

SECTIONAL CURVATURE OF STANDARD STATIC SPACE-TIMES

A THESIS

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By

Bengi Ruken Yavuz

January, 2013

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Bülent Ünal (Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Metin Gürses

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Konstantin Zheltukhin

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

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Bengi Ruken Yavuz

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Supervisor: Asst. Prof. Dr. Bülent Ünal

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Curvature related geometric properties of warped product manifolds are given. The casual structure of Lorentzian warped products is reviewed. In particular, two well-known examples of warped product space-times models, i.e, generalized Robertson-Walker and standard static space-times are studied. The sectional curvature of a standard static space-time is established and some conditions are obtained to have nonnegative sectional curvature so that applications of singularity theorems to a standard static space-time can be considered.

Keywords: semi-Riemannian geometry, warped products, Generalized Robertson-Walker space-times, Standard Static space-times, sectional curvature, geodesic.

ÖZET

TEK BİÇİMLİ STATİK UZAY-ZAMANLARIN KESİTSEL EĞRİLİĞİ

Bengi Ruken Yavuz

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Tez Yöneticisi: Yrd. Doç Dr. Bülent Ünal

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Eğrilmiş çarpım manifoldlarının eğrilik ile ilgili geometrik özellikleri verildi. Lorentz eğrilmiş çarpımlarının nedensel yapısı incelendi. Özellikle, eğrilmiş çarpım uzay-zaman modellerinden en iyi bilinen iki örneği, yani, genelleştirilmiş Robertson-Walker uzay-zamanları ve tek biçimli statik uzay-zamanlar çalışıldı. Tek biçimli statik uzay-zamanların kesitsel eğrilikleri belirlendi ve negatif olmayan kesitsel eğrilğe sahip olabilmeleri için gerekli koşullar elde edildi. Bunların sonucu olarak, tekillik teoremleri tek biçimli statik uzay-zamanlara uygulanabilir.

Anahtar sözcükler: yarı-Riemann geometri, eğrilmiş çarpımlar, Genelleştirilmiş Robertson-Walker uzay-zamanları, Tek biçimli statik uzay-zamanları, kesitsel eğrilik, jeodesik.

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Chapter 1

Introduction

In 1915, *Einstein's field equations* represented as tensor equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu}$$

(see [1]) where $R_{\mu\nu}$ is the *Ricci curvature tensor*, R the *scalar curvature*, $g_{\mu\nu}$ is the *metric tensor*, Λ is the *cosmological constant*, G is *Newton's gravitational constant*, c is the speed of light in vacuum, and $T_{\mu\nu}$ is the stress-energy tensor. Via these equations, it is possible to equate the stress-energy tensor with space-time curvature. Warped product space-times are important from both geometrical and physical point of view, especially in Lorentzian geometry and general relativity, since they comprise a wide variety of exact solutions to Einstein's field equations [2, 3, 4, 5]. Most important examples include: Bertotti-Robinson, Robertson-Walker, Schwarzschild, Reissner-Nordstrom, de Sitter, anti de Sitter, static, etc. Warped products were first introduced by O'Neill and Bishop in [6] in order to obtain a manifold with negative curvature. Afterwards, O'Neill discussed curvature of warped products and give explicit curvature formulas in terms of the base and the fiber of a warped product manifold. Furthermore, O'Neill discussed Robertson-Walker, static, Schwarzschild and Kruskal space-times as warped products. In [2], Beem and Ehrlich stated causal properties and completeness of warped products, and how they are related to the causality and completeness of its components. Generalizations of warped products are also important in both geometry and physics. *Doubly warped products* were investigated in [7] and [8].

Curvature properties of *multiply warped products* were considered in [9]. *Twisted warped products*, where the warping function may depend on the points of both components, were investigated in [10] and [11]. In the present work we study *generalized Robertson-Walker space-times* and *standard static space-times* where the first one is the generalization of Robertson-Walker space-times and the second one is the generalization of Einstein static universe which is the first relativistic cosmological model. A Lorentzian warped product of the form $M = (t_1, t_2) \times_f F$ where $-\infty \leq t_1 < t_2 \leq \infty$ with $((t_1, t_2), -dt^2)$ is the base, (F, g_F) is the fiber which is s -dimensional connected Riemannian manifold and the warping function is any positive function $f > 0$ on (a, b) . A *standard static space-time* $I_f \times F$ is a Lorentzian warped product with dimension $m(= s + 1)$ furnished with the metric $g = -f^2 dt^2 \oplus g_F$, where (F, g_F) is a Riemannian manifold of dimension s , $f : F \rightarrow (0, \infty)$ is smooth, and $-\infty \leq t_1 < t_2 \leq \infty$. The fiber (F, g_F) of a standard static space-time is always assumed to be connected. Two important examples of standard static space-times are Minkowski space-time and the Einstein static universe. These examples are given in [2, 12].

Because of their theoretical importance, standard static space-times have been studied in a variety of papers. Geodesic structure and curvature properties of these kind of space-times have been studied by many authors. For example in [13], the geodesic structure of standard static spacetimes is studied. In [14] and [15] geodesic equations, geodesic completeness and causal structures of these space-times were discussed. Moreover, in [16] the authors give global characterization of killing vector fields of a standard static space-time. And in [17], conditions are found which guarantee that standard static space-times either satisfy or else fail to satisfy certain curvature conditions from general relativity.

In Chapter 2, we give a brief summary of the Lorentzian Geometry. We give the definition of a metric on a manifold and then we introduce some differential operators which will be useful in the proceeding chapters. Causal structure, which gives a simpler approximation to metric structure, is studied. Physically, the causal relations between points in the manifold describes which events in space-time can influence which other events. Also connection and curvature formulas are given for general semi-Riemannian manifolds.

In chapter 3, we introduce warped product manifolds which has a great importance both in geometry and in general relativity. We give linear connections and geodesic equations for warped products in terms of its base and fiber. Then we consider causal properties of warped products by considering the relation of causal structure of the warped manifold with its components. At the end of this chapter, in *Theorem A.0.8*, we give sectional curvature formula for a non-degenerate plane section in a warped product manifold with a shortened proof. The generalized version of this theorem to doubly warped products with an explicit proof can be found in [7]. Then, we introduce one of the most important solution to Einstein's field equations Generalized Robertson-Walker space-times which need not to have a fiber with constant sectional curvature. The curvature and connection formulas for these type of space-times by adapting the formulas given in Chapter 3. Finally we give geodesic equations for Generalized Robertson-Walker space-times from [18].

In Chapter 5, we explain standard static space-times. First we give the definition of *static space-times*. Afterwards we give Ricci curvature and sectional curvature of these kind of space-times by using *Theorem A.0.8*. Finally we obtain necessary and sufficient conditions for a standard static space-time to have non-degenerate time-like sectional curvatures.

Chapter 2

Introduction to semi-Riemannian Geometry

In this chapter, we shall give a brief summary of the semi-Riemannian geometry and the Lorentzian geometry. We shall first present necessary definitions, properties and results about semi-Riemannian and Lorentzian Geometry as an introduction to succeeding sections. Next we shall define a linear connection on a manifold and which is a key tool for calculating curvature tensors and geodesics. We shall give a special attention to sectional curvature. In the succeeding section we introduce Killing vector fields which can be interpreted as an infinitesimal isometry on a Riemannian or semi-Riemannian manifold. Finally, we shall introduce causal structure of Lorentzian manifolds which defines a partial order on the events of space-time. Furthermore, we give the definition of *Lorentzian distance function* which possess many different properties when compared to the Riemannian one. We shall use the standard notations and some facts specialized in [2], [19] and [3]. The content in this chapter is standard and can be found in any books about semi-Riemannian Geometry.

2.1 Definitions and some differential operators

In this section we restrict our attention to Lorentzian Geometry and review some of its basic properties. We start with introducing notations of tangent space at a point and the set of all vector fields on a manifold which is followed by definitions.

Set of all tangent vectors to a point $p \in M$ is denoted by $T_p(M)$. The set of all tangent vectors of the manifold M is denoted by $T(M)$

$$T(M) = \bigcup_{p \in M} T_p(M).$$

The set of all vector fields on M is denoted by $\mathfrak{X}(M)$.

Definition 2.1.1. A *metric tensor* g on a smooth manifold M is a non-degenerate symmetric (0,2) tensor field on M that has fixed index. Alternatively, $g_p \in \mathcal{T}_2^0(M)$ assigns a *scalar product* g_p on the tangent space $T_p(M)$ to each point p in M

Definition 2.1.2. Let M be an m dimensional smooth manifold with a metric tensor g . Let the metric tensor g has a set of negative eigenvalues with dimension r . This implies that g has $s = m - r$ positive eigenvalues. Then *index* of g is r and *signature* of g is (r, s) .

Lemma 2.1.3. *Every second countable smooth manifold admits a Riemannian metric tensor.*

Proof. See p.140 of [3]. □

We can represent the metric tensor in local coordinates or in the matrix form as follows:

Let U be an open subset of M and (x^1, x^2, \dots, x^n) be the local coordinates in U . Then

$$g|_U = \sum_{i,j=1}^n g_{ij}(x) dx^i \otimes dx^j.$$

Definition 2.1.4. A *semi-Riemannian manifold* is a smooth manifold M that is furnished with a metric tensor g .

- (i) A semi-Riemannian manifold M with zero index is called *Riemannian manifold*. Riemannian manifolds have metrics of signature $(+, \dots, +)$ and the induced metric on the tangent space of a Riemannian manifold is Euclidean.
- (ii) If a semi-Riemannian manifold M of dimension greater than 2 has index 1 then, it is called a *Lorentzian manifold*. Here signature of the metric is $(-, +, \dots, +)$ and the tangent spaces of a Lorentzian manifold have induced Minkowskian metrics.

Example 2.1.5. In Cartesian coordinates (x_1, \dots, x_n) , \mathbb{R}^n is a smooth manifold with a single chart $(\mathbb{R}^n, x_1, \dots, x_n)$ and the Minkowski metric $g = -((dx)_1)^2 + \dots + ((dx)_n)^2$. (\mathbb{R}^n, g) is a Lorentzian manifold.

Definition 2.1.6. The *Lie Bracket* of $X, Y \in \mathfrak{X}(M)$ is a vector field $[X, Y] \in \mathfrak{X}(M)$ such that $[X, Y](f) = X(Y(f)) - Y(X(f))$.

We can write the above definition more explicitly by using a orthonormal frame field on M . Let $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}$ then

$$[X, Y] = \sum_{i=1}^n \sum_{j=1}^n \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

The following proposition interprets *Lie Bracket* as the rate of change of Y under the flow of X .

Proposition 2.1.7. [3] If $X, Y \in \mathfrak{X}(M)$, let ψ be a local flow of X near $p \in M$. Then

$$[X, Y]_p = \lim_{t \rightarrow 0} \frac{1}{t} [d\psi_{-t}(Y_{\psi_t p}) - Y_p]. \quad (1.1)$$

Definition 2.1.8. For a vector field $V \in \mathfrak{X}(M)$ the tensor derivation L_X satisfying the following conditions

- (i) $L_X(f) = Xf$ for all $f \in C^\infty(M)$,

(ii) $L_X(Y) = [X, Y]$ for all $Y \in \mathfrak{X}(M)$,

is called the *Lie derivative* relative to X .

Definition 2.1.9. Let (M, g) be an m dimensional semi-Riemannian manifold. A *connection* ∇ on M is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ satisfying the following conditions:

- (i) $\nabla_X Y$ is $C^\infty(M)$ -linear in X ,
- (ii) $\nabla_X Y$ is \mathbb{R} -linear in Y ,
- (iii) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$ for $f \in C^\infty(M)$.

Here $\nabla_X Y$ is the *covariant derivative* of Y with respect to X for the connection ∇ .

Definition 2.1.10. The *gradient* of a function $f \in C^\infty(M)$ is defined by

$$X(f) = df(X) = g(\text{grad}(f), X) \quad (1.2)$$

where $df \in \mathfrak{T}_1^0(M)$.

Definition 2.1.11. The *hessian* of a function $f \in C^\infty(M)$ is the symmetric $(0, 2)$ tensor such that

$$\text{hess}^f(X, Y) = g(\nabla_X \text{grad}(f), Y) \quad (1.3)$$

Definition 2.1.12. The *Laplacian* of f is

$$\Delta(f) = \text{div}(\text{grad}(f)) \quad (1.4)$$

Definition 2.1.13. The *curl* of $V \in \mathfrak{X}(M)$ is a skew-symmetric $(0, 2)$ tensor field and defined by $(\text{curl } V)(X, Y) = g(\nabla_X(V), Y) - g(\nabla_Y(V), X)$.

2.2 Connections and Curvature on Lorentzian Manifolds

In this section we will state some formulas and properties of connections and curvatures from [3] and [2].

For an m dimensional semi-Riemannian manifold (M, g) of arbitrary signature $(-, \dots, -, +, \dots, +)$ there is a unique torsion free connection which is called as the *Levi-Civita connection* and denoted by ∇ . The Levi-Civita connection, curvature, Ricci curvature, scalar curvature and sectional curvature of a semi-Riemannian manifold satisfies the same formal relations in the same way of a Riemannian manifold.

Theorem 2.2.1. [3] *Let (M, g) be a semi-Riemannian manifold. Then there is a unique connection ∇ , called the Levi-Civita connection of (M, g) , which is torsion free,*

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$

and metric compatible,

$$Vg(X, Y) = g(\nabla_V X, Y) + (X, \nabla_V Y)$$

for all $X, Y, V \in \mathfrak{X}(M)$.

The action of the connection ∇ can be represented in local coordinates as follows: If

$$X = X^i(x) \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = Y^i(x) \frac{\partial}{\partial x^i},$$

then

$$\nabla_X Y = \left(X^j \frac{\partial Y^k}{\partial x^j} + \Gamma_{ji}^k X^j Y^i \right) \frac{\partial}{\partial x^k}.$$

The vector field $\nabla_X Y$ is called as the *covariant derivative* of Y with respect to X and the connection coefficients

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{m=1}^n g^{mk} \left(\frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g_{mj}}{\partial x^i} \right)$$

is called as the *Christoffel symbols*.

Definition 2.2.2. Let $X, Y, Z \in \mathfrak{X}(M)$. Then the *Riemannian curvature* is a $(3, 1)$ tensor field such that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Let e_1, \dots, e_n be an orthonormal frame field and $X, Y \in \mathfrak{X}(M)$ then the *Ricci curvature* is given by the following formula

$$\text{Ric}(X, Y) = \sum_{i=1}^n g(e_i, e_i) g(R(e_i, Y)X, e_i)$$

and the trace of the Ricci curvature gives the *scalar curvature* τ where

$$\tau = \sum_{i=1}^n g(e_i, e_i) \text{Ric}(e_i, e_i).$$

2.2.1 Sectional Curvature

Let (M, g) be a semi-Riemannian manifold. A two dimensional linear subspace E of $T_p(M)$ is called a *plane section*. If for each nontrivial vector $X_1 \in E$ there exists $Y_1 \in E$ such that $g(X_1, Y_1) \neq 0$ then g is non-degenerate. This condition is the same with requiring that $g_p|_E$ be a non-degenerate inner product on E . Let X_1 and Z_1 be a basis for the plane section E . Then

$$g(X_1, X_1)g(Z_1, Z_1) - [g(X_1, Z_1)]^2 \neq 0$$

if and only if E is non-degenerate. The plane E is

- *timelike* if the signature of $g_p|_E$ is $(-, +)$.
- *spacelike* if the signature of $g_p|_E$ is $(+, +)$.

For a Lorentzian manifold, degenerate planes are called either *null* or *lightlike* and have signature $(0, +)$. A null plane in $T_p(M)$ is a plane tangent to the null cone in $T_p(M)$. For Lorentzian manifolds there is exactly one generator of the null cone in degenerate plane. We can classify the plane sections of Lorentzian manifolds generated by basis vectors X_1 and Z_1 as follows:

- The plane is *timelike* if $g(X_1, X_1)g(Z_1, Z_1) - [g(X_1, Z_1)]^2 < 0$.
- *degenerate* if $g(X_1, X_1)g(Z_1, Z_1) - [g(X_1, Z_1)]^2 = 0$.
- *spacelike* if $g(X_1, X_1)g(Z_1, Z_1) - [g(X_1, Z_1)]^2 > 0$.

Definition 2.2.3. Let $X_p, Y_p \in T_p(M)$ and E be the the nondegenerate plane section generated by X_p and Y_p . Then the *sectional curvature* $K(E)$ of E is given by the following formula:

$$K(p, E) = \frac{g(R(Y_p, X_p)X_p, Y_p)}{g(X_p, X_p)g(Y_p, Y_p) - [g(X_p, Y_p)]^2}$$

If we say $P(p, E) = g(R(Y_p, X_p)X_p, Y_p)$ and $Q(p, E) = g(X_p, X_p)g(Y_p, Y_p) - [g(X_p, Y_p)]^2$ then

$$K(E) = \frac{P(E)}{Q(E)}$$

If a semi-Riemannian manifold (M, g) has the same sectional curvature on all non-degenerate plane sections then it has *constant curvature*. If (M, g) has constant curvature c then

$$R(X, Y)Z = c[g(Y, Z)X - g(X, Z)Y]$$

where $X, Y, Z \in \mathfrak{X}(M)$ (see [20]).

Lemma 2.2.4. [20] *All non-degenerate planes have sectional curvature c if and only if $R(X, Y)Z = c[g(Y, Z)X - g(X, Z)Y]$.*

We will state some of the most prominent results about sectional curvatures without proofs. Let (M, g) be a Lorentzian manifold where $\dim(M) \geq 3$ and let the sectional curvatures of timelike planes are bounded both above and below. Then (M, g) has constant curvature (see [21], [22]). Families of Lorentzian manifolds which are conformal to Lorentzian manifolds of constant curvature can be constructed that have all timelike sectional curvatures bounded in one direction (see [23]). If $\dim(M) \geq 3$ and the sectional curvatures of all nondegenerate planes are bounded either from above or from below, then the sectional curvature is constant (see [24]).

2.3 Killing Vector Fields

In *Definition 2.1.8* we defined *Lie derivative* L_X applied to a vector field Y and in *Proposition 2.1.7* we gave the interpretation of this bracket as the rate of change of Y under the flow of X . A similar interpretation holds for L_X applied to any tensor field A , for simplicity we take A to be covariant.

Proposition 2.3.1. [3] *If $X \in \mathfrak{X}(M)$ and $A \in \mathcal{T}_s^0(M)$, then*

$$L_X(A) = \lim_{t \rightarrow 0} \frac{1}{t} [\psi_t^*(A) - A], \quad (3.1)$$

where ψ_t is the flow of X . For local flows, the equation holds locally.

Definition 2.3.2. A *Killing vector field* on a semi-Riemannian manifold is a vector field X for which the Lie derivative of the metric tensor vanishes, i.e., $L_X(g) = 0$.

Hence under the flow of X the metric tensor does not change. In other words, the flow generates a symmetry, in the sense that moving each point on an object the same distance in the direction of the Killing vector field will not distort distances on the object.

Proposition 2.3.3. [3] *A vector field X is Killing if and only if the stages ψ_t of all its (local) flows are isometries.*

The above proposition interprets a Killing vector field as an *infinitesimal isometry*.

Recall that the covariant differential of a vector field is the $(1, 1)$ tensor field ∇X such that $(\nabla X)V = \nabla_V X$ for all $\nabla \in \mathfrak{X}(M)$. Thus at each $p \in M$, $(\nabla X)_p$ is the linear operator on $T_p(M)$ sending v to $\nabla_v X$.

Proposition 2.3.4. [3] *The following conditions on a vector field X are equivalent.*

- (i) X is Killing.

(ii) $Xg(V, W) = g([X, V], W) = g([X, V], W) + g(V, [X, W])$ for all $V, W \in \mathfrak{X}(M)$.

(iii) ∇X is skew-adjoint relative to g ; that is, $g(\nabla_V X, W) + g(\nabla_W X, V) = 0$ for all $V, W \in \mathfrak{X}(M)$.

Lemma 2.3.5. *Let X be a Killing vector field on a connected semi-Riemannian manifold M . If $X_p = 0$ and $(\nabla X)_p = 0$ for some one point p of M , then $X = 0$.*

The above lemma implies that if X and Y are Killing vector fields such that $X_p = Y_p$ and $\nabla X_p = \nabla Y_p$ for some one point p , then $X = Y$.

2.4 Causal Structure of Lorentzian Manifolds

In this section we give basic definitions of causality theory and then define Lorentzian distance function. At the end of this section we give statement of a theorem from [2] which says that distance realizing geodesics exist for the class of globally hyperbolic space-times.

Definition 2.4.1. Let (M, g) be a Lorentzian manifold. A nonzero tangent vector $X_p \in T_p(M)$ can be classified as follows: X_p is

- (i) timelike if $g(X_p, X_p) < 0$,
- (ii) null if $g(X_p, X_p) = 0$,
- (iii) causal or nonspacelike if $g(X_p, X_p) \leq 0$,
- (iv) spacelike $g(X_p, X_p) > 0$.

(M, g) is said to be *time-oriented* if there exists a continuous, nonvanishing, timelike vector field $X \in \mathfrak{X}(M)$. This vector field separates the nonspacelike vectors into two classes that are *future directed* and *past directed*. For example, for the space in *Example 2.1.5*, the given manifold is time-oriented where $X = \frac{\partial}{\partial t}$.

Definition 2.4.2. Let $p, q \in M$.

- (i) If there exists a future directed, piecewise smooth, timelike curve from p to q then we denote it by $p \ll q$.
- (ii) If there exists a future directed, piecewise smooth, nonspacelike curve from p to q or $p = q$ then we denote it by $p \leq q$.

Definition 2.4.3. Let (M, g) be a *Lorentzian manifold* and let $p, q \in M$. Then

- (i) The set $\{q \in M | p \ll q\}$ is called as the *chronological future* of p and it is denoted by $I^+(p)$.
- (ii) The set $\{q \in M | p \leq q\}$ is called as the *causal future* of p and it is denoted by $J^+(p)$.

The *chronological past* $I^-(p)$ and *causal past* $J^-(p)$ of p can be defined dually.

Definition 2.4.4. A time-oriented Lorentzian manifold (M, g) is called as *space-time*. The *causal structure* of the space-time (M, g) can be defined as the collection of past and future sets at all points of M together with their properties.

- A space-time (M, g) with no closed timelike curves, i.e., $p \notin I^+(p)$ for all $p \in M$, is *chronological*.
- A space-time with no closed nonspacelike curves is *causal*. Alternatively there is no pair of distinct points $p, q \in M$ with $p \leq q \leq p$.
- An open set U in a space-time is *causally convex* if no nonspacelike curve intersects U in a disconnected set.
- Let $p \in M$, the space-time (M, g) is called *strongly causal* at p if p has arbitrarily small causally convex neighborhoods.
- If a space-time is strongly causal at each point then it is *strongly causal*.

- A space-time (M, g) is *globally hyperbolic* if it is strongly causal and $J^-(p) \cap J^+(q)$ is compact for all $p, q \in M$. Globally hyperbolic space-times can be characterized by using Cauchy surfaces where a *Cauchy surface* S is a subset of M which every inextendible nonspacelike curve intersects exactly one.

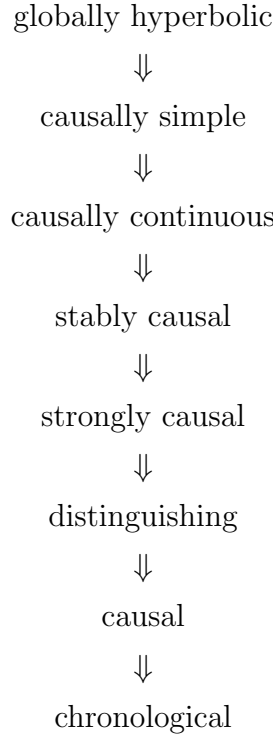
In [12], it is shown that a space-time is globally hyperbolic if and only if it admits a Cauchy surface. In a complete Riemannian manifold, any two points can be joined by a geodesic of minimal length. The Lorentzian analogue of this result is the following:

Theorem 2.4.5. [2] *Let (M, g) be globally hyperbolic and $p \leq q$. Then there is a non-spacelike geodesic from p to q whose length is greater than or equal to that of any future directed non-spacelike curve from p to q . Moreover, this geodesic is not necessarily unique.*

Remark 2.4.6. [25] In general relativity, spacetimes are usually assumed to be chronological because of physical reasons. They are also assumed to be non-compact, since any compact Lorentzian manifold (M, g) contains closed timelike curves.

Proof. [25] Let assume M is time-oriented. It is easy to see that $\{I^+(p)\}_{p \in M}$ is an open cover for M . If M is compact then there exists $p_1, \dots, p_N \in M$ such that $\{I^+(p_1), \dots, I^+(p_N)\}$ is a finite subcover of M . If $p_1 \in I^+(p_i)$ for $i \neq 1$ then $I^+(p_1) \subset I^+(p_i)$ and we can exclude $I^+(p_1)$ from the subcover. Therefore we can assume without loss of generality $p_1 \in I^+(p_1)$, hence there exists closed timelike curve starting and ending at p_1 . \square

The following diagram illustrates the relation between causality conditions on a Lorentzian manifold:



Then we can define the *Lorentzian distance* function $d = d(g) : M \times M \rightarrow [0, \infty]$ on an arbitrary space-time (M, g) . If $c : [0, 1] \rightarrow M$ is a piecewise smooth nonspacelike curve differentiable except at $0 = t_1 < \dots < t_n = 1$ then the length $L(c)$ of c is given by

$$L(c) = \sum_{i=1}^{n-1} \int_{t=t_i}^{t=t_{i+1}} \sqrt{-g(c'(t), c'(t))} dt.$$

Then, let fix $p \in M$ and let $p \leq q$, define $C(p, q)$ as the set of all future directed piecewise smooth nonspacelike curves from p to q , i.e., $c(0) = p$ and $c(1) = q$.

$$d(p, q) = \begin{cases} 0 & \text{if } q \notin J^+(p) , \\ \sup\{L(c) | c \in C(p, q)\} & \text{if } q \in J^+(p) \end{cases}$$

From the definition, we can deduce that

$$d(p, q) > 0 \text{ if and only if } q \in I^+(p).$$

Also we can deduce that Lorentzian distance function determines the chronological past and future of any point. But in general, the Lorentzian distance function

does not determine causal past and future sets of p since $d(p, q) = 0$ does not imply $q \in J^+(p) - I^+(p)$. But at least if $q \in J^+(p) - I^+(p)$ then $d(p, q) = 0$. In a Lorentzian manifold, a reverse triangle inequality holds, i.e., if $p \leq q \leq r$ then $d(p, q) + d(q, r) \leq d(p, r)$. Lorentzian distance function may take infinite values but for a globally hyperbolic space-time the Lorentzian distance function is finite and continuous. Also note that the Lorentzian distance function is only positive for points connected by timelike curves.

Definition 2.4.7. A future directed non-spacelike curve γ from p to q is said to be *maximal* if $L(\gamma) = d(p, q)$.

Theorem 2.4.8. [26] *Let (M, g) be a globally hyperbolic space-time. Then given any $p, q \in M$ with $q \in J^+(p)$, there is a maximal geodesic segment $c \in C(p, q)$, i.e., a future directed non-spacelike geodesic c from p to q with $L(c) = d(p, q)$.*

Chapter 3

Warped Products and Spacetimes

The main object of this chapter is to present warped product manifolds, their curvatures and covariant derivatives. We also give causal properties of warped products. For details see [2] and [3].

3.1 Warped Products

In this section we briefly express relation of a warped product manifold with its base and fiber. We give covariant derivative and curvature formulas for warped product manifold. We recommend [3] for the omitted proofs.

Definition 3.1.1. Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds with dimensions r and s respectively. And let $b : B \rightarrow (0, \infty)$ be a smooth function. $M = B \times_b F$ is the *warped product manifold* $B \times F$ with the metric tensor $g = \pi^*(g_B) \oplus (b\sigma)^2\sigma^*(g_F)$ and $\dim(M) = m = r + s$. Here $\pi : M \rightarrow B$ and $\sigma : M \rightarrow F$ are the usual projection maps where $\pi^*(g_B)$ and $\sigma^*(g_F)$ are pullbacks of g_B and g_F , respectively.

Warped products were first introduced in 1969 by Bishop and O'Neill in [6], in order to construct Riemannian manifolds with negative sectional curvature. Then in [27], it was indicated that many exact solutions to Einstein's field equation can

be expressed in terms of Lorentzian warped products. They are important cosmological models because of their simplicity and symmetry advantages. Not only expanding universes but also static ones can be formulated as warped products. The simplicity of warped products arise from the fact that their geometric properties can be derived from base and fiber manifolds. For example in [3], B. O’Neill showed that curvature and geodesic structures of a warped product can be expressed in terms of its base and fiber. Also in [2], it was indicated that causality and completeness of a warped product manifold can be derived from its base and fiber.

The geometry of the warped product manifold $M = B \times_b F$ is different than the usual product manifold $B \times F$ since the metrics on the two manifolds are different because of the warping function b . But they have the same tangent spaces: Let $m = (p, q) \in M = B \times_b F$, we get the natural isomorphism

$$T_m(M) = T_m(B \times F) \cong T_p(B) \times T_q(F). \quad (1.1)$$

The symmetry of g is obvious. Since g_B and g_F are non-degenerate and $b > 0$ then g is also non-degenerate metric tensor.

Let $X_{(p,q)} \in T_{(p,q)}(M)$. Then

$$g(X_{(p,q)}, X_{(p,q)}) = g_B(d\pi(X_{(p,q)}), d\pi(X_{(p,q)})) + b^2(p)g_F(d\sigma(X_{(p,q)}), d\sigma(X_{(p,q)}))$$

In order to understand the geometry of warped products more explicitly, we express it in terms of geometry of the *base* B , the *fiber* F and the *warping function* b . In fact the relation of the base with the warped product manifold is simple as for semi-Riemannian products, but the relation of the fiber is a little complicated due to the warping function. The *fibers* $(\pi)^{-1}(p) = \{p\} \times F$ and the *leaves* $(\sigma)^{-1}(q) = B \times \{q\}$ are semi-Riemannian submanifolds of M . The vectors that are tangent to fibers are *vertical* and the vectors that are tangent to leaves are *horizontal*.

Remark 3.1.2. The warped metric is characterized by

- (i) For each $q \in F$, the map $\pi|_{B \times \{q\}}$ is an isometry onto B .

- (ii) For each $p \in B$, the map $\sigma|_{\{p\} \times F}$ is a positive homothety onto F with scale factor $1/b(p)$.
- (iii) For each $(p, q) \in M$, the leaf $B \times \{q\}$ and the fiber $\{p\} \times F$ are orthogonal at (p, q) .

Definition 3.1.3. The procedure of carrying functions, tangent vectors and vector fields from components of warped product to the warped product manifold is called as *lift*. Let $b: B \rightarrow \mathbb{R} \in C^\infty(B)$ then the lift of b to $B \times F$ is $\tilde{b} = b \circ \pi \in C^\infty(B \times F)$, where $C^\infty(B)$ is the set of all smooth real-valued functions on B . Let $X_p \in T_p(B)$ and $q \in F$ then the lift $\tilde{X}_{(p,q)}$ of X_p is the unique tangent vector in $T_{(p,q)}(B \times \{q\})$ such that $d\pi_{(p,q)}(\tilde{X}_{(p,q)}) = X_p$ and $d\sigma_{(p,q)}(\tilde{X}_{(p,q)}) = 0$. The set of all lifts of tangent vectors of B is denoted by $L_{(p,q)}(B)$. Let $X \in \mathfrak{X}(B)$ then the lift of X to $B \times F$ is the vector field $\tilde{X} \in \mathfrak{X}(B \times F)$ whose value at each (p, q) is the lift of X_p to (p, q) . The set of all lifts of vector fields of B is denoted by $\mathfrak{L}(B)$.

Remark 3.1.4. It is possible to define lifts of tensors as follows: Let T be a covariant tensor on the base B . Its lift \tilde{T} is its pullback $\pi^*(T)$ where $\pi: M \rightarrow B$ is the usual projection map. Since this procedure has nothing to do with the warping function b , then it is also valid for the lifts from the fiber F .

The Levi-Civita connection of the warped product M can be expressed in terms of the Levi-Civita connections B and F (see *Theorem 2.2.1*) in the following way:

Proposition 3.1.5. [3] Let $M = B \times_b F$, $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, then

- (i) $\nabla_X Y \in \mathfrak{L}(B)$ is the lift of $\nabla_X Y$ on B .
- (ii) $\nabla_X V = \nabla_V X = (X(b)/b)V$.
- (iii) $\text{nor}\nabla_V W = II(V, W) = -(g(V, W)/b)\text{grad}_B(b)$.
- (iv) $\text{tan}\nabla_V W \in \mathfrak{L}(F)$ is the lift of $\nabla_V W$ on F .

Corollary 3.1.6. [3] *Let $M = B \times_b F$ be a warped product manifold. The leaves $B \times \{q\}$ are totally geodesic; the fibers $\{p\} \times F$ are totally umbilic submanifolds of M .*

In the following proposition we give the expression of geodesic equations of a warped product in terms of its base and fiber manifolds. The proof of the proposition is omitted and can be found in [3].

Let $M = B \times_b F$ and $\gamma(s)$ be a curve defined on some interval $I \subseteq \mathbb{R}$. Then $\gamma(s)$ can be written as $\gamma(s) = (\alpha(s), \beta(s))$ where α and β are projections of γ into B and F respectively.

Proposition 3.1.7. [3] *A curve $\gamma(s) = (\alpha(s), \beta(s))$ in $M = B \times_b F$ is a geodesic if and only if*

$$(i) \quad \alpha(s)'' = g_F(\beta', \beta')(\text{grad}_B b)(b \circ \alpha)$$

$$(ii) \quad \beta(s)'' = \frac{-2}{(b \circ \alpha)} \frac{d(b \circ \alpha)}{ds} \beta' \text{ in } F.$$

Remark 3.1.8. Let $\gamma(s) = (\alpha(s), \beta(s))$ be a geodesic in $M = B \times_b F$. Since β is a pregeodesic in F , and the function $(b \circ \alpha)^4$ is constant C (see [3]) with zero derivative. Thus the property (i) in the above proposition becomes

$$\alpha(s)'' = \frac{C}{(b \circ \alpha)^3} \text{grad}(b) = -\text{grad}\left(\frac{C}{2f^2}\right),$$

by reparametrization it can be assumed that $C/2$ is $-1, 0$ or $+1$ depending on the causal character of β .

3.2 Causal Properties of Warped Products

In this section we express causal properties of warped products and its relations with the base and the fiber of the warped product. The omitted proofs can be found in [2].

Proposition 3.2.1. [2] *Let $M = B \times {}_bF$ be a Lorentzian doubly warped product with the metric $g = g_B \oplus b^2 g_F$. Then if $(p, q) \in M$ then $d\pi_{(p,q)}: T_{(p,q)}(B \times F) \rightarrow T_p(B)$ maps nonspacelike vectors of $T_{(p,q)}(B \times F)$ to nonspacelike vectors of $T_p(B)$ and $\pi: B \times F \rightarrow B$ maps nonspacelike curves of $B \times F$ to nonspacelike curves of B .*

Proposition 3.2.2. [2] *The Lorentzian warped product $M = B \times {}_bF$ is time orientable if and only if (B, g_B) is time orientable (if $r \geq 2$) or (B, g_B) is a one-dimensional manifold with a negative definite metric.*

Proof. Suppose that $B \times {}_bF$ is time orientable. If $\dim(B) = 1$, then (B, g_B) has negative definite metric by Definition 2.1.4. Now let assume $\dim(B) \geq 2$. Since $B \times {}_bF$ is time orientable, there exists a continuous timelike vector field X for $B \times {}_bF$. Since $b > 0$ and g_F is positive definite, we have $g_B(\pi_*(X), \pi_*(X)) \leq g(X, X) < 0$. Thus the vector field $\pi_*(X)$ provides a time orientation for (B, g_B) . For the converse part there are two cases, first suppose that $\dim(B) \geq 2$ and (B, g_B) is time oriented by the timelike vector field Y . Then we can take the lift of Y from B to M , which is also a timelike vector field \tilde{Y} and satisfies $\pi_*(\tilde{Y}) = Y$ and $\sigma_*(\tilde{Y}) = 0$. Let fix $m = (p, q) \in M = B \times {}_bF$. Then we can define \tilde{Y} at m by setting $\tilde{Y}(m) = (Y(p), 0_q)$ by the isomorphism in (1.1). From the definition of Lorentzian warped product it is clear that $g(\tilde{Y}, \tilde{Y}) = g_B(Y, Y) < 0$. Therefore, \tilde{Y} time orients $M = B \times {}_bF$.

Secondly let assume that $\dim(B) = 1$. Let Z be a smooth vector field on B with $g_B(Z, Z) = -1$. If we define $\tilde{Z}(m) = V(\pi(m), 0_{\sigma(m)})$ as above, we have $\sigma_*(\tilde{Z}) = 0$, so that \tilde{Z} time orients M . \square

For a spacetime (H, h) , a C^0 -function $f: H \rightarrow \mathbb{R}$ is a *global time function* if f is strictly increasing along each future directed nonspacelike curve. A spacetime (H, h) admits a global time function if and only if it is stably causal (see [28]). However, there is generally no natural choice of a time function for a stably causal space-time.

Lemma 3.2.3. *Let (F, g_F) be an arbitrary Riemannian manifold and $I = (t_1, t_2)$ for $-\infty \leq t_1 < t_2 \leq \infty$ is given the negative definite metric $-dt^2$. Then for any*

smooth function $b : I \rightarrow (0, \infty)$, the Lorentzian warped product $M = (t_1, t_2) \times {}_bF$ with the metric $g = -dt^2 \oplus b^2g_F$ is stably causal.

Proof. The projection map $\pi : (t_1, t_2) \times F \rightarrow (t_1, t_2)$ serves as a time function. \square

Corollary 3.2.4. *Let (F, g_F) be an arbitrary Riemannian manifold, and let $(I = (t_1, t_2), -dt^2)$ with $-\infty \leq t_1 < t_2 \leq +\infty$ be a Lorentzian manifold. Then for any smooth function $b : I \rightarrow (0, \infty)$, the Lorentzian warped product $M = (t_1, t_2) \times {}_bF$ with the metric $g = -dt^2 \oplus b^2g_F$ is chronological, causal, distinguishing and strongly causal by the causal diagram in the previous chapter.*

Lemma 3.2.5. [2] *Let $M = B \times {}_bF$ be a Lorentzian warped product with the metric $g = g_B \oplus b^2g_F$. Suppose that $(p_1, q_1), (p_2, q_2) \in M$. Also let (p_1, q) and (p_2, q) are points in in the same leaf $\sigma^{-1}(q)$ of M then*

- (i) *if $(p_1, q_1) \ll (p_2, q_2)$ then $p_1 \ll p_2$ in (B, g_B) .*
- (ii) *if $(p_1, q_1) \leq (p_2, q_2)$ then $p_1 \leq p_2$ in (B, g_B) .*
- (iii) *if $p_1 \ll p_2$ then $(p_1, q) \ll (p_2, q)$.*
- (iv) *if $p_1 \leq p_2$ then $(p_1, q) \leq (p_2, q)$.*

Lemma 3.2.5 implies that each leaf $\sigma^{-1}(q)$, $q \in F$, has the same chronology and causality as (B, g_B) . Also this lemma imply that (M, g) has closed time-like or non-spacelike curve if and only if (B, g_B) has closed timelike or nonspacelike curve, respectively.

Proposition 3.2.6. *Let (B, g_B) be a space-time and (F, g_F) be an arbitrary Riemannian manifold. The Lorentzian warped product $M = B \times {}_bF$ with the metric $g = g_B \oplus b^2g_F$ is chronological (respectively causal) if and only if the space-time (B, g_B) is chronological (respectively, causal).*

Proposition 3.2.7. [2] *Let (B, g_B) be a space-time and (F, g_F) be an arbitrary Riemannian manifold. The Lorentzian warped product $M = B \times {}_bF$ with the metric $g = g_B \oplus b^2g_F$ is strongly causal if and only if the space-time (B, g_B) is strongly causal.*

Proposition 3.2.8. *Let (B, g_B) be a space-time and (F, g_F) be an arbitrary Riemannian manifold. The Lorentzian warped product $M = B \times_b F$ with the metric $g = g_B \oplus b^2 g_F$ is stably causal if and only if the space-time (B, g_B) is stably causal.*

Proof. The proof follows from the identification $T_m(B \times F) \cong T_p(B) \times T_q(F)$ for all $m = (p, q) \in B \times F$.

□

The preceding proposition verifies the equivalence of stable causality for (B, g_B) and $M = B \times_b F$ where $\dim(B) \geq 2$. Also from the last three propositions we can deduce that the basic causal properties of $M = B \times_b F$ are determined by those of (B, g_B) .

Theorem 3.2.9. *Let (F, g_F) be an arbitrary Riemannian manifold and let $I = (t_1, t_2)$ where $-\infty \leq t_1 < t_2 \leq \infty$ be given with the negative definite metric $-dt^2$. Then the Lorentzian warped product $M = I \times_b F$ is globally hyperbolic if and only if (F, g_F) is complete.*

Theorem 3.2.10. [2] *Let (B, g_B) be a space-time and let (F, g_F) be a Riemannian manifold. Then the Lorentzian warped product $M = B \times_b F$ with the metric $g = g_B \oplus b^2 g_F$ is globally hyperbolic if and only if both of the following conditions are satisfied*

- (i) (B, g_B) is a globally hyperbolic space-time
- (ii) (F, g_F) is a complete Riemannian manifold.

Theorem 3.2.11. [2] *Let (F, g_F) be a Riemannian manifold. Let the Lorentzian warped product $\mathbb{R} \times F$ be given with the metric $g = -dt^2 \oplus g_F$. Then the following are equivalent*

- (i) (F, g_F) is geodesically complete.
- (ii) $(\mathbb{R} \times F, -dt^2 \oplus g_F)$ is geodesically complete.
- (iii) $(\mathbb{R} \times F, -dt^2 \oplus g_F)$ is globally hyperbolic.

3.3 Curvature of Warped Products

In this chapter we give curvature formulas that express curvature of the warped product in terms of the curvatures of components and the warping function.

Proposition 3.3.1. [3] *Let $M = B \times_b F$ be a warped product with the warped product metric $g = g_B \oplus b^2 g_F$. Let R be the Riemannian curvature tensor on M . If $X, Y, Z \in \mathfrak{L}(B)$ and $U, V, W \in \mathfrak{L}(F)$, then*

- (i) $R(X, Y)Z \in \mathfrak{L}(B)$ is the lift of $R^B(X, Y)Z$ on B .
- (ii) $R(V, X)Y = \frac{\text{hess}_B^b(X, Y)}{b}V$.
- (iii) $R(X, Y)W = R(V, W)X = 0$.
- (iv) $R(X, V)W = \frac{g_F(V, W)}{b}\nabla_X \text{grad}(b)$.
- (v) $R(V, W)U = R^F(V, W)U - \frac{g_B(\text{grad}_B(b), \text{grad}_B(b))}{b^2}[g_F(V, U)W - g_F(W, U)V]$.

These tensor equations are valid for tangent vectors.

Proof. (i) Let R^B be the lift of the Riemannian curvature tensor of B to M . Since $\pi : M \rightarrow B$ is an isometry on each leaf, R^B gives the Riemannian curvature tensor of each leaf. Then by Gauss' equation we have

$$\begin{aligned} g_B(R^B(V, W)X, Y) &= g(R^M(V, W)X, Y) + g(II(V, X), II(W, Y)) \\ &\quad - g(II(V, Y), II(W, Z)) \end{aligned}$$

Since leaves are totally geodesic submanifolds of M the last two terms vanishes (see [3], p.104). Hence

$$g_B(R^B(V, W)X, Y) = g(R^M(V, W)X, Y)$$

- (ii) $R(V, X)Y = \nabla_V \nabla_X Y - \nabla_X \nabla_V Y - \nabla_{[V, X]}Y$. Since $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$ then $[X, V] = 0$. This implies

$$R(V, X)Y = \nabla_V \nabla_X Y - \nabla_X \nabla_V Y.$$

Then by *Proposition 3.1.5*, we obtain

$$\begin{aligned} \nabla_X \nabla_V Y &= -(\nabla_X(Y(b)/b)V) = X(Y(b)/b)V + (V(b)/b)\nabla_X V \\ &= [X(Y(b))/b + Y(b)X(1/b)]V + (Y(b)/b)(X(b)/b)V. \end{aligned}$$

since $X(1/b) = X(b)/b^2$, $\nabla_X \nabla_V Y = [X(Y(b))/b]V$. Hence, we have

$$R(V, X)Y = [((\nabla_X Y)b)/b]V - X(Y(b))V = (\text{hess}^b(X, Y)/b)V.$$

since $\nabla_X Y \in \mathfrak{L}(B)$ and $\nabla_V \nabla_X Y = (\nabla_X Y(b))/bV$

$$(iii) \quad R(V, W)X = \nabla_V \nabla_W X - \nabla_W \nabla_V X \text{ since } [V, W] = 0.$$

$$\nabla_V \nabla_W X = \nabla_V(X(b)/b)W = V(X(b)/b)W + X(b)/b\nabla_V W.$$

On the fibers $X(b)/b$ is constant, therefore $V(X(b)/b) = 0$. Hence

$$R(V, W)X = (X(b)/b)(\nabla_V W - \nabla_W) = (X(b)/b)[V, W] = 0.$$

By the symmetry properties of R we have,

$$g(R(X, Y)V, W) = g(R(V, W)X, Y) = 0.$$

And by (i) we have

$$g(R(X, Y)V, Z) = -g(R(X, Y)Z, V) = 0.$$

These equations hold for all $Z \in \mathfrak{L}(B)$ and $W \in \mathfrak{L}(F)$, hence $R(X, Y)V = 0$.

$$(iv) \quad g(R(X, V)W, U) = g(R(W, U)X, V) = 0 \text{ by (iii), this implies that } R(X, V)W \text{ is horizontal. Since } R(V, W)X = 0, \text{ the Jacobi identity implies that } R(X, V)W = R(X, W)V. \text{ Hence}$$

$$\begin{aligned} g(R(X, V)W, Y) &= g(R(V, X)Y, W) \\ &= \text{hess}^b(X, Y)[g(V, W)/b] = [g(V, W)/b]g(\nabla_X(\text{grad}(b)), Y). \end{aligned}$$

by the property (ii). Then since $R(X, V)W$ is horizontal and the *Equation 3.1* holds for all Y , the result follows.

(v) By the property (iii) we have $g(R(V, W)U, X) = -g(R(V, W)X, U) = 0$, hence $R(V, W)U$ is vertical. Since σ is a homothety on fibers $R^F(V, W)U \in \mathfrak{L}(F)$ is the application to V, W, U of the curvature tensor of each fiber. Then we can use the Gauss equation:

$$\begin{aligned} g_F(R^F(V, W)U, Y) &= g(R^M(V, W)U, Y) + g(II(V, U), (W, Y)) \\ &\quad - g(II(V, Y), II(W, U)). \end{aligned}$$

Since the shape tensor of the fibers is given by

$$II(V, W) = -(b^2 g_F(V, W)/b) \text{grad}(b)$$

the result follows. □

It is possible to express the Ricci curvature tensor Ric of warped product in terms of lifts Ric^B and Ric^F of the base B and the fiber F respectively.

Corollary 3.3.2. [3] *Let $M = B \times_b F$ be a warped product manifold with $s = \dim(F) > 1$ and the metric $g = \pi^*(g_B) \oplus (b\sigma)^2 \sigma^*(g_F)$. Let $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. Then*

$$(i) \text{ Ric}(X, Y) = Ric^B(X, Y) - (s/b) \text{hess}_B^b(X, Y).$$

$$(ii) \text{ Ric}(X, V) = 0.$$

$$(iii) \text{ Ric}(V, W) = Ric^F(V, W) - g(V, W)b^\#, \text{ where}$$

$$b^\# = \frac{\Delta_B b}{b} + (s-1) \frac{g(\text{grad}_B(b), \text{grad}_B(b))}{b^2}.$$

(see Definition 1.4 for $\Delta_B(b)$).

Proof. (i) Let $e_1, \dots, e_r, e_{r+1}, \dots, e_{r+s}$ be a local frame field on M where $\dim(B) = r$, $\dim(F) = s$ and $e_1, \dots, e_r, X, Y \in \mathfrak{L}(B)$ also

$e_{r+1}, \dots, e_{r+s}, V, W \in \mathfrak{L}(F)$. By applying the Ricci curvature formula in *Definition 2.2.2* we get

$$\begin{aligned}
\text{Ric}(X, Y) &= \sum_{i=1}^{r+s} g(e_i, e_i) g(R(e_i, Y)X, e_i) \\
&= \sum_{i=1}^r g(e_i, e_i) g(R(e_i, Y)X, e_i) \\
&\quad + \sum_{i=r+1}^{r+s} g(e_i, e_i) g(R(e_i, Y)X, e_i) \\
&= \text{Ric}^B(X, Y) - \frac{1}{b} \sum_{i=r+1}^{r+s} g(e_i, e_i) g(\text{hess}^b(X, Y)e_i, e_i) \\
&= \text{Ric}^B(X, Y) - \frac{s}{b} \text{hess}^b(X, Y).
\end{aligned}$$

(ii) It follows from the symmetry properties of the Riemannian curvature tensor R and 3.3.1 (iii).

(iii)

$$\begin{aligned}
\text{Ric}(V, W) &= \sum_{i=r+1}^{r+s} g(e_i, e_i) g(R^F(e_i, V)W, e_i) \\
&\quad - g(\text{grad}(b), \text{grad}(b))/b^2 \left[\sum_{i=r+1}^{r+s} g(e_i, e_i) g(V, W)g(e_i, e_i) \right. \\
&\quad \left. - \sum_{i=r+1}^{r+s} g(e_i, e_i) g(e_i, W)g(e_i, V) \right] \\
&\quad - \sum_{i=r}^{r+s} g(e_i, e_i) (g(V, W)/b) (\nabla_{e_i} \text{grad}(b), e_i) \\
&= \text{Ric}^F(V, W) + (s-1)g(\text{grad}(b), \text{grad}(b))/b^2 \\
&\quad g(V, W)(g(V, W)/b)(\nabla_{e_i} \text{grad}(b), e_i).
\end{aligned}$$

□

The manifold $B_f \times {}_bF$ with the metric $f^2g_B \oplus b^2g_F$ is called *doubly warped product*. In the following proposition we give some properties of the Hessian

tensor for doubly warped products from [7]. The case for singly warped products can be obtained by assuming $f \equiv 1$.

Proposition 3.3.3. [7] *Let $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. Then*

Let $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. Then

(i) *If $X, Y \in \mathfrak{L}(B)$ then*

$$\begin{aligned} \text{hess}^b(X, Y) &= \text{hess}_B^b(X, Y) \\ \text{hess}^f(X, Y) &= \frac{f}{b^2} g_B(X, Y) g_F(\text{grad}_F(f), \text{grad}_F(f)) \end{aligned}$$

(ii) *If $V, W \in \mathfrak{L}(F)$ then*

$$\begin{aligned} \text{hess}^f(V, W) &= \text{hess}_F^f(V, W) \\ \text{hess}^b(V, W) &= \frac{b}{f^2} g_F(V, W) g_B(\text{grad}_B(b), \text{grad}_B(b)) \end{aligned}$$

(iii) *If $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$ then*

$$\begin{aligned} \text{hess}^b(X, V) &= -\frac{V(f)X(b)}{(f \circ \sigma)} \\ \text{hess}^f(X, V) &= -\frac{X(b)V(f)}{(b \circ \pi)} \end{aligned}$$

(iv) *If $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$ then*

$$\begin{aligned} \text{hess}^b(V, X) &= -\frac{V(f)X(b)}{(f \circ \sigma)} \\ \text{hess}^f(V, X) &= -\frac{X(b)V(f)}{(b \circ \pi)} \end{aligned}$$

Proof. Let $V, W \in \mathfrak{L}(B)$ then we have

$$\begin{aligned} \text{hess}^f(V, W) &= V(W(f)) - (\nabla_V W)(f) \\ &= V(W(f)) - \left(\nabla_V^F W - \frac{g(V, W)}{(b)} \text{grad}_B(b) \right)(f) \\ &= V(W(f)) - (\nabla_V^F W)(f) + \frac{g(V, W)}{(b)} \text{grad}(b)(f) \\ &= V(W(f)) - (\nabla_V^F W)(f) + \\ &+ \frac{g(V, W)}{(b)} g(\text{grad}_B(b), \text{grad}(f)) \\ &= \text{hess}_F^f(V, W) \end{aligned}$$

Because $g_B(\text{grad}(b), \text{grad}(f)) = g(\frac{\text{grad}_B(b)}{f^2}, \frac{\text{grad}_F(f)}{b^2}) = 0$ and $\text{grad}_B(b) \in \mathfrak{L}(B)$, $\text{grad}_F(f) \in \mathfrak{L}(F)$

Moreover, we have

$$\begin{aligned}
\text{hess}^b(V, W) &= VW(b \circ \pi) - (\nabla_V W)(b) \\
&= V(W(b)) - ((\widetilde{\nabla_V^F W}) + \frac{-g(V, W) \text{grad}_B(b)}{(b)})(b) \\
&= V(0) - (\nabla_V^F W)(b) + \frac{b^2 g_F(V, W)}{b} \frac{(\text{grad}_B(b))(b)}{f^2} \\
&= 0 - 0 + \frac{b g_F(V, W)}{f^2} (\text{grad}_B(b)(b)) \\
&= + \frac{b}{f^2} g_F(V, W) g_B(\text{grad}_B(b), \text{grad}_B(b))
\end{aligned}$$

The other cases can be easily proved by using the symmetry of the *doubly warped product* of the form $M = {}_f B \times {}_b F$ with the metric $g = f^2 g_B \oplus b^2 g_F$.

□

The following theorem gives the sectional curvature formula for a non-degenerate plane section $E = \text{span}(\{\bar{X}, \bar{Y}\})$ of an arbitrary warped product of the form $M = B \times {}_b F$.

Theorem 3.3.4. [7] *Let $M = B \times {}_b F$ be an arbitrary warped product with the metric $g = g_B \oplus b^2 g_F$. Let $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. If $\bar{X} = X + V \in \mathfrak{X}(M)$ and $\bar{Y} = Y + W \in \mathfrak{X}(M)$ and $E = \text{span}(\{\bar{X}, \bar{Y}\})$ is a non-degenerate plane section then*

$$K(E) = \frac{P(E)}{Q(E)}$$

where

$$\begin{aligned}
P(E) &= P_B(\{X, Y\}) + b^2 P_F(\{V, W\}) \\
&- b g_F(V, V) \text{hess}_B^b(Y + W, Y + W) \\
&- b g_F(W, W) \text{hess}_B^b(X + V, X + V) \\
&+ 2b g_F(V, W) \text{hess}_B^b(X + V, Y + W) \\
&+ b^2 g_F(V, V) g_F(W, W) g_B(\text{grad}_B(b), \text{grad}_B(b)) \\
&- b^2 g_F(V, W)^2 g_B(\text{grad}_B(b), \text{grad}_B(b))
\end{aligned}$$

and

$$\begin{aligned} Q(E) &= Q(\{X, Y\}) + Q(\{V, W\}) + Q(\{X, W\}) \\ &+ Q(\{Y, V\}) - 2b^2 g_B(X, Y)g_F(V, W) \end{aligned}$$

Proof. See Appendix. □

3.4 Robertson-Walker Space-Times and Isotropy

In the previous sections we stated the general expression of the Riemannian curvature tensor, Ricci curvature tensor and geodesic equations of a general warped product in terms of its base, fiber and the warping function. Here we apply these results to GRW spacetimes. We omit the proofs since all of the results can be obtained by applying the same procedure that is used in the previous chapter. For more detailed study of geodesic connectedness, geodesic completeness see [9], [29] and [30]. The causal properties of general warped products are valid for Generalized Robertson-Walker space-times and given in *Section 3.2* In order to understand the main difference between Robertson Walker space-times and Generalized Robertson Walker space-times we will recall some concepts from the theory of two-point homogeneous Riemannian manifolds and isotropic Riemannian manifolds. We will see that the GRW space-times generalizes the Robertson-Walker ones with no assumption on the fiber.

Definition 3.4.1. Let (F, g_F) be a Riemannian manifold and $I(F)$ be the isometry group of F and $d_F : F \times F \rightarrow \mathbb{R}$ be the Riemannian distance function of (F, g_F) . Then the Riemannian manifold (F, g_F) is said to be *homogeneous* if $I(F)$ acts transitively on F , i.e., given any $q_1, q_2 \in F$, there is an isometry $\phi \in I(F)$ with $\phi(q_1) = q_2$. Furthermore, (F, g_F) is said to be *two-point homogeneous* if given any $q_1, q_2, q'_1, q'_2 \in F$ with $d_F(q_1, q_2) = d_F(q'_1, q'_2)$, there is an isometry $\phi \in I(F)$ with $\phi(q_1) = q'_1$ and $\phi(q_2) = q'_2$. It is obvious that a two-point homogeneous Riemannian manifold is also homogeneous.

Lemma 3.4.2. [2] *If (F, g_F) is a homogeneous Riemannian manifold, then it is*

complete. Generally this conclusion is false for homogeneous Lorentzian manifolds (see [31]).

Definition 3.4.3. A Riemannian manifold (F, g_F) is said to be *isotropic* at p if $I_p(F)$ acts transitively on the unit sphere $S_p(F)$ of $T_p(F)$, i.e., there is an isometry $\phi \in I_p(F)$ for any given $X_p, Y_p \in S_p(F)$ with $\phi_*(X_p) = Y_p$. And the Riemannian manifold (F, g_F) is said to be isotropic if it is isotropic at every point.

Proposition 3.4.4. [2] *A Riemannian manifold (F, g_F) is isotropic if and only if it is two-point homogeneous.*

Hence the class of isotropic Riemannian manifolds coincides with the class of two-point homogeneous Riemannian manifolds (see [31]).

Corollary 3.4.5. [2] *Any isotropic Riemannian manifold is homogeneous and complete.*

Astronomical observations indicate that the spatial universe is approximately spherically symmetric about the earth. This suggests that the spatial universe should be modeled as a three-dimensional isotropic Riemannian manifold.

Remark 3.4.6. [32] Any three-dimensional Riemannian manifold (F, g_F) has constant sectional curvature.

Here we define Robertson Walker space-times which is a Lorentzian manifold modelling an expanding universe.

Definition 3.4.7. A *Robertson-Walker* space-time (M, g) is any Lorentzian manifold which can be written in the form of a Lorentzian warped product $(I \times_b F, g)$ where $(I = (t_1, t_2), -dt^2)$ is the base for $-\infty \leq t_1 < t_2 \leq \infty$ and (F, g_F) is a three-dimensional isotropic Riemannian manifold and $b > 0$ is a smooth function on I . Hence F has constant sectional c curvature by *Remark 3.4.6* and complete.

In the above definition the standard choices for F are the three-sphere $S^3(c)$, the Euclidean three-space \mathbb{E}^3 , and the hyperbolic three-space $H^3(c)$, with curvature $c > 0$, $c = 0$ and $c < 0$, respectively. The family of Robertson-Walker

space-times includes, for instance, the de Sitter space-time, Friedmann cosmological models, Minkowski space-time, the static Einstein space-time and the anti-de Sitter space-time as well. Robertson-Walker space-times provide good first descriptions of the universe, except in the earliest era and the final era (see [12] and [3]).

Generalized Robertson-Walker space-times extend classical Robertson-Walker ones to include the cases in which the fiber has not constant sectional curvature, i.e, the fiber of the generalized Robertson-Walker space-time is not assumed to be of constant sectional curvature. Thus our ambient space-times widely extend to those that are classically called Robertson-Walker space-times. Recall that this family includes the usual big-bang cosmological models. Contrary to these space-times, our ambient space-times are not necessarily spatially-homogeneous which is a very strong assumption. Note that, to be spatially-homogeneous, which is reasonable as a first approximation of the large scale structure of the universe, could not be appropriate when we consider a more accurate scale. On the other hand, small deformations of the metric on the fiber of classical Robertson-Walker space-times fit into the class of generalized Robertson-Walker space-times.

Definition 3.4.8. A *generalized Robertson-Walker space-time* is a Lorentzian warped product space-time $(t_1, t_2) \times_b F$ where $-\infty \leq t_1 < t_2 \leq \infty$ with $((t_1, t_2), -dt^2)$ is the base, (F, g_F) is the fiber which is s -dimensional connected Riemannian manifold and the warping function is any positive function $b > 0$ on (t_1, t_2) . In other words *GRW spacetime* is the product manifold $M = (t_1, t_2) \times F$ with the Lorentzian metric

$$g = -dt^2 + b^2 g_F \tag{4.1}$$

where $b : I = (t_1, t_2) \rightarrow (0, \infty)$ where the natural projections $\pi : M \rightarrow (t_1, t_2)$ and $\sigma : M \rightarrow F$ are omitted. Also, it is a spatially homogeneous space-time (see [2]).

Note that the fiber is not assumed to be of constant sectional curvature. When this assumption holds and the dimension of the space-time is 3, the GRW space-time is a (classical) Robertson-Walker space-time. Thus, GRW space-times

widely extend Robertson-Walker space-times. Besides, small deformations on the fiber of Robertson-Walker space-times fit into the class of GRW space-times (see [12]).

A generalized Robertson-Walker space-time is said to be *spatially closed* when the fiber F is compact, and *static* when the warping function b is constant.

The lift of $d/dt \in I$ to M is $\frac{\partial}{\partial t} \in \mathfrak{L}(I)$ and we denote it by ∂_t . Besides ∂_t is globally defined time-like vector field which determines the time orientation in $I \times F$.

$$g(\partial_t, \partial_t) = -1. \quad (4.2)$$

The GRW product manifold $M = I \times F$ has two natural orthogonal foliations, say, the foliation by the bases by the bases $I_q = I \times \{q\}$ for $q \in F$ and the foliation by the fibers $F_t = \{t\} \times F$ for $t \in I$. In general, the geometry of GRW is studied by the geometric properties of these foliations.

Particularly, we can express the Levi-Civita connection ∇ of M in terms of the connections of I and F by using *Proposition 3.1.5*.

Corollary 3.4.9. *Let $\partial_t \in \mathfrak{L}(I)$ and $U, V \in \mathfrak{L}(F)$, then*

- (i) $\nabla_{\partial_t} \partial_t = 0$
- (ii) $\nabla_{\partial_t} U = \nabla_U \partial_t = (b'/b)U$
- (iii) $\text{nor} \nabla_U V = \text{II}^F(U, V) = g(U, V)(b''/b)\partial_t$
- (iv) $\text{tan} \nabla_U V$ is the lift of $\text{tan} \nabla_U V$ on F .

Proof. (i) Directly follows from *Proposition 3.1.5(i)*.

(ii) By *Proposition 3.1.5(ii)*, $\nabla_{\partial_t} U = \nabla_U \partial_t = (\partial_t b)/b = (b'/b)U$.

(iii) By (1.2) and *Proposition 3.1.5(iii)* we get the result.

(iv) By *Proposition 3.1.5(iv)*.

□

Also, it is not difficult to check that the bases I_q are *totally geodesic*, and the fibers F_t are *totally umbilical* submanifolds of M .

Remark 3.4.10. The GRW metric (4.1) can be written as follows:

$$g = b^2(t)(-b^{-2}(t)dt^2 + g_F) \equiv b^2(s)(-ds^2 + g_F). \quad (4.3)$$

Here the variable t is changed by the variable s and $ds = dt/b(t)$. Hence, the warped product metric g is conformal to the product metric $h = -dt^2 + g_F$. This fact has the following consequence:

Since the projection π_I is a time function, (M, h) is globally hyperbolic iff F is complete by *Theorem 3.2.10*; in this case, the fibers F_t are Cauchy surfaces for all $t \in I$. Since the causal character of a space-time is a conformal invariant, (M, g) and (M, h) has the same causal character. Further results about geodesics and killing vector fields of GRW space-times can be found in [29].

We can express Riemann and Ricci curvatures of GRW space-times in terms of curvatures of the base I and the fiber F . The proof of the following two corollaries are very similar to *Proposition 3.3.1* and *Corollary 3.3.2*.

Corollary 3.4.11. [9]

Let $\partial_t \in \mathfrak{L}(I)$ and $U, V, W \in \mathfrak{L}(F)$ then,

(i) $R(\partial_t, \partial_t)\partial_t$ is the lift of $R^I(\partial_t, \partial_t)\partial_t$ on I .

(ii) $R(U, \partial_t)\partial_t = (b''/b)U$,

(iii) $R(\partial_t, \partial_t)U = R(U, V)\partial_t = 0$.

(iv) $R(\partial_t, U)V = (b''/f)g(U, V)\partial_t$

(v) $R(U, V)W = R^F(U, V)W - \frac{g(\text{grad}(b), \text{grad}(b))}{b^2}(g(U, W)V - g(V, W)U)$

Corollary 3.4.12. [9] Let $\partial_t \in \mathfrak{L}(I)$ and $U, V \in \mathfrak{L}(F)$ then,

$$(i) \text{ Ric}(\partial_t, \partial_t) = -s(b''/b),$$

$$(ii) \text{ Ric}(\partial_t, V) = 0,$$

$$(iii) \text{ Ric}(V, W) = \text{Ric}_F(V, W) + (b.b'' + (s-1)(b')^2)g_F(V, W)$$

Proof. See Corollary 3.3.2. □

3.5 Geodesics of GRW

Proposition 3.5.1. [3] A curve $\gamma(s) = (\alpha(s), \beta(s))$ in $M = I \times_b F$ is a geodesic if and only if

$$(i) \frac{d^2\alpha}{d\alpha^2} + g_F(\beta', \beta') \frac{db}{d\alpha} b(\alpha) = 0$$

$$(ii) \beta'' + 2 \frac{db}{d\alpha} \frac{1}{b(\alpha)} \frac{d\alpha}{ds} \beta' = 0$$

Proof. Since $\text{grad}(f) = \frac{db}{d\alpha} \partial\alpha$ and $\frac{d}{d\alpha}(b(\alpha)) = \frac{d}{d\alpha} \frac{d\alpha}{ds}$ we get the result by Proposition 3.1.7. □

We give the geodesic equations for GRW space-times from [30]

Let $\gamma : \mathfrak{J} \rightarrow I \times F$, $\gamma(s) = (\alpha(s), \beta(s))$ be a smooth curve on the interval \mathfrak{J} . Then γ is a geodesic with respect to g if and only if

$$\frac{d^2\alpha}{d\alpha^2} = -\frac{c}{b^3 \circ \alpha} \frac{db}{d\alpha} \circ \alpha \tag{5.1}$$

$$\frac{\nabla_F d\beta}{ds} \frac{d\beta}{ds} = -\frac{2}{b \circ \alpha} \frac{d(b \circ \alpha)}{d\alpha} \frac{d\beta}{ds} \tag{5.2}$$

on \mathfrak{J} , where $\frac{\nabla_F}{ds}$ denotes the covariant derivative associated to β and $c = (b^4 \circ \alpha)g_F(\frac{d\beta}{ds}, \frac{d\beta}{ds})$. From *equation* (5.1),

$$\frac{d\alpha}{ds} = \epsilon \left[-g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) + \frac{c}{b^2 \circ \alpha} \right]^{1/2} \quad (5.3)$$

where $\epsilon = \mp 1$.

Note that if $c = 0$ then $d^2\alpha/ds^2 \equiv 0$, i.e. the geodesics on the base I are naturally lifted to geodesics of the GRW space-time as in any warped product.

3.6 Geometry of Standard Static Space-Times

An *observer field* on an arbitrary space-time, say M , is a time-like, future directed unit vector field V . In fact each integral curve of V is an observer. V is *irrotational* if the curl of U , denoted by $\text{curl}U$ (see *Definition* 2.1.13), is zero on vector fields X, Y that are orthogonal to V . Then we are ready to give definition of *static space-time*:

Definition 3.6.1. A space-time M is *static* relative to an observer field V if V is irrotational and there is a smooth function $h > 0$ on M such that hV is a Killing vector field.

Now we will prove that Killing vector field K of a manifold is not determined univocally; nevertheless, the other static Killing fields are constant multiples of K .

Lemma 3.6.2. *Let K be nowhere zero Killing vector field and let h be a C^∞ function. If hK is also a Killing field, then h is a constant.*

Proof. Let fix $p \in M$. Then by *Proposition* 2.3.4(iii),

$$g(\nabla_Y(hK), V) + g(Y, \nabla_V(hK)) = 0 = (Vh)g(Y, X),$$

for any $V \in T_p(M)$. Since $K_p \neq 0$ we can choose V such that $g(V, K) \neq 0$ and conclude $Yh = 0$. Suppose $g(Y, K) \neq 0$. Choosing $V = Y$ gives $Yh = 0$ again. Thus $Yh = 0$ for all $V \in T_p(M)$. Since M is connected and p is arbitrary, $h = \text{constant}$. \square

By *Lemma 3.6.2*, we can identify the Killing field of a static space-time with ∂_t . More information about static space-times can be found in [2]. Here we give the definition of *standard static space-time* which is our main interest. A *standard static space-time* can be considered as a Lorentzian warped product where the warping function defined on the fiber which is a Riemannian manifold and acts on the negative definite metric on the base which is an open interval of real numbers. More formally:

Definition 3.6.3. A *standard static space-time* $I_f \times F$ is a Lorentzian warped product with dimension $m(= s + 1)$ furnished with the metric $g = -f^2 dt^2 \oplus g_F$, where (F, g_F) is a Riemannian manifold of dimension s , $f : F \rightarrow (0, \infty)$ is smooth, and $I = (t_1, t_2)$ for $-\infty \leq t_1 < t_2 \leq \infty$. We always assume that the fiber (F, g_F) of a standard static space-time is connected.

The following proposition shows that any static space-time is locally isometric to a standard one:

Proposition 3.6.4. [3] *A space-time M is static relative to an observer field V if and only if for each $p \in M$ there is a V -preserving isometry of a standard static space-time onto a neighborhood of p .*

The following Ricci curvature formula can be obtained from *Corollary 3.3.2* and using the fact that $(t_1, t_2)_f \times F$ and $F \times_f (t_1, t_2)$ are symmetric.

Proposition 3.6.5. *Let $M = (t_1, t_2)_f \times F$ be a standard static space-time with the metric $g = -f^2 dt^2 \oplus g_F$. Also let $V, W \in \mathfrak{L}(F)$ and $\partial_t \in \mathfrak{L}((t_1, t_2))$. Then*

$$\text{Ric}(\partial_t + V, \partial_t + W) = \text{Ric}_F(V, W) + f\Delta_F(f) - \frac{1}{f} \text{hess}_F^f(V, W),$$

where Ric and Ric_F denote the Ricci curvatures of M and F respectively, and hess_F is the Hessian tensor on F .

Proof. Since $(t_1, t_2)_f \times F$ and $F \times_f (t_1, t_2)$ are symmetric, we can interchange the role of the fiber and the base in order to apply *Corollary 3.3.2*. By bilinearity of the Ricci tensor we have

$$\text{Ric}(\partial_t + V, \partial_t + W) = \text{Ric}(\partial_t, \partial_t) + \text{Ric}(\partial_t, W) + \text{Ric}(\partial_t, V) + \text{Ric}(V, W).$$

The terms

$$\text{Ric}(\partial_t, W) = \text{Ric}(\partial_t, V) = 0.$$

by *Corollary 3.3.2*,(ii). We can compute $\text{Ric}(\partial_t, \partial_t)$ by inserting in *Corollary 3.3.2*,(iii):

$$\text{Ric}(\partial_t, \partial_t) = \text{Ric}_{(t_1, t_2)}(\partial_t, \partial_t) - g_{(t_1, t_2)}(\partial_t, \partial_t)[(1-1)g_{(t_1, t_2)}(\text{grad}_F(f), \text{grad}_F(f))] + f\Delta_F(f).$$

Then we get

$$\text{Ric}(\partial_t, \partial_t) = f\Delta_F(f). \quad (6.1)$$

$$\text{Ric}(V, W) = \text{Ric}_F(V, W) - \frac{1}{f} \text{hess}_F^f(V, W) \quad (6.2)$$

Hence the result follows from *equations (6.1), (6.2)*. \square

Proposition 3.6.6. [13] *A smooth curve $\gamma(s) = (\alpha(s), \beta(s)) : I \rightarrow (t_1, t_2) \times F$ in a standard static space-time of the form $M = (t_1, t_2)_f \times F$ with the metric $g = -f^2 dt^2 \oplus g_F$ is a geodesic if and only if the following hold:*

$$(i) \quad \alpha(s)'' = \frac{-2}{f \circ \beta} \frac{d(f \circ \beta)}{dt} \alpha',$$

$$(ii) \quad \beta(s)'' = -(f \circ \beta)(\alpha')^2 \text{grad}_F(f)|_{\beta(t)}.$$

Remark 3.6.7. [13] If $\gamma(s) = (\alpha(s), \beta(s))$ is a geodesic in a standard static space-time, then $-(f \circ \beta)^4 (\alpha')^2 \equiv C$ is constant and $-(f \circ \beta)^2 (\alpha')^2 + g_F(\beta', \beta') \equiv D$, i.e, the constant speed of the geodesic. Moreover, α turns out to be a pre-geodesic on (t_1, t_2) with the metric $-dt^2$.

Chapter 4

Standard Static Space-Times

In this chapter, we obtain some conditions to have nonnegative sectional curvature so that applications of singularity theorems to a standard static space-time can be considered. In [20], Graves and Nomizu stated some important results on sectional curvatures of indefinite metrics, including Lorentzian metric. Moreover, in [21] the results found in [20] are extended and clarified.

4.1 Sectional Curvature

In this section we will prove necessary and sufficient conditions for a standard static space-time to have non-negative time-like sectional curvatures.

From *Chapter 2*, for a semi-Riemannian manifold (M, g) , a *plane section* is two dimensional linear subspace E of $T_p(M)$. If for each nontrivial vector $X_1 \in E$ there exists $Y_1 \in E$ such that $g(X_1, Y_1) \neq 0$ then g is non-degenerate. This condition is the same with requiring that $g_p|_E$ be a non-degenerate inner product on E . Let X_1 and Z_1 be a basis for the plane section E . Then

$$g(X_1, X_1)g(Z_1, Z_1) - [g(X_1, Z_1)]^2 \neq 0$$

if and only if E is non-degenerate.

Proposition 4.1.1. *Let $M = (t_1, t_2)_f \times F$ be standard static space-time with the metric $g = -f^2 dt^2 \oplus g_F$ and $\dim(F) \geq 2$. Also let $V, W \in \mathfrak{L}(F)$ and $\partial_t \in \mathfrak{L}(t_1, t_2)$. If $\bar{X} = \partial_t + \alpha V \in \mathfrak{X}(M)$ and $\bar{Y} = W \in \mathfrak{X}(M)$ and $-1 < \alpha < 1$, $g_{(t_1, t_2)}(\partial_t, \partial_t) = -1$, $g_F(V, V) = f^2$, $g_F(V, W) = 0$, $g_F(W, W) > 0$ then $E = \text{span}(\{\bar{X}, \bar{Y}\})$ is a non-degenerate time-like plane section and $K(E) = P(E)/Q(E)$ where*

$$Q(E) = f^2(\alpha^2 - 1)g_F(W, W) \text{ and } P(E) = \alpha^2 P_F(V, W) + f \text{hess}_F^f(W, W).$$

Proof. By Theorem A.0.8 and symmetry of $(t_1, t_2)_f \times F$ and $F \times_f (t_1, t_2)$ we have:

$$\begin{aligned} P(E) &= P(\bar{X}, \bar{Y}) \\ &= P_F(\alpha V, W) + f^2 P_{(t_1, t_2)}(\partial_t, 0) - f g_B(\partial_t, \partial_t) \text{hess}_F(f)(W, W) \\ &\quad - f g_B(0, 0) \text{hess}_F^f(\alpha V, \alpha V) - f^2 Q_{(t_1, t_2)}(\partial_t, 0) g_F(\text{grad}(f), \text{grad}(f)) \\ &= P_F(\alpha V, W) - (-f) \text{hess}_F^f(W, W) \\ &= \alpha^2 P_F(V, W) + f \text{hess}_F^f(W, W). \end{aligned} \tag{1.1}$$

$$\begin{aligned} Q(E) &= Q_F(\alpha V, W) + f^4 Q_{(t_1, t_2)}(\partial_t, 0) + f^2 g_F(\alpha V, \alpha V) g_{(t_1, t_2)}(0, 0) \\ &\quad + f^2 g_F(W, W) g_{(t_1, t_2)}(\partial_t, \partial_t) - 2f^2 g_F(\alpha V, W) g_{(t_1, t_2)}(\partial_t, 0) \\ &= g_F(\alpha V, \alpha V) g_F(W, W) - g_F(\alpha V, W)^2 - f^2 g_F(W, W) \\ &= \alpha^2 f^2 g_F(W, W) - f^2 g_F(W, W) \\ &= f^2 g_F(W, W)(\alpha^2 - 1). \end{aligned} \tag{1.2}$$

□

Notice that we only consider time-like plane sections E as in the above form because we can decompose any vector field on M by decomposing it into tangential and normal components, and by using *General Axes Theorem* we can deduce that every non-degenerate time-like plane section of a standard static space-time can be generated by tangent vectors as the form mentioned in *Proposition 4.1.1*.

Moreover, notice that $E_F = \text{span}\{V, W\}$ is a non-degenerate plane section in F because $Q_F(E_F) = f^2 g_F(W, W)$. Thus $K(E)$ can be expressed as follows:

$$K(E) = \frac{1}{\alpha^2 - 1} (\alpha^2 K_F(E_F) + \frac{\text{hess}_F^f(W, W)}{f g_F(W, W)}).$$

Proposition 4.1.2. *Let $M = (t_1, t_2)_f \times F$ be standard static space-time with the metric $g = -f^2 dt^2 \oplus g_F$ and $\dim(F) \geq 2$. If the fiber (F, g_F) has non-positive sectional curvature, i.e., $K(E_F) \leq 0$, for every non-degenerate plane section E_F in F and $\text{hess}_F^f(W, W) \leq 0$ for any vector field W on F , then the space-time (M, g) has non-negative time-like sectional curvature, i.e., $K(E) \geq 0$, for every non-degenerate time-like plane section E in M .*

Proof. Since $|\alpha| < 1$, $1/(\alpha^2 - 1) < 0$, $\alpha^2 K_F(E_F) \leq 0$ and $\text{hess}_F^f(W, W)/(f g_F(W, W)) \leq 0$ then $K(E) \geq 0$. □

Proposition 4.1.3. *Let $M = (t_1, t_2)_f \times F$ be standard static space-time with the metric $g = -f^2 dt^2 \oplus g_F$ and $\dim(F) \geq 2$. If the fiber (F, g_F) has non-negative sectional curvature, i.e., $K(E_F) \geq 0$, for every non-degenerate plane section E_F in F and $\text{hess}_F^f(W, W) \geq k f g_F(W, W)$ for some $k \in (-\infty, 0)$ and for any vector field W on F , then the space-time (M, g) has time-like sectional curvature bounded by k from above, i.e., $K(E) \leq k < 0$, for every non-degenerate time-like plane section E in M .*

Proof. We have that $|\alpha| < 1$ and $1/(\alpha^2 - 1) < 0$. If we assume $K_F(E_F) = 0$ then we get $K(E) \leq k < 0$. For the case $K_F(E_F) \neq 0$, we obtain values which are smaller than K . □

The following proposition gives the sectional curvature formula for a non-degenerate plane section $E = \text{span}(\{\bar{X}, \bar{Y}\})$ of an arbitrary warped product of the form $M = B_f \times F$ furnished with $g = f^2 g_B \oplus g_F$ for $f \in C_{>}^\infty(F)$.

Proposition 4.1.4. *Let $M = B_f \times F$ be an arbitrary warped product with the metric $g = f^2 g_B \oplus g_F$. Let $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. If $\bar{X} = X + V \in$*

$\mathfrak{X}(M)$, and $\bar{Y} = Y + W \in \mathfrak{X}(M)$ and $E = \text{span}(\{\bar{X}, \bar{Y}\})$ is a non-degenerate plane section then

$$K(E) = \frac{P(E)}{Q(E)},$$

where

$$\begin{aligned} P(E) &= P_F(\{V, W\}) \\ &+ f(h_2^2 \text{hess}_F^f(V, V) + h_1^2 \text{hess}_F^f(W, W) \\ &- 2fh_1h_2 \text{hess}_F^f(V, W)), \end{aligned}$$

and

$$\begin{aligned} Q(E) &= Q_F(V, W) - f^2[h_2^2g_F(V, V) - 2h_1h_2g_F(V, W) \\ &+ h_1^2g_F(W, W)], \end{aligned}$$

where $g_B = -dt^2$, $B = I$, $X = h_1\partial_t$ and $Y = h_2\partial_t$ for $h_1, h_2 \in C^\infty(I)$.

Proof. The result can be obtained by applying *Theorem A.0.8* and substituting the following values $g_B = -dt^2$, $B = I$, $X = h_1\partial_t$ and $Y = h_2\partial_t$ for $h_1, h_2 \in C^\infty(I)$. Then we have,

$$\begin{aligned} Q(E) &= Q_F(V, W) + f^4Q_B(X, Y) \\ &+ f^2(g_F(V, V)g_B(Y, Y) + g_F(W, W)g_B(X, X)) \\ &- 2f^2g_F(V, W)g_B(X, Y), \end{aligned}$$

$$\begin{aligned} Q_B(h_1\partial_t, h_2\partial_t) &= g_B(h_1\partial_t, h_1\partial_t)g_B(h_2\partial_t, h_2\partial_t) \\ &- g_B(h_1\partial_t, h_2\partial_t)^2 \\ &= (-h_1^2)(-h_2^2) - (-h_1h_2)^2 \\ &= 0. \end{aligned}$$

So,

$$\begin{aligned} Q(E) &= Q_F(V, W) \\ &- f^2(h_1^2g_F(W, W) + h_2^2g_F(V, V)) \\ &+ 2h_1h_2f^2g_F(V, W), \end{aligned}$$

$$\begin{aligned}
P(E) &= P_F(\{V, W\}) \\
&+ f^2 P_B(X, Y) - f^2 Q_B(X, Y) g_F(\text{grad}_F(f), \text{grad}_F(f)) \\
&- f(\text{hess}_F^f(V, V) g_B(Y, Y) + \text{hess}_F^f(W, W) g_B(X, X)) \\
&+ 2f \text{hess}_F^f(V, W) g_B(X, Y).
\end{aligned}$$

Now, $Q_B(h_1 \partial_t, h_2 \partial_t) = 0$ as before. Then,

$$\begin{aligned}
P_B(X, Y) &= g_B(R^B(X, X)X, Y) \\
&= g_B(R(h_2 \partial_t, h_1 \partial_t), h_1 \partial_t, h_2 \partial_t) \\
&= h_1^2 h_2^2 g_B(R(\partial_t, \partial_t) \partial_t, \partial_t) \\
&= 0.
\end{aligned}$$

Then the result follows. □

Remark 4.1.5. In the above proposition, if $V \perp W$, i.e., $g_F(V, W) = 0$ then,

$$Q(E) = Q_F(V, W) - f^2[h_2^2 g_F(V, V) + h_1^2 g_F(W, W)],$$

and

$$\begin{aligned}
P(E) &= P_F(\{V, W\}) \\
&+ f(h_2^2 \text{hess}_F^f(V, V) + h_1^2 \text{hess}_F^f(W, W)) \\
&- 2fh_1 h_2 \text{hess}_F^f(V, W).
\end{aligned}$$

In the following proposition, we express the situation that when a non-degenerate time-like plane section of a standard static space-time has constant sectional curvature implies the fiber has constant sectional curvature.

Proposition 4.1.6. *If $\bar{X} = V \in \mathfrak{X}(M)$ and $\bar{Y} = W \in \mathfrak{X}(M)$, i.e., $h_1 \equiv 0$ and $h_2 \equiv 0$ then,*

$$\begin{aligned}
K(\{V, W\}) &= P(\{V, W\})/Q(\{V, W\}) \\
&= P_F(\{V, W\})/Q_F(\{V, W\}) \\
&= c,
\end{aligned}$$

if $K = c$.

Proof. Take $h_1 \equiv 0$ and $h_2 \equiv 0$ and substitute in *Proposition 4.1.4*. Then we have

$$\begin{aligned} Q(E) &= Q_F(V, W) \\ &= g_F(V, V)g_F(W, W) - g_F(V, W)^2 \end{aligned}$$

and $P(\{V, W\}) = P_F(\{V, W\})$. □

Proposition 4.1.7. *Let $h_1 \equiv h$ and $h_2 \equiv 0$. If c is the constant sectional curvature of (F, g_F) (i.e., $P_F = cQ_F$) and $\text{hess}_F^f = -c f g_F$ then, standard static space-time has constant sectional curvature c .*

Proof. By *Proposition 4.1.4* we have, $Q(\{h\partial_t + V, W\}) = Q_F(\{V, W\}) - f^2 h^2 g_F(W, W)$ and

$$P(\{h\partial_t + V, W\}) = P_F(\{V, W\}) + f h^2 \text{hess}_F^f(W, W).$$

If $P_F = cQ_F$ and $\text{hess}_F^f = -c f g_F$ then $K = c$. □

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Appendix A

Theorem A.0.8. [7] Let $M = B \times_b F$ be an arbitrary warped product with the metric $g = g_B \oplus b^2 g_F$. Let $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. If $\bar{X} = X + V \in \mathfrak{X}(M)$ and $\bar{Y} = Y + W \in \mathfrak{X}(M)$ and $E = \text{span}(\{\bar{X}, \bar{Y}\})$ is a non-degenerate plane section then

$$K(E) = \frac{P(E)}{Q(E)}$$

where

$$\begin{aligned} P(E) &= P_B(\{X, Y\}) + b^2 P_F(\{V, W\}) \\ &\quad - b g_F(V, V) \text{hess}_B^b(Y + W, Y + W) \\ &\quad - b g_F(W, W) \text{hess}_B^b(X + V, X + V) \\ &\quad + 2b g_F(V, W) \text{hess}_B^b(X + V, Y + W) \\ &\quad + b^2 g_F(V, V) g_F(W, W) g_B(\text{grad}_B(b), \text{grad}_B(b)) \\ &\quad - b^2 g_F(V, W)^2 g_B(\text{grad}_B(b), \text{grad}_B(b)) \end{aligned}$$

and

$$\begin{aligned} Q(E) &= Q(\{X, Y\}) + Q(\{V, W\}) + Q(\{X, W\}) \\ &\quad + Q(\{Y, V\}) - 2b^2 g_B(X, Y) g_F(V, W) \end{aligned}$$

Proof. Here we write b instead of $b \circ \pi$ for the lift \tilde{b} of b from B to M . Let $E = \text{span}(\{\bar{X}, \bar{Y}\})$ be the non-degenerate plane section where $\bar{X} = X + V \in \mathfrak{X}(M)$

and $\bar{Y} = Y + W \in \mathfrak{X}(M)$ then

$$\begin{aligned}
Q(E) &= Q(\{X + V, Y + W\}) \\
&= g(X + V, X + V)g(Y + W, Y + W) - (g(X + V, Y + W))^2 \\
&= (g(X, X) + 2g(X, V) + g(V, V))(g(Y, Y) + 2g(Y, W) + g(W, W)) \\
&\quad - (g(X, Y) + g(V, W) + g(X, W) + g(V, Y))^2 \\
&= (g_B(X, X) + b^2g_F(V, V))(g_B(Y, Y) + b^2g_F(W, W)) \\
&\quad - (g_B(X, Y) + g_F(V, W))^2 \\
&= g_B(X, X)g_B(Y, Y) + b^2g_B(X, X)g_F(W, W) + b^2g_B(Y, Y)g_F(V, V) \\
&\quad + b^4g_F(V, V)g_F(W, W) - (g_B(X, Y))^2 - 2b^2g_B(X, Y)g_F(V, W) \\
&\quad - b^4(g_F(V, W))^2 \\
&= (g_B(X, X)g_B(Y, Y) - (g_B(X, Y))^2) \\
&\quad + b^4(g_F(V, V)g_F(W, W) - (g_F(V, W))^2) \\
&\quad + b^2(g_B(X, X)g_F(W, W) + g_B(Y, Y)g_F(V, V) - 2g_B(X, Y)g_F(V, W)) \\
&= Q_B(\{X, Y\}) + b^4Q_F(\{V, W\}) + b^2g_B(X, X)g_F(W, W) \\
&\quad + b^2g_B(Y, Y)g_F(V, V) - 2b^2g_B(X, Y)g_F(V, W) \\
&= Q(\{X, Y\}) + Q(\{V, W\}) + Q(\{X, W\}) + Q(\{Y, V\}) \\
&\quad - 2b^2g_B(X, Y)g_F(V, W) \\
&= Q(\{X, Y\}) + Q(\{V, W\}) + Q(\{X, W\}) + Q(\{Y, V\}) - 2g(X, Y)g(V, W)
\end{aligned}$$

$$\begin{aligned}
P(E) &= P(\{X + V, Y + W\}) \\
&= g(R(Y + W, X + V)(X + V), Y + W) \\
&= g(\widetilde{R_B(Y, X)}X, Y) \\
&+ \widetilde{R_F(W, V)}V - \frac{g(V, V)}{(b)} \nabla_W(\text{grad}_B(b)) \\
&+ \frac{g(W, V)}{(b)} \nabla_V(\text{grad}_B(b)) \\
&+ Y\left(\frac{X(b)}{(b)}\right)V + \frac{Y(b)X(b)}{(b)^2}V \\
&+ \frac{g(W, V)}{(b)} \nabla_X(\text{grad}_B(b)) \\
&+ Y\left(\frac{X(b)}{(b)}\right)V \\
&- X\left(\frac{Y(b)}{(b)}\right)V - \frac{[Y, X](b)}{(b)}V \\
&+ \frac{(\widetilde{\nabla_X^B X})(b)}{(b)}W \\
&- X\left(\frac{X(b)}{(b)}\right)W - \frac{X(b)X(b)}{(b)^2}W \\
&- \frac{g(V, V)}{(b \circ \pi)} \nabla_Y(\text{grad}(b \circ \pi))
\end{aligned}$$

$$\begin{aligned}
&= g_B(R_B(Y, X)X, Y) \\
&+ g(R_F(\widetilde{W}, \widetilde{V})V, Y) - \frac{g(V, V)}{(b)}g(\nabla_W(\text{grad}_B(b)), Y) \\
&+ \frac{g(W, V)}{(b)}g(\nabla_V(\text{grad}(b)), Y) \\
&+ Y\left(\frac{X(b)}{(b \circ \pi)}\right)g(V, Y) + \frac{Y(b)X(b)}{(b)^2}g(V, Y) \\
&- \frac{(\widetilde{\nabla_Y^B X})(b)}{(b)}g(V, Y) \\
&+ \frac{g(W, V)}{(b)}g(\nabla_X(\text{grad}_B(b)), Y) \\
&+ Y\left(\frac{X(b)}{(b)}\right)g(V, Y) \\
&- X\left(\frac{Y(b)}{(b)}\right)g(V, Y) - \frac{[Y, X](b)}{(b)}g(V, Y) \\
&+ \frac{(\widetilde{\nabla_X^B X})(b)}{(b)}g(W, Y) \\
&- X\left(\frac{X(b)}{(b)}\right)g(W, Y) - \frac{X(b)X(b)}{(b)^2}g(W, Y) \\
&- \frac{g(V, V)}{(b)}g(\nabla_Y(\text{grad}_B(b)), Y)
\end{aligned}$$

$$\begin{aligned}
& + g(R_B(\widetilde{Y, X})X, W) \\
& + b^2 g_F(R_F(W, V)V, W) - \frac{g(V, V)}{(b)} g(\nabla_W(\text{grad}_B(b)), W) \\
& + \frac{g(W, V)}{(b)} g(\nabla_V(\text{grad}_B(b)), W) \\
& + Y\left(\frac{X(b)}{(b)}\right) g(V, W) \\
& - \frac{(\widetilde{\nabla_Y^B X})(b)}{(b)} g(V, W) + bg(W, W) \\
& + \frac{g(W, V)}{(b)} g(\nabla_X(\text{grad}_B(b)), W) \\
& + Y\left(\frac{X(b)}{(b)}\right) g(V, W) \\
& - X\left(\frac{Y(b)}{(b)}\right) g(V, W) \\
& - \frac{[Y, X](b)}{(b)} g(V, W) \\
& + \frac{(\widetilde{\nabla_X^B X})(b)}{(b)} g(W, W) - X\left(\frac{X(b)}{b}\right) g(W, W) \\
& - \frac{X(b)X(b)}{b^2} g(W, W) + \frac{g(V, V)}{b} g(\nabla_Y(\text{grad}_B(b)), W)
\end{aligned}$$

$$\begin{aligned}
&= P_B(\{X, Y\}) + b^2 P_F(\{V, W\}) \\
&- g_F(V, V) \left(b \text{hess}^b(W, Y) + b \text{hess}^b(W, W) + \right. \\
&+ b \text{hess}^b(Y, Y) + b \text{hess}^b(Y, W) \left. \right) \\
&+ g_F(W, W) \left(-b \text{hess}^b(X, V) - b \text{hess}^b(X, V) + \right. \\
&+ b(\nabla_X^B X)(b) - b(X(X(b))) + X(b)X(b) - X(b)X(b) \left. \right) \\
&+ g_F(V, W) \left(b \text{hess}^b(V, Y) + \text{hess}^b(X, Y) + b \text{hess}^b(V, W) + \right. \\
&+ b(Y(X(b))) - Y(b)X(b) + Y(b)X(b) - b(\nabla_Y^B X)(b) - \\
&+ b \text{hess}^b(X, W) + b(Y(X(b))) - Y(b)X(b) - \\
&- b(X(Y(b))) + X(b)Y(b) - b(Y(X(b))) + b(X(Y(b))) + \\
&+ b \text{hess}^b(X, W) + b \text{hess}^b(V, Y) \left. \right) \\
&= P_B(\{X, Y\}) + b^2 P_F(\{V, W\}) \\
&- g_F(V, V) \left(b \text{hess}^b(W, Y) + b \text{hess}^b(W, W) + \right. \\
&+ b \text{hess}^b(Y, Y) + b \text{hess}^b(Y, W) \left. \right) \\
&+ g_F(W, W) \left(-b \text{hess}^b(X, V) - b \text{hess}^b(X, V) - \right. \\
&- b(X(X(b)) - (\nabla_X^B X)(b)) \left. \right) \\
&+ g_F(V, W) \left(b \text{hess}^b(V, Y) + b \text{hess}^b(X, Y) + b \text{hess}^b(V, W) + \right. \\
&+ b(Y(X(b)) - (\nabla_Y^B X)(b)) - \\
&+ b \text{hess}^b(X, W) + b \text{hess}^b(X, W) + b \text{hess}^b(V, Y) \left. \right) \\
&= P_B(\{X, Y\}) + b^2 P_F(\{V, W\}) \\
&- g_F(V, V) \left(b \text{hess}^b(W, Y) + b \text{hess}^b(W, W) + \right. \\
&+ b \text{hess}^b(Y, Y) + b \text{hess}^b(Y, W) \left. \right) \\
&+ g_F(W, W) \left(-b \text{hess}^b(X, V) - b \text{hess}^b(X, V) - \right. \\
&- b \text{hess}^b(X, X) \left. \right) \\
&+ g_F(V, W) \left(b \text{hess}^b(V, Y) + b \text{hess}^b(X, Y) + \right. \\
&+ b \text{hess}^b(V, W) + b \text{hess}^b(Y, X) - \\
&+ b \text{hess}^b(X, W) + b \text{hess}^b(X, W) + b \text{hess}^b(V, Y) \left. \right)
\end{aligned}$$

In order to obtain a symmetric equation we add and subtract the following equation to $P(E)$:

$$bg_F(W, W) \text{hess}^b(V, V) + bg_F(V, W) \text{hess}^b(V, W)$$

Then by *Proposition 3.3.3* with $f \equiv 1$ and bilinearity of hessian operator we have

$$\begin{aligned} P(E) &= P_B(\{X, Y\}) + b^2 P_F(\{V, W\}) \\ &\quad - bg_F(V, V) \text{hess}_B^b g(Y + W, Y + W) \\ &\quad - bg_F(W, W) \text{hess}_B^b g(X + V, X + V) \\ &\quad + 2bg_F(V, W) \text{hess}_B^b g(X + V, Y + W) \\ &\quad + bg_F(W, W)(bg_F(V, V)g_B(\text{grad}_B(b), \text{grad}_B(b))) \\ &\quad - bg_F(V, W)(bg_F(V, W)g_B(\text{grad}_B(b), \text{grad}_B(b))) \end{aligned}$$

Hence,

$$\begin{aligned} P(E) &= P_B(\{X, Y\}) + b^2 P_F(\{V, W\}) \\ &\quad - bg_F(V, V) \text{hess}_B^b g(Y + W, Y + W) \\ &\quad - bg_F(W, W) \text{hess}_B^b g(X + V, X + V) \\ &\quad + 2bg_F(V, W) \text{hess}_B^b g(X + V, Y + W) \\ &\quad + b^2 g_F(V, V)g_F(W, W)g_B(\text{grad}_B(b), \text{grad}_B(b)) \\ &\quad - b^2 g_F(V, W)^2 g_B(\text{grad}_B(b), \text{grad}_B(b)) \end{aligned}$$

Then the result follows. □