ISOMORPHISM THEOREMS OF LINEAR GROUPS

by

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ABSTRACT

ISOMORPHISM THEOREMS OF LINEAR GROUPS

The goal in this thesis is the isomorphism theory of linear groups over fields as illustrated by the theorem

$$\mathrm{H}(V_1) \cong \mathrm{H}(V_2) \Leftrightarrow \dim V_1 = \dim V_2 \text{ and } F_1 \cong F_2$$

where F_1 and F_2 are the underlying fields of finite dimensional vector spaces V_1 and V_2 respectively and H is any subset of the (projective) collinear transformations, (projective) general linear groups or (projective) special linear groups for dimension ≥ 3. The theory that follows is typical of much of the research between the years 50's and 60's on the isomorphisms of the classical groups over rings. The thesis will start from the basic facts of calculus of residues and transvections. Then, in particular, the fundamental theorem of projective geometry will be proved and whatever is needed from projective geometry will be developed. Via reorganizing the literature on the isomorphisms of the classical groups, it will be possible to extend the known theory from groups of linear transformations to groups of collinear transformations, and also to improve the isomorphism theory from dimension ≥ 5 to dimension ≥ 3.

ÖZET

LİNEER GRUPLARIN İZOMORFİZMA TEOREMLERİ

Bu tezin amacı aşağıdaki teoremde de gösterildiği gibi cisimler üzerindeki lineer grupların izomorfizma teorisidir:

$$\mathrm{H}(V_1) \cong \mathrm{H}(V_2) \Leftrightarrow \dim V_1 = \dim V_2 \text{ and } F_1 \cong F_2$$

Burada F_1 ve F_2, sırasıyla sonlu boyutlu V_1 ve V_2 vektör uzaylarının üzerinde bulundukları cisimleri, H ise (projektif) kolineer transformasyonların, (projektif) genel lineer grupların veya (projektif) özel lineer grupların boyutu 3'e eşit veya daha büyük olan alt uzayını göstermektedir. Sunulacak olan teori 50'li ve 60'lı yıllarda halkalar üzerindeki klasik grupların izomorfizmaları üzerine yapılan çalışmalarla benzerlikler göstermektedir. Bu tez çalışması reziü hesabı ve transveksiyonların temel özellikleriyle başlayacaktır. Ardından özelde Projektif Geometri'nin Temel Teoremi ispatlanacak ve projektif geometriden ihtiyacımız olan neyse geliştirilecektir. Klasik grupların izomorfizmaları üzerine var olan materyalin yeniden düzenlenmesiyle lineer transformasyon gruplarındaki bilinen teorinin kolineer transformasyon gruplarına genişletilmesi ve yine izomorfizma teorisinin, boyutun 5'e eşit veya büyük olması koşulundan, 3'e eşit veya büyük olma koşuluna geliştirilmesi mümkün olmaktadır.

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LIST OF SYMBOLS / ABBREVIATIONS

F	Field
$\dot{F}$	Set of nonzero elements of F
$\mathbf{F}_q$	Finite field of q elements
card F	Cardinality of a field F
char F	Characteristic of a field F
V	n-dimensional vector space over a field F with $1 \leq n \leq \infty$
V_1	n_1-dimensional vector space over a field F_1 with $1 \leq n \leq \infty$
V'	Dual space of V
$\langle a \rangle$	The group generated by a
$C_A(X)$	The centralizer in A of a nonempty subset X
$[p,q]$	The commutator $pqp^{-1}q^{-1}$ of the elements p, q of X
DX	The commutator subgroup of X
S^0	The annihilator in V' of a subset S of V
T^0	The annihilator in V of a subset T of V'
$P(V)$	Projective space of V (the set of all subspaces of V)
$P^n(V)$	The set of all n-dimensional subspaces of V
$\Gamma\mathrm{L}_n(V)$	The set of collinear transformations of V
$\mathrm{GL}_n(V)$	General linear group of V (the set of all invertible transformation of V)
$\mathrm{SL}_n(V)$	The set of all invertible transformations with determinants are 1
$\mathrm{PGL}_n(V)$	The projective general linear group (subgroup of the group of projectivities of V)
$\mathrm{PSL}_n(V)$	The projective special linear group (subgroup of the group of projectivities of V)
$\mathrm{RL}_n(V)$	The set of radiations of V
σ, Σ, τ	Linear transformations
R	The residual space of a linear transformation

P	The fixed space of a linear transformation
res σ	The dimension of residual space of a linear transformation σ
Λ, Ψ	Group isomorphisms

1. INTRODUCTION

One of the most important problems in group theory is *the description of isomorphisms (automorphisms)* for a given class of groups. The corresponding problem for the class of linear groups is even of greater importance due to the prominent position which linear groups occupy in mathematics.

The theory of isomorphisms of linear groups was initiated in a paper [12] of 1928 by Schreier and van der Waerden in which they described the isomorphisms of finite-dimensional projective special linear groups over fields. It turned out that the isomorphism type of the projective linear group PSL(V) of a vector space V of finite dimension ≥ 3 over a field F was fully determined by *the dimension of V* and *the* (*isomorphism type of*) *underlying field F*. In other words, if V_1 and V_2 are vectors spaces of dimension ≥ 3 over fields $F_1$and F_2 respectively, then

$$\mathrm{PSL}(V_1) \cong \mathrm{PSL}(V_2) \Leftrightarrow \dim V_1 = \dim V_2 \text{ and } F_1 \cong F_2.$$

Suppose that V is a finite-dimensional vector space of dimension ≥ 3 over a field F. The main (and fundamental!) idea of the quoted paper [12] is that

the geometry of V can be reconstructed in PSL(V) *by means of group theory*.

Namely, Schreier and van der Waerden suggested to 'code' lines and hyperplanes of V by suitable pairs of transvections. Recall that a linear transformation $\tau \in \mathrm{GL}(V)$ is a *transvection* if there is a linear functional $\delta\colon V \to F$ and a vector $a \in V$ such that

$$\tau x = x + \delta(x)\, a$$

for all $x \in V$. Recall also that the projective image of τ in the group PGL(V) is called a *projective transvection*.

Thus $\bar{\tau}$ (the projective image of τ) determines a line of V (namely $\langle a \rangle$) and a hyperplane of V (namely the kernel of δ) Schreier and van der Waerden show then that the product $\tau_1\tau_2$ of (projective) transvections is a (projective) transvection if and only if τ_1 and τ_2 have a mutual line or a mutual hyperplane. Accordingly, if one has a group–theoretic description of projective transvections in PSL(V), —and Schreier and van der Waerden showed that such a description did exist— then any isomorphism Λ: PSL(V_1) → PSL(V_2) takes transvections to transvections and thereby *induces* a map Λ^* from the projective space P(V_1) over V_1 onto the projective space P(V_2) over V_2 or onto the projective space P(V_2'), where V_2' is the dual space of V_2. The first case is realized when Λ takes pairs of projective transvections with mutual lines to pairs of projective transvections with mutual lines, and the second one when pairs with mutual lines go to the pairs with mutual hyperplanes and vice versa.

Furthermore, it can be proved that Λ^* can be extended to a projectivity from P(V_1) onto P(V_2) or onto P(V_2') and so the projective space P(V_1) is isomorphic to P(V_2) or to P(V_2') (the latter two spaces being isomorphic.) By applying the Fundamental Theorem of Projective Geometry, one shows finally that the dimensions of V_1 and V_2 are the same and the underlying fields F_1 and F_2 are isomorphic.

It was understood in the 20-40s of the last century that the approach by Schreier-van der Waerden could be applied for the study of isomorphisms of the other types of linear groups. However, instead of the use of transvections the authors that followed Schreier and van der Waerden preferred to use *involutions*, that is, the linear transformations of order two. The reason here is obvious: involutions can be immediately described by group-theoretic means in any linear group (by the condition $x^2 = 1$), whereas a group-theoretic characterization of transvections can be rather tricky. An involution σ of the general linear group GL(V) of a vector space V determines (as a tranvections does) a pair of subspaces of V. For instance, in the case when the characteristic of underlying field F is not 2, these subspaces are eigenspaces of σ corresponding to eigenvalues 1 and −1.

Mackey [6] in his study of isomorphism types of the automorphisms linear normed spaces found a group-theoretic condition which was satisfied by a pair of involutions if and only if they had a mutual line or a mutual hyperplane. Later Dieudonné [1] and Rickart [9, 10] adapted Mackey's ideas to the description of isomorphism types of various linear groups over division rings. In 1951 Rickart [11] also started the program of description of isomorphism types *infinite-dimensional* linear groups (which had been not completed till 1977.)

The use of involutions also lead to a significant progress in the study of the isomorphism types of linear groups over principal ideal domains: Hua and Reiner described in [3] isomorphisms of general linear groups of free modules of finite rank over the ring of integers (unimodular groups), Landin and Reiner considered then the linear groups over the ring of Gaussian numbers [4] and then obtained the general description of isomorphisms of linear groups over principal ideal domains [5].

However, methods that were based upon the properties of involutions a given linear group did not work for certain important types of linear groups and/or their subgroups. For instance, O'Meara found `large' linear groups that contained *no* involutions. To overcome this and other difficulties, O'Meara developed in a series of papers the calculus of the residues (see Section 1.2 for details) of linear transformations. He also once again placed the main stress on the properties of transvections of a given linear group. This enabled him to develop a very impressive general theory of isomorphisms of finite-dimensional linear group over large class of rings and to complete the program of description of isomorphisms of some important types of infinite-dimensional linear groups started by Rickart; all these results were published in a paper [8] of 1977.

The aim of the present work is to provide an enlarged version of the 'core' part of the famous work 'Lectures on Linear Groups' [7] by O'Meara in which his main ideas and methods are reproduced in a rather concise, though a very elegant way. One quickly finds that O'Meara's work is mainly addressed to professional mathematicians rather than, for instance, to graduate students. To follow the author of [7] is sometimes hard enough even as we think for a professional mathematician: some proofs are just sketched, it is required of the reader to work out the material *very thoroughly* indeed in order to follow author's steps etc. Thus we make an effort to supply missing details and to rework some of the proofs of results in [7] to make the important work by O'Meara understandable to *senior graduate students*. We believe that the present thesis can be used in teaching of an (advanced) course in linear groups for graduate students. Our work presents a complete proof of the fact that the isomorphism type of a linear group H(V) of a vector space V of dimension ≥ 3 over a field F which is full of (projective) transvections is fully determined by dim$V$ and the isomorphism type of F. In particular, it gives the description of isomorphisms of linear groups of types H = ΓL, PΓL, GL, PGL, SL and PSL and, as a consequence, the following classical theorem:

$$H(V_1) \cong H(V_2) \Leftrightarrow \text{ dim } V_1 = \text{dim } V_2 \quad \text{and} \quad F_1 \cong F_2,$$

where F_1 and F_2 are the underlying fields of V_1 and V_2, respectively provided that the dimension of both V_1 and V_2 is at least three.

1.1. Radiations

For any nonzero α in F define the linear transformation r_α by

$$r_\alpha\, x = \alpha\, x \qquad \forall\, x \in V \tag{1.1}$$

Thus r_α is in $\mathrm{GL}_n(V)$. Any σ in GL_n which has the form $\sigma = r_\alpha$ for some such α will be called *a radiation of V*. The set of radiations of V is a normal subgroup of $\mathrm{GL}_n(V)$ which will be written $\mathrm{RL}_n(V)$. The isomorphism $\mathrm{RL}_n \rightarrow \dot{F}$ is obvious.

1.1.1. *Let σ be any element of* $\mathrm{GL}_n(V)$. *Then σ is in* $\mathrm{RL}_n(V)$ *if and only if σL=L for all lines L in V. In particular*

$$\ker(P|\,\mathrm{GL}_n) = \mathrm{RL}_n\,, \quad \ker(P|\,\mathrm{SL}_n) = \mathrm{SL}_n \cap \mathrm{RL}_n \tag{1.2a}$$

and

$$\mathrm{PGL}_n \cong \mathrm{GL}_n / \mathrm{RL}_n, \quad \mathrm{PSL}_n \cong \mathrm{SL}_n / \mathrm{SL}_n \cap \mathrm{RL} \tag{1.2b}$$

PROOF. Fix z in $\dot{V}$. There is then a β in $\dot{F}$ such that $\sigma z = \beta z$. We have to prove that $\sigma x = \beta x$ for a typical x in $\dot{V}$. By hypothesis, $\sigma x = \alpha x$ for some α in $\dot{F}$. If x is in Fz, then x has the form $x = \lambda z$, so

$$\sigma x = \sigma(\lambda z) = \lambda(\sigma z) = \lambda\beta z = \beta x. \tag{1.3}$$

If x is not in Fz, then

$$\alpha x + \beta z = \sigma(x+z) = \gamma(x+z), \tag{1.4}$$

so $\alpha = \gamma = \beta$ by the independence of x and z. □

1.2. Residues

Consider $\sigma \in \mathrm{GL}_n(V)$. We define the *residual space* $R=R(\sigma)$ by

$$R = (\sigma - 1_V)V = \{ \sigma x - x : x \in V\}, \tag{1.5}$$

the *fixed space* $P = P(\sigma)$ by

$$P = \ker(\sigma - 1_V) = \{ x \in V : \sigma x = x \}, \tag{1.6}$$

and *the residual index* res σ by

$$\operatorname{res} \sigma = \dim R = \operatorname{codim} P. \tag{1.7}$$

The latter equation follows from the theorem on the sum of rank and nullity of a given linear transformation ρ of V:

$$\operatorname{rank} \rho + \operatorname{nullity}(\rho) = \dim(\rho V) + \dim(\ker \rho) = \dim V. \tag{1.8}$$

The subspaces $R(\sigma)$ and $P(\sigma)$ are called the *spaces* of σ. Clearly, both subspaces of σ are σ–invariant:

$$\sigma R(\sigma) = R(\sigma) \text{ and } \sigma P(\sigma) = P(\sigma). \tag{1.9}$$

Let us check, for instance, the first statement. We have

$$\sigma R(\sigma) = \sigma (\sigma - 1_V) V = (\sigma - 1_V) (\sigma V) = (\sigma - 1_V) V, \tag{1.10}$$

since σ is in $\mathrm{GL}_n(V)$ and hence surjective which means that $\sigma V = V$. It is evident that

$$\text{res } \sigma = 0 \iff \sigma = 1_V. \tag{1.11}$$

Note also that the residual space (fixed space) of the inverse transformation σ^{-1} coincides with that one of σ:

$$R(\sigma^{-1}) = R(\sigma) \text{ and } P(\sigma^{-1}) = P(\sigma). \tag{1.12}$$

For instance, for the residual spaces we have

$$R(\sigma^{-1}) = (\sigma^{-1} - 1_V) V = (\sigma^{-1} - 1_V) (\sigma V) = (\sigma - 1_V) V; \tag{1.13}$$

the justification of the latter equation is as follows: for all $x \in V$

$$(\sigma^{-1} - 1_V)(\sigma x) = \sigma^{-1} (\sigma x) - \sigma x = x - \sigma x = -(\sigma x - x) \in (\sigma - 1_V) V. \tag{1.14}$$

Convention:

> whenever a $\sigma \in \mathrm{GL}_n(V)$ is under discussion, the letter R automatically refer to its residual and the letter P to its fixed space.

1.2.1. Let σ_1 and σ_2 be elements of $\mathrm{GL}_n(V)$ and put $\sigma = \sigma_1\sigma_2$. Then

$$R \subseteq R_1 + R_2, \qquad P \supseteq P_1 \cap P_2, \tag{1.15}$$

$$\text{res}\,\sigma_1\sigma_2 \leq \text{res}\,\sigma_1 + \text{res}\,\sigma_2. \tag{1.16}$$

PROOF. We have

$$\sigma_1\sigma_2(x) - x = \{\sigma_1(\sigma_2 x) - \sigma_2 x\} + \{\sigma_2 x - x\}; \tag{1.17}$$

the first element in the curly brackets in the right-hand side belongs to R_1 and the second one to R_2. Hence $R \subseteq R_1 + R_2$, whence

$$\text{res}\,\sigma = \dim R \leq \dim(R_1 + R_2) \leq \dim R_1 + \dim R_2 = \text{res}\,\sigma_1 + \text{res}\,\sigma_2. \tag{1.18}$$

The statement about the fixed spaces is trivial. □

1.2.2. *Let* σ_1 *and* σ_2 *be elements of* $\text{GL}_n(V)$ *and put* $\sigma = \sigma_1\sigma_2$. *Then*

(1) $V = P_1 + P_2 \Rightarrow R = R_1 + R_2,$ (1.19)

(2) $R_1 \cap R_2 = 0 \Rightarrow P = P_1 \cap P_2.$ (1.20)

PROOF. Let us prove the part (1) first.

$$R_1 = (\sigma_1 - 1_V)V = (\sigma_1 - 1_V)\,(P_1 + P_2). \tag{1.21}$$

Since P_1 is the kernel of $\sigma_1 - 1_V$ we get

$$R_1 = (\sigma_1 - 1_V)\,P_2. \tag{1.22}$$

Now $\sigma_2 P_2 = P_2$ and then

$$R_1 = (\sigma_1 - 1_V)\,P_2 = (\sigma_1\sigma_2 - 1_V)\,P_2 \subseteq (\sigma_1\sigma_2 - 1_V)\,V = (\sigma - 1_V)\,V = R. \tag{1.23}$$

So $R_1 \subseteq R$. In other words,

$$R(\sigma_1) \subseteq R(\sigma_1\sigma_2). \tag{1.24}$$

By symmetry our argument can be applied to the transformations σ_2^{-1} and σ_1^{-1} (because $P(\sigma_1^{-1}) = P_1$ and $P(\sigma_2^{-1}) = P_2$) and their product $\sigma_2^{-1}\sigma_1^{-1}$. This
means that

$$R(\sigma_2^{-1}) \subseteq R(\sigma_2^{-1}\,\sigma_1^{-1}) \tag{1.25}$$

or

$$R_2 \subseteq R(\sigma^{-1}) = R. \tag{1.26}$$

Therefore $R_1 + R_2 \subseteq R$. By 1.2.1 we obtain that $R = R_1 + R_2$.

Let us prove (2). Take $x \in P$ and consider the element

$$y = \sigma_1\sigma_2(x) - \sigma_2(x) = \sigma_1(\sigma_2 x) - \sigma_2\, x = (\,\sigma_1 - 1_V)\,\sigma_2\, x. \tag{1.27}$$

On one hand it is an element of R_1. On the other hand since $x \in P$

$$y = \sigma_1\sigma_2(x) - \sigma_2(x) = \sigma(x) - \sigma_2(x) = -\,(\sigma_2 x - x). \tag{1.28}$$

and hence $y \in R_2$. Then $y \in R_1 \cap R_2$ and by the condition $y = 0$. Then (1.28) implies that $P \subseteq P_2$. Similarly, let y' be in $P = P(\sigma_2^{-1}\sigma_1^{-1}) = P(\sigma^{-1})$

$$x' = \sigma_2^{-1}\sigma_1^{-1}(y') - \sigma_1^{-1}(y') = \sigma_2^{-1}(\sigma_1^{-1}y') - \sigma_1^{-1}y' = (\sigma_2^{-1} - 1_V)\,\sigma_1^{-1}y'. \tag{1.29}$$

Then $x' \in R_2$. And again,

$$x' = \sigma_2^{-1}\sigma_1^{-1}(y') - \sigma_1^{-1}(y') = \sigma^{-1}(y') - \sigma_1^{-1}(y') = -\,(\sigma_1^{-1}(y') - y'). \tag{1.30}$$

and hence $x' \in R_1$. Then $x' \in R_1 \cap R_2$ and by the condition $x' = 0$. Then (1.30) implies that $P \subseteq P_1$. And as a conclusion $P \subseteq P_1 \cap P_2$. Applying 1.2.1. once again we have that $P = P_1 \cap P_2$ as we promised. □

1.2.3. *Let* σ *and* Σ *be elements of* $\mathrm{GL}_n(V)$. *Then the residual and fixed spaces of* $\Sigma\sigma\Sigma^{-1}$ *are* ΣR *and* ΣP *respectively. In particular*, $\mathrm{res}(\Sigma\sigma\Sigma^{-1}) = \mathrm{res}\ \sigma$; *and if* $\sigma\Sigma = \Sigma\sigma$ *implies that* $\Sigma R = R$ *and* $\Sigma P = P$.

1.2.4. *Let* σ_1 *and* σ_2 *be elements of* $\mathrm{GL}_n\ (V)$. *Then* $R_1 \subseteq P_2$ *and* $R_2 \subseteq P_1$ *makes* $\sigma_1\sigma_2 = \sigma_2\sigma_1$, *that is*, σ_1 *and* σ_2 *are commuting linear transformations*.

PROOF. First, note that the condition $R_2 \subseteq P_1$ implies that

$$\sigma_1(\sigma_2 x - x) = \sigma_2 x - x \tag{1.31}$$

for every $x \in V$. Then for all $x \in V$

$$\begin{aligned}\sigma_1\sigma_2 x &= \sigma_1(\sigma_2 x - x) + \sigma_1 x = \sigma_2 x - x + \sigma_1 x \\ &= \sigma_1 x - x + \sigma_2 x = \sigma_2(\sigma_1 x - x) + \sigma_2 x \\ &= \sigma_2\sigma_1 x.\end{aligned} \tag{1.32}$$

□

1.2.5. *Let* σ_1 *and* σ_2 *be elements of* $\mathrm{GL}_n(V)$ *with* $\sigma_1\sigma_2 = \sigma_2\sigma_1$. *Then*

$$R_1 \subseteq P_2 \text{ and } R_2 \subseteq P_1, \tag{1.33}$$

provided that either $R_1 \cap R_2 = 0$ *or* $V = P_1 + P_2$.

PROOF. Since σ_1 and σ_2 are commuting we have by 1.2.3 that, for instance,

$$\sigma_1 R_2 = R_2 \text{ and } \sigma_1 P_2 = P_2. \tag{1.34}$$

Suppose first that $R_1 \cap R_2 = 0$. By (1.34)

$$(\sigma_1 - 1_V)R_2 \subseteq R_2; \tag{1.35}$$

evidently, $(\sigma_1 - 1_V)R_2 \subseteq R_1$. Then

$$(\sigma_1 - 1_V)R_2 \subseteq R_2 \cap R_1 = 0. \tag{1.36}$$

This implies that R_2 is in the kernel of $\sigma_1 - 1_V$, that is, in P_1: $R_2 \subseteq P_1$. By symmetry $R_1 \subseteq P_2$.

Now assume that $V = P_1 + P_2$. We have

$$R_1 = (\sigma_1 - 1_V)\, V = (\sigma_1 - 1_V)\, (P_1 + P_2) = (\sigma_1 - 1_V)\, P_2. \tag{1.37}$$

By (1.34) $\sigma_1 P_2 = P_2$ and hence

$$R_1 = (\sigma_1 - 1_V)\, P_2 \subseteq P_2. \tag{1.38}$$

Similarly, $R_2 \subseteq P_1$. □

1.2.6. *Let σ be an element of* $\mathrm{GL}_n(V)$. *Then* $\sigma^2 = 1_V$ *if and only if* $\sigma|_R$, *the restriction of σ on R, is* -1_R.

PROOF.

$$\begin{aligned} \sigma^2 = 1_V &\Leftrightarrow \sigma^2 x = x \qquad \forall\ x \in V, \\ &\Leftrightarrow \sigma(\sigma x - x) = -(\sigma x - x) \qquad \forall\ x \in V, \\ &\Leftrightarrow \sigma y = -y \qquad \forall\ y \in R, \\ &\Leftrightarrow \sigma|_R = -1_R. \end{aligned} \tag{1.39}$$

□

1.2.7. *Suppose that V is finite−dimensional. Then for every $\sigma \in$* $\mathrm{GL}_n(V)$

$$\det \sigma = \det(\sigma|_R). \tag{1.40}$$

PROOF. Assume $\dim V = n$. Let $e_1, \ldots , e_s$ be a base of R. Extend it to a base of V

$$e_1, \ldots , e_s, e_{s+1}, \ldots , e_n. \tag{1.41}$$

Now for every base vector e_k with $k > s$

$$\sigma e_k - e_k \in R \tag{1.42}$$

and hence

$$\sigma e_k = e_k + r_k \tag{1.43}$$

where r_k is a linear combination of vectors $e_1, \dots, e_s$. Then the matrix $[\sigma]$ of σ in the base $e_1, \dots, e_s, e_{s+1}, \dots, e_n$ is as follows

$$\begin{pmatrix} M & * \\ 0 & E \end{pmatrix} \tag{1.44}$$

where M is an $s \times s$ matrix (in fact the matrix of $\sigma|_R$ in the base $e_1, \dots, e_s$ of R), **0** denote a zero block, and E is $(n - s)\times(n - s)$ identity matrix. Therefore

$$\det \sigma = \det[\sigma] = \det M \cdot \det E = \det [\sigma|_R] = \det (\sigma|_R), \tag{1.45}$$

as desired. □

1.2.8. *If* $V = V_1 \oplus V_2$ *and* $\sigma = \sigma_1 \oplus \sigma_2$ *with* $\sigma_1 \in \mathrm{GL}_{n_1}(V_1)$ *and* $\sigma_2 \in \mathrm{GL}_{n_2}(V_2)$. *Then*

$$R = R_1 \oplus R_2 \ \ and \ \ P = P_1 \oplus P_2. \tag{1.46}$$

PROOF. Suppose that $x = x_1 + x_2$, where $x_1 \in V_1$ and $x_2 \in V_2$. Then according to the definition of the map $\sigma = \sigma_1 \oplus \sigma_2$,

$$\sigma(x) = \sigma_1 x_1 + \sigma_2 x_2. \tag{1.47}$$

Therefore

$$\sigma x - x = \sigma_1 x_1 + \sigma_2 x_2 - (x_1 + x_2) = (\sigma_1 x_1 - x_1) + (\sigma_2 x_2 - x_2) \quad (1.48)$$

and hence $R = R_1 \oplus R_2$. □

1.2.9. *Let σ be an element of* $\mathrm{GL}_n(V)$ *and let W be a subspace of V with $R \subseteq W$ or $P \subseteq W$. Then $\sigma W = W$.*

PROOF. For instance, the condition $R \subseteq W$ means that for all $y \in W$

$$\sigma y - y \in W, \quad (1.49)$$

whence $\sigma y \in W$ for all $y \in W$. □

1.3. Transvections

Let σ be an element of $\mathrm{GL}_n(V)$. We say that σ is a *transvection*, if either $\sigma = 1_V$, or

$$\mathrm{res}\,\sigma = 1,\ \det \sigma = 1. \quad (1.50)$$

We say that σ is a *dilation* if

$$\mathrm{res}\,\sigma = 1,\ \det \sigma \neq 1. \quad (1.51)$$

In view of 1.2.3 it is clear that any conjugate $\Sigma\sigma\Sigma^{-1}$ of a transvection (dilation) σ by an element Σ of $\mathrm{GL}_n(V)$ is also a transvection (dilation). Note also the condition res $\sigma = 1$ implies that P is a hyperplane.

1.3.1. *Let $n \geq 2$ and $\sigma \in$* $\mathrm{GL}_n(V)$ *is of residue* 1. *Then*

1) *σ is a transvection if and only if $R \subseteq P$*;

2) *σ is a dilation if and only if $V = R \oplus P$*;

3) *if σ is a transvection, then the* (*only*) *eigenspace of σ is P and the only eigenvalue of σ is* 1;

4) *if σ is a dilation, then the set of all eigenvectors of σ is* $\dot{R} \cup \dot{P}$, *and* 1 *and* det *σ are the eigenvalues of σ.*

PROOF. Since σ is a transvection res$\sigma = 1$. Then suppose that $R = \langle a \rangle$ and that $e_1, \ldots, e_{n-1}$ is a base of P.

1) By 1.2.7 we have that

$$\det \sigma|_R = \det \sigma = 1. \tag{1.52}$$

Now since $\sigma|_R$ is a radiation (because we have $\sigma R = R$ or by the notation of transvections $\sigma a = \tau_{a,\rho}\, a = a + \rho(a)a = a(1+\rho(a))$ and since $1+\rho(a) \in F$ then $\sigma a = \alpha a$, σ is a radiation where $\alpha = 1 + \rho(a)$), we have

$$\sigma a = (\det\sigma|_R)a = a. \tag{1.53}$$

Then $a \in P$, whence $R \subseteq P$. Conversely, if $R \subseteq P$, then

$$\sigma a = a. \tag{1.54}$$

It then follows that $\det \sigma|_R = 1$, and, again by 1.2.7 $\det \sigma = 1$. Since we have res$\sigma = 1$ then σ is a transvection.

The second part 2) is an immediate consequence of 1): as σ is a dilation and P is a hyperplane then $\det \sigma \neq 1$ and $\sigma a = \alpha a$ where $\alpha \neq 1$. Then $a \notin P$ and $R \not\subset P$ as by 1), then $V = R \oplus P$, as required.

3) Extend the above chosen base of P to a base of V by a vector b. Since

$$\sigma b = \beta\, b + p \tag{1.55}$$

for suitable $p \in P$, we get that

$$\det\,[\sigma] = \begin{vmatrix} \beta & 0 & 0 & \dots & 0 \\ * & 1 & 0 & \dots & 0 \\ * & 0 & 1 & \dots & 0 \\ * & \vdots & \vdots & \ddots & 0 \\ * & 0 & \dots & 0 & 1 \end{vmatrix} = \beta = 1. \tag{1.56}$$

Then the matrix $[\sigma]$ of σ in the base $(b, e_1, \ldots, e_{n-1})$ is so–called lower unitriangular. This means that all diagonal elements are 1 and the elements above the main diagonal are 0. It then easily follows that the only eigenvalue of σ is 1 and all eigenvectors of σ are non–zero elements of P.

To prove 4) we consider the matrix of σ in a base $e_1, \ldots, e_{n-1}, a$. The result then follows easily. □

Notation: Consider a vector $a \in V$ and a (non–zero) linear functional $\rho \in V'$. We define the map $\tau_{a,\rho}$, a linear operator of V:

$$\tau_{a,\rho}(x) = x + \rho(x)\mathrm{a}, \qquad \forall\ x \in V. \tag{1.57}$$

The check that $\tau_{a,\rho}$ is linear is straightforward.

Since

$$\tau_{a,\rho}\, x - x = \rho(x)\, a \tag{1.58}$$

we have that $R = R(\tau_{a,x}) \subseteq \langle a \rangle$. Then if $\tau_{a,\rho}$ is in $\mathrm{GL}_n(V)$ its residual space is one–dimensional. Hence $\tau_{a,\rho}$ is either a transvection, or a dilation.

Recall one useful fact we shall frequently use in this section.

1.3.2. *Let* ρ: $V \to F$ *be a non–zero linear functional. Then the kernel* kerρ *of* ρ *is a hyperplane of* V.

PROOF. Suppose that $a \in V$ is such that $\rho a \neq 0$. Let now $a, e_1, \ldots, e_{n-1}$ be a base of V. Then for appropriate scalars $\lambda_1, \ldots, \lambda_{n-1}$ we have that

$$\rho(e_1) = \lambda_1 \rho a, \ldots, \rho(e_{n-1}) = \lambda_{n-1} \rho a. \tag{1.60}$$

This implies that

$$\rho(e_1 - \lambda_1 a) = \ \ldots \ = \rho(e_{n-1} - \lambda_{n-1} a) = 0. \tag{1.61}$$

It is clear that the vectors

$$e_1 - \lambda_1 a, \ldots, e_{n-1} - \lambda_{n-1} a \tag{1.62}$$

are linearly independent. Thus dim ker$\rho \geq n-1$. Since ρ is non–zero, hence ker$\rho \neq V$ and the result follows. □

We shall use the following formulae:

If we have $(\tau_{a,\rho} - 1_V)V \subseteq Fa$, so $\tau_{a,\rho}$ is either a transvection or a dilation when it is invertible (det $\tau \neq 0$) as we said above. Note that

$$\tau_{a,\rho} = 1_V \Leftrightarrow a = 0 \quad \text{or} \quad \rho = 0 \tag{1.63}$$

and

$$\tau_{\lambda a,\rho}(x) = (x) + \rho(x)\lambda a = (x) + \lambda\rho(x)a = \tau_{a,\lambda\rho} \qquad \forall \lambda \in \mathrm{F} \tag{1.64}$$

So we have this statement:

1.3.3. *Suppose a, a´ are nonzero vectors, and ρ, ρ´ are nonzero linear functionals, and* $\rho a = \rho' a' = 0$. *So* $\tau_{a,\rho}$ *and* $\tau_{a',\rho'}$ *are elements of* $\mathrm{GL}_n(V)$ *which are not equal to* 1_V. *Then* $\tau_{a,\rho} = \tau_{a',\rho'}$ *if and only if there is a* λ *in* $\dot{F}$ *with* $a' = \lambda a$ *and* $\rho' = \lambda^{-1}\rho$.

PROOF. $\tau_{a,\rho}$ is either a transvection or a dilation then we have

$$\tau_{\lambda a,\rho} = \tau_{a,\rho\lambda} \quad \forall \lambda \in \mathrm{F} \tag{1.65}$$

then

$$\tau_{\lambda a,\rho} = \tau_{\lambda a',\rho'} = \tau_{a',\lambda\rho'} \qquad (1.66)$$

then

$$a' = \lambda a \quad \text{and} \quad \rho' = \lambda^{-1}\rho \qquad (1.67)$$

Converse is the same □

1.3.4. det $\tau_{a,\rho} = 1 + \rho a$.

PROOF. If ρ is the zero functional, then $\tau_{a,\rho} = 1_V$ and the conclusion of the proposition is trivially true.

Suppose that $\rho \neq 0$. By 1.3.2 the kernel H of ρ is a hyperplane of V. Consider a base $e_1, \dots, e_{n-1}$ of this hyperplane.

We have two cases to consider.

1) $\rho(a) = 0$. Then we extend the system $e_1, \dots, e_{n-1}$ to a base of V by a vector e_n. By the definition

$$\tau_{a,\rho}\, e_n = e_n + \rho(e_n)a. \qquad (1.68)$$

As $a \in$ H and by $\tau_{a,\rho}\, e_n = e_n + \rho(e_n)a$, since $\langle a\rangle = R \subseteq P$ we get $a = \beta_1 e_1 + \dots + \beta_{n-1}e_{n-1}$ for appropriate $\beta_i \in F$ where $1 \le i \le n-1$. And since $e_n \notin H$, $\rho(e_n) \neq 0$ then we have

$$\tau_{a,\rho}\, e_n = e_n + \rho(e_n)a = e_n + \alpha_1 e_1 + \dots + \alpha_{n-1} e_{n-1} \qquad (1.69)$$

for a suitable $\alpha_1, \dots, \alpha_{n-1} \in F$. Then the matrix of our operator in the base $(e_n, e_1, \dots, e_{n-1})$ is a lower unitriangular. Hence its determinant is 1, that is, $1 + \rho(a) = 1 + 0$, as desired.

2) $\rho(a) \neq 0$. If so, $a \notin H$ and then $(e_1, \dots, e_{n-1}, a)$ is a base of V. We have

$$\tau_{a,\rho}\, a = a + \rho(a)a = (1+\rho(a))\, a. \tag{1.70}$$

Since, furthermore, $\tau_{a,\rho}\, e_k = e_k$ where k = 1, ... ,n−1, we see that $\tau_{a,\rho}$ is diagonalizable in our base, that is,

$$[\tau_{a,\rho}] = \begin{pmatrix} 1+\rho(a) & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{1.71}$$

Hence det $\tau_{a,\rho} = 1 + \rho a$ □

Therefore $\tau_{a,\rho}$ is in $GL_n(V)$ if and only if $\rho a \neq -1$; it is a transvection if and only if $\rho a = 0$; and it is a dilation if and only if $\rho a \neq 0, -1$. If σ is an element of $GL_n(V)$ with $\sigma \neq 1_V$ and $\sigma = \tau_{a,\rho}$, then $R = Fa$ and $P = \rho^{-1}(0)$. In general,

$$\tau_{a,\rho}\tau_{b,\phi} = \{x + (\rho x)a + (\phi x)b\} + (\phi x)(\rho b)a \tag{1.72}$$

If $\tau_{a,\rho}$ and $\tau_{b,\rho}$ are transvection, then

$$\tau_{a,\rho}\tau_{b,\rho} = \tau_{a+b,\rho}, \tag{1.73}$$

and if $\tau_{a,\rho}$ and $\tau_{a,\phi}$ are transvections, then

$$\tau_{a,\rho}\tau_{a,\phi} = \tau_{a,\rho+\phi} \tag{1.74}$$

In particular, if $\tau_{a,\rho}$ is a transvection and n is a rational integer, then

$$\tau^n_{a,\rho} = \tau_{na,\rho}. \tag{1.75}$$

For any σ in $GL_n(V)$,

$$\sigma \tau_{a,\rho} \sigma^{-1} = \tau_{\sigma a, \rho\sigma^{-1}} \tag{1.76}$$

Notation: In the case when P is a hyperplane of V we shall denote by P^0 the set of all linear functionals whose kernel is P.

1.3.5. *Let σ be an element of* GL(V) *with* res$\sigma = 1$. *So R is a line and P is a hyperplane.*
If $\rho \in P^0$, then there is an $a \in R$ such that $\sigma = \tau_{a,\rho}$.
If $b \in R$, then there is a $\phi \in P^0$ such that $\sigma = \tau_{b,\phi}$.

PROOF. Suppose that $\rho \in P^0$. Since the image of V under ρ is F, then there is a vector $z \in V$ with $\rho z = 1$. Clearly, $z \notin P$. Put $a = \sigma z - z$. Then it is immediate to check that σ and $\tau_{a,\rho}$ agree both on R and P. Hence $\sigma = \tau_{a,\rho}$.

Now take a nonzero b in R. Since R is a line, in particular, $b = \lambda a$. We have already shown that $\sigma = \tau_{a,\rho}$ then

$$\sigma = \tau_{a,\rho} = \tau_{\lambda^{-1}(\lambda a), \rho} = \tau_{\lambda a, \lambda^{-1}\rho} = \tau_{b,\phi} \tag{1.77}$$

where $\phi = \lambda^{-1}\rho$. □

1.3.6. *Let τ_1, τ_2 be transvections in* GL$_n$(V) *and let α be a scalar. Then $\tau_2 = \alpha\,\tau_1$ if and only if $\alpha = 1$ with $\tau_1 = \tau_2$. In particular, $\alpha\,\tau_1$ is not a transvection when $\alpha \neq 1$.*

PROOF. Suppose that $\tau_2 = \alpha\,\tau_1$. The only eigenvalue of a transvection is 1. The condition implies therefore that α is an eigenvalue of τ_2, whence $\alpha = 1$. □

1.3.7. *Let σ_1, σ_2 be elements of* GL$_n$(V) *of residue* 1 *and the product $\sigma_1\sigma_2$ is non−trivial. Then*

$$\text{res}\ \sigma_1\sigma_2 = 1 \Leftrightarrow R_1 = R_2 \quad or \quad P_1 = P_2. \tag{1.78}$$

PROOF. Put $\sigma = \sigma_1\sigma_2$.

($\Leftarrow$). We know that $R \subseteq R_1 + R_2$. Then if $R_1 = R_2$ we have $R = R_1 = R_2$ and then res$\sigma = 1$. Suppose that $P_1 = P_2$. Now, dually to the first case, we have $P \supseteq P_1 \cap P_2$. Since $\sigma \neq 1_V$ we have $P = P_1 = P_2$. Now P is a hyperplane and therefore res$\sigma = 1$, as required.

($\Rightarrow$). Let res$\sigma = 1$. If $P_1 = P_2$, we are done. Assume $P_1 \neq P_2$. This implies that $V = P_1 + P_2$. Hence by 1.2.2 we have that $R = R_1 + R_2$. But R is a line, and then R_1 and R_2 are to be equal lines. □

1.3.8. *Let σ_1, σ_2 be transvections of* $\mathrm{GL}_n(V)$. *Then $\sigma_1\sigma_2$ is a transvection if and only if $R_1 = R_2$ or $P_1 = P_2$.*

PROOF. The result is an immediate corollary of 1.3.7. One has only to take into account that the product $\sigma_1\sigma_2$ is of determinant 1 (i.e. det $\sigma_1\sigma_2 = \det\sigma_1 \det\sigma_2 = 1 \cdot 1 = 1$). □

1.3.9. Let X be a subgroup of $\mathrm{GL}_n(V)$ all whose elements are transvections. Then either the elements of X have the same residual line, or they all have the same fixed hyperplane.

PROOF. In the case when all elements of X have the same residual space we are through.

Suppose then that there σ_1, σ_2 in X such that $R_1 \neq R_2$. Since X is a subgroup, $\sigma_1\sigma_2 \in X$ and a transvection. Then by 1.3.8 we get that $P_1 = P_2$. Take any nontrivial element σ of X. We want to show $P = P_1 = P_2$. Let $P \neq P_1$, it is impossible that $P \neq P_1$, since X is a subgroup again and by applying 1.3.8 to the pairs σ, σ_1 and σ, σ_2, are transvections and we see that $R = R_1$ and $R = R_2$. This implies that $R_1 = R_2$, which is absurd. □

1.3.10. *Two* nontrivial *transvections $\sigma_1, \sigma_2 \in$* $\mathrm{GL}_n(V)$ *are commuting if and only if*

$$R_1 \subseteq P_2 \text{ and } R_2 \subseteq P_1. \tag{1.79}$$

PROOF. The sufficiency part follows from 1.2.4. Since σ_k is a tranvection, we have $R_k \subseteq P_k$, where $k = 1,2$. In the case when $R_1 \cap R_2 = 0$, or, in other words, in the case when R_1 are distinct lines we have the desired by 1.2.5. If $R_1 = R_2$ we have that $R_1 = R_2 \subseteq P_2$ and $R_2 = R_1 \subseteq P_1$. □

1.3.11. *Let x, y be distinct vectors of V. Let further H be a hyperplane that contains* $y - x$ *but not x. Then there is a transvection with* $P = H$ *and* $R = \langle y - x \rangle$ *which takes x to y*: $\sigma x = y$.

PROOF. The conditions imply that $n > 1$. Pick a linear functional from V to F as ρ whose kernel is H and such that $\rho x = 1$. As $y - x \in H$, $\rho(y - x) = 0$. Consider the linear map $\sigma = \tau_{y-x,\rho}$. It is easy to check that $P = H$ and $R = \langle y - x \rangle$; furthermore, σ is a transvection by 1.3.4. Finally,

$$\sigma x = \tau_{y-x,\rho}\, x = x + \rho(x)\,(y - x) = x + 1(y - x) = y. \tag{1.80}$$

□

Our next result is a dual version to 1.3.11: lines are replaced by hyperplanes and vice versa.

1.3.12. *Let H, H' be hyperplanes, and let L be a line that is contained neither in H, nor in H'. Then there is a transvection* τ *whose residual space is L and which takes H to H'*: $\tau H = H'$.

PROOF. Suppose that $x_1, \dots, x_{n-1}$ is a base of H and $L = \langle a \rangle$. The conditions imply that

$$V = H \oplus L = H' \oplus L. \tag{1.81}$$

The second equality implies that for each $i = 1, \dots, n-1$ there exist a vector x_i' of H' and a scalar λ_i of F with

$$x_i = x'_i + \lambda_i a. \tag{1.82}$$

It is clear that the vectors $x'_1, \dots, x'_{n-1}$ are linearly independent, and so they form a base of H'.

Consider the following linear functional in V':

$$\rho a = 0 \text{ and } \rho x_i = -\lambda_i, \qquad i = 1, \dots, n-1. \tag{1.83}$$

One then easily checks that $\sigma = \tau_{a,\rho}$ is a non–trivial transvection with the residual line $L = \langle a \rangle$. To complete the proof we see that $\sigma H = H'$. But

$$\sigma x_i = \tau_{a,\rho} x_i = x_i + \rho(x_i)\, a = x_i - \lambda_i\, a = x'_i \tag{1.84}$$

for all $i = 1, \dots, n-1$. Then σ takes the base $x_1, \dots, x_{n-1}$ to the base $x'_1, \dots, x'_{n-1}$, and hence H to H'. □

2. COLLINEAR TRANSFORMATION AND PROJECTIVE GEOMETRY

A geometric transformation g of V onto V_1 is a bijection g: $V \rightarrow V_1$ which has the following property for all subsets X of V: X is a subspace of V if and only if gX is a subspace of V_1. And a projectivity π of V onto V_1 is a bijection π: $P(V) \rightarrow P(V_1)$ (P as the set of all subspaces) which has the following property for all U, W in $P(V)$: $U \subseteq W$ if and only if $\pi U \subseteq \pi W$.

2.1. The Fundamental Theorem of Projective Geometry

2.1.1. *Let π be a bijection of the lines of V onto the lines of V_1, i.e. let π: $P^1(V) \rightarrow P^1(V_1)$. Suppose π satisfies*

$$L_1 \subseteq L_2 + L_3 \;\Leftrightarrow\; \pi L_1 \subseteq \pi L_2 + \pi L_3 \tag{2.1}$$

for all L_1, L_2, L_3 in $P^1(V)$. Then π can be extended uniquely to a projectivity
Π: $P(V) \rightarrow P(V_1)$.

PROOF. Existence, by induction on r that

$$L \subseteq L_1 + \ldots + L_r \Leftrightarrow \pi L \subseteq \pi L_1 + \ldots + \pi L_r \tag{2.2}$$

Define $\Pi 0 = 0$. For any U in $P(V)$ with $U \neq 0$ express U as the sum of lines

$$U = L_1 + \ldots + L_r \tag{2.3}$$

and define

$$\Pi U = \pi L_1 + \ldots + \pi L_r \tag{2.4}$$

We find that Π is a well–defined, order preserving, bijection of $P(V)$ onto $P(V_1)$ that induces π on lines. □

2.1.2. *Let* $\pi : P(V) \to P(V_1)$ *be a bijection such that*

$$U \subseteq W \Rightarrow \pi U \subseteq \pi W \tag{2.5}$$

Suppose $\dim_F V = \dim_{F1} V_1$. *Then* π *is a projectivity*.

PROOF. We have to show $\pi U \subseteq \pi W \Rightarrow U \subseteq W$ (Because if π is a projectivity then $U \subseteq W \Leftrightarrow \pi U \subseteq \pi W$). Write $\pi W = \pi U \oplus \pi T$. Then

$$\pi (U \cap T) \subseteq \pi U \cap \pi T = 0. \tag{2.6}$$

Since $\pi(U \cap T) = 0$ and $\pi 0 = 0$ then $U \cap T = 0$ and obviously $U + T = U \oplus T$. We obtain $\pi(U \oplus T) \supseteq \pi U \oplus \pi T = \pi W$; hence $\pi(U \oplus T) = \pi W$ by dimension argument, and hence $U \subseteq U \oplus T = W$. □

2.1.3. *Let* π *be a bijection of the lines of* V *onto the lines of* V_1. *Suppose* $\dim_F V = \dim_{F1} V_1$, *and*

$$L_1 \subseteq L_2 + L_3 \Rightarrow \pi L_1 \subseteq \pi L_2 + \pi L_3. \tag{2.7}$$

Then π *can be extended uniquely to a projectivity* $\Pi: P(V) \to P(V_1)$.

PROOF. (1) By induction,

$$L \subseteq L_1 + ... + L_r \Rightarrow \pi L \subseteq \pi L_1 + ... + \pi L_r. \tag{2.8}$$

Hence

$$V = L_1 + ... + L_n \Rightarrow V_1 = \pi L_1 + ... + \pi L_n. \tag{2.9}$$

Hence

$$L_1, \ \dots , L_r \qquad \text{independent} \Rightarrow \pi L_1, \ \dots , \pi L_r \qquad \text{independent.} \tag{2.10}$$

(2) Define $\Pi 0 = 0$. For any U in $P(V)$ with $U \neq 0$ express $U = L_1 + \dots + L_r$ and define $\Pi U = \pi L_1 + \dots + \pi L_r$. Then Π is well−defined prolongation of π to $P(V)$ by step (1). And it is clear that π preserves + and dim. It is easily verified that π is surjective, also that

$$U \subseteq W \Rightarrow \Pi U \subseteq \Pi W. \tag{2.11}$$

Therefore, in the light of 2.1.2., it remains for us to prove injectivity, i.e. that $\Pi U = \Pi W$ implies $U = W$. It is enough to show that $\Pi L \subseteq \Pi W$ implies $L \subseteq W$. And this is true since

$$\dim(W + L) = \dim \Pi(W + L) = \dim(\Pi W + \Pi L) = \dim \Pi W = \dim W. \tag{2.12}$$

□

2.1.4. *Let* π *be a bijection of the lines of* V *onto the lines of* V_1, *and let* $\dim_F V = \dim_{F1} V_1$. *Suppose there is a fixed* p $(2 \leq p \leq n-1)$ *such that for each* p*−dimensional subspace* U *of* V, *all the lines* πL (*with* $L \subseteq U$) *fall in a* p*−dimensional subspace of* V_1. *Then* π *can be extended uniquely to a projectivity* $\Pi: P(V) \to P(V_1)$

PROOF. (1) If $p = 2$ the results follows easily from 2.1.3. So let $(3 \leq p \leq n-1)$. We will prove that the property then holds for $p-1$, so ultimately for 2, and we will be through.

(2) For any subspace X of V, let X_* denote the subspace of V_1 generated by πL (with $L \subseteq X$). Clearly

$$\dim X \leq p \Rightarrow \dim X_* \leq p \tag{2.13}$$

We have to show that

$$\dim U = p-1 \Rightarrow \dim U_* \leq p-1. \tag{2.14}$$

Suppose not. Then we would have a U with dim $U = p - 1$, dim $U_* = p$. Pick a line $K \subseteq V$ with $K_* \not\subset U_*$. Then $K \not\subset U$. So dim $(K + U) = p$, so we claim that $(K + U)_* \supseteq K_* + U_*$. Take a line N as $N \subseteq K_* + U_*$ then $\exists N_1$, N_2 such that $N_1 \subseteq K_*$, $N_2 \subseteq U_*$. Then we have $N \subseteq N_1 + N_2$ and then $\pi N \subseteq \pi N_1 + \pi N_2$. So $L_1 \subseteq K$ and $L_2 \subseteq U$

$$\dim(K + U)_* \geq \dim(K_* + U_*) = p + 1, \tag{2.15}$$

which is absurd. □

2.1.5. *Suppose $\dim_F V \geq 3$. Then every projectivity of V onto V_1 is a projective collinear transformation.*

PROOF. (1) For any a in $\dot{V}$, $\langle a \rangle$ will be the line Fa; for any a' in $\dot{V}_1$, $\langle a' \rangle$ will be the line $F_1 a'$. Let π: $P(V) \rightarrow P(V_1)$ be a given projectivity of V onto V_1. Fix a base $x_1, ..., x_n$ for V. It is easily seen that there is a base $x'_1, ..., x'_n$ for V_1 such that

$$\pi \langle x_i \rangle = \langle x_i' \rangle \qquad (1 \leq i \leq n), \tag{2.16}$$

$$\pi \langle x_1 + x_i \rangle = \langle x_1' + x_i' \rangle \qquad (2 \leq i \leq n). \tag{2.17}$$

(2) Since π is a projectivity, each α in F determines an element α' of F_1 such that $\pi \langle x_1 + \alpha x_2 \rangle = \langle x_1' + \alpha' x_2' \rangle$. Clearly $0' = 0$, $1' = 1$, and $\alpha' = \beta'$ implies $\alpha = \beta$. So we have an injection, easily seen to be a bijection,

$$' : F \rightarrow F_1 \tag{2.18}$$

(3) Let us show that $\pi \langle x_1 + \alpha x_i \rangle = \langle x_1' + \alpha' x_i' \rangle$ for any $i \geq 2$. By step (2) we have $\sim$: $F \rightarrow F_1$ such that $\pi \langle x_1 + \alpha x_i \rangle = \langle x_1' + \tilde{\alpha} x_i' \rangle$ and $\tilde{0} = 0$, $\tilde{1} = 1$. Now

$$\langle \alpha x_2 - \alpha x_i \rangle \subseteq \begin{cases} \langle x_2 \rangle + \langle x_i \rangle \\ \langle x_1 + \alpha x_2 \rangle + \langle x_1 + \alpha x_i \rangle \end{cases} \tag{2.19}$$

so

$$\pi \langle \alpha\, x_2 - \alpha\, x_i \rangle \subseteq \begin{cases} \langle x_2' \rangle + \langle x_i' \rangle \\ \langle x_1' + \alpha' x_2' \rangle + \langle x_1' + \tilde{\alpha} x_i' \rangle. \end{cases} \tag{2.20}$$

Hence $\pi\langle \alpha\, x_2 - \alpha\, x_i \rangle = \langle \alpha' x_2' - \tilde{\alpha}\, x_i' \rangle$. In particular $\pi\langle x_2 - x_i \rangle = \langle x_2' - x_i' \rangle$. But $\pi\langle \alpha x_2 - \alpha\, x_i \rangle = \pi\, \langle x_2 - x_i \rangle$. Hence $\alpha' = \tilde{\alpha}$.

(4) Next we observe that

$$\pi\langle x_1 + \alpha_2\, x_2 + \ldots + \alpha_n\, x_n \rangle = \langle x'_1 + \alpha'_2\, x'_2 + \ldots + \alpha'_n\, x'_n \rangle. \tag{2.21}$$

For

$$\pi\langle x_1 + \alpha_2\, x_2 + \ldots + \alpha_n\, x_n \rangle = \langle x'_1 + *_2\, x'_2 + \ldots + *_n\, x'_n \rangle. \tag{2.22}$$

$$\pi\langle x_1 + \alpha_2\, x_2 + \ldots + \alpha_n x_n \rangle \subseteq \langle x'_1 + \alpha'_i\, x'_i \rangle + \langle x'_2 \rangle \ldots + \langle x'_n \rangle. \tag{2.23}$$

(omit $\langle x'_i \rangle$), whence $*_i = \alpha'_i$ as required.

(5) We also have

$$\pi\langle \alpha_2\, x_2 + \ldots + \alpha_n\, x_n \rangle = \langle \alpha'_2\, x'_2 + \ldots + \alpha'_n\, x'_n \rangle. \tag{2.24}$$

since $\pi\langle \alpha_2\, x_2 + \ldots + \alpha_n\, x_n \rangle = \langle *_2\, x'_2 + \ldots + *_n\, x'_n \rangle$. and $\pi\langle \alpha_2 x_2 + \ldots + \alpha_n\, x_n \rangle \subseteq \langle x'_1 + \alpha'_2 x'_2 + \ldots + \alpha'_n x'_n \rangle + \langle x'_1 \rangle$, whence $*_i = \alpha'_i$, can be arranged.

(6) The bijection $'\colon F \to F_1$ is in fact a field isomorphism. For

$$\langle x'_1 + (\alpha + \beta)' x'_2 + x'_3 \rangle = \pi\langle x_1 + (\alpha + \beta) x_2 + x_3 \rangle \subseteq \langle x'_1 + \alpha' x'_2 \rangle + \langle \beta' x'_2 + x'_3 \rangle. \tag{2.25}$$

whence $(\alpha+\beta)' = \alpha'+\beta'$, and $\langle x'_1+(\alpha\beta)'x'_2+\beta'x'_3\rangle = \pi\langle x_1+(\alpha\beta)x_2+x_3\rangle \subseteq$ $\langle x'_1\rangle + \langle \alpha'x'_2+x'_3\rangle$ whence $(\alpha\beta)' = \alpha'\beta'$.

(7) If we now use from O'Meara §4.1 to define a collinear transformation k: $V\to V_1$ with respect to $'$ by

$$k(\alpha_1 x_1+ \ldots +\alpha_n x_n) = (\alpha'_1 x'_1+ \ldots +\alpha'_n x'_n) \tag{2.26}$$

we find that $\bar{k}$ and π agree on lines, hence π is the projective collinear transformation $\bar{k}$.

2.2. The Isomorphisms Φ_g

We now introduce group isomorphisms Φ_g where g is first a collinear transformation g: $V\to V_1$ of V onto V_1, and secondly a projective collinear transformation g: $P(V)\to P(V_1)$ of V onto V_1.

First consider a collinear transformation g: $V\to V_1$. Let μ: $F\to F_1$ be the associated field isomorphisms. Here $n = n_1$ follows. Then it is clear that the Φ_g defined by

$$\Phi_g\, k = gkg^{-1} \qquad \forall\, k \in \Gamma L_n(V) \tag{2.27}$$

is actually a group isomorphism

$$\Phi_g : \Gamma L_n(V)\to \Gamma L_{n1}(V_1). \tag{2.28}$$

Under composition and inversion,

$$\Phi_{g1\,g} = \Phi_{g1}\, \Phi_g, \qquad \Phi^{-1}_g = \Phi_{g^{-1}}. \tag{2.29}$$

We find that Φ_g induces

$$\Phi_g : GL_n(V)\to GL_{n1}(V_1), \tag{2.30}$$

$$\Phi_g : \mathrm{SL}_n(V) \to \mathrm{SL}_{n1}(V_1), \tag{2.31}$$

$$\Phi_g : \mathrm{RL}_n(V) \to \mathrm{RL}_{n1}(V_1). \tag{2.32}$$

If σ is in $\mathrm{GL}_n(V)$, then

$$\det(\Phi_g\sigma) = (\det\sigma)^{\mu}; \tag{2.33}$$

and the residual and fixed spaces of $\Phi_g\sigma$ are gR and gP respectively; in particular

$$\mathrm{res}\,\Phi_g\,\sigma = \mathrm{res}\,\sigma. \tag{2.34}$$

If H is a hyperplane and L is a line with $L \subseteq H$, then gL is a line contained in the hyperplane gH of V_1, and Φ_g carries the set of transvections with spaces $L \subseteq H$ onto the set of transvections with spaces $gL \subseteq gH$. If σ is the transvection $\sigma = \tau_{a,\rho}$ in usual form, then

$$\Phi_g \tau_{a,\rho} = \tau_{ga,\mu\rho g^{-1}}\ . \tag{2.35}$$

Now consider a projective collinear transformation g: $P(V) \to P(V_1)$ of V onto V_1. We again have $n = n_1$. This time define

$$\Phi_g k = gkg^{-1} \quad \forall\, k \in \mathrm{P\Gamma L}_n(V) \tag{2.36}$$

and obtain a group isomorphism

$$\Phi_g : \mathrm{P\Gamma L}_n(V) \to \mathrm{P\Gamma L}_{n1}(V_1). \tag{2.37}$$

Under composition and inversion,

$$\Phi_{g1\,g} = \Phi_{g1}\,\Phi_g, \qquad \Phi^{-1}_{\ g} = \Phi_{g^{-1}} \tag{2.38}$$

Since g is a projective collinear transformation it is of the form $g = \overline{h}$ for some collinear transformation h: $V \rightarrow V_1$. We find that

$$\Phi_g \overline{j} = \Phi_{\overline{h}} \overline{j} = \overline{\Phi_h j} \qquad \forall j \in \Gamma L_n(V). \tag{2.39}$$

We conclude that ϕ_g induces

$$\phi_g: \mathrm{PGL}_n(V) \rightarrow \mathrm{PGL}_{n1}(V_1), \tag{2.40}$$

$$\phi_g: \mathrm{PSL}_n(V) \rightarrow \mathrm{PSL}_{n1}(V_1), \tag{2.41}$$

and also that ϕ_g carries the set of projective transvections with spaces $L \subseteq H$ onto the set with spaces $gL \subseteq gH$.

2.2.1. *Suppose* $n = n_1 \geq 2$. *If* g_1 *and* g_2 *are collinear transformations of* V *onto* V_1, *then the following statements are equivalent:*

(1) $\phi_{\overline{g_1}} = \phi_{\overline{g_2}}$.

(2) $\overline{g}_1 = \overline{g}_2$.

(3) $g_1 = g_2 r$ *for some* r *in* $\mathrm{RL}_n(V)$.

(4) $g_1 = r_1 g_2$ *for some* r_1 *in* $\mathrm{RL}_{n1}(V_1)$.

2.3. The Contragradient

Consider a semilinear mapping k: $V \rightarrow V_1$ with respect to the field isomorphism μ: $F \rightarrow F_1$. For each $\rho_1 \in V_1'$ it is clear that $\mu^{-1}\rho_1 k \in V'$. So each semilinear k defines a mapping

$$k^t: V_1' \rightarrow V', \tag{2.42}$$

called the transpose of k, whose action is determined by sending a typical ρ_1 in V_1' to $\mu^{-1}\rho_1 k$ in V'. Thus

$$k^t(\rho_1) = \mu^{-1}\rho_1 k. \tag{2.43}$$

In other words, each k determines exactly one map k^t such that

$$\langle x, k^t\rho_1\rangle^{\mu} = \langle kx, \rho_1\rangle \qquad \forall\, x\in V, \rho_1\in V_1'. \tag{2.44}$$

This equation is the defining equation of k^t.

We find that k^t: $V_1' \to V'$ is semilinear with respect to μ^{-1}: $F_1 \to F$. For any two semilinear maps k and l of V into V_1 we have

$$k^t = 0 \Leftrightarrow k = 0, \tag{2.45}$$

$$k^t = l^t \Leftrightarrow k = 1. \tag{2.46}$$

If k: $V \to V_1$ and k_1: $V_1 \to V_2$ are semilinear, then k_1k: $V \to V_2$ is semilinear with $(k_1k)^t = k^t k_1{}^t$. If we fix bases $\mathcal{X}$ and $\mathcal{D}$ for V and V_1 respectively, if $\mathcal{X}'$ and $\mathcal{D}'$ denote the corresponding dual bases for V' and V_1' respectively, if A is the matrix of k with respect to $\mathcal{X},\mathcal{D}$, and if B is the matrix of k^t with respect to $\mathcal{D}'$, $\mathcal{X}'$, then

$$B^{\mu} = A^t. \tag{2.47}$$

To prove this, just establish the equations

$$a_{ij} = \langle \sum_{\lambda} a_{\lambda j}\, y_{\lambda}, y_i' \rangle = \langle kx_j, y_i'\rangle = \langle x_j, k^t y_i'\rangle^{\mu} \tag{2.48}$$

$$= \langle x_j, \sum_{\lambda} b_{\lambda i}\, x_{\lambda}' \rangle \mu = b_{ji}{}^{\mu} \tag{2.49}$$

In particular,

$$k \text{ bijective} \Leftrightarrow k^t \text{ bijective} \tag{2.50}$$

If k: $V \to V_1$ is bijective, then $(k^{-1})^t$ and $(k^t)^{-1}$ are both semilinear bijections of V' onto V_1' with respect to μ: $F \to F_1$ and, in fact, we find that

$$(k^{-1})^t = (k^t)^{-1}. \qquad (2.51)$$

Accordingly, the contragradient $\breve{k}$ is defined for any collinear transformation k by the equation

$$\breve{k} = \left(k^{-1}\right)^t. \qquad (2.52)$$

The associated field isomorphism is the same for $\breve{k}$ as for k and we have the diagrams

$$k: V \to V_1, \qquad \mu: F \to F_1, \qquad \breve{k}: V' \to V_1'. \qquad (2.53)$$

Behaviour under composition and inversion is given by

$$\breve{\overline{k_1 \cdots k_t}} = \breve{k}_1 \ldots \breve{k}_t \text{ and } (\breve{k}^{-1}) = (\breve{k})^{-1}. \qquad (2.54)$$

Now fix V and consider the action of the contragradient on the collinear transformations of V, i.e. on $\Gamma L_n(V)$. Then it is easily seen that we have an isomorphism

$$\breve{}: \Gamma L_n(V) \to \Gamma L_n(V') \qquad (2.55)$$

which preserves associated field automorphisms and which induces

$$\breve{}: GL_n(V) \to GL_n(V'), \qquad (2.56)$$

$$\breve{}: SL_n(V) \to SL_n(V'), \qquad (2.57)$$

$$\breve{}: RL_n(V) \to RL_n(V'). \qquad (2.58)$$

Furthermore we have

$$\text{mat}_{\mathcal{X}} k = A \Leftrightarrow \text{mat}_{\mathcal{X}'} \breve{k} = \breve{A} \qquad (2.59)$$

where $\mathcal{X}'$ denotes the dual base of $\mathcal{X}$, and $\breve{A}$ is defined for the invertible matrix A by

$$\breve{A} = \left(A^{-1}\right)^t. \qquad (2.60)$$

And for any k in $\Gamma L_n(V)$ and any subspace U of V we have

$$\breve{k}\, U^0 = (kU)^0. \qquad (2.61)$$

We call $\breve{\ }: \Gamma L_n(V) \rightarrow \Gamma L_n(V')$ the contragradient isomorphism of V

2.3.1. *Let* $\breve{\ }$ *be the contragradient isomorphism of V and consider a typical* σ *in* $GL_n(V)$. *Then*

(1) *The residual space of* $\breve{\sigma}$ *is* P^0.

(2) *The fixed space of* $\breve{\sigma}$ *is* R^0 .

(3) res $\breve{\sigma}$ = res σ.

(4) $\breve{\ }$ *carries the set of transvections with spaces* $L \subseteq H$ *onto the set of transvections with spaces* $H^0 \subseteq L^0$.

(5) *If* σ *is the transvection* $\sigma = \tau_{a,\rho}$ *in the usual form, then* $\breve{\tau}_{a,\rho} = \tau_{\rho,-\tilde{a}}$, where $\tilde{a} \in V''$ is defined by $\langle \phi, \tilde{a} \rangle = \langle a, \phi \rangle$.

PROOF. It is enough to prove (1)–(4) for σ^t instead of $\breve{\sigma}$, and to prove $\tau^t_{a,\rho} = \tau_{\rho,\tilde{a}}$ instead of (5). Let R_t and P_t denote the residual and fixed spaces of σ^t. If $\rho \in R^0$, then for any x in V we have

$$\langle x, \sigma^t\rho - \rho\rangle = \langle \sigma x - x, \rho\rangle \in \langle R, R^0\rangle = 0; \qquad (2.62)$$

so $\sigma^t\rho = \rho$; so $R^0 \subseteq P_t$. On the other hand, if $\rho \in P_t$, then, for typical x in V,

$$\langle x, \sigma^t\rho - \rho\rangle = \langle \sigma x - x, \rho\rangle = 0; \tag{2.63}$$

so $P_t \subseteq R^0$. Therefore $P_t = R^0$. This proves (2). Hence (3). Now for any $p \in P$, $\rho \in V'$,

$$\langle p, \sigma^t\rho - \rho\rangle = \langle \sigma p - p, \rho\rangle = 0; \tag{2.64}$$

so $R_t \subseteq P^0$; so $R_t = P^0$ by dimensions. We now have (1), (2) and (3). And (4) is a consequence of (1) and (2). For (5), establish the equations

$$\langle x, \tau^t_{a,\rho}\, \phi\rangle = \langle x + (\rho x)a, \phi\rangle = \langle x, \phi\rangle + \langle x, \rho\rangle\langle a, \phi\rangle$$

$$= \langle x, \phi\rangle + \tilde{a}(\phi)\langle x, \rho\rangle = \langle x, \phi + \tilde{a}(\phi)\rho\rangle = \langle x, \tau_{\rho,\tilde{a}}\, \phi\rangle. \tag{2.65}$$

□

2.4. Comments

The following diagrams illustrate our results for $n \geq 2$ (but excluding $n = 2$ with $F = \mathbf{F}_2$, $\mathbf{F}_3$).

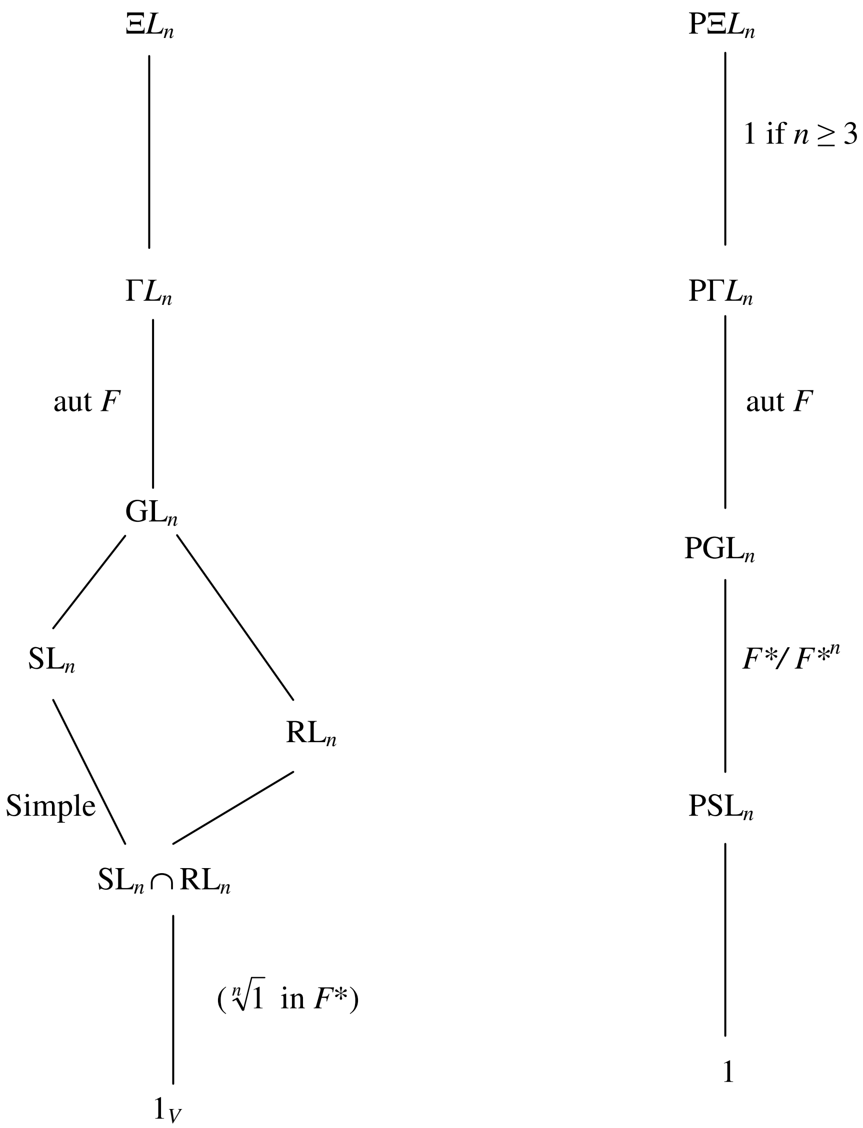

Figure 2.1. Inclusion scheme

3. THE ISOMORPHISMS OF LINEAR GROUPS

3.1. Preliminaries

We use $C_A(X)$ denote the centralizer in A of a nonempty subset X of an abstract group A (i.e. $C_A(X) = \{a\in A \mid \; ax = xa \quad \forall x\in X\}$). Thus $C_A(X)$ is a subgroup of A, and

$$X_1 \subseteq X_2 \Rightarrow \mathrm{C}_A(X_1) \supseteq \mathrm{C}_A(X_2), \tag{3.1}$$

$$X \subseteq \mathrm{C}_A\mathrm{C}_A(X). \tag{3.2}$$

If ϕ is an isomorphism of a onto ϕA, then

$$\phi\ \mathrm{C}_A(X) = C_{\phi \mathrm{A}}(\phi X), \tag{3.3}$$

For short we put $C_V(X) = C_{GL_n(V)}(X)$ whenever we are working in $\mathrm{GL}_n(V)$, and $C_V(X) = C_{PGL_n(V)}(X)$ whenever we are working in $\mathrm{PGL}_n(V)$. The symbol C will be reserved for the centralizers C_G and C_Δ of groups G and Δ to be defined later.

3.1.1. *If* $\sigma\in \mathrm{GL}_2(V) - \mathrm{RL}_2(V)$, *then* $DC_V(\sigma) = 1_V$.

PROOF. σ has to move a line since $\sigma \notin \mathrm{RL}_2$. Hence there is a base $\mathcal{X}$ for V in which σ has matrix of the form $\begin{pmatrix} 0 & \beta \\ 1 & \alpha \end{pmatrix}$. By matrix calculation we find that the matrix of any Σ in $\mathrm{C}_V(\sigma)$ has the form

$$\begin{pmatrix} p & \beta r \\ r & p+\alpha r \end{pmatrix} \tag{3.4}$$

in the base $\mathcal{X}$. Any two such matrices commute. □

3.1.2. *If* $n = 2$, *if* L *and* K *are distinct lines in* V, *and if* τ_L *and* τ_K *are transvections with residual lines* L *and* K *respectively, then* $\tau_L\tau_K\tau_L^{-1}\tau_K^{-1}$ *is an element of* $\mathrm{GL}_2-\mathrm{RL}_2$ *with residual space* V. *If* T_K *is a transvection, distinct from* τ_K, *with residual line* K, *then* $\tau_L\tau_K\tau_L^{-1}\tau_K^{-1}$ *and* $\tau_L\mathrm{T}_K\,\tau_L^{-1}\mathrm{T}_K^{-1}$ *do not permute.*

PROOF. (1) All characteristic vectors of τ_L fall in L, so $J = \tau_L K$ is a line distinct from K. But $\tau_J = \tau_L\tau_K\tau_L^{-1}$ is a transvection with residual line J by 1.2.3. So the residual space of $\tau_L\tau_K\tau_L^{-1}\tau_K^{-1} = \tau_J\tau_K^{-1}$ is equal to $V=J+K$ because we know that any product of different transvections has a residual space as $R_1 + R_1$. Since $\tau_L\tau_K\tau_L^{-1}\tau_K^{-1}$ is a product of two transvections, it cannot be in RL_2. This proves the first part.

(2) Now choose a base x_1, x_2 with $Fx_1 = L$ and $Fx_2 = \mathrm{K}$. In this base

$$\tau_L \sim \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix}, \qquad \tau_K \sim \begin{pmatrix} 1 & \\ \beta & 1 \end{pmatrix}, \qquad \mathrm{T}_K \sim \begin{pmatrix} 1 & \\ \gamma & 1 \end{pmatrix} \tag{3.5}$$

with $\alpha\beta\gamma \neq 0$ and $\beta \neq \gamma$. By computing with matrices we find that the given commutators do not permute. □

Recall from linear algebra that $\sigma \in \mathrm{GL}_n(V)$ is called *unipotent* if $(\sigma - 1_V)^k = 0$ holds for some $k > 0$, i.e. if $(\sigma - 1_V)$ is nilpotent. If σ is unipotent and U is a nonzero subspace of V for which $\sigma U = U$, then $(\sigma|U)$ is also unipotent. The following statements which can be found, for instance, in [3] are equivalent:

(1) σ is unipotent.

(2) All characteristic roots of σ are 1

(3) There is a base for V in which σ is upper–triangular with 1's on the diagonal.

3.1.3. *If* $\mathrm{char}\, F = p > 0$, *then* $\sigma \in \mathrm{GL}_n(V)$ *is unipotent if and only if* $\sigma^{p^{\nu}} = 1_V$ *for some* $\nu \geq 0$.

PROOF. By the binomial theorem for permuting linear transformations,

$$(\sigma - 1_V)^{p^{\nu}} = \sigma^{p^{\nu}} - 1_V. \tag{3.6}$$

So $\sigma^{p^{\upsilon}} = 1_V$ implies that σ is unipotent . Conversely, if $(\sigma - 1_V)^k = 0$ for some $k > 0$, then $(\sigma - 1_V)^{p^{\upsilon}} = 0$ for all $p^{\nu} > k$, so $\sigma^{p^{\upsilon}} = 1_V$. □

We call an element Σ of $P\Gamma L_n(V)$ a projective unipotent transformation if it is of the form $\Sigma=\overline{\sigma}$ for some unipotent transformation σ in $GL_n(V)$ (σ as a unipotent representative of Σ and it is unique.)

We note that all transvections are unipotent transformations and all projective transvections are projective unipotent transformations. Indeed, as we saw above any transvection has an upper−triangular matrix in a suitable base of V (see the proof of 1.3.1.)

We say that two elements k_1 and k_2 of $\Gamma L_n(V)$ *permute projectively* if $\overline{k}_1$ and $\overline{k}_2$ permute. Evidently,

$$\text{Permutability} \Rightarrow \text{projective permutability}, \tag{3.7}$$

but the converse statement is not true in general. Below we examine a number of natural conditions under which projective permutability in $GL_n(V)$ becomes the usual permutability.

3.1.4. *Let σ be any element of* $GL_n(V)$ *which satisfies any one of the following conditions:*

(1) $\operatorname{res}\sigma < \frac{1}{2}n$

(2) $\operatorname{res}\sigma = \frac{1}{2}n$ *with σ not a big dilation.*

(3) *σ has exactly one characteristic root in F.*

(4) *σ is unipotent.*

Then if σ permutes projectively with Σ in $GL_n(V)$, *it permutes with Σ.*

PROOF. We assume (in the proofs of all our statements) that σ permutes projectively with Σ of $GL_n(V)$ and

$$\Sigma\, \sigma\, \Sigma^{-1} = \alpha\, \sigma, \tag{3.8}$$

where $\alpha \in F^*$ is a non−zero scalar.

(1) If the residue of σ is less than $n/2$, then the dimension of the fixed space $P(\sigma)$ of σ is greater than $n/2$ due to

$$\dim R(\sigma) + \dim P(\sigma) = n. \tag{3.9}$$

Let $e_1, \dots, e_k$, where $k > n/2$ are linearly independent vectors of $P(\sigma)$.

(3.8) implies that

$$\text{res}\,(\Sigma\, \sigma\, \Sigma^{-1}) = \text{res}\, \sigma = \text{res}(\alpha\, \sigma). \tag{3.10}$$

We consider the vectors

$$a_1 = (\alpha\, \sigma - 1_V)\, (e_1), \dots, a_k = (\alpha\sigma - 1_V)(e_k) \tag{3.11}$$

in $R(\alpha\sigma)$. Clearly,

$$a_1 = (\alpha - 1)e_1, \dots, a_k = (\alpha - 1)\, e_k. \tag{3.12}$$

If $\alpha \neq 1$, then the system $a_1, \dots, a_k$ is a linearly independent system of vectors of $R(\alpha\sigma)$, and hence

$$\text{res}(\alpha\sigma) \geq k > n/2, \tag{3.13}$$

which is absurd, since $\text{res}(\alpha\sigma) < n/2$. Then α=1 and σ commutes with Σ.

(2) The condition $\text{res}(\sigma) = n/2$ means that the dimension of the fixed space is also $n/2$. Suppose that

$$\Sigma\, \sigma\, \Sigma^{-1} = \alpha\, \sigma, \tag{3.14}$$

where $\alpha \neq 1$. For all $x \in P(\sigma)$ we have

$$\sigma\, \Sigma^{-1}(x) = \alpha\, \Sigma^{-1} \sigma(x) = \alpha\, \Sigma^{-1}(x). \tag{3.15}$$

It follows that the subspace $S = \Sigma^{-1}P(\sigma)$ of dimension $n/2$ is the eigenspace of σ corresponding to the characteristic value $\alpha \neq 1$. Then $V = P(\sigma) \oplus S$ and σ is a big dilation, a contradiction.

(3) Recall that spec(σ) denotes the set of all characteristic roots of σ. We know that any conjugate transformation $\Sigma\, \sigma\, \Sigma^{-1}$ has the same characteristic roots as σ does. Let λ be the only characteristic value of σ. Supposing that $\alpha \neq 1$ in (3.8) we get

$$\{\lambda\} = \mathrm{spec}(\Sigma\, \sigma\, \Sigma^{-1}) = \mathrm{spec}(\sigma) = \mathrm{spec}(\alpha\sigma) = \{\alpha\, \lambda\}, \tag{3.16}$$

which is impossible.

(4) By (3), since any unipotent transformation has exactly one characteristic root. □

3.1.5. *If* σ *and* Σ *are elements of* $\mathrm{GL}_n(V)$, *and if* σ *permutes projectively with* Σ, *then* σ^n *permutes with* Σ.

PROOF. Suppose that

$$\Sigma\, \sigma\, \Sigma^{-1} = \alpha\, \sigma, \tag{3.17}$$

where $\alpha \in$ F*. It follows

$$\Sigma\, \sigma\, \Sigma^{-1}\sigma^{-1} = \alpha \cdot 1_V. \tag{3.18}$$

In the left hand side we have a transformation from $\mathrm{SL}_n(V)$ (that is, of determinant 1). If so, we have

$$1 = \det(\alpha \cdot 1_V) = \alpha^n. \tag{3.19}$$

Clearly, for every integer m

$$(\Sigma\, \sigma\, \Sigma^{-1})^m = \Sigma\, \sigma^m\, \Sigma^{-1}. \tag{3.20}$$

Then for $m=n$ we have

$$\Sigma\, \sigma^n\, \Sigma^{-1} = (\alpha\sigma)^n = \alpha^n \sigma^n = \sigma^n \tag{3.21}$$

So $\sigma^n\, \Sigma = \Sigma\, \sigma^n$, as desired. □

3.1.6. **Example**. Let us consider the case $n = 3$ with σ an element of $GL_3(V)$ with res$\sigma = 2$. We claim there is a Σ in $GL_3(V)$ which does not permute, but which does permute projectively, with σ if and only if σ satisfies

$$\det \sigma = 1, \qquad \sigma \text{ diagonalizable over F}, \qquad \sigma^3 = 1 \tag{3.22}$$

First suppose the conditions are satisfied. Then there is a base for V in which σ has matrix

$$\sigma \sim \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} \qquad (\omega \neq 1,\ \omega^3=1). \tag{3.23}$$

Let Σ be the transformation in $GL_3(V)$ defined by

$$\Sigma \sim \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Sigma^{-1} \sim \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.24}$$

Then $\Sigma\, \sigma\, \Sigma^{-1} = \omega^2\sigma$, so Σ does not permute, but does permute projectively, with σ. Conversely, suppose we have a Σ and an α with $\alpha \neq 0,1$, such that $\alpha(\Sigma\, \sigma\, \Sigma^{-1}) = \sigma$. Clearly $\alpha^3=1$. We have a nonzero x in V such that $\sigma x = x$ since res$\sigma = 2$. But $\sigma(\Sigma x) = \alpha(\Sigma x)$ and $\sigma(\Sigma^2 x) = \alpha^2(\Sigma^2 x)$. Hence

$$\sigma \sim \begin{pmatrix} 1 & & \\ & \alpha & \\ & & \alpha^2 \end{pmatrix} \tag{3.25}$$

since 1, α, α^2 are distinct. So the conditions are satisfied. Note that if our σ in $GL_3(V)$ with res $\sigma = 2$ satisfies the above conditions, then $\langle(CV(\overline{\sigma}))^3\rangle$ is abelian. To see this fix a base x_1, x_2, x_3 in which σ is diagonal. So $\sigma\, x_i = \alpha_i\, x_i$ with distinct α_i for i = 1, 2, 3. Consider a representative Σ of a typical $\overline{\Sigma}$ in $C_V(\overline{\sigma})$. Then Σ permutes projectively with σ, so Σ^3 permutes with σ by 3.1.5. So each element $\overline{\phi}$ of $\langle(CV(\overline{\sigma}))^3\rangle$ has a representative ϕ which permutes with σ, i.e. $\phi\sigma\phi^{-1} = \sigma$. Now Fx_i is the only line on which σ has characteristic root α_i. But $\sigma(\phi x_i) = \alpha_i(\phi x_i)$. So $\phi(Fx_i) = Fx_i$ for $i = 1, 2, 3$. Thus all the ϕ's permute among themselves. So $\langle(CV(\overline{\sigma}))^3\rangle$ is abelian, as asserted.

3.2. Full Groups

A subgroup G of $\Gamma L_n(V)$ is *full of transvections* if $n \geq 2$ and for each hyperplane H of V and each line $L \subseteq H$, there is at least one transvection σ in G with $R=L$ and $P=H$.

Similarly, a subgroup Δ of $P\Gamma L_n(V)$ is said to be *full of projective transvections* if $n \geq 2$ and for each hyperplane H of V and each line $L \subseteq H$, there is at least one projective transvection σ in Δ with $R=L$ and $P=H$.

Example. Clearly, $SL_n(V)$ is full of transvections, and $PSL_n(V)$ is full of projective transvections. It is then evident that every which contains $SL_n(V)$ (resp. $PSL_n(V)$) is full of transvection (resp. of projective transvections.)

From now on G will denote a subgroup of $\Gamma L_n(V)$ *that is full of transvections, and* Δ *will denote a subgroup of* $P\Gamma L_n(V)$ *that is full of projective transvections. And* G_1 *and* Δ_1 *will denote similar groups in the* V_1, n_1, F_1 *situation.* Λ *will denote a group isomorphism* $\Lambda: \Delta \rightarrow \Delta_1$.

Λ preserves the projective transvection σ in Δ if $\Lambda\sigma$ is a projective transvection in Δ_1, it preserves the projective transvection σ_1 in Δ_1 if $\Lambda^{-1}\sigma$ is a projective transvection in Δ, and it preserves projective transvections if it preserves all projective transvections in Δ and Δ_1.

3.2.1. *The groups* $\breve{G}$ *and* $\breve{\Delta}$ *are also full, or, in other words, the fullness is preserved under the contragradient isomorphism.*

PROOF. Recall from Section 2.3. that if K is a subspace of V, then K^0 is the subspace of all linear functionals of V' that send all vectors of K to zero. Let us take a line L and a hyperplane H of V with $L \subseteq H$. As G is full of transvection then there exists at least one transvection $\sigma \in$ G with $R=L$ and $P=H$. According to the results in the section 2.3, the subspaces of the transformation $\breve{\sigma} \in \breve{G}$ are then L^0 and H^0 and $H^0 \subseteq L^0$ and hence $\breve{G}$ is also full. □

3.2.2. *Suppose that* $n \geq 3$. *The commutator subgroup DG of G* (*resp.* Δ) *is also full.*

PROOF. Let us first demonstrate that the fact that DG is full implies that $D\Delta$ is full. Really, consider the preimage $P^{-1}\Delta$ of Δ in $GL_n(V)$. Evidently, $P^{-1}\Delta$ is full, and hence $DP^{-1}\Delta$ is full, whence we get that $D\Delta = PDP^{-1}\Delta$ is full.
Now let L be a line and H a hyperplane of V with $L \subseteq H$. We have to find a transvection $\tau \in DG$, whose subspaces are L and H. There is a base $x_1, \dots ,x_{n-1}, x_n$ of V such that

$$L = \langle x_1 \rangle \text{ and } H = \sigma\langle x_1, \dots ,x_{n-1}\rangle. \tag{3.26}$$

Let $\rho_1, \dots ,\rho_n$ be the dual base of V' of the base $x_1, \dots ,x_n$. Since G is full there is a transvection τ_1 with the subspaces $\sigma\langle x_1\rangle \subseteq \ker(\rho_2)$ and a transvection τ_2 with the subspaces $\langle x_2\rangle \subseteq \ker(\rho_n)$. Then for suitable non−zero scalars α, β of F we have

$$\tau_1 = \tau_{x_1,\alpha\rho_2} \text{ and } \tau_2 = \tau_{x_2,\beta\rho_n}. \tag{3.27}$$

We claim that

$$\sigma = [\tau_1, \tau_2] = \tau_{x_1, \alpha\beta\rho_n}. \tag{3.28}$$

It implies that $R(\sigma) = \langle x_1 \rangle = L$ and $P(\sigma) = \ker(\rho_n) = H$, as desired.

We have

$$\sigma = \tau_1 \tau_2 \tau_1^{-1} \tau_2^{-1}. \tag{3.29}$$

By the results of Section 1.3

$$\tau_1 \ \tau_2 \ \tau_1^{-1} = \tau_1 \tau_{x_2, \beta\rho_n} \ \tau_1^{-1} = \tau \ \tau_{\tau_1(x_2), \beta\rho_n \tau_1^{-1}} \tag{3.30}$$

Now

$$\tau_1(x_2) = \tau_{x_1, \alpha\rho_2}(x_2) = x_2 + \alpha\rho_2(x_2)x_1 = x_2 + \alpha\, x_1 \tag{3.31}$$

and for all $x \in V$

$$\beta\, \rho_n \tau_1^{-1}(x) = \beta\rho_n \ \tau_{-x_1, \alpha\rho_2}(x_2) = \beta\rho_n(x - \alpha\rho_2(x)x_1). \tag{3.32}$$

Since $\rho_n(x_1) = 0$, we arrive at the equation

$$\beta\rho_n \tau_1^{-1}(x) = \beta\rho_n(x) \qquad \forall\, x \in V. \tag{3.33}$$

Therefore

$$\sigma = \tau_1 \tau_2 \tau_1^{-1} \tau_2^{-1} = \tau_{x_2 + \alpha x_1, \beta\rho_n} \ \tau_{-x_2, \beta\rho_n} = \tau_{x_2 + \alpha x_1 - x_2, \beta\rho_n} = \tau_{\alpha x_1, \beta\rho_n} = \tau_{x_1, \alpha\beta\rho_n} \tag{3.34}$$

(we used formulas from Section 1.3; see 1.3.3). □

3.2.3 *Let* R_0 *and* P_0 *be any two subspaces of* V *with* $\dim R_0 + \dim P_0 = n$. *If* R_0 *is a line, assume in addition that* $R_0 \subseteq P_0$. *Then there is a product* σ *of* $\dim R_0$ *transvections in* G *such that* $R = R_0$ *and* $P = P_0$.

PROOF. If R_0 is 0 or is a line then the result is obvious.

Now let R_0 be a plane. We can always choose lines L_1 , L_2 and hyperplanes H_1 , H_2 such that

$$L_1 \subseteq H_1, \qquad L_2 \subseteq H_2, \tag{3.35}$$

$$R_0 = L_1 + L_2, \qquad P_0 = H_1 \cap H_2. \tag{3.36}$$

Indeed, there are three cases to consider: when $R_0 \cap P_0$ is respectively, 0, a line and a plane.

a) If $R_0 \cap P_0 = 0$, then L_1 and L_2 can be taken as the spans of the vectors of some base of R_0 and

$$H_1 = L_1 \oplus P_0 \text{ and } H_2 = L_2 \oplus P_0. \tag{3.37}$$

b) If $R_0 \cap P_0$ is a line, then the subspace $R_0 + P_0$ is of codimension one. Hence there is a line L with

$$V = L \oplus (R_0 + P_0). \tag{3.38}$$

Take any vector $a \in R_0 \setminus P_0$. Then our choice of L_1, L_2, H_1, H_2 is as follows:

$$L_1 = \langle a \rangle, \qquad H_1 = R_0 + P_0, \tag{3.39}$$

$$L_2 = R_0 \cap P_0, \qquad H_2 = L \oplus P_0. \tag{3.40}$$

c) The last case is the case when $R_0 \subseteq P_0$. Then we take two distinct lines L_1, L_2 of R_0 with $R_0 = L_1 \oplus L_2$ and take some subspaces K_1, K_2 with

$$V = K_1 \oplus P_0 \text{ and } P_0 = R_0 \oplus K_2. \tag{3.41}$$

Now

$$H_1 = K_1 \oplus L_1 \oplus K_2 \text{ and } H_2 = K_1 \oplus L_2 \oplus K_2. \tag{3.42}$$

Since G is full of transvections, we can pick transvections σ_1 and σ_2 in G with R_1=L_1, P_1=H_1, R_2=L_2, P_2=H_2. Put σ=$\sigma_1\sigma_2$. According to 1.2.2 we have the following implications:

(1) $V = P_1+P_2 \Rightarrow R = R_1 + R_2$, (3.43)

(2) $R_1 \cap R_2 = 0 \Rightarrow P = P_1 \cap P_2$, (3.44)

Thus the transformation σ does the job.

Now we apply induction on dim R_0, the cases dim $R_0 \leq 2$ being considered as the induction base. Suppose that dim $R_0 \geq 3$. Then dim $P_0 \leq n-3$.

Now let P_1 be a hyperplane that contains P_0. If so, for some line L we have

$$V = L \oplus P_1. \tag{3.45}$$

Put $P_2 = L \oplus P_0$. In particular, $V = P_1 + P_2$ (once again, we are going to apply 1.2.2.)

The intersection $R_0 \cap \mathrm{P}_1$ is of dimension at least 2; it is then possible to pick up a line R_1 in this intersection. Choose then a subspace R_2 of R_0 which is a direct complement of R_1 to R_0 :

$$R = R_1 \oplus R_2 \quad (\Rightarrow R_1 \cap R_2 = 0). \tag{3.46}$$

Being full of transvections, G has a transvection σ_1 with the subspaces $R_1 \subseteq P_1$. By the induction hypothesis, a transformation σ_2, product of at most $\leq$ dim R_2 transvections has the subspaces R_2, P_2. Application of 1.2.2 completes the proof. □

As a corollary we get the following fact.

3.2.4. *Let $n \geq 2$. Then DG contains an element σ with $R(\sigma) = V$.*

PROOF. If $n \geq 3$ we use 3.2.2 and 3.2.3. For the case when $n = 2$ we apply 3.1.2. □

3.2.5. *Suppose* $n \geq 2$. *Then*
(1) *The centralizer of G in* ΓL_n *is contained in* RL_n, *in particular*

$$G \cap \text{cen } \Gamma L_n \subseteq \text{cen } G \subseteq \text{cen } GL_n. \tag{3.47}$$

(2) *The centralizer of* Δ *in* $P\Gamma L_n$ *is trivial. In particular,* Δ *is centerless.*

PROOF. (1) As G is full, then for every line L there is a transvection τ with $R(\tau) = L$. A transformation of $GL_n(V)$ which commutes with τ must stabilize $R(\tau)$. So any element of the centralizer of G stabilizes all lines of V, and hence it is a radiation. This proves (1).

(2) is an immediate consequence of (1). □

3.2.6. *Suppose* $n \geq 2$. *Then* $C_V(G) = RL_n(V)$, *and* $\text{cen } G = G \cap RL_n(V)$.

PROOF. It is merely a reformulation of the previous result. □

3.2.7. *Let* $n \geq 3$ *and* $F \neq \mathbf{F}_2$. *Then for each hyperplane H in V and each line* $L \subseteq H$ *there are at least two distinct transvections in G, and at least two distinct projective transvections in* Δ*, with residual line L and fixed hyperplane H.*

PROOF. Let $\rho \in V'$ describe H, that is $H = \ker \rho$. Pick a line L of H; then since G is full, there is an element a, a base vector of L such $\tau_{a,\rho}$ is in G. Also an element $b \in H$ with $b \notin \langle a \rangle$ can be found such that $\tau_{b,\rho} \in$ G. The product of the tranvections we have found, the transvection $\tau_{a+b,\rho}$ is also in G.

Since $F \neq \mathbf{F}_2$ there is a line $\langle \lambda a + \mu b \rangle$ in the plane $\langle a, b \rangle$ that is distinct from the three lines $\langle a \rangle$, $\langle b \rangle$, $\langle a+b \rangle$, and is such that $\tau_{\lambda a+\mu b,\rho} \in G$.

a) If $\mu = 1$, then $\lambda \neq 0, 1$; hence

$$\tau_{(1-\lambda)a,\rho} = \tau_{a,\rho}\,\tau^{-1}{}_{\lambda a+\mu b,\,\rho}\,\tau_{b,\rho} \in G \tag{3.48}$$

with $1-\lambda \neq 0,1$.

b) So let $\mu \neq 1$ and let σ be a transvection G whose residual line is L and whose fixed hyperplane contains L but not b. Since σ is a transvection whose line is the linear span of a, then

$$\sigma b = b + v\,a, \tag{3.49}$$

where $v \neq 0$, since $b \notin P(\sigma)$. Now for any X in V, the vector $\sigma x - x$, an element of $R(\sigma) = L$ is in H and hence

$$\rho(\sigma x - x) = 0 \Rightarrow \rho(\sigma x) = \rho\, x. \tag{3.50}$$

Thus $\rho\sigma = \rho$, whence $\rho\sigma^{-1} = \rho$. Therefore

$$\tau_{va,\rho} = \tau_{\sigma b-b,\,\rho} = \sigma\,\tau_{b,\rho}\,\sigma^{-1}\tau^{-1}{}_{b,\rho} \in G \tag{3.51}$$

(recall that $\pi\,\tau_{c,\psi}\,\pi^{-1} = \tau_{\pi c,\psi\pi^{-1}}$, for all $c \in V$ and $\psi \in V'$; see Section 1.3).

Similarly,

$$\begin{aligned}\tau_{\mu va,\,\rho} = \tau_{\mu(\sigma b-b),\rho} &= \tau_{\sigma(\lambda a+\mu b),\rho}\,\tau^{-1}{}_{\lambda a+\mu b,\rho}\\ &= \sigma\,\tau_{\lambda a+\mu b,\rho}\,\sigma^{-1}\tau^{-1}{}_{\lambda a+\mu b,\rho}\,;\end{aligned} \tag{3.52}$$

since all the transformations σ, $\tau_{\lambda a+\mu b,\rho}, \tau^{-1}{}_{\lambda a+\mu b,\rho}$ are in G, we obtain that $\tau_{\mu va,\rho}$ is also in G. As $\mu \neq 1$, $\tau_{va,\rho}$ and $\tau_{\mu va,\rho}$ are distinct transvections in G with spaces $L \subseteq H$. Their projective images are distinct, too; this proves the result for Δ. □

3.2.8. If $G \subseteq \mathrm{GL}_n$, then

(1) $C_V(DG) = \mathrm{RL}_n(V)$ if $n \geq 3$,

(2) $C_V(DG) = \mathrm{RL}_n(V)$ if $n \geq 2$ and G contains at least two distinct transvections having the same residual line,

(3) the only unipotent transformation in $C_V(DG)$ is 1_V

PROOF. (1) If $n \geq 3$ we know that by 3.2.2 and 3.2.6

(2) DG is not abelian by 3.1.1. Then for any $\sigma \in C_V(DG)$,

$$C_V(\sigma) \supseteq C_V(C_V(DG)) = C_V C_V(DG) \supseteq DG, \tag{3.53}$$

so $C_V(\sigma)$ is not abelian, so $DC_V(\sigma) \neq 1_V$, so $\sigma \in \mathrm{RL}_2$ by 3.1.1. Then $C_V(DG) = \mathrm{RL}_2$.

(3) If $C_V(DG) = \mathrm{RL}_n$ the result holds. We need therefore just consider the situation $n = 2$. Here a typical unipotent σ in $C_V(DG)$ is a transvection, say with the residual line L. Let K be a line in V, distinct from L. Transvections τ_L and τ_K in G with residual lines L and K. Then σigm$a$ permutes with $\tau_L \tau_K \tau_L^{-1} \tau_K^{-1}$, it permutes with τ_L, so it permutes with the transvection $\tau_K \tau_L^{-1} \tau_K^{-1}$ of the residual line $\tau_K L$. So σ fixes each of the lines L and $\tau_K L$. But these lines are distinct since $L \neq K$. Hence $\sigma = 1$ since σ is unipotent. □

3.3. CDC in the Linear Case

If just Δ is given and we define G to be the set of representatives of the elements of Δ in $\Gamma \mathrm{L}_n(V)$ (If we define G to be the inverse image of Δ under the homomorphism $(P|\Gamma \mathrm{L}_n)$), then G is a subgroup of $\Gamma \mathrm{L}_n(V)$ that is full of transvections, and the theory in the previous section (3.2) will apply to it. If Δ satisfies $\Delta \subseteq \mathrm{PGL}_n(V)$, the G just constructed will satisfy $G \subseteq \mathrm{GL}_n(V)$. Throughout this and the next chapter we assume that our G and Δ have these additional properties:

$$\Delta \subseteq \mathrm{PGL}_n, \qquad G = P^{-1}\Delta \cap \Gamma \mathrm{L}_n, \qquad G \subseteq \mathrm{GL}_n \tag{3.54}$$

and $\breve{G}$ and $\breve{\Delta}$ are related in the same way over V'

$$\breve{\Delta} \subseteq \mathrm{PGL}_n \qquad \breve{G} = P^{-1}\breve{\Delta} \cap \Gamma \mathrm{L}_n, \qquad \breve{G} \subseteq \mathrm{GL}_n \tag{3.55}$$

C denotes the centralizer C_Δ, C_g, $C_{\breve{\Delta}}$, $C_{\breve{G}}$ when we are working respectively in Δ, G, $\breve{\Delta}$, $\breve{G}$.

3.3.1. *For any element σ of G ,*

$$\overline{C(\sigma)} \subseteq C(\overline{\sigma}), \qquad \overline{DC(\sigma)} \subseteq DC(\overline{\sigma}). \tag{3.56}$$

If σ has the additional property that it permutes with an element of G whenever it permutes projectively with it, then

$$\overline{C(\sigma)} = C(\overline{\sigma}), \qquad \overline{DC(\sigma)} = DC(\overline{\sigma}). \tag{3.57}$$

For any two subspaces U and W of V we define

$$G(U, W) = \{\sigma\in G \mid R\subseteq U, P\supseteq W\}, \quad \Delta(U, W) = \overline{G(U,W)}. \tag{3.58}$$

By 1.2.1. both $G(U, W)$ and $\Delta(U, W)$ are subgroups of G and Δ respectively; and $\Delta(U, W)$ consists of those Σ in Δ which have at least one representative σ with $R\subseteq U$ and $P \supseteq W$. Note that

$$\sigma U = U, \qquad \sigma W = W \qquad \forall\, \sigma \in G(U, W) \tag{3.59}$$

and

$$\Sigma U = U, \qquad \Sigma W = W \qquad \forall\, \Sigma \in \Delta(U, W). \tag{3.60}$$

Example. If H is a hyperplane of V and L is a line with $L\subseteq H$, then $G(L, H)$ is the group consisting of all transvections in G with residual line L and fixed hyperplane H, plus 1_V; while $\Delta(L, H)$ is the group consisting of all projective transvections in Δ with residual line L and fixed hyperplane H, plus 1.

3.3.2. *If U and W are subspaces of V, then*

$$\breve{G(U,W)} = \breve{G}(W^0,U^0), \qquad \breve{\Delta(U,W)} = \breve{\Delta}(W^0,U^0). \tag{3.61}$$

3.3.3. *Let σ_1 and σ_2 be nontrivial transvections in G. Then the following statements are equivalent*:

(1) $R_1=R_2$ *and* $P_1=P_2$.

(2) $C(\sigma_1) = C(\sigma_2)$.

(3) $C(\overline{\sigma}_1) = C(\overline{\sigma}_2)$.

PROOF. (1) $\Rightarrow$ (2). Here $R_1=R_2$ and $P_1=P_2$. Consider $\sigma \in C(\sigma_1)$. Write $\sigma_1=\tau_{a,\rho}$ and $\sigma_2=\tau_{\alpha a,\rho}$ in the usual way. We have

$$\tau_{a,\rho}= \Sigma\,\tau_{a,\rho}\,\Sigma^{-1}=\tau_{\Sigma a,\rho\Sigma^{-1}} \tag{3.62}$$

and so $\Sigma a=\lambda a$, $\rho\Sigma^{-1}=\lambda^{-1}\rho$ for some λ in $\dot{F}$, whence

$$\Sigma\ \tau_{\alpha a,\rho}\,\Sigma^{-1} = \tau_{\alpha\Sigma a,\rho\Sigma^{-1}} = \tau_{\alpha\lambda a,\lambda^{-1}\rho}=\tau_{\alpha a,\rho} \tag{3.63}$$

so $\sigma\in C(\sigma_2)$. Hence $C(\sigma_1) = C(\sigma_2)$. (2) $\Rightarrow$ (3). Just an application of 3.3.1. (3) $\Rightarrow$ (1). Suppose if possible that $P_1\neq P_2$. Then there is a line L with $L\subseteq P_2$ but $L\not\subset P_1$. Since G is full of transvections there is a transvection σ_3 with $R_3=L$ and $P_3=P_2$. Then $\sigma_3\in C(\sigma_2)$ and $\sigma_3\notin C(\sigma_1)$ by 1.3.10, hence $\overline{\sigma}_3\in C(\overline{\sigma}_2)$ and $\overline{\sigma}_3\notin C(\overline{\sigma}_1)$ by 3.1.4., hence $C(\overline{\sigma}_2) \neq C(\overline{\sigma}_1)$, contrary to hypothesis. So $P_1= P_2$. And $R_1 = R_2$ follows by applying $\breve{}$. □

3.3.4. *If $n \geq 3$ and σ is a nontrivial transvection in G*, *then*

$$\mathrm{G}(R, P) \cap DC(\sigma) \neq 1_V, \tag{3.64}$$

$DC(\sigma)$ contains a nontrivial transvection with the same spaces as the given transvection σ

PROOF. There is a base $x_1,\dots, x_n$ for V with dual base $\rho_1,\dots, \rho_n$ such that $\sigma=\tau_{x_1,\rho_n}$. So $R=Fx_1$ and $P=\ker\rho_n$. Since G is full of transvections we have α,β in $\dot{F}$ such that $\tau_{x_1,\alpha\rho_2}\in G$ and $\tau_{x_2,\beta\rho_n}\in G$. Then $\tau_{x_1,\alpha\rho_2}\in C(\sigma)$ and $\tau_{x_2,\beta\rho_n}\in C(\sigma)$. Put $\Sigma = \tau_{x_1,\alpha\beta\rho_n}$. So Σ is a transvection in G with residual line R and fixed hyperplane P . But

$$\Sigma = [\tau_{x_1,\alpha\rho_2}, \tau_{x_2,\beta\rho_n}] \tag{3.65}$$

So $\Sigma \in DC(\sigma)$. □

3.3.5. *If* $n \geq 3$ *and H is a hyperplane of V and L is a line in H, then there is a nontrivial transvection* τ *in G with spaces* $L \subseteq H$ *such that* $\tau \in DC(\tau)$.

PROOF. Let σ be a transvection in G with spaces $L \subseteq H$. By previous proposition we have a transvection τ in G with spaces $L \subseteq H$ such that $\tau \in DC(\sigma)$. But $C(\tau) = C(\sigma)$ by 3.3.3. □

3.3.6. *If* $n \geq 4$ *and* σ *is a nontrivial transvection in G, then*

$$G(L, P) \cap DC(\sigma) \neq 1_V, \tag{3.66}$$

for all lines L in P .

PROOF. Fix a line K in P with $K \not\subset R+L$. Let M be a hyperplane of V containing $R+L$ but not K. Let τ_L be a transvection in G with spaces $L \subseteq M$, let τ_K be a transvection in G with spaces $K \subseteq P$. By 1.3.10, τ_L and τ_K are in $C(\sigma)$. Put $\Sigma = \tau_L \tau_K \tau^{-1}{}_L \tau^{-1}{}_K$. Clearly $\Sigma \in DC(\sigma)$. Now $\tau_L K \neq K$ since $K \not\subset M$, hence $\tau_L \tau_K \tau^{-1}{}_L$ is a transvection with residual line $\tau_L K$ distinct from K, and with fixed hyperplane $\tau_L P = P$, hence $\Sigma = (\tau_L \tau_K \tau^{-1}{}_L)\tau^{-1}{}_K$ is a nontrivial transvection with fixed hyperplane P. Similarly $\Sigma = \tau_L(\tau_K \tau^{-1}{}_L \tau^{-1}{}_K)$ has residual line L. Hence Σ is a nontrivial element of $G(L, P) \cap DC(\sigma)$. □

3.3.7. *Let* σ *be a nontrivial transvection in G. Then*

$$CC(\overline{\sigma}) = \Delta(R, P) \quad \textit{if} \quad n \geq 2, \tag{3.67}$$

$$CDC(\overline{\sigma}) = \Delta(R, P) \quad \textit{if} \quad n \geq 4, \tag{3.68}$$

PROOF. (1) If $\bar{\Sigma}$ is a typical nontrivial element of Δ(*R, P*), then $C(\bar{\Sigma}) = C(\bar{\sigma})$ by 3.3.3. Hence

$$\bar{\Sigma} \in CC(\bar{\Sigma}) = CC(\bar{\sigma}); \tag{3.69}$$

hence $\Delta(R, P) \subseteq CC(\bar{\sigma})$.

(1*a*) First let $n \geq 3$. Consider $\Sigma \in G$ with $\bar{\Sigma} \in CC(\bar{\sigma})$. Then for each line $L \subseteq P$ there is a projective transvection in Δ with residual line *L* and fixed hyperplane *P*; this projective transvections is in $C(\bar{\sigma})$ by from O'Meara §1.6; hence $\bar{\Sigma}$ permutes with it; hence Σ permutes with its representative transvection; hence $\Sigma L = L$ for all *L* in *P*; hence there is an α in $\dot{F}$ such that the fixed space of $\alpha\Sigma$ contains *P*. Applying this result to $\breve{\Sigma}$ and $\breve{\sigma}$ gives us $\beta \in \dot{F}$ such that the residual space of $\beta\Sigma$ is contained in *R*. It is easily seen that $\alpha = \beta$. Then $\alpha\Sigma \in G(R, P)$. Hence $\bar{\Sigma} \in \Delta(R, P)$. Hence $CC(\bar{\sigma}) = \Delta(R, P)$ is established for $n \geq 3$.

(1*b*) Now *n*=2. Consider $\Sigma \in G$ with $\bar{\Sigma} \in CC(\bar{\sigma})$. Then $\bar{\sigma} \in C(\bar{\sigma})$, so $\bar{\Sigma} \in C(\bar{\sigma})$, so $\Sigma \in C(\sigma)$ by 3.1.4. If we take a base for *V* in which σ has matrix $\begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix}$ we find that the matrix of Σ in the base will have to be of the form $\begin{pmatrix} p & q \\ & p \end{pmatrix}$ ($\bar{\Sigma}$ will have a representative whose matrix has the form $\begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix}$, i.e. $\bar{\Sigma} \in \Delta(R, P)$.) Therefore $CC(\bar{\sigma}) = \Delta(R, P)$ when $n = 2$.

(2) Now let $n \geq 4$. Clearly $\Delta(R, P) = CC(\bar{\sigma}) \subseteq \mathrm{CDC}(\bar{\sigma})$. To show the reverse inclusion we shall use 3.3.6 and (1*a*).

3.3.8. *Assume* $n \geq 3$. *Let σ be an element of G with* res$\sigma = 2$ *and suppose that* $(\sigma|R)$ *is not a radiation. Exclude the situation where* $n = 3$, det$\sigma = 1$, *σ is diagonalizable over F*, $\sigma^3 = 1$. *Then* $\Delta(R, P) \subseteq \mathrm{CDC}(\bar{\sigma})$.

PROOF. (1) First suppose $DC(\sigma) \subseteq G(P, R)$. Consider typical $\bar{\Sigma}$ in $DC(\bar{\sigma})$. Then $DC(\bar{\sigma}) = \overline{DC(\sigma)}$ by 3.1.4, 3.1.6, 3.3.1; so $\bar{\Sigma}$ has a representative Σ in *DC*(σ). Then $\Sigma \subseteq G(P, R)$ by our assumption. So, for each $\phi \in G(R, P)$,

$$R_\Sigma \subseteq P \subseteq P\phi, \qquad P_\Sigma \supseteq R \supseteq R\phi, \tag{3.70}$$

each element ϕ of $G(R, P)$ permutes with Σ, i.e. each element $\bar{\phi}$ of $\Delta(R, P)$ permutes with each element $\bar{\Sigma}$ of $DC(\bar{\phi})$ $\{\bar{\phi} \in \mathrm{CDC}(\bar{\sigma})\}$ therefore $\Delta(R, P) \subseteq \mathrm{CDC}(\bar{\sigma})$.

(2) If σ satisfies the given conditions with G, then $\breve{\sigma}$ will satisfy the given conditions with $\breve{G}$ (to prove this consider the possibility that $(\breve{\sigma}|P^0)$ is first a nontrivial radiation and then a trivial one). If we can prove that $\sigma_3 \in DC(\sigma) \Rightarrow R \subseteq P_3$, we will have $P \supseteq R_3$ by duality, hence $DC(\sigma) \subseteq G(P, R)$ and we will be through. Consider the implication, then $(\sigma|R) \in \mathrm{GL}_2(R) - \mathrm{RL}_2(R)$ by hypothesis; hence $DC_R(\sigma|R) = 1_R$ by 3.1.1. But it is easily verified that

$$DC(\sigma)|R \subseteq DC_R(\sigma|R). \tag{3.71}$$

Hence $(\sigma_3|R) = 1_R$. Hence $R \subseteq P_3$, as required. □

3.3.9. *Assume* $n \geq 4$. *Let* σ *be an element of* G *with* $\mathrm{res}\sigma = 2$ *and suppose that* $R \cap P = 0$ *with* σ *not a big dilation. Exclude the situation* $n = 4$ *with* $F = \mathbf{F}_2$. *Then* $\Delta(R, P) = \mathrm{CDC}(\bar{\sigma})$.

PROOF. By 3.3.8 we have to show that $\mathrm{CDC}(\bar{\sigma}) \subseteq \Delta(R, P)$.

(1) For each hyperplane H of P and each line L in H fix a transvection $\tau_{L,H}$ in G with residual line L and fixed hyperplane $R + H$. (When $n = 4$ we have $F \neq \mathbf{F}_2$ so by 3.2.7 we can, and do, fix two distinct $\tau_{L,H}$ and $\tau'_{L,H}$ for each such L and H.) Clearly $\tau_{L,H}$ stabilizes both R and P, and $(\tau_{L,H}|P)$ is a transvection with spaces $L \subseteq H$, and $(\tau_{L,H}|R) = 1_R$. (Similarly with $\tau'_{L,H}$.) Let G_p denote the subgroup of $\mathrm{GL}_{n-2}(P)$ that is generated by all the $(\tau_{L,H}|P)$ and $(\tau'_{L,H}|P)$. It is obvious that G_p is full of transvections (doubly full when necessary), and

$$1_R \oplus G_P \subseteq C(\sigma), \qquad 1_R \oplus DG_p = D(1_R \oplus G_P) \subseteq DC(\sigma), \quad C_P(DG_P) = \mathrm{RL}_{n-2}(P), \tag{3.72}$$

the last equation being a consequence of 3.2.8.

(2) First assume $n \geq 5$. Consider a typical $\overline{\Sigma}$ in CDC($\overline{\sigma}$) and let Σ be one of its representatives. Then Σ permutes projectively with each element of $1_R \oplus DG_P$. Now for each line L in P there is a transvection in DG_P with residual line L since DG_P is full of transvections by 3.2.2; hence $1_R \oplus DG_P$ contains a transvection with residual line L; but Σ will then permute with it by 3.1.4; hence $\Sigma L = L$ for all L in P . In particular, $\Sigma P = P$ and $(\Sigma|P) \in \mathrm{RL}_{n-2}(P)$. By duality, $\Sigma R = R$. Hence $\overline{\Sigma} \in \Delta(R, P)$.

(3) Now $n = 4$. Here our $\tau_{L,H}$'s can be written τ_L. Again consider typical $\overline{\Sigma}$ in CDC($\overline{\sigma}$) and let Σ be one of its representatives. Let L and K be any two distinct lines in P . Then

$$(\tau_L \tau_K \tau^{-1}{}_L \tau^{-1}{}_K)|P = (\tau_L|P)(\tau_K|P)(\tau_L|P)^{-1}(\tau_K|P)^{-1} \tag{3.73}$$

is an element of residual index 2 in $\mathrm{GL}_2(P) - \mathrm{RL}_2(P)$, by 3.1.2. Hence $\tau_L \tau_K \tau^{-1}{}_L \tau^{-1}{}_K$ has residual space P and fixed space R, it is not a big dilation, and it belongs to $DC(\sigma)$. Hence Σ permutes projectively, indeed permutes, with $\tau_L \tau_K \tau^{-1}{}_L \tau^{-1}{}_K$ and so $\Sigma R = R$ and $\Sigma P = P$. Now Σ permutes projectively with all elements of

$$1_R \oplus DG_P \subseteq DC(\sigma); \tag{3.74}$$

hence Σ permutes with all elements of $1_R \oplus DG_P$ which are not big dilations of residue 2 by 3.1.4; but it obviously permutes with all big dilations of residue 2 in this group; hence Σ permutes with all elements of $1_R \oplus DG_P$, so

$$(\sigma\,|P) \in C_P(DG_P) = \mathrm{RL}_2(P) \tag{3.75}$$

hence $\overline{\Sigma} \in \Delta(R, P)$. □

3.3.10. *Assume* $n \geq 4$. *Let* σ *be an element of* G *with* $\mathrm{res}\,\sigma = 2$ *and suppose that* $R \cap P = 0$ *with* σ *not a big dilation. Then every projective unipotent transformation in* CDC($\overline{\sigma}$) *is a projective transvection in* $\Delta(R, P)$.

PROOF. If $n \geq 5$, or if $n = 4$ with $F \neq \mathbf{F}_2$, apply the result of 3.3.8. If $n = 4$ and $F = \mathbf{F}_2$, proceed as in the proof of 3.3.8, using the third part of 3.2.8. □

3.3.11. *Assume* $n \geq 4$. *Let* σ *be an element of* G *with* $\operatorname{res}\sigma = 2$ *and suppose that* $R \cap P = 0$ *with* σ *not a big dilation. Then* $\overline{\sigma} \notin DC(\overline{\sigma})$.

3.3.12. *If* Σ *is an element of* G *such that* $\Sigma \in DC(\sigma)$, *then* $\Sigma^{n!}$ *is unipotent*.

PROOF. A moment's reflection will show that there is no loss of generality in assuming that F is algebraically closed and $G = \mathrm{GL}_n(V)$. Let us make these assumptions.

Let $\alpha, \beta, \ldots$ be the distinct characteristic roots of Σ. The Jordan canonical form of Σ then provides parallel splittings

$$V = V_\alpha \oplus V_\beta \oplus \ldots, \qquad \Sigma = \Sigma_\alpha \oplus \Sigma_\beta \oplus \ldots, \tag{3.76}$$

such that all the roots of Σ_α are α, those of Σ_β are β, etc. Note that $\det \Sigma_\alpha = \alpha^{n_\alpha}$ where $n_\alpha = \dim V_\alpha$, etc. Now $V_\alpha = \{x \in V \mid (\Sigma - \alpha\, 1_V)^k\, x = 0 \text{ for some } k > 0\}$, etc., and from this it follows that $\mathbf{T} \in C(\Sigma) \Rightarrow \mathbf{T}V_\alpha = V_\alpha$, etc. Hence any Ψ in $DC(\Sigma)$ will have the form

$$\Psi = \Psi_\alpha \oplus \Psi_\beta \oplus \ldots \tag{3.77}$$

with $\Psi_\alpha \in \mathrm{SL}_{n_\alpha}(V_\alpha)$ etc. In particular, $\Sigma_\alpha \in \mathrm{SL}_{n_\alpha}(V_\alpha)$, so $\alpha^{n_\alpha} = 1$, so $\alpha^{n!} = 1$, etc. Hence the characteristic roots of $\Sigma^{n!}$ are all 1. □

3.3.13. Assume $n = 3$. Let σ be an element of DG with $\overline{\sigma} \in DC(\overline{\sigma})$. Then

(1) σ is a transvection if it is unipotent.

(2) σ^{18} is a transvection.

(3) σ^2 is a transvection if char $F = 3$.

(4) σ^9 is a transvection if char $F = 2$.

PROOF. We can assume that $\sigma \neq 1_V$.

(1) Here σ is unipotent, so resσ is 1 or 2, say. Assume if possible that resσ = 2. We have $\sigma R = R$ with $(\sigma|R) \in \mathrm{GL}_2(R) - \mathrm{RL}_2(R)$, again since σ is unipotent. Now each element of $C(\sigma)$ acts on R , so it follows from 3.1.1 that each element of $DC(\sigma)$ has action 1_R on R. On the other hand $\overline{\sigma} \in DC(\overline{\sigma}) = \overline{DC(\sigma)}$ by 3.1.4 and 3.3.1. So $\alpha\sigma$ has action 1_R on R for some αa, but σ is unipotent, so $\alpha = 1$ and σ has action 1_R on R, contradicting the fact that resσ = 2. Hence resσ is indeed 1. But detσ = 1 since $\sigma \in DG$. So σ is a transvection as asserted.

(2) By 3.1.5 we have $C(\overline{\sigma}) \subseteq \overline{C(\sigma^3)}$, so $\sigma \in DC(\overline{\sigma}) \subseteq \overline{DC(\sigma^3)}$, so $\alpha\sigma \in DC(\sigma^3)$, but $\alpha^3=1$ by determinants, so $\sigma^3 \in DC(\sigma^3)$ so σ^{18} is unipotent by 3.3.12. But $\overline{\sigma}^{18} \in DC(\overline{\sigma}) \subseteq DC(\overline{\sigma}^{18})$. Apply step (1).

(3) By step (2), $(\sigma^2)^{3^2}$ is a transvection; hence $(\sigma^2)^{3^3} = 1_V$ since the characteristic is 3; hence σ^2 is unipotent by 3.1.3. But $\overline{\sigma}^2 \in DC(\overline{\sigma}) \subseteq DC(\overline{\sigma}^2)$. Apply step (1).

(4) As in step (3) □

3.4. Preservation of Projective Transvections in the Linear Case

Recall that in §3.3 and 3.4 we are assuming that G and Δ have these additional properties:

$$\Delta \subseteq \mathrm{PGL}_n, \qquad G = P^{-1}\Delta \cap \Gamma\mathrm{L}_n, \qquad G \subseteq \mathrm{GL}_n. \tag{3.78}$$

In order to apply the results of §3.3 to the groups G_1 and Δ_1 as well as to the groups G and Δ we assume throughout §3.4 that G_1 and Δ_1 also have the additional properties:

$$\Delta_1 \subseteq \mathrm{PGL}_{n1}, \qquad G_1 = P^{-1}\Delta_1 \cap \Gamma\mathrm{L}_{n1}, \qquad G_1 \subseteq \mathrm{GL}_{n1}. \tag{3.79}$$

Our object in §3.4 is to show that, under these assumptions, any isomorphism $\Lambda: \Delta \to \Delta_1$ preserves projective transvections whenever the underlying dimensions are ≥ 3. We start by proving, so to speak, that Λ preserves residue 2 at least once.

3.4.1. *Let $n \geq 3$, $n_1 \geq 3$. Exclude the possibility of* char $F \neq 2$ *with* $F_1 = \mathbf{F}_2$. *Then there are elements $\sigma \in DG$, $\sigma_1 \in DG_1$ with* resσ = 2 = resσ_1 *such that $\Lambda\overline{\sigma} = \overline{\sigma}_1$. Indeed, given any transvection $\tau \in$* G *with spaces $L \subseteq H$, σ and σ_1 can be so chosen such that $\sigma\tau \neq \tau\sigma$ with*

$$L \subseteq R, \qquad H \supseteq P, \qquad (\sigma|R) \notin \mathrm{RL}_2(R) \tag{3.80}$$

and

$$R_1 \cap P_1 = 0, \qquad (\sigma_1|R_1) \notin \mathrm{RL}_2(R_1). \tag{3.81}$$

PROOF. (1) Write $\Lambda\bar{\tau} = \bar{\Phi}$ with $\Phi \in G_1$. Since $\bar{\Phi}$ is nontrivial, there will be a line $L_1 = F_1 a$ in V_1 with $\Phi L_1 \neq L_1$. Pick a hyperplane H_1 of V_1 such that $L_1 \subseteq H_1$, $\Phi L_1 \not\subset H_1$, $\Phi^{-1} L_1 \not\subset H_1$. We have ,

$$L_1 \not\subset \Phi H_1, \qquad H_1 \neq \Phi H_1 \tag{3.82}$$

and

$$\dim(L_1 + \Phi L_1) = 2, \qquad \dim(H_1 \cap \Phi H_1) = n_1 - 2 \tag{3.83}$$

and

$$V_1 = (L_1 + \Phi L_1) \oplus (H_1 \cap \Phi H_1). \tag{3.84}$$

(2) Fix $\rho \in V'_1$ with $\ker \rho = \mathrm{H}_1$. Of course there are several nonzero a in L_1 with $\tau_{a,\rho} \in G_1$ since G_1 is full of transvections. We claim there is at least one such a for which Φ will not permute projectively with $\tau_{a,\rho}\Phi^{-1}\tau^{-1}_{a,\rho}$. Suppose this does not hold for a first choice of a. Then there is a scalar α in F_1 such that

$$\Phi\tau_{a,\rho}\Phi^{-1}\tau^{-1}_{a,\rho} = \alpha\tau_{a,\rho}\Phi^{-1}\tau^{-1}_{a,\rho}\Phi, \tag{3.85}$$

i.e. $\tau_{-\Phi a,\rho\Phi^{-1}}\tau_{-a,\rho} = \alpha\tau_{a,\rho}\tau_{-\Phi^{-1}a,\rho\Phi}$. Hence

$$(\alpha - 1)x + ((\alpha + 1)(\rho x) - \alpha(\rho\Phi x)(\rho\Phi^{-1}a))a + ((\rho x)(\rho\Phi^{-1}a) - (\rho\Phi^{-1}x))\Phi a = \alpha(\rho\Phi x)\Phi^{-1}a \tag{3.86}$$

for all X in V_1. Putting $x = a$ shows that a, $\Phi a, \Phi^{-1}a$ are dependent, i.e. they all fall in a plane; taking x outside this plane shows that $\alpha = 1$; so

$$(2(\rho\, x) - (\rho\Phi\, x)(\rho\Phi^{-1}a))a + ((\rho x)(\rho\Phi^{-1}a) - (\rho\Phi^{-1}x))\Phi a = (\rho\Phi x)\Phi^{-1}a. \qquad (3.87)$$

If $F_1 \neq \mathbf{F}_2$ we can replace a by λa for some $\lambda \neq 0$, 1 by 3.2.7. Together these equations (for a and $\lambda\, a$) then yield

$$(\rho\Phi\, x)(\rho\Phi^{-1}a) = \lambda\,(\rho\Phi\, x)(\rho\Phi^{-1}a) \qquad (3.88)$$

which is absurd since $\lambda \neq 1$ and $\rho\Phi^{-1}a \neq 0$. Therefore when $F_1 \neq \mathbf{F}_2$, if a does not work, then λa will. On the other hand, when $F_1 = \mathbf{F}_2$ we have char $F = 2$ by hypothesis; so $\Phi^2 = 1_{V_1}$ since τ is a transvection and 1 is the only nonzero scalar in F_1; so our equality becomes

$$(\rho\Phi x)(\rho\Phi a)a + (\rho x)(\rho\Phi a)\Phi a = 0 \qquad (3.89)$$

and this contradicts the independence of a and $\Phi\, a$. Our claim is established.

(3) We now have $\rho \in V'_1$ with ker $\rho = H_1$ and a nonzero a in L_1 such that $\tau_{a,\rho} \in G_1$ with Φ not permuting projectively with $\tau_{a,\rho}\Phi^{-1}\tau^{-1}{}_{a,\rho}$. Choose $\psi \in G$ with $\Lambda\overline{\psi} = \overline{\tau}_{a,\rho}$ and define

$$\sigma = \tau\psi\,\tau^{-1}\psi^{-1} \in DG, \qquad \sigma_1 = \Phi\,\tau_{a,\rho}\Phi^{-1}\tau^{-1}{}_{a,\rho} \in DG_1 \qquad (3.90)$$

Clearly $\Lambda\overline{\sigma} = \overline{\sigma}_1$. And τ does not permute with $\psi\tau^{-1}\psi^{-1}$ so $\sigma\tau \neq \tau\sigma$

(4) As far as σ_1 is concerned, it is enough to verify that

$$R_1 = L_1 + \Phi L_1, \qquad P_1 = H_1 \cap \Phi H_1. \qquad (3.91)$$

For then res σ_1=2 and $R_1 \cap P_1$=0 are obviously true, while $(\sigma_1|R_1) \notin RL_2(R_1)$ is a consequence of the fact that a big dilation of residue 2 cannot be expressed as a product of two transvections, by O'Meara §2.1. Now σ_1 is a product of two transvections with spaces $\phi L_1 \subseteq \Phi H_1$ and $L_1 \subseteq H_1$; and $\Phi H_1 + H_1 = V_1$, so $R_1 = L_1 + \Phi L_1$ by 1.2.2; and $L_1 \cap \Phi L_1 = 0$, so $P_1 = H_1 \cap \Phi H_1$ again by 1.2.2.

(5) Now consider σ. Here it is enough to verify that

$$\psi L \neq L, \qquad R = L + \psi L, \qquad P = H \cap \psi H, \qquad R \not\subset P. \tag{3.92}$$

For then res$\sigma = 2$, $L \subseteq R$, $H \supseteq P$ are obvious, while $(\sigma|R) \notin \mathrm{RL}_2(R)$ follows from O'Meara §2.1 as above. Clearly $\psi L \neq L$, for otherwise the transvection τ would permute with the transvection $\psi \tau^{-1} \psi^{-1}$; ditto $\psi H \neq H$; so R and P have the desired form; and $R \not\subset P$ since otherwise $L + \psi L \subseteq H \cap \psi H$, i. e. $L \subseteq \psi H$ and $\psi L \subseteq H$, i.e. τ would permute with $\psi \tau^{-1} \psi^{-1}$ by 1.3.10.

Next we prove that Λ preserves residue 1, at least once. □

3.4.2. *If* $n \geq 3$, $n_1 \geq 3$ *there are elements* $\tau \in DG$, $\tau_1 \in DG_1$ *with* res$\tau = 1 =$ resτ_1 *such that* $\Lambda \bar{\tau} = \bar{\tau}_1$.

Proof. (1) Both dimensions ≥ 4. By interchanging V, n, F and V_1, n_1, F_1 and considering Λ^{-1} instead of Λ, if necessary, we can assume that char $F = 2$ if $F_1 = \mathbf{F}_2$. Consider an arbitrary nontrivial transvection $\mathbf{T}$ in G which satisfies $\mathbf{T} \in DC(\mathbf{T})$. Let $L \subseteq H$ denote the spaces of $\mathbf{T}$. By 3.4.1 we have $\sigma \in DG$, $\sigma_1 \in DG_1$ with res$\sigma = 2 =$ resσ_1 such that $\Lambda \bar{\sigma} = \bar{\sigma}_1$ with $\sigma \mathbf{T} \neq \mathbf{T} \sigma$ and

$$L \subseteq R, \qquad H \supseteq P, \qquad (\sigma|R) \notin \mathrm{RL}_2(R), \tag{3.93}$$

$$R_1 \cap P_1 = 0, \qquad (\sigma_1|R_1) \notin \mathrm{RL}_2(R_1). \tag{3.94}$$

It is clear that $\bar{T}$ is a nontrivial projective transvection in $\Delta(R, P)$; but $\Delta(R, P) \subseteq \mathrm{CDC}(\bar{\sigma})$, by 3.3.8; hence $\bar{T} \in \mathrm{CDC}(\bar{\sigma})$; but $\bar{\sigma}\,\bar{T} \neq \bar{T}\,\bar{\sigma}$ and $\bar{\sigma} \in \Delta(R, P) \subseteq \mathrm{CDC}(\bar{\sigma})$, so $\bar{T}$ is a noncentral element of $\mathrm{CDC}(\bar{\sigma})$ such that $\bar{T} \in DC(\bar{T})$. Therefore $\Lambda \bar{T}$ is a noncentral element of $\mathrm{CDC}(\bar{\sigma}_1)$ with $\Lambda \bar{T}$ in $DC(\Lambda \bar{T})$.

(la) First suppose that we do not have $F_1 = \mathbf{F}_2$ with $n_1 = 4$. Then by 3.3.9, $\Lambda\overline{T} \in \mathrm{CDC}(\overline{\sigma}_1) = \Lambda(R_1, P_1)$. So $\Lambda\overline{T}$ has a representative σ_3 in G_1 with $R_3 \subseteq R_1$ and $P_3 \supseteq P_1$. In fact $R_3 \subset R_1$; for if not, we would have $R_3 = R_1$ and $P_3 = P_1$; if σ_3 were not a big dilation, then $\overline{\sigma}_3 \notin DC(\overline{\sigma}_3)$ by 3.3.11, and this contradicts $\Lambda\overline{T} \in DC(\Lambda\overline{T})$; if σ_3 were a big dilation it would be central in $G_1(R_1, P_1)$, contradicting the fact that $\Lambda\overline{T}$ is noncentral in $\mathrm{CDC}(\overline{\sigma}_1)$. So indeed $R_3 \subset R_1$. We have therefore shown that if $\mathbf{T}$ is an arbitrary transvection in G with $\mathbf{T} \in DC(\mathbf{T})$, then $\Lambda\overline{T} = \overline{\sigma}_3$ for some σ_3 in G_1 with res$\sigma_3 = 1$. Now for given $L \subseteq H$ you always have a transvection $\mathbf{T}$ in G with spaces $L \subseteq H$ and $\mathbf{T} \in DC(\mathbf{T})$ by 3.3.5; looking at elementary transvections we can easily find transvections $\mathbf{T}_i$ $(1 \leq i \leq 3)$ in G with $\mathbf{T}_i \in DC(\mathbf{T}_i)$ such that $\mathbf{T}_1 = [\mathbf{T}_2, \mathbf{T}_3]$. Then, by what we have just proved, each $\Lambda\overline{T}_i$ has a representative ϕ_i in G_1 with res$\phi_i = 1$. So $\Lambda\overline{T}_1$ has a representative in G_1 of residue 1, and another in DG_1 of residue ≤ 2. Since $n_1 \geq 4$, these representatives must be equal. In other words, $\Lambda\overline{T}_1 = \overline{\tau}_1$ for some $\tau_1 \in DG_1$ with res$\tau_1 = 1$. Put $\tau = \mathbf{T}_1$.

(1b) We must complete the excluded situation $F_1 = \mathbf{F}_2$ with $n_1 = 4$. This of course makes char $F = 2$. So $\overline{T}^2 = 1$ since $\mathbf{T}$ is a transvection in characteristic 2, so $\Lambda\overline{T}$ has a representative σ_3 in G_1 with $\sigma_3^2 = 1$ since $F_1 = \mathbf{F}_2$, i.e. $\Lambda\overline{T}$ is a projective unipotent transformation in $\mathrm{CDC}(\overline{\sigma}_1)$. This makes $\Lambda\overline{T}$ a projective transvection by 3.3.10. Now we always have a nontrivial transvection $\mathbf{T}$ in G which satisfies $\mathbf{T} \in DC(\mathbf{T})$ by 3.3.5. For this $\mathbf{T}$, $\Lambda\overline{T}$ must therefore have a representative τ_1 with τ_1 a transvection in G_1. In particular res$\tau_1 = 1$. But $G_1 = SL_4(V_1)$ since $F_1 = \mathbf{F}_2$. So $\tau_1 \in G_1 = DG_1$ by O'Meara §3.3. Put $\tau = \mathbf{T}$.

(2) *One dimension* = 3, *the other* ≥ 4. By reversing the situation, if necessary, we can assume that $n = 3$ with $n_1 \geq 4$.

(2a) First suppose char $F = 2$ if $F_1 = \mathbf{F}_2$. the procedure here is exactly the same as in step (1) except for the possibility that the σ that turns up may be an element of residue 2 with $n = 3$, det $\sigma = 1$, σ is diagonalizable over F, $\sigma^3 = 1$, in which case 3.3.8 cannot be applied. Actually this cannot happen. For suppose it did. If char $F_1 = 3$, then $\overline{\sigma}^3 = 1$. So $\overline{\sigma}_1^3 = 1$, so $\sigma_1^3 = 1_{V1}$ since res$\sigma_1 = 2$, so $(\sigma_1|R_1)^3 = 1_{R1}$, so $(\sigma_1|R_1)$ is unipotent on the plane R_1, so $(\sigma_1|R_1)$ is a transvection, so σ_1 is a transvection since $R_1 \cap P_1 = 0$, and this is absurd since res$\sigma_1 = 2$. On the other hand, if char $F_1 \neq 3$, we use the fact that $\langle (CV(\overline{\sigma}))^3 \rangle$ is abelian by 3.1.6; then $\langle (C(\overline{\sigma}))^3 \rangle$ is abelian; hence $\langle (C(\overline{\sigma}_1))^3 \rangle$ is abelian; but $C(\sigma_1)$ clearly contains transvections which do not permute, their cubes do not permute since the characteristics is not 3, so the projective images of their cubes do not permute, therefore $\langle (C(\overline{\sigma}_1))^3 \rangle$ is not abelian; this is also absurd.

(2b) Now char $F \neq 2$ with $F_1 = \mathbf{F}_2$. Fix a hyperplane H_1 in V_1 and let L_1 be a variable line in H_1. We have a transvection τ_1 in G_1 with spaces $L_1 \subseteq H_1$ and $\overline{\tau}_1 \in D\Delta_1$ since $D\Delta_1$ is full of projective transvections, then $\Lambda^{-1}\overline{\tau}_1$ belongs to $D\Delta = \overline{DG}$, so $\Lambda^{-1}\overline{\tau}_1$ has a representative σ in DG, in particular with det $\sigma = 1$. Now $\tau^2_1 = 1_{V1}$ since τ_1 is a transvection in characteristic 2, so $\sigma^2 = \alpha\, 1_V$ with $\alpha^3 = 1$. Replacing τ_1 by its cube allows us to assume that in fact $\sigma^2 = 1_V$. Now H_1 contains at least four distinct lines. We can therefore find distinct transvections $\tau_1, ..., \tau_5$ in G_1(with exactly one of them, say τ_5, trivial) which are pairwise permutable such that the corresponding $\sigma_1, ..., \sigma_5$ (with $\sigma_5 = 1_V$ are involutions of determinant 1 which permute projectively in pairs. Now $-\sigma_1, ..., -\sigma_4, \sigma_5$ are easily seen to be involutions of residue ≤ 1 by 1.2.6, hence $\sigma_1, ..., \sigma_5$ permute in pairs by 3.1.4. Therefore

$$\text{card}\,(\overline{\sigma}_1, ..., \overline{\sigma}_5) \leq 2^{3-1} = 4 \qquad (3.95)$$

by O'Meara §1.6 and this is absurd since $\overline{\tau}_1, ..., \overline{\tau}_5$ are distinct.

(3) *Both dimensions* 3. By 3.3.5 there is a transvection τ in G with $\tau \in DC(\tau)$. In particular $\tau \in DG$ with res τ=1. Then $\overline{\tau} \in DC(\overline{\tau})$, so

$$\Lambda\overline{\tau} \in DC(\Lambda\overline{\tau}) \subseteq D\Delta_1 = \overline{DG_1}\,. \qquad (3.96)$$

We can therefore pick $\tau_1 \in DG_1$ such that $\Lambda\overline{\tau} = \overline{\tau}_1$ and $\overline{\tau}_1 \in DC(\overline{\tau}_1)$.

(3a) If at least one of the characteristic $\neq 2, 3$, we can assume that in fact char $F \neq 2$, 3. Then $\tau^{18}{}_1$ is a transvection by 3.3.13. Replace τ by τ^{18}.

(3b) If both characteristics are 3, then $\tau^2{}_1$ is a transvection by 3.3.13. Replace τ by τ^2.

(3c) If both characteristics are 2, do the same thing with τ^9.

(3d) We are left with one characteristic 3, the other 2. In fact we can assume that char F = 3 and char F_1 = 2. If $F_1 \neq \mathbf{F}_2$, proceed as in step (2b). So let $F_1 = \mathbf{F}_2$. Then Δ_1 = $PSL_3(V_1)$, so card Δ_1 = 168 by O'Meara §3.1. If $F = \mathbf{F}_3$ we have $PSL_3(V) \subseteq \Delta \subseteq PGL_3(V)$ with card $PSL_3(V)$ = 5616, and so $\Lambda: \Delta \to \Delta_1$ is impossible here. If card $F > 3$ (with char F = 3), then V contains $q^2 + q + 1$ lines with q = card F, so V has at least 91 lines, so Δ has at least 182 projective transvections, therefore $\Lambda: \Delta \to \Delta_1$ is again impossible. □

3.4.3 Λ *preserves projective transvections when* $n \geq 3$, $n_1 \geq 3$

PROOF. (1) First note that if Λ preserves the projective transvection $\overline{\sigma}$ in Δ and if $\overline{\tau}$ is any projective transvection in Δ with the same spaces as $\overline{\sigma}$, then Λ preserves $\overline{\tau}$, and the spaces of $\Lambda\overline{\tau}$ are the same spaces of $\Lambda\overline{\sigma}$. This follows from

$$\Lambda\overline{\tau} \in \Lambda\Delta(R, P) = \Lambda CC(\overline{\sigma}) = CC(\Lambda\overline{\sigma}) = \Delta_1(R_1, P_1) \tag{3.97}$$

($\Lambda\overline{\sigma} = \overline{\sigma}_1$) which comes from 3.3.7

(2) Next we observe that if Λ preserves the projective transvections $\overline{\sigma}$ and $\overline{\tau}$ in Δ, and if $\overline{\sigma}$ and $\overline{\tau}$ have the same fixed hyperplane, then $\Lambda\overline{\sigma}$ and $\Lambda\overline{\tau}$ either have the same fixed hyperplane or they have the same residual line. To prove this we can assume, by step (1), that $\overline{\sigma}$ and $\overline{\tau}$ have distinct residual lines. We can then find a base $x_1, \ldots, x_n$ for V with dual base $\rho_1, \ldots, \rho_n$ such that $\overline{\sigma} = \overline{\tau}_{x_1, \rho_n}$, $\overline{\tau} = \overline{\tau}_{x_2, \rho_n}$. Let $\overline{\tau}_{\alpha x_1, \rho_2}$ be a nontrivial projective transvection in Δ. Then

$$\overline{\tau}_{\alpha x_1, \rho_n} = [\overline{\tau}_{\alpha x_1, \rho_2}, \overline{\tau}_{x_2, \rho_n}] \tag{3.98}$$

is a nontrivial projective transvection in Δ with the same spaces as $\overline{\sigma}$; hence, by step (1), $\Lambda\overline{\tau}_{\alpha x_1, \rho_n}$ is a projective transvection in Δ_1 with the same spaces as $\Lambda\overline{\sigma}$. But

$$(\Lambda\overline{\tau}_{\alpha x_1, \rho_n}) \cdot (\Lambda\overline{\tau}) = (\Lambda\overline{\tau}_{\alpha x_1, \rho_2}) \cdot (\Lambda\overline{\tau}) \cdot (\Lambda\overline{\tau}_{\alpha x_1, \rho_2})^{-1} \tag{3.99}$$

by the above commutator relation, so this expression is a projective transvection, so the projective transvections $\Lambda\bar{\tau}_{\alpha x_1,\rho_n}$ and $\Lambda\bar{\tau}$ have the same fixed hyperplane or the same residual line by O'Meara §1.6, so $\Lambda\bar{\sigma}$ and $\Lambda\bar{\tau}$ do.

(3) Now let us show that if Λ preserves the nontrivial projective transvection $\bar{\sigma}$ in Δ, then Λ preserves all projective transvections in Δ with the same fixed hyperplane as $\bar{\sigma}$. Let $\bar{\tau}$ be such a transvection. We can again assume that we have a base in which $\bar{\sigma}=\bar{\tau}_{x_1,\rho_n}, \bar{\tau}=\bar{\tau}_{x_2,\rho_n}$. Let $\bar{\tau}_{\alpha x_2,\rho_1}$ be a nontrivial projective transvection in Δ. Then

$$\begin{aligned}\bar{\tau}_{\alpha x_2,\rho_n} &= [\bar{\tau}_{\alpha x_2,\rho_1}, \bar{\tau}_{x_1,\rho_n}] \\ &= (\bar{\tau}_{\alpha x_2,\rho_1}\bar{\tau}_{x_1,\rho_n}\bar{\tau}^{-1}_{\alpha x_2,\rho_1})\bar{\tau}^{-1}_{x_1,\rho_n} \\ &= \bar{\tau}_{x_1+\alpha x_2,\rho_n}\bar{\tau}^{-1}_{x_1,\rho_n}. \qquad (3.100)\end{aligned}$$

It is obvious that $\bar{\tau}_{x_1+\alpha x_2,\rho_n}$ and $\bar{\tau}_{x_1,\rho_n}=\bar{\sigma}$ are conjugate projective transvections with the same fixed hyperplane in Δ, and $\Lambda\bar{\sigma}$ is a projective transvection by hypothesis, so $\Lambda\bar{\tau}_{x_1+\alpha x_2,\rho_n}$ is a projective transvection by conjugacy, and it either has the same residual line or the same fixed hyperplane as $\Lambda\bar{\sigma}$ by step (2). Hence $\Lambda\bar{\tau}_{\alpha x_2,\rho_n}$, being a product of projective transvections with the same line or hyperplane, is a projective transvection. Hence $\Lambda\bar{\tau}=\Lambda\bar{\tau}_{x_2,\rho_n}$ is a projective transvection by step (1).

(4) If Λ preserves the nontrivial projective transvection $\bar{\sigma}$ in Δ, then it preserves all projective transvections in Δ with the same residual line as $\bar{\sigma}$. proof by the duality

(5) Λ preserves at least one nontrivial projective transvection $\bar{\sigma}$ in Δ, by 3.4.2. Let $\bar{\tau}$ be any other nontrivial projective transvection in Δ. Let $L\subseteq H$ be the spaces of $\bar{\tau}$. Then Λ preserves $\bar{\sigma}$ so it preserves a projective transvection in Δ with the same fixed hyperplane as $\bar{\sigma}$ and with residual line contained in $P \cap H$ by step (3), hence it preserves a projective transvection in Δ having this as its residual line and H as its fixed hyperplane by step (4), hence it preserves $\bar{\tau}$ by step (3). So Λ preserves all projective transvections in Δ. And Λ^{-1} preserves all in Δ_1. So Λ preserves projective transvections. □

3.5. The Isomorphism Theorems in General

We now return to the general situation, i.e. G is an arbitrary subgroup of $\Gamma L_n(V)$ that is full of transvections, and Δ is an arbitrary subgroup of $P\Gamma L_n(V)$ that is full of projective transvections. Similarly in the V_1, n_1, F_1, G_1, Δ_1 situation. And Ψ: $G \to G_1$, Λ: $\Delta \to \Delta_1$ are group isomorphisms.

We let $\mathcal{L}$, $\mathcal{H}$, $\mathcal{X}$ be the subsets

$$\mathcal{L} = P^1(V) \qquad \mathcal{H} = P^{n-1}(V) \qquad \mathcal{X} = \mathcal{L} \cup \mathcal{H} \tag{3.101}$$

of the projective space $P(V)$, i.e. $\mathcal{L}$ is the set of lines in V, $\mathcal{H}$ the set of hyperplanes, $\mathcal{X}$ their union. Of course $\mathcal{L} \cup \mathcal{H} = \varnothing$ if $n \geq 3$ and $\mathcal{L} = \mathcal{H}$ if $n = 2$. We ignore the case $n = 1$ since full groups are not defined there. For each $L \in \mathcal{L}$, $H \in \mathcal{H}$ with $L \subseteq H$ define $\Delta(L, H)$ as the group consisting of all projective transvections in Δ with spaces $L \subseteq H$ plus 1. This is consistent with the use of $\Delta(L, H)$ in the special situation of §3.3 and 3.4. For any $L \in \mathcal{L}$ define $\Delta(L)$ as the group consisting of all projective transvections in Δ with residual line L, plus 1; and $\Delta(H)$ as the group consisting of all projective transvections in Δ with fixed hyperplane H, plus 1; for any X in $\mathcal{X}$ define $\Delta(X)$ by $\Delta(X) = \Delta(L)$ if $X = L \in \mathcal{L}$, and by $\Delta(X) = \Delta(H)$ if $X = H \in \mathcal{H}$. The two definitions of $\Delta(X)$ clearly coincide when $n = 2$.

In keeping with the convention used in the special situation of §3.3 and 3.4 we use C to denote the centralizer C_Δ, C_G, $C_{\breve{\Delta}}$, $C_{\breve{G}}$ when we are working, respectively, in Δ, G, $\breve{\Delta}$, $\breve{G}$.

The quantities $\mathcal{L}_1$, $\mathcal{H}_1$, $\mathcal{X}_1$, $\Delta_1(L_1,H_1)$, $\Delta_1(L_1)$, $\Delta_1(H_1)$, $\Delta_1(X_1)$, C are defined in the same way for the Δ_1 situation.

If $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are nontrivial projective transvections in Δ, then it follows by applying 3.3.3 to the full group $\Delta \cap PGL_n(V)$ that

$$C(\overline{\sigma}_1) = C(\overline{\sigma}_2) \Rightarrow R_1 = R_2 \quad \text{and} \quad P_1 = P_2 \tag{3.102}$$

3.5.1. *Suppose* $n \geq 3$. *If* Δ^* *is a subgroup of* $\Delta \cap \mathrm{PGL}_n(V)$ *that is full of projective transvections, and if* $\bar{\sigma}$ *is a nontrivial projective transvection in* Δ^*, *then* $C_\Delta C_{\Delta^*}(\bar{\sigma}) = \Delta(R, P)$. *In particular,* $CC(\bar{\sigma}) \subseteq \Delta(R, P)$ *for any nontrivial projective transvection in* Δ.

PROOF. Put $\Delta^{**} = \Delta \cap \mathrm{PGL}_n(V)$. Then Δ^{**} is full of projective transvections, and $\Delta^* \subseteq \Delta^{**}$. So, by 3.3.7,

$$\Delta(R, P) = \Delta^{**}(R, P) = C_{\Delta^{**}}C_{\Delta^{**}}(\bar{\sigma}) \subseteq C_\Delta C_{\Delta^{**}}(\bar{\sigma}) \subseteq C_\Delta C_{\Delta^*}(\bar{\sigma}). \tag{3.103}$$

To reverse this inequality, proceed as in the proof of 3.3.7, and by O'Meara §4.3 and 4.4. Finally, if $\bar{\sigma}$ is any nontrivial projective transvection in Δ, then $\bar{\sigma} \in \Delta^{**}$, and $C(\bar{\sigma}) \supseteq C_{\Delta^{**}}(\bar{\sigma})$; so

$$CC(\bar{\sigma}) \subseteq C_\Delta C_{\Delta^{**}}(\bar{\sigma}) = \Delta(R, P). \tag{3.104}$$

□

3.5.2. EXAMPLE. Let us show that the results

$$C(\bar{\sigma}_1) = C(\bar{\sigma}_2) \Leftrightarrow R_1 = R_2 \quad \text{and} \quad P_1 = P_2 \tag{3.105}$$

and

$$CC(\bar{\sigma}_1) = \Delta(R, P) \tag{3.106}$$

for projective transvections in the special situation of §3.3 and 3.4 do not hold here. To this end consider $\Delta = \mathrm{P\Gamma L}_n(V)$ with $n \geq 3$ over a field F which possesses a nontrivial field automorphism μ. Let α be an element of F for which $\alpha^\mu \neq \alpha$, let $x_1, \ldots, x_n$ be a base for V with dual $\rho_1, \ldots, \rho_n$, and let k be the element of $\Gamma\mathrm{L}_n(V)$ with associated field automorphism μ and with matrix $\mathrm{diag}(\alpha, 1, \ldots, 1, \alpha)$ in $x_1, \ldots, x_n$. It is easily verified that

$$\mu\rho_n k^{-1} = \alpha^{-1}\rho_n \tag{3.107}$$

whence, by O'Meara §4.4,

$$k\tau_{x_1,\rho_n}k^{-1}=\tau_{x_1,\rho_n} \qquad \text{and} \qquad k\tau_{\alpha x_1,\rho_n}k^{-1}=\tau_{\alpha^{\mu}x_1,\rho_n}\neq\tau_{\alpha x_1,\rho_n}. \tag{3.108}$$

In other words, $\bar{k}$ permutes with $\bar{\tau}_{x_1,\rho_n}$ but not with $\bar{\tau}_{\alpha x_1,\rho_n}$. In particular, we have two projective transvections in Δ with the same spaces but with different centralizers. Furthermore, $\bar{\tau}_{\alpha x_1,\rho_n}$ is clearly in $\Delta(R, P)$ (where $\bar{\sigma}=\bar{\tau}_{x_1,\rho_n}$) but it is not in $CC(\bar{\sigma})$ since it does not permute with $\bar{k}\in(\bar{\sigma})$.

3.5.3. *If* $n\geq 3$, $n_1\geq 2$, *there is a subgroup* Δ^0 *of* Δ *that is still full of projective transvections and satisfies*

$$\Delta^0\subseteq \mathrm{PSL}_n(V), \qquad \Lambda\Delta^0\subseteq \mathrm{PSL}_{n1}(V_1). \tag{3.109}$$

PROOF. (1) First let us show that Λ sends at least one nontrivial projective transvection in Δ into $\mathrm{PSL}_{n1}(V_1)$. Start with a nontrivial projective transvection $\bar{\tau}$ in Δ. Since $\Lambda\Delta=\Delta_1$ with Δ_1 full, there is an element $\bar{\psi}$ of Δ such that $\Lambda\bar{\psi}$ is an element of $\mathrm{PSL}_{n1}(V_1)$ (in fact a projective transvection in Δ_1) which does not permute with $\Lambda\bar{\tau}$ (apply 3.2.5 to the group generated by all projective transvections in Δ_1). Let $\bar{\sigma}$ be the element $\bar{\sigma}=\bar{\psi}\,\bar{\tau}\,\bar{\psi}^{-1}\bar{\tau}^{-1}$ of Δ. Then $\bar{\sigma}$ is in $\mathrm{PSL}_n(V)$ and $\Lambda\bar{\sigma}$ is in $\mathrm{PSL}_{n1}(V_1)$ by O'Meara §4.3; and $\bar{\sigma}$ and $\Lambda\bar{\sigma}$ are nontrivial by choice of $\Lambda\bar{\psi}$. Obviously $\bar{\sigma}$ has a representative σ in $\mathrm{SL}_n(V)$ with $1\leq \mathrm{res}\,\sigma\leq 2$. If $\mathrm{res}\,\sigma=1$ we are through. So assume $\mathrm{res}\,\sigma=2$. By adapting the "second simplicity trick" of the proof of Theorem 3.4.1 from O'Meara we can find $\bar{\tau}_{a,\rho}\in\Delta$ such that $\bar{\sigma}\,\bar{\tau}_{a,\rho}\,\bar{\sigma}^{-1}\bar{\tau}^{-1}_{a,\rho}$ is a nontrivial projective transvection in Δ. Then

$$\Lambda(\bar{\sigma},\bar{\tau}_{a,\rho},\bar{\sigma}^{-1},\bar{\tau}^{-1}_{a,\rho})\in \mathrm{PSL}_{n1}(V_1) \tag{3.110}$$

and again we are through.

(2) Next we note that if Λ sends a nontrivial projective transvection $\overline{\sigma}$ in Δ into $\mathrm{PSL}_{n1}(V_1)$, then for each line L in P there is at least one nontrivial projective trasvection $\overline{\tau}$ in Δ with spaces $L \subseteq P$ such that $\Lambda\overline{\tau}$ falls in $\mathrm{PSL}_{n1}(V_1)$. To verify this we can assume that $L \neq Fx_1$, and then take a base $x_1, \ldots, x_n$ for V with dual $\rho_1, \ldots, \rho_n$ such that $\overline{\sigma} = \overline{\tau}_{x_1,\rho_n}$, $L = Fx_2$. Let $\alpha \in \dot{F}$ be such that $\overline{\tau}_{\alpha x_2,\rho_1} \in \Delta$. The result then follows from the commutator relation

$$\overline{\tau}_{\alpha x_2,\rho_n} = [\,\overline{\tau}_{\alpha x_2,\rho_1}\,,\ \overline{\tau}_{x_1,\rho_n}\,]. \tag{3.111}$$

(3) By duality, if Λ sends a nontrivial projective transvection $\overline{\sigma}$ in Δ into $\mathrm{PSL}_{n1}(V_1)$, then for each hyperplane H containing R there is at least one nontrivial projective transvection $\overline{\tau}$ in Δ with spaces $R \subseteq H$ such that $\Lambda\overline{\tau}$ falls in $\mathrm{PSL}_{n1}(V_1)$.

(4) The result now follows easily using the argument of step (5) of the proof of 3.4.3. □

3.5.4. *If $n \geq 3$, $n_1 \geq 2$, there is a subgroup G^0 of G that is still full of transvections and satisfies*

$$G^0 \subseteq \mathrm{SL}_n(V), \qquad \Psi G^0 \subseteq \mathrm{SL}_{n1}(V_1) \tag{3.112}$$

PROOF. The proof is similar to the proof of 3.5.3. Just take care to choose the non–trivial transvection τ in G at the beginning of step (1) in such a way that $\Psi\tau \notin \mathrm{RL}_{n1}(V_1)$. The existence of such a τ follows easily from the commutator relations for elementary transvections. □

3.5.5. Λ *preserves projective transvections when $n \geq 3$, $n_1 \geq 3$.*

PROOF. (1) Applying 3.5.3 to Λ gives a subgroup Δ^0 of $\Delta \cap \mathrm{PSL}_n(V)$ that is full of projective transvections such that $\Lambda\Delta^0 \subseteq \Delta_1 \cap \mathrm{PSL}_{n1}(V_1)$. Applying it to Λ^{-1} gives a subgroup $\Delta^0{}_1$ of $\Delta_1 \cap \mathrm{PSL}_{n1}(V_1)$ that is full of projective transvections such that $\Lambda^{-1}\Delta^0{}_1 \subseteq \Delta \cap \mathrm{PSL}_n(V)$. Then the groups $\Delta^* = \langle \Delta^0, \Lambda^{-1}\Delta^0{}_1 \rangle$, $\Delta^*{}_1 = \langle \Lambda\,\Delta^0, \Delta^0{}_1 \rangle$ satisfy

$$\Delta^0 \subseteq \Delta^* \subseteq \Delta \cap \mathrm{PSL}_n(V), \tag{3.113}$$

$$\Delta^0{}_1 \subseteq \Delta^*{}_1 \subseteq \Delta_1 \cap \mathrm{PSL}_{n1}(V_1) \tag{3.114}$$

and are, in particular, full of projective transvections. Furthermore Λ: $\Delta^* \to \Delta^*{}_1$.

(2) Now consider a typical projective transvection $\overline{\sigma}$ in Δ. Let $\overline{\sigma}^*$ be a projective transvection in Δ^* with the same spaces $R \subseteq P$ as $\overline{\sigma}$. Then $\overline{\sigma}^*{}_1 = \Lambda\ \overline{\sigma}^*$ is a projective transvection by 3.4.3. And by 3.5.1 we have

$$\Lambda\overline{\sigma} \in \Lambda\ \Delta(R, P) = \Lambda C_\Delta C_{\Delta^*}(\overline{\sigma}^*) = C_{\Delta_1}\ C_{\Delta^*_1}(\overline{\sigma}\,{}_1^*) = \Delta_1(R_1^*, P_1^*). \tag{3.115}$$

So $\Lambda\overline{\sigma}$ is a projective transvection. So Λ preserves each projective transvection in Δ. So by symmetry Λ^{-1} preserves each projective transvection in Δ_1. So Λ preserves projective transvections. □

3.5.6. $\Psi(\mathrm{G} \cap \mathrm{RL}_n(V)) = \mathrm{G}_1 \cap \mathrm{RL}_{n1}(V_1)$ *when* $n \geq 3$, $n_1 \geq 3$.

PROOF. Proceeding as in step (1) of the proof of 3.5.4 we can find subgroups $G^* \subseteq G \cap \mathrm{SL}_n(V)$ and $G_1^* \subseteq G_1 \cap \mathrm{SL}_{n1}(V_1)$ that are full of transvections such that Ψ: $G^* \to G_1^*$. Then for any σ in $G \cap \mathrm{RL}_n(V)$ we have $\sigma \in C(G^*)$. Hence $\Psi\sigma \in C(G_1^*)$, so $\Psi\sigma$ is in the centralizer of G_1^* in $\Gamma\mathrm{L}_{n1}(V_1)$; but G_1^* is full of transvections, so $\Psi\sigma$ is in $\mathrm{RL}_{n1}(V_1)$ by 3.2.5. Hence $\Psi(G \cap \mathrm{RL}_n(V)) \subseteq G_1 \cap \mathrm{RL}_{n1}(V_1)$. Equality follows by considering Ψ^{-1} instead of Ψ. □

Throughout these lectures V is an n–dimensional vector space over the field F with $1 \leq n < \infty$, and Δ is a subgroup of $\mathrm{P}\Gamma\mathrm{L}_n(V)$ (or $\mathrm{PGL}_n(V)$ in §3.3 and §3.4) that is full of projective transvections. In order to simplify the statement of exceptional situations we will say, for example, that Δ is PSL_2 over $\mathbf{F}_7$ if $F = \mathbf{F}_7$, $n = 2$, and Δ is equal to $\mathrm{PSL}_2(V)$. Similarly with PSL_3 over $\mathbf{F}_2$, and so on.

Let us recall that a power of any transvection is calculated via the formula

$$\tau^n_{a,\rho} = \tau_{na,\rho}, \tag{3.116}$$

which means that if char $F = 2$, then the square of any transvection (as then of a projective transvection) is the identity map:

$$\tau^2_{a,\rho} = \tau_{2a,\rho} = \tau_{0,\rho} = 1_V. \tag{3.117}$$

3.5.7. *There can be no isomorphism* $\Lambda\colon \Delta \to \Delta_1$ *when* $n \geq 3$ *with* $n_1=2$ *except, possibly, when* Δ *is* PSL_3 *over* $\mathbf{F}_2$ *with* Δ_1 *equal to* PSL_2 *over* $\mathbf{F}_7$.

PROOF. For suppose we have an isomorphism $\Lambda\colon \Delta \to \Delta_1$ with $n \geq 3$ with $n_1 = 2$. By 3.5.3 there is a subgroup Δ^0 of $\Delta \cap \mathrm{PSL}_n(V)$ that is full of projective transvections such that $\Lambda\Delta^0 \subseteq \mathrm{PSL}_2(V_1)$.

(1) *Suppose* char $F \neq 2$. Take a nontrivial projective transvection $\overline{\sigma}$ in Δ^0. Then by the projective version of 3.3.5 we have $\overline{\sigma} \in DC_{\Delta^0}(\overline{\sigma})$ and, in particular, $DC_{\Delta^0}(\overline{\sigma}) \neq 1$.

Since $\Lambda\Delta^0 \subseteq \mathrm{PSL}_2(V_1)$ there is therefore an element σ_1 in $\mathrm{SL}_2(V_1)$ such that $\Lambda\overline{\sigma} = \overline{\sigma}_1$ with $DC_{V_1}(\overline{\sigma}_1) \neq 1$. Clearly,

$$C_{V_1}(\overline{\sigma}_1) \subseteq C_{V_1}(\overline{\sigma}_1^2) = \overline{C_{V_1}(\sigma_1^2)}, \tag{3.118}$$

where the latter equality is based upon the fact that all transvections are unipotent and all projective transvections are projective unipotent transvections. But $\overline{\sigma}^2 \neq 1$ since $\overline{\sigma}$ is a projective transvection and char $F \neq 2$, so $\sigma^2_1 \in \mathrm{GL}_2(V_1) - \mathrm{RL}_2(V_1)$, so $DC_{V_1}(\sigma^2_1) = 1$ by 3.1.1, so $DC_{V_1}(\overline{\sigma}_1) = 1$. This is absurd. So the case when char $F \neq 2$ cannot occur.

(2) Suppose char $F = 2$, char $F_1=2$. Again let $\overline{\sigma}$ be a nontrivial projective transvection in Δ^0 such that $DC_{\Delta^0}(\overline{\sigma}) \neq 1$. Again we have $\Lambda\Delta^0 \subseteq \mathrm{PSL}_2(V_1)$ and σ_1 in $\mathrm{SL}_2(V_1)$ with $\Lambda\overline{\sigma} = \overline{\sigma}_1$. This time $\overline{\sigma}^2 = 1$, so $\sigma_1^2 = \alpha\ 1_{V_1}$ with $\alpha^2 = (\det \sigma_1)^2 = 1$, since $\sigma \in \mathrm{SL}_n(V)$. Due to the fact that characteristic of F_1 is two, we get that $\alpha = 1$. So σ_1 is a

nontrivial element in $SL_2(V_1)$ with $\sigma_1{}^2 = 1_{V_1}$. Since char F_1=2, σ_1 is unipotent and therefore a nontrivial transvection. By 3.3.1 and 3.1.1,

$$DC_{\Lambda\Delta^0}(\overline{\sigma}_1) \subseteq DC_{V_1}(\overline{\sigma}_1) = \overline{DC_{V_1}(\sigma_1)} = 1 \qquad (3.119)$$

which contradicts the fact that $DC_{\Delta^0}(\overline{\sigma}) \neq 1$. So this situation cannot occur either.

(3) *Suppose* char F = 2, char $F_1 \neq 2$, *but exclude* n = 3 *with* F = $\mathbf{F}_2$. Consider any nontrivial projective transvection $\overline{\sigma}$ in Δ^0. Then $\overline{\sigma}^2 = 1$ with $\overline{\sigma} \neq 1$. Express $\Lambda\overline{\sigma} = \overline{\sigma}_1$ with σ_1 in $SL_2(V_1)$. We have $\overline{\sigma}_1{}^2$=1 with $\overline{\sigma}_1 \neq 1$. Thus $\overline{\sigma}_1{}^2 = \alpha 1_{V_1}$ for some αa in $\dot{F}_1$. But detσ_1 = 1, so α^2 = 1, i.e. $\alpha = \pm 1$. So $\sigma_1{}^2 = \pm 1_{V_1}$. Now we cannot have $\sigma_1{}^2 = 1_{V_1}$ for this equation would imply, by O'Meara §1.6 and the fact that detσ_1 = 1, that $\overline{\sigma}$=1 (the quoted result states that any involution is diagonalizable; since detσ_1 = 1 the only choices here are $\sigma_1 = \pm 1_{V_1}$).

Therefore $\sigma_1{}^2 = -1_{V_1}$. In other words, with each nontrivial projective transvection $\overline{\sigma}$ in Δ^0 we can associate an element σ_1 of $SL_2(V_1)$ such that

$$\overline{\sigma}_1 = \Lambda\overline{\sigma} \neq 1, \qquad (3.120)$$

$$\sigma^2{}_1 = -1_{V_1}, \quad \det \sigma_1 = 1. \qquad (3.121)$$

We suggested that $F \neq \mathbf{F}_2$; keeping in mind that F is of characteristic two, we have that $|F| \geq 2^2 = 4$. Then each hyperplane of V contains at least five distinct lines, so we have five distinct non–trivial pairwise permuting projective transvections in Δ^0; so we have σ_1, σ_2, σ_3, σ_4, σ_5 in $SL_2(V_1)$ such that the corresponding σ_i are distinct, nontrivial, permute in pairs, and such that

$$\sigma_i{}^2 = -1_{V_1}, \quad \det \sigma_i = 1 \quad (1 \leq i \leq 5). \qquad (3.122)$$

The projective permutability conditions give

$$\sigma_i\,\sigma_j = \pm\, \sigma_j\,\sigma_i \qquad (3.123)$$

for all i and j:

$$\sigma_i\,\sigma_j\,\sigma_i^{-1} = \beta\,\sigma_j \Rightarrow \sigma_j^2 = -1_{V_1} = -\beta^2\,1_{V_1} \tag{3.124}$$

and $\beta = \pm\ 1$. The fact that $\sigma_1{}^2 = -1_{V_1}$ implies that there is a base for V_1 in which σ_1 has matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.125}$$

Indeed, if there is a nonzero $x \in V_1$ such that the vectors x, σx are linearly independent, we are done. Suppose that for all $x \in V_1$ the Vectors x, $\sigma\,x$ are linearly dependent:

$$\sigma\,x = \gamma\,x \Rightarrow \sigma(\sigma\,x) = \gamma\,\sigma\,x \Rightarrow -x = \gamma\,\sigma\,x, \tag{3.126}$$

whence $\gamma = -\gamma^{-1}$ or $\gamma^2 = -1$. Now if x, y is a base of V_1 then σ_1 has the matrix

$$\begin{pmatrix} \gamma & 0 \\ 0 & \pm\gamma \end{pmatrix} \tag{3.127}$$

in this base. Then case when the second element on the main diagonal is γ is impossible, for in this case $\overline{\sigma} = 1$. In the either case the vectors

$$x + y \quad \text{and} \quad \sigma_1(x + y) = \gamma\ x - \gamma\ x \tag{3.128}$$

are clearly linearly independent, a contradiction.

Consider now the condition for a matrix of $GL_2(F_1)$ to commute with the matrix of σ_1:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.129}$$

This implies that $b + c = 0$ and $a = d$. Now consider the square of a matrix commuting with the matrix of σ_1:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^2 = \begin{pmatrix} a^2 - b^2 & 2ab \\ -2ab & a^2 - b^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.130}$$

Then $ab = 0$. In the case when $a = 0$ and $b \neq 0$ we get the matrix

$$\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \tag{3.131}$$

and clearly a linear transformation π which has such a matrix has the same projective image as σ_1 has. In the either case the projective image of π is trivial. Then by (3.123) the only possibility for σ_i, $i = 2, \ldots, 5$ is to commute with σ_1 projectively:

$$\sigma_i \, \sigma_1 = -\sigma_1 \, \sigma_i. \tag{3.132}$$

Arguing as before one quickly sees that the matrices of σ_i in our base must be of the form

$$\begin{pmatrix} p_i & q_i \\ q_i & -p_i \end{pmatrix} \tag{3.133}$$

for suitable scalars in F_1, for $2 \le i \le 5$. At most one of the p_i $(2 \le i \le 5)$ can be 0 since the $\overline{\sigma}_j$ are distinct, so in fact we can assume that $\overline{\sigma}_2$, $\overline{\sigma}_3$, $\overline{\sigma}_4$ have representatives σ_2', σ_3', σ_4' with matrices

$$\begin{pmatrix} 1 & \alpha \\ \alpha & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \beta \\ \beta & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \gamma \\ \gamma & -1 \end{pmatrix} \tag{3.134}$$

with α, β, γ distinct. We can then assume, in addition, that $1 + \alpha\beta \neq 0$. Indeed, otherwise we would have that

$$1+\alpha\ \beta = 0, \tag{3.135}$$

$$1+\alpha\ \gamma = 0, \tag{3.136}$$

$$1+\ \beta\ \gamma = 0 \tag{3.137}$$

whence assuming, for instance, that $\alpha \neq 0$, we deduce

$$1+\alpha\beta = 1+\alpha\gamma \Rightarrow \alpha(\beta-\gamma) = 0 \tag{3.138}$$

and $\beta=\gamma$, contradicting to the choice above. But then $\overline{\sigma}_2$ and $\overline{\sigma}_3$ do not permute: one easily checks that the condition $1+\alpha\beta = 0$ is necessary for the projective permutability of σ_2 and σ_3 (recall that $\sigma_2\sigma_3 = -\sigma_3\sigma_2$.) This is absurd.

So the situation in (3) also cannot arise.

(4) *Finally suppose* $n = 3$, $F= \mathbf{F}_2$, char $F_1 \neq 2$. Since the field $\mathbf{F}_2$ has no nontrivial automorphisms, we have $P\Gamma L_3(V) = PSL_3(V)$; furthermore, in this situation we clearly have $PSL_3(V) = \Delta=\Delta^0$.

Next, by a well–known formula

$$|PSL_n(\mathbf{F}q)| = \frac{q^{n(n-1)/2}\Pi_1^n\left(q^i-1\right)}{(q-1)\gcd(q-1,n)}. \tag{3.139}$$

In particular, straightforward calculation shows that card Δ=168.

And $\Lambda\Delta = \Delta_1$, is a subgroup of $PSL_2(V_1)$ that is full of projective transvections. Put p = char F_1, q = card F_1. Clearly $q < \infty$, so $p > 0$. Let G_1 be a subgroup of $SL_2(V_1)$ that is full of transvections and for which $PG_1 = \Delta_1$. Now V_1 has $(q + 1)$ distinct lines, and we see by taking powers that there are at least $(p - 1)$ distinct nontrivial transvections in G_1 on each given line, so G_1 contains at least, $(p - 1)(q + 1)$ distinct nontrivial transvections. Now if you fix a line L and form $\tau_L\tau_K$ as τ_L runs through all the nontrivial transvections in G_1 with line L and τ_K runs through all nontrivial transvections in G_1 with lines K distinct from L, you get at least $(p-1)^2q$ distinct elements, none of them a transvection (apply 1.3.8). Therefore

$$\text{card } G_1 \geq (p-1)(q+1)+(p-1)^2 q+1. \tag{3.140}$$

But the kernel of $P|\mathrm{SL}_2(V_1)$ has 2 elements, so

$$\text{card } \Delta_1 \geq \frac{1}{2}\, p\,((p-1)\,q+1) = f\,(p,\,q). \tag{3.141}$$

But then $f\,(11, 3) = 341/2 > 168$, and, in particular $P \leq 7$ since card $\Delta_1 = 168$. Indeed, F_1 can only be $\mathbf{F}_3$, $\mathbf{F}_5$, $\mathbf{F}_7$, $\mathbf{F}_9$, $\mathbf{F}_{27}$. But card Δ_1 must divide card $\mathrm{PSL}_2(V_1)$, i.e. 168 must divide 12, 60, 168, 360, 9828. The only case in which this happens is the third.

(5) Therefore, by the process of elimination, we have shown that if you have an isomorphism Λ: $\Delta \rightarrow \Delta_1$ then $n = 3$, $F = \mathbf{F}_2$ and $F_1 = \mathbf{F}_7$. Of course $n_1 = 2$ by hypothesis. Now $\mathrm{PSL}_3(V)$ is the only full group in 3–dimensions over $\mathbf{F}_2$, so Δ is PSL_3 over $\mathbf{F}_2$. In particular card $\Delta = 168$, so card Δ_1=168. On the other hand, every full group in 2–dimensions over $\mathbf{F}_7$ must contain $\mathrm{PSL}_2(V_1)$ since F_7 is a prime field; hence $\Delta_1 \supseteq \mathrm{PSL}_2(V_1)$; but card $\Delta_1 = 168$ = card $\mathrm{PSL}_2(V_1)$; hence Δ_1 is PSL_2 over $\mathbf{F}_7$. □

3.5.8. *There can be no isomorphism* Ψ: $G \rightarrow G_1$ *when* $n \geq 3$ *with* $n_1 = 2$.

PROOF. For suppose we had an isomorphism Ψ: $G \rightarrow G_1$ with $n \geq 3$ and $n_1 = 2$. By 3.5.4 there is a subgroup G^0 of $G \cap \mathrm{SL}_n(V)$ that is full of transvections such that $\Psi G^0 \subseteq \mathrm{SL}_2(V_1)$. By considering two transvections in G^0 which do not permute we see that there must be a nontrivial transvection σ in G^0 with $\Psi\sigma \notin \mathrm{RL}_2(V_1)$, for a radiation commutes with every element of . By 3.3.4 $DC_{G^0}(\sigma) \neq 1_V$. Then $\sigma_1 = \Psi\sigma$ is an element of $\mathrm{SL}_2(V_1)$ with $DC_{\Psi G^0}(\sigma_1) \neq 1_V$, i.e. σ_1 is an element of $\mathrm{GL}_2(V_1) - \mathrm{RL}_2(V_1)$ with $DC_{V_1}(\sigma_1) \neq 1_{V_1}$ and this is impossible by 3.1.1. □

3.5.9. *Let* $X, Y \in \mathcal{X}$; *and* $L, K \in \mathcal{L}$ *and* $H, J \in \mathcal{H}$; *with* $L \subseteq H$ *and* $K \subseteq J$. *Then*

(1) $\Delta(X) = \Delta(Y) \Leftrightarrow X = Y$.

(2) $\Delta(L, H) = \Delta(K, J) \Leftrightarrow L = K$ *and* $H = J$.

(3) $\breve{\Delta}(X) = \breve{\Delta}(X^0)$

(4) $\breve{\Delta}(L, H) = \breve{\Delta}(H^0, L^0)$

(5) $\Delta(X) \cap \Delta(Y) \supset 1 \Leftrightarrow X \subseteq Y$ *or* $Y \subseteq X$.

(6) $\Delta(X)$ *is a maximal group of projective transvections in* Δ.

(7) *Every maximal group of projective transvections in* Δ *is a* $\Delta(X)$.

PROOF. The observations in (1–7) are rather easy consequences of the results from Chapter 1 and Chapter 3. We shall remark only (for the proof of (7)) that by 1.3.9 a given family of projective transvections is subgroup of $PGL_n(V)$ if and only if the elements has either the same fixed hyperplane or the same residual line. The result then follows easily. □

When $n \geq 3$, $n_1 \geq 3$, we can derive a mapping π: $\mathcal{X} \Rightarrow \mathcal{X}_1$ from the isomorphism Λ as follows. For each $X \in \mathcal{X}$, $\Delta(X)$ is a maximal group of projective transvections in Δ by 3.5.9; hence $\Lambda\Delta(X)$ is a maximal group of projective transvections in Δ_1 by 3.5.5; hence $\Lambda\Delta(X)$ has the form $\Lambda\Delta(X) = \Delta_1(X_1)$ for some unique X_1 in $\mathcal{X}_1$ by 3.5.9. Define $\pi X = X_1$.

3.5.10. *The mapping* π *derived from* Λ *in the above way* $n \geq 3$, $n_1 \geq 3$, *satisfies the following properties*:

(1) π: $\mathcal{X} \rightarrow \mathcal{X}_1$ is bijective.

(2) Its defining equation is $\Lambda\Delta(X) = \Delta_1(\pi X)$ for all X in $\mathcal{X}$.

(3) $X \subseteq Y$ or $Y \subseteq X \Leftrightarrow \pi X \subseteq \pi Y$ or $\pi Y \subseteq \pi X$

(4) ($\pi\mathcal{L} = \mathcal{L}_1$ and $\pi\mathcal{H} = \mathcal{H}_1$) or ($\pi\mathcal{L} = \mathcal{H}_1$ and $\pi\mathcal{H} = \mathcal{L}_1$)

PROOF. The first two results are immediate, the third follows from (5) of \ref{5.5.9}. So let us prove the fourth.

Suppose $\pi L \in \mathcal{L}_1$ for some $L \in \mathcal{L}$; then for any hyperplane $H \supseteq L$ we have $\pi L \subseteq \pi H$ or $\pi L \supseteq \pi H$ by step (3); but $\pi L \neq \pi H$ by injectivity; hence $\pi L \subseteq \pi H$ since πL is a line; in other words, if $\pi L \in \mathcal{L}_1$, for some $L \in \mathcal{L}$, then $\pi H \in \mathcal{H}_1$ for all hyperplanes H containing L. Dual reasoning shows that if $\pi H \in \mathcal{H}_1$ for some hyperplane H of V, then $\pi L \in \mathcal{L}_1$ for all lines $L \subseteq H$. Similarly, if $\pi L \in \mathcal{H}_1$ for some $L \in \mathcal{L}$, then $\pi H \in \mathcal{L}_1$ for all hyperplanes H containing L. And if $\pi H \in \mathcal{L}_1$ for some $H \in \mathcal{H}$ then $\pi L \in \mathcal{H}_1$, for all lines L, contained in H.

Suppose now that there is an L_0 in $\mathcal{L}$ with $\pi L_0 \in \mathcal{L}_1$. Then by surrounding L_0 and an arbitrary line L in V by a common hyperplane and applying the above facts, we see that $\pi\mathcal{L}\subseteq\mathcal{L}_1$. By applying the above to a typical hyperplane and one of its lines, we see that $\pi\mathcal{H} \subseteq \mathcal{H}_1$. But $\pi(\mathcal{L} \cup \mathcal{H})=\mathcal{L}_1 \cup \mathcal{H}_1$. Hence $\pi\mathcal{L}=\mathcal{L}_1$ and $\pi\mathcal{H}=\mathcal{H}_1$.

We may therefore suppose that $\pi\mathcal{L}\subseteq\mathcal{H}_1$. The above reasoning then makes $\pi\mathcal{H}\subseteq\mathcal{L}_1$. Hence $\pi\mathcal{L}=\mathcal{H}_1$ and $\pi\mathcal{H}=\mathcal{L}_1$. □

3.5.11. *An isomorphism* Λ: $\Delta\rightarrow\Delta_1$ *with* $n \geq 2$, $n_1 \geq 2$ *makes* $n = n_1$ *except, possibly, when one of the* Δ*'s is* PSL_3 *over* $\mathbf{F}_2$ *and the other is* PSL_2 *over* $\mathbf{F}_7$.

Proof. By 3.5.7 we can assume that $n \geq 3$, $n_1 \geq 3$. By interchanging the two groups if necessary we can assume that $n \geq n_1 \geq 3$. In particular the mapping π: $\mathcal{X}\rightarrow \mathcal{X}_1$ is now available. By passing to the dual

$$\Delta \xrightarrow{\Lambda} \Delta_1 \longrightarrow \breve{\Delta}_1 \tag{3.142}$$

if necessary we can assume that $\pi\mathcal{L} = \mathcal{L}_1$, $\pi\mathcal{H} = \mathcal{H}_1$. For any subspace U of V define

$$\Pi U = \sum_{L\subseteq U} \pi L \tag{3.143}$$

Then Π agrees with π on $\mathcal{X} = \mathcal{L}\cup\mathcal{H}$ by 3.5.10. And

$$U\subseteq W \Rightarrow \pi U \subseteq \pi W. \tag{3.144}$$

By considering a strictly ascending chain of n+1 subspaces of V we see that we will be through if we can verify that $U \subset \mathrm{W} \Rightarrow \pi U \subset \pi W$. To this end consider $U\subset W$ and pick a line L and a hyperplane H, both in V, with $L\subseteq W$, $U\subseteq H$, $L\not\subset H$. Then

$$\pi L\subseteq\pi W, \qquad \pi U \subseteq\pi H, \qquad \pi L\not\subset\pi H, \tag{3.145}$$

since Π agrees with π on L and H. If we had $\Pi U = \Pi W$ we would have $\Pi L \subseteq \Pi W = \Pi U \subseteq \Pi H$, and this is absurd. So $\Pi U \subset \Pi W$ as required. □

3.5.12. *Suppose* $n \geq 3$, $n_1 \geq 3$ *and that the mapping* π *associated with* Λ *satisfies* $\pi\mathcal{L} = \mathcal{L}_1$ *and* $\pi\mathcal{H} = \mathcal{H}_1$. *Let* Φ *be an isomorphism of* Δ *into* $\mathrm{P\Gamma L}_{n1}(V_1)$ *such that every element of* $\Phi\,\Delta(\mathrm{L})$ *is a projective transvection with residual line* πL, *for each* L *in* $\mathcal{L}$. *Then* $\Phi = \Lambda$.

Proof. Let k be a typical element of Δ. We must show that $\Phi k = \Lambda k$. Consider a typical line L in V. Then πL is a typical line in V_1. Let τ_L denote a projective transvection in Δ with residual line L. Then, by O'Meara §4.4, $k\tau_L\, k^{-1}$ is a projective transvection in Δ with residual line kL and we write it τ_{kL}. Now $\Phi\,\tau_L$ is a projective transvection in $\mathrm{P\Gamma L}_{n1}(V_1)$ with residual line πL; accordingly it can be written in the form $\Phi\,\tau_L = \tau_{\pi L}$. Similarly $\Phi\,\tau_{kL}$ is a projective transvection in $\mathrm{P\Gamma L}_{n1}(V_1)$ with residual line $\pi(kL)$ and can be written $\Phi\,\tau_{kL} = \tau_{\pi\,(kL)}$. We have

$$\tau_{\pi\,(kL)} = \Phi(\tau_{kL}) = \Phi(k\tau_L k^{-1}) = (\Phi k)(\tau_{\pi L})(\Phi k)^{-1} \tag{3.146}$$

and so

$$(\Phi k)\,(\pi L) = \pi\,(kL) \tag{3.147}$$

by O'Meara §4.4. Now Λ is a Φ so

$$(\Lambda k)(\pi L) = \pi(kL) \tag{3.148}$$

Hence

$$(\Phi K)(\pi L) = (\Lambda k)(\pi L) \tag{3.149}$$

In other words, Φk and Λk agree on the lines of V_1. Therefore $\Phi k = \Lambda k$. Therefore $\Phi = \Lambda$ □

3.5.13. THEOREM. *Let* Δ *and* Δ_1 *be subgroups of* $\mathrm{P\Gamma L}_n(V)$ *and* $\mathrm{P\Gamma L}_{n1}(V_1)$ *respectively that are full of projective transvections, and suppose* $n \geq 3$ *and* $n_1 \geq 3$. *Then each isomorphism* $\Lambda: \Delta \to \Delta_1$ *has exactly one of the forms*:

$$\Lambda k = gkg^{-1} \qquad \forall\, k \in \Delta \tag{3.150}$$

for a unique projective collinear transformation g of V onto V_1; *or*

$$\Lambda k = h\breve{k}\,h^{-1} \qquad \forall\, k \in \Delta \tag{3.151}$$

for a unique projective collinear transformation H of V' onto V_1.

PROOF. By 3.5.11 we have $n = n_1 \geq 3$.

(1) By considering the dual situation

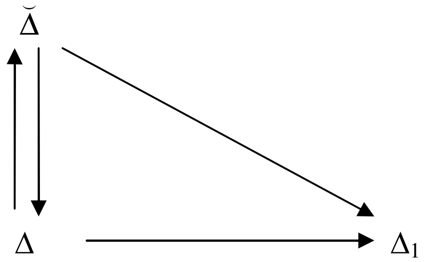

Figure 3.1: The dual situation

we see that, as far as existence is concerned, it will be enough to assume that the mapping π of 3.5.10 satisfies $\pi\mathcal{L}=\mathcal{L}_1$ and $\pi\mathcal{H}=\mathcal{H}_1$ and then deduce the existence of a g for which $\Lambda k = gkg^{-1}$ for all k in Δ.

Now, by 3.5.10, for any $L\in \mathcal{L}$, $H\in \mathcal{H}$ with $L\subseteq H$ we have $\pi L\subseteq \pi H$. So 2.1.4 applies and π can be extended uniquely to a projectivity g: $P(V)\to P(V_1)$. By the Fundamental Theorem of Projective Geometry, g is a projective collinear transformation. Then the restriction

$$\Phi_g k = gkg^{-1} \qquad \forall\, k \in \Delta \tag{3.152}$$

of the mapping Φ_g of O'Meara §4.4 is an isomorphism of Δ into $P\Gamma L_{n1}(V_1)$. And every element of $\Phi_g(\Delta(L))$ is a projective transvection in $P\Gamma L_{n1}(V_1)$ with residual line $gL = \pi L$, by O'Meara §4.4. Hence $\Phi_g = \Lambda$ by 3.5.12, i.e.

$$\Lambda k = gkg^{-1} \qquad \forall\, k \in \Delta \tag{3.153}$$

as required.

(2) Now the question of uniqueness. If we have two projective collinear transformations g and j of V onto V_1 such that $gkg^{-1} = \Lambda k = jkj^{-1}$ $\forall\, k \in \Delta$, then for any line L in $\mathcal{L}$ we have

$$g\tau_L g^{-1} = j\tau_L j^{-1} \tag{3.154}$$

for a nontrivial projective transvection τ_L in Δ with residual line L. So $gL = jL$ by §4.4. But g and j, being projectivities, are determined by their values on lines, hence $g = j$. The uniqueness of h then follows by applying the uniqueness of g to the dual situation $\breve{\Delta} \rightarrow \Delta_1$. Finally, we cannot have

$$gkg^{-1} = h\breve{k}\,h^{-1} \qquad \forall\, k \in \Delta \tag{3.155}$$

To see this consider projective transvections τ_1 and τ_2 in Δ with the same residual lines but distinct fixed hyperplanes; then $g\tau_1 g^{-1}$ and $g\tau_2 g^{-1}$ have the same property; while $h\breve{\tau}_1 h^{-1}$ and $h\breve{\tau}_2 h^{-1}$ do not. □

Now we consider a number of important corollaries of 3.5.13

3.5.13A THEOREM. *Isomorphic projective groups of collinear transformations that are full of projective transvections have the same underlying dimension except, possibly, when one of the groups is* PSL_3 *over* $\mathbf{F}_2$ *and the other is* PSL_2 *over* $\mathbf{F}_7$.

PROOF. By 3.5.11 and 3.5.13. □

3.5.13B THEOREM. *Isomorphic projective groups of collinear transformations that are full of projective transvections have isomorphic underlying fields when their common underlying dimension is* ≥ 3.

PROOF. The fact that groups Δ and Δ_1 are isomorphic implies by 3.5.13 existence of a projective collinear transformation from $P(V)$ or $P(V')$ onto $P(V_1)$ both over vector spaces of dimension ≥ 3. The Fundamental Theorem of Projective Geometry implies then that the fields F and F_1 are isomorphic (see O'Meara the proof of 4.5.2). □

3.5.13C THEOREM. *Isomorphisms between subgroups of* $\mathrm{P\Gamma L}_n(V)$ *that are full of projective transvections are induced by automorphisms of* $\mathrm{P\Gamma L}_n(V)$ *when* $n \geq 3$.

PROOF. Indeed, an isomorphism between two full subgroups of $\mathrm{P\Gamma L}_n(V)$ is determined by a projective collineation. The latter one determines in turn an automorphism of $\mathrm{P\Gamma L}_n(V)$.□

As usual, Aut(X)stands for the group of automorphisms of an arbitrary group X , and Inn(X) stands for the normal subgroup of Aut(X) consisting of all inner automorphisms of X . For any X the group Inn(X) is a normal subgroup of Aut(X).

3.5.13D THEOREM. (Aut $\mathrm{P\Gamma L}_n(V)$: Inn $\mathrm{P\Gamma L}_n(V)$) = 2 *when* $n \geq 3$.

PROOF. The inner automorphisms $\mathrm{P\Gamma L}_n(V)$ are maps of the first form (3.150) among the automorphism given in Theorem 3.5.13:

$$k \to gkg^{-1}, \tag{3.156}$$

where $g \in \mathrm{P\Gamma L}_n(V)$. Thus it remains to prove that two automorphisms of $\mathrm{P\Gamma L}_n(V)$ of the second kind described by the formula (3.151) are congruent modulo Inn $\mathrm{P\Gamma L}_n(V)$.

First of all, let us note that for any projective collineation h: $V' \to V$, the mapping ${}^t h$ is formally a map from V' onto V''. It is known however that $V'' \cong V$ and hence we can identify V'' and V. Thus we consider $\breve{h}$ as a map from V' to V. Now we claim that the inverse map of an automorphism of the form

$$\Phi\, k = h \breve{k}\, h^{-1}, \qquad \forall\, k \in \mathrm{P\Gamma L}_n(V) \tag{3.157}$$

is

$$\Psi k = {}^t h\ \breve{k}\ ({}^t h^{-1}), \qquad \forall\, k \in\ \mathrm{P\Gamma L}_n(V) \tag{3.158}$$

Really, keeping in mind that according to our identification of V and V'', $\breve{k} = k$ for all $k \in\ \mathrm{P\Gamma L}_n(V)$ we have that

$$\begin{aligned}\Phi\,(\Psi\, k) &= h\,({}^t h \breve{k}\ ({}^t h^{-1}))^{\breve{}}\ h^{-1}\\ &= h\,({}^t h)^{\breve{}}\ (\breve{k}\,)^{\breve{}}({}^t(h^{-1}))^{\breve{}}\ h^{-1}\\ &= h\, h^{-1} k\, h\, h^{-1} = k \qquad \forall\, k \in\ \mathrm{P\Gamma L}_n(V).\end{aligned} \tag{3.159}$$

In particular, our considerations imply that the inverse of an automorphism of the form (3.151) is an automorphism of the same form. Finally, it is now easy to check that the product of two automorphisms of the form (3.151) is an automorphism of the form (3.150), that is an inner automorphism.

Suppose that

$$\Phi_1(k) = h_1 \breve{k}\, h_1^{-1} \ \text{ and } \ \Phi_2(k) = h_2 \breve{k}\, h_2^{-1} \tag{3.160}$$

for all $k \in\ \mathrm{P\Gamma L}_n(V)$. Now

$$\Phi_1(\Phi_2\, k) = h_1(h_2 \breve{k}\, h_2^{-1})^{\breve{}}\ h_1^{-1} = h_1 \breve{h}_2 k\, (h_2^{-1})^{\breve{}}\ h_1. \tag{3.161}$$

Clearly, the transformation $h_1 \breve{h}_2$ is a projective collineation from V to V and we proved the desired. □

3.5.14 THEOREM. *Let G and G_1 be subgroups of $\Gamma\mathrm{L}_n(V)$ and $\Gamma\mathrm{L}_{n1}(V)_1$ respectively that are full of transvections, and suppose $n \geq 3$ and $n_1 \geq 3$. Then each isomorphism Ψ: $G \rightarrow G_1$ has exactly one of the forms*:

$$\Psi k = \chi(k) g k g^{-1} \qquad \forall\, k \in G \tag{3.162}$$

for a mapping χ *of* G *into* $\mathrm{RL}_{n1}(V_1)$ *and a collinear transformation* g *of* V *onto* V_1; *or*

$$\Psi k = \chi(k) h \breve{k} h^{-1} \qquad \forall\, k \in G \tag{3.163}$$

for a mapping χ *of* G *into* $\mathrm{RL}_{n1}(V_1)$ *and a collinear transformation* H *of* V' *onto* V_1.

PROOF. It is clear that the groups $\overline{G}$ and $\overline{G_1}$ are full of projective transvections. If we define $\overline{\Psi}$ by

$$\overline{\Psi}\,\overline{k} = \overline{\Psi k} \qquad \forall\, \overline{k} \in \overline{G} \tag{3.164}$$

then, by 3.5.6, $\overline{\Psi}$ is a well–defined isomorphism $\overline{\Psi}:\ \overline{G} \to \overline{G_1}$. Therefore, by Theorem 3.5.13, $\overline{\Psi}$ has exactly one of the forms

$$\overline{\Psi}\,\overline{k} = \overline{g}\,\overline{k}\,\overline{g}^{-1} \qquad \forall\, \overline{k} \in G \tag{3.165}$$

for some projective collinear transformation $\overline{g}$ of V onto V_1; or

$$\overline{\Psi}\,\overline{k} = \overline{h}\,\{\,\overline{k}\,\}^{\smile}\,\overline{h}^{-1} \qquad \forall\, \overline{k} \in G. \tag{3.166}$$

for some projective collinear transformation $\overline{h}$ of V' onto V_1. In the first case we have a collinear transformation g of V onto V_1 such that the elements Ψk and gkg^{-1} of $\Gamma\mathrm{L}_{n1}(V_1)$ satisfy

$$\overline{\Psi k} = \overline{gkg^{-1}} \qquad \forall\, \overline{k} \in G. \tag{3.167}$$

There is accordingly an element $\chi(k)$, in the kernel $\mathrm{RL}_{n1}(V_1)$ and dependent on k such that

$$\Psi k = \chi(k) g k g^{-1} \qquad \forall\, k \in G. \tag{3.168}$$

Similarly with the h situation. By applying Theorem 3.5.13 we see that Ψ cannot have both the g – and the h–forms. □

As before we consider a number of important corollaries of the last theorem.

3.5.14A. THEOREM. *Isomorphic groups of collinear transformations that are full of transvections have the same underlying dimension, when both underlying dimensions are* ≥ 2.

PROOF. By 3.5.14 if n, $n_1 \geq 3$ we are done. By 3.5.8 the case when $n \geq 3$ and $n_1 = 2$ is impossible. □

3.5.14B THEOREM. *Isomorphic groups of collinear transformations that are full of transvections have isomorphic underlying fields when their common underlying dimension is*≥ 3.

PROOF. Similarly to 3.5.13B. We just note that the condition n, $n_1 \geq 3$ was essential in the course of the proof of the result we refer to. □

3.5.14C REMARK. If the groups G and G_1 of Theorem 3.5.14 are groups of linear transformations, i.e. if they are contained in $GL_n(V)$ and $GL_{n1}(V_1)$ respectively, then: χ is a group homomorphism; and χ is uniquely determined by Ψ; and g (resp. h) is unique up to premultiplication by a radiation of V_1.

3.5.14D REMARK. If the groups G and G_1 of Theorem 3.5.14 are not only linear but also satisfy $DG = G$ and $DG_1 = G_1$ (for example if $G= SL_n(V)$ and $G_1= SL_{n1}(V_1)$, then the χ function is trivial, i. e.

$$\Psi k = gkg^{-1} \quad \forall\, k \in G \qquad \text{or} \qquad \Psi k = h \breve{k}\, h^{-1} \quad \forall\, k \in G \tag{3.169}$$

It easily follows from the fact that χ is a homomorphism and the fact that F is commutative:

$$\chi(k) = \chi([k_1,k_2]) = \chi(k_1^{-1})\, \chi(k_2^{-1})\, \chi(k_1)\, \chi(k_2) = \chi(k_1^{-1})\, \chi(k_1)\, \chi(k_2^{-1})\, \chi(k_2) = 1. \qquad (3.170)$$

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