PIATETSKI-SHAPIRO PRIME NUMBER THEOREM AND CHEBOTAREV DENSITY THEOREM

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ABSTRACT

PIATETSKI-SHAPIRO PRIME NUMBER THEOREM AND CHEBOTAREV DENSITY THEOREM

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Let K be a finite Galois extension of the field $\mathbb{Q}$ of rational numbers. In this thesis, we derive an asymptotic formula for the number of the Piatetski-Shapiro primes not exceeding a given quantity for which the associated Frobenius class of automorphisms coincide with any given conjugacy class in the Galois group of $K/\mathbb{Q}$. Applying this theorem to appropriate field extensions, we conclude that there are infinitely many Piatetski-Shapiro primes lying in a given arithmetic progresion and furthermore there are infinitely many primes that can be expressed as a sum of a square and a fixed positive integer multiple of another square.

Keywords: Chebotarev density theorem, Piatetski-Shapiro prime number theorem, exponential sums over ideals.

ÖZET

PIATETSKI SHAPIRO ASAL SAYI TEOREMI VE CHEBOTAREV YOĞUNLUK TEOREMI.

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K rasyonel sayı cisminin bir Galois genişlemesi olsun. Bu tezde, Frobenius otomorfizması C konjuge sınıfına denk gelen Piatetski-Shapiro asallarının asimptotiği incelenmiştir. Elde edilen asimptotik bağıntı bazı cisim genişlemelerine uygulanarak ilk önce verilmiş bir n pozitif doğal sayısı için $a^2 + nb^2$ şeklindeki Piatetski-Shapiro asallarının asimptotiği; sonrasinda arithmetik dizilerdeki Piatetski-Shapiro asallarının asimptotiği hesaplanmıştır.

Anahtar sözcükler: Chebotarev yoğunluk teoremi, Piatetski-Shapiro asal sayıteoremi, idealler üzerine üssel toplamlar .

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Contents

Notation

Throughout this thesis, we use Vinogradov's notation $f \ll g$ to mean that $|f(x)| \leqslant Cg(x)$, where g is a positive function and $C > 0$ is a constant. Similarly, we define $f \gg g$ to mean $|f| \geqslant Cg$ and $f \asymp g$ to mean both $f \ll g$ and $f \gg g$. We write $n \sim N$ to mean that n lies in a subinterval of $(N, 2N]$. We will use $\varepsilon > 0$ to denote a quantity which may be taken arbitrarily small and not the same at each occurence. Morever, $c > 1$ is a real number, and $\delta = 1/c$.

- For a fixed $c > 1$, we set

  $$A_c(x) = \{\lfloor n^c \rfloor \leqslant x |\ n \in \mathbb{N}\},$$

  where $\lfloor x \rfloor$ is the floor of x defined to be largest integer not exceeding x.

- For any $x \geqslant 2$ and $1 \leqslant a \leqslant q$ with $\gcd(a, q) = 1$, we set

  $$\pi(x; q, a) = \#\{p \leqslant x : p \text{ prime}, \ p \equiv a \bmod (q)\}.$$

- For any function f, we put

  $$\Delta f(x) = f(-(x+1)^\delta) - f(-x^\delta), \quad (x > 0).$$

- For any subset $\mathbb{P}$ of primes, we denote by $\langle \mathbb{P} \rangle$ the subset of natural numbers that are composed solely of primes from $\mathbb{P}$.

We write $e(z)$ for $\exp(2\pi i z)$. We use the notation $\psi(x)$ for $x - \lfloor x \rfloor - \frac{1}{2}$.

For any finite field extension $L/\mathbb{Q}$, we shall write Δ_L for its absolute discriminant and d_L for its degree $[L : \mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of real embeddings of L and $2r_2$ is the number of complex embeddings. We denote the ring of integers of L by $\mathfrak{O}_L$ and the absolute norm of an ideal $\mathfrak{a}$ is denoted by $\mathfrak{N}\mathfrak{a}$.

The letter p always denotes an ordinary prime number. Similarly, we use the letters $\mathfrak{p}$, $\mathfrak{P}$ for prime ideals.

Chapter 1

Introduction and Statement of Results

In 1953 Ilya Piatetski-Shapiro proved in [12] an analog of the prime number theorem for primes of the form $\lfloor n^c \rfloor$, where n runs through positive integers and $c > 0$ is fixed. He showed therein that such primes constitute a thin subset of the primes; more precisely, that the number $\pi_c(x)$ of these primes not exceeding a given number x is asymptotic to $x^{1/c}/\log x$ provided that $c \in (1, 12/11)$. Since then, the admissible range of c has been extended by many authors and the result is currently known for $c \in (1, 2817/2426)$ (cf. [13]).

A related question is to determine the asymptotic behavior of a particular subset of these primes; for example, those belonging to a given arithmetic progression, or those of the form $a^2 + nb^2$. The former was considered by Leitmann and Wolke (cf. [8]) in 1974, and it has been used in a recent paper by Roger et al. (cf. [1]) to show the existence of infinitely many Carmichael numbers that are products of the Piateski-Shapiro primes.

For both of the aforementioned examples, the problem can be interpreted as counting the Piatetski-Shapiro primes that belong to a particular Chebotarev class of some number field (see Theorem 1 and the remark following Theorem 2). Motivated by this observation, we study in this thesis the following more general problem:

Take a finite Galois extension $K/\mathbb{Q}$ and a conjugacy class C in the Galois group $G = Gal(K/\mathbb{Q})$. Put

$$\pi(K, C) = \{p \text{ prime}: \ \gcd(p, \Delta_K) = 1; \ [K/\mathbb{Q}, p] = C\}$$

where Δ_K is the discriminant of K, and the Artin symbol $[K/\mathbb{Q}, p]$ is defined as the conjugacy class of the Frobenius automorphism associated with any prime ideal $\mathfrak{P}$ of K above p. Recall that the Frobenius automorphism is the generator of the decomposition group of $\mathfrak{P}$, which is the cyclic subgroup of automorphisms of G that fixes $\mathfrak{P}$. The Chebotarev Density Theorem as given by Lemma 10 below states that the natural density of primes in $\pi(K, C)$ is $|C|/|G|$; that is,

$$\pi(K, C, x) \sim \frac{|C|}{|G|} \operatorname{li}(x) \qquad (x \to \infty)$$

where $\pi(K, C, x) = \#\{p \leqslant x : p \in \pi(K, C)\}$ and $\operatorname{li}(x) = \int_2^x (\log t)^{-1} dt$ is the logarithmic integral.

Our intent in this thesis is to find an asymptotic formula for the number of the Piatetski-Shapiro primes that belong to $\pi(K, C)$. To this end, we define the counting function

$$\pi_c(K, C, x) = \#\{p \leqslant x \ : \ p \in \pi(K, C); \quad p = \lfloor n^c \rfloor \text{ for some } n \in \mathbb{N}\}.$$

The first result we prove in this direction is for abelian extensions $K/\mathbb{Q}$. By the Kronecker-Weber Theorem this problem easily reduces to counting the Piatetski-Shapiro primes in an arithmetic progression, which was handled in [8] as we have mentioned above. We do, however, reprove their theorem here in a slightly different manner following a more recent method given in [4, §4.6] that utilizes Vaughan's identity.

Before stating our first result, we recall that the conductor $\mathfrak{f}$ of an abelian extension $K/\mathbb{Q}$ is the modulus of the smallest ray class field $K^{\mathfrak{f}}$ containing K.

Theorem 1. *Let $K/\mathbb{Q}$ be an abelian extension of conductor $\mathfrak{f}$. Take any automorphism σ in the Galois group $G = Gal(K/\mathbb{Q})$. Then, there exists an absolute constant $D > 0$ and a constant $x_0(\mathfrak{f})$ such that for any fixed $c \in (1, 12/11)$ and $x \geqslant x_0(\mathfrak{f})$, we have*

$$\pi_c(K, \{\sigma\}, x) = \frac{1}{c|G|} \operatorname{li}(x^{1/c}) + O\big(x^{1/c} \exp(-D\sqrt{\log x})\big)$$

where the implied constant depends only on c.

Next, we consider a non-abelian Galois extension $K/\mathbb{Q}$. Given a conjugacy class C in G, take any representative $\sigma \in C$ and put $d_L = [G : \langle\sigma\rangle] = [L : \mathbb{Q}]$, where L is the fixed field corresponding to the cyclic subgroup $\langle\sigma\rangle$ of G generated by σ. Note that $d_L \geqslant 2$. As in the abelian case, we obtain a similar asymptotic formula, only this time the range of c depends on the size of d_L (not on L, hence σ). This is due to the nature of an exponential sum that appears in the estimate of one of the error terms. In this case, we prove the following result:

Theorem 2. *Let K, C, G and d_L be as defined above. Then, there exists an absolute constant $D > 0$, and a constant x_0 which depends on the degree d_K and the discriminant Δ_K of K such that for $x \geqslant x_0$ and for c that satisfies*

$$1 < c < 1 + \begin{cases} (2^{d_L+1} d_L + 1)^{-1} & \text{if} \quad d_L \leqslant 10, \\ \big(6({d_L}^3 + {d_L}^2)\log(125 d_L) - 1\big)^{-1} & \text{otherwise,} \end{cases}$$

we have

$$\pi_c(K, C, x) = \frac{|C|}{c|G|}\mathrm{li}(x^{1/c}) + O\big(x^{1/c}\exp\big(-D|\Delta_K|^{-1/2}(\log x)^{1/2}\big)\big)$$

where the implied constant depends on c, the degree d_L and the discriminant Δ_L of the intermediate field L defined above.

The asymptotic formula above follows from the effective version of the Chebotarev density theorem (see Lemma 10) coupled with an adaptation of the method in [4, §4.6] to our case using an analog of Vaughan's identity for number fields (see Lemma 2). The main difference from [4, §4.6] here is that one has to deal with the estimate of an exponential sum that runs over the integral ideals of L (see §3.1.0.2, §3.3) and most of chapter 4 is devoted to the estimate of this sum. The main idea in a nutshell to handle the exponential sum in §3.1.0.2 is to split it into ray classes, then choose an integral basis for each class, and finally use van der Corput's method for small values of d_L, and Vinogradov's Method for the rest on one of the integer variables.

Although the above theorems yield non-trivial ranges for the permissible values of c for which the associated asymptotics hold, these values are squeezed in a small portion of $(1, 2)$. The following theorem asserts that the exceptional set of values of $c \in (1, 2)$ for which theorems above do not hold is of measure zero in the sense of Lebesque measure.

Theorem 3. *The conclusions of both Theorems 1 and 2 hold for almost all $c \in (1, 2)$.*

1.1 Applications

We consider the ring class field L_n (see e.g., [3, §9]) of the order $\mathbb{Z}[\sqrt{-n}]$ in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-n})$ where n is a positive integer. It follows from [3, Lemma 9.3] that L_n is a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $Gal(L_n/K) \rtimes (\mathbb{Z}/2\mathbb{Z})$, where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $Gal(L_n/K)$ by sending σ to its inverse σ^{-1}. For example, $Gal(L_{27}/\mathbb{Q}) \simeq S_3$ is non-abelian, while $Gal(L_3/\mathbb{Q})$ is abelian since $L_3 = \mathbb{Q}(\sqrt{-3})$. In any case, we have from [3, Theorem 9.4] that if p is an odd prime not dividing n then $p = a^2 + nb^2$ for some integers a, b if and only if p splits completely in L_n, which occurs exactly when $[L_n/\mathbb{Q}, p]$ is the identity automorphism $\mathbf{1}_G$ of $G = Gal(L_n/\mathbb{Q})$. Therefore, as a corollary of the theorems above we see that the number of Piatetski-Shapiro primes up to x that are of the form $a^2 + nb^2$ is asymptotic to $(c|G|)^{-1}\mathrm{li}(x^{1/c})$ as $x \to \infty$ for any c in the range given by the relevant Theorem above depending on whether $L_n/\mathbb{Q}$ is abelian.

Chapter 2

Preliminaries and Technical Preparation

In this chapter, we state some of the core lemmas and theorems that will be frequently used in the proof of Theorems 1 and 2. We refer the reader to [5] and [6] for the tools that are not presented here.

2.1 Analytical Tools

We first start with the following *partial summation formula* whose proof may be found in [9, §A]:

Lemma 1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers, y positive real number, and*

$$A(x) = \sum_{n \leqslant x} a_n,$$

where $A(x) = 0$ if $x < y$. Assume that f has a continuous derivative on the interval $[y, x]$. Then, we have

$$\sum_{y < n \leqslant x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

We next state the Siegel-Walfisz theorem. For the proof, we refer the reader to [9, Corrollary 11.19].

Theorem 4. *Let $A > 0$ be fixed and $x \geqslant x(A)$. Then for $q \leqslant \log^A x$, and for $a < q$ such that $(a,q) = 1$, there is some $D > 0$ such that*

$$\pi(x;q,a) = \frac{\mathrm{li}(x)}{\phi(q)} + O\left(x \exp(-D\sqrt{\log x})\right).$$

The following lemma lies at the heart of the proofs of Theorems 1 and 2. It allows one to decompose Von Mangoldt function and its number field generalizations into more amenable arithmetical functions. There are similar decompositions due to various authors (see e.g., [5, §6]). The one that we present here is more suitable for our purposes.

Lemma 2. *Let $u, v \geqslant 1$. Let $L/\mathbb{Q}$ be a finite extension, for any ideal $\mathfrak{a} \subseteq \mathfrak{O}_L$ with $\mathfrak{N}\mathfrak{a} > u$,*

$$\Lambda_L(\mathfrak{a}) = \sum_{\substack{\mathfrak{b}\mathfrak{c}=\mathfrak{a} \\ \mathfrak{N}\mathfrak{b}\leqslant v}} \mu_L(\mathfrak{b}) \log \mathfrak{N}\mathfrak{c} - \sum_{\substack{\mathfrak{b}\mathfrak{c}\mathfrak{d}=\mathfrak{a} \\ \mathfrak{N}\mathfrak{b}\leqslant v, \mathfrak{N}\mathfrak{c}\leqslant u \\ \mathfrak{N}\mathfrak{b}\mathfrak{c}\leqslant v}} \mu_L(\mathfrak{b})\Lambda_L(\mathfrak{c})$$
$$- \sum_{\substack{\mathfrak{b}\mathfrak{c}\mathfrak{d}=\mathfrak{a} \\ \mathfrak{N}\mathfrak{b}\leqslant v, \mathfrak{N}\mathfrak{c}\leqslant u \\ \mathfrak{N}\mathfrak{b}\mathfrak{c}> v}} \mu_L(\mathfrak{b})\Lambda_L(\mathfrak{c}) - \sum_{\substack{\mathfrak{c}\mathfrak{e}=\mathfrak{a} \\ \mathfrak{N}\mathfrak{c}>u, \mathfrak{N}\mathfrak{e}>v}} \Lambda_L(\mathfrak{c}) \sum_{\substack{\mathfrak{b}\mathfrak{d}=\mathfrak{e} \\ \mathfrak{N}\mathfrak{b}\leqslant v}} \mu_L(\mathfrak{b}),$$

where

$$\mu_L(\mathfrak{a}) = \begin{cases} (-1)^k & \text{if } \mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_k, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\Lambda_L(\mathfrak{a}) = \begin{cases} \log \mathfrak{N}\mathfrak{p} & \text{if } \mathfrak{a} = \mathfrak{p}^k \text{ for some } k \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use the identity

$$\Lambda_L(\mathfrak{a}) = \sum_{\mathfrak{b}\mathfrak{c}=\mathfrak{a}} \mu_L(\mathfrak{b}) \log \mathfrak{N}\mathfrak{c} \tag{2.1}$$

and then follow the argument preceding [5, proposition 13.4]. Finally, note that

$$\begin{aligned}\sum_{\substack{\mathfrak{b}\mathfrak{c}\mathfrak{d}=\mathfrak{a}\\ \mathfrak{N}\mathfrak{b}>v,\mathfrak{N}\mathfrak{c}>u}} \mu_L(\mathfrak{b})\Lambda_L(\mathfrak{c}) &= \sum_{\substack{\mathfrak{c}\mathfrak{e}=\mathfrak{a}\\ \mathfrak{N}\mathfrak{c}>u}} \Lambda_L(\mathfrak{c}) \sum_{\substack{\mathfrak{b}\mathfrak{d}=\mathfrak{e}\\ \mathfrak{N}\mathfrak{b}>v}} \mu_L(\mathfrak{b}) \\ &= \sum_{\substack{\mathfrak{c}\mathfrak{e}\\ =\mathfrak{a}\\ \mathfrak{N}\mathfrak{c}>u,\mathfrak{N}\mathfrak{e}>v}} \Lambda_L(\mathfrak{c})\left(\sum_{\mathfrak{b}\mathfrak{d}=\mathfrak{e}} \mu_L(\mathfrak{b}) - \sum_{\substack{\mathfrak{b}\mathfrak{d}=\mathfrak{e}\\ \mathfrak{N}\mathfrak{b}\leqslant v}} \mu_L(\mathfrak{b})\right) \\ &= -\sum_{\substack{\mathfrak{c}\mathfrak{e}=\mathfrak{a}\\ \mathfrak{N}\mathfrak{c}>u,\mathfrak{N}\mathfrak{e}>v}} \Lambda_L(\mathfrak{c}) \sum_{\substack{\mathfrak{b}\mathfrak{d}=\mathfrak{e}\\ \mathfrak{N}\mathfrak{b}\leqslant v}} \mu_L(\mathfrak{b}).\end{aligned}$$

□

Next, we state several lemmata needed for exponential sum estimates. The proof of the first one can be found in [15, Theorem 2a.], and proofs of the next four can be found in [4, Theorem 2.8], [4, Theorem 2.9] and [4, Lemma 4.13], respectively.

Lemma 3. *Let $n \geqslant 11$. Let N and P be positive real numbers, P being large.Let $f(x)$ be a real function, defined for $x \in I = [N, N+P]$. Suppose on I, f has a continuous $(n+1)$ th derivative satisfying*

$$\frac{f^{(n+1)}(x)}{(n+1)!} \asymp \frac{1}{A}$$

where

$$P \ll A \ll P^{2+2^{\frac{1}{n}}}.$$

Then

$$\sum_{n\in I} e(f(n)) \ll P^{1-\frac{1}{3n^2\log 125n}}.$$

Lemma 4. *Let q be a positive integer. Suppose that f is a real valued function with $q+2$ continuous derivatives on some interval I. Suppose also that for some $\lambda > 0$ and for some $\alpha > 1$,*

$$\lambda \leqslant |f^{(q+2)}(x)| \leqslant \alpha\lambda$$

on I. Let $Q = 2^q$. Then,

$$\sum_{n\in I} e(f(n)) \ll |I|(\alpha^2\lambda)^{1/(4Q-2)} + |I|^{1-1/(2Q)}\alpha^{1/(2Q)} + |I|^{1-2/Q+1/Q^2}\lambda^{-1/(2Q)}$$

where the implied constant is absolute.

If $q = 0$, then Lemma 4 is readily simplified to:

Lemma 5. *Suppose that f is a real valued function with two continous derivations on I. Suppose also that there is some $\lambda > 0$ such that*

$$|f''(x)| \asymp \lambda$$

on I. Then

$$\sum_{n \in I} e(f(n)) \ll |I|\lambda^{1/2} + \lambda^{-1/2}.$$

Here we remark that when f grows fast, Lemma 3 is superior to Lemmma 4, since it yields a polynomial saving while Lemma 4 yields an exponential saving. However, for slowly growing f, Lemma 4 is superior.

We next state the following lemma in order to estimate double exponential sums (see e.g., [4, Lemma 4.13]).

Lemma 6. *Suppose $\alpha(m)$ and $\beta(n)$ are sequences supported on subintervals of the intervals $(X, 2X]$ and $(Y, 2Y]$ respectively. Suppose further that*

$$\sum_{n} |\alpha(n)|^2 \ll X \log^{2A} X, \quad \sum_{m} |\beta(n)|^2 \ll Y \log^{2B} Y$$

Let j be a positive real number, and set $F = jX^\delta Y^\delta$. Finally assume that $XY \asymp N$. Then

$$\sum_{n}\sum_{m} \alpha(n)\beta(m)e(jm^\delta n^\delta)$$
$$\ll (F^{1/6}X^{2/3}Y^{5/6} + NF^{-1/2} + XY^{1/2} + YX^{3/4}) \log^{A+B+1} N.$$

The following lemma will be used to estimate exponential integrals (see e.g., [4, Lemma 3.1.]).

Lemma 7. *Assume that f and g are differentiable on $[a, b]$. Assume moreover that g/f' is monotonic and that $\left|f'(x)/g(x)\right| \geqslant \lambda$ on $[a, b]$. Then*

$$\int_a^b g(x)e(f(x))dx \ll \frac{1}{\lambda}$$

where the implied constant is absolute.

The following result due to Vaaler gives an approximation to $\psi(x)$ (see, for example, [4, Appendix]).

Lemma 8. *Let $H \geqslant 1$ be a real number. Then there exists a trigonometric polynomial*

$$\psi^*(x) = \sum_{1 \leqslant |h| \leqslant H} a_h e(hx), \qquad (a_h \ll |h|^{-1})$$

such that for any real x,

$$|\psi(x) - \psi^*(x)| \leqslant \sum_{|h| < H} b_h e(hx), \qquad (b_h \ll H^{-1}).$$

The following lemma together with Lemma 8 is to be used in order to study weighted sums over Piatetski-Shapiro sequences.

Lemma 9. *Fix $c \in (1, 2)$. Let $z_1, z_2, ...$ be a uniformly bounded sequence of complex numbers. Then*

$$\sum_{\substack{k \leqslant x \\ k = \lfloor n^c \rfloor}} z_k = \delta \sum_{k \leqslant x} z_k k^{\delta - 1} + \sum_{k \leqslant x} z_k \Delta\psi(x) + O(\log x).$$

Proof. The equality $k = \lfloor n^c \rfloor$ holds precisely when $k \leqslant n^x < k+1$, or equivalently, when $-(k+1)^\delta \leqslant n < -k^\delta$. Hence

$$\sum_{\substack{k \leqslant x \\ k = \lfloor n^c \rfloor}} z_k = \sum_{k \leqslant x} z_k \left(\lfloor -k^\delta \rfloor - \lfloor -(k+1)^\delta \rfloor \right).$$

The desired result follows on recalling the fact that $(k+1)^\delta - k^\delta = \delta k^{\delta-1} + O(k^{\delta-2})$ and that $\sum_{k \leqslant x} z_k k^{\delta-2} \ll \log x$. $\square$

2.2 Algebraic Tools

We first start with Chebotarev Density theorem that forms the backbone of our motivation (see e.g., [7]).

Lemma 10 (Chebotarev density theorem)**.** *Let $K/\mathbb{Q}$ be a Galois extension and C a conjugacy class in the Galois group G. If $d_K > 1$, there exists an absolute, effectively computable constant D and a constant $x_0 = x_0(d_K, \Delta_K)$ such that if $x \geqslant x_0$, then*

$$\pi(K, C, x) = \frac{|C|}{|G|}\,\mathrm{li}(x) + O\big(x \exp(-D|\Delta_K|^{-1/2}\sqrt{\log x})\big)$$

where the implied constant is absolute.

We refer reader to [6, Statement 2.15] for the following result.

Lemma 11. *Let L be a number field of degree d_L, then there is a number k depending only on L such that*

$$\sum_{\substack{\mathfrak{a}\subset\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{a}\leqslant x}} 1 = kx + O(x^{1-\frac{1}{d_L}}).$$

The proof of the following result can be found in [2, Lemma 2].

Lemma 12. *Let $L/\mathbb{Q}$ be a finite extension of degree d_L and discriminant Δ_L. For each ideal $\mathfrak{a}$ of L, there exists a basis $\alpha_1, \ldots, \alpha_{d_L}$ such that for any embedding τ of L,*

$$A_1^{-d_L+1}(\mathfrak{N}\mathfrak{a})^{1/(2d_L)} \leqslant |\tau\alpha_j| \leqslant A_1(\mathfrak{N}\mathfrak{a})^{1/d_L} \tag{2.2}$$

where $A_1 = {d_L}^{d_L}|\Delta_L|^{1/2}$.

For the proof of the next lemma, see for example [6, Theorem 11.8].

Lemma 13. *Let L be a finite extension and $\mathfrak{U}$ be a nonzero ideal in the ring of integers $\mathfrak{O}_L$. There exists an element $\alpha \neq 0$ in $\mathfrak{U}$ such that*

$$\mathfrak{N}(\alpha\mathfrak{U}^{-1}) \leqslant \frac{d_L!}{{d_L}^{d_L}}\left(\frac{4}{\pi}\right)^{r_2}|\Delta_L|^{1/2},$$

where $2r_2$ is the number of complex embeddings of L.

Chapter 3

Proof of Theorem 2

Initial steps of our treatment for Thereoms 1, 2 and 3 are similar. Thus, most of the calculations will be done only in this chapter and will be quoted later.

We first appeal to Lemma 9 with the obvious choice

$$z_k = \begin{cases} 1 & \text{if} \quad k \in \pi(K,C), \\ 0 & \text{otherwise.} \end{cases}$$

to derive

$$\pi_c(K,C,x) = \sum_{\substack{p \leqslant x \\ p \in \pi(K,C)}} \delta p^{\delta - 1} + \sum_{\substack{p \leqslant x \\ p \in \pi(K,C)}} \Delta\psi(p) + O(\log x).$$

Using partial summation, it follows from Lemma 10 that for $x \geqslant x_0 = x_0(d_K, |\Delta_K|)$,

$$\sum_{\substack{p \leqslant x \\ p \in \pi(K,C)}} \delta p^{\delta - 1} = \frac{|C|}{c|G|} \mathrm{li}(x^{1/c}) + O(x^{1/c} \exp(-D|\Delta_K|^{-1/2}\sqrt{\log x})$$

where the implied constant is absolute.

The rest of this chapter deals with the estimate of the sum involving ψ. Using dyadic division yields

$$\sum_{\substack{p \leqslant x \\ p \in \pi(K,C)}} \Delta\psi(p) = \sum_{\substack{1 \leqslant N < x \\ N = 2^k}} \sum_{\substack{N < p \leqslant N_1 \\ p \in \pi(K,C)}} \Delta\psi(p)$$

where $N_1 = \min(x, 2N)$. By Lemma 8, we can approximate $\psi(x)$ with the function

$$\psi^*(x) = \sum_{1 \leqslant |h| \leqslant H} a_h e(hx),$$

where the coefficients satisfy $a_h \ll h^{-1}$ and the error $\psi(x) - \psi^*(x) \ll \Delta(x)$ holds for some non-negative function Δ given by

$$\Delta(x) = \sum_{|h|<H} b(h) e(hx)$$

with $b(h) \ll 1/H$. Using definition of Δ, it follows from Lemma 5 that

$$\sum_{\substack{N<p\leqslant N_1 \\ p\in\pi(K,C)}} \Delta(\psi - \psi^*)(p) \ll \sum_{N<n\leqslant N_1} \Delta(-n^\delta) \ll NH^{-1} + N^{\delta/2} H^{1/2}.$$

Thus, taking

$$H = N^{1-\delta+\varepsilon} \tag{3.1}$$

yields

$$\sum_{p\in\pi(K,C,x)} \Delta(\psi - \psi^*)(p) \ll x^\delta \exp(-D|\Delta_K|^{-1/2}\sqrt{\log x})$$

provided that $1 < c < 2$ and $\varepsilon > 0$ is sufficiently small, both of which are assumed in what follows.

Having dealt with the error term, we now turn to the sum involving ψ^*. Using partial summation we obtain

$$\sum_{\substack{N<p\leqslant N_1 \\ p\in\pi(K,C)}} \Delta\psi^*(p) \ll \frac{1}{\log N} \max_{N'\in(N,N_1]} \left| \sum_{\substack{N<n\leqslant N' \\ n\in\langle\pi(K,C)\rangle}} \Delta\psi^*(n)\Lambda(n) \right| + O(\sqrt{N}).$$

Recalling the definition of ψ^* above we derive that

$$\begin{aligned} \sum_{\substack{N<n\leqslant N' \\ n\in\langle\pi(K,C)\rangle}} \Delta\psi^*(n)\Lambda(n) &= \sum_{1\leqslant|h|\leqslant H} a_h \sum_{\substack{N<n\leqslant N' \\ n\in\langle\pi(K,C)\rangle}} \Delta e(-hn^\delta)\Lambda(n) \\ &\ll \sum_{1\leqslant|h|\leqslant H} h^{-1} \left| \sum_{\substack{N<n\leqslant N' \\ n\in\langle\pi(K,C)\rangle}} e(hn^\delta)\phi_h(n)\Lambda(n) \right| \end{aligned}$$

where $\phi_h(x) = 1 - e\left(h\left((x+1)^\delta - x^\delta\right)\right)$. Using the bounds

$$\phi_h(x) \ll hx^{\delta-1}, \quad \phi_h'(x) \ll hx^{\delta-2},$$

and partial summation yield

$$\sum_{\substack{N<n\leqslant N' \\ n\in\langle\pi(K,C)\rangle}} e(hn^\delta)\phi_h(n)\Lambda(n) \ll hN^{\delta-1} \max_{N'\in(N,N_1]} \left| \sum_{\substack{N<n\leqslant N' \\ n\in\langle\pi(K,C)\rangle}} e(hn^\delta)\Lambda(n) \right|.$$

We note at this point that to finish the proof of Theorem 2 it is enough to show that

$$\sum_h \max_{N'\in(N,2N]} \left| \sum_{\substack{N<n\leqslant N' \\ n\in\langle\pi(K,C)\rangle}} e(hn^\delta)\Lambda(n) \right| \ll N\exp(-D|\Delta_K|^{-1/2}\sqrt{\log N}).$$

Lemma 14. *Take a representative $\sigma \in C$. Let L be the fixed field of the cyclic group $\langle\sigma\rangle$ generated by σ. Then, for $N' \leqslant N_1 \leqslant 2N$,*

$$\sum_{\substack{N<n\leqslant N' \\ n\in\langle\pi(K,C)\rangle}} e(hn^\delta)\Lambda(n) = \frac{|C|}{|G|}\sum_\psi \overline{\psi(\sigma)}$$

$$\cdot \sum_{\substack{\mathfrak{a}\subseteq\mathfrak{O}_L \\ N<\mathfrak{N}\mathfrak{a}\leqslant N'}} \psi([K/L,\mathfrak{a}])\Lambda_L(\mathfrak{a})e\left(h(\mathfrak{N}\mathfrak{a})^\delta\right) + O(\sqrt{N})$$

where the first summation is taken over all characters of $Gal(K/L)$ and the second is over powers of prime ideals of L that are unramified in K.

Proof. Since K/L is abelian we obtain by the orthogonality of characters of $\mathrm{Gal}(K/L)$, the expression

$$\sum_\psi \overline{\psi(\sigma)} \sum_{\substack{\mathfrak{a}\subseteq\mathfrak{O}_L \\ N<\mathfrak{N}\mathfrak{a}\leqslant N'}} \psi([K/L,\mathfrak{a}])\Lambda_L(\mathfrak{a})e\left(h(\mathfrak{N}\mathfrak{a})^\delta\right)$$

equals

$$\mathrm{ord}_G(\sigma) \sum_{\substack{\mathfrak{a}\subseteq\mathfrak{O}_L \\ N<\mathfrak{N}\mathfrak{a}\leqslant N' \\ [K/L,\mathfrak{a}]=\sigma}} \Lambda_L(\mathfrak{a})e\left(h(\mathfrak{N}\mathfrak{a})^\delta\right).$$

Removing prime ideals $\mathfrak{p}$ of L with $\deg \mathfrak{p} > 1$ and powers of prime ideals $\mathfrak{p}^k$ with $k > 1$, the last sum can be written as

$$\sum_{\substack{N < \mathfrak{N}\mathfrak{p} \leqslant N' \\ [K/L,\mathfrak{p}]=\sigma \\ \mathfrak{N}p \text{ is prime}}} e\left(h(\mathfrak{N}\mathfrak{p})^{\delta}\right) \log \mathfrak{N}\mathfrak{p} + O(\sqrt{N}),$$

or

$$\sum_{N < p \leqslant N'} \Bigg(\sum_{\substack{\mathfrak{p} \subseteq \mathfrak{O}_L \\ [K/L,\mathfrak{p}]=\sigma \\ \mathfrak{N}\mathfrak{p}=p}} 1 \Bigg) e\left(hp^{\delta}\right) \log p + O(\sqrt{N}).$$

If p is a prime that is unramified in K and $\mathfrak{p}$ is a prime ideal of L above p satisfying $[K/L, \mathfrak{p}] = \sigma$, then $\mathfrak{p}$ remains prime in K and

$$[K/L, \mathfrak{p}] = \sigma \text{ and } \mathfrak{N}\mathfrak{p} = p \Longleftrightarrow [K/\mathbb{Q}, \mathfrak{p}\mathfrak{O}_K] = \sigma.$$

In particular, $[K/\mathbb{Q}, p] = C$. Furthermore, the number of prime ideals $\mathfrak{P}$ of K above such a prime p with $[K/\mathbb{Q}, \mathfrak{P}] = \sigma$ equals $[C_G(\sigma) : \langle\sigma\rangle]$, where $C_G(\sigma)$ is the centralizer of σ in G. The result now follows by observing that $|C_G(\sigma)| = |G|/|C|$ and noting that

$$\sum_{\substack{N<n\leqslant N' \\ n \in \langle \pi(K,C) \rangle}} e(hn^{\delta})\Lambda(n) = \sum_{\substack{p\in\pi(K,C) \\ N<p\leqslant N'}} e(hp^{\delta}) \log p + O(\sqrt{N}).$$

□

Remark 1. From now on we shall write $\chi(\mathfrak{a})$ for the composition $\Psi([K/L, \mathfrak{a}])$. Note that since K/L is abelian, χ is a character of the ray class group $J^{\mathfrak{f}}/P^{\mathfrak{f}}$ (see, e.g., [10, p. 525]) where $\mathfrak{f}$ is the conductor of the extension K/L. Furthermore, we shall require that $\chi(\mathfrak{a}) = 0$ whenever $\mathfrak{a}$ is not coprime to $\mathfrak{f}$. This way, we can assume that the inner sum in the lemma above runs over all integral ideals of L.

Our current objective is to prove that

$$\sum_{h} \max_{N' \in (N,2N]} \Bigg| \sum_{\substack{\mathfrak{a}\subseteq\mathfrak{O}_L \\ N<\mathfrak{N}\mathfrak{a}\leqslant N'}} \chi(\mathfrak{a})\Lambda_L(\mathfrak{a}) e\left(h(\mathfrak{N}\mathfrak{a})^{\delta}\right) \Bigg| \ll N \exp(-D|\Delta_K|^{-1/2}\sqrt{\log N}).$$

3.1 Exponential Sums over Ideals

At this point we appeal to Lemma 2 and assume from now onwards that $u < N$. Hence

$$\sum_{\substack{\mathfrak{a}\subseteq\mathfrak{O}_L\\ N<\mathfrak{N}\mathfrak{a}\leqslant N'}} \chi(\mathfrak{a})\Lambda_L(\mathfrak{a})e(h(\mathfrak{N}\mathfrak{a})^\delta) = S_1 + S_2 + S_3 + S_4$$

where

$$S_1 = -\sum_{\substack{\mathfrak{a}\subseteq\mathfrak{O}_L\\ N<\mathfrak{N}\mathfrak{a}\leqslant N'}} \chi(\mathfrak{a})e(h(\mathfrak{N}\mathfrak{a})^\delta) \sum_{\substack{\mathfrak{c}\mathfrak{e}=\mathfrak{a}\\ \mathfrak{N}\mathfrak{c}>u,\mathfrak{N}\mathfrak{e}>v}} \Lambda_L(\mathfrak{c}) \sum_{\substack{\mathfrak{b}\mathfrak{d}=\mathfrak{e}\\ \mathfrak{N}\mathfrak{b}\leqslant v}} \mu_L(\mathfrak{b}),$$

$$S_2 = \sum_{\substack{\mathfrak{a}\subseteq\mathfrak{O}_L\\ N<\mathfrak{N}\mathfrak{a}\leqslant N'}} \chi(\mathfrak{a})e(h(\mathfrak{N}\mathfrak{a})^\delta) \sum_{\substack{\mathfrak{b}\mathfrak{c}=\mathfrak{a}\\ \mathfrak{N}\mathfrak{b}\leqslant v}} \mu_L(\mathfrak{b}) \log \mathfrak{N}\mathfrak{c},$$

$$S_3 = -\sum_{\substack{\mathfrak{a}\subseteq\mathfrak{O}_L\\ N<\mathfrak{N}\mathfrak{a}\leqslant N'}} \chi(\mathfrak{a})e(h(\mathfrak{N}\mathfrak{a})^\delta) \sum_{\substack{\mathfrak{b}\mathfrak{c}\mathfrak{d}=\mathfrak{a}\\ \mathfrak{N}\mathfrak{b}\leqslant v,\mathfrak{N}\mathfrak{c}\leqslant u\\ \mathfrak{N}\mathfrak{b}c>u}} \mu_L(\mathfrak{b})\Lambda_L(\mathfrak{c}),$$

and

$$S_4 = -\sum_{\substack{\mathfrak{a}\subseteq\mathfrak{O}_L\\ N<\mathfrak{N}\mathfrak{a}\leqslant N'}} \chi(\mathfrak{a})e(h(\mathfrak{N}\mathfrak{a})^\delta) \sum_{\substack{\mathfrak{b}\mathfrak{c}\mathfrak{d}=\mathfrak{a}\\ \mathfrak{N}\mathfrak{b}\leqslant v,\mathfrak{N}\mathfrak{c}\leqslant u\\ \mathfrak{N}\mathfrak{b}c\leqslant u}} \mu_L(\mathfrak{b})\Lambda_L(\mathfrak{c}).$$

3.1.0.1 Estimate of S_1 and S_3

We first need an auxiliary result.

Lemma 15. *Let X, Y be positive integers and*

$$\begin{aligned}\alpha(m) &= -\sum_{\substack{\mathfrak{c}\subseteq\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{c}=m}} \chi(\mathfrak{c})\Lambda_L(\mathfrak{c}),\\ \beta(n) &= \sum_{\substack{\mathfrak{e}\subseteq\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{e}=n}} \chi(\mathfrak{e}) \sum_{\substack{\mathfrak{b}\mathfrak{d}=\mathfrak{e}\\ \mathfrak{N}\mathfrak{b}\leqslant v}} \mu_L(\mathfrak{b}).\end{aligned} \tag{3.2}$$

Then,

$$\sum_{X<m\leqslant 2X} |\alpha(m)|^2 \ll X \log^{2d_L-1} X, \qquad \sum_{Y<n\leqslant 2Y} |\beta(n)|^2 \ll Y(\log Y)^{4{d_L}^2}.$$

Proof. By Cauchy-Schwartz inequality

$$\sum_{Y<n\leqslant 2Y}|\beta(n)|^2 \leqslant \sum_{Y\leqslant n\leqslant 2Y}\Bigg(\sum_{\substack{\mathfrak{e}\subseteq\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{e}=n}}1\Bigg)\sum_{\substack{\mathfrak{e}\subseteq\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{e}=n}}\Bigg(\sum_{\substack{\mathfrak{b}\mathfrak{d}=\mathfrak{e}\\ \mathfrak{N}\mathfrak{b}\leqslant v}}\mu_L(\mathfrak{b})\Bigg)^2 \leqslant \sum_{Y\leqslant n\leqslant 2Y}g(n)$$

where $g(n)$ is the multiplicative function defined by

$$g(n)=\Bigg(\sum_{\substack{\mathfrak{e}\subseteq\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{e}=n}}1\Bigg)\sum_{\substack{\mathfrak{e}\subseteq\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{e}=n}}\tau^2(\mathfrak{e})$$

and $\tau(\mathfrak{e})$ is the number of integral ideals of L that divide $\mathfrak{e}$. Note that for any prime $p\geqslant 2$, $g(p)\leqslant 4{d_L}^2$, while for $k>1$ we see that the number of ideals $\mathfrak{e}$ with $\mathfrak{N}\mathfrak{e}=p^k$ is bounded by

$$\binom{d_L+k-1}{d_L-1}=e^{\sum_{m=1}^{k}\log\left(1+\frac{d_L-1}{m}\right)}\leqslant e^{\sum_{m=1}^{k}\frac{d_L-1}{m}}\leqslant (ek)^{d_L-1}$$

and $\tau^2(\mathfrak{e})\leqslant (k+1)^2\leqslant 4k^2$. Thus, $g(p^k)\leqslant 4e^{d_L-1}k^{d_L+1}$. It follows that

$$\begin{aligned}\log\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\cdots\right)&=\log\left(1+\frac{g(p)}{p}\right)+O(1/p^2)\\ &\leqslant\frac{4{d_L}^2}{p}+O(1/p^2)\end{aligned}$$

where the implied constant depends on d_L. Therefore,

$$\begin{aligned}\sum_{Y\leqslant n\leqslant 2Y}g(n)\leqslant 2Y\sum_{Y\leqslant n\leqslant 2Y}\frac{g(n)}{n}&\leqslant 2Ye^{\sum_{p\leqslant 2Y}\log\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\cdots\right)}\\ &\leqslant 2Ye^{O(1)+4{d_L}^2\sum_{p\leqslant 2Y}\frac{1}{p}}\ll_{d_L}Y(\log Y)^{4{d_L}^2}.\end{aligned}$$

As for the other sum, we obtain

$$\begin{aligned}\sum_{X<m\leqslant 2X}|\alpha(m)|^2&\leqslant\sum_{X\leqslant m\leqslant 2X}\sum_{\substack{\mathfrak{c}\subseteq\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{c}=m}}1\cdot\sum_{\substack{\mathfrak{c}\subseteq\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{c}=m}}(\Lambda_L(\mathfrak{c}))^2\\ &=\sum_{X\leqslant m\leqslant 2X}(\Lambda(m))^2\Bigg(\sum_{\substack{\mathfrak{c}\subseteq\mathfrak{O}_L\\ \mathfrak{N}\mathfrak{c}=m}}1\Bigg)^2\\ &\ll_{d_L}(\log X)^2\sum_{X\leqslant p^k\leqslant 2X}k^{2(d_L-1)}\\ &\ll(\log X)^{2d_L}\sum_{X\leqslant p^k\leqslant 2X}1\ll X(\log X)^{2d_L-1},\end{aligned}$$

as claimed. □

We are now ready to estimate S_1. First, rewrite S_1 as

$$S_1 = -\sum_{\substack{\mathfrak{c},\mathfrak{e}\\ \mathfrak{N}\mathfrak{e}>v;\,\mathfrak{N}\mathfrak{c}>u\\ N\leqslant\mathfrak{N}(\mathfrak{c}\mathfrak{e})\leqslant N'}} \chi(\mathfrak{e})\Bigg(\sum_{\substack{\mathfrak{b}\mathfrak{d}=\mathfrak{e}\\ \mathfrak{N}\mathfrak{b}\leqslant v}} \mu_L(\mathfrak{b})\Bigg)\chi(\mathfrak{c})\Lambda_L(\mathfrak{c})e(h(\mathfrak{N}\mathfrak{c}\mathfrak{e})^\delta)$$

$$= \sum_{\substack{n,m\\ n>v;\,m>u\\ N<nm\leqslant N'}}\sum \alpha(m)\beta(n)e(h(nm)^\delta)$$

where $\alpha(m)$ and $\beta(n)$ are given by (3.2). Let

$$u = v = N^{\delta-1+\eta} \tag{3.3}$$

and split the ranges of m and n into $\ll \log^2 N$ subintervals of the form $[X, 2X]$ and $[Y, 2Y]$ such that $N/4 \leqslant XY \leqslant 2N$, $v < X, Y < N'/v$. Summing over $h \leqslant H$ we conclude from Lemma 6 and Lemma 15 that the contribution of each subinterval is

$$\ll \big(H^{7/6}N^{\delta/6+5/6}\min(X^{-1/6}, Y^{-1/6})$$
$$+ HN^{1/2}\max(X,Y)^{1/2}\big)(\log N)^{2d_L{}^2+d_L+1/2}$$
$$\ll \Big(N^{2-1/12-\delta} + N^{5/2-3\delta/2-\eta/2}\Big)N^{8\varepsilon/6}.$$

Finally, summing over X and Y we conclude that the estimate

$$\sum_h |S_1| \ll \Big(N^{2-1/12-\delta} + N^{5/2-3\delta/2-\eta/2}\Big)N^{2\varepsilon}$$
$$\ll N\exp(-D|\Delta_K|^{-1/2}\sqrt{\log N})$$

holds provided that

$$1-\delta < \min\Big(\frac{1}{12}, \frac{\eta}{3}\Big), \tag{3.4}$$

and $\varepsilon > 0$ is sufficiently small, both of which we shall assume in what follows.

To estimate S_3, we first note that

$$S_3 = -\sum_{\substack{\mathfrak{d},\mathfrak{e}\\ v<\mathfrak{N}\mathfrak{e}\leqslant v^2\\ N<\mathfrak{N}(\mathfrak{d}\mathfrak{e})\leqslant N'}}\sum \chi(\mathfrak{d})\chi(\mathfrak{e})\Bigg(\sum_{\substack{\mathfrak{b}\mathfrak{c}=\mathfrak{e}\\ \mathfrak{N}\mathfrak{b}\leqslant v,\mathfrak{N}\mathfrak{c}\leqslant u}} \mu_L(\mathfrak{b})\Lambda_L(\mathfrak{c})\Bigg)e\left(h(\mathfrak{N}(\mathfrak{d}\mathfrak{e}))^\delta\right)$$

$$= \sum_{\substack{n,m\\ v<m\leqslant v^2\\ N<nm\leqslant N'}}\sum \alpha(m)\beta(n)e\left(h(nm)^\delta\right)$$

with

$$\alpha(m) = \sum_{\substack{\mathfrak{e} \\ \mathfrak{N}\mathfrak{e}=m}} \chi(\mathfrak{e}) \Bigg(\sum_{\substack{\mathfrak{b}\mathfrak{c}=\mathfrak{e} \\ \mathfrak{N}\mathfrak{b}\leqslant v, \mathfrak{N}\mathfrak{c}\leqslant u}} \mu_L(\mathfrak{b}) \Lambda_L(\mathfrak{c}) \Bigg)$$

$$\beta(n) = \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}\mathfrak{d}=n}} \chi(\mathfrak{d}).$$

and split the ranges of m and n as we did for S_1 with the only difference that we now have $v < X \leqslant v^2$ and $N/v^2 < Y < N'/v$ in addition to $N/4 \leqslant XY \leqslant 2N$. Furthermore, an analog of Lemma 15 can easily be established for the coefficients $\alpha(m)$ and $\beta(n)$ and will be omitted here. Using Lemma 6 once again we see that the estimate

$$\sum_{h\leqslant H} |S_3| \ll \left(N^{2-\delta-1/12} + N^{2-\delta} v^{-1/2} + N^{3/2-\delta} v\right) N^{2\varepsilon}$$

$$\ll N \exp(-D|\Delta_K|^{-1/2} \sqrt{\log N})$$

holds if we assume (3.4), that $\varepsilon > 0$ is sufficiently small and that

$$3\eta \leqslant 1. \tag{3.5}$$

3.1.0.2 Estimate of the sums S_2 and S_4

We first use the identity

$$\log \mathfrak{N}\mathfrak{b} = \sum_{\mathfrak{d}|\mathfrak{b}} \Lambda_L(\mathfrak{d})$$

to derive that

$$S_4 = -\sum_{\substack{\mathfrak{e} \\ \mathfrak{N}\mathfrak{e}\leqslant v}} \chi(\mathfrak{e}) \Bigg(\sum_{\substack{\mathfrak{b}\mathfrak{c}=\mathfrak{e} \\ \mathfrak{N}\mathfrak{b}\leqslant v, \mathfrak{N}\mathfrak{c}\leqslant u}} \mu_L(\mathfrak{b}) \Lambda_L(\mathfrak{c}) \Bigg) \sum_{\substack{\mathfrak{d} \\ N<\mathfrak{N}(\mathfrak{d}\mathfrak{e})\leqslant N'}} \chi(\mathfrak{d}) e\left(h(\mathfrak{N}(\mathfrak{d}\mathfrak{e}))^\delta\right)$$

$$\ll \log N \max_{N'\in(N,N_1]} \sum_{\substack{\mathfrak{e} \\ \mathfrak{N}\mathfrak{e}\leqslant v}} \Bigg| \sum_{\substack{\mathfrak{d} \\ N<\mathfrak{N}(\mathfrak{d}\mathfrak{e})\leqslant N'}} \chi(\mathfrak{d}) e\left(h(\mathfrak{N}(\mathfrak{d}\mathfrak{e}))^\delta\right) \Bigg|,$$

also by partial summation

$$S_2 \ll \log N \max_{N'\in(N,N_1]} \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}\mathfrak{d}\leqslant v}} \Bigg| \sum_{\substack{\mathfrak{c} \\ N/\mathfrak{N}\mathfrak{d}<\mathfrak{N}\mathfrak{c}\leqslant N'/\mathfrak{N}\mathfrak{d}}} \chi(\mathfrak{c}) e\left(h(\mathfrak{N}\mathfrak{c}\mathfrak{d})^\delta\right) \Bigg|.$$

Thus it is suffices to estimate one of them, say S_2. To this end, we shall estimate

$$S = \sum_{\substack{\mathfrak{c} \\ N/\mathfrak{N}\mathfrak{d} < \mathfrak{N}\mathfrak{c} \leqslant N'/\mathfrak{N}\mathfrak{d}}} \chi(\mathfrak{c}) e\left(h(\mathfrak{N}\mathfrak{c}\mathfrak{d})^{\delta}\right).$$

for all $N < N' \leqslant 2N$.

Recall that χ is a ray class character of modulus $\mathfrak{f}$. Splitting S into ray classes $\mathfrak{K}$ we obtain $S = \sum_{\mathfrak{K}} \chi(\mathfrak{K}) S_{\mathfrak{K}}$ where

$$S_{\mathfrak{K}} = \sum_{\substack{\mathfrak{c} \in \mathfrak{K} \\ N/\mathfrak{N}\mathfrak{d} < \mathfrak{N}\mathfrak{c} \leqslant N'/\mathfrak{N}\mathfrak{d}}} e\left(h(\mathfrak{N}\mathfrak{c}\mathfrak{d})^{\delta}\right).$$

Since there are only finitely many classes it is enough to consider a fixed class $\mathfrak{K}$. Let $\mathfrak{b}$ be an integral ideal in the inverse class $\mathfrak{K}^{-1}$. Any integral ideal $\mathfrak{c} \in \mathfrak{K}$ is given by $\alpha\mathfrak{b}^{-1}$ for some $\alpha \in \mathfrak{b} \cap L_{\mathfrak{f},1}$, where

$$L_{\mathfrak{f},1} := \{x \in L^* : x \equiv 1 \bmod \mathfrak{f}, \text{ and } x \text{ is totally positive}\}.$$

Thus, we have

$$S_{\mathfrak{K}} = \sum_{\substack{\alpha\mathfrak{a} \\ \alpha \in \mathfrak{b} \cap L_{\mathfrak{f},1} \\ P^{d_L} < \mathfrak{N}(\alpha\mathfrak{D}_L) \leqslant (P')^{d_L}}} e\left(h(\mathfrak{N}(\alpha\mathfrak{a}\mathfrak{d}))^{\delta}\right)$$

where $\mathfrak{a} = \mathfrak{b}^{-1}$,

$$P = \left(\frac{N}{\mathfrak{N}(\mathfrak{a}\mathfrak{d})}\right)^{1/d_L} \quad \text{and} \quad P' = \left(\frac{N'}{\mathfrak{N}(\mathfrak{a}\mathfrak{d})}\right)^{1/d_L}. \tag{3.6}$$

Since $\mathfrak{f}$ and $\mathfrak{b}$ are coprime ideals, we can find an $\alpha_0 \in \mathfrak{b}$ such that $\alpha_0 \equiv 1 \bmod \mathfrak{f}$. Hence, the condition that $\alpha \in \mathfrak{b} \cap L_{\mathfrak{f},1}$ is equivalent to the conditions that $\alpha \equiv \alpha_0 \bmod \mathfrak{f}\mathfrak{b}$ and that α is totally positive.

Define a linear transformation T from L to the Minkowski space $L_{\mathbb{R}} := \{(z_\tau) \in L_{\mathbb{C}} : z_\tau = \overline{z_{\overline{\tau}}}\}$ by

$$T\alpha = (\tau_1\alpha, \ldots, \tau_{d_L}\alpha)$$

where $L_{\mathbb{C}} := \prod_\tau \mathbb{C}$ and $\tau_1, \ldots, \tau_{d_L}$ are the embeddings of L with the first r_1 embeddings being real and the first r_1+r_2 corresponding to the different archimedean valuations of L.

Note that $\alpha, \beta \in \mathfrak{b} \cap L_{\mathfrak{f},1}$ generate the same ideal if and only if they differ by a unit $u \in \mathfrak{O}_L^* \cap L_{\mathfrak{f},1}$. Since $\mathfrak{O}_L^* \cap L_{\mathfrak{f},1}$ is of finite index in $\mathfrak{O}_L^*$, its free part is of rank $r = r_1 + r_2 - 1$. Let $\xi_1, \dots, \xi_r$ be a system of fundamental units for $\mathfrak{O}_L^* \cap L_{\mathfrak{f},1}$ and E the invertible $r \times r$ matrix whose rows are given by $\ell(T\xi_1), \dots, \ell(T\xi_r)$ where $\ell : L_{\mathbb{C}}^* = \prod_\tau \mathbb{C}^* \to \mathbb{R}^r$ is defined by

$$\ell(z_1, \dots, z_{d_L}) = (\log|z_1|, \dots, \log|z_r|).$$

If L contains exactly ω roots of unity, then for any $t \in \mathbb{R}^*$, $\ell(T(t\alpha)) = \ell(T(t\beta))$ holds for exactly ω associates α of a given $\beta \in L^*$. Thus, in order to pick a representative $\alpha \in \mathfrak{b} \cap L_{\mathfrak{f},1}$ for the ideal $\alpha\mathfrak{a} \in \mathfrak{K}$ that is unique up to multiplication by roots of unity in L, we impose the condition that $\ell(T\alpha)E^{-1} \in [0,1)^r$. At this point, we define the set

$$\Gamma_0 := \{\mathbf{z} \in L_{\mathbb{C}}^* : 1 < \mathfrak{N}\mathbf{z} \leqslant N'/N; \ell(\mathbf{z})E^{-1} \in [0,1)^r; z_1, \dots, z_{r_1} > 0\}$$

where norm $\mathfrak{N}\mathbf{z} = \mathfrak{N}(z_1, \dots, z_{d_L}) := \prod_i z_i$. Recalling the definition of $S_{\mathfrak{K}}$ above and noting that $\mathfrak{N}T\alpha = \mathfrak{N}_{L/\mathbb{Q}}(\alpha)$ for $\alpha \in L^*$, we see that

$$\omega S_{\mathfrak{K}} = \sum_{\substack{\alpha \in \alpha_0 + \mathfrak{f}\mathfrak{b} \\ T\alpha \in P\Gamma_0}} e\left(h(\mathfrak{N}(\alpha\mathfrak{a}\mathfrak{d}))^\delta\right).$$

Fix a $\mathbb{Z}$-basis $\{\alpha_1, \dots, \alpha_{d_L}\}$ for the integral ideal $\mathfrak{f}\mathfrak{b}$ that satisfies (2.2) and let M be the invertible matrix whose rows are given by $T\alpha_1, \dots, T\alpha_{d_L}$. Since for $\alpha \in \alpha_0 + \mathfrak{f}\mathfrak{b}$, $T\alpha$ can be written as $T\alpha_0 + \mathbf{n}M$ for some unique $\mathbf{n} \in \mathbb{Z}^{d_L}$, we see that $\omega S_{\mathfrak{K}} = \sum_{\mathbf{n} \in \mathbb{Z}^{d_L}} f(\mathbf{n})$, where $f : \mathbb{R}^{d_L} \to \mathbb{R}$ is given by

$$f(\mathbf{x}) = \begin{cases} e\big(D(\mathfrak{N}(\mathbf{x}_0 + \mathbf{x}M))^\delta\big) & \text{if } \mathbf{x}_0 + \mathbf{x}M \in P\Gamma_0, \\ 0 & \text{otherwise}, \end{cases}$$

$\mathbf{x}_0 = T\alpha_0$, and $D = h\left(\mathfrak{N}(\mathfrak{a}\mathfrak{d})\right)^\delta$. Partitioning $\mathbb{R}^{d_L}$ into a disjoint union of translates B of $[0,Y)^{d_L}$, where $Y \geqslant 1$ is an integer to be chosen later, we obtain

$$\sum_{\mathbf{n} \in \mathbb{Z}^{d_L}} f(\mathbf{n}) = \sum_B \sum_{\mathbf{n} \in B \cap \mathbb{Z}^{d_L}} f(\mathbf{n}).$$

Note that the condition $\ell(\mathbf{z})E^{-1} \in [0,1)^r$ in the definition of Γ_0 above implies the existence of positive constants $c_1 = c_1(d_L, \Delta_L)$ and $c_2 = c_2(d_L, \Delta_L)$ such that for any $\alpha \in L^*$ with $T\alpha \in P\Gamma_0$ and any embedding τ of L, we have

$$c_1 P < |\tau\alpha| < c_2 P. \tag{3.7}$$

Let $\mathbf{R}$ be the region $\{(z_1,\ldots,z_{d_L}) \in L_{\mathbb{R}} : c_1P < |z_i| < c_2P\}$. Suppose that f is not identically zero on $B \cap \mathbb{Z}^{d_L}$ for some B. If $\mathbf{x}_0 + BM$ is partially contained in $\mathbf{R}$ then it must be intersecting the boundary of $\mathbf{R}$. Thus, we see that the contribution of such B to the sum $\sum_{\mathbf{n}} f(\mathbf{n})$ is $O(YP^{d_L-1})$. For the rest of the boxes B for which $f(B\cap\mathbb{Z}^{d_L}) \not\equiv 0$, we necessarily have that $\mathbf{x}_0+BM \subseteq \mathbf{R}$. From now on, we assume that B is such a box. By the arguments in §3.3, there exist constants $C_1 = C_1(k, d_L, \Delta_L)$, $C_2 = C_2(k, d_L, \Delta_L)$ and a matrix $U \in SL(d_L, \mathbb{Z})$ such that for $N \geqslant C_1$, $1 \leqslant Y \leqslant C_2P$ and any $\mathbf{x} = (x_1,\ldots,x_{d_L}) \in BU^{-1}$, we have

$$\left|\frac{\partial^k}{\partial x_1^k} g_U(\mathbf{x})\right| \asymp P^{\delta d_L - k} \qquad \text{and} \qquad \frac{\partial \lambda_i}{\partial x_1}(\mathbf{x}) \gg P^{-1} \tag{3.8}$$

where g_U is given by (3.14), λ_i's are determined by the condition $\ell(\mathbf{x}_0+\mathbf{x}UM) = (\lambda_1(\mathbf{x}),\ldots,\lambda_r(\mathbf{x}))E$, and the implied constants depend on k (only if relevant) and on d_L and Δ_L. After a change of variable we obtain

$$\sum_{\mathbf{n}\in B\cap\mathbb{Z}^{d_L}} f(\mathbf{n}) = \sum_{\mathbf{n}\in BU^{-1}\cap\mathbb{Z}^{d_L}} f(\mathbf{n}U) = \sum\cdots\sum_{(n_2,\ldots,n_{d_L})\in\mathbb{Z}^{d_L}} \sum_{\substack{n_1\in\mathbb{Z}\\ \mathbf{n}\in BU^{-1}\cap\mathbb{Z}^{d_L}}} f(\mathbf{n}U) \tag{3.9}$$

where $\mathbf{n} = (n_1,\ldots,n_{d_L})$. Since $f(B \cap \mathbb{Z}^{d_L}) \not\equiv 0$ there is at least one tuple $(n_2,\ldots,n_{d_L})$ such that $f(\mathbf{n}U) \not\equiv 0$ for $n_1 \in \mathbb{Z}$ and $\mathbf{n} \in BU^{-1} \cap \mathbb{Z}^{d_L}$. Fix such a tuple. It follows from (3.8) with $k = 1$ that both λ_i's and the norm function are monotonic and thus there is an interval $I = I(n_2,\ldots,n_{d_L})$ of length at most $O(Y)$ such that the function $f(x; n_2,\ldots,n_{d_L}) \neq 0$ for $x \in I$. We are now ready to estimate (3.9). We shall do so in what follows using different methods according to the size of the degree d_L of the extension $L/\mathbb{Q}$.

3.1.0.3 Vinogradov's Method - Large degree

Assume that $d_L \geqslant 11$. It follows from (3.8) that there exist positive constants $C_3 = C_3(d_L, \Delta_L)$ and $C_4 = C_4(d_L, \Delta_L)$ such that

$$\frac{1}{A_0} \leqslant \left|\frac{\partial^{d_L+1}}{\partial x_1^{d_L+1}}\left(Dg_U(\mathbf{x})\right)\right| \leqslant \frac{C_4}{A_0},$$

where

$$A_0 = \frac{P^{d_L(1-\delta)+1}}{C_3D} = \frac{N^{1-\delta+1/d_L}}{C_3h\left(\mathfrak{N}(\mathfrak{a}\mathfrak{d})\right)^{1+1/d_L}}.$$

Using (3.1) and (3.3) we see that

$$\frac{N^{1/d_L-\varepsilon-(1+1/d_L)(\eta+\delta-1)}}{C_3\left(\mathfrak{N}(\mathfrak{a})\right)^{1+1/d_L}} < A_0 \leqslant \frac{P^{d_L(1-\delta)+1}}{C_3\left(\mathfrak{N}(\mathfrak{a})\right)^{\delta}}.$$

Therefore, assuming that $\eta < 1/(1+d_L)$ and ε is sufficiently small it follows from Lemma 13 that for sufficiently large N, we have $A_0 > 1$. Put $\rho = 1/(3{d_L}^2\log(125d_L))$ and take

$$Y = A_0^{1/((2+2/d_L)(1-\rho))}. \tag{3.10}$$

Using equation (3.4), the upper bound for A_0 above and the inequality $(1+1/d_L)(1-\rho) > 1$, we obtain for sufficiently large N that

$$A_0^{1/(2+2/d_L)} < Y \leqslant \min\left(C_2 P, A_0\right). \tag{3.11}$$

If the interval I in (3.9) satisfies

$$A_0^{1/(2+2/d_L)} \ll |I|,$$

we derive from (3.11) and [15, Theorem 2a, p. 109] that

$$\sum_{\substack{n_1\in I\\ \mathbf{n}\in BU^{-1}\cap\mathbb{Z}^{d_L}}} e\left(Dg_U(\mathbf{n})\right) \ll |I|^{1-\rho} \ll Y^{1-\rho}.$$

For smaller intervals I, trivially estimating the sum yields a contribution $\ll Y^{1-\rho}$ due to the choice of Y in (3.10). Since the number of tuples $(n_2,\ldots,n_{d_L}) \in \mathbb{Z}^{d_L-1}$ such that $\mathbf{n} \in BU^{-1}\cap\mathbb{Z}^{d_L}$ is $O(Y^{d_L-1})$ we obtain

$$\sum_{\mathbf{n}\in B\cap\mathbb{Z}^{d_L}} f(\mathbf{n}) \ll Y^{d_L-\rho}.$$

Therefore, the contribution to the sum in (3.9) of those B for which $f(B\cap\mathbb{Z}^{d_L}) \not\equiv 0$ and $\mathbf{x}_0 + BM \subseteq R$ is $\ll P^{d_L}Y^{-\rho}$, and this is already larger than the contribution from the rest of the boxes B.

Using (3.6) and partial summation and then summing over the ray classes $\mathfrak{K}$ we see that the sum

$$\sum_{\substack{\mathfrak{c}\\ N/\mathfrak{N}\mathfrak{d}<\mathfrak{N}\mathfrak{c}\leqslant N'/\mathfrak{N}\mathfrak{d}}} \chi(\mathfrak{c})e\left(h(\mathfrak{N}\mathfrak{c}\mathfrak{d})^{\delta}\right)\log\mathfrak{N}\mathfrak{c}$$

is

$$\ll \frac{N}{\mathfrak{N}\mathfrak{d}} \left(\frac{N^{1-\delta+1/d_L}}{h(\mathfrak{N}\mathfrak{d})^{1+1/d_L}} \right)^{-\frac{\rho}{(2+2/d_L)(1-\rho)}} \log N$$
$$= N^{1-\frac{\rho(1-\delta+1/d_L)}{(2+2/d_L)(1-\rho)}} (\mathfrak{N}\mathfrak{d})^{\frac{\rho}{2(1-\rho)}-1} h^{\frac{\rho}{(2+2/d_L)(1-\rho)}} \log N.$$

Finally, summing over ideals $\mathfrak{d}$ with $\mathfrak{N}\mathfrak{d} \leqslant v$ by Lemma 11 and then summing over h with $h \leqslant H$ we obtain from (3.1) and (3.3) that

$$\sum_{h \leqslant H} |S| \ll N^{1-\frac{\rho(1-\delta+1/d_L)}{(2+2/d_L)(1-\rho)}} v^{\frac{\rho}{2(1-\rho)}} H^{1+\frac{\rho}{(2+2/d_L)(1-\rho)}} \log N \ll N^{1+q+2\varepsilon}$$

where

$$q = \frac{1}{2(1-\rho)} \left(-\frac{\rho}{d_L+1} + (1-\delta)(2-3\rho) + \rho\eta \right).$$

Thus, assuming (3.4) and choosing

$$\frac{\eta}{3} = \frac{\rho}{2(d_L+1)} = \frac{1}{6(d_L+1){d_L}^2 \log(125 d_L)} \tag{3.12}$$

we see that both (3.5) and the inequality $q < 0$ hold. We conclude that for sufficiently large N and sufficiently small $\varepsilon > 0$,

$$\sum_{h \leqslant H} |S_2| \ll N \exp(-D|\Delta_K|^{-1/2}\sqrt{\log N})$$

provided that $d_L \geqslant 11$.

3.1.0.4 Van Der Corput's Method - Small degree

By Lemma 4 and (3.8) we obtain

$$\sum_{\substack{n_1 \\ \mathbf{n} \in BU^{-1} \cap \mathbb{Z}^{d_L}}} e\left(Dg_U(\mathbf{n})\right) \ll Y\lambda^{1/(2^{k+2}-2)} + Y^{1-1/2^{k+1}}$$
$$+ Y^{1-1/2^{k-1}+1/2^{2k}} \lambda^{-1/2^{k+1}}$$

where $\lambda := DP^{d_L\delta-(k+2)}$. Note that this bound is no better than the trivial estimate unless $\lambda < 1$. Therefore, we shall require that $\eta < 1/(d_L+1)$. With this assumption, we obtain that for $k \geqslant d_L - 1$, for sufficiently large N and sufficiently small $\varepsilon > 0$, both of the inequalities $k+2 > d_L\delta$ and $\lambda < 1$ hold, since by (3.1), (3.3) and (3.4) we have

$$\begin{aligned}\lambda = DP^{d_L\delta-(k+2)} &= \frac{h(\mathfrak{N}(\mathfrak{a}\mathfrak{d}))^{\delta}}{(N/(\mathfrak{N}\mathfrak{a}\mathfrak{d}))^{(k+2-d_L\delta)/d_L}} \\ &\ll \frac{HN^{\delta}}{(N/v)^{(k+2)/d_L}} \\ &\ll N^{1+\frac{k+2}{d_L}(\eta+\delta-2)+\varepsilon}.\end{aligned}$$

We derive as before that the contribution from the boxes B for which $f(B\cap\mathbb{Z}^{d_L}) \not\equiv 0$ and $\mathbf{x}_0 + BM \subseteq R$ is

$$\ll P^{d_L}\left(\lambda^{1/(2^{k+2}-2)} + Y^{-1/2^{k+1}} + Y^{-1/2^{k-1}+1/2^{2k}}\lambda^{-1/2^{k+1}}\right),$$

while that from the rest of the boxes B is $O(YP^{d_L-1})$. Combining these estimates yields the bound $S_{\mathfrak{K}} \ll P^{d_L}\left(\lambda^{1/(2^{k+2}-2)} + G(Y)\right)$, where

$$G(Y) = Y^{-1/2^{k+1}} + Y^{-1/2^{k-1}+1/2^{2k}}\lambda^{-1/2^{k+1}} + YP^{-1}.$$

Using [4, Lemma 2.4] it follows that for some $Y \in [1, C_2P]$,

$$\begin{aligned}G(Y) &\ll P^{-1/(1+2^{k+1})} + \left(P^{-1/2^{k-1}+1/2^{2k}}\lambda^{-1/2^{k+1}}\right)^{1/(1+1/2^{k-1}-1/2^{2k})} \\ &\qquad + P^{-1} + P^{-1/2^{k+1}} + \lambda^{-1/2^{k+1}}P^{-1/2^{k-1}+1/2^{2k}} \\ &\ll P^{-1/(1+2^{k+1})} + \left(P^{-1/2^{k-1}+1/2^{2k}}\lambda^{-1/2^{k+1}}\right)^{1/(1+1/2^{k-1}-1/2^{2k})}.\end{aligned}$$

Note that in order to have $P^{-1/2^{k-1}+1/2^{2k}}\lambda^{-1/2^{k+1}} < 1$ one needs that $k < d_L + 2$, which can be seen using (3.1), (3.3), (3.4), (3.6) and that $\eta < 1/(d_L+1)$. Using equation (3.6), the fact that $\lambda = DP^{d_L\delta-(k+2)}$ and partial summation we derive that the sum

$$(\log N)^{-1} \sum_{\substack{\mathfrak{c} \\ N/\mathfrak{N}\mathfrak{d} < \mathfrak{N}\mathfrak{c} \leqslant N'/\mathfrak{N}\mathfrak{d}}} \chi(\mathfrak{c})e\left(h(\mathfrak{N}\mathfrak{c}\mathfrak{d})^{\delta}\right)\log\mathfrak{N}\mathfrak{c}$$

is

$$\begin{aligned}&\ll h^{1/(2^{k+2}-2)}\mathfrak{N}(\mathfrak{d})^{\frac{k+2}{d_L(2^{k+2}-2)}-1}N^{1+\frac{d_L\delta-(k+2)}{d_L(2^{k+2}-2)}} \\ &\quad + N^{1+\frac{1+2^{k-1}(k-2-d_L\delta)}{d_L(2^{2k}+2^{k+1}-1)}}(\mathfrak{N}\mathfrak{d})^{-\frac{1+2^{k-1}(k-2)}{d_L(2^{2k}+2^{k+1}-1)}-1}h^{-\frac{1}{2^{k+1}+4-2^{1-k}}} \\ &\quad + (N/\mathfrak{N}\mathfrak{d})^{1-\frac{1}{d_L(1+2^{k+1})}}.\end{aligned}$$

Summing over ideals $\mathfrak{d}$ with $\mathfrak{N}\mathfrak{d} \leqslant v$, followed by summation over $h \leqslant H$ yields

$$(\log N)^{-1} \sum_{h \leqslant H} |S_2| \ll H^{1+1/(2^{k+2}-2)} v^{\frac{k+2}{d_L(2^{k+2}-2)}} N^{1+\frac{d_L\delta-(k+2)}{d_L(2^{k+2}-2)}}$$
$$+ HN^{1-\frac{1}{d_L(1+2^{k+1})}} v^{\frac{1}{d_L(1+2^{k+1})}} + N^{1+\frac{1+2^{k-1}(k-2-d_L\delta)}{d_L(2^{2k}+2^{k+1}-1)}} H^{1-\frac{1}{2^{k+1}+4-2^{1-k}}}$$
$$\ll N^{1+q_1(k)+2\varepsilon} + N^{1+q_2(k)+\varepsilon} + N^{1+q_3(k)+\varepsilon}$$

where, assuming (3.5), it follows that the exponents $q_i(k)$ satisfy

$$\begin{aligned} q_1(k) &= (1-\delta)\left(1+\frac{1}{2^{k+2}-2}\right) + (\delta-1+\eta)\frac{k+2}{d_L(2^{k+2}-2)} + \frac{d_L\delta-(k+2)}{d_L(2^{k+2}-2)} \\ &< \frac{1}{d_L(2^{k+2}-2)}\left(\frac{\eta}{3}\left(d_L(2^{k+2}-2)+2k+4\right) + d_L - k - 2\right), \\ q_2(k) &= 1-\delta-\frac{1}{d_L(1+2^{k+1})} + (\delta-1+\eta)\frac{1}{d_L(1+2^{k+1})} \\ &< \frac{1}{d_L(1+2^{k+1})}\left(\frac{\eta}{3}\left(d_L(1+2^{k+1})+2\right)-1\right), \end{aligned}$$

and

$$\begin{aligned} q_3(k) &= \frac{1+2^{k-1}(k-2-d_L\delta)}{d_L(2^{2k}+2^{k+1}-1)} + (1-\delta)\left(1-\frac{1}{2^{k+1}+4-2^{1-k}}\right) \\ &< \frac{1+2^{k-1}(k-2-d_L)}{d_L(2^{2k}+2^{k+1}-1)} + \frac{\eta}{3}. \end{aligned}$$

Thus, for sufficiently small ε, the estimate $\sum_h |S_2| \ll N\exp(-D|\Delta_K|^{-1/2}\sqrt{\log N})$ holds provided that for $1 \leqslant d_L - 1 \leqslant k \leqslant d_L + 1$,

$$\begin{aligned} \frac{\eta}{3} = \min\Bigg(&\frac{1}{3(d_L+1)+\varepsilon}, \frac{k+2-d_L}{d_L(2^{k+2}-2)+2k+4}, \\ &\frac{1}{d_L(1+2^{k+1})+2}, \frac{2^{k-1}(d_L+2-k)-1}{d_L(2^{2k}+2^{k+1}-1)}\Bigg). \end{aligned} \tag{3.13}$$

3.2 Conclusion of Theorem 2

Upon comparing (3.12) and (3.13) we conclude that for $2 \leqslant d_L < 11$, the maximum value for $\eta/3$ (hence the widest range for δ) is obtained via Van Der Corput's Method when $k = d_L - 1$ is substituted into the function

$$\frac{k+2-d_L}{d_L(2^{k+2}-2)+2k+4},$$

while for $d_L \geqslant 11$ one needs to use Vinogradov's method; in this case, we obtain

$$\frac{\eta}{3} = \frac{1}{6(d_L+1){d_L}^2 \log(125 d_L)}.$$

With the above choice of η, the claimed range for c in Theorem 2 follows easily from (3.4).

Remark 2. To estimate S_2, one may also use [14, Lemma 6.12] for $d_L \geqslant 7$, but the result is worse than what we have already obtained.

3.3 Derivative of the Norm Function

In this section we prove some auxiliary Lemmas used in the estimate of S_2.

Lemma 16. *Let $V \in GL(d_L, \mathbb{R})$, $\mathbf{n} \in \mathbb{Z}^{d_L}$ and $\mathbf{x}, \mathbf{u} \in \mathbb{R}^{d_L}$. Put*

$$g_V(\mathbf{x}) = |\mathfrak{N}(\mathbf{x}_0 + \mathbf{x}VM)|^\delta, \qquad \tilde{g}_{\mathbf{u}}(t) = |\mathfrak{N}(\mathbf{x}_0 + \mathbf{n}M + t\mathbf{u}M)|^\delta. \quad (3.14)$$

Then, for any $k \geqslant 1$,

$$\frac{\partial^k g_V}{\partial x_1^k}\Big|_{\mathbf{x}=\mathbf{n}V^{-1}} = \frac{d^k}{dt^k}\tilde{g}_{V_1}(0) = \sum_{\substack{i_1,\dots,i_k \\ 1\leqslant i_j \leqslant d_L}} \cdots \sum D_{i_1} \dots D_{i_k} F(\mathbf{x}_0 + \mathbf{n}M) v_{i_1} \cdots v_{i_k} \quad (3.15)$$

where $F(z_1, \dots, z_{d_L}) = \prod_{i=1}^{d_L} z_i^\delta$, $D_i = \frac{\partial}{\partial z_i}$, v_i is the ith component of the vector $V_1 M$ and V_1 is the first row of V.

Proof. The result easily follows by induction and chain rule for derivatives. □

Lemma 17. *Given $\mathbf{a} \in \mathbf{R}$, there exists $\mathbf{v} = \mathbf{v}(\mathbf{a}) \in \mathbb{R}^{d_L}$ and a positive constant $\tilde{c}_1 = \tilde{c}_1(k, d_L, \Delta_L)$ such that for any $k \geqslant 1$,*

$$\left|\frac{d^k}{dt^k}\tilde{g}(0)\right| \geqslant \tilde{c}_1 P^{\delta d_L - k}$$

where $\tilde{g}(t) = |\mathfrak{N}(\mathbf{a} + t\mathbf{v}M)|^\delta$.

Proof. Assume first that L has no real embeddings and that the first two coordinates in $L_{\mathbb{R}}$ correspond to conjugate embeddings. Write $\mathbf{a} = (a_1, a_2, \dots, a_{d_L})$

and take $\mathbf{v}(\mathbf{a}) = \big(\frac{a_1}{|a_1|}, \frac{a_2}{|a_2|}, 0, \dots, 0\big)M^{-1}$. Note that $a_1 = \overline{a}_2$ since $\mathbf{a} \in L_{\mathbb{R}}$. Using Lemma 16 with $V_1 = \mathbf{v}$ and $\mathbf{x}_0 + \mathbf{n}M = \mathbf{a}$ we see that

$$\begin{aligned}\frac{d^k}{dt^k}\tilde{g}(0) &= \sum_{\substack{i_1,\dots,i_k \\ 1\leqslant i_j \leqslant d_L}} D_{i_1}\dots D_{i_k}F(\mathbf{a})v_{i_1}\dots v_{i_k} \\ &= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} D_1^j D_2^{k-j} F(\mathbf{a}) \left(\frac{a_1}{|a_1|}\right)^j \left(\frac{a_2}{|a_2|}\right)^{k-j} \\ &= \frac{k! F(\mathbf{a})}{|a_1|^k} \sum_j \binom{\delta}{j}\binom{\delta}{k-j} = \frac{k! F(\mathbf{a})}{|a_1|^k}\binom{2\delta}{k}\end{aligned}$$

where $\binom{\delta}{j}$ is the coefficient of x^j in the Taylor series expansion of $(1+x)^\delta$ and the last equality follows by writing $(1+x)^{2\delta} = (1+x)^\delta \cdot (1+x)^\delta$ in two ways as series and comparing the coefficients of x^k. Since $\mathbf{a} \in \mathbf{R}$, $c_1 P < |a_i| < c_2 P$ for each i. We thus obtain

$$\left|\frac{d^k}{dt^k}\tilde{g}(0)\right| \geqslant c_1^{\delta d_L} c_2^{-k} P^{\delta d_L - k} k! \left|\binom{2\delta}{k}\right|.$$

If L has at least one real embedding, take $\mathbf{v} = (1, 0, \dots, 0)M^{-1}$. In this case, Lemma 16 gives

$$\left|\frac{d^k}{dt^k}\tilde{g}(0)\right| = \left|\delta(\delta-1)\cdots(\delta-k+1)F(\mathbf{a})a_1^{-k}\right| \geqslant c_1^{\delta d_L} c_2^{-k} P^{\delta d_L - k} k! \left|\binom{\delta}{k}\right|.$$

Since $\delta \in (1/2, 1)$ and is fixed, we obtain the claimed lower bound. □

Lemma 18. *Given* $\mathbf{a} = \mathbf{x}_0 + \mathbf{n}M \in \mathbf{R}$ *where* $\mathbf{n} \in \mathbb{Z}^{d_L}$, *there exists a matrix* $U \in SL(d_L, \mathbb{Z})$ *such that for any* $k \geqslant 1$,

$$\frac{\partial^k g_U(\mathbf{n}U^{-1})}{\partial x_1^k} \gg P^{\delta d_L - k}, \qquad \textit{and} \qquad \frac{\partial \lambda_i(\mathbf{n}U^{-1})}{\partial x_1} \gg P^{-1} \qquad \forall i = 1, \dots, r$$

where g_U *is given by* (3.15) *and the implied constants depend on* d_L *and* Δ_L, *with the first one also depending on* k.

Proof. Using Lemma 17 we find a vector $\tilde{\mathbf{v}} = (\tilde{v}_1, \dots, \tilde{v}_{d_L}) \in \mathbb{R}^{d_L}$. Put $\mathbf{v} = \tilde{\mathbf{v}}M = (v_1, \dots, v_{d_L})$. Suppose that for some $Q > 0$, there exists $\tilde{\mathbf{u}} = (\tilde{u}_1, \dots, \tilde{u}_{d_L}) \in \mathbb{Z}^{d_L}$

such that $|\tilde{u}_i - Q\tilde{v}_i| < 1$. Put $\mathbf{u} = \tilde{\mathbf{u}}M$ and $\mathbf{w} = \mathbf{u} - Q\mathbf{v} = (w_1, \ldots, w_{d_L})$. By Lemma 16 we see that

$$\begin{aligned}
\frac{d^k}{dt^k}\tilde{g}_{\tilde{\mathbf{u}}}(0) &= \sum_{\substack{i_1,\ldots,i_k \\ 1\leqslant i_j\leqslant d_L}} D_{i_1}\ldots D_{i_k}F(\mathbf{a})\prod_{l=1}^{k}(Qv_{i_l} + w_{i_l}) \\
&= \sum_{\substack{i_1,\ldots,i_k \\ 1\leqslant i_j\leqslant d_L}} D_{i_1}\ldots D_{i_k}F(\mathbf{a})\Big(Q^k v_{i_1}\cdots v_{i_k} + \sum_{l=1}^{k} Q^{k-l}A_l(\mathbf{v},\mathbf{w})\Big) \\
&= Q^k\frac{d^k}{dt^k}\tilde{g}_{\tilde{\mathbf{v}}}(0) + \sum_{l=1}^{k} Q^{k-l}\sum_{\substack{i_1,\ldots,i_k \\ 1\leqslant i_j\leqslant d_L}} D_{i_1}\ldots D_{i_k}F(\mathbf{a})A_l(\mathbf{v},\mathbf{w}).
\end{aligned}$$

Let's write $D_{i_1}\ldots D_{i_k}F(\mathbf{a})$ by grouping the same indices as $D_{j_1}^{l_1}\ldots D_{j_r}^{l_r}F(\mathbf{a})$ with j_i's distinct and $\sum_i l_i = k$. Since $\mathbf{a} \in \mathbf{R}$, $c_1P < |a_i| < c_2P$ for each i. Thus, we have

$$\begin{aligned}
|D_{j_1}^{l_1}\ldots D_{j_r}^{l_r}F(\mathbf{a})| &= |F(\mathbf{a})|\prod_i \frac{|\delta(\delta-1)\cdots(\delta-l_i+1)|}{|a_i|^{l_i}} \\
&\leqslant (c_2P)^{\delta d_L}\prod_i \frac{|\delta(\delta-1)\cdots(\delta-l_i+1)|}{(c_1P)^{l_i}} \leqslant c_3P^{\delta d_L-k}
\end{aligned}$$

for some constant $c_3 = c_3(k, d_L, \Delta_L) > 0$. Owing to the way $\tilde{\mathbf{v}}$ is constructed in Lemma 17, each $|v_i| \leqslant 1$. Furthermore, each w_i is bounded only in terms of d_L and Δ_L. Therefore, there exists a constant $c_4 = c_4(k, d_L, \Delta_L)$ such that $|A_l(\mathbf{v},\mathbf{w})| \leqslant c_4$. We thus conclude from Lemma 17 that

$$\begin{aligned}
\left|\frac{d^k}{dt^k}\tilde{g}_{\tilde{\mathbf{u}}}(0)\right| &\geqslant Q^k\left|\frac{d^k}{dt^k}\tilde{g}_{\tilde{\mathbf{v}}}(0)\right| - \sum_{l=1}^{k}Q^{k-l}\left|\sum_{\substack{i_1,\ldots,i_k \\ 1\leqslant i_j\leqslant d_L}} D_{i_1}\ldots D_{i_k}F(\mathbf{a})A_l(\mathbf{v},\mathbf{w})\right| \\
&\geqslant P^{\delta d_L-k}\big(\tilde{c}_1Q^k - C_{k-1}Q^{k-1} - \ldots - C_1Q - C_0\big)
\end{aligned}$$

for some constants $C_i = C_i(k, d_L, \Delta_L) > 0$.

Next, let $G_U(\mathbf{x}) = \ell(\mathbf{x}_0 + \mathbf{x}UM)E^{-1}$. Note that $\lambda_i(\mathbf{x})$ is the ith coordinate of this function. Writing $\mathbf{a} = (a_1, \ldots, a_{d_L})$ and $\mathbf{u} = (u_1, \ldots, u_{d_L})$ we derive that

$$\frac{\partial G_U(\mathbf{x})}{\partial x_1}\Big|_{\mathbf{x}=\mathbf{n}U^{-1}} = \Big(\mathrm{Re}\Big(\frac{u_1}{a_1}\Big), \ldots, \mathrm{Re}\Big(\frac{u_r}{a_r}\Big)\Big)E^{-1}$$

where $\mathrm{Re}(z)$ denotes the real part of z. Recalling that $u_i = Qv_i + w_i$ we conclude as before that

$$\left|\frac{\partial\lambda_i(\mathbf{n}U^{-1})}{\partial x_1}\right| \geqslant P^{-1}(\tilde{C}_1Q - \tilde{C}_0)$$

for some positive constants $\tilde{C}_1$ and $\tilde{C}_0$ that depend only on d_L and Δ_L.

It follows that there exists a constant $Q_0 = Q_0(k, d_L, \Delta_L) > 0$ such that both polynomials in Q above are positive for $Q > Q_0$. If all the components of $\tilde{\mathbf{v}}$ are equal we fix some $Q > Q_0$ and let $\tilde{u}_1 = \lceil Q\tilde{v}_1 \rceil$ and $\tilde{u}_i = \lfloor Q\tilde{v}_1 \rfloor$ (if any $\tilde{u}_i$ turns out to be zero, we can instead choose all $\tilde{u}_i = 1$). Otherwise, find the first index i_0 such that $|\tilde{v}_{i_0}| = \max_i |\tilde{v}_i|$ and choose $Q = (p - 1/2)/|\tilde{v}_{i_0}|$, where p is the smallest prime $> Q_0|\tilde{v}_{i_0}|$. Choose $\tilde{u}_{i_0} = \pm p$ depending on the sign of $\tilde{v}_{i_0}$, and the rest of the $\tilde{u}_j$'s as either the ceiling or the floor of $Q\tilde{v}_j$ so that $0 < |\tilde{u}_j| < |\tilde{u}_{i_0}| = p$ for $j \neq i_0$. In either case, we can find a vector $\tilde{\mathbf{u}} \in \mathbb{Z}^{d_L}$ that satisfies $|\tilde{u}_i - Q\tilde{v}_i| < 1$ and that $\gcd(\tilde{u}_1, \ldots, \tilde{u}_{d_L}) = 1$. It follows from [11, Corollary II.1] that $\tilde{\mathbf{u}}$ then can be completed to a matrix $U \in SL(d_L, \mathbb{Z})$ with $\tilde{\mathbf{u}}$ as the first row. Thus, the claimed lower bound follows by noting that s

$$\frac{\partial^k g_U(\mathbf{n}U^{-1})}{\partial x_1^k} = \frac{d^k}{dt^k}\tilde{g}_{\tilde{\mathbf{u}}}(0) \gg P^{\delta d_L - k}.$$

□

Suppose now that $\mathbf{x}_0 + \mathbf{n}M \in P\Gamma_0$ for some $\mathbf{n} \in B \cap \mathbb{Z}^{d_L}$. It follows from Lemma 18 with $\mathbf{a} = \mathbf{x}_0 + \mathbf{n}M$ that there exists a matrix U such that the inequality

$$\left|\frac{\partial^k}{\partial x_1^k} g_U(\mathbf{x})\right| \geqslant c_3 P^{\delta d_L - k}$$

holds for some positive constant $c_3 = c_3(k, d_L, \Delta_L)$ where $\mathbf{x} = \mathbf{n}U^{-1}$. If $\mathbf{x}'$ is any other point in BU^{-1} it follows from the Mean Value Theorem for integrals, Lemma 16 and the fact that $\mathbf{x}_0 + BM \subseteq \mathbf{R}$ that

$$\begin{aligned}\frac{\partial^k}{\partial x_1^k} g_U(\mathbf{x}) - \frac{\partial^k}{\partial x_1^k} g_U(\mathbf{x}') &= \int_0^1 \frac{d}{dt}\left(\frac{\partial^k}{\partial x_1^k} g_U\left(t\mathbf{x} + (1-t)\mathbf{x}'\right)\right) dt \\ &\ll Y P^{\delta d_L - k - 1}\end{aligned}$$

where the implied constant, say c_4, depends on k, d_L, and Δ_L. In particular, it does not depend on the choice of $\mathbf{x}' \in BU^{-1}$. Thus, for any point $\mathbf{x}' \in BU^{-1}$, the lower bound

$$\left|\frac{\partial^k}{\partial x_1^k} g_U(\mathbf{x}')\right| \geqslant \frac{c_3}{2} P^{\delta d_L - k}$$

holds provided that $1 \leqslant Y \leqslant c_3 P/(2c_4)$. This condition imposes a further restriction on N; namely, that $N^{2-\delta-\eta} \geqslant \mathfrak{Na}(2c_4/c_3)^{d_L}$. Assuming $\eta < 1/d_L$ and that

$\mathfrak{Na}$ is bounded (follows from Lemma 13), it follows that for sufficiently large N, and all $\mathbf{x}' \in BU^{-1}$,

$$\frac{\partial^k}{\partial x_1^k} g_U(\mathbf{x}') \asymp P^{\delta d_L - k}$$

where the implied constants depend only on k, d_L and Δ_L provided $1 \leqslant Y \ll P$. Using the same argument we can also show that λ_i's are monotonic in the first variable on BU^{-1}.

Chapter 4

Proof of Theorem 1

By the definition of the conductor (cf. [10, Ch. VI - 6.3 and 6.4]), $K^{\mathfrak{f}}/K$ is the smallest ray class field containing the abelian extension $K/\mathbb{Q}$. Furthermore, every ray class field over $\mathbb{Q}$ corresponds to a cyclotomic extension. In particular, it follows from [10, Proposition 6.7] that there is an integer q such that $\mathfrak{f} = (q)$ and $K^{\mathfrak{f}}$ is the qth cyclotomic extension of $\mathbb{Q}$.

Fix $\sigma_0 \in G$ and put $A_0 = \{\sigma \in Gal(L/\mathbb{Q}) : \sigma_{|_K} = \sigma_0\}$, where $\sigma_{|_K}$ is the restriction of σ to K. Then, it follows from [6, Ch. 3, Property 2.4] that the set $\pi(K, \{\sigma_0\})$ is the *disjoint* union of the sets $\pi(L, \{\sigma\})$ for $\sigma \in A_0$. Therefore, we conclude that

$$\pi_c(K, \{\sigma_0\}, x) = \sum_{\sigma \in A_0} \pi_c(L, \{\sigma\}, x).$$

Since each $\sigma \in A_0$ corresponds to some $a_\sigma \in \left(\mathbb{Z}/q\mathbb{Z}\right)^*$, we have $\pi_c(L, \{\sigma\}, x) = \pi_c(x; q, a_\sigma)$, where the latter counts the Piatetski-Shapiro primes not exceeding x that are congruent to a_σ modulo q.

By Theorem 4 and partial summation, there exists an absolute constant $D > 0$ and a constant $x_0(\mathfrak{f})$ such that for $x \geqslant x_0(\mathfrak{f})$, we have

$$\sum_{\substack{p \leqslant x \\ p \equiv a_\sigma \bmod q}} \left((p+1)^\delta - p^\delta\right) = \frac{\delta}{\varphi(q)} \mathrm{li}(x^\delta) + O\big(x^\delta \exp(-D\sqrt{\log x})\big)$$

where the implied constant is absolute. Furthermore, as in the proof of Theorem

2, choosing $H = N^{1-\delta+\varepsilon}$ we derive that the difference

$$\pi_c(x; q, a_\sigma) - \sum_{\substack{p \leqslant x \\ p \equiv a_\sigma \bmod q}} \left((p+1)^\delta - p^\delta\right)$$

is

$$\ll \sum_{\substack{1 \leqslant N < x \\ N = 2^k}} N^{\delta - 1} \max_{N' \in (N, N_1]} \sum_{h \leqslant H} \left| \sum_{\substack{N < n \leqslant N' \\ n \equiv a_\sigma \bmod q}} e(hn^\delta)\Lambda(n) \right| + x^\delta \exp(-D\sqrt{\log x})$$

where D is the same constant above. Thus, to finish the proof it suffices to show that for any $N' \in (N, N_1]$,

$$\sum_{h \leqslant H} \left| \sum_{\substack{N < n \leqslant N' \\ n \equiv a_\sigma \bmod q}} e(hn^\delta)\Lambda(n) \right| \ll N \exp(-D\sqrt{\log N}).$$

Using Lemma 2 with $L = \mathbb{Q}$ and assuming that $v = u < N$, we obtain

$$\sum_{\substack{N < n \leqslant N' \\ n \equiv a_\sigma \bmod q}} e(hn^\delta)\Lambda(n) = S_1 + S_2 + S_3 + S_4$$

where

$$S_1 = -\sum_{\substack{N < n \leqslant N' \\ n \equiv a_\sigma \bmod q}} e(hn^\delta) \sum_{\substack{n = cd \\ c, d > v}} \Lambda(c) \sum_{\substack{d = ab \\ b \leqslant v}} \mu(b),$$

$$S_2 = \sum_{\substack{N < n \leqslant N' \\ n \equiv a_\sigma \bmod q}} e(hn^\delta) \sum_{\substack{n = ab \\ b \leqslant v}} \mu(b) \log a,$$

$$S_3 = -\sum_{\substack{N < n \leqslant N' \\ n \equiv a_\sigma \bmod q}} e(hn^\delta) \sum_{\substack{n = abc \\ b, c \leqslant v \\ bc \leqslant v}} \mu(b)\Lambda(c),$$

and

$$S_4 = -\sum_{\substack{N < n \leqslant N' \\ n \equiv a_\sigma \bmod q}} e(hn^\delta) \sum_{\substack{n = abc \\ b, c \leqslant v \\ bc > v}} \mu(b)\Lambda(c).$$

Using Dirichlet characters χ modulo q (for a concrete definition of Dirichlet characters see [9, §4]) we obtain

$$S_1 = -\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a_\sigma)} \sum_{\substack{N < cd \leqslant N' \\ c, d > v}} \chi(d) \left(\sum_{\substack{d = ab \\ b \leqslant v}} \mu(b) \right) \chi(c)\Lambda(c) e(h(cd)^\delta),$$

where φ is Euler's totient function. By Lemma 6, we conclude as in the non-abelian case that

$$N^{-4\varepsilon/3}\sum_h |S_1| \ll N^{2-1/12-\delta} + N^{2-\delta}v^{-1/2}.$$

Similarly, applying Lemma 6 once again we conclude as we did for S_1 above that

$$N^{-4\varepsilon/3}\sum_h |S_3| \ll N^{2-\delta-1/12} + N^{2-\delta}v^{-1/2} + N^{3/2-\delta}v.$$

To estimate S_2, we use additive characters modulo q to obtain

$$S_2 = \frac{1}{q}\sum_{k=0}^{q-1} e(-ka_\sigma/q)\sum_{b\leqslant v}\mu(b)\sum_{\substack{a\\ N/b<a\leqslant N'/b}} e(f(a))\log a,$$

where $f(x) = hb^\delta x^\delta + kbx/q$. Since $|f''(x)| \asymp hb^2N^{\delta-2}$ for $N/b < x \leqslant N'/b$ we conclude by Lemma 5 that

$$\sum_{\substack{a\\ N/b<a\leqslant N'/b}} e(h(ab)^\delta + kab/q) \ll N^{\delta/2}h^{1/2} + h^{-1/2}b^{-1}N^{1-\delta/2}.$$

Using partial summation and then summing over $b \leqslant v$ followed by $h \leqslant H$ we obtain

$$\sum_h |S_2| \ll \left(N^{\delta/2}H^{3/2}v + H^{1/2}N^{1-\delta/2}\right)\log^2 N \ll N^{3/2-\delta+2\varepsilon}v.$$

Finally, as indicated in Theorem 2, S_4 can be handled exactly the same way as S_2. Choosing $v = N^{\delta-1/2-3\varepsilon}$ with a sufficiently small ε and combining all the estimates obtained above we see that

$$\sum_{h\leqslant H}\left|\sum_{\substack{N<n\leqslant N'\\ n\equiv a_\sigma \bmod q}} e(hn^\delta)\Lambda(n)\right| \ll N\exp(-D\sqrt{\log N}),$$

as desired, provided that $c \in (1, 12/11)$.

The proof of Theorem 1 is thus completed by noting that the number of elements in A_0 equals $|Gal(L/K)| = \varphi(q)|\Delta_K|^{-1}$.

Chapter 5

Proof of Theorem 3

We start with a simple but useful lemma of which the proof of Theorem 3 is an immediate corrollary.

Lemma 19. *Let* $\{c_n\}_{n=1}^{\infty}$ *be a bounded sequence of complex numbers. Let* $c > 0$ *and* $0 \leqslant \beta < 1/4$ *be fixed. Then, for almost all* $\delta \in (1/2 + 2\beta, 1)$ *one has*

$$\sum_{n \in \mathcal{A}_c(x)} c_n = \delta \sum_{n \leq x} c_n n^{\delta - 1} + o\left(x^{\delta-\beta} exp\left(-c\sqrt{\log x}\right)\right). \tag{5.1}$$

Here we note that Theorem 3 now follows by taking c_n to be the indicator function of the related set of primes either in Theorem 2 or in Theorem 1.

Proof of Lemma 19. Let A denote the subset of $(1/2+2\beta, 1)$ for which (5.1) holds. We shall prove that the complement of A has Lebesque measure zero. Note that it is enough to work on the smaller interval

$$\mathcal{I} = (1/2 + 2\beta + \varepsilon, 1)$$

for any $\varepsilon > 0$ fixed. Following the same methodology in Theorem 2, choosing $H_N = N^{1-\delta+\beta} exp\left(c\sqrt{\log N}\right) \log N$, we see that

$$\sum_{n \in \mathcal{A}_c(x)} c_n - \delta \sum_{n \leq x} c_n n^{\delta - 1}$$

is

$$\ll o\left(x^{\delta-\beta} exp\left(-c\sqrt{\log x}\right)\right) + y$$

$$+ \sum_{\substack{y<N\leqslant x\\ N=2^k}} \sum_{1\leqslant|h|\leqslant H} h^{-1}\left| \sum_{N<n\leqslant N'} e(hn^{\delta})\phi_h(n)c_n \right| \tag{5.2}$$

where $N' = \min\{2N, x\}$, and $1 \leqslant y < x$ is an arbitrary number.

Put $E = \exp\left(c\sqrt{\log N}\right)\log N$. Defining the sets

$$\mathcal{A}(N) = \left\{\delta \in \mathcal{I} : \sum_{1\leqslant|h|\leqslant H} h^{-1}\left| \sum_{N<n\leqslant N'} c_n e(hn^{\delta})\phi_h(n) \right| > N^{\delta-\beta}E^{-1}\right\},$$

we observe that

$$\mathcal{I} \setminus \bigcup_{y<N\leqslant x} \mathcal{A}(N) \subseteq \mathcal{A} \cap \mathcal{I}.$$

Thus, it is sufficient to show that for arbitrary $y > 1$,

$$\sum_{N=2^l>y} \mu(\mathcal{A}(N)) \ll_{\mathcal{S}} y^{-\varepsilon} \tag{5.3}$$

where μ denotes the Lebesque measure.

Observe that

$$\mu(\mathcal{A}(N)) < N^{2\beta-2\delta}E^2 \int_{\mathcal{I}} \left(\sum_{1\leqslant|h|\leqslant H} h^{-1}\left| \sum_{N<n\leqslant N'} c_n e(hn^{\delta})\phi_h(n) \right| \right)^2 d\delta.$$

The bounds $\phi_h(x) \ll hx^{\delta-1}$ and $\delta > 1/2 + 2\beta + \varepsilon$, together with a proper use of Cauchy-Schwartz inequality yield

$$\mu(\mathcal{A}(N)) \ll N^{-2\varepsilon}E^3 + N^{3\beta-1-\delta}E^3 \sum_h \sum_{\substack{m,n\\ m>n}} \left| \int_{\mathcal{I}} e(h(m^{\delta} - n^{\delta}))d\delta \right|.$$

We set $m = n + q$ for some q, to derive

$$\sum_h \sum_{\substack{m,n\\ m>n}} \left| \int_{\mathcal{I}} e(h(m^{\delta} - n^{\delta}))d\delta \right| = \sum_h \sum_n \sum_{1\leqslant q\leqslant N-n} \left| \int_{\mathcal{I}} e(h((n+q)^{\delta} - n^{\delta}))d\delta \right|.$$

Using Lemma 7, one gets

$$\int_{\mathcal{I}} e(h((n+q)^\delta - n^\delta))d\delta \ll \frac{1}{q|h|N^\delta \log N},$$

and thus

$$\mu(A(N)) \ll N^{-\varepsilon}.$$

Setting $N = 2^k$ and summing over k, we arrive at (5.3), hence Lemma 19 follows.

□

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