ROBUST AUCTION DESIGN UNDER MULTIPLE PRIORS

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Robust Auction Design Under Multiple Priors
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ROBUST AUCTION DESIGN UNDER MULTIPLE PRIORS

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In optimal auction design literature, it is a common assumption that valuations of buyers are independently drawn from a unique distribution. In this thesis, we study auctions with ambiguity for an environment where valuation distribution is uncertain itself and introduce a linear programming approach to robust auction design problem. We develop an algorithm that gives the optimal solution to the problem under certain assumptions when the seller is ambiguity averse with prior set P and the buyers are ambiguity neutral with a prior $f \in P$. Also, we consider the case where the buyers are ambiguity averse as the seller and formulate this problem as a mixed integer programming problem. Then, we propose a hybrid algorithm that enables to achieve a good solution for this problem in a reduced time.

Keywords: robust optimization, auction design, mechanism design, ambiguity aversion.

ÖZET

GÜRBÜZ İHALE TASARIMI

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İhale tasarımı literatüründe yapılan genel bir varsayım, alıcıların satılan ürüne verdikleri değerlerin bağımsız olarak tek bir dağılımdan geldikleridir. Bu tezde, biz bu değerlerin bağlı olduğu dağılımın kendisinin de belirsiz olduğu varsayımını yapıyoruz ve gürbüz ihale tasarımı problemini doğrusal programlama bakış açısıyla çözüyoruz. Belli varsayımlar altında, satıcının doğru değer dağılımını bilmediği, belirsizlikten kaçındığı ve alıcıların ise doğru değer dağılımını bildiği çevre için, probleme en iyi sonucu veren bir algoritma geliştiriyoruz. Ayrıca, alıcıların da belirsizlik karşıtı olduğu durumu da göz önünde bulunduruyoruz ve bu problem için bir karışık tamsayı formülasyonu veriyoruz. Daha sonra, probleme kısa bir süre içerisinde iyi bir sonuç veren, hibrit bir algoritma öneriyoruz.

Anahtar sözcükler: gürbüz, optimizasyon, ihale tasarımı, mekanizma tasarımı, belirsizlik karşıtlığı .

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

An auction is a process of selling a single/multiple goods. A common aspect of auctions is that they collect bids from buyers and the outcome is the determination of an allocation rule specifying who gets the object and a payment rule describing how much every bidder must pay. Determining the most profitable auction rule is important because auctions have been used for a significant volume of economic transactions both in public and private sector [1].

In auctions, each buyer has a valuation -willingness to pay- assigned to goods on sale. The major reason for holding auctions is the seller's lack of knowledge about these valuations. In optimal auction design literature, it is mostly assumed that buyers' valuations are independently drawn from a unique distribution. However, in reality, it is more likely that there will occur some estimation errors and this valuation distribution will be uncertain itself.

In this thesis, we study auctions for an environment where valuation distribution comes from a set of distributions P and introduce a linear programming approach to robust auction design problem where a single object is sold to potential buyers. To have a finite number of equations in our formulation, we let the valuation distribution to be discrete as well as the set P. Moreover, we allow ambiguity about valuation distribution. In the literature, it is showed that the decision makers mostly develop ambiguity averse behaviour [2]. In this study, we consider the seller to be ambiguity averse in a way that she tries to maximize the worst case expected revenue. Hence, we adopt a more realistic approach to formulate auction design problem compared to the studies with unique valuation distribution assumption.

The remainder of this thesis is organized as follows: *Chapter 2* provides a literature review in auction design. Also, some important concepts related to our study are introduced. In *Chapter 3*, we define robust action design problem when the seller is ambiguity averse and the buyers are ambiguity neutral. Note that ambiguity neutrality of buyers leads them to give the same importance to all possible realizations of the valuation distribution. We reformulate this problem as a linear programming problem. Then, we develop an algorithm which gives the optimal solution under certain assumptions. Moreover, we make analyses and conclusions on the optimal mechanism derived from the algorithm. In *Chapter 4*, we introduce robust auction design problem when the buyers are ambiguity averse too. We give a reformulation of the problem as a mixed integer programming problem. To solve the given problem, we propose an algorithm which almost always enables to achieve an optimal solution in a reduced time. We support our claim by computational results. Finally, we give concluding remarks and extensions of robust auction design problem in *Chapter 5*.

Contributions of this thesis are as follows:

1. In Chapter 3, we give a specific and applicable optimal mechanism for the robust auction design problem with ambiguity averse seller and ambiguity neutral buyers under certain assumptions, which is the only detailed optimal mechanism in the literature to our knowledge. Our optimal mechanism is easy to understand due to similarity to well-known Vickrey auction, and it is reasonable and fair from participants' perspective because only the winner makes a payment which never exceeds his own bid.

2. In Chapter 4, the MIP formulation is new, to the best of our knowledge, as well as the algorithm. With the MIP formulation, we allow robust auction design problem with ambiguity averse seller and buyers to be trackable in a way that it is solvable by Operations Research softwares. Also, to shorten the solution time,

we propose an algorithm and prove that it is useful by computational results.

Chapter 2

Literature Review

In this section, we give a literature review related to our work. For a more detailed one, see [1]. Also, we recommend [3] as an introductory book.

Auction design entered the economics literature relatively recently. In 1961, William Vickrey wrote the first game theoretical analysis of auctions [4]. This was the first occurrence of well-known second price sealed-bid auctions in which buyers simultaneously report sealed bids to the seller, the highest bidder wins the object and pays second highest bid. Today, second price sealed-bid auctions are also called Vickrey auctions.

Myerson, in 1981, came up with *the Revelation Principle* [5].

The outcomes resulting from any equilibrium of any mechanism can be replicated by a truthful equilibrium of some direct mechanism.

By the Revelation Principle, Myerson concluded that restricting attention to only direct mechanisms, a mechanism where all the buyers report their true valuations, does not cause loss of generality under certain assumptions [5]. Utilizing this result, he also showed that the second price auction with a reserve price is an optimal mechanism to classical auction design problem when hazard function, ratio of density function to survival function -one minus cumulative distribution function-, is monotone [5]. In classical auction design problem, there is a risk neutral seller with a single good which she desires to sell to a number of risk neutral buyers. Each buyer has a private valuation assigned to the good. Buyers' valuations are assumed to be independently drawn with respect to a unique continuous distribution function over a finite interval.

In 1981, simultaneously, Myerson [5],and Riley and Samuelson [6] extended Vickrey's results regarding expected revenue equivalence in different auctions and led to the famous *Revenue Equivalence Principle*.

Given certain conditions, any auction mechanism that results in the same outcomes (i.e. allocates items to the same bidders) also has the same expected revenue.

Myerson also analyzed optimal auctions when the monotone hazard function and symmetric buyers assumptions are relaxed [5].

When risk aversion is introduced to the auction design problems, the Revelation Principle is not valid for most of the cases. For analyses of how risk aversion affects the Revelation Principle and literature in risk aversion, we direct the reader to [1]. In this thesis, we assume that the seller and the buyers are risk neutral.

Recently, Rakesh V.Vohra [7] published a paper showing the close relation between linear programming and auction design when valuations of buyers are discrete. In this publication, he used standard results from linear programming to solve a wide class of auction design problems. His work has been a motivation for us to use linear programming in robust auction design problem.

Moreover, although auction problems have been widely studied in the literature, results on robust auction design are somehow limited due to the complexity of the problem. In [8], Gilboa and Schmeidler modeled ambiguity aversion using maxmin expected utility (MMEU). In MMEU, decision maker is characterized by a utility function and a set of priors and the chosen act maximizes the minimal expected utility over the prior set. In this thesis, we follow their work to formulate robust auction design problem.

There have been a few studies on auction design allowing ambiguity in prior distribution but most of them look at some specific auctions, such as first price auction and second price auction, rather than an optimal auction [9], [10].

Bandi and Bertsimas studied optimal design for multi-item auction from a robust optimization perspective but this study is slightly different from our work [11]. They define allocation and payment rule variables for all possible realizations of the prior. In other words, they present an auction mechanism that is carried out when the prior is realized while in our problem, the seller announces a single allocation rule and a single payment rule before the realization.

Reference [12] is closer to our work. The difference from our approach is that valuation distribution f is assumed to be continuous over a finite interval and the prior set P is infinite. Under monotone hazard function assumption,in [12], it is proved that when the seller is ambiguity averse and the bidders are ambiguity neutral, an auction that fully insures the seller is in the set of optimal mechanisms. In Chapter 3, we derive an optimal mechanism for robust auction design problem and claim that this is the unique optimal mechanism. In fact, the mechanism we observed does not fully insure the seller. Hence, the result of [12] is not valid for our setting.

We also point out that, under certain assumptions some properties of optimal mechanism were given in [12] when buyers are also ambiguity averse. Bose et al. [12] showed that when the bidders face more ambiguity than the seller in a way that buyers' prior set contains the seller's prior set, the seller can always increase revenue by switching to an auction providing full insurance to all types of bidders, and in general neither the first nor the second price auction is optimal.

In this thesis, we developed an algorithm which gives an applicable optimal mechanism for robust auction design problem when the seller is ambiguity averse and buyers are ambiguity neutral. Some properties of the optimal mechanism to the robust auction design problem was stated in the literature but such a detailed one was never driven to the best of our knowledge. Also, we considered the case where buyers are ambiguity averse too. We do not derive an optimal mechanism for this problem but we formulate the problem as a mixed integer programming problem which is trackable. Then, we propose an algorithm which enables to solve this problem in a reduced time. We think that our results will ease further studies for robust auction design problem with ambiguity averse seller and buyers.

Chapter 3

Auction Design Problem with Ambiguity Averse Seller

3.1 Problem Definition

The environment of our problem consists of a single ambiguity averse seller with prior set P and n number of ambiguity neutral buyers (agents) with a prior $f \in P$. Note that ambiguity aversion occurs due to seller's lack of knowledge about the maximum amount each buyer is willing to pay, which we call valuation (type) of agent. An agent of course knows his own valuation and he also believes that others' valuations are independently drawn from a finite set T with respect to distribution function f. On the other hand, the seller has the knowledge of a prior set P with finite number of distributions in it. The set P includes the true valuation distribution function f. Both the seller and the agents are risk neutral. In other words, they have linear utility functions.

The seller desires to sell a single good to the agents. Since the seller is ambiguity averse, the objective is to maximize her worst case expected revenue.

To formulate this problem, we invoke the Revelation Principle and restrict our attention to only direct mechanisms in which agents simultanously report their true valuations. Recall from Chapter 2, the Revelation Principle states that the outcomes resulting from any equilibrium of any mechanism can be replicated by a truthful equilibrium of some direct mechanism.

3.2 Formulation

Before problem formulation, let us give the notation. We use $t \in T^n$ to denote a profile vector which is constructed by reports of all agents. The symbols a and p are defined to be allocation and payment rule, respectively. The symmetry assumption allows focusing on one agent, say agent 1. Therefore, we let $a(i, t^{-1})$ be the allocation to agent 1 and $p(i, t^{-1})$ be the payment done by agent 1 when he reports his type as $i \in T$ and all other agents report $t^{-1} \in T^{n-1}$. The probability of agents having types that give rise to the profile t^{-1} is denoted by $\pi_f(t^{-1})$ forall $f \in P$. The number of agents with type i in profile t is shown by $n_i(t)$.

Interim (expected) allocations and payments are denoted accordingly:

$$A_f(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1}) \pi_f(t^{-1}) \quad \forall f \in P,$$
$$P_f(i) = \sum_{t^{-1} \in T^{n-1}} p_i(i, t^{-1}) \pi_f(t^{-1}) \quad \forall f \in P.$$

To clarify, $A_f(i)$ denotes expected allocation to agent 1 and $P_f(i)$ is the payment of agent 1 if he reports type i where $f \in P$.

We face the following constrained maximization problem (opt^1):

$$\max \quad \left\{ \min_{f \in P} \sum_{i \in T} f_i P_f(i) \right\} \tag{3.1}$$

$$\text{s.t.} \quad iA_f(i) - P_f(i) \geq iA_f(j) - P_f(j) \qquad \forall i, j \in T \qquad \forall f \in P \tag{3.2}$$

$$iA_f(i) - P_f(i) \geq 0 \qquad \forall i \in T \qquad \forall f \in P \tag{3.3}$$

$$A_f(i) = \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1}) \pi_f(t^{-1}) \qquad \forall i \in T \qquad \forall f \in P \tag{3.4}$$

$$\sum_{i \in T} n_i(t) a_i(t) \leq 1 \qquad \forall t \in T^n \tag{3.5}$$

$$a_i(t) \geq 0 \qquad \forall i \in T, \forall t \in T^n \tag{3.6}$$

The objective of our problem is to maximize seller's worst case expected revenue (3.1). In other words, since the seller does not know which member of P is the true valuation distribution function, she tries to maximize the minimum expected revenue over $f \in P$ due to ambiguity aversion.

Constraints (3.2) are called Bayes-Nash Incentive Compatibility (BNIC) constraints in literature. These constraints ensure that, for an agent, misreporting the valuation will always result in expected utility which is less than or equal to the one when the type is truthfully reported. Note that we are only interested in direct mechanisms and, by BNIC, a risk neutral agent's optimal strategy is to truthfully report his valuation.

With constraints (3.3), each agent will choose to participate in the auction because he will gain a nonnegative expected payoff in every possible outcome of profiles. This type of constraints is known as Individual Rationality (IR) constraints.

Constraints (3.4) satisfy the consistency between interim allocations and allocation rule variables. Obviously, constraints (3.5) and (3.6) ensure that at most one good is allocated for each profile outcome and no agent receives a negative amount.

Next, we associate shortest path problems with BNIC and IR constraints to reformulate (opt^1).

3.2.1 Network Association

In this section, we follow Vohra's approach [7] and benefit from shortest path problems and duality theory.

Consider (3.2) and (3.3). They can be rewritten as follows:

$$iA_f(i) - iA_f(j) \geq P_f(i) - P_f(j) \qquad \forall i, j \in T \qquad \forall f \in P \qquad (3.2),$$
$$iA_f(i) \geq P_f(i) \qquad \forall i \in T \qquad \forall f \in P \qquad (3.3).$$

For each $f \in P$, we can associate system (3.2) and (3.3) with the following network:

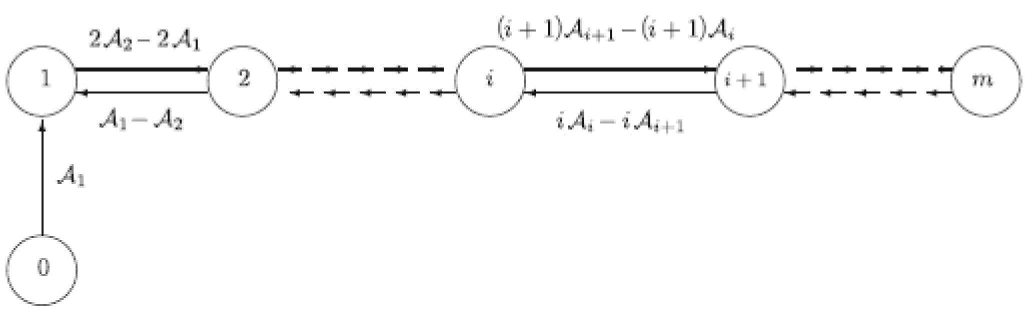

Figure 3.1: Network of Valuations

In Figure 3.1, each vertex corresponds to a type in T. Dummy type of value 0 -with $A_f(0)$ and $P_f(0)$ equal to 0 forall $f \in P$- is introduced to the network to include IR constraints (3.3) to network association. There is a directed edge of length $iA_f(i) - iA_f(j)$ between every ordered pair of types (j, i).

Now, consider the following shortest path problem from vertex 0 to m:

$$\min \sum_{i\in T}\sum_{j\in T}(iA_f(i) - iA_f(j))x_{ji}$$

$$\text{s.t.} \sum_{j\in T} x_{ji} - \sum_{j\in T} x_{ij} = \begin{cases} 1 & \text{if } i = m \\ -1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$x_{ij} \in \{0,1\} \qquad \forall i,j \in T$$

We can let x_{ij}'s take continuous values and the optimal solution to the relaxed shortest path problem will still be an integer solution due to totally unimodularity property of feasible set. Note that we consider the relaxed shortest path problem from this point.

For fixed interim allocation values, if we introduce $P_f(i)$'s to be dual variables corresponding to each constraint of the shortest path problem then we observe that (3.2) and (3.3) are the constraints of the dual problem. Hence, system (3.2) and (3.3) is feasible if and only if the network has no negative length cycles. Otherwise, the shortest path problem is unbounded, which leads the corresponding dual problem to be infeasible.

Theorem 1. *The system (3.2) and (3.3) is feasible if and only if interim allocations are monotonic, i.e., if $i \leq j$, then $A_f(i) \leq A_f(j)$ for all $f \in P$.*

For proof, see [7].

Moreover, note that to have no negative length cycles, the length of the edge from i to $i+2$ must be at least as large as the sum of the lengths of edges $(i, i+1)$ and $(i+1, i+2)$. This implies that Figure 3.1 includes all shortest paths from vertex 0 to m.

Also, we observe that, in absence of negative cycles, the shortest path from vertex 0 to i gives the tightest upper bound for each $P_f(i)$. Since the objective is to maximize sum of $P_f(i)$'s with nonnegative multipliers, it is reasonable to set them equal to their tighest upper bounds. Therefore, we can rewrite the objective as follows:

$$\sum_{i\in T} f_i P_f(i) = \sum_{i\in T} f_i \sum_{k=1}^{i} kA_f(k) - kA_f(k-1) = \sum_{i\in T} f_i(iA_f(i) - \sum_{k=1}^{i} A_f(k-1))$$
$$= \sum_{i\in T} f_i i A_f(i) - (1-F(i))A_f(i) = \sum_{i\in T} f_i(i - \frac{1-F(i)}{f_i})A_f(i)$$

We let $\nu_f(i) = i - \frac{1-F(i)}{f_i}$.

Using the development so far, (opt^1) can be reformulated as follows:

$$\max \quad \Big\{\min_{f\in P} \sum_{i\in T} f_i \nu_f(i) A_f(i)\Big\} \tag{3.7}$$

$$\text{s.t.} \quad 0 \leq A_f(1) \leq \cdots \leq A_f(m) \qquad \forall f \in P \tag{3.8}$$

$$A_f(i) = \sum_{t^{-1}\in T^{n-1}} a_i(i, t^{-1})\pi_f(t^{-1}) \qquad \forall i \in T \qquad \forall f \in P$$

$$\sum_{i\in T} n_i(t) a_i(t) \leq 1 \qquad \forall t \in T^n$$

$$a_i(t) \geq 0 \qquad \forall i \in T, \forall t \in T^n.$$

While the objective function takes a new form (3.7), monotonicity of expected allocations (3.8) replaces BNIC (3.2) and IR (3.3).

Vohra's [7] next step is to take out allocation rule variables and solve the problem only over interim allocations. However, we will take out interim allocations instead because otherwise, we cannot really find a useful formulation to ensure existence of a corresponding allocation rule.

3.2.2 The Reason for Projecting Out Expected Allocations

We shall take the same step as Vohra [7] and show his reformulation won't ensure feasibility of expected allocations in our problem. Vohra uses the following theorem to reduce the auction design problem without ambiguity to a polymatroid optimization problem.

Theorem 2. *Border's Theorem [7]* *The expected allocation $A(i)$ is feasible if and only if*

$$n\sum_{i\in S} f_i A(i) \leq 1 - (\sum_{i\notin S} f_i)^n \qquad\qquad \forall S \subseteq T$$

The proof follows from reformulating (3.4), (3.5) and (3.6) as a transportation problem and standard maxflow-mincut characterization of feasibility [13]. Note that in Vohra's problem definition, it is assumed that buyers' valuations depend on a unique distribution function. Hence, (3.4), (3.5) and (3.6) refer to only one f.

In our formulation, since expected allocations differ for each $f \in P$, we need to write inequlities from Border's theorem for all:

$$\begin{array}{lll} \max & \Big\{\min_{f\in P} \sum_{i\in T} f_i \nu_f(i) A_f(i)\Big\} & \\ \text{s.t.} & 0 \leq A_f(1) \leq \cdots \leq A_f(m) & \forall f \in P \\ & n\sum_{i\in S} f_i A_f(i) \leq 1 - (\sum_{i\notin S} f_i)^n \qquad \forall S \subseteq T & \forall f \in P. \end{array}$$

This formulation decomposes for each $f \in P$. The solutions from the decomposed problems will yield several allocation rules which may not be implementable at the same time. Hence, this approach does not answer our problem of maximizing the minimum expected revenue.

3.2.3 Final Form of the Formulation

We take out expected allocation variables and reformulate the problem accordingly:

$$
\begin{aligned}
\max_{a} \quad & \Big\{ \min_{f \in P} \sum_{i \in T} f_i \nu_f(i) \sum_{t^{-1} \in T^{n-1}} a_i(i, t^{-1}) \pi_f(t^{-1}) \Big\} \\
\text{s.t.} \quad & 0 \leq \sum_{t^{-1} \in T^{n-1}} a_1(1, t^{-1}) \pi_f(t^{-1}) \leq \cdots \leq \sum_{t^{-1} \in T^{n-1}} a_m(m, t^{-1}) \pi_f(t^{-1}) && \forall f \in P \\
& \sum_{i \in T} n_i(t) a_i(t) \leq 1 && \forall t \in T^n \\
& a_i(t) \geq 0 && \forall i \in T, \forall t \in T^n.
\end{aligned}
$$

Introducing a new variable z, we can linearize this problem. Below, the final form of the formulation can be found.

$$
\begin{aligned}
\max_{a,z} \quad & z && && (3.9) \\
\text{s.t.} \quad & z \leq \sum_{i \in T} \sum_{t^{-1} \in T^{n-1}} \nu_f(i) a_i(i, t^{-1}) \pi_f(i, t^{-1}) && \forall f \in P && (3.10) \\
& 0 \leq \sum_{t^{-1} \in T^{n-1}} a_1(1, t^{-1}) \pi_f(t^{-1}) \leq \ldots \\
& \leq \sum_{t^{-1} \in T^{n-1}} a_m(m, t^{-1}) \pi_f(t^{-1}) && \forall f \in P && (3.11) \\
& \sum_{i \in T} n_i(t) a_i(t) \leq 1 && \forall t \in T^n && (3.12) \\
& a_i(t) \geq 0 && \forall i \in T, \forall t \in T^n && (3.13)
\end{aligned}
$$

This is a linear programming problem. Hence, it is easy to solve.

3.3 The Solution Approach

To derive an optimal mechanism to final formulation, we focus on the case where there are two agents and the type distribution set is equal to P = {f , g}. Also, we assume that monotone hazard condition holds, which leads $\nu(i)$ to be non-decreasing in $i \in T$. If we ignore monotonicity of interim allocations (3.11), following two propositions and stated results in between hold.

Proposition 1. *Optimal allocation rule satisfies $a^*_{i'}(i', j') \geq a^*_{j'}(j', i')$ for all $\forall(i', j') \in T^2$ such that $i' \geq j'$.*

Proof. Let us show why this is the case by analyzing multipliers of $a_{i'}(i', j')$ and $a_{j'}(j', i')$ in the objective function:
We aim to *maximize* z such that

$$z \leq \sum_{i \in T} \sum_{j \in T} f_i a_i(i, j) \nu_f(i) f_j \tag{3.14}$$

$$z \leq \sum_{i \in T} \sum_{j \in T} g_i a_i(i, j) \nu_g(i) g_j \tag{3.15}$$

For arbitrary i' and j', (3.14) and (3.15) can be rewritten as

$$z \leq ... + f_{i'} f_{j'} a_{i'}(i', j') \nu_f(i') + f_{i'} f_{j'} a_{j'}(j', i') \nu_f(j'),$$
$$z \leq ... + g_{i'} g_{j'} a_{i'}(i', j') \nu_g(i') + g_{i'} g_{j'} a_{j'}(j', i') \nu_g(j').$$

Assume $i' \geq j'$. Then $\nu_f(i') f_{i'} f_{j'} \geq \nu_f(j') f_{i'} f_{j'}$ and $\nu_g(i') g_{i'} g_{j'} \geq \nu_g(j') g_{i'} g_{j'}$ which states a unit increase in $a_{i'}(i', j')$ improves objective function by a larger quantity compared to the same amount of increase in $a_{j'}(j', i')$. Considering the constraint $a_{i'}(i', j') + a_{j'}(j', i') \leq 1$ and allocation variables being nonnegative, it is concluded that $a_{i'}(i', j') \geq a_{j'}(j', i')$ $\forall i' \geq j'$ at optimal solution. □

In fact, it is direct to see that this result is independent from the number of agents participating in the auction and the number of distribution functions contained in P. The interpretation is that, for a profile outcome, allocating the good to the highest bidder is always more profitable if the monotone hazard condition holds.

Remark 1. *By proof of Proposition 1, we can conclude that the optimal allocation rule obeys $a^*_{j'}(j', i') = 0$ $\forall(i', j') \in T^2$ such that $j' \leq i'$ since increasing $a_{i'}(i', j')$ is always preferable to increasing $a_{j'}(j', i')$ and their sum is upperbounded by 1.*

Proposition 2. *If* $f_{i'}\nu_f(i') \geq f_{j'}\nu_f(j')$ $\forall(i',j') \in T^2$ *such that* $i' \geq j'$ $\forall f \in P$, *the optimal allocation rule fulfills the condition* $a^*_{i'}(i',k) \geq a^*_{j'}(j',k)$ $\forall i' \geq j'$.

Proof. Take arbitrary i' and j'.
Case 1: $i', j' < k$ then $a^*_{i'}(i',k) = a^*_{j'}(j',k) = 0$ by Remark 1.
Case 2: $j' < k$ then $a^*_{i'}(i',k) \geq a^*_{j'}(j',k) = 0$.
Case 3: $i', j' \geq k$
For arbitrary i' and j', (3.14) and (3.15) can be rewritten as

$$z \leq ... + f_{i'}f_k a_{i'}[i',k]\nu_f(i') + f_{j'}f_k a_{j'}[j',k]\nu_f(j'),$$
$$z \leq ... + g_{i'}g_k a_{i'}[i',k]\nu_g(i') + g_{j'}g_k a_{j'}[j',k]\nu_g(j').$$

Note that it is assumed $\nu_f(i')f_{i'} \geq \nu_f(j')f_{j'}$ and $\nu_g(i')g_{i'} \geq \nu_g(j')g_{j'}$. Since the objective function multiplier of $a_{i'}(i',k)$ is higher in above equations, a unit increment in $a_{i'}[i',k]$ leads to a greater improvement in objective function value than a unit rise in $a_{j'}(j',k)$ would. With the fact that both $a_{i'}(i',k)$ and $a_{j'}(j',k)$ are upperbounded by 1 (by remark 1), the result is proved. □

Although we prove Proposition 2 for two agents and two distribution functions case, it is obvious that this result is valid for the general case. For the implication of Proposition 2, think of two profile outcomes where only highest bid differs and all other reported types are identical. If the seller allocates the good to the highest bidder which is the lowest in these two profile outcomes, she also sells the good in case of the second profile outcome.

Proposition 3. *If* $f_{i'}\nu_f(i') \geq f_{j'}\nu_f(j')$ $\forall(i',j') \in T^2$ *such that* $i' \geq j'$ $\forall f \in P$ *and hazard function is monotone, then optimal solution ignoring monotonicity constraints is feasible to the final form of the formulation.*

Proof. If $f_{i'}\nu_f(i') \geq f_{j'}\nu_f(j')$ $\forall(i',j') \in T^2$ such that $i' \geq j'$ $\forall f \in P$ and hazard function is monotone, the optimal allocation rule obeys $a^*_{i'}(i',j') \geq a^*_{j'}(j',i')$ and $a^*_{i'}(i',k) \geq a^*_{j'}(j',k)$ for all $\forall(i',j') \in T^2$ such that $i' \geq j'$ by proposition 1 and 2. Hence, it is direct to see that monotonicity constraints (3.11) are satisfied. □

Theorem 3. *If all ν_f's corresponding to $f \in P$ start taking non-negative values from type $i' \in T$ such that*

$$\begin{array}{lll} \nu_f(i) \geq 0 & \forall f \in P & \forall i \in T \ st. \ i \geq i' \\ \nu_f(i) < 0 & \forall f \in P & \forall i \in T \ st. \ i < i' \end{array}$$

then optimal solution of the final formulation has the following structure:

$$a_i^*(i,j) = \begin{cases} 1 & \textit{if } i \geq i' \ \wedge \ i > j \\ 0.5 & \textit{if } i \geq i' \ \wedge \ i = j \\ 0 & o.w. \end{cases} \qquad \forall (i,j) \in T^2$$

The proof follows from the following idea. If we project out allocation rule variables in (opt^1) and decompose the observed formulation for each $f \in P$ as explained before in this chapter, then we would obtain optimal interim allocations for each decomposed problem which are feasible with respect to given allocation rule in Theorem 3; see knapsack solution approach of Vohra [13] for solution of decomposed subproblems.

In this case, i' denotes the reserve price and the good is allocated with equal probability to the highest bidders if the highest bid exceeds the reserve price.

To analyze optimal structure under different circumstances, we make the following assumption. Note that this assumption does not cause loss of generality if the hazard function and respectively ν are monotone.

Assumption 1. $x_f, x_g \in T$ such that $x_f > x_g$ and,

$$\nu_f(i) \text{ is } \begin{cases} \text{nonnegative,} & \text{if } i \geq x_f \\ \text{negative,} & \text{if } i < x_f \end{cases}$$

$$\nu_g(i) \text{ is } \begin{cases} \text{nonnegative,} & \text{if } i \geq x_g \\ \text{negative,} & \text{if } i < x_g \end{cases}$$

Assumption 1 is valid for Theorems 4, 5 and 6.

We introduce the following inequality as a useful condition:

$$\sum_{i=x_f}^{m}\sum_{j}^{i-1}\nu_f(i)f_if_j + \sum_{i=x_f}^{m}0.5\nu_f(i)f_i^2 \leq \sum_{i=x_f}^{m}\sum_{j}^{i-1}\nu_g(i)g_ig_j + \sum_{i=x_f}^{m}0.5\nu_g(i)g_i^2. \quad (3.16)$$

Theorem 4. *If condition (3.16) is met, the optimal solution is in the following structure:*

$$a_i^*(i,j) = \begin{cases} 1 & \text{if } i \geq x_f \ \wedge \ i > j \\ 0.5 & \text{if } i \geq x_f \ \wedge \ i = j \\ 0 & o.w. \end{cases} \qquad \forall (i,j) \in T^2$$

Proof. We aim to maximize the minimum expected revenue over distributions f and g. Solution a^* gives the maximum expected revenue if distribution f is known to be true valuation distribution[13]. Since maximum expected revenue with respect to f is the minimum over set P in the case of (3.16), a^* is an optimal solution. □

We also need the following:

$$\sum_{i=x_g}^{m}\sum_{j}^{i-1}\nu_f(i)f_if_j + \sum_{i=x_g}^{m}0.5\nu_f(i)f_i^2 \geq \sum_{i=x_g}^{m}\sum_{j}^{i-1}\nu_g(i)g_ig_j + \sum_{i=x_g}^{m}0.5\nu_g(i)g_i^2. \quad (3.17)$$

Theorem 5. *When condition (3.17) is satisfied, the optimal solution has the following form:*

$$a_i^*(i,j) = \begin{cases} 1 & \text{if } i \geq x_g \ \wedge \ i > j \\ 0.5 & \text{if } i \geq x_g \ \wedge \ i = j \\ 0 & o.w. \end{cases} \qquad \forall (i,j) \in T^2$$

Proof. Solution a^* gives the maximum expected revenue for the distribution g [13] which is the minimum in the case of (3.17). □

Now, we propose an algorithm to solve the robust auction design problem with ambiguity averse seller when (3.16) and (3.17) fail to hold.

Algorithm 1

Initialize:

$$
\begin{aligned}
& x \leftarrow x_f \\
& a_i^*(i,j) = \begin{cases} 1 & \text{if } i \geq x_f \text{ and } i > j \\ 0.5 & \text{if } i \geq x_f \text{ and } i = j \\ 0 & \text{ow.} \end{cases} \qquad \forall (i,j) \in T^2 \\
& obj_f = \sum_{i=x}^{m} \sum_{j}^{i-1} \nu_f(i) f_i f_j + \sum_{i=x}^{m} (0.5) \nu_f(i) f_i^2 \\
& obj_g = \sum_{i=x}^{m} \sum_{j}^{i-1} \nu_g(i) g_i g_j + \sum_{i=x}^{m} (0.5) \nu_g(i) g_i^2 \\
& \Gamma = \{(i,j) \in T^2 | x \geq i \geq x_g \text{ and } i \geq j\} \\
& \Gamma(i,j) = \frac{v_g(i) g_i g_j}{v_f(i) f_i f_j} \qquad \forall (i,j) \in \Gamma
\end{aligned}
$$

while Γ is not empty & $obj_f > obj_g$ **do**
    Determine $(i', j') \in \Gamma$ st. $\Gamma(i', j') = \min_{(i,j)\in\Gamma} \Gamma(i,j)$
    if $i' \neq j'$ **then**
        if $\nu_f(i') f_{i'} f_{j'} + obj_f > \nu_g(i') g_{i'} g_{j'} + obj_g$ **then**
            $a_{i'}^*(i', j') \leftarrow 1$
            $obj_f \leftarrow obj_f + \nu_f(i') f_{i'} f_{j'}$
            $obj_g \leftarrow obj_g + \nu_g(i') g_{i'} g_{j'}$
        else $\nu_f(i') f_{i'} f_{j'} + obj_f \leq \nu_g(i') g_{i'} g_{j'} + obj_g$
            $a_{i'}^*(i', j') \leftarrow \frac{obj_f - obj_g}{\nu_f(i') f_{i'} f_{j'} - \nu_g(i') g_{i'} g_{j'}}$
            Stop at the current solution.
        end if
    end if
    if $i' = j'$ **then**
        if $(0.5)\nu_f(i') f_{i'} f_{j'} + obj_f > (0.5)\nu_g(i') g_{i'} g_{j'} + obj_g$ **then**
            $a_{i'}^*(i', j') \leftarrow 0.5$
            $obj_f \leftarrow obj_f + (0.5)\nu_f(i') f_{i'} f_{j'}$
            $obj_g \leftarrow obj_g + (0.5)\nu_g(i') g_{i'} g_{j'}$
        else $(0.5)\nu_f(i') f_{i'} f_{j'} + obj_f \leq (0.5)\nu_g(i') g_{i'} g_{j'} + obj_g$
            $a_{i'}^*(i', j') \leftarrow \frac{obj_f - obj_g}{(0.5)\nu_f(i') f_{i'} f_{j'} - (0.5)\nu_g(i') g_{i'} g_{j'}}$
            Stop at the current solution.
        end if
    end if
    Exclude (i', j') from Γ
end while
Stop at the current solution.

In initialization, Algorithm 1 fixes a^* for profile outcomes in which both ν_f and ν_g values of the highest bid reported are nonnegative and leads to an allocation rule that allocates the good to the highest bidders with equal probability. All other allocation variables take initial value 0. The algorithm calculates right hand side values of (3.10) with the initial a^* as obj_f and obj_g. If obj_f is lower than or equal to obj_g then the algorithm stops at the current solution. Also, the algorithm determines $\Gamma(t)$ values for t profile outcomes such that ν_f is negative but ν_g takes a value greater than or equal to 0 at the highest bid reported. If there is no such t profile, the algorithm again stops at the current solution. Otherwise, from 2, Algorithm 1 checks whether the objective value z can be improved. Starting from minimum $\Gamma(t)$ value over t profile outcomes as described before, the algorithm changes a^* in such a way that the highest bid in t wins the object and continues with a profile giving the next minimum $\Gamma(t)$ value until obj_f is equal to obj_g or all allocation variables are set to their upperbound 1 forall t profiles.

Theorem 6. *If neither (3.16) nor (3.17) hold, the Algorithm 1 gives an optimal solution when $\nu_f(i)f_i$ and $\nu_g(i)g_i$ are non-decreasing in $i \in T$ and hazard function is monotone.*

Proof. Assume that a^* from the algorithm violates monotonicity of interim allocations. Then $\exists i'$ such that at least one of $\sum_{j=1}^m a^*_{i'-1}(i'-1,j)f_j > \sum_{j=1}^m a^*_{i'}(i',j)f_j$ or $\sum_{j=1}^m a^*_{i'-1}(i'-1,j)g_j > \sum_{j=1}^m a^*_{i'}(i',j)g_j$ holds. Note that f_j and g_j are positive $\forall j \in T$. Once we prove that $a^*_{i'}(i',j) \geq a^*_{i'-1}(i'-1,j)\ \forall j \in T$, this creates a contradiction.

For arbitrary $j' \in T$, consider $\Gamma(i',j')$ and $\Gamma(i'-1,j')$. By assumption, $v_g(i')g_{i'}g_{j'} \geq v_g(i'-1)g_{i'-1}g_{j'} \geq 0$ and $0 \geq v_f(i')f_{i'}f_{j'} \geq v_f(i'-1)f_{i'-1}f_{j'}$. Therefore, we should have $\Gamma(i',j') \leq \Gamma(i'-1,j')$. Hence, the algorithm increases $a^*_{i'}(i',j')$ before $a^*_{i'-1}(i'-1,j')$.

If $i' \neq j'$,

Case 1.1: $a^*_{i'}(i',j') = 1$. Then, $a^*_{i'}(i',j') > a^*_{i'-1}(i'-1,j')$.

Case 1.2: $a^*_{i'}(i',j') = \frac{obj_f - obj_g}{\nu_f(i')f_{i'}f_{j'} - \nu_g(i')g_{i'}g_{j'}} \geq 0$. Then, the algorithm stops so that $a^*_{i'-1}(i'-1,j') = 0$.

Else if $i' = j'$,
Since $i' - 1 < j'$, the algorithm sets $a^*_{i'-1}(i'-1, j') = 0$.

This proves that a^* yields monotonic interim allocations.

The algorithm considers $a_i(i, j)$ values only if $i > j$ and always assigns values between 1 and 0. Therefore, a^* is feasible.

Now assume that $\exists a' \neq a^*$ such that it is feasible and gives $z' > z^*$. Lets consider the constraint on z.

$$z^* \leq \mu^*_f + obj_f \qquad \forall f \in P$$

where

$$\sum_{j \in T} \sum_{i=1}^{x_g - 1} \nu_f(i) a^*_i(i, j) \pi_f(i, j) = 0 \qquad \forall f \in P$$

$$\sum_{j \in T} \sum_{i=x_g}^{x_f - 1} \nu_f(i) a^*_i(i, j) \pi_f(i, j) = \mu^*_f \qquad \forall f \in P$$

$$\sum_{j \in T} \sum_{i=x_f}^{m} \nu_f(i) a^*_i(i, j) \pi_f(i, j) = obj_f \qquad \forall f \in P$$

The point a^* follows the structure in Theorem 3 for profiles where highest type is greater than or equal to x_f or both reported types are less than x_g. Therefore, it is obvious that a^* and a' are equal for these profile outcomes.

Let's first assume a^* leads to $\mu^*_f + obj_f = \mu^*_g + obj_g$. Note that this is a stop condition for the algorithm. In this case, if $z' > z^*$, we have the folowing:

$$\mu'_f + obj_f > \mu^*_f + obj_f$$
$$\mu'_g + obj_g > \mu^*_g + obj_g$$

Then $\mu'_f > \mu^*_f$ and $\mu'_g > \mu^*_g$ should be satisfied so that we have:

$$\mu^*_f = \sum_{j \in T} \sum_{i=x_g}^{x_f - 1} \nu_f(i) a^*_i(i, j) \pi_f(i, j) < \sum_{j \in T} \sum_{i=x_g}^{x_f - 1} \nu_f(i) a'_i(i, j) \pi_f(i, j) = \mu'_f$$

$$\mu^*_g = \sum_{j \in T} \sum_{i=x_g}^{x_f - 1} \nu_g(i) a^*_i(i, j) \pi_g(i, j) < \sum_{j \in T} \sum_{i=x_g}^{x_f - 1} \nu_g(i) a'_i(i, j) \pi_g(i, j) = \mu'_g$$

However, $\nu_f(i) < 0$ and $\nu_g(i) \geq 0$ $\forall i \in T$ such that $x_g \leq i < x_f$. This creates a contradiction to $\mu'_f > \mu^*_f$ and $\mu'_g > \mu^*_g$.

Now assume $\mu^*_f + obj_f \neq \mu^*_g + obj_g$. If $\mu^*_f + obj_f < \mu^*_g + obj_g$, obtained solution a^* is optimal by Theorem 4. Otherwise, $\mu^*_f + obj_f > \mu^*_g + obj_g$. To let $z' > z^*$, we should have

$$\mu'_g + obj_g > \mu^*_g + obj_g$$

This requires $\mu'_g > \mu^*_g$. Note that Γ is empty in this case. Hence, $a^*_i(i,j)$'s $\forall (i,j) \in T^2$ such that $x_g \leq i < x_f$ and $i \geq j$ are at their upper bound. One can increase $a^*_j(i,j)$ values. However, this increase leads to same amount of decrease in corresponding $a^*_i(i,j)$'s which have a higher opportunity cost. This creates a contradiction to existence of an optimal a' and completes the proof. □

Theorem 7. *For a given allocation rule a^*,*

$$p^*_i(i,j) = i a^*_i(i,j) - \sum_{k<i} a^*_k(k,j) \qquad \forall (i,j) \in T^2 \tag{3.18}$$

is a corresponding payment rule.

Proof. Recall in section 3.2.1, we set expected payments to their tightest upper-bounds:

$$P_f(i) = iA_f(i) - \sum_{j=0}^{i-1} A_f(j) \qquad \forall i \in T, \forall f \in P \tag{3.19}$$

Also, by definition, we have:

$$P_f(i) = \sum_{j \in T} p_i(i,j) f(i) f(j) \qquad \forall i \in T, \forall f \in P \tag{3.20}$$

$$A_f(i) = \sum_{j \in T} a_i(i,j) f(i) f(j) \qquad \forall i \in T, \forall f \in P \tag{3.21}$$

Now, (3.19) together with (3.20) and (3.21) gives (3.18). □

We do not claim that p^* in Theorem 7 is the unique optimal payment rule. In certain cases, it is likely to have multiple payment rules consistent with allocation rule a^*. However, a^* is the unique optimal allocation rule as it is seen in proof of Theorem 6. We show that there is no such a' providing a higher expected revenue to the seller but it is also clear that any other allocation rule cannot lead to the objective value resulting from a^*.

In the optimal mechanism, under assumptions of Theorem 6, only the highest bidder has a chance to win the object. Also, an agent makes a payment only if he gets the object and this payment does not exceed agent's type. If x_f denotes a threshold in the optimal mechanism, for profile outcomes where the highest bid exceeds or equal to x_f, we observe a mechanism which resembles the Vickrey auction. The highest bidder wins the object and pays an amount between the second highest bid and his own bid. When the highest bid reported is less than x_f but bigger than or equal to x_g, for certain profile outcomes -detected by the algorithm-, the good is allocated to the highest bidder. The winner pays at most what he reported. If reported types are less than x_g, then the seller keeps the object.

On the other hand, if we relax the assumption $\nu_f(i)f_i$ and $\nu_g(i)g_i$ being non-decreasing in $i \in T$, a buyer who didn't report the highest bid may have the object for certain profile outcomes. In this case, the seller makes a payment to the highest bidder.

In conclusion, we derived an applicable optimal mechanism for robust auction design problem. Our mechanism does not require payments higher than what an agent offered and only the winner makes a payment to the seller, which are reasonable and fair from buyers' perspective. Moreover, the mechanism we proposed is easy to understand and it resembles the well-known Vickrey auction so that the implementation will not lead to much increased complexity.

Chapter 4

Auction Design Problem with Ambiguity Averse Seller and Buyers

4.1 Problem Definition

There is a single ambiguity averse seller with a single good which she wishes to sell to n ambiguity averse buyers (agents). Each agent has a valuation(type) assigned to the good and this is private information to the agent. Agents are assumed to be symmetric and type of each agent is an independent drawn from a finite set T according to a distribution function f. The seller and agents do not know this distribution function f but they have the information that f is a member of a set of distributions P which is common knowledge.

The objective of the problem remains identical to our setting in Chapter 3, and it is to maximize the seller's worst case expected revenue.

To formulate this problem, we again invoke the Revelation Principle and focus on direct mechanisms.

4.2 Formulation

We use the notation given in Chapter 3. The robust auction design problem with ambiguity averse seller and buyers is formulated as follows:

$$
\begin{aligned}
\max_{p,a} \quad & \Big\{ \min_{f \in P} \sum_{i \in T} f(i) \sum_{j \in T} p_i(i,j) f(j) \Big\} && && (4.1)\\
\text{s.t.} \quad & \min_{f \in P} \Big\{ i \sum_{j \in T} a_i(i,j) f(j) - \sum_{j \in T} p_i(i,j) f(j) \Big\} \geq 0 && \forall i \in T && (4.2)\\
& \min_{f \in P} \Big\{ i \sum_{k \in T} a_i(i,k) f(k) - \sum_{k \in T} p_i(i,k) f(k) \Big\} \geq && && (4.3)\\
& \min_{f \in P} \Big\{ i \sum_{k \in T} a_j(j,k) f(k) - \sum_{k \in T} p_j(j,k) f(k) \Big\} && \forall i,j \in T &&\\
& \sum_{i \in T} n_i(t) a_i(t) \leq 1 && \forall t \in T^2 && (4.4)\\
& a_i(i,j) \geq 0 && \forall i,j \in T && (4.5)\\
& p_i(i,j) \geq 0 && \forall i,j \in T. && (4.6)
\end{aligned}
$$

For ease of notation, we give the formulation for the case where there are two agents.

Individual Rationality constraints (4.2) ensure that each agent gains at least zero payoff from participation. Incentive Compability constraints, which force agents to truthfully report their types, are given as in (4.3) because the bidders consider the worst case payoffs due to ambiguity aversion. No more than one unit of good can be allocated by (4.4). Allocations and payments take nonnegative values by (4.5) and (4.6).

This model can be reformulated as the following Mixed Integer Programming (MIP) Problem:

$$
\begin{aligned}
\max \quad & z \\
\text{s.t.} \quad & z \leq \sum_{i \in T} f(i) \sum_{j \in T} p_i(i,j) f(j) && \forall f \in P && (4.7) \\
& i \sum_{j \in T} a_i(i,j) f(j) - \sum_{j \in T} p_i(i,j) f(j) \geq 0 && \forall f \in P, \forall i \in T && (4.8) \\
& D_{ij} \leq i \sum_{k \in T} a_j(j,k) f(k) - \sum_{k \in T} p_j(j,k) f(k) && \forall f \in P, \forall i,j \in T && (4.9) \\
& D_{ij} + M b_f(i,j) \geq && && (4.10) \\
& i \sum_{k \in T} a_j(j,k) f(k) - \sum_{k \in T} p_j(j,k) f(k) && \forall f \in P, \forall i,j \in T && \\
& \sum_{f \in P} b_f(i,j) \leq |P| - 1 && \forall i,j \in T && (4.11) \\
& i \sum_{k \in T} a_i(i,k) f(k) - \sum_{k \in T} p_i(i,k) f(k) \geq D_{ij} && \forall f \in P, \forall i,j \in T && (4.12) \\
& \sum_{i \in T} n_i(t) a_i(t) \leq 1 && \forall t \in T^2 && \\
& a_i(i,j) \geq 0 && \forall i,j \in T && \\
& p_i(i,j) \geq 0 && \forall i,j \in T && \\
& b_f(i,j) \in 0,1 && \forall f \in P, \forall i,j \in T &&
\end{aligned}
$$

where M is a sufficiently large number.

Since we try to maximize z, it takes the value of minimum expected revenue over $f \in P$ by (4.7). Also, note that dummy variables D_{ij} take the value of minimum expected payoff over $f \in P$ if an agent with true valuation i reports j by (4.9), (4.10) and (4.11).

The high dimension of the problem is an important factor that makes this problem hard to solve. The number of distributions in P leads to a large increase in the number of constraints. Next, we introduce an algorithm for this problem which usually enables to achieve optimal solution with a reduced version of the formulation.

4.3 A Hybrid Algorithm

Without loss of generality, assume $P = \{f_0, f_1, ..., f_m\}$. Let P' be a subset of P and MIP(P') is a reformulation of MIP in which set P is replaced by P'. In other words, we take a subset of distributions in P and eliminate constraints and variables corresponding to remaining distributions.

According to our computational study, Algorithm 2 (almost) always gives an exact solution to MIP formulation.

The algorithm starts by solving MIP only over one distribution function, f_0. Using the optimal solution obtained, rows $5-9$ check whether the constraint type (4.7) is satisfied by remaining distribution functions in P and determine the most violated one. The distribution function which causes the most violated constraint is added to P'. Violation in Individual Rationality constraints (4.8) is detected in $10-15$. Again, detected distribution function is added to P' if it is not already in it. Note that it is possible to observe identical distribution functions from $5-9$ and $10-15$.

The algorithm doesn't consider constraints (4.9) to (4.11) in MIP. The reason is that it also updates right hand side of constraint (4.11) according to P'. This corresponds to the fact that D_{ij}'s now take the value of minimum expected payoff over $f \in P'$ instead of $f \in P$ if an agent with true valuation i reports j. This causes a restriction rather than a relaxation.

From row 16 to row 25, the algorithm detects violation in constraint (4.12). However, as the right hand side, observed D_{ij}'s are not considered for previously explained reasoning. The algorithm calculates the right hand side of each constraint as $IC^r_{min}(i,j)$, minimum expected payoff with observed optimal solution values over $f \in P$ if an agent with true valuation i reports j. Using these right hand side values, the most violated constraint is determined and corresponding distribution function is included in P'.

If at least one distribution function is added to P', the algorithm repeats the

Algorithm 2

Initialize:

$P' = \{f_0\}$

while $exit = false$ **do**

Set $P_{initial} = P'$

Solve $MIP(P')$: power * denotes optimal solution

Set $z_{min} = \min_{f \in P} \sum_{i \in T} f(i) \sum_{j \in T} p^*(i,j) f(j)$

Determine $\dot{f} \in P$ st. $z_{min} = \sum_{i \in T} \dot{f}(i) \sum_{j \in T} p^*(i,j) \dot{f}(j)$

if $z_{min} < z^*$ **then**

Update $P' = P' \cup \dot{f}$

end if

Set $IR_{min} = \min_{f \in P, i \in T} i \sum_{j \in T} a^*(i,j) f(j) - \sum_{j \in T} p^*(i,j) f(j)$

Determine $\ddot{f} \in P$ st.

$IR_{min} = \min_{i \in T} i \sum_{j \in T} a^*(i,j) \ddot{f}(j) - \sum_{j \in T} p^*(i,j) \ddot{f}(j)$

if $\ddot{f} \notin P'$ and $IR_{min} < 0$ **then**

Update $P' = P' \cup \ddot{f}$

end if

Set

$IC^r_{min}(i,j) = \min_{f \in P} i \sum_{k \in T} a^*(j,k) f(k) - \sum_{k \in T} p^*(j,k) f(k) \ \forall (i,j) \in T^2$

$IC^l_{min}(i) = \min_{f \in P} i \sum_{k \in T} a^*(i,k) f(k) - \sum_{k \in T} p^*(i,k) f(k) \ \forall i \in T$

$IC_{min} = \min_{i \in T, j \in T} IC^l_{min}(i) - IC^r_{min}(i,j)$

Determine

$\bar{i} \in T$ st. $IC_{min} = \min_{j \in T} IC^l_{min}(\bar{i}) - IC^r_{min}(\bar{i}, j)$

$\bar{f} \in P$ st. $IC^l_{min}(\bar{i}) = \bar{i} \sum_{k \in T} a^*(\bar{i},k) \bar{f}(k) - \sum_{k \in T} p^*(\bar{i},k) \bar{f}(k)$

if $\bar{f} \notin P'$ and $IC_{min} < 0$ **then**

Update $P' = P' \cup \bar{f}$

end if

if $P' = P_{initial}$ and $\min_{i \in T, j \in T} IC^r_{min}(i,j) - D^*_{ij} \geq 0$ **then**

Set $exit = true$

else $P' = P_{initial}$ and $\min_{i \in T, j \in T} IC^r_{min}(i,j) - D^*_{ij} < 0$

Set $D_{min}(i,j) = \min_{i \in T, j \in T} IC^r_{min}(i,j) - D^*_{ij} = IC^r_{min}(\bar{i}, \bar{j}) - D^*_{\bar{i}\bar{j}}$

Determine $\tilde{f} \in P$ st.

$IC^r_{min}(\bar{i}, \bar{j}) = \bar{i} \sum_{k \in T} a^*(\bar{j},k) \tilde{f}(k) - \sum_{k \in T} p^*(\bar{j},k) \tilde{f}(k)$

Update $P' = P' \cup \tilde{f}$

end if

end while

Stop at current solution

process starting from row 3. If P' remains the same, it is concluded that observed solution is feasible to original formulation. Since the algorithm also restricts the problem, it may not be optimal. Therefore, this restriction is questioned by looking at the difference between D^*_{ij} and $IC^r_{min}(i,j)$ for all $(i,j) \in T^2$. The distribution function causing the highest difference is added to P' and the process is repeated from 3 until no restriction or violation is detected.

In each step, the algorithm gives a bound to the problem but it is somehow hard to determine if it is a lower or an upper bound because some constraints of MIP formulation are relaxed while some are restricted. However, under certain conditions, we can tell more about the bound observed. If P' remains the same until row 25, previously observed solution is feasible to MIP. Therefore, it is a lower bound for the problem.

The algorithm can be adjusted to obtain an upper bound for MIP formulation. If $|P|-1$ in constraint (4.11) is not updated depending on P' and remains the same throughout the algorithm, then we observe an upper bound in each step. It is reasonable to expect that the upper bound from adjusted algorithm will be close to the upper bound from LP Relaxation of the problem because all binary variables introduced to the MIP(P') will take value 1 until $P' = P$. In the LP Relaxation, binary variables take fractional values when relaxed, and big M still will be large enough to ignore Incentive Compability constraints.

4.3.1 A Numerical Example

We give the following numerical example to clarify how the algorithm works. Note that there are two buyers in this example as well as in all instances we solved in Computational Results.

	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
1	$0,12$	$0,112$	$0,088$	$0,116$	$0,095$	$0,088$	$0,078$	$0,106$	$0,071$	$0,119$
2	$0,18$	$0,176$	$0,144$	$0,165$	$0,169$	$0,158$	$0,164$	$0,144$	$0,157$	$0,13$
3	$0,2$	$0,241$	$0,245$	$0,234$	$0,225$	$0,247$	$0,244$	$0,223$	$0,213$	$0,231$
4	$0,23$	$0,272$	$0,261$	$0,25$	$0,242$	$0,262$	$0,245$	$0,264$	$0,27$	$0,236$
5	$0,27$	$0,197$	$0,26$	$0,232$	$0,266$	$0,242$	$0,267$	$0,26$	$0,286$	$0,281$

Table 4.1: Input of Numerical Example

In Table 4.1, each column represents a distinct distribution function in set P and rows are valuations in T. (eg., according to distribution function f_0, 0.12 is the probability that an agent's true valuation is 1)

$$\text{MIP Formulation} \begin{cases} \text{objective value } = 1.1678898913027 \\ \text{solution time } = 8.594 \text{ seconds} \end{cases}$$

Algorithm

$P' = \{f_0\}$

$4:$ **solve MIP**(P') objective value $= 1.1925000000000$

solution time $= 0.016$ seconds

5-9: (4.7) is violated at f_8

10-15:(4.8) is violated at f_9

16-25:(4.12) is violated at f_9

$P' = \{f_0, f_8, f_9\}$

$4:$ **solve MIP**(P') objective value $= 1.2155299999999$

solution time $= 0.125$ seconds

5-9:(4.7) is violated at f_1

$P' = \{f_0, f_8, f_9, f_1\}$

$4:$ **solve MIP**(P') objective value $= 1.1695663606954$

solution time $= 0.297$ seconds

5-9:(4.7) is violated at f_6

16-25:(4.12) is violated at f_6

$P' = \{f_0, f_8, f_9, f_1, f_6\}$

$4:$ **solve MIP**(P') objective value $= 1.1678898913027$

solution time $= 0.578$ seconds

The MIP Formulation is solved using CPLEX Studio 12.6.1 and the solution time is 8.594 seconds while the algorithm solves the same problem in 1.016 seconds in total.

4.3.2 Computational Results

In Table 4.2 and Table 4.3, each row corresponds to an instance result. Solution times are given in seconds. $|P|$ is the total number of distributions in set P and $|T|$ is the number of valuations. Recursion column shows how many times the algorithm repeated itself to obtain the solution. $|P'|$ represents the final number of distributions in set P'. P sets belonging to instances grouped in double lines are randomly generated within an $(f_0 \pm \epsilon)$ interval from the same given f_0 and ϵ. While randomizing input data, we ensure that distribution values do not take negative values. Input data can be seen in Tables A.1 and A.2.

In almost all rows, we give the optimal objective value and total solution time for MIP. However, for the values written in italic, as in row 14, we set a time limit and observed the best integer solution within the allotted time.

Note that we used NetBeans IDE 8.0.1 and CPLEX Studio 12.6.1 for solving all instances.

			MIP Formulation		Algorithm			
	$\|P\|$	$\|T\|$	Objective Value	Solution Time	Objective Value	Solution Time	Recursion	$\|P'\|$
1	20	5	1,293066	17,914	1,293066	10,583	7	12
2	20	5	1,290019	23,493	1,290019	5,304	6	10
3	20	5	1,281526	34,728	1,281526	9,345	6	10
4	20	5	1,22677	22,275	1,22677	15,883	8	12
5	20	5	1,223931	25,897	1,223931	28,167	9	13
6	20	5	1,222235	51,394	1,222235	2,559	5	7
7	20	5	1,346207	8,332	1,346207	4,289	5	10
8	20	5	1,304475	52,233	1,304475	9,357	7	11
9	20	5	1,293792	9,43	1,293792	7,221	6	10
10	20	6	1,714115	884,722	1,714115	970,904	10	18
11	20	6	1,75066	919,362	1,75066	112,23	8	15
12	20	6	1,724949	1535,607	1,724949	788,868	11	15
13	20	6	1,637257	6070,151	1,637257	596,016	7	11
14	20	6	*1,672065*	*900,02*	1,672065	142,404	6	11
15	20	6	*1,661494*	*900,02*	1,661494	255,527	7	11
16	20	6	1,076266	375,062	1,076266	318,536	11	13
17	20	6	1,082449	518,958	1,082449	462,247	13	16
18	20	6	1,089408	408,33	1,089408	164,907	10	12
19	20	6	*1,124627*	*900,03*	*1,124869*	*663,018*	8	12
20	20	6	*1,10295*	*900,02*	1,10295	102,121	6	10
21	20	6	*1,116765*	*900,02*	1,116765	453,76	9	13

Table 4.2: Numerical Results 1

			MIP Formulation		Algorithm			
	$\lvert P\rvert$	$\lvert T\rvert$	Objective Value	Solution Time	Objective Value	Solution Time	Recursion	$\lvert P'\rvert$
22	20	6	1,064647	263,756	1,064647	149,095	8	13
23	20	6	1,078787	67,818	1,078787	34,016	7	11
24	20	6	1,073474	156,101	1,073538	81,204	8	13
25	15	6	1,092282	835,478	1,092282	28,419	6	9
26	15	6	1,105905	196,086	1,105905	101,928	6	10
27	15	6	1,092442	59,869	1,092442	22,295	5	8
28	15	6	1,103278	381,92	1,103278	25,897	6	8
29	15	6	1,114296	332,558	1,114296	170,605	8	11
30	15	6	1,111806	238,398	1,111806	335,704	8	11
31	15	6	1,084282	78,431	1,084282	51,805	7	11
32	15	6	1,041195	142,567	1,041202	83,884	9	11
33	15	6	1,050686	157,032	1,050686	78,218	7	9
34	10	7	0,765615	95,12	0,765615	10,727	3	5
35	10	7	0,746313	70,392	0,746313	141,371	5	6
36	10	7	0,777228	14263,43	0,777228	120,816	4	6
37	10	7	1,052553	33,193	1,052553	75,399	6	9
38	10	7	1,070767	960,996	1,070767	417,991	6	8
39	10	7	1,085262	227,791	1,085262	74,282	6	8
40	10	7	1,027729	156,642	1,027727	92,235	5	8
41	10	7	1,065472	209,731	1,065472	34,3	6	6
42	10	7	1,04994	226,327	1,04994	223,999	6	8

Table 4.3: Numerical Results 2

From Table 4.2 and Table 4.3, we see that the hybrid algorithm for auction design problem leads to time efficiency and obtains optimal solution for almost all instances. Note that there are fractional differences between objective values from MIP formulation and the algorithm in row 24, 32 and 40. However, this observation does not create a counterexample to the claim that the algorithm always gives the optimal solution because this deviation in numerical values may occur due to tolerances in Java Programming Language.

Recursion and $|P'|$ have a marked effect on the improvement that our algorithm brings. Consider instances 4 to 6. For solving row 6, the algorithm repeats itself 5 times and the final number of distributions included in P' is 7, when compared to instances 4 and 5, very low. Hence, as expected, the reduction in solution time by the algorithm is noticeably higher than rows 4 and 5 both in percentage and net amount.

Total solution times seem to depend on all randomized distributions in P rather than only given f_0 and ϵ. For example, although P sets corresponding to instances 34 to 36 are randomized in a similar fashion, 36 has a huge solution time compared to others. This difference is reduced for our algorithm solution time even though recursion and $|P'|$ values of 36 are not the lowest in this sample. This tells us how many times the algorithm repeated itself, and the final number of distributions are not the only elements determining the time efficiency caused by the algorithm.

From this computational study, we cannot make a precise conclusion on the class of problems for which the algorithm will be efficient. Even though the input is randomized in a similar fashion, we observe significant differences in both MIP solution time and the algorithm solution time. However, it can be concluded that the algorithm effectively reduces the solution time in most cases.

Chapter 5

Conclusion

In this thesis, we focused on the auction design problem with discrete valuations for a single good when buyers' valuation distribution comes from a set of distributions P rather than being unique. Recall that we assumed both the seller and the buyers are risk neutral.

In Chapter 3, we gave a formulation for robust auction design problem with an ambiguity averse seller and n ambiguity neutral buyers. Then, we reformulated the problem with the help of standard results from linear programming. Our final model was linear. We developed an algorithm which gives the optimal solution for the case where there are two buyers and and P consists two discrete distributions under certain assumptions. In the optimal mechanism, the highest bidders win the object with equal probability until the highest bid reported falls under a threshold. Only the winner makes a payment and he pays an amount between his own bid and second highest bid. Under the threshold, there may be allocation to the highest bid for some profile outcomes and these are determined by the algorithm.

Although there have been studies in the literature underlying few properties of the optimal mechanism to robust auction design problem [12], a specific mechanism was never driven. The optimal mechanism we derived is both detailed and applicable. It is easy to understand because it resembles the well-known Vickrey auction and it does not require payments which exceed the buyer's offer. Also, only the winner makes a payment, reasonable and fair from buyers' perspective. Hence, the implementation of our study will not lead to much increased complexity.

In Chapter 4, we analyzed the same problem when the buyers are also ambiguity averse. This problem is known to be very complex and consequently the literature is very limited [12]. We formulate the problem as a mixed integer programming problem and to the best of our knowledge our formulation is novel. Then, we propose an algorithm which enables to solve the problem in a reduced time. Our computational results show that the algorithm leads to time efficiency and achieves the optimal solution for most of the instances we solved.

There are several research directions arising from our study. In Chapter 3, while deriving an optimal mechanism, we assumed that there are two distinct distributions in set P. Although, we proved our results for this environment, from our emprical results, we observe that the optimal structure seems to be preserved for the general case. Perhaps, under certain assumptions, it might be possible to extend the set P. For example, necessary assumptions for including all convex combinations of these two distinct distributions can be determined. Morever, the effect of additional constraints such as budget constraints on the optimal mechanism can be considered as a future work.

The optimal mechanism we achieved in Chapter 3, differs from the results of [12]. Our results can be extended to the continuos distribution case to analyze what this difference is caused by in the future.

Throughout Chapter 3 and Chapter 4, we invoke the Revelation Principle holds and formulate problems with only direct mechanisms. One can also consider indirect mechanisms in an environment where ambiguity is introduced. In this case, the optimal strategy of the buyers should be analyzed.

For the robust auction design problem with ambiguity averse seller and buyers, the structure of an optimal mechanism can be examined.

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Appendix A

Input Data

	ϵ	$f_0(1)$	$f_0(2)$	$f_0(3)$	$f_0(4)$	$f_0(5)$	$f_0(6)$	$f_0(7)$
1	0,08	0,12	0,18	0,2	0,23	0,27		
2	0,08	0,12	0,18	0,2	0,23	0,27		
3	0,08	0,12	0,18	0,2	0,23	0,27		
4	0,04	0,12	0,18	0,2	0,23	0,27		
5	0,04	0,12	0,18	0,2	0,23	0,27		
6	0,04	0,12	0,18	0,2	0,23	0,27		
7	0,1	0,12	0,18	0,2	0,23	0,27		
8	0,1	0,12	0,18	0,2	0,23	0,27		
9	0,1	0,12	0,18	0,2	0,23	0,27		
10	0,08	0,08	0,11	0,14	0,17	0,2	0,3	
11	0,08	0,08	0,11	0,14	0,17	0,2	0,3	
12	0,08	0,08	0,11	0,14	0,17	0,2	0,3	
13	0,04	0,08	0,11	0,14	0,17	0,2	0,3	
14	0,04	0,08	0,11	0,14	0,17	0,2	0,3	
15	0,04	0,08	0,11	0,14	0,17	0,2	0,3	
16	0,08	0,11	0,17	0,29	0,2	0,13	0,1	
17	0,08	0,11	0,17	0,29	0,2	0,13	0,1	
18	0,08	0,11	0,17	0,29	0,2	0,13	0,1	
19	0,04	0,11	0,17	0,29	0,2	0,13	0,1	
20	0,04	0,11	0,17	0,29	0,2	0,13	0,1	
21	0,04	0,11	0,17	0,29	0,2	0,13	0,1	

Table A.1: Input Data 1

	ϵ	$f_0(1)$	$f_0(2)$	$f_0(3)$	$f_0(4)$	$f_0(5)$	$f_0(6)$	$f_0(7)$
22	0,1	0,11	0,17	0,29	0,2	0,13	0,1	
23	0,1	0,11	0,17	0,29	0,2	0,13	0,1	
24	0,1	0,11	0,17	0,29	0,2	0,13	0,1	
25	0,08	0,11	0,17	0,29	0,2	0,13	0,1	
26	0,08	0,11	0,17	0,29	0,2	0,13	0,1	
27	0,08	0,11	0,17	0,29	0,2	0,13	0,1	
28	0,04	0,11	0,17	0,29	0,2	0,13	0,1	
29	0,04	0,11	0,17	0,29	0,2	0,13	0,1	
30	0,04	0,11	0,17	0,29	0,2	0,13	0,1	
31	0,1	0,11	0,17	0,29	0,2	0,13	0,1	
32	0,1	0,11	0,17	0,29	0,2	0,13	0,1	
33	0,1	0,11	0,17	0,29	0,2	0,13	0,1	
34	0,08	0,28	0,2	0,17	0,1	0,1	0,08	0,07
35	0,08	0,28	0,2	0,17	0,1	0,1	0,08	0,07
36	0,08	0,28	0,2	0,17	0,1	0,1	0,08	0,07
37	0,08	0,13	0,18	0,25	0,15	0,12	0,09	0,08
38	0,08	0,13	0,18	0,25	0,15	0,12	0,09	0,08
39	0,08	0,13	0,18	0,25	0,15	0,12	0,09	0,08
40	0,1	0,13	0,18	0,25	0,15	0,12	0,09	0,08
41	0,1	0,13	0,18	0,25	0,15	0,12	0,09	0,08
42	0,1	0,13	0,18	0,25	0,15	0,12	0,09	0,08

Table A.2: Input Data 2