \\
&= M_\varphi.
\end{aligned}$$

Thus, M_φ has a biset functor structure.

Furthermore, the simplicity of M_φ, as a biset functor, follows from the surjectivity of the map $f : \mathrm{Aut}(H) \twoheadrightarrow \mathrm{Aut}(\langle s \rangle)$ defined above.

Note that $M_\varphi = S_{H,\mathbb{C}_\varphi}$ on $\mathcal{F}_{|H|}$ since $M_\varphi(H) \cong \mathbb{C}_\varphi$ and H is the minimal group such that $M_\varphi(H) \neq 0$. Thus, we find on the full-subcategory $\mathcal{F}_{|H|}$, for every $\varphi \in \mathrm{Out}(\langle s \rangle)$, $S_{H,\mathbb{C}_\varphi}$ is a simple composition factor of $\mathbb{C}pp_k$.

Now, by finite reduction principle for biset functors, it follows that they are also simple composition factors of $\mathbb{C}pp_k$ on $\mathbb{C}\mathcal{C}$, as required. □

Now, note that with this setting, by Proposition 4.3.3, and surjectivity of the map f above, we have for every $i \in (\mathbb{Z}/l\mathbb{Z})^\times$, $(P, s) \simeq (P, s^i)$. Therefore,

$$dim_{\mathbb{C}} S^{p-perm.}_{H,W_{P,s}}(H) = dim_{\mathbb{C}} W_{P,s} = \sum_{\substack{(Q,t)\simeq(P,s) \\ \langle Qt \rangle = \langle Ps \rangle}} 1 = \phi(l).$$

On the other hand,

$$\sum_{\forall \varphi \in \mathrm{Out}(\langle s \rangle)} \dim_{\mathbb{C}} S_{H,\mathbb{C}_\varphi}(H) = \sum_{\forall \varphi \in \mathrm{Out}(\langle s \rangle)} \dim_{\mathbb{C}} \mathbb{C}_\varphi = \phi(l).$$

This observation tells us that restricting the simple p-permutation factor $S^{p-perm.}_{H,W_{P,s}}$, it partially decomposes into precisely the simple biset functors $S_{H,\mathbb{C}_\varphi}$ for every $\varphi \in \mathrm{Out}(\langle s \rangle)$ for this specific type of p-hypo-elementary B-group H.

Bibliography

[1] S. Bouc, *Biset Functors for Finite Groups.* Springer Berlin Heidelberg, 2010.

[2] M. Baumann, "The composition factors of the functor of permutation modules," *Journal of Algebra*, vol. 344, no. 1, pp. 284–295, 2011.

[3] M. Baumann, *Le foncteur de bi-ensembles des modules de p-permutation.* PhD thesis, École Polytechnique Fédérale de Lausanne, 2012.

[4] M. Baumann, "The functor of p-permutation modules for abelian groups," *Journal of Algebra*, vol. 392, pp. 142–157, 2013.

[5] M. Ducellier, *Foncteurs de p-permutation.* PhD thesis, Université de Picardie Jules Verne, 2015.

[6] S. Bouc and J. Thévenaz, "The primitive idempotents of the p-permutation ring," *Journal of Algebra*, vol. 323, no. 10, pp. 2905–2915, 2010.

[7] J. Thévenaz, *G-Algebras and Modular Representation Theory.* Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.

[8] D. J. Benson, *Representations and Cohomology I.* Cambridge University Press (CUP), 1991.

[9] M. Broué, "On scott modules and p-permutation modules: An approach through the brauer morphism," *Proceedings of the American Mathematical Society*, vol. 93, no. 3, p. 401, 1985.

[10] S. Bouc, "Burnside rings," in *Handbook of Algebra*, pp. 739–804, Elsevier BV, 2000.

[11] R. Stanley, *Enumerative Combinatorics.* Cambridge, NY: Cambridge University Press, 2012.

[12] L. Barker, "Fibred permutation sets and the idempotents and units of monomial burnside rings," *Journal of Algebra*, vol. 281, no. 2, pp. 535–566, 2004.

[13] R. Boltje, "Representation rings of finite groups, their species, and idempotent formulae." preprint (2001).

[14] C. Okay, "The monomial Burnside functor," Master's thesis, Bilkent University.

[15] R. Boltje, "Linear source modules and trivial source modules," in *Group Representations: Cohomology, Group Actions and Topology*, vol. 63, pp. 7–30, American Mathematical Society, 1998.

[16] J. Rotman, *Advanced Modern Algebra.* Providence, R.I: American Mathematical Society, 2010.

[17] D. Gorenstein, *Finite Groups.* New York: Chelsea Pub. Co, 1980.