RECURSION OPERATORS AND CLASSIFICATION OF INTEGRABLE NONLINEAR EQUATIONS

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ABSTRACT

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Recursion operators, if they exist, of nonlinear partial differential equations map symmetries to symmetries of these equations. It is this property that the integrable nonlinear partial differential equations possess infinitely many symmetries. In this work we studied two other properties of recursion operators. We shall use the recursion operators as Lax operators in the Gelfand-Dikii formalism and to classify certain integrable equations.

Keywords: symmetries, Gelfand-Dikii formalism, recursion operator, classification.

ÖZET

ADIM SİMETRİ OPERATÖRLERİ VE İNTEGRE EDİLEBİLİR LİNEER OLMAYAN DENKLEMLERİN SINIFLANDIRILMASI

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Lineer olmayan kısmi diferansiyel denklemlerin, eğer varsa, simetri adım operatörlerı denklemin simetrilerini simetrilerine gönderirler. Bu yüzden integre edilebilir lineer olmayan kısmi diferansiyel denklemler sonsuz simetriye sahiptirler. Bu tezde simetri adım operatörlerinin diğer iki özelliğini çalıştık. Simetri adım operatörlerini Gelfand-Dikii formülasyonunda Lax operatörü olarak ve integre edilebilir denklemleri sınıflandırmak için kullandık.

Anahtar sözcükler: simetriler, Gelfand-Dikii formülasyonları, simetri adım operatörü, sınıflandırma.

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Contents

Chapter 1

Introduction

A symmetry group of differential equation produces solutions from a known solution. That is, if G is a symmetry group of some system of differential equations and if f is a solution to the system then for every $g \in G$, $g.f$ is also a solution to this system. To deduce a given group of transformations is a symmetry group of the system we use the infinitesimal generators of the group actions and they are vector fields over the space $X \times U$, where X is the space of independent variables and U is the space of dependent variables.
Using symmetry conditions for group of transformations we define recursion operator which is first introduced by Olver in 1977 [10]. In chapter 2, we study the symmetries of differential equations and recursion operators. We consider the evolutionary type of equations

$$u_t = F(x, u, u_x, u_{xx}, ..., u_n). \tag{1.1}$$

An operator R is the recursion operator of (1.1) if the following is satisfied

$$R_t = [F^*, R], \tag{1.2}$$

where F^* is the Frechet derivative of (1.1). For example, the operator

$$R = D_x^2 + 4u + 2u_x D_x^{-1} \tag{1.3}$$

is the recursion operator of the KdV equation

$$u_t = u_{xxx} + 6uu_x.$$

where $F^* = D_x^3 + 6uD_x + 6u_x$.
Here in this chapter, we introduce the concept of pseudo-differential operator.

In chapter 3, we study Gel'fand-Dikii Formalisms which make use of pseudo differential algebra. Operators of pseudo-differential algebra are defined as follows. [1]

Definition 1.0.1. *A pseudo-differential operator of order n is a infinite series*

$$L = \sum_{i=-\infty}^{n} \mathrm{P}_i[u] D_x^i, \tag{1.4}$$

where $\mathrm{P}_i[u]$ is a differentiable function and the operator D_x^{-1} is the formal inverse of D_x (i.e, $D_x.D_x^{-1} = D_x^{-1}.D_x = 1$).

We will consider equations of the form which were introduced by Peter D. Lax in 1968. [11]

$$L_t = [A, L], \tag{1.5}$$

where commutator $[A, L]$ is defined as

$$[A, L] = A.L - L.A.$$

Equation (1.5) is called Lax equation and the operators L and A are called Lax pair. From Lax equations we can get evolution equations for suitable A and L. Pseudo-differential algebra uses the Lax operators of type

$$L = D_x^m + u_{m-2}(x,t)D_x^{m-2} + ... + u_1(x,t)D_x + u_0,$$

where u_{m-2}, u_{m-3}, ... , u_1, u_0 are functions of x and t. Next we consider the fractional powers of L and the hierarchy

$$L_{t_n} = [A_n, L] \qquad where \quad A_n = (L^{\frac{n}{N}})_{\geq k}, \tag{1.6}$$

where n is a positive integer $n = 0, 1, 2...$ and N is the order of L, $(L^{\frac{n}{N}})_{\geq k}$ is the differential operator with powers $\geq$ k. The case $k = 0$ was first introduced by Gelfand in 1976 [7] and the cases $k = 1, 2$ were introduced by Kuperschmidt in 1988. [8]
The operator $L = D_x^2 + u$ with $A = (L^{\frac{3}{2}})_{\geq 0}$ gives KdV equation in (1.6) and the Lax pair $L = D_x^2 + 2uD_x$, $A = (L^{\frac{3}{2}})_{\geq 1}$ gives the MKdV equation [4]

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}u^2u_x.$$

Moreover, in section 3.2, we analyze the form of L in Lax hierarchy so that (1.6) is consistent for all $n \geq 1$. Blaszak [4] found that (1.6) is consistent if and only if k is either 0, 1 or 2 and he provided the forms of L for each case

$$L = c_m D_x^m + c_{m-1} D_x^{m-1} + ... + u_0 + u_{-1} D_x^{-1} + ..., \qquad k = 0$$

$$L = c_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_0 + u_{-1} D_x^{-1} + ..., \qquad k = 1$$

$$L = u_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_0 + u_{-1} D_x^{-1} + ..., \qquad k = 2$$

where c_m and c_{m-1} are functions which do not depend on variable t.

In chapter 4, we present a method to find Recursion operator of (1.1) if its Lax representation (1.6) is given. This method was introduced by Gürses, Karasu and Sokolov in 1999 citeon construction.We use the equation

$$L_{t_{n+m}} = L.L_{t_n} + [R_n, L], \tag{1.7}$$

which is called the recursion relation and the term R_n is called the remainder and it is defined as $R_n = (L.(L^{\frac{n}{m}})_-)_+$ and $ord(R_n) \leq m - 1$. The cases where L is a differential operator and L is a pseudo-differential operator are considered and in each case the formula to find recursion operator is given in [2]. For example, using (1.7) and the Lax pair

$$L = D_x^2 + u \quad with \quad A = (L^{\frac{3}{2}})_{\geq 0}$$

we find that KdV equation admits the recursion operator (1.3).
Recursion operators are also Lax operators because they satisfy the equation

$$R_t = [F^*, R]. \tag{1.8}$$

In chapter 5, we discuss the case when the recursion operator is a Lax operator in Gelfand-Dikii formalism. We can use it in Gelfand-Dikii formalism if

$$F^* = (R^{\frac{n}{m}})_{\geq k}, \tag{1.9}$$

is satisfied for some $k = 0, 1, 2$. Furthermore, the adjoint operators should satisfy the condition

$$(F^*)^+ = -((R^+)^{\frac{n}{m}})_{\geq k}, \tag{1.10}$$

for some $k = 0, 1, 2$. We check (1.9) and (1.10) for the recursion operators of the KdV, MKdV, Harry Dym and Burgers' equations. We show that recursion operators of KdV and MKdV satisfy these conditions but recursion operators of Harry Dym and Burgers' fail to satisfy conditions (1.9) or (1.10). Recursion operators should also produce themselves as recursion operators since they satisfy the Lax equation (1.8). This property will be used for the purpose of classification of certain PDEs.

In section 5.3, we consider the 2nd order recursion operators of the type

$$L = D_x^2 + \alpha + \beta D_x^{-1} \tag{1.11}$$

where α and β are some functions of (x, t), using the Lax equation the Lax equation

$$L_{t_n} = [L, A] \quad where \quad A = (L^{\frac{3}{2}})_+,$$

we find α and β must satisfy the following conditions

$$\beta = \frac{1}{2}\alpha_x \quad and \quad \alpha_t = \alpha_{xxx} + \frac{3}{2}\alpha\alpha_x.$$

Hence we deduce that α satisfies the KdV eqation.
Secondly, since $\beta = \frac{1}{2}\alpha_x$ the pseudo differential operator (1.11) is itself a recursion operator and also the Lax operator.

There are certain ways of classifying the hierarchies of evolution equations $u_{t_n} = R^n(0)$, $u_{t_n} = R^n(u_x)$ or $u_{t_n} = R^n(c)$ where c is a nonzero constant and R is the recursion operator of the evolution equation, $u_{t_1} = R(0)$, $u_{t_1} = R(u_x)$ or $u_{t_1} = R(c)$ respectively. In chapter 6, we classifying 2nd order partial differential equations of the type $u_t = R(0)$ with recursion operator of the first degree with coefficients depend at most 2nd order derivatives. Similar classification was done before.[5] We find all equations of the type

$$u_t = \beta(u, u_x, u_{xx})$$

with the recursion operators of the form

$$R = \gamma D_x + \alpha + \beta D_x^{-1}(\rho),$$

where γ, α and ρ are all functions of u, u_x and u_{xx}. We use (1.2) to find the formulas for γ, α, β and ρ and we assume $\rho \neq 0$ to get evolution equation directly from recursion operator. By solving necessary equations we obtain 3 conditions,
i. $\beta_{u_{xx}u_{xx}} \neq 0$,
ii. $\beta_{u_{xx}} \neq 0$ but $\beta_{u_{xx}u_{xx}} = 0$,
iii. $\beta_{u_{xx}} = 0$.

Neither case i nor case iii produces any class of equations because in each case we find $\rho = 0$. However, case ii produces evolution equations of the type

$$u_t = c(u)u_{xx} - \frac{1}{2}c(u)_u u_x^2 + \frac{c(u)c(u)_{uu}}{c(u)_u}u_x^2 + \frac{c_4}{c_2}c(u)u_x, \tag{1.12}$$

which admits recursion operators of the type

$$R = 2c_2 c(u)^{\frac{1}{2}} D_x + 2c_4 c(u)^{\frac{1}{2}} + (2c_2 \frac{c(u)^{\frac{1}{2}} c(u)_{uu}}{c(u)_u} - c_2 \frac{c(u)_u}{c(u)^{\frac{1}{2}}}) u_x + u_t D_x^{-1} (\frac{c_2 c(u)_u}{c(u)^{\frac{3}{2}}}).$$

Diffusion equation

$$u_t = u^2 u_{xx} \quad with \quad R = uD_x + u^2 u_{xx} D_x^{-1} (\frac{1}{u^2}),$$

and nonlinear diffusion equations

$$u_t = D_x(\frac{u_x}{u^2}) \quad with \quad R = \frac{1}{u} D_x - 2\frac{u_x}{u^2} + (\frac{2u_x^2}{u^3} - \frac{u_{xx}}{u^2}) D_x^{-1}$$

are in this class with $c(u) = u^2$, $c_2 = \frac{1}{2}$, $c_4 = 0$ and $c(u) = \frac{1}{u^2}$, $c_2 = \frac{1}{2}$, $c_4 = 0$, respectively.

In the Appendix, a list of integrable nonlinear partial differential equations with their recursion operators are given.

Chapter 2

Symmetries and Recursion Operators

In this chapter we review the symmetries of partial differential equations and their recursion operators. The main reference on this subject is the book of P. Olver [1].

2.1 Symmetry Groups and Infinitesimal Generators

Consider a system of differential equations,

$$\triangle_k(x, u^{(n)}) = 0, \qquad k = 1, ..., l, \tag{2.1}$$

which depend on the indenpendent variables $x = (x^1, ..., x^p) \in X$ and the derivatives of dependent variables $u = (u^1, ..., u^q) \in U$ with respect to x up to order n. The space $X \times U^n$, which has independent variables and the derivatives of the dependent variables up to order n as coordinates is called the n-th order jet space. The functions $\triangle(x, u^{(n)}) = (\triangle_1(x, u^{(n)}), ..., \triangle_l(x, u^{(n)}))$ are smooth maps from $X \times U^n$ to $\mathbb{R}^\ell$. Hence we have,

$$\triangle : X \times U^n \to \mathbb{R}^\ell. \tag{2.2}$$

Definition 2.1.1. *A symmetry group of the system* (2.1) *is defined to be a local group of transformations G acting on $X \times U$ which transforms solutions of the system to other solutions of the same system. That is, if f is a solution to* (2.1) *then for every $g \in G$ $g.f$ is also a solution to* (2.1).

Rather than studying group actions, it is more easier to study with infinitesimal generators of group actions. Infinitesimal generators are vector fields that act on the space $X \times U$. By using them we can easily check whether a group of transformation is a symmetry group of a system of differential equations or not.

Example 2.1.2. *Consider the rotation group SO(2) acting on $X \times U$*

$$\theta(x, u) = (x \cos\theta - u \sin\theta, x \sin\theta + u \cos\theta), \tag{2.3}$$

infinitesimal generator of this group is

$$v = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}. \tag{2.4}$$

Definition 2.1.3. *Let v be infinitesimal generator of a one-parameter group G. Then n-th prolongation of v is defined as the infinitesimal generator of the corresponding n-th prolongation of the one parameter group G and denoted by $pr^{(n)}v$.*

Theorem 2.1.4. **[1]** *Let*

$$v = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{2.5}$$

be a vector field. Then the n-th prolongation of v is

$$pr^{(n)}v = v + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x, u^n)\frac{\partial}{\partial u_J^\alpha}, \tag{2.6}$$

that acts on the jet space $X \times U^n$ and the second sum is over all indices $J =$

$(j_1, ..., j_k)$ *with* $1 \leq j_k \leq p$, $1 \leq k \leq n$ *and we have the formula for each* ϕ_α^J,

$$\phi_\alpha^J(x, u^{(n)}) = D_j(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha, \tag{2.7}$$

where $u_i^\alpha = \dfrac{\partial u^\alpha}{\partial x^i}$ *and* $u_{J,i}^\alpha = \dfrac{\partial u_J^\alpha}{\partial x^i}$.

Example 2.1.5. *If we compute the 2nd prolongation of the infinitesimal generator of the SO(2) in Example* (2.1.2) *we get*

$$pr^{(2)}v = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1 + u_x^2)\frac{\partial}{\partial u_x} + (3u_x u_{xx})\frac{\partial}{\partial u_{xx}}.$$

Now we state a theorem that gives a condition for a group to be a symmetry group of a system.

Theorem 2.1.6. **[1]** *Suppose*

$$\triangle_k(x, u^{(n)}) = 0, \qquad k = 1, ..., l \tag{2.8}$$

is a system of differential equations. Then G is a symmetry group of the system if G is a local group of transformations acting over $X \times U$ *and*

$$pr^{(n)}v[\triangle_k(x, u^{(n)})] = 0, \qquad k = 1, ..., l \quad \textit{whenever} \quad \triangle(x, u^{(n)}) = 0 \tag{2.9}$$

for every infinitesimal generator v of G.

We can use theorem (2.1.5) to find symmetry groups of a given system. Firstly, using (2.9), we can find infinitesimal generators of group actions and then we find the corresponding symmetry groups of the system. The set of all infinitesimal symmetries of the system also forms a Lie algebra of vector fields over $X \times U$. Now we give examples of symmetry groups of KdV and heat equations, see [1]. Here we use the notation ∂_x instead of $\dfrac{\partial}{\partial x}$.

Example 2.1.7. *KdV equation* $u_t + u_{xxx} + uu_x = 0$ *has 4 symmetry groups with* *infinitesimal generators,*

$$
\begin{aligned}
v_1 &= \partial_x && \textit{space translation},\\
v_2 &= \partial_t && \textit{time translation},\\
v_3 &= t\partial_x + \partial_u && \textit{Galilean boost},\\
v_4 &= x\partial_x + 3t\partial_t - 2u\partial_u && \textit{scaling},
\end{aligned}
$$

and the corresponding group actions are

$$
\begin{aligned}
G_1 &: (x,t,u) \to (x+\epsilon, t, u),\\
G_2 &: (x,t,u) \to (x, t+\epsilon, u),\\
G_3 &: (x,t,u) \to (x+\epsilon t, t, u+\epsilon),\\
G_4 &: (x,t,u) \to (e^{\epsilon}x, e^{3\epsilon}t, e^{-2\epsilon}u).
\end{aligned}
$$

Example 2.1.8. *Heat equation* $u_t = u_{xx}$ *has 7 symmetry groups with infinitesimal generators,*

$$
\begin{aligned}
v_1 &= \partial_x,\\
v_2 &= \partial_t,\\
v_3 &= u\partial_u,\\
v_4 &= x\partial_x + 2t\partial_t,\\
v_5 &= 2t\partial_x - xu\partial_u,\\
v_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u,\\
v_7 &= \alpha(x,t)\partial_u,
\end{aligned}
$$

where α *is any solution to the heat equation and the corresponding group actions* *are*

$$
\begin{aligned}
&G_1 : (x,t,u) \to (x+\epsilon, t, u),\\
&G_2 : (x,t,u) \to (x, t+\epsilon, u),\\
&G_3 : (x,t,u) \to (x, t, e^{\epsilon}u),\\
&G_4 : (x,t,u) \to (e^{\epsilon}x, e^{2\epsilon}t, u),\\
&G_5 : (x,t,u) \to \Big(x + 2\epsilon t, t, u \exp\big\{-\epsilon x - \epsilon^2 t\big\}\Big),\\
&G_6 : (x,t,u) \to \left(\frac{x}{1-4\epsilon t}, \frac{t}{1-4\epsilon t}, u\sqrt{1-4\epsilon t}\exp\left\{\frac{-\epsilon x^2}{1-4\epsilon t}\right\}\right),\\
&G_7 : (x,t,u) \to (x, t, u + \epsilon\alpha(x,t)).
\end{aligned}
$$

2.2 Generalized Symmetries

In the previous section, symmetries(infinitesimal generators of group actions) is defined in the space $X \times U$, that is

$$v = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{2.10}$$

and ξ and ϕ are functions of x, u.

In this section, we will consider more general case of this symmetries. ξ and ϕ will also depend on derivatives of u. We denote the space of smooth functions $P(x, u^n)$ by $\mathcal{A}$ and the functions in $\mathcal{A}$ are called differential functions. We will use the notation $P[u] = P(x, u^n)$. Also, the space of ℓ-tuples of differential functions $P[u] = (P_1[u], ..., P_l[u])$ where each $P_j \in \mathcal{A}$ will be denoted by $\mathcal{A}^\ell$. Now we define what a generalized vector field is. [1]

Definition 2.2.1. *An expression of the form*

$$v = \sum_{i=1}^{p} \xi^i[u]\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha[u]\frac{\partial}{\partial u^\alpha} \tag{2.11}$$

is called a generalized vector field.

Definition 2.2.2. *A generalized vector field v is a generalized infinitesimal symmetry of a system of differential equations,*

$$\triangle_k[u] = \triangle_k(x, u^{(n)}) = 0, \qquad k = 1, ..., l, \tag{2.12}$$

if and only if

$$prv[\triangle_k] = 0, \qquad k = 1, ..., l \tag{2.13}$$

for every smooth solution u of (2.12).

Definition 2.2.3. *Let $Q[u] = (Q_1[u], ..., Q_q[u]) \in \mathcal{A}^q$ be a q-tuple of differential functions. An evolutionary vector field is a generalized vector field of the form*

$$v = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha}, \tag{2.14}$$

and Q is called its characteristic.

For every generalized vector field of the form (2.11) we have an evolutionary representative v_Q, where its characteristic Q is defined by (2.14) and it is given by the formula

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha, \quad \alpha = 1, ..., q, \tag{2.15}$$

where $u_i^\alpha = \dfrac{\partial u^\alpha}{\partial x^i}$.

Corollary 2.2.4. *A generalized vector field v is a symmetry of a system of differential equations if and only if its evolutionary representative v_Q is a symmetry.*

Computation of evolutionary symmetries of a given system of differential equations is more than the method in section (2.1). However, it is almost impossible to find all evolutionary symmetries because it is not easy to find all higher order evolutionary symmetries.

For example, for the heat equation the symmetry $u_x\partial_u$ is evolutionary representative of the space translational symmetry generator $-\partial_x$. Similarly, Galilean generator $-2t\partial_x + xu\partial_u$ has evolutionary representative $(2tu_x + xu)\partial_u$.

Example 2.2.5. *Burgers' equation $u_t = u_{xx} + u_x^2$ has 6 second order generalized symmetries and characteristic of their evolutionary representatives are given by*

$$
\begin{aligned}
Q_0 &= 1,\\
Q_1 &= u_x,\\
Q_2 &= tu_x + \frac{1}{2}x,\\
Q_3 &= u_{xx} + u_x^2,\\
Q_4 &= t(u_{xx} + u_x^2) + \frac{1}{2}xu_x,\\
Q_5 &= t^2(u_{xx} + u_x^2) + xtu_x + (\frac{1}{2}t + \frac{1}{4}x^2)
\end{aligned}
$$

From now on we will consider differential equations of type

$$\frac{\partial u}{\partial t} = P[u], \tag{2.16}$$

and they are called evolution equations. The differential functions $P[u]$ in (2.16) are only functions of x and derivatives of u with respect to x.

Definition 2.2.6. *Let v_Q and v_R be evolutionary vector fields. Then their Lie bracket is defined as*

$$[v_Q, v_R] = v_S, \tag{2.17}$$

where v_S is also an evolutionary vector field with characteristic

$$S = prv_Q(R) - prv_R(Q). \tag{2.18}$$

Proposition 2.2.7. *An evolutionary vector field v_Q is a symmetry of the system of evolution equations $u_t = P[u]$ if and only if*

$$\frac{\partial v_Q}{\partial t} = [v_P, v_Q], \tag{2.19}$$

where $\frac{\partial v_Q}{\partial t}$ is the evolutionary vector field with characteristic $\frac{\partial Q}{\partial t}$.

2.3 Pseudo-Differential Algebra

Pseudo differential algebra is very useful tool to study the symmetries of partial differential equations, to use the Gelfand-Dikii formalism and to construct the recursion operators. The main reference we used is [3] to review this subject.

Definition 2.3.1. *A differential operator of order n is a finite sum*

$$\mathcal{D} = \sum_{i=0}^{n} \mathrm{P}_i[u] D_x^i, \tag{2.20}$$

where the coefficients $\mathrm{P}_i[u]$ are differentiable functions.

To multiply two differential operators we have the following equation,

$$D_x^i . D_x^j = D_x^{i+j}, \tag{2.21}$$

where $i, j \geq 0$
For the operator D_x, derivational property is given by the Leibniz rule

$$D_x . Q = Q_x + Q D_x, \tag{2.22}$$

where Q is a differentiable function.

Definition 2.3.2. *A pseudo-differential operator of order n is a infinite series*

$$\mathcal{D} = \sum_{i=-\infty}^{n} \mathrm{P}_i[u] D_x^i, \tag{2.23}$$

where $\mathrm{P}_i[u]$ is a differentiable function and the operator D_x^{-1} is the formal inverse of D_x ($D_x . D_x^{-1} = D_x^{-1} . D_x = 1$).

Corollary 2.3.3. *We have the following formula for the operator D_x^{-1},*

$$D_x^{-1} . Q = \sum_{k=0}^{\infty} (-1)^k \mathrm{Q}^k \mathrm{D}_x^{-k-1}. \tag{2.24}$$

Proof. If we multiply (2.22) on both left and the right by D_x^{-1}, it reduces to

$$D_x^{-1}.Q = QD_x^{-1} - D_x^{-1}.Q^{'}D_x^{-1}, \tag{2.25}$$

where $Q^{'} = D_x Q$. If we apply (2.25) to the second term on the right hand side of (2.25), we get

$$D_x^{-1}.Q = QD_x^{-1} - Q^{'}D_x^{-2} + D_x^{-1}.Q^{''}D_x^{-2}.$$

Applying (2.25) to the third term on the right hand side of (2.3) and continuing this process we find

$$D_x^{-1}.Q = QD_x^{-1} - Q^{'}D_x^{-2} + Q^{''}D_x^{-3} - .... = \sum_{k=0}^{\infty}(-1)^k Q^k D_x^{-k-1}. \tag{2.26}$$

□

The reason why we define pseudo-differential operator is that we want to take roots of the differential operators. It is not possible to take roots of differential operators in the set of differential operators but as we will prove we can take roots of any pseudo-differential operator.

Lemma 2.3.4. *Every nonzero pseudo-differential operator of order $n > 0$ has an n-th root.*

Proof. Let $\mathcal{D}$ be a pseudo differential operator such that,

$$\mathcal{D} = \sum_{i=-\infty}^{n} P_i[u]D_x^i \quad , \qquad P_n \neq 0.$$

Then the n-th root of $\mathcal{D}$ will be a first order pseudo-differential operator. Let $\mathcal{D}^{'} = \mathcal{D}^{\frac{1}{n}}$. Then $\mathcal{D}^{'}$ has the form,

$$\mathcal{D}^{'} = (P_n)^{\frac{1}{n}}D_x + Q_0 + Q_{-1}D_x^{-1} + ...$$

if we take the n-th power of $\mathcal{D}^{'}$, we can compare the coefficients of $(\mathcal{D}^{'})^n$ and $\mathcal{D}$ and we can find Q_k for each $k = ..., -2, -1, 0$ in terms of P_i's , $i = ..., -1, 0, ..., n$.

□

Example 2.3.5. *Consider the operator $\mathcal{D} = D_x^2 + u$ which corresponds to the KdV equation. Then the square root of $\mathcal{D}$ is*

$$\mathcal{D}^{\frac{1}{2}} = D_x + \frac{1}{2}uD_x^{-1} - \frac{1}{4}u_x D_x^{-2} + \frac{1}{8}(u_{xx} - u^2)D_x^{-3} + ...$$

2.4 Recursion Operators

It is impossible to find all generalized symmetries of system of evolution equations by using the method in Section (2.2). Hence, in this section we present a new method to find new symmetries of evolution equations using recursion operators. Recursion operators provide a method to find infinite hierarchies of generalized symmetries. On the other hand, it may not be possible to find all symmetries using recursion operator.

Definition 2.4.1. *Let $\triangle$ be a system of differential equations. A recursion operator for $\triangle$ is a linear operator $R : \mathcal{A}^q \to \mathcal{A}^q$ in the space of q-tuples of differential functions that satisfies the following condition, if v_Q is an evolutionary symmetry of $\triangle$ then $v_{\widetilde{Q}}$ is also a symmetry where $\widetilde{Q} = RQ$.*

Hence if we know the recursion operator of a system then we can generate infinitely many symmetries by applying R to a known symmetry. In other words, if Q_0 is a symmetry of the system then $Q_n = R^n Q_0$ for $n = 1, 2, ..,$ are also a symmetries of this system and they constitute an infinite family of symmetries of the system.

Now the question arises : how can we check whether a given differential operator is a recursion operator of the system? To solve this problem, we define Frechet derivatives.

Definition 2.4.2. *The Frechet derivative of $P[u] \in \mathcal{A}^r$ is the differential operator $D_P : \mathcal{A}^q \to \mathcal{A}^r$ which is defined as*

$$D_P(Q) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} P[u + \epsilon Q[u]] \tag{2.27}$$

for any $Q \in \mathcal{A}^q$.

Also we can say that D_P is a $q \times r$ matrix differential operator with entries

$$(D_P)_{\mu\nu} = \sum_J \left(\frac{\partial P_\mu}{\partial u_J^\nu} \right) D_J, \quad \mu = 1, ..., r, \quad \nu = 1, ..., q, \tag{2.28}$$

where the sum is taken over all multi-indices J.

Proposition 2.4.3. *If* $P \in \mathcal{A}^r$ *and* $Q \in \mathcal{A}^q$, *then we have*

$$D_P(Q) = prv_Q(P). \tag{2.29}$$

One can obtain from Proposition (2.4.3) that if $D_\triangle(P) = 0$ then P is a symmetry of the system

$$\triangle_v[u] = \triangle_v(x, u^{(n)}) = 0, \qquad v = 1, ..., l.$$

Theorem 2.4.4. *Suppose* $\triangle[u] = 0$ *is a system of differential equations. If for all solutions* u *of* $\triangle$, $R : \mathcal{A}^q \to \mathcal{A}^q$ *is a linear operator such that*

$$D_\triangle . R = \widetilde{R} . D_\triangle, \tag{2.30}$$

where $\widetilde{\mathbb{R}} : \mathcal{A}^q \to \mathcal{A}^q$ *is a linear differential operator, then* R *is a recursion operator for the system.*

As a special case of Theorem 2.4.4, we consider evolution equations

$$\triangle[u] = u_t - K[u] = 0.$$

Then we have $D_\triangle = D_t - D_K$. Also, in this case we have $\widetilde{R} = R$ and the condition (2.30) is equivalent to

$$R_t = [D_K, R]. \tag{2.31}$$

The equality (2.31) has the form of a Lax equation but in most cases recursion operators are not usual operators appearing in Lax pairs. In Chapter 5, we will concentrate on the usage of recursion operators as Lax operators.

Example 2.4.5. *The operator*

$$R = D_x^2 + 4u + 2u_x D_x^{-1}, \tag{2.32}$$

is a recursion operator for the KdV equation

$$u_t = u_{xxx} + 6uu_x. \tag{2.33}$$

Proof. We will check the condition (2.31). First we find the Frechet derivative of $K[u] = u_{xxx} + uu_x$

$$D_K = D_x^3 + uD_x + u_x, \tag{2.34}$$

then we find the required commutator as

$$[D_K, R] = \frac{2}{3}(u_{xxx} + uu_x) + \frac{1}{3}(u_{xxxx} + uu_{xx} + u_x^2)D_x^{-1}, \tag{2.35}$$

which is consistent with the condition (2.31). □

Hence we can use this recursion operator to find more symmetries of KdV equation. We begin with the space translational symmetry which corresponds to characteristic $Q_0 = u_x$, we find

$$Q_1 = RQ_0 = u_{xxx} + uu_x, \tag{2.36}$$

which corresponds to the characteristic of time translational symmetry. Next we obtain

$$Q_2 = RQ_1 = u_{xxxxx} + \frac{5}{3}uu_{xxx} + \frac{10}{3}u_x u_{xx} + \frac{5}{6}u^2 u_x, \tag{2.37}$$

by continuing this process one can find infinite number of symmetries of the KdV equation.

In the following chapters, instead of D_F we will use the notation F^* for Frechet derivative of differential function F . That is,

$$R_t = [F^*, R] \tag{2.38}$$

will be the condition for R to be a recursion operator of evolution equation

$$u_t = F[u].$$

Chapter 3

Gel'fand-Dikii Formalism

Using the pseudo-differential algebra in Gel'fand-Dikii formalism, given the Lax representation, we are able to obtain classes of nonlinear partial differential equations. It was recently shown that recursion operators can also be constructed by the use of this formalism.

3.1 Lax Representations

Let L be a differential operator of order m and A be a differential operator having coefficients, which are functions of x and t. Consider the Lax representation of the form.

$$L_t = [A, L], \tag{3.1}$$

where commutator $[A, L]$ is defined as

$$[A, L] = A.L - L.A.$$

Let

$$L = D_x^m + u_{m-2}(x,t)D_x^{m-2} + ... + u_1(x,t)D_x + u_0, \tag{3.2}$$

consider $L^{\frac{1}{m}}$ and its fraction $L^{\frac{n}{m}}$, where $n \in \mathbb{Z}$ and $n \neq am$. Let

$$L^{\frac{n}{m}} = \sum_{i=-\infty}^{n} v_i D_x^i = (L^{\frac{n}{m}})_+ + (L^{\frac{n}{m}})_-,$$

where

$$(L^{\frac{n}{m}})_+ = \sum_{i=0}^{n} v_i D_x^i \qquad \textit{and} \qquad (L^{\frac{n}{m}})_- = \sum_{i=-\infty}^{-1} v_i D_x^i.$$

Since $[L, L^{\frac{n}{m}}] = 0$, we have

$$[L, (L^{\frac{n}{m}})_+] = -[L, (L^{\frac{n}{m}})_-]. \tag{3.3}$$

The left hand side of (3.3) is a differential operator of order $\leq n + m - 1$, but the right hand side is series of order $\leq m - 1$. Therefore, there are n number of terms canceling each other and that gives us system of evolution equations for $u_i(x,t)$'s for $i = 0, 1, ..., m-2$. To have a hierarchy of nonlinear system of differential equations at (3.3), define [2]

$$A_n := (L^{\frac{n}{m}})_+, \tag{3.4}$$

and consider

$$L_{t_n} = [A_n, L]. \tag{3.5}$$

As an example for Lax pairs of form, we will give the following (3.5)

Example 3.1.1. *KdV equation has the following two Lax representations. [2]*
i. $L = D^2 + u, \quad A = (L^{\frac{3}{2}})_+,$
ii. $L = (D^2 + u)D_x^{-1}, \quad A = (L^3)_+.$
Firstly, we will show each of these lax pairs gives the KdV equation
i) $L = D^2 + u$**,** $\quad A = (L^{\frac{3}{2}})_+$
We have

$$L_t = [A, L], \tag{3.6}$$

and

$$L^{\frac{1}{2}} = D_x + \frac{1}{2} u D_x^{-1} - \frac{1}{4} u_x D_x^{-2} + ...$$

$$A = D_x^3 + \frac{1}{2}uD_x + \frac{3}{4}u_x, \tag{3.7}$$

using (3.6) *and* (3.7) *we find*

$$[A, L] = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x,$$

and so we have

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x,$$

which is the KdV equation.

ii) $L = (D^2 + u)D_x^{-1}$**,** $\quad A = (L^3)_+$

We have

$$L_t = [A, L], \tag{3.8}$$

and

$$A = D_x^3 + 3uD_x + 3u_x, \tag{3.9}$$

using (3.8) *and* (3.9) *we find*

$$[A, L] = (u_{xxx} + 6uu_x)D_x^{-1},$$

and so we have,

$$u_t = u_{xxx} + 6uu_x,$$

which is the KdV equation.

3.2 Consistent Lax Hierarchies

Consider the Lax equations for $n = 1, 2, ...$

$$L_{t_n} = [A_n, L] \qquad \textit{where} \quad A_n = (L^{\frac{n}{N}})_{\geq k}, \tag{3.10}$$

what type of Lax operator can be used in (3.10) to obtain a consistent evolution equation? Let us consider the Lax operators in the following general form

$$L = u_N D_x^N + u_{N-1} D_x^{N-1} + ... + u_0 + u_{-1} D_x^{-1} + ... \qquad (3.11)$$

where u_k's are functions of x and t.
Firstly to obtain a consistent Lax equation, order of the right hand side of the Lax equation (3.10) should not exceed the order of L, which is N. Since $[L, L^{\frac{n}{N}}] = 0$ we have

$$[L, (L^{\frac{n}{N}})_{\geq k}] = -[L, (L^{\frac{n}{N}})_{<k}], \qquad (3.12)$$

where $(L^{\frac{n}{N}})_{<k} = a_{k-1} D_x^{k-1} + a_{k-2} D_x^{k-2} + ...$
Now let's look at the equation,

$$L_{t_n} = -[L, (L^{\frac{n}{N}})_{<k}] = [u_N D_x^N + ..., a_{k-1} D_x^{k-1}], \qquad (3.13)$$

the right hand side of (3.13) has order at most $N+k-2$ but the left hand side has order N. Therefore, the cases $k = 0, 1, 2$ can produce consistent Lax hierarchies. Hence there are 3 consistent Lax hierarchies that satisfy the Lax equation,

$$L_{t_n} = [A_n, L] \qquad where \quad A_n = (L^{\frac{n}{N}})_{\geq k} \quad k = 0, 1, 2. \qquad (3.14)$$

For each $k = 0, 1, 2$ we have some restrictions on the form of the Lax operator L so that (3.14) is consistent.
To have a consistent Lax pair in each case L should be in the following form, [4]

$$L = c_m D_x^m + c_{m-1} D_x^{m-1} + ... + u_0 + u_{-1} D_x^{-1} + ..., \qquad k = 0 \qquad (3.15)$$

$$L = c_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_0 + u_{-1} D_x^{-1} + ..., \qquad k = 1 \qquad (3.16)$$

$$L = u_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_0 + u_{-1} D_x^{-1} + ..., \qquad k = 2 \qquad (3.17)$$

where c_m and c_{m-1} are functions which do not depend on t.
If number of u_i's are finite, then L should be in the following form, [4]

$$L = c_m D_x^m + c_{m-1} D_x^{m-1} + ... + u_0, \qquad k=0$$
$$L = c_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_0 + D_x^{-1}(u_{-1}), \qquad k=1$$
$$L = u_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_0 + D_x^{-1}(u_{-1}) + D_x^{-2}(u_{-2}) k=2$$

where c_m and c_{m-1} are functions which do not depend on t.
For the case $k=1$ there can be reductions on formula (3.16), and they are given by [4]

$$L = c_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_1 D_x + u_0,$$
$$L = c_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_1 D_x + \lambda,$$

where λ is a constant parameter.
For the case $k=2$ there can be reductions on formula (3.17), and they are given by, [4]

$$L = u_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_1 D_x + u_0 + D_x^{-1}(u_{-1}),$$
$$L = u_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_1 D_x + u_0,$$
$$L = u_m D_x^m + u_{m-1} D_x^{m-1} + ... + u_1 D_x + (\lambda_1 + \lambda_2 x),$$
$$L = u_m D_x^m + u_{m-1} D_x^{m-1} + ... + (\lambda_1 + \lambda_2 x) D_x + \lambda_3,$$

where λ_1, λ_2 and λ_3 are constant parameters. As an example for each case, [4]
i. $k=0$, $L = D_x^2 + u$, $A = (L^{\frac{3}{2}})_{\geqslant 0}$ gives KdV equation
ii. $k=1$, $L = D_x^2 + 2uD_x$, $A = (L^{\frac{3}{2}})_{\geqslant 1}$ gives MKdV equation
iii. $k=2$, $L = u^2 D_x^2$, $A = (L^{\frac{3}{2}})_{\geqslant 2}$ gives Harry Dym equation

In the previous section we derived KdV equation which corresponds to the case $k=0$. Now we will derive MKdV and Harry-Dym equations which correspond to the cases $k=1$ and $k=2$, respectively.

MKdV equation

Let

$$L = D_x^2 + 2uD_x i \tag{3.18}$$

$$L_t = [A, L] \quad \text{where} \quad A = (L^{\frac{3}{2}})_{\geq 1}. \tag{3.19}$$

We find

$$L^{\frac{1}{2}} = D_x + u - \frac{1}{2}(u_x + u^2)D_x^{-1} + ...$$

and

$$A = D_x^3 + 3uD_x^2 + \frac{3}{2}(u_x + u^2)D_x.$$

Then we find

$$[A, L] = (\frac{1}{2}u_{xxx} - 3u^2u_x)D_x = 2u_tD_x.$$

Hence we get the MKdv equation,

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}u^2u_x.$$

Harry-Dym Equation

We have the Lax operator

$$L = w^2D_x^2,$$

and

$$L_t = [A, L] \quad where \quad A = (L^{\frac{3}{2}})_{\geq 2}, \tag{3.20}$$

we find

$$L^{\frac{1}{2}} = wD_x - \frac{1}{2}w_x + ...$$

and

$$A = w^3D_x^3 + \frac{3}{2}w^2w_xD_x^2, \tag{3.21}$$

by using (3.21) in (3.20), we find

$$[A, L] = \frac{1}{2}w^4w_{xxx}D_x^2 = 2ww_t,$$

hence we get

$$w_t = \frac{1}{2}w^3w_{xxx}.$$

which is Harry-Dym eqaution.

Now we will derive some system of differential equations, [4]

1) Boussinesq Equation

Consider

$$L = D_x^3 + uD_x + v,$$

and

$$L_t = [A, L] \quad \text{with} \quad A = (L^{\frac{2}{3}})_{\geq 0},$$

we find

$$A = D_x^2 + \frac{2}{3}u,$$

and we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -u_{xx} + 2v_x \\ -\frac{2}{3}u_{xxx} + v_{xx} - \frac{2}{3}uu_x \end{pmatrix} \tag{3.22}$$

using (3.22) we find that u satisfies the Boussinesq equation

$$u_{tt} + \frac{1}{3}(u_{xxx} + 4uu_x)_x = 0.$$

2) Consider

$$L = D_x^4 + uD_x^2 + vD_x + w,$$

and

$$L_t = [A, L] \quad with \quad A = (L^{\frac{2}{4}})_{\geq 0},$$

we find,

$$A = D_x^2 + \frac{1}{2}u,$$

and we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = \begin{pmatrix} -2u_{xx} + 2v_x \\ -2u_{xxx} + v_{xx} + 2w_x - uu_x \\ -\frac{1}{2}u_{xxxx} + w_{xx} - \frac{1}{2}uu_{xx} - \frac{1}{2}u_x v \end{pmatrix}.$$

3) Consider

$$L = D_x + u + D_x^{-1}v, \tag{3.23}$$

and

$$L_t = [A, L] \quad with \quad A = (L^2)_{\geq 1},$$

we find,

$$A = D_x^2 + 2uD_x,$$

and we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xx} + 2v_x + 2uu_x \\ -v_{xx} + 2(uv)_x \end{pmatrix}.$$

In (3.23) if we choose $v = 0$, then we get Burgers' equation

$$u_t = u_{xx} + 2uu_x,$$

with the following Lax pair

$$L = D_x + u \quad \text{with} \quad A = (L^2)_{\geq 1}.$$

4) Consider

$$L = D_x^2 + uD_x + v + D_x^{-1}w, \tag{3.24}$$

and

$$L_t = [A, L] \quad \text{with} \quad A = (L^{\frac{3}{2}})_{\geq 1},$$

we find,

$$A = D_x^3 + \frac{3}{2}uD_x^2 + \frac{3}{8}(2u_x + u^2 + 4v)D_x,$$

and we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = \begin{pmatrix} \frac{1}{4}u_{xxx} + \frac{3}{2}v_{xx} + 3w_x - \frac{3}{8}u^2u_x + \frac{3}{2}(uv)_x \\ v_{xxx} + \frac{3}{2}u(v)_{xx} + 6u_xv_x + \frac{3}{2}u_xw + 3uw_x + \frac{3}{2}vv_x + \frac{3}{8}u^2v_x \\ w_{xxx} - \frac{3}{2}u(w)_{xx} - \frac{9}{4}u_xw_x - \frac{3}{4}u_{xx}w + \frac{3}{2}(vw)_x + \frac{3}{8}(u^2w)_x \end{pmatrix}. \tag{3.25}$$

If we consider the cases $w = v = 0$ in (3.24) ,the system (3.25) reduces to MKdV equation

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{8}u^2u_x.$$

5) Consider

$$L = uD_x + v + D_x^{-1}w + D_x^{-1}z, \tag{3.26}$$

and

$$L_t = [A, L] \quad \text{with} \quad A = (L^2)_{\geq 2},$$

we find,

$$A = u^2 D_x^2,$$

and we have

$$\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix}_t = \begin{pmatrix} u^2 u_{xx} + 2u^2 v_x \\ u^2 v_{xx} 2u(uw)_x \\ -(u^2 w)_{xx} + 2(u^2 z)_x \\ -(u^2 z)_{xx} \end{pmatrix}.$$

If we choose $z = 0$ in (3.26), then we have the following reduction,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = \begin{pmatrix} u^2 u_{xx} + 2u^2 v_x \\ u^2 v_{xx} + 2u(uw)_x \\ -(u^2 w)_{xx} \end{pmatrix},$$

if $z = w = 0$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u^2 u_{xx} + 2u^2 v_x \\ u^2 v_{xx} \end{pmatrix},$$

if $z = w = 0$ and $v = \lambda x$, we get the partial differential equations,

$$u_t = u^2 u_{xx} + 2\lambda u^2,$$

where λ is a constant.

Chapter 4

A Method to Find Recursion Operator

In chapter 3, we considered the hierarchy

$$L_{t_n} = [A_n, L],$$

where

$$A_n := (L^{\frac{n}{m}})_+.$$

By using this hierarchy it is possible to derive recursion operators of evolution equations. This method was introduced by Gürses, Karasu and Sokolov [2]

Proposition 4.0.1. *For any n*

$$A_{n+m} = L.A_n + R_n, \tag{4.1}$$

where R_n is a differentiable operator and $ord(R_n) \leq m - 1$.

Proof.

$$A_{n+m} = (L.L^{\frac{n}{m}})_+ = (L.[(L^{\frac{n}{m}})_+ + (L^{\frac{n}{m}})_-])_+,$$

then

$$A_{n+m} = L.(L^{\frac{n}{m}})_+ + (L.(L^{\frac{n}{m}})_-)_+ = L.A_n + R_n,$$

since $(L.(L^{\frac{n}{m}})_+)_+ = L.(L^{\frac{n}{m}})_+$ and $R_n = (L.(L^{\frac{n}{m}})_-)_+$. Furthermore, $ordR_n \leq m-1$ because $(L^{\frac{n}{m}})_-$ does not have any differential part. $\square$

Corollary 4.0.2. *For any n*

$$L_{t_{n+m}} = L.L_{t_n} + [R_n, L], \tag{4.2}$$

where R_n is as defined in Proposition (4.0.1)*.*

Proof. From Proposition (4.0.1),

$$L_{t_{n+m}} = [A_{n+m}, L] = [L.A_n + R_n, L] = L.[A_n, L] + [R_n, L] = L.L_{t_n} + [R_n, L]. \quad \square$$

The equation (4.2) is called the recursion relation and R_n is called the remainder. If we write $A_{n+m} = (L^{\frac{n}{m}}.L)_+$, then we get a new formula

$$L_{t_{n+m}} = L.L_{t_n} + [\overline{R_n}, L], \tag{4.3}$$

where $\overline{R_n} = ((L^{\frac{n}{m}})_-L)_+$ is a differential operator and $ord(\overline{R_n}) \leq m-1$.
If we equate the coefficients of powers of D_x in (4.2), we can find R_n in terms of L and L_{t_n} from D_x^i for $i = 2m-2, ..., m-1$. For the case $i = m-2, ..., 0$ the comparison of coefficients gives us the relation

$$\begin{pmatrix} u_0 \\ u_1 \\ . \\ . \\ u_{m-2} \end{pmatrix}_{t_{n+m}} = R \begin{pmatrix} u_0 \\ u_1 \\ . \\ . \\ u_{m-2} \end{pmatrix}_{t_n} \tag{4.4}$$

where R is the *recursion operator*. Also the recursion operators for (4.2) and (4.3) are the same. [2]

4.1 Symmetric and Skew-Symmetric Reduction of a Differential Operator

Definition 4.1.1. *Let L be a differential operator given by*

$$L = \sum_{i=0}^{m} \mathrm{a}_i D_x^i. \tag{4.5}$$

Then its adjoint L^+ is defined as

$$L^+ = \sum_{i=0}^{m} (-D_x)^i.\mathrm{a}_i. \tag{4.6}$$

If $L^+ = L$, in which case $ord(L) = m$ must be even, then (3.4) and Proposition (4.0.1) remains valid, but if $L^+ = -L$ then m should be odd and we take

$$A_{n+2m} = (L^{\frac{n+2m}{m}})_+ = (L^2 L^{\frac{n}{m}})_+. \tag{4.7}$$

In this case we have the following proposition

Proposition 4.1.2. *If $L^+ = -L$ then*

$$A_{n+2m} = L^2 A_n + R_n, \tag{4.8}$$

and we have the recursion relation

$$L_{t_{n+2m}} = L^2.L_{t_n} + [R_n, L] \tag{4.9}$$

where $R_n = (L^2(L^{\frac{n}{m}})_-)_+$ and $ord(R_n) \leq 2ord(L) - 1$. [2]

Also we can use the following recursion relations instead of (4.9)

$$L_{t_{n+2m}} = L.L_{t_n}.L + [\widehat{R_n}, L], \tag{4.10}$$

$$L_{t_{n+2m}} = L_{t_n}.L^2 + [\widetilde{R_n}, L]. \tag{4.11}$$

If L is a differential operator, recursion operators obtained from (4.9), (4.10) and (4.11) are same. However, in pseudo-differential case, they are not equivalent. Let us consider the case $L = MD^{-1}$, where M is a differential operator and define $L^{\dagger} = D_x L^{+} D_x^{-1}$. Now we have a lemma that shows how to find recursion operators in some special cases.

Lemma 4.1.3. *Let $L^{\dagger} = \epsilon L$ where $\epsilon = \pm 1$. Then*

$$R_n = a_{m-1} D_x^{m-1} + \ldots + a_0, \qquad for \quad \epsilon = 1, \tag{4.12}$$

where R_n is defined by Proposition (4.0.1)

$$\widehat{R_n} = a_{2m-1} D_x^{2m-1} + \ldots + a_{-1} D_x^{-1}, \qquad for \quad \epsilon = -1, \tag{4.13}$$

where $\widehat{R_n}$ is defined by (4.10). *[2]*

Now we will derive the recursion operators of some evolution equations from their Lax representations. Firstly, we will find recursion operator of the KdV equation by using two different Lax representations. [2]

1. $L = D^2 + u$,
2. $L = (D^2 + u) D_x^{-1}$.

1) $L = D_x^2 + u$

To find the recursion operator, we use

$$L_{t_{n+2}} = L.L_{t_n} + [R_n, L],$$

we choose R_n as in Proposition (4.0.1) and we take $R_n = a_n D_x + b_n$. Since $L_{t_n} = u_{t_n}$, we have

$$u_{t_{n+2}} = (D_x^2 + u).u_{t_n} + [a_n D_x + b_n, D_x^2 + u]. \tag{4.14}$$

We compare the coefficients in (4.14)

$$a_n = \frac{1}{2} D_x^{-1}(u_{t_n}), \tag{4.15}$$

$$b_n = \frac{3}{4} u_{t_n}, \tag{4.16}$$

then by using (4.15) and (4.16) in (4.14) we find

$$u_{t_{n+2}} = (\frac{1}{4} D_x^2 + u + \frac{1}{2} u_x D_x^{-1}).u_{t_n},$$

which gives us the recursion operator

$$R = D_x^2 + 4u + 2u_x D_x^{-1}.$$

2) $L = D_x + uD_x^{-1}$
We find

$$L^+ = -D_x - D_x^{-1}(u),$$

and

$$L^\dagger = -L,$$

Then we have $\epsilon = -1$ and we will use the recursion relation (4.10) and we will take R_n as (4.13)

$$L_{t_{n+2}} = L.L_{t_n}.L + [\widehat{R_n}, L], \tag{4.17}$$

with $\widehat{R_n} = a_n D_x + b_n + c_n D_x^{-1}$. Then we find

$$a_n = D_x^{-1}(u_{t_n}),$$

$$b_n = u_{t_n},$$

$$c_n = -u_{t_{n_x}} - uD_x^{-1}(u_{t_n}),$$

if we use a_n, b_n and c_n in (4.17) , we get

$$u_{t_{n+2}} = u_{t_{n_{xx}}} + 4uu_{t_n} + 2u_x D_x^{-1}(u_{t_n}),$$

Hence we find,

$$R = D_x^2 + 4u + 2u_x D_x^{-1}.$$

3) We will now find the recursion operator of MKdV equation from its lax representation. Recall that Lax operator of MKdV is

$$L = D_x^2 + 2uD_x.$$

To find the recursion operator, we use

$$L_{t_{n+2}} = L.L_{t_n} + [R_n, L], \tag{4.18}$$

with $R_n = c_n D_x^2 + a_n D_x + b_n$. By replacing R_n into (4.18) we find

$$c_n = D_x^{-1}(u_{t_n}),$$

$$a_n = \frac{3}{2}u_{t_n} - D_x^{-1}(uu_{t_n}) + 2uD_x^{-1}(u_{t_n}),$$

$$b_n = 0.$$

Hence we get

$$2u_{t_n+2} = \frac{1}{2}u_{t_n xx} - 2u^2 u_{t_n} - 2u_x D_x^{-1}(uu_{t_n}),$$

and we find

$$R = D_x^2 - 4u^2 - 4u_x D_x^{-1}(u).$$

4) Recursion operator of the Harry-Dym equation from its Lax representation :

$$L = w^2 D_x^2, \tag{4.19}$$

we use

$$L_{t_{n+2}} = L.L_{t_n} + [R_n, L], \tag{4.20}$$

with $R_n = d_n D_x^3 + c_n D_x^2 + a_n D_x + b_n$. Using R_n and (4.19) in (4.20) we find

$$d_n = w^3 D_x^{-1}(\frac{w_{t_n}}{w^2}),$$

$$c_n = \frac{3}{2}w^2 w_x D_x^{-1}(\frac{w_{t_n}}{w^2}) + \frac{3}{2}ww_{t_n}, \tag{4.21}$$

$$a_n = -2b_{n_x},$$

$$b_{n_{xx}} = 0,$$

Hence we can take $a_n = b_n = 0$. Then, substitutingc_n and d_n into (4.20)

$$w_{t_{n+2}} = \frac{1}{2}w^3 w_{xxx} D_x^{-1}\Big(\frac{w_{t_n}}{w^2}\Big) + \Big(-\frac{1}{2}ww_x w_{t_n x} + \frac{1}{2}ww_{xx}w_{t_n} + \frac{1}{2}w^2 w_{t_n xx}\Big),$$

hence we find

$$R = w^2 D_x^2 - ww_x D_x + ww_{xx} + w^3 w_{xxx} D_x^{-1}\Big(\frac{1}{w^2}\Big),$$

5) Recursion operator of Sawada-Kotera equation [2] :

$$u_t = u_{5x} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x,$$

from its Lax representation

$$L = D_x^3 + uD_x, \qquad A = (L^{\frac{5}{3}})_+.$$

We find

$$L^+ = -D_x^3 - uD_x - u_x,$$

and

$$L^\dagger = -L,$$

Then we have $\epsilon = -1$ and we will use the recursion relation (4.10) and we will take R_n as (4.13)

$$L_{t_{n+6}} = L.L_{t_n}.L + [R_n, L], \tag{4.22}$$

with $R_n = a_n D_x^5 + b_n D_x^4 + c_n D_x^3 + d_n D_x^2 + e_n D_x + f_n$. Then we find

$$a_n = \frac{1}{3}D_x^{-1}(u_{t_n}),$$

$$b_n = \frac{5}{3}u_{t_n},$$

$$c_n = \frac{1}{9}(5uD_x^{-1}(u_{t_n}) + 29u_{t_{n_x}}),$$

$$d_n = \frac{1}{9}(5u_x D_x^{-1}(u_{t_n}) + 14uu_{t_n} + 26u_{t_{n_{xx}}}),$$

$$\begin{aligned} e_n &= \frac{1}{27}(10u_{xx}D_x^{-1}(u_{t_n}) - 2D_x^{-1}(u_{xx}u_{t_n}) - D_x^{-1}(u^2u_{t_n}) + 5u^2D_x^{-1}(u_{t_n}) \\ &\quad +28u_{t_{n_{xxx}}} + 32uu_{t_{n_x}} - 32u_xu_{t_n}), \end{aligned}$$

$$f_n = 0.$$

Hence we find recursion operator

$$\begin{aligned} R &= D_x^6 + 6uD_x^4 + 9u_xD_x^3 + 9u^2D_x^2 + 11u_{xx}D_x^2 + 10u_{xxx}D_x + 12uu_xD_x \\ &\quad +4u^3 + 16uu_{xx} + 6u_x^2 + 5u_{4x} \\ &\quad +u_xD_x^{-1}(2u_{xx} + u^2) + u_tD_x^{-1}. \end{aligned}$$

6) Recursion operator of the DSIII (Drinfeld-Sokolov III) system [2]

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x + 6v_x, \\ v_t &= \quad 2v_{xxx} - 6uv_x. \end{aligned} \quad .$$

from its Lax representation

$$L = (D_x^5 - 2uD_x^3 - 2D_x^3u - 2D_xw - 2wD_x)D_x^{-1}, \qquad A = (L^{\frac{3}{4}})_+,$$

where $w = v - u_{2x}$. We find

$$L^+ = D_x^{-1}(D_x^5 - 2uD_x^3 - 2D_x^3u - 2D_xw - 2wD_x),$$

and

$$L^\dagger = L.$$

Then applying lemma (4.1.3) with $\epsilon = 1$ we use the recursion relation (4.2) and we will take R_n as in (4.12)

$$L_{t_{n+4}} = L.L_{t_n} + [R_n, L], \tag{4.23}$$

with $R_n = a_n D_x^3 + b_n D_x^2 + c_n D_x + d_n$. By replacing R_n and L into (4.23) we find

$$a_n = -D_x^{-1}(u_{t_n}),$$

$$b_n = -4u_{t_n},$$
$$c_n = \frac{1}{2}(6uD_x^{-1}(u_{t_n}) - 11u_{t_{n,x}} - 2D_x^{-1}(uu_{t_n}) - 2D_x^{-1}(v_{t_n})),$$
$$d_{n_x} = \frac{1}{2}(6u_{xx}D_x^{-1}(u_{t_n}) + 10u_x u_{t_n} - 5u_{t_{n,xxx}} + 4uu_{t_{n,x}} - 6v_{t_{n,x}}).$$

Hence we get the equation

$$\begin{pmatrix} u_{t_{n+4}} \\ v_{t_{n+4}} \end{pmatrix} = \mathbb{R} \begin{pmatrix} u_{t_n} \\ v_{t_n} \end{pmatrix}$$

with the recursion operator

$$\mathbb{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \tag{4.24}$$

where

$$\begin{aligned}
R_{11} &= D_x^4 - 8uD_x^2 - 8u_{2x} + 16v - 12u_x D_x + 16u^2 + (12uu_x - 2u_{3x} + 12v_x)D_x^{-1} \\
&\quad + 4u_x D_x^{-1} u, \\
R_{12} &= 10D_x^2 + 8u + 4u_x D_x^{-1}, \\
R_{21} &= 12v_{2x} + 10v_x D_x + (4v_{3x} - 12uv_x)D_x^{-1} + 4v_x D_x^{-1} u, \\
R_{22} &= -4D_x^4 + 16uD_x^2 + 8u_x D_x + 16v + 4v_x D_x^{-1}.
\end{aligned} \tag{4.25}$$

7) Recursion operator of Boussinesq System. [2], Boussinesq equation

$$u_{tt} = -\frac{1}{3}(u_{4x} + 2(u^2)_{xx}),$$

can be represented by the system

$$\begin{aligned} u_t &= v_x, \\ v_t &= -\tfrac{1}{3}(u_{xxx} + 8uu_x). \end{aligned}$$

Its Lax representation is given by

$$L = D_x^3 + 2uD_x + u_x + v, \qquad A = (L^{\frac{2}{3}})_+,$$

We use recursion relation (4.2) and we will take R_n as in (4.0.1)

$$L_{t_{n+3}} = L.L_{t_n} + [R_n, L], \tag{4.26}$$

with $R_n = a_n D_x^2 + b_n D_x + c_n$. By replacing R_n and L into (4.26) we find

$$a_n = \frac{2}{3} D_x^{-1}(u_{t_n}),$$

$$b_n = \frac{1}{3}(5u_{t_n} + D_x^{-1}(v_{t_n})),$$

$$c_n = \frac{1}{9}(6v_{t_n} + 8uD_x^{-1}(u_{t_n}) + 10u_{t_{n,x}}),$$

$$d_{n_x} = \frac{1}{2}(6u_{xx}D_x^{-1}(u_{t_n}) + 10u_x u_{t_n} - 5u_{t_{n,xxx}} + 4uu_{t_{n,x}} - 6v_{t_{n,x}}).$$

Hence we obtain the recursion operator

$$\mathbb{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \tag{4.27}$$

where

$$R_{11} = 3v + 2v_x D_x^{-1},$$
$$R_{12} = D_x^2 + 2u + u_x D_x^{-1},$$
$$R_{21} = -(\tfrac{1}{3}D_x^4 + \tfrac{10}{3}uD_x^2 + 5u_x D_x + 3u_{xx} + \tfrac{16}{3}u^2 + (\tfrac{2}{3}u_{xxx} + \tfrac{16}{3}uu_x)D_x^{-1}),$$
$$R_{22} = 3v + v_x D_x^{-1}.$$

Chapter 5

Recursion Operators in Gelfand-Dikii Formalism

5.1 Recursion Operators as Lax Operators in Gelfand-Dikii Formalism

Let

$$u_t = F(u, u_x, u_{xx}, ...), \tag{5.1}$$

be an integrable nonlinear partial differential equation where u is a function of x and t. Then we know that if

$$R_t = [F^*, R] \tag{5.2}$$

is satisfied then R is the recursion operator of (5.1). Hence, recursion operators are also Lax operators of the evolutionary type partial differential equations. In this chapter we discuss if we can use recursion operators in Gelfand-Dikii formalism to derive the evolution equation itself.

Remark 5.1.1. [4] *If the Lax equation*

$$R_t = [F^*, R] \tag{5.3}$$

is satisfied then the adjoint equation is given by

$$(R^+)_t = [-(F^*)^+, R^+], \tag{5.4}$$

where $(F^*)^+$ and R^+ are both adjoint operators.

The consistent Lax operators has the form

$$L_t = [A_n, L], \qquad A_n = (L^{\frac{n}{m}})_{\geq k}, \tag{5.5}$$

where k is either 0 or 1 or 2. For recursion operators we have F^* instead of $(L^{\frac{n}{m}})_{\geq k}$. Hence we have

$$F^* = (R^{\frac{n}{m}})_{\geq k}, \tag{5.6}$$

for some $k = 0, 1, 2$ we can use R in Gelfand-Dikii formalism. Also the adjoint equation should be consistent, that is we should have

$$(F^*)^+ = -((R^+)^{\frac{n}{m}})_{\geq k}, \tag{5.7}$$

for some $k = 0, 1, 2$.

5.2 Recursion Operators Produce Themselves

Since recursion operators are also Lax operators, we can use them in Proposition 4.0.1 or in Lemma 4.1.3. Hence they should reproduce themselves as recursion operators. That is, using recursion relation (4.2),

$$L_{t_{n+m}} = L.L_{t_n} + [R_n, L] \tag{5.8}$$

we should find $R = L$.

Now we will check (5.6) and (5.7) for the recursion operators of KdV, MKdV, Burgers' and Harry Dym equations. Firstly, we will list all required operators and their adjoints and then we will check if (5.6) and (5.7) are satisfied for any

$k = 0, 1, 2$ or not. If both of them are satisfied we will conclude that, that recursion operator can be used as Lax operator in Gelfand-Dikii formalism. But if any of them is not satisfied we will conclude that, that recursion operator cannot be used as Lax operator in Gelfand-Dikii formalism. Also we will show that recursion operators of these PDEs reproduce themselves.

1) KdV Equation : $u_t = u_{xxx} + 6uu_x$
Here we list all required operators and their adjoints,
i) R
$R = D_x^2 + 4u + 2u_x D_x^{-1}$,
$F^* = D_x^3 + 6uD_x + 6u_x$,
$R^{\frac{3}{2}} = D_x^3 + 6uD_x + 6u_x + (2u_{xx} + 6u^2)D_x^{-1} + ...$
ii) $\mathbf{R^+}$
$R^+ = D_x^2 + 4u - D_x^{-1}(2u_x)$,
$(F^*)^+ = -D_x^3 - 6uD_x$,
$(R^+)^{\frac{3}{2}} = D_x^3 + 6uD_x + ..$
By using the above list, we see that for the Lax equation,

$$F^* = (R^{\frac{3}{2}})_{\geq 0}$$

is satisfied and for the adjoint Lax equation,

$$(F^*)^+ = -((R^+)^{\frac{3}{2}})_{\geq 1}$$

is satisfied and both of these Lax equations are consistent. Hence we can use the recursion operator of KdV in Gelfand-Dikii formalism.

a) Recursion Operator is the Lax operator, L = R

$$L = D_x^2 + 4u + 2u_x D_x^{-1},$$

$$L_t = [A, L] \quad \text{and} \quad A = (L^{\frac{3}{2}})_+. \tag{5.9}$$

We find

$$A = D_x^3 + 6uD_x + 6u_x,$$

If we compare coefficients in (5.9), we find

$$u_t = u_{xxx} + 6uu_x,$$

which is KdV equation.

b) Adjoint Lax equation, $\mathbf{L = R^+}$

$$L = D_x^2 + 4u - D_x^{-1}(2u_x)$$

$$L_t = [A, L] \quad \text{and} \quad A = (L^{\frac{3}{2}})_{\geq 1}i \tag{5.10}$$

we find

$$A = D_x^3 + 6uD_x.$$

If we compare coefficients in (5.10), we find

$$u_t = u_{xxx} + 6uu_x,$$

which is KdV equation.

Now if we take the recursion operator of KDV as a Lax operator, we must find its recursion operator as itself.

c) Recursion Operator Produces Itself, $\mathbf{L = R}$

Recursion operator of KdV is

$$L = D_x^2 + 4u + 2u_x D_x^{-1}$$

we find

$$L^+ = D_x^2 + 4u - 2D_x^{-1}(u_x),$$

and

$$L^\dagger = L.$$

Then applying Lemma (4.1.3) with $\epsilon = 1$ and we use the recursion relation (4.2). Also we will take R_n as in (4.12)

$$L_{t_{n+2}} = L.L_{t_n} + [R_n, L], \qquad R_n = a_n D_x + b_n. \tag{5.11}$$

Then we find

$$a_n = 2D_x^{-1}(u_{t_n}),$$

$$b_n = 4u_{t_n}.$$

If we put a_n and b_n into (5.11), we get

$$u_{t_{n+2}} = u_{t_{n_{xx}}} + 4uu_{t_n} + 2u_x D_x^{-1}(u_{t_n}).$$

Then we find

$$R = D_x^2 + 4u + 2u_x D_x^{-1},$$

which is the initial lax operator.

2) MKdV Equation : $u_t = u_{xxx} - 6u^2 u_x$

Here we list all required operators and their adjoints,

i) R

$R = D_x^2 - 4u^2 - 4u_x D_x^{-1}(u)$,
$F^* = D_x^3 - 6u^2 D_x - 12uu_x$,
$R^{\frac{3}{2}} = D_x^3 - 6u^2 D_x - 12uu_x + (6u^4 - 4uu_{xx} + 2u_x^2)D_x^{-1} + ...$

ii) $\mathbf{R^+}$

$R^+ = D_x^2 + -4u^2 + 4uD_x^{-1}(u_x)$,
$(F^*)^+ = -D_x^3 + 6u^2 D_x$,
$(R^+)^{\frac{3}{2}} = D_x^3 - 6u^2 D_x - 8uu_x + ..$

By using the above list, we see that for the Lax equation

$$F^* = (R^{\frac{3}{2}})_{\geq 0}$$

is satisfied and for the adjoint Lax equation

$$(F^*)^+ = -((R^+)^{\frac{3}{2}})_{\geq 1}$$

is satisfied. Both of these Lax equations are consistent. Hence we can use the recursion operator of MKdV in Gelfand-Dikii formalism.

a) Recursion Operator is the Lax operator, L = R

$$L = D_x^2 - 4u^2 - 4u_x D_x^{-1}(u),$$

$$L_t = [A, L] \quad \text{where} \quad A = (L^{\frac{3}{2}})_+. \tag{5.12}$$

We find

$$A = D_x^3 - 6u^2 D_x - 12uu_x. \tag{5.13}$$

By using (5.12) and (5.13) we find

$$u_t = u_{xxx} - 6u^2 u_x,$$

which is MKdV equation.

b) Adjoint Lax equation, $\mathbf{L} = \mathbf{R}^+$

$$L = D_x^2 + -4u^2 + 4uD_x^{-1}(u_x),$$

$$L_t = [A, L] \quad \text{and} \quad A = (L^{\frac{3}{2}})_{\geq 1}. \tag{5.14}$$

We find

$$A = D_x^3 - 6u^2 D_x.$$

If we compare coefficients in (5.14), we find

$$u_t = u_{xxx} - 6u^2 u_x,$$

which is MKdV equation

c) Recursion Operator Produces Itself, $\mathbf{L} = \mathbf{R}$

Recursion operator of MKdV equation is

$$L = D_x^2 - 4u^2 - 4u_x D_x^{-1}(u),$$

we find

$$L^+ = D_x^2 - 4u^2 + 4uD_x^{-1}(u_x),$$

and

$$L^\dagger = L.$$

Then applying Lemma (4.1.3) with $\epsilon = 1$ we use the recursion relation (4.2) and we will take R_n as in (4.12)

$$L_{t_{n+2}} = L.L_{t_n} + [R_n, L], \tag{5.15}$$

with $R_n = a_n D_x + b_n$. By replacing R_n and L into (5.15) we find

$$a_n = -4D_x^{-1}(uu_{t_n}),$$

$$b_n = -8uu_{t_n}.$$

Hence we get

$$u_{t_n+2} = 2u_{t_nxx} - 8u^2u_{t_n} - 8u_xD_x^{-1}(uu_{t_n}),$$

this yields to

$$R = D_x^2 - 4u^2 - 4u_xD_x^{-1}(u).$$

3) Burgers' Equation : $u_t = u_{xx} + 2uu_x$

Here we list all required operators and their adjoints,

i) R

$R = D_x + u + u_xD_x^{-1}$,

$F^* = D_x^2 + 2uD_x + 2u_x$,

$R^2 = D_x^2 + 2uD_x + (3u_x + u^2) + (2uu_x + u_{xx})D_x^{-1}$

ii) $\mathbf{R^+}$

$R^+ = -D_x + u - D_x^{-1}(u_x)$,

$(F^*)^+ = D_x^2 - 2uD_x$,

$(R^+)^2 = D_x^2 - 2uD_x + (u_x + u^2) + ...$

By using the above list, we see that for the Lax equation

$$F^* = (R^2)_{\geq k}$$

is not satisfied for any $k = 0, 1, 2$ and for the adjoint Lax equation

$$(F^*)^+ = -((R^+)^2)_{\geq k}$$

is not satisfied for any $k = 0, 1, 2$. Hence we cannot use recursion operator of Burgers' equation in Gelfand-Dikii formalism as the Lax operator.

There is an interesting situation here. We have $(F^*)^+ = ((R^+)^2)_{\geq 1}$ but for a consistent Lax representation we should have $(F^*)^+ = -((R^+)^2)_{\geq k}$ for some k. We will check if this interesting case in adjoint case is consistent or not. Firstly we show the inconsistency in Lax equation.

a) Recursion Operator is the Lax operator , L = R

$$L = D_x + u + u_x D_x^{-1},$$

$$L_t = [A, L] \quad \text{and} \quad A = (L^{\frac{3}{2}})_{\geq 1}. \tag{5.16}$$

We find

$$A = D_x^2 + 2uD_x..$$

If we compare coefficients in (5.16), we find

$$u_t = 3u_{xx} + 2uu_x, \tag{5.17}$$

but coefficients of D_x^{-1} gives

$$u_{tx} = u_{xxx} + 2uu_{xx} + 2u_x^2,$$

which is not compatible with (5.17). Also the left hand side of (5.16) include terms with powers D_x^{-2}, D_x^{-3},... but the right hand side does not have terms with these powers.

b) Adjoint Lax equation, L = R$^+$

$$L = -D_x + u - D_x^{-1}(u_x),$$

we do not have $(F^*)^+ = -((L^+)^2)_{\geq k}$ for any k but if we take $A = ((L^+)^2)_{\geq 1}$

$$L_t = [A, L] \quad \text{and} \quad A = (L^{\frac{3}{2}})_{\geq 1},$$

we find

$$u_t = -u_{xx} - 2uu_x,$$

which is not Burgers' equation. Thus, there is a minus difference between adjoint Lax equation and corresponding Gelfand-Dikii formalism.

c) Recursion Operator Produces Itself, L = R

Recursion operator of Burgers's equation is

$$L = D_x + u + u_x D_x^{-1},$$

$$L_{t_{n+1}} = L.L_{t_n} + [R_n, L], \tag{5.18}$$

with $R_n = a_n D_x + b_n$. By replacing R_n and L into (5.18) we find

$$a_n = D_x^{-1}(u_{t_n}),$$

$$b_n = u_{t_n}.$$

Hence we get

$$u_{t_{n+1}} = u u_{t_n} + u_{t_n x} + u_x D_x^{-1}(u_{t_n}),$$

then we find

$$R = D_x + u + u_x D_x^{-1}.$$

4) Harry-Dym Equation : $w_t = w^3 w_{xxx}$

Here we list all required operators and their adjoints,

i) R

$R = w^2 D_x^2 - w w_x D_x + w w_{xx} + w^3 w_{xxx} D_x^{-1}(\frac{1}{w^2})$,
$F^* = w^3 D_x^3 + 3w^2 w_{xxx}$,
$R^{\frac{3}{2}} = w^3 D_x^3 + (w^2 w_{xx} - \frac{1}{2} w w_x^2) D_x + (2w^2 w_{xxx} - w w_x w_{xx} + \frac{1}{2} w_x^3) + \ldots$

ii) $\mathbf{R^+}$

$R^+ = w^2 D_x^2 + 3w w_x D_x + (3w_x^2 + 4w w_{xx}) - \frac{1}{w^2} D_x^{-1}(w^3 w_{xxx})$,
$(F^*)^+ = -w^3 D_x^3 - 9w^2 w_x D_x^2 - (19 w w_x^2 + 9w^2 w_{xx}) D_x - (18 w w_x w_{xx} + 6w_x^3)$,
$(R^+)^{\frac{3}{2}} = w^3 D_x^3 + 6w^2 w_x D_x^2 + (\frac{17}{2} w^2 w_{xx} + 10 w w_x^2) D_x + (\frac{7}{4} w^2 w_{xxx} - \frac{49}{4} w w_x w_{xx}$
$\quad + 4w_x^3) + \ldots$

By using the above list, we see that for the Lax equation

$$F^* = (R^2)_{\geq k}$$

is not satisfied for any $k = 0, 1, 2$ and for the adjoint Lax equation

$$(F^*)^+ = -((R^+)^2)_{\geq k}$$

is not satisfied for any $k = 0, 1, 2$. Hence we cannot use recursion operator of Harry-Dym equation in Gelfand-Dikii formalism as the Lax operator.

Let us show the inconsistency in Lax equation if we take recursion operator as the Lax operator.

a) Recursion Operator is the Lax operator, L = R

$$L = w^2 D_x^2 - w w_x D_x + w w_{xx} + w^3 w_{xxx} D_x^{-1} (\frac{1}{w^2}),$$

$$L_t = [A, L] \quad \text{and} \quad A = (L^{\frac{3}{2}})_{\geq 2}. \tag{5.19}$$

We find

$$A = w^3 D_x^3.$$

If we compare coefficients in (5.19), D_x^2 gives

$$w_t = w^3 w_{xxx}, \tag{5.20}$$

which is Harry-Dym equation, but coefficients of D_x gives

$$-w^4 w_{xxxx} - 4w^3 w_x w_{xxx} = 5w^4 w_{xxxx} + 8w^3 w_x w_{xxx} + 3w^3 w_x w_{xx} - 3w^3 w_{xx}^2,$$

which is not compatible with (5.20).

5.3 A General Type Lax Operator of Order Two

Now we consider a recursion operator of order two

$$L = D_x^2 + \alpha + \beta D_x^{-1}.$$

where α and β are function of x and t

a) If L satisfies the Lax equation,

$$L_{t_n} = [L, A] \quad \text{where} \quad A = (L^{\frac{3}{2}})_+,$$

what conditions α and β must satisfy ? First we find

$$L^{\frac{1}{2}} = D_x + \frac{1}{2}\alpha D_x^{-1} + (\frac{1}{2}\beta - \frac{1}{4}\alpha_x)D_x^{-2} + ...$$

and

$$A = D_x^3 + \frac{3}{2}\alpha D_x + (\frac{3}{4}\alpha_x + \frac{3}{2}\beta).$$

We get,

$$\begin{aligned}
[A, L] &= (\frac{1}{4}\alpha_{xxx} + \frac{3}{2}\beta_{xx} + \frac{3}{2}\alpha\alpha_x) + (\beta_{xxx} + \frac{3}{2}\beta\alpha_x + \frac{3}{2}\alpha\beta_x)D_x^{-1} + \\
&\quad +(\frac{3}{2}\beta\beta_x - \frac{3}{4}\beta\alpha_{xx})D_x^{-2} + (\frac{3}{4}\beta\alpha_{xxx} - \frac{3}{2}\beta\beta_{xx})D_x^{-3} + ... \\
&= \alpha_t + \beta_t D_x^{-1}. \qquad (5.21)
\end{aligned}$$

From the coefficients of D_x^{-2} in (5.21)

$$\beta = \frac{1}{2}\alpha_x \quad \text{when} \quad \beta \neq 0,$$

and we have

$$\alpha_t = \alpha_{xxx} + \frac{3}{2}\alpha\alpha_x.$$

Hence α satisfies the KdV equation and we get $\alpha = u$. Also, the rest of (5.21) is also satisfied.

b) If this pseudo-differential operator has itself as a recursion operator, what conditions α and β must satisfy ? We have the recursion relation,

$$L_{t_{n+2}} = L.L_{t_n} + [R_n, L],$$

with $R_n = a_n D_x + b_n$ and $L_{t_n} = \alpha_{t_n} + \beta_{t_n} D_x^{-1}$. We have

$$\begin{aligned} L.L_{t_n} + [R_n, L] &= (\alpha_{t_n} - 2a_{n_x})D_x^2 + (2\alpha_{t_n x} + \beta_{t_n} - a_{n_{xx}} - 2b_{n_x})D_x \\ &\quad +(\alpha_{t_n xx} + 2\beta_{t_n x} + \alpha\alpha_{t_n} + a_n\alpha_x - b_{n_{xx}}) \\ &\quad +(a_n\beta_x + \beta a_{n_x} + \beta_{t_n xx} + \alpha\beta_{t_n} + \beta\alpha_{t_n})D_x^{-1} \\ &\quad +(\beta b_{n_x} + \beta\beta_{t_n} - \beta a_{n_{xx}} - \beta\alpha_{t_n x})D_x^{-2} + \ldots \\ &= \alpha_{t_{n+2}} + \beta_{t_{n+2}} D_x^{-1}. \end{aligned} \tag{5.22}$$

If we compare the coefficients of D_x^n in (5.22),

$$a_n = \frac{1}{2} D_x^{-1}(\alpha_{t_n}) \quad (D_x^2),$$

$$b_n = \frac{3}{4}\alpha_{t_n} + \frac{1}{2} D_x^{-1}(\beta_{t_n}) \quad (D_x),$$

$$\alpha_{t_{n+2}} = \alpha_{t_n xx} + 2\beta_{t_n x} + \alpha\alpha_{t_n} + a_n\alpha_x - b_{n_{xx}} \quad (D_x^0), \tag{5.23}$$

$$\beta_{t_{n+2}} = a_n\beta_x + \beta a_{n_x} + \beta_{t_n xx} + \alpha\beta_{t_n} + \beta\alpha_{t_n} \quad (D_x^{-1}), \tag{5.24}$$

$$\beta b_{n_x} + \beta\beta_{t_n} - \beta a_{n_{xx}} - \beta\alpha_{t_n x} = 0 \quad (D_x^{-2}). \tag{5.25}$$

If we put a_n and b_n into (5.25), we get

$$\beta = \frac{1}{2}\alpha_x.$$

If we insert a_n, b_n and β into (5.23) we find

$$\alpha_{t_{n+2}} = \alpha_{t_n xx} + \frac{1}{2}\alpha_x D_x^{-1}(\alpha_{t_n}) + \alpha\alpha_{t_n},$$

hence we have

$$\alpha_{t_{n+2}} = (D_x^2 + \alpha + \frac{1}{2}\alpha_x D_x^{-1})\alpha_{t_n}.$$

and we find, $\alpha = u$. Note that (5.24) is also satisfied.

As a result, recursion operators of nonlinear PDE's can be used in Gelfand-Dikii Formalism if they satisfy certain conditions (5.6) and (5.7). That is, if we have an evolution equation

$$u_t = F(u, u_x, u_{xx}, ...), \tag{5.26}$$

and

$$R_t = [F^*, R]$$

is satisfied. Hence R is recursion operator of (5.26). Also If

$$F^* = (R^{\frac{n}{m}})_{\geq k},$$

for some $k = 0, 1, 2$ and

$$(F^*)^+ = -((R^+)^{\frac{n}{m}})_{\geq k},$$

for some $k = 0, 1, 2$, then we can use this recursion operator in Gelfand-Dikii formalism. For the examples we considered, recursion operators of KdV and MKdV equations both satisfies this conditions for some k's. Hence we can use them in Gelfand-Dikii formalism.

However, recursion operators of Burgers' and Harry Dym equations fail to satisfy these conditions. Hence we cannot use them in Gelfand-Dikii formalism.

Chapter 6

Classification Problem

6.1 Classification of Nonlinear PDEs by the Use of Recursion Operators

In this chapter, we will classify all evolution equations that have specific type of recursion operators of order 1. Firstly, we will consider most general case and then we will go through all sub-cases. Mainly there are two ways of classifying the hierarchies of evolution equations, by considering the hierarchies $u_{t_n} = R^n(0)$ or $u_{t_n} = R^n(u_x)$, where R is the recursion operator of the evolution equation, $u_{t_1} = R(0)$ or $u_{t_1} = R(u_x)$ respectively. The criteria for an operator to be a recursion operator of a evolution equation was given in (2.38),

$$R_t = [F^*, R]. \tag{6.1}$$

Hence we will use (6.1) to classify evolution equations that admits some specific type of recursion operators.

6.2 Classification for Burgers' Type Equations

In this section, we will classify 2nd order partial differential equations of the type $u_t = R(0)$ with recursion operator of the first order. Similar classification was considered in thesis by Jon Haggblad [5]. We will consider the case when evolution equation depends on at most 2nd order derivatives and also coefficients of recursion operator will also depend on at most 2nd order derivatives. We will consider evolution equations of order two that admits recursion operators of order one and using (6.1) we will classify them. That is

$$u_t = \beta(u, u_x, u_{xx}),$$

which have recursion operator of the form

$$R = \gamma D_x + \alpha + \beta D_x^{-1}(\rho),$$

where γ, α and ρ are all functions of u, u_x and u_{xx}. This is the most general recursion operator of order 1. In this case, to have a hierarchy of evolution equations and to get evolution equation directly from recursion operator we must have

$$\rho \neq 0,$$

In this case, we have $F^* = \beta_{u_{xx}} D_x^2 + \beta_{u_x} D_x + \beta_u$.
Let's take $B = \beta_{u_{xx}}$, $E = \beta_{u_x}$ and $K = \beta_u$. We have

$$F^* = BD_x^2 + ED_x + K.$$

First we calculate $[F^*, R]$:

$$\begin{aligned}
[F^*, R] &= (2BD_x\gamma - \gamma D_x B)D_x^2 \\
&\quad +(ED_x\gamma + BD_x^2\gamma + 2BD_x\alpha - \gamma D_x E)D_x \\
&\quad +(ED_x\alpha + BD_x^2\alpha + 2B\rho D_x\beta + 2B\beta D_x\rho - \gamma D_x K + \beta\rho D_x(B)) \\
&\quad +\beta D_x^{-1}(D_x(\rho E) - \beta K - D_x^2(\rho B)) \\
&\quad +(\beta K + ED_x\beta + BD_x^2\beta)D_x^{-1}(\rho) \\
&= R_t \\
&= D_t\gamma D_x + D_t\alpha + D_t\beta D_x^{-1}(\rho) + \beta D_x^{-1}(D_t(\rho)).
\end{aligned}$$

Hence we get 5 equations for γ, α, β and ρ :

$$2BD_x\gamma - \gamma D_x B = 0, \tag{6.2}$$

$$ED_x\gamma + BD_x^2\gamma + 2BD_x\alpha - \gamma D_x E = D_t\gamma, \tag{6.3}$$

$$ED_x\alpha + BD_x^2\alpha + 2B\rho D_x\beta + B\beta D_x\rho - \gamma D_x K + \beta D_x(\rho B)) = D_t\alpha, \tag{6.4}$$

$$\beta K + ED_x\beta + BD_x^2\beta = D_t\beta, \tag{6.5}$$

$$D_x(\rho E) - \beta K - D_x^2(\rho B) = D_t\rho. \tag{6.6}$$

From equation (6.2) we have

$$2BD_x\gamma - \gamma D_x B = 0,$$

$$2B(\gamma_u u_x + \gamma_{u_x} u_{xx} + \gamma_{u_{xx}} u_{xxx}) = \gamma(B_u u_x + B_{u_x} u_{xx} + B_{u_{xx}} u_{xxx}). \tag{6.7}$$

We equate the coefficients of u_{xxx}, u_{xx}, u_x in (6.7) .
(u_{xxx})

$$2B\gamma_{u_{xx}} = \gamma B_{u_{xx}} \qquad \Longrightarrow \qquad \gamma = w(u, u_x)B^{\frac{1}{2}}.$$

If we insert γ into equation (6.7), we get
(u_{xx})

$$w_{u_x} = 0 \qquad \Longrightarrow \qquad \gamma = w(u)B^{\frac{1}{2}},$$

(u_x)

$$w_u = 0 \qquad \Longrightarrow \qquad \gamma = wB^{\frac{1}{2}} \qquad w: \text{constant}.$$

We have,
$\beta = \beta(u, u_x, u_{xx}), \qquad \alpha = \alpha(u, u_x, u_{xx}),$
$\rho = \rho(u, u_x, u_{xx}),, \ \gamma = wB(u, u_x, u_{xx})^{\frac{1}{2}}.$
Now we look at equation (6.3)

$$ED_x\gamma + BD_x^2\gamma + 2BD_x\alpha - \gamma D_xE = D_t\gamma,$$

and we compare the coefficients derivatives
(u_{xxx}^2)

$$\beta_{u_{xx}}\gamma_{u_{xx}u_{xx}} = \gamma_{u_{xx}}\beta_{u_{xx}u_{xx}},$$

that equals to

$$\frac{1}{2}\beta_{u_{xx}}^{\frac{1}{2}}\beta_{u_{xx}u_{xx}u_{xx}} = \frac{3}{4}\beta_{u_{xx}}^{-\frac{1}{2}}\beta_{u_{xx}u_{xx}}, \tag{6.8}$$

$\beta_{u_{xx}} = 0$ and $\beta_{u_{xx}u_{xx}} = 0$ are solutions to this equation. Hence there are 3 cases,
i. $\beta_{u_{xx}u_{xx}} \neq 0$,
ii. $\beta_{u_{xx}} \neq 0$ but $\beta_{u_{xx}u_{xx}} = 0$,
iii. $\beta_{u_{xx}} = 0$.

6.3 Case 1: $\beta_{u_{xx}u_{xx}} \neq 0$

By solving the equation (6.8) we find

$$\beta = \frac{-4}{(c_1(u, u_x)u_{xx} + c_2(u, u_x)c_1(u, u_x))} + c_3(u, u_x), \tag{6.9}$$

and the coefficient of next highest derivative of u gives
(u_{xxx})

$$\alpha_{u_{xx}} = \frac{w}{2}\frac{\beta_{u_{xx}u_{xx}}}{\beta_{u_{xx}u_{xx}}^{\frac{1}{2}}},$$

by using (6.9) in here we find α

$$\alpha = \frac{2w(c_2(u,u_x)_{u_x} - c_1(u,u_x)_u u_x)}{(c_1(u,u_x)u_{xx} + c_2(u,u_x))^2} + c_4(u,u_x).$$

But we cannot go further from here and for now we have,

$$\beta = \frac{-4}{(c_1(u,u_x)u_{xx} + c_2(u,u_x)c_1(u,u_x)} + c_3(u,u_x),$$

$$\alpha = \frac{2w(c_2(u,u_x)_{u_x} - c_1(u,u_x)_u u_x)}{(c_1(u,u_x)u_{xx} + c_2(u,u_x))^2} + c_4(u,u_x),$$

$$\gamma = \frac{2w}{c_1(u,u_x)u_{xx} + c_2(u,u_x)},$$

$$\rho = \rho(u,u_x,u_{xx}).$$

To be able to solve all these equations we make some assumptions $c_1 = 2$, $c_2 = 0$, $c_4 = 0$. Then γ, α, β, ρ reduces to,
$\gamma = \frac{w}{u_{xx}}$, $\quad \alpha = 0$, $\quad \beta = -\frac{1}{u_{xx}} + c_3(u,u_x)$, $\quad \rho = \rho(u,u_x,u_{xx})$
with $B = \dfrac{1}{u_{xx}^2}$, $E = c_3(u,u_x)_{u_x}$ and $K = c_3(u,u_x)_u$.
From the rest of the equation (6.3), we get

$$c_{3_{uu_x}} u_x u_{xx} + c_{3_u} u_{xx} = 0,$$

and

$$c_{3_{uu}} = 0.$$

By solving them we get

$$c_3(u,u_x) = c_5 \frac{u}{u_x} + c_6(u_x) \qquad c_5 : \text{constant}.$$

Equation (6.6) gives
(u_{xxxx})

$$\rho = c_7(u,u_x)u_{xx},$$

(u_{xxx})

$$\rho = c_7(u_x)u_{xx},$$

(u_{xx})

$$\frac{c_5 c_7(u_x)}{u_x} + c_{7_{u_x u_x}} + c_5 c_{7_{u_x}} = 0. \tag{6.10}$$

We cannot solve (6.10) but equation (6.4) reduces to
(u_{xxx})

$$c_7(u_x) = 0 \qquad \Longrightarrow \qquad \rho = 0.$$

This is a contradiction with the assumption $\rho \neq 0$ so we move to the next case.

6.4 Case 2: $\beta_{u_{xx}u_{xx}} = 0$, $\beta_{u_{xx}} \neq 0$

In this case we have

$$\beta = c(u, u_x)u_{xx} + d(u, u_x).$$

Lemma 6.4.1. *If $\beta = c(u, u_x)u_{xx} + d(u, u_x)$, then we have $\rho = \rho(u)$.*

Proof. We use the equation (6.6)

$$D_x(\rho E) - \beta K - D_x^2(\rho B) = D_t\rho.$$

We have, $B = c$, $E = c_{u_x}u_{xx} + d_{u_x}$ and $K = c_u u_{xx} + d_u$ and we compare the coefficients of derivatives of u in (6.4),
(u_{xxxx})

$$-(\rho c)_{u_{xx}} = c\rho_{u_{xx}} \qquad \Longrightarrow \qquad \rho_{u_{xx}} = 0,$$

(u_{xxx})

$$\rho c_{u_x} = -(c\rho)_{u_x} \qquad \Longrightarrow \qquad \rho_{u_x} = 0.$$

Hence we have $\rho = \rho(u)$. $\square$
In which case we have $\gamma = wc(u, u_x)^{\frac{1}{2}}$

and we start with (6.3)

$$ED_x\gamma + BD_x^2\gamma + 2BD_x\alpha - \gamma D_x E = D_t\gamma,$$

(u_{xxx})

$$\alpha_{u_{xx}} = \frac{wc_{u_x}}{2c^{\frac{1}{2}}} \qquad \Longrightarrow \qquad \alpha = \frac{wc(u,u_x)_{u_x}}{2c^{\frac{1}{2}}}u_{xx} + c_1(u,u_x),$$

(u_{xx}^2)

$$-\frac{3}{4}w\frac{c_{u_x}^2}{c^{\frac{1}{2}}} + \frac{w}{2}c^{\frac{1}{2}}c_{u_xu_x} = 0, \tag{6.11}$$

$c_{u_x} = 0$ is a solution to (6.11), but if $c_{u_x} \neq 0$ we get,

$$c(u,u_x) = \frac{4}{(c_3(u) - c_2(u)u_x)^2},$$

(u_{xx})

$$\begin{aligned} &-\frac{16wc_2(u)c_3(u)_u u_x}{(c_3(u)-c_2(u)u_x)^5} + \frac{16wc_2(u)c_2(u)_u u_x^2}{(c_3(u)-c_2(u)u_x)^5} \\ &+\frac{8c_1(u,u_x)_{u_x}}{(c_3(u)-c_2(u)u_x)^2} + \frac{16wc_2(u)_u u_x}{(c_3(u)-c_2(u)u_x)^4} - \frac{2wd(u,u_x)_{u_xu_x}}{c_3(u)-c_2(u)u_x} = 0. \end{aligned} \tag{6.12}$$

But we cannot solve (6.12) so we now consider two subcases,
i. $c_u = 0$,
ii. $c_{u_x} = 0$.

6.4.1 Case i . $c_u = 0$

In this case, $\beta = c(u_x)u_{xx} + d(u,u_x)$, $B = c$, $E = c_{u_x}u_{xx} + d_{u_x}$ and $K = d_u$. Also $\gamma = wc(u_x)^{\frac{1}{2}}$.
From equation (6.3)
(u_{xxx})

$$\alpha_{u_{xx}} = \frac{w}{2}\frac{c_{u_x}}{c^{\frac{1}{2}}} \qquad \Longrightarrow \qquad \alpha = \frac{w}{2}\frac{c(u,u_x)_{u_x}}{c^{\frac{1}{2}}}u_{xx} + c_1(u,u_x),$$

(u_{xx}^2)

$$-\frac{3}{4}w\frac{c_{u_x^2}}{c^{\frac{1}{2}}}+\frac{w}{2}c^{\frac{1}{2}}c_{u_xu_x}=0. \tag{6.13}$$

By solving (6.13) we get

$$c(u_x)=\frac{4}{(c_3-c_4u_x)^2} \qquad c_3,c_4=\text{constant},$$

(u_{xx})

$$\frac{4c_1u_x}{(c_3-c_4u_x)^2}-\frac{wd_{u_xu_x}}{c_3-c_4u_x}=0. \tag{6.14}$$

Solving (6.14) gives

$$c_1=\frac{w}{4}(c_3-c_4u_x)+\frac{w}{4}c_4d(u,u_x)+c_5 \qquad c_5=\text{constant}, \tag{6.15}$$

we can drop constant term c_5 in (6.15) because in all equations we only have $D_x\alpha$. Hence we have

$$\beta=\frac{4u_{xx}}{c_3-c_4u_x}+d(u,u_x),$$

$$\alpha=\frac{2wc_u}{(c_3-c_4u_x)^2}u_{xx}+\frac{w}{4}(c_3-c_4u_x)d(u,u_x)_{u_x}+\frac{wc_4}{4}d(u,u_x).$$

Now we consider the equation (6.6) and analyze the coefficients of derivatives of u.

(u_{xx})

$$d_{u_xu_x}=8\frac{\rho_u}{\rho(c_3-c_4u_x)^2}-8\frac{c_4\rho_uu_x}{\rho(c_3-c_4u_x)^3}. \tag{6.16}$$

By solving (6.16) we get

$$d(u,u_x)=c_5(u)u_x+c_6(u)+\frac{4c_3}{c_4^2(c_3-c_4u_x)}\frac{\rho(u)_u}{\rho(u)}. \tag{6.17}$$

In (6.17), we suppose $\rho(u)\neq 0$

(u_x)

$$\rho_ud_{u_x}u_x+d_{u_xu}\rho u_x-\rho d_u-\rho\frac{4}{(c_3-c_4u_x)^2}u_x^2=\rho_ud, \tag{6.18}$$

replacing (6.17) into (6.18) we get

$$\frac{4}{c_4^2}\rho_{uu}+(\rho c_6)_u=0,$$

that reduces to

$$\rho_u + \frac{c_4^2}{4} c_6 \rho = c_7. \tag{6.19}$$

To find ρ in (6.19) we make the substitution

$$\rho = h(u) e^{-\xi} \qquad \text{where} \quad \xi = \frac{c_4^2}{4} \int c_6(u)\, \mathrm{d}u,$$

then we find ρ as,

$$\rho = e^{-\frac{c_4^2}{4} \int c_6(u)\, \mathrm{d}u} . \left(\int c_7 e^{\frac{c_4^2}{4} \int c_6(u)\, \mathrm{d}u} \mathrm{d}u + c_8 \right),$$

Now we have,

$$\gamma = \frac{2w}{c_3 - c_4 u_x},$$

$$\beta = \frac{4u_{xx}}{(c_3 - c_4 u_x)^2} - \frac{c_4 c_6(u) u_x}{c_3 - c_4 u_x} + c_5(u) u_x + \frac{4 c_3 c_7 u_x}{c_4^2 (c_3 - c_4 u_x)} . \frac{e^{\frac{c_4^2}{4} \int c_6(u)\, \mathrm{d}u}}{\int c_7 e^{\frac{c_4^2}{4} \int c_6(u)\, \mathrm{d}u} \mathrm{d}u + c_8},$$

$$\begin{aligned} \alpha \quad &= \quad \frac{2wc_4}{(c_3 - c_4 u_x)^2} u_{xx} + \frac{w}{4} \left(c_3 c_5(u) - \frac{(c_3 + c_4 u_x) c_4 c_6(u)}{c_3 - c_4 u_x} \right. \\ &\left. + \frac{8 c_3 c_7}{c_4 (c_3 - c_4 u_x)} . \frac{e^{\frac{c_4^2}{4} \int c_6(u)\, \mathrm{d}u}}{\int c_7 e^{\frac{c_4^2}{4} \int c_6(u)\, \mathrm{d}u} \mathrm{d}u + c_8} \right), \end{aligned}$$

$$\rho = e^{-\frac{c_4^2}{4} \int c_6(u)\, \mathrm{d}u} . \left(\int c_7 e^{\frac{c_4^2}{4} \int c_6(u)\, \mathrm{d}u} \mathrm{d}u + c_8 \right).$$

We make the substitution

$$c_6 \to \frac{4}{c_4^2} \frac{c_6(u)_{uu}}{c_6(u)_u}.$$

Then α, β, γ, ρ reduces to,

$$\gamma = \frac{2w}{c_3 - c_4 u_x},$$

$$\beta = \frac{4u_{xx}}{(c_3 - c_4u_x)^2} - \frac{4}{c_4(c_3 - c_4u_x)}\frac{c_6(u)_{uu}}{c_6(u)_u}u_x + c_5(u)u_x + \frac{4c_3c_7c_6(u)_u}{c_4^2(c_3 - c_4u_x)(c_7c_6(u) + c_8)},$$

$$\alpha = \frac{2wc_4}{(c_3 - c_4u_x)^2}u_{xx} + \frac{w}{4}(c_3c_5(u) - \frac{4(c_3 + c_4u_x)}{c_4(c_3 - c_4u_x)}\frac{c_6(u)_{uu}}{c_6(u)_u} + \frac{4c_3c_7c_6(u)_u}{c_4(c_3 - c_4u_x)(c_7c_6(u) + c_8)},$$

$$\rho = \frac{c_7c_6(u) + c_8}{c_6(u)_u}.$$

We look at the equation (6.4) (u_{xxx})

$$\frac{32\rho}{(c_3 - c_4u_x)^3} = 0 \qquad \Longrightarrow \qquad \rho = 0.$$

This is a contradiction with the assumption $\rho \neq 0$ and we move to the next case.

6.4.2 Case ii . $c_{u_x} = 0$

In this case, $\beta = c(u)u_{xx} + d(u, u_x)$ $B = c(u)$, $E = d(u, u_x)_{u_x}$ and $K = c(u, u_x)_u u_{xx} + d(u, u_x)_u$. Also $\gamma = wc(u)^{\frac{1}{2}}$.
From equation (6.3)
(u_{xxx})

$$2c\alpha_{u_{xx}} = 0 \qquad \Longrightarrow \qquad \alpha = \alpha(u, u_x),$$

(u_{xx})

$$2c\alpha_{u_x} = wc^{\frac{1}{2}}d_{u_xu_x} \qquad \Longrightarrow \qquad \alpha = \frac{w}{2c^{\frac{1}{2}}}d(u, u_x)_{u_x} + c_1(u), \tag{6.20}$$

Now we insert (6.20) into (6.3) and continue with
(u_x)

$$d = -\frac{1}{2}c_u u_x^2 + \frac{cc_{uu}}{c_u}u_x^2 + \frac{4}{w}\frac{c^{\frac{3}{2}}}{c_u}c_{1_u}u_x. \tag{6.21}$$

In (6.21) we suppose $c_u \neq 0$ and later we will consider the case $c_u = 0$. Hence we have

$$\gamma = wc(u)^{\frac{1}{2}},$$

$$\alpha = -\frac{w}{2}\frac{c(u)_u}{c(u)^{\frac{1}{2}}}u_x + w\frac{c(u)^{\frac{1}{2}}c(u)_{uu}}{c(u)_u}u_x + 2\frac{c(u)c_1(u)_u}{c(u)_u} + c_1(u),$$

$$\beta = c(u)u_{xx} - \frac{1}{2}c(u)_u u_x^2 + \frac{c(u)c(u)_{uu}}{c(u)_u}u_x^2 + \frac{4}{w}\frac{c(u)^{\frac{3}{2}}c_1(u)_u}{c(u)_u}u_x,$$

$$\rho = \rho(u).$$

Now we look at the equation (6.6)

(u_{xx})

$$-3c_u\rho + 2\rho\frac{cc_{uu}}{c_u} - 2c\rho_u = 0, \tag{6.22}$$

$\rho = 0$ is a solution to (6.22) but first we assume $\rho \neq 0$. Hence we find

$$\rho = \frac{c_2 c(u)_u}{c(u)^{\frac{3}{2}}} \qquad c_2 = \text{constant},$$

the rest of (6.6) is fully satisfied and next we look at the equation (6.4)

(u_{xxx})

$$w = 2c_2,$$

(u_{xx})

$$-2cc_{1_u} + 4\frac{c^2}{c_u^2}c_{1_u}c_{uu} - 4\frac{c^2}{c_u}c_{1_{uu}} = 0, \tag{6.23}$$

(6.23) is linear in c_1, hence we find

$$c_1(u) = c_4 c(u)^{\frac{1}{2}} + c_5 \qquad c_4, c_5 = constant.$$

Finally we have

$$\gamma = 2c_2 c(u)^{\frac{1}{2}},$$

$$\alpha = -\frac{w}{2}\frac{c(u)_u}{c(u)^{\frac{1}{2}}}u_x + w\frac{c(u)^{\frac{1}{2}}c(u)_{uu}}{c(u)_u}u_x + 2c_4 c(u)^{\frac{1}{2}} + c_5,$$

$$\beta = c(u)u_{xx} - \frac{1}{2}c(u)_u u_x^2 + \frac{c(u)c(u)_{uu}}{c(u)_u}u_x^2 + \frac{c_4}{c_2}c(u)u_x,$$

$$\rho = \frac{c_2 c(u)_u}{c(u)^{\frac{3}{2}}}.$$

Hence we get the equation

$$u_t = c(u)u_{xx} - \frac{1}{2}c(u)_u u_x^2 + \frac{c(u)c(u)_{uu}}{c(u)_u}u_x^2 + \frac{c_4}{c_2}c(u)u_x, \tag{6.24}$$

with the recursion operator

$$R = 2c_2 c(u)^{\frac{1}{2}} D_x + 2c_4 c(u)^{\frac{1}{2}} + \Big(2c_2 \frac{c(u)^{\frac{1}{2}} c(u)_{uu}}{c(u)_u} - c_2 \frac{c(u)_u}{c(u)^{\frac{1}{2}}}\Big) u_x + u_t D_x^{-1} \Big(\frac{c_2 c(u)_u}{c(u)^{\frac{3}{2}}}\Big).$$

6.4.2.1 If $c_u = 0$

In this case, $\beta = cu_{xx} + d(u, u_x)$, $B = c$, $E = d(u, u_x)_{u_x}$ and $K = d(u, u_x)_u$. Also γ is a constant since B is constant. From equation (6.3)

$$2cD_x\alpha - \gamma D_x d_{u_x} = 0.$$

Then we have

$$2c(\alpha_u u_x + \alpha_{u_x} u_{xx}) + \alpha_{u_{xx}} u_{xxx}) = \gamma(d_{u_x u} u_x + d_{u_x u_x} u_{xx}) = 0,$$

and so

(u_{xxx})

$$\alpha_{u_{xx}} = 0 \qquad \Longrightarrow \qquad \alpha = \alpha(u, u_x),$$

(u_{xx})

$$2c\alpha_{u_x} = \gamma d_{u_x u_x} \qquad \Longrightarrow \qquad \alpha = \frac{\gamma}{2c} d(u, u_x)_{u_x} + d_1(u),$$

(u_x)

$$2c\alpha_u = \gamma d_{u_x u} \qquad \Longrightarrow \qquad \alpha = \frac{\gamma}{2c} d(u, u_x)_{u_x} + d_1 \qquad d_1 : \text{constant.} \tag{6.25}$$

Now we solve equation (6.6), if we replace (6.25) into (6.6) we get

$$(\rho d_{u_x})_u u_x + (\rho d_{u_x})_{u_x} u_{xx} - \rho d_u - c\rho_{uu} u_x^2 - c\rho_u u_{xx} = \rho_u(cu_{xx} + d),$$

and so

(u_{xx})

$$\rho d_{u_x u_x} = 2c\rho_u \qquad \Longrightarrow \qquad d_{u_x u_x} = 2c \frac{\rho(u)_u}{\rho(u)}, \tag{6.26}$$

In (6.26) we assume $\rho \neq 0$, we have

$$d(u, u_x) = c\frac{\rho_u}{\rho}u_x^2 + c_1(u)u_x + c_2(u), \tag{6.27}$$

replacing (6.27) into (6.6) again, we get

$$(\rho c_2(u))_u = 0 \qquad \Longrightarrow \qquad \rho = \frac{c_3}{c_2(u)} \qquad c_3 : \text{constant}. \tag{6.28}$$

In (6.28) we assume $c_2(u) \neq 0$, we have

$$d(u, u_x) = -c\frac{c_2(u)_u}{c_2(u)}u_x^2 + c_1(u)u_x + c_2(u).$$

Hence we get the following equalities :

$$\gamma = constant,$$

$$\beta = cu_{xx} - c\frac{c_2(u)_u}{c_2(u)}u_x^2 + c_1(u)u_x + c_2(u),$$

$$\alpha = -\gamma\frac{c_2(u)_u}{c_2(u)}u_x + \frac{\gamma}{2c}c_1(u) + c_2(u),$$

$$\rho = \frac{c_3}{c_2(u)}.$$

Finally we solve the equation (6.4)
(u_{xxx})

$$-c\gamma\frac{c_2(u)_u}{c_2(u)} + 2c\frac{c_3}{c_2(u)} = -c\gamma\frac{c_2(u)_u}{c_2(u)} \qquad \Longrightarrow \qquad c_3 = 0 \qquad \Longrightarrow \qquad \rho = 0,$$

this is a contradiction because we have assumed $\rho \neq 0$ at first.

Now we go back to the case, $c_2(u) = 0$. In this case, we have

$$\gamma = constant,$$

$$\beta = cu_{xx} + c\frac{\rho(u)_u}{\rho(u)}u_x^2 + c_1(u)u_x,$$

$$\alpha = \gamma \frac{\rho(u)_u}{\rho(u)} u_x + \frac{\gamma}{2c} c_1(u) + c_2(u),$$

$$\rho = \rho(u) \neq 0.$$

At last we solve the equation (6.4)
(u_{xxx})

$$2c^2 \rho = 0 \qquad \Longrightarrow \qquad \rho = 0,$$

which is a contradiction to the assumption $\rho \neq 0$ and so we analyze the next case.

6.5 Case 3: $\beta_{u_{xx}} = 0$

In this case we have $\beta = \beta(u, u_x)$, $B = 0$ so the equations (6.2), (6.3), (6.4), (6.5) and (6.6) reduces to

$$ED_x\gamma - \gamma D_x E = D_t\gamma, \tag{6.29}$$

$$ED_x\alpha - \gamma D_x K = D_t\alpha, \tag{6.30}$$

$$\beta K + ED_x\beta = D_t\beta, \tag{6.31}$$

$$D_x(\rho E) - \beta K = D_t\rho. \tag{6.32}$$

To be able to solve these equations, we assume γ has the form

$$\gamma = m(u, u_x)u_{xx} + n(u, u_x).$$

Equation (6.29) gives
(u_{xx}^2)

$$2m\beta_{u_x u_x} = 0,$$

then $m = 0$ or $\beta_{u_x u_x} = 0$ but in either case we have $\beta_{u_x u_x} = 0$. Hence we take

$$\beta = c_1(u)u_x + c_2(u),$$

and continue with equation (6.29). Also to go further we assume

$$m(u, u_x) = m_1(u)u_x + m_2(u),$$

and continue to analyze the coefficients of derivatives of u : $(u_{xx}u_x^2)$

$$m_1 c_{1_u} = 0, \tag{6.33}$$

$(u_{xx}u_x)$

$$m_{1_u} c_2 + 2m_1 c_{2_u} = 0, \tag{6.34}$$

(6.33) and (6.34) yield to $c_{1_u} = 0$ because $m \neq 0$
(u_{xx})

$$(c_2 m_2)_u = 0,$$

to be able to go further we assume

$$n(u, u_x) = n_1(u)u_x + n_2(u),$$

and we have
(u_x^3) : $m_1 c_{2_{uu}} = 0$
(u_x^2) : $m_2 c_{2_{uu}} = 0$
(u_x) : $(n_1 c_2)_u = 0$
(u_x^0) : $c_2 n_{2_u} = 0$
$c_2 = 0$ satisfies all these equations so we have

$$\beta = c_1 u_x, \tag{6.35}$$

and the rest of equation (6.29) is fully satisfied. Also equation (6.30), (6.31) and (6.32) are also fully satisfied. Actually any operator has the form $R = \gamma D_x + \alpha + \beta D_x^{-1}(\rho)$ is the recursion opearator of the evolution equation (6.35). If $c_2 \neq 0$, by solving the rest we get

$$\beta = c_1 u_x + c_3 u,$$

and

$$\gamma = \frac{c_4}{u^2} u_{xx} + \frac{c_5}{u} u_{xx} + \frac{c_6}{u} u_x + c_7 u + c_8.$$

Moreover, solving equations (6.30) and (6.32) we get

$$\rho = -c_1 u_x - c_3 u,$$

$$\alpha = -c_1 u_x - c_3 u.$$

Hence we get the evolution equation

$$u_t = c_1 u_x + c_3 u,$$

with the recursion operator

$$R = (\frac{c_4}{u^2}u_{xx} + \frac{c_5}{u}u_{xx} + \frac{c_6}{u}u_x + c_7 u + c_8)D_x + (c_1 u_x + c_3 u) + u_t D_x^{-1}(c_1 u_x + c_3 u).$$

In [6], there are four partial differential equations that have recursion operators of type

$$R = \gamma D_x + \alpha + \beta D_x^{-1}(\rho).$$

1) Burgers' Equation

$$u_t = u_{xx} + 2uu_x \quad \text{with} \quad R = D_x + u + u_x D_x^{-1}$$

2)Potential Burgers' Equation

$$u_t = u_{xx} + u_x^2 \quad \text{with} \quad R = D_x + u_x$$

2) Diffusion Equation

$$u_t = u^2 u_{xx} \quad \text{with} \quad R = uD_x + u^2 u_{xx} D_x^{-1}\Big(\frac{1}{u^2}\Big)$$

4) Nonlinear Diffusion Equation

$$u_t = D_x\Big(\frac{u_x}{u^2}\Big) \quad \text{with} \quad R = \frac{1}{u}D_x - 2\frac{u_x}{u^2} - D_x\Big(\frac{u_x}{u^2}\Big)D_x^{-1}$$

Diffusion equation and nonlinear diffusion equation can be obtained by the recursion operators of type (6.25).

• Diffusion equation can be obtained by choosing $c(u) = u^2$, $c_2 = \dfrac{1}{2}$ and $c_4 = 0$.

• Nonlinear diffusion equation can be obtained by choosing $c(u) = \dfrac{1}{u^2}$, $c_2 = \dfrac{1}{2}$ and $c_4 = 0$.

• Potential Burgers' equation and Burgers' equation belong to the case $u_t = R(u_x)$. But we we did not consider these types of equations.

Chapter 7

Conclusion

We studied symmetries of nonlinear partial differential equations and stated a criteria for an operator to be a recursion operator of a given nonlinear partial differential equations. In this thesis, we focused on recursion operators in the framework of Gel'fand-Dikii Formalism and used the recursion operators classification of evolution equations. We provided consistent Lax hierarchies and we presented Lax operators for each consistent case. We studied a method to construct recursion operators of evolution equations from the Gel'fand-Dikii Formalism. Then we find out whether we can use recursion operators as Lax operators in Gel'fand-Dikii Formalism or not. In some cases, answer is affirmative but in some cases, answer is negative, because there are certain conditions for both recursion operator and evolution equation be satisfied.

We stated a classification problem of second order evolution equations that admits first order recursion operators of specific type, then we found a class of evolution equations with recursion operators.

Appendix A

Well Known PDEs and Recursion Operators

In this Appendix we give some evolution type of nonlinear partial differential equations with their recursion operators [6]

1) Burgers' Equation [1]

$u_t = u_{xx} + 2uu_x = R(u_x)$

$R = D_x + u + u_x D_x^{-1}.$

2) Potential Burgers' Equation [1]

$u_t = u_{xx} + u_x^2 = R_1(u_x)$

$R_1 = D_x + u_x$

$R_2 = tD_x + tu_x + \frac{1}{2}x.$

3) Diffusion Equation [12]

$u_t = u^2 u_{xx} = R(0)$

$R = uD_x + u^2 u_{xx} D_x^{-1}(\dfrac{1}{u^2}).$

4) Nonlinear Diffusion Equation [1]

$u_t = D_x(\dfrac{u_x}{u^2}) = R(0)$

$R = D_x^2(\dfrac{1}{u})D_x^{-1} = \dfrac{1}{u}D_x - 2\dfrac{u_x}{u^2} - D_x(\dfrac{u_x}{u^2})D_x^{-1}.$

5) KdV Equation [1]

$u_t = u_{xxx} + 6uu_x = R(u_x)$

$R = D_x^2 + 4u + 2u_x D_x^{-1}.$

6) Potential KdV Equation [14]

$u_t = u_{xxx} + 3u_x^2 = R(u_x)$

$R = D_x^2 + 4u_x - 2D_x^{-1}(u_{xx}).$

7) Modified KdV Equation [12]

$u_t = u_{xxx} - 6u^2 u_x = R(u_x)$

$R = D_x^2 - 4u^2 - 4u_x D_x^{-1}(u).$

8) Potential Modified KdV Equation [1]

$u_t = u_{xxx} + 3u_x^3 = R(u_x)$

$R = D_x^2 + 2u_x^2 - 2u_x D_x^{-1}(u_{xx}).$

9) Harry Dym Equation [12]

$u_t = u^3 u_{xxx} = R(0)$

$R = u^3 D_x^3(u) D_x^{-1}(\frac{1}{u^2}) = u^2 D_x^2 - uu_x D_x + uu_{xx} + u^3 u_{xxx} D_x^{-1}(\frac{1}{u^2}).$

10) Krichever-Novikov Equation [14]

$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} = R(u_x)$

$R = D_x^2 - 2\frac{u_{xx}}{u_x} D_x + (\frac{u_{xxx}}{u_x^3} - 3\frac{u_{xx}^2}{u_x^4}) D_x - u_x D_x^{-1}(\alpha) \quad , \quad \alpha = 3\frac{u_{xx}^3}{u_x^4} - 4\frac{u_{xx}u_{xxx}}{u_x^3} + \frac{u_{4x}}{u_x^2}$

11) Cavalcante-Tenenblat Equation [15]

$u_t = D_x^2(u_x^{-\frac{1}{2}}) + u_x^{\frac{3}{2}} = R(u_x)$

$R = \frac{1}{u_x} D_x^2 - \frac{3u_{xx}}{2u_x^2} D_x - (\frac{1}{2}\frac{u_{xxx}}{u_x^2} + \frac{3}{4}\frac{u_{xx}^2}{u_x^3} - u_x) + \frac{u_t}{2} D_x^{-1}(u_x^{-\frac{3}{2}} u_{xx}).$

12) Sine-Gordon Equation [1]

$u_{xt} = sinu$

$R = D_x^2 + u_x^2 - u_x D_x^{-1}(u_{xx}).$

13) Liouville Equation [14]

$u_{xt} = e^u$

$R = D_x^2 - u_x^2 - u_x D_x^{-1}(u_{xx}).$

14) Klein-Gordon Equation [16], [17]

$u_{xt} = \alpha e^{-2u} + \beta e^{u}$

$$\begin{aligned} R \;=\;& D_x^6 + 6(u_{xx} - u_x^2)D_x^4 + 9(u_{xxx} - 2u_x u_{xx})D_x^3 \\ &+(5u_{4x} - 22u_x u_{xxx} - 13u_{xx}^2 - 6u_x^2 u_{xx} + 9u_x^4)D_x^2 \\ &+(u_{5x} - 8u_x u_{4x} - 15u_{xx}u_{xxx} - 3u_x^2 u_{xxx} - 6u_x u_{xx}^2 + 18u_x^3 u_{xx})D_x \\ &-4u_x u_{5x} + 20u_x^3 u_{xxx} - 20u_x u_{xx} u_{xxx} + 20u_x^2 u_{xx}^2 - 4u_x^6 \\ &+2u_x D_x^{-1}(u_{6x} + 5u_{xx}u_{4x} + 5u_{xxx}^2 - 5u_x^2 u_{4x} \\ &-20u_x u_{xx} u_{xxx} - 5u_{xx}^3 + 5u_x^4 u_{xx}) \\ &+2(u_{5x} + 5u_{xx}u_{xxx} - 5u_x^2 u_{xxx} - 5u_x u_{xx}^2 + u_x^5)D_x^{-1}(u_{xx}). \end{aligned}$$

15) Kupershmidt Equation [9], [18]

$u_t = u_{5x} + 5u_x u_{xxx} + 5u_{xx}^2 - 5u^2 u_{xxx} - 20uu_x u_{xx} - 5u_x^3 + 5u^4 u_x = R(0)$

$$\begin{aligned} R \;=\;& D_x^6 + 6(u_x - u^2)D_x^4 + 15(u_{xx} - 2uu_x)D_x^3 \\ &+(9u_{4x} - 6u^2 u_x - 40uu_{xx} - 31u_x^2 + 14u_{xxx})D_x^2 \\ &+(6u_{4x} - 9u^2 u_{xx} + 54u^3 u_x - 18uu_x^2 - 6u_x u_{xx}^2 - 30uu_{xxx} - 63u_x u_{xx})D_x \\ &+u_{5x} - 12uu_{4x} - 3u^2 u_{xxx} - 23u_x u_{xxx} - 15u_{xx}^2 \\ &-38uu_x u_{xx} + 38u^3 u_{xx} + 74u^2 u_x^2 - 4u^6 \\ &-2u_x D_x^{-1}(u_{4x} + 5u^2 u_{xx} - 5uu_x^2 + 5u_x u_{xx} + u^5) + 2u_t D_x^{-1}(u). \end{aligned}$$

16) Sawada-Kotera Equation [12], [18]

$u_t = u_{5x} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x = R(0)$

$$\begin{aligned} R \;=\;& D_x^6 + 6uD_x^4 + 9u_x D_x^3 + 9u^2 D_x^2 + 11u_{xx}D_x^2 + 10u_{xxx}D_x + 12uu_x D_x \\ &+4u^3 + 16uu_{xx} + 6u_x^2 + 5u_{4x} + u_x D_x^{-1}(2u_{xx} + u^2) + u_t D_x^{-1}. \end{aligned}$$

17) Potential Sawada-Kotera Equation [9], [18]

$u_t = u_{5x} + 5u_x u_{xxx} + \frac{5}{3}u_x^3 = R(1)$

$$\begin{aligned} R \;=\;& D_x^6 + 6u_x D_x^4 + 3u_{xx}D_x^3 + 8u_{xxx}D_x^2 + 9u_x^2 D_x^2 \\ &+(2u_{4x} + 3u_{xx}u_x)D_x + 3u_{5x} + 13u_{xxx}u_x + 3u_{xx}^2 + 4u_x^3 \\ &-2u_x D_x^{-1}(u_{4x} + u_{xx}u_x) - 2D_x^{-1}(u_{6x} + 3u_{4x}u_x + 6u_{xxx}u_{xx} + 2u_{xx}u_x^2). \end{aligned}$$

18) Kaup-Kupershmidt Equation [9], [18]

$u_t = u_{5x} + 5uu_{xxx} + \frac{25}{2}u_x u_{xx} + 5u^2 u_x = R(0)$

$$\begin{aligned} R &= D_x^6 + 6uD_x^4 + 18u_x D_x^3 + (9u^2 + \tfrac{49}{2}u_{xx})D_x^2 \\ &\quad +(30uu_x + \tfrac{35}{2}u_{xxx})D_x + 4u^3 + \tfrac{41}{2}uu_{xx} + \tfrac{69}{4}u_x^2 + \tfrac{13}{2}u_{4x} \\ &\quad \tfrac{1}{2}u_x D_x^{-1}(u_{xx} + 2u^2) + u_t D_x^{-1}. \end{aligned}$$

19) Potential Kaup-Kupershmidt Equation[9], [18]

$u_t = u_{5x} + 10u_x u_{xxx} + \frac{15}{2}u_{xx}^2 + \frac{20}{3}u_x^3 = R(\frac{1}{2})$

$$\begin{aligned} R &= D_x^6 + 12u_x D_x^4 + 24u_{xx}D_x^3 + (25u_{xxx} + 36u_x^2)D_x^2 \\ &\quad (10u_{4x} + 48u_x u_{xx})D_x + 3u_{5x} + 21u_{xx}^2 + 34u_x u_{xxx} + 32u_x^3 \\ &\quad -2u_x D_x^{-1}(u_{4x} + 8u_x u_{xx}) - D_x^{-1}(u_{6x} + 12u_x u_{4x} + 24u_{xx}u_{xxx} + 32u_x^2 u_{xx}). \end{aligned}$$

20) Dispersiveless Long Wave System [16], [19]

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_x v + uv_x \\ u_x + vv_x \end{pmatrix} = \mathrm{R}\begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

$$R = \begin{pmatrix} v & 2u + u_x D_x^{-1} \\ 2 & v + v_x D_x^{-1} \end{pmatrix}.$$

21) Diffusion System [12]

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xx} + v^2 \\ v_{xx} \end{pmatrix} = \mathrm{R}\begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

$$R = \begin{pmatrix} D_x & vD_x^{-1} \\ 0 & D_x \end{pmatrix}.$$

22) Sine-Gordon Equation in the Laboratory Coordinates [20]

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v \\ u_{xx} - sin(u) \end{pmatrix}$$

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

$R_{11} = 4D_x^2 - 2cos(u) + (u_x + v)^2 - (u_x + v)D_x^{-1}(u_{xx} + v_x - sin(u)$

$R_{12} = D_x + (u_x + v)D_x^{-1}(u_x + v)$

$$R_{21} = 4D_x^3 + (u_x + v)^2 D_x - 4cos(u)D_x + 2u_x sin(u) + (u_{xx} + v_x)(u_x + v)$$
$$-(u_{xx} + v_x - sin(u))D_x^{-1}(u_{xx} + v_x - sin(u))$$
$$R_{22} = 4D_x^2 + (u_x + v)^2 - 2cos(u) + (u_{xx} + v_x - sin(u))D_x^{-1}(u_x + v).$$

23) AKNS System [12]

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -u_{xx} + 2u^2 v \\ v_{xx} - 2v^2 u \end{pmatrix} = \mathrm{R} \begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

$$R = \begin{pmatrix} -D_x + 2uD_x^{-1}(v) & 2uD_x^{-1}(u) \\ -2vD_x^{-1}(v) & D_x - 2vD_x^{-1}(u) \end{pmatrix}.$$

24) Nonlinear Schrdinger Equation [12], [14]

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v_{xx} \mp v(u^2 + v^2) \\ -u_{xx} \pm u(u^2 + v^2) \end{pmatrix} = \mathrm{R} \begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

$$R = \begin{pmatrix} \mp 2vD_x^{-1}(u) & D_x \mp 2vD_x^{-1}(v) \\ -D_x \pm 2uD_x^{-1}(u) & \pm 2uD_x^{-1}(v) \end{pmatrix}.$$

25 Boussinesq System [1]

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v_x \\ \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x \end{pmatrix} = \mathrm{R} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R = \begin{pmatrix} 3v + 2v_x D_x^{-1} & D_x^2 + 2u + u_x D_x^{-1} \\ R_{21} & 3v + v_x D_x^{-1} \end{pmatrix}$$

$$R_{21} = \frac{1}{3}D_x^4 + \frac{10}{3}uD_x^2 + 5u_x D_x + 3u_{xx} + \frac{16}{3}u^2 + 2v_t D_x^{-1}.$$

26)Modified Boussinesq System [21]

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 3v_{xx} + 6uv_x \\ -u_{xx} - 6vv_x + 2uu_x \end{pmatrix} = \mathrm{R} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

$$
\begin{aligned}
R_{11} &= 6vD_x^2 + 9v_xD_x + 3v_{xx} - 12u_xv_x - 2u_tD_x^{-1}(u) - 6u_xD_x^{-1}(2uv + v_x) \\
R_{12} &= 3D_x^3 + 6uD_x^2 + 9u_xD_x - 3u^2D_x - 9v^2D_x + 3u_{xx} - 6u^3 - 36vv_x \\
&\quad -18uv^2 - 6u_tD_x^{-1}(v) + 6u_xD_x^{-1}(u_x - u^2 + 3v^2) \\
R_{21} &= -D_x^3 + 2uD_x^2 + u^2D_x + 3u_xD_x + 3v^2D_x + u_{xx} - 6uv^2 - 2u^3 + 4uu_x \\
&\quad -2v_tD_x^{-1}(u) - 6v_xD_x^{-1}(v_x + 2uv) \\
R_{22} &= -6vD_x^2 - 9v_xD_x - 12uv^2 + 12u_xv - 3v_{xx} + 36v^3 \\
&\quad -6v_tD_x^{-1}(v) + 6v_xD_x^{-1}(u_x - u^2 + 3v^2).
\end{aligned}
$$

27) The Symmetrically-coupled KdV System [22]

$$
\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xxx} + v_{xxx} + 6uu_x + 4uv_x + 2u_xv \\ u_{xxx} + v_{xxx} + 6vv_x + 4vu_x + 2v_xu \end{pmatrix} = \mathrm{R}\begin{pmatrix} u_x \\ v_x \end{pmatrix}
$$

$$
R = \begin{pmatrix} D_x^2 + 4u + 2u_xD_x^{-1} & D_x^2 + 4u + 2u_xD_x^{-1} \\ D_x^2 + 4v + 2v_xD_x^{-1} & D_x^2 + 4v + 2v_xD_x^{-1} \end{pmatrix}.
$$

28) The Complexly-coupled KdV System [22]

$$
\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xxx} + 6uu_x + 6vv_x \\ v_{xxx} + 6uv_x + 6vu_x \end{pmatrix} = \mathrm{R}\begin{pmatrix} u_x \\ v_x \end{pmatrix}
$$

$$
R = \begin{pmatrix} D_x^2 + 4u + 2u_xD_x^{-1} & 4v + 2v_xD_x^{-1} \\ 4v + 2v_xD_x^{-1} & D_x^2 + 4u + 2u_xD_x^{-1} \end{pmatrix}.
$$

29) Coupled Nonlinear Wave System (Ito System) [23], [24], [14]

$$
\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xxx} + 6uu_x + 2vv_x \\ 2uv_x + 2u_xv \end{pmatrix} = \mathrm{R}\begin{pmatrix} u_x \\ v_x \end{pmatrix}
$$

$$
R = \begin{pmatrix} D_x^2 + 4u + 2u_xD_x^{-1} & 2v \\ 2v + 2v_xD_x^{-1} & 0 \end{pmatrix}.
$$

30) Benney System [25], [24]

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = \begin{pmatrix} vv_x + 2D_x(uw) \\ 2u_x + D_x(vw) \\ 2v_x + 2ww_x \end{pmatrix} = \mathrm{R} \begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix}$$

$$R = \begin{pmatrix} w & v & 2u + u_x D_x^{-1} \\ 2 & 0 & v + v_x D_x^{-1} \\ 0 & 2 & w + w_x D_x^{-1} \end{pmatrix}.$$

31) Dispersive Water Wave System [24]

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = \begin{pmatrix} D_x(uw) \\ -v_{xx} + 2D_x(vw) + uu_x \\ w_{xx} - 2v_1 + 2ww_x \end{pmatrix} = \mathrm{R} \begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & 0 & u + u_x D_x^{-1} \\ u & -D_x + w & 2v + v_x D_x^{-1} \\ 0 & -2 & D_x + w + w_x D_x^{-1} \end{pmatrix}.$$

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