TULCZYJEW’S CONSTRUCTION OF LEGENDRE TRANSFORMATIONS

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ABSTRACT

TULCZYJEW'S CONSTRUCTION OF LEGENDRE TRANSFORMATIONS

Being equivalent to Newton's equations, the dynamics of a system can either be formulated by Lagrangian function on tangent bundle of configuration space or by Hamiltonian function on cotangent bundle. Two formulations are equivalent via Legendre transformation provided the Lagrangian function satisfies certain non degeneracy conditions which become even more complicated if the system possesses various constraints.

By investigating of natural geometric structures underlying Lagrangian and Hamiltonian formulations it is shown that, the dynamics can be represented as a Lagrangian submanifold of certain symplectic structures on higher order bundles over configuration space. The objects generating such manifolds turn out to be not unique. Hamiltonian and Lagrangian formulations are, in fact, two realizations of the same Lagrangian submanifold by different generating objects. In this sense, the Legendre transformation becomes a passage between two different realizations of the same Lagrangian submanifold.

ÖZET

TULCZYJEW'İN LEGENDRE TRANSFORMASYONLARINI YAPILANDIRMASI

Bir dinamik sistem, konfigürasyon uzayının tanjant demeti üzerinde tanımlı Lagrange fonksiyonları veya kotanjant demeti üzerinde tanımlı Hamilton fonksiyonları aracılığıyla iki farklı şekilde formülize edilebilir. Legendre transformasyonları bu iki temel yaklaşım arasındaki transformasyonlardır, ki bu transformasyonların varlığı Lagrange fonksiyonlarının bazı dejenere olmama koşullarını sağlamasıyla mümkün olabilmektedir.

Hamilton ve Lagrange formülasyonları bir takım geometrik yapılar barındırmaktadırlar. Tezde bu gerçekten hareketle, dinamik, konfigürasyon uzayı üzerindeki yüksek mertebeden demetlerin barındırdığı simplektik yapıların Lagrange altkatmanları olarak tanımlanmıştır. Lagrange altkatmanları doğurucu objeler tarafından üretilir. Bir Lagrange altkatmanının farklı objeler tarafından üretilmesi de mümkündür. Bu ise Hamilton ve Lagrange formülasyonlarının aynı Lagrange altkatmanını, diğer bir ifadeyle aynı dinamiği, üreten farklı doğurucu objeler olarak tanımlanmasını sağlamıştır. Ayrıca, verilen bir doğurucu objeden aynı Lagrange altkatmanını üreten diğer bir doğurucu objeye geçiş Legendre transformasyonu olarak adlandırılmıştır.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

Q, P, R	Manifolds
S, N, X	Submanifolds
TQ	Tangent bundle of the manifold Q
T^*Q	Cotangent bundle of the manifold Q
τ_Q	Tangent fibration
π_Q	Cotangent fibration
θ_Q	Liouville 1-form
ω_Q	Canonical 2-form
φ	Symplectomorphism between two manifolds
$\langle .,. \rangle$	Pairing between T^*Q and TQ
d	Exterior derivative
$\Omega(Q)$	Graded algebra of the differential forms on Q
$[.,.]$	Lagrange bracket
VR	Vertical bundle of the manifold Q
U	Function on the manifold R
ρ	Fibration from R to X
$Cr(U,\rho)$	Critical set
V	Vector field over reals
H	Hamiltonian function
L	Lagrangian function
$\overset{o}{TQ}$	Tangent bundle with zero section removed
Λ	The Legendre Relation

1. INTRODUCTION

Around 1790, in his investigations of Newton's equations $m\ddot{x}^i = X^i$ written in Euclidean coordinates $x^i = x^i(q^1, q^2, ..., q^m)$, Lagrange (1736-1813) introduced generalized coordinates $(q^1, q^2, ..., q^m)$ and their velocities $(\dot{q}^1, \dot{q}^2, ..., \dot{q}^m)$. Starting from the fact that, the right hand side of the equation; that is, the forces acting on the system is the rate of change of work done on the system, he was able to write Newton's equations in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^j}\right) - \frac{\partial T}{\partial q^j} = F_j, \tag{1.1}$$

where F_j and T are the generalized forces and kinetic energy given, in coordinates, as

$$F_j = \delta_{ij} X^i \frac{\partial x^j}{\partial q^k}, \quad T = \frac{1}{2} m \delta_{ij} \dot{q}^i \dot{q}^j, \tag{1.2}$$

δ_{ij} is the Kronecker delta and we employ the summation over repeated indices. For conservative force fields $F_i = -\partial V / \partial q^i$, the Lagrangian form of Newton's equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \tag{1.3}$$

and the function $L = T - V$ is called the **Lagrangian function**.

About 1830, Hamilton (1805-1865) realized that these equations can be obtained by requiring the generalized coordinates q^i minimize the functional

$$I = \int_{t_1}^{t_2} L\left(t, q^i, \dot{q}^j\right) dt \tag{1.4}$$

along with some appropriate boundary conditions. That is, Eqs (1.3) are the Euler-Lagrange equations for the functional in Eq (1.4). This is called **Hamilton's principle** or **principle of least action**. Due to generalized coordinates introduced by

Lagrange, these equations do not depend on specific choice of coordinates, in other words; Euler-Lagrange equations are invariant with respect to coordinate transformations. This important property of Lagrangian formulation of dynamics is called **covariance**.

Introducing the **generalized momenta**

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad i = 1, ...m \tag{1.5}$$

the second order Euler-Lagrange equations, which are m in number, can be written as a system of $2m$ first order equations

$$\dot{p}_i = \frac{\partial L}{\partial q^i}, \;\; p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad i = 1, ...m \tag{1.6}$$

provided $L = L\left(q^i, \dot{q}^j\right)$ and the second set of m equations can be solved for generalized velocities $\dot{q}^i$ in terms of momenta p_i. This imposes the non-degeneracy condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0, \tag{1.7}$$

on the Lagrangian function $L = L\left(q, \dot{q}\right)$. In this case, L is called **(hyper)regular**.

The regularity condition brings serious restrictions on the conversion of the second order Euler-Lagrange equations to a system of first order equations and is in fact the main problem to be addressed in this thesis. The problem of explicit dependence of L on time can easily be handled by introducing energy as a canonically conjugate variable to t.

Once the condition of the regularity is satisfied, one can introduce the **Legendre transformation** of the function L,

$$H = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L \tag{1.8}$$

which is called the **Hamiltonian function**. With this function, the system of the first order equations takes the form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{1.9}$$

of **canonical Hamilton's equations**. These first order equations, which can be considered as a vector field in the canonical coordinates (q^i, p_j) of the cotangent space T_q^*Q are the most basic for the geometrization of mechanics. However, there is a pay off to obtain this geometric picture due to the distinguished role of time as an evolution parameter; the loss of covariance of the Euler-Lagrange equations.

Although, Eqs (1.9) are named to Hamilton, they first appeared in a work of Lagrange on celestial mechanics published in 1808. Lagrange used Hamilton's equations for computational purposes only, thereby not realizing the structural aspects of them. The geometric and algebraic structures underlying the Hamilton's equations were developed by Hamilton, Jacobi (1804-1851), Lie(1842-1899), Poisson(1781-1840), Dirac(1902-1984), Arnold(1937-...), Weinstein(1937-...), Lichnerowicz(1915-1998) and many others.

We briefly summarize the present day status of Hamilton's equations: On the cotangent bundle T^*Q of any manifold Q of dimension m, there is a natural, non-degenerate, closed two-form which is called **canonical symplectic structure**. In canonically conjugate coordinates this is $\omega_Q = dq^i \wedge dp_i$. In the framework of symplectic geometry, the Hamiltonian dynamics is governed by a choice of function H on T^*Q via the Hamilton's equation

$$i_{X_H}(\omega_Q) = dH, \quad X_H = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i}. \tag{1.10}$$

The correspondence between H and X_H is one to one provided ω_Q is non-degenerate. Being coordinate independent, this structure can be generalized to any manifold P with a non-degenerate two form ω. The pair (P, ω) is called a **symplectic manifold**. The vector fields on P which can be written in the form $i_{X_H}(\omega) = dH$ for some

H constitute the infinite dimensional Lie algebra of Hamiltonian vector fields in the tangent bundle TP. This is isomorphic to the algebra of functions on P with the **Poisson bracket**

$$\{H, G\} = \omega\left(X_H, X_G\right). \tag{1.11}$$

The Poisson bracket algebra of functions on P can, in turn, be associated with a dual contravariant structure on P, one defined by a non-degenerate bi-vector Λ on P. This is called a **Poisson structure**. One can now release the condition for degeneracy because, in this picture, the dynamics $X_H = \Lambda\left(dH\right)$ for a given function H is unique. This time, the degeneracy, if any, brings ambiguity in the choice of Hamiltonian function H. Namely, the kernel of the bi-vector Λ consists of Casimir functions, which are conserved quantities of X_H.

1.1. Content of The Work

In the next section, we shall briefly summarize the differential geometric construction of tangent and cotangent bundles as well as mappings between them. We shall introduce the higher order tangent and cotangent bundles and fix the local coordinates which will be used throughout the work. Of particular importance is the construction of canonical isomorphism κ_Q on TTQ .

In section 3, we shall explain the natural exact symplectic structure that exists on any cotangent bundle. The canonical one form arises from two different fibrations of the tangent bundle TT^*Q of the cotangent bundle. For future reference, we shall fix the ones on T^*Q, T^*TQ and T^*T^*Q.

In section 4, we shall construct the derivations $i_T : \Omega\left(Q\right) \rightarrow \Omega\left(TQ\right)$ and $d_T : \Omega\left(Q\right) \rightarrow \Omega\left(TQ\right)$ of exterior algebras over Q and TQ of degree -1 and 0, respectively. We shall use these derivations to construct an exact symplectic structure on TT^*Q

with two different canonical one-forms, namely, $d_T\theta_Q$ and $i_T\omega_Q$, where $\omega_Q = -d\theta_Q$ is the symplectic two-form on T^*Q. The fact that these 1-forms are not related by any symmetry of symplectic geometry will be crucial in constructing symplectic structure on product spaces.

In section 5, we shall show that the symplectic structure on TT^*Q is naturally diffeomorphic to the canonical symplectic structure on T^*T^*Q. To show that TT^*Q is also symplectomorphic to T^*TQ, we shall construct the isomorphism $\alpha_Q : TT^*Q \to T^*TQ$ which turns out to be the required symplectic diffeomorphism. Construction of α_Q relies on the fibrations of the spaces dual to the ones in the isomorphism $\kappa_Q : TTQ \to TTQ$. These symplectomorphisms are indeed special cases of a more abstract construction that we shall explain in the next section.

In section 6, we shall define a special symplectic structure for a symplectic manifold $(P, \omega = -d\theta)$ to be a fibration $\pi : P \to Q$ which is symplectomorphic with $\alpha : P \to T^*Q$ to the fibration $T^*Q \to Q$ with canonical symplectic structure. Such a structure implies that the collection $(TP, TQ, T\pi, d_T\theta, \alpha_Q \circ T\alpha)$ is also a special symplectic manifold. Choosing $P = T^*Q$, it turns out that the symplectomorphisms of TT^*Q to T^*T^*Q and to T^*TQ can equivalently be realized as two different special symplectic structure of the same underlying symplectic manifold $(TT^*Q, d_T\omega_Q)$. The totality of structures on the above three spaces is called a Tulczyjew's triplet.

For dynamical interpretations of these structures, we shall introduce, in section 7, Lagrangian submanifolds and their generating objects. We shall first explain how to construct generating objects of Lagrangian submanifolds of cotangent bundles. Combined with special symplectic structures, we shall be able to obtain generating objects of Lagrangian submanifolds of general symplectic manifolds. In particular, we shall obtain Hamiltonian and Lagrangian dynamics as Lagrangian submanifolds of TT^*Q with generating functions $-H$ and L.

In section 8, we shall consider Lagrangian submanifolds of products of symplectic manifolds which can alternatively be characterized as graphs of symplectomorphisms of the manifolds involving the product. We shall investigate relations between generating functions of various Lagrangian submanifolds.

In section 9, we shall define Legendre transformation in the framework of geometric structures developed so far. The main result of this section is the geometric interpretation of Legendre transformation as a relation between generating objects of Lagrangian submanifolds of special symplectic manifolds and of their products. We shall conclude this section with discussions of special cases and various examples.

2. PRELIMINARIES

2.1. Tangent and Cotangent Bundles

Let Q be an m-dimensional differentiable manifold and

$$\left(\varphi^{i}\right) : Q \to \mathbb{R}^{m} : x \to \varphi\left(x\right) = \left(q^{1}, q^{2}, ..., q^{m}\right) \tag{2.1}$$

a coordinate system around $x \in Q$. A function f on Q is **differentiable**, if the coordinate function

$$\tilde{f} = f \circ \varphi^{-1} : \mathbb{R}^{m} \to \mathbb{R},$$

is differentiable. In this case, the partial derivatives of f are defined by

$$\partial_{i} f\left(x\right) = \frac{\partial}{\partial q^{i}} f\left(x\right) = \frac{\partial \tilde{f}}{\partial q^{i}} \circ \varphi\left(x\right) = \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial \varphi^{i}\left(x\right)} \circ \varphi\left(x\right). \tag{2.2}$$

We shall use the letter D to denote derivatives of vector valued functions. For example, for $f : Q \to \mathbb{R}$ and $\gamma : \mathbb{R} \to Q$ we have

$$D(f \circ \gamma)\left(t\right) = \partial_{i} f\left(\gamma\left(t\right)\right) \frac{d\gamma^{i}}{dt}\left(t\right). \tag{2.3}$$

A **curve** on Q through the point x is a function $\gamma : \mathbb{R} \to Q$ with $\gamma\left(0\right) = x$. Two curves γ and $\tilde{\gamma}$ are called **equivalent** if they agree at x and if the directional derivatives of functions along them at x are the same, namely,

$$\gamma\left(0\right) = \tilde{\gamma}\left(0\right), \ D(f \circ \gamma)\left(0\right) = D\left(f \circ \tilde{\gamma}\right)\left(0\right) \tag{2.4}$$

for all functions $f : Q \to \mathbb{R}$.

A **tangent vector** $v(x)$ at x is an equivalence class of curves at x. This class will be denoted by $t\gamma(0)$ if $\gamma(0) = x \in Q$. The set of all equivalence classes of curves, that is the set of all tangent vectors at x is the **tangent space** T_xQ at $x \in Q$.

A basis for the tangent space T_xQ is induced by the coordinate system (φ^i) around x by their action on functions

$$\frac{\partial}{\partial q^i} f(x) = \frac{\partial}{\partial \varphi(x)} \left(f \circ \varphi^{-1}\right)(\varphi(x)), \tag{2.5}$$

where the right hand side is evaluated in $\mathbb{R}^m$ and defines the left hand side. Thus, a tangent vector can be written in the form $v(x) = a^i \partial/\partial q^i$ for some real numbers a^i.

Let γ be a representative of $v(x) \in T_xQ$, then $q^i = \varphi^i(t\gamma(0))$ and $\dot{q}^i = D(\varphi^i \circ \gamma)(0)$. So the induced coordinates of T_xQ are

$$(\varphi, \dot{\varphi}) : T_xQ \to \mathbb{R}^{2m} : v \to (q, \dot{q}). \tag{2.6}$$

By definition, the coordinate functions $\varphi : Q \to \mathbb{R}$ and $\varphi : T_xQ \to \mathbb{R}$ are different due to their domains, but we shall use the same letter for simplicity.

The sum $TQ = \bigcup\limits_{x \in Q} T_xQ$ of all tangent spaces T_xQ as x varies on Q is the total space of the **tangent bundle** of Q with fibers being T_xQ.

The **tangent fibration** $\tau_Q : TQ \to Q$ is defined on each fiber T_xQ by $\tau_Q(t\gamma(0)) = \gamma(0)$ or $\tau_Q(v(x)) = x$.

A **section** of the tangent fibration is a map $v : Q \to TQ$ such that $\tau_Q \circ v : id_Q$ where id_Q is the identity map on Q. A **vector field** on Q is a section of the tangent fibration.

For any two manifolds Q and P, the **tangent map** $T\psi : TQ \to TP$ of a differentiable mapping $\psi : Q \to P$ is defined by requiring the following diagram to be commutative

$$\begin{tikzcd} TQ \arrow[r, "T\psi"] \arrow[d, "\tau_Q"] & TP \arrow[d, "\tau_P"] \\ Q \arrow[r, "\psi"'] & P \end{tikzcd}$$

Figure 2.1. Tangent map

namely, if $\gamma : \mathbb{R} \to Q$ is a representative of a vector $v(x) \in T_xQ$, then $\psi \circ \gamma : \mathbb{R} \to P$ is a representative of the vector $T\psi \circ v(x) \in T_{\psi(x)}P$ and the commutativity condition becomes

$$T\psi(t\gamma(0)) = t(\psi \circ \gamma). \tag{2.7}$$

Written in terms of sections $v : Q \to TQ$, above diagram defines a vector field

$$\begin{tikzcd} TQ \arrow[r, "T\psi"] & TP \\ Q \arrow[r, "\psi"'] \arrow[u, "v"'] & P \arrow[u, "v_P"'] \end{tikzcd}$$

Figure 2.2. Vector field

$v_P : P \to TP$ called the **push-forward** of v,

$$\psi_* v = v_P = T\psi \circ v \circ \psi^{-1}. \tag{2.8}$$

In local coordinates $(\varphi^i) : Q \to \mathbb{R}^m$ and $\left(\phi^i\right) : P \to \mathbb{R}^n$, with $\phi^i(y) = p^i$ for

$y \in P$. The tangent mapping $T\psi$ is given by

$$\left(\varphi^i, \dot{\varphi}^j\right) \circ T\psi = \left(\phi^i \circ \psi \circ \tau_Q, \left(\frac{\partial\left(\phi^j \circ \psi\right)}{\partial q^i} \circ \tau_Q\right)\dot{q}^i\right) \tag{2.9}$$

from which we read the components of the push-forward v_P at $y = \psi(x) \in P$ as

$$\left(v_P(y)\right)^j = \frac{\partial \psi^j(x)}{\partial q^i} v^i(x). \tag{2.10}$$

In particular, let $P = \mathbb{R}$ and consider the function $f : Q \to \mathbb{R}$, then from previous paragraph the tangent map of f at x is given by

$$\begin{aligned} T_x f(t\gamma(0)) &= D(f \circ \gamma)(0) = D\left(f \circ \varphi^{-1} \circ \varphi \circ \gamma\right)(0) \\ &= D\left(f \circ \varphi^{-1}\right)(\varphi(x))\, D(\varphi \circ \gamma)(0) \\ &= \frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial q^i}(q)\, \dot{q}^i(x) \end{aligned} \tag{2.11}$$

hence the map $T_x f$ can be identified with numbers $\left(\partial\left(f \circ \varphi^{-1}\right)/\partial q^i\right)(q)$. This defines a linear map $df(x) : T_x Q \to \mathbb{R}^m$ which can be interpreted as a **linear functional** on tangent vectors. Thus, the **differential** df of a function on Q is an element of the dual space $(T_x Q)^*$ which can also be written as $T_x^* Q$. This is the **cotangent space** at x. It follows that the basis of $T_x^* Q$ dual to the basis $\{\partial/\partial q^i\}$ of $T_x Q$ is $\{dq^i\}$. In this basis, we have the differentials $df = \left(\partial f/\partial q^i\right) dq^i$.

2.2. Differential Forms

Elements of $T_x^* Q$ of the form $\theta(x) = \theta_i(x)\, dq^i$ are called **differential 1-forms** at x.The collection of all cotangent spaces $T^* Q = \bigcup_{x \in Q} T_x^* Q$ is the total space of the **cotangent bundle** of Q with the projection $\pi_Q : T^* Q \to Q$ defined by $\pi_Q(\theta(x)) = x$. Sections $\theta : Q \to T^* Q$ of cotangent bundle are 1-forms on Q satisfying $\pi_Q \circ \theta = id_Q$.

In general, a **differential** p**-form** on Q is a skew-symmetric multilinear functional on TQ and can be uniquely represented in local coordinates by

$$\omega(x) = \frac{1}{p!}\omega_{i_1 i_2 \ldots i_p}(x)\, dq^{i_1} \wedge dq^{i_2} \wedge \ldots \wedge dq^{i_p} \tag{2.12}$$

where $\wedge$ denotes the **wedge product** defined as an alternating **tensor product**

$$dq^i \wedge dq^j = \frac{1}{2}\left(dq^i \otimes dq^j - dq^j \otimes dq^i\right) \tag{2.13}$$

and extended by linearity. In particular, a 2-form on Q at x has the representative

$$\omega(x) = \frac{1}{2}\omega_{ij}(x)\, dq^i \wedge dq^j. \tag{2.14}$$

On the space $\Lambda(T^*Q) = \Omega(Q)$ of all differential forms of degree p, $0 \leq p \leq m$ on Q the **exterior derivative** is defined on functions as their differentials, on a p-form ω by the property $d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^p \omega \wedge d\alpha$ for any form α, and extended by linearity. A p-form ω is **closed** if $d\omega = 0$ and is **exact** if there exist a $(p-1)$-form α such that $\omega = d\alpha$.

Since d and exterior product carry the coordinate transformations and their Jacobians intrinsically, differential forms transform covariantly and formulation with these covariant objects are coordinate free.

2.3. Higher Order Tangent and Cotangent Bundles

It is possible to give a description of manifold structures on the total spaces TQ and T^*Q induced from that of Q. Then, regarding TQ and T^*Q as manifolds, one can proceed to construct higher order tangent and cotangent bundles TTQ, TT^*Q, T^*TQ and T^*T^*Q, as well as to assign induced local coordinates on fibers of these bundles [15], [16]. For future reference we shall give a list of local coordinates on these bundles.

We shall use the coordinates $\{q^i\}$ on Q around x and **canonical basis** $\{e_i(x) = \partial/\partial q^i\}$, $\{e^i(x) = dq^i\}$ for T_xQ and T_x^*Q respectively, together with the **canonical pairing** $\langle e^i, e_j\rangle(x) = \delta^i_j$. In the adapted basis the elements of T_xQ and T_x^*Q can be written in the form

$$v(x) = v^i(x)\frac{\partial}{\partial q^i}, \quad \alpha(x) = p_i(x)\,dq^i \tag{2.15}$$

respectively. Equivalently, we may represent $v(x)$ and $\alpha(x)$ by their base and fiber coordinates (q^i, v^j) and (q^i, p_j). In such a representation we shall employ the following coordinates for higher order bundles:

$$\left(q^i, p_j; v^k, V_l\right) \in T_{\alpha(x)}T_x^*Q, \tag{2.16}$$

$$\left(q^i, v^j; \alpha_k, \beta_l\right) \in T^*_{v(x)}T_xQ, \tag{2.17}$$

$$\left(q^i, p_j; a_k, b^l\right) \in T^*_{\alpha(x)}T_x^*Q, \tag{2.18}$$

$$\left(q^i, v^j; \xi^k, \eta^l\right) \in T_{v(x)}T_xQ. \tag{2.19}$$

Note that each of these spaces are $4m$ dimensional if Q is m dimensional.

2.4. Dual Tangent Rhombic

Elements of the iterated tangent space $T_{v(x)}T_xQ$ are equivalence classes of curves in T_xQ. The total space TTQ has two fibrations over TQ. One is the natural tangent bundle fibration $\tau_{TQ} : TTQ \to TQ$ and the other is induced from $\tau_Q : TQ \to Q$ as a tangent mapping $T\tau_Q : TTQ \to TQ$. To find local representatives of these fibrations we first note that the coordinates (q, v) on T_xQ are representatives of the tangent map $T\varphi : TQ \to \mathbb{R}^{2m}$ of the coordinate function $(\varphi) : Q \to \mathbb{R}^m$. From the diagram

$$
\begin{tikzcd}
TQ \arrow[r, "T\varphi"] \arrow[d, "\tau_Q"] & IR^{2m} \\
Q \arrow[r, "\varphi"'] & IR^{m}
\end{tikzcd}
$$

Figure 2.3. Local representative of projection tau

we have the local representative of τ_Q

$$\left(\varphi \circ \tau_Q \circ T\varphi^{-1}\right)(q, v) = q. \tag{2.20}$$

By construction of TTQ, if φ is a coordinate function around $x \in Q$, then $T\varphi$ is a coordinate function around $v\,(x) \in T_xQ$. Replacing φ with $T\varphi$ in above formula we get the local representative of τ_{TQ},

$$\left(T\varphi \circ \tau_{TQ} \circ TT\varphi^{-1}\right)(q, v; \xi, \eta) = (q, v)\,. \tag{2.21}$$

For the local representative of $T\tau_Q$ we replace τ_Q with $T\tau_Q$ and φ with $T\varphi$ to get

$$\left(T\varphi \circ T\tau_Q \circ TT\varphi^{-1}\right)(q, v; \xi, \eta) = T\left(\varphi \circ \tau_Q \circ T\varphi^{-1}\right)(q, v; \xi, \eta) \tag{2.22}$$

which is actually the tangent map of the representative of τ_Q in Eq (2.20). Hence, it gives the vector over x with components from $T_{v(x)}T_xQ$ that is

$$\left(T\varphi \circ T\tau_Q \circ TT\varphi^{-1}\right)(q, v; \xi, \eta) = (q, \xi)\,. \tag{2.23}$$

From now on we shall not distinguish the projections and their coordinate representations. In other words, we shall use, for example, instead of Eqs (2.21) and (2.23)

$$\tau_{TQ}\,(q, v; \xi, \eta) = (q, v)\,, \quad T\tau_Q\,(q, v; \xi, \eta) = (q, \xi) \tag{2.24}$$

to avoid proliferation of notations. The iterated tangent bundle with two projections

$$\begin{tikzcd} & TTQ \arrow[dl, "T\tau_Q"'] \arrow[dr, "\tau_{TQ}"] & \\ TQ \arrow[dr, "\tau_Q"'] & & TQ \arrow[dl, "\tau_Q"] \\ & Q & \end{tikzcd}$$

Figure 2.4. Dual tangent rhombic

is known as **dual tangent rhombic**. This structure implies the existence of an isomorphism κ_Q on TTQ such that the following diagram commutes

$$\begin{tikzcd} TTQ \arrow[rr, "\kappa_Q"] \arrow[dr, "T\tau_Q"'] & & TTQ \arrow[dl, "\tau_{TQ}"] \\ & TQ & \end{tikzcd}$$

Figure 2.5. Kappa mapping

that is κ_Q may be defined as to satisfy $T\tau_Q = \tau_{TQ} \circ \kappa_Q$ which implies together with Eqs (2.24), the coordinate representations

$$\kappa_Q\left(q, v; \xi, \eta\right) = \left(q, \xi; v, \eta\right). \tag{2.25}$$

3. CANONICAL SYMPLECTIC STRUCTURES

In this section we shall describe the natural symplectic structures on T^*Q, T^*TQ and T^*T^*Q. These are exact symplectic manifolds on which canonical 1-forms arises naturally from two different fibrations of their tangent bundles. We shall show the construction for T^*Q explicitly and present the results for the other two. In case of T^*Q, the fibration of TT^*Q over T^*Q and TQ together with the canonical pairing lead us to a natural definition of canonical or Liouville 1-form on T^*Q.

If $\alpha(x) = (q,p) \in T^*_x Q$ and $v(x) = (q,v) \in T_x Q$ we can write a general $1-$form as

$$\theta_Q(\alpha(x)) = a_i(q,p)\, dq^i + b^i(q,p)\, dp_i \in T^*_{\alpha(x)} T^*_x Q$$

for arbitrary functions a_i and b^i.

Based on the commutativity of the following diagram

$$\begin{tikzcd}
TT^*Q \arrow[r, "\tau_{T^*Q}"] \arrow[d, "T\pi_Q"'] & T^*Q \arrow[d, "\pi_Q"] \\
TQ \arrow[r, "\tau_Q"'] & Q
\end{tikzcd}$$

Figure 3.1. Fibrations of TT^*Q

the canonical or Liouville $1-$form on T^*Q is defined by requiring θ_Q to satisfy

$$\langle \theta_Q, v_{T^*Q} \rangle := \langle \tau_{T^*Q} \circ v_{T^*Q}, T\pi_Q \circ v_{T^*Q} \rangle \tag{3.1}$$

for all $v_{T^*Q} : T^*Q \to TT^*Q$. Locally, if

$$v_{T^*Q}\left(\alpha\left(x\right)\right) = \left(q, p; v, V\right) \tag{3.2}$$

then we have the projections

$$\left(\tau_{T^*Q} \circ v_{T^*Q}\right)\left(\alpha\left(x\right)\right) = \left(q, p\right), \quad \left(T\pi_Q \circ v_{T^*Q}\right)\left(\alpha\left(x\right)\right) = \left(q, v\right) \tag{3.3}$$

and Eq (3.1) gives

$$a_i v^i + b^i V_i = p_i v^i. \tag{3.4}$$

Therefore, $b^i = 0$, $a_i = p_i$ and the local representative of the canonical 1-form on T^*Q becomes

$$\theta_Q\left(q, p\right) = p_i dq^i$$

The exterior derivative of θ_Q

$$\omega_Q\left(q, p\right) = -d\theta_Q\left(q, p\right) = dq^i \wedge dp_i \tag{3.5}$$

is the **canonical symplectic 2-form** on T^*Q. Being a bilinear, skew-symmetric functional on TQ, ω_Q satisfies the condition of non-degeneracy, that is,
$\omega_Q\left(v_{T^*Q}, u_{T^*Q}\right) = 0$ for all $u_{T^*Q} : T^*Q \to TT^*Q$ implies $v_{T^*Q} = 0$.

A similar construction can be applied to the higher cotangent bundles T^*TQ and T^*T^*Q. In the adapted coordinates $\left(q^i, v^j; \alpha_k, \beta_l\right) \in T^*_{v(x)}T_xQ$ and $\left(q^i, p_i; a_i, b^i\right) \in T^*_{\alpha(x)}T^*Q$ the canonical 1-forms are

$$\begin{aligned} \theta_{TQ}\left(q, v; \alpha, \beta\right) &= \alpha_i dq^i + \beta_i dv^i, \\ \theta_{T^*Q}\left(q, p; a, b\right) &= a_i dq^i + b^i dp_i \end{aligned} \tag{3.6}$$

and the symplectic forms can be obtained by derivatives of them.

We note that the 1-form defining the exact symplectic structure is not unique. In fact, every canonical symplectic structure can be obtained from

$$\theta(x) = \theta_Q(x) + d\phi(x) \tag{3.7}$$

for arbitrary function ϕ on Q. This is the **gauge invariance** of symplectic geometry. In the next section, we shall construct two 1-forms on TT^*Q not related as in Eq (3.7) which, however, give rise to the same symplectic structures.

4. EXACT SYMPLECTIC STRUCTURES ON TT^*Q

We shall first introduce two derivations i_T, $d_T : \Omega(Q) \to \Omega(TQ)$ of exterior algebras over Q and TQ. We then apply these derivations to canonical 1-form θ_Q and symplectic 2-form ω_Q on T^*Q to obtain 1-forms $d_T\theta_Q$ and $i_T\omega_Q$ on TT^*Q. The derivative of these 1-forms will give the same symplectic 2-form on TT^*Q. Hence, TT^*Q will be shown to be an exact symplectic manifold with two different 1-forms.

4.1. Derivatives of Exterior Algebras

Let α be a 1-form on Q. Define the function $i_T\alpha$ on TQ by

$$(i_T\alpha)(q,v) = \langle \alpha, v \rangle \quad \text{for all } v \in TQ. \tag{4.1}$$

In coordinates, if $\alpha(x) = \alpha_i(x)\,dq^i$ and $v(x) = v^i(x)\,\partial/\partial q^i$ then $i_T\alpha(q,v) = \alpha_i(q)\,v^i$. In particular, on basis 1-forms $i_T(dq^i) = v^i$. To define the operator i_T on 2-forms recall from the construction of dual tangent rhombic that if $v_{TQ} : TQ \to TTQ$ then we have

$$(\tau_{TQ} \circ v_{TQ}, T\tau_Q \circ v_{TQ}) \in TQ \times_Q TQ \tag{4.2}$$

since $\tau_Q \circ \tau_{TQ} = \tau_Q \circ T\tau_Q$. The action of i_T on a 2-form ω on Q is a 1-form on TQ defined by

$$\langle i_T(\omega), v_{TQ} \rangle = \omega(\tau_{TQ} \circ v_{TQ}, T\tau_Q \circ v_{TQ}) \tag{4.3}$$

In coordinates, let $\omega(x) = \frac{1}{2}\omega_{ij}(q)\, dq^i \wedge dq^j,\ \upsilon_{TQ}(q,v) = \xi^i(q,v)\frac{\partial}{\partial q^i} + \eta^i(q,v)\frac{\partial}{\partial v^i}$, from computation related to dual tangent rhombic we recall that

$$\begin{aligned}(\tau_{TQ} \circ \upsilon_{TQ})(q,v,\xi,\eta) &= (q,v) = v^i \frac{\partial}{\partial q^i} \\ (T\tau_Q \circ \upsilon_{TQ})(q,v,\xi,\eta) &= (q,\xi) = \xi^i(q,v)\frac{\partial}{\partial q^i}.\end{aligned}$$

Hence, if $(i_T\omega)(x) = \nu_i(q,v)\, dq^i + \mu_i(q,v)\, dv^i$ for some functions $\nu_i(q,v),\ \mu_i(q,v)$ we have

$$\begin{aligned}i_T\omega(\upsilon_{TQ}(q,v)) &= \left(\frac{1}{2}\omega_{ij}(q)\, dq^i \wedge dq^j\right)\left(v^i\partial/\partial q^i, \xi^i(q,v)\,\partial/\partial q^i\right) \\ \nu_i(q,v)\,\xi^i + \mu_i(q,v)\,\eta^i &= \omega_{ij}(q)\, v^j \xi^i(q,v) \qquad (4.4)\end{aligned}$$

for all $\xi^i(q,v)$ and $\eta^i(q,v)$. This implies

$$(i_T\omega)(q,v) = \omega_{ij}(q)\, v^j dq^i. \qquad (4.5)$$

By multilinearity, the action of i_T can be extended to all of $\Omega(Q)$. The operator $i_T : \Omega(Q) \to \Omega(TQ)$ can be shown, by appendix, to be a derivation of degree -1 with its action on functions being 0.

Using i_T and exterior derivative d define the operator $d_T : \Omega(Q) \to \Omega(TQ)$ by

$$d_T = i_T d + d i_T$$

Since

$$dd_T = di_T d + d^2 i_T = di_T d + i_T d^2 = (i_T d + d i_T)\, d = d_T d \qquad (4.6)$$

we see that d and d_T commutes. The action of d_T on a function on Q is

$$\begin{aligned} d_T f(q) &= i_T df(q) + di_T f(q) = i_T df(q) \\ &= i_T\left(\frac{\partial f(q)}{\partial q^i}dq^i\right) = \frac{\partial f(q)}{\partial q^i}v^i \end{aligned} \tag{4.7}$$

which is the directional derivative of f in the direction of $v(x)$. On a one-form $\alpha(x) = \alpha_i(q)\, dq^i$ on Q,

$$\begin{aligned} d_T\alpha(x) &= di_T\alpha(x) + i_T d\alpha(x) \\ &= d\left(\alpha_i(q)\, v^i\right) + i_T\left(\frac{\partial \alpha_i}{\partial q^j}dq^j \wedge dq^i\right) \\ &= \alpha_i(q)\, dv^i + \frac{\partial \alpha_i}{\partial q^j}v^j dq^i. \end{aligned} \tag{4.8}$$

As usual, the action of d_T on $p-$forms on Q can be obtained by linearity. $d_T : \Omega(Q) \to \Omega(TQ)$ is a derivation of degree 0. The derivations i_T and d_T are introduced and studied in detail in references [8] and [16].

4.2. Construction of 1-forms

We shall apply these derivations to canonical 1-form θ_Q and symplectic 2-form ω_Q on T^*Q to obtain 1-forms $d_T\theta_Q$ and $i_T\omega_Q$ on TT^*Q. The derivatives of these 1-forms give the same symplectic 2-form on TT^*Q. Hence TT^*Q will be shown to be an exact symplectic manifold with two different 1-forms.

If we replace Q with T^*Q then d_T on canonical 1-form θ_Q and i_T on symplectic 2-form ω_Q give 1-form on TT^*Q. More explicitly, using coordinates (q, p, v, V) of

$T_{\alpha(x)}T_x^*Q$, we have

$$\begin{aligned}
d_T\theta_Q &= i_T d\theta_Q + d i_T \theta_Q \\
&= i_T d\left(p_j dq^j\right) + d i_T\left(p_j dq^j\right) \\
&= i_T\left(dp_j \wedge dq^j\right) + d\left(p_j v^j\right) \\
&= \left[i_T dp_j \wedge dq^j\right] - \left[i_T dq^j \wedge dp_j\right] + v^j dp_j + p_j dv^j \\
&= V_j dq^j - v^j dp_j + v^j dp_j + p_j dv^j \\
&= V_j dq^j + p_j dv^j. \qquad (4.9)
\end{aligned}$$

and for the symplectic 2-form ω_Q on T^*Q

$$\begin{aligned}
i_T\left(\omega_Q\right) &= i_T\left(dq^i \wedge dp_i\right) \\
&= v^i dp_i - V_i dq^i.
\end{aligned}$$

The exterior derivatives of these 1-forms give the same symplectic 2-form

$$-d\left(d_T\theta_Q\right) = d\left(i_T\left(\omega_Q\right)\right) = dq^j \wedge dV_j + dv^j \wedge dp_j \qquad (4.10)$$

on TT^*Q. We observe that although the sum of above 1-forms is exact

$$d_T\theta_Q + i_T\omega_Q = d\left(p_i v^i\right) \qquad (4.11)$$

their difference is not

$$d_T\theta_Q - i_T\omega_Q = 2V_i dq^i + p_i dv^i - v^i dp_i. \qquad (4.12)$$

This observation will be used to define a non-trivial exact symplectic structure on the product space $TT^*Q \times TT^*Q$.

5. SYMPLECTIC DIFFEOMORPHISMS

Let (P_1, ω_1) and (P_2, ω_2) be two symplectic manifolds. A diffeomorphism $\varphi : P_1 \to P_2$ is called **symplectic diffeomorphism** or **symplectomorphism** of (P_1, ω_1) into (P_2, ω_2) if $\varphi^* \omega_2 = \omega_1$. Using definition of pull-back (see appendix), this can be written as

$$\omega_1 (u_P, v_P) = \omega_2 (T\varphi \circ u_P, T\varphi \circ v_P), \tag{5.1}$$

for all $u_P(p), v_P(p) \in \tau_{P_1}^{-1}(p)$ and for all $p \in P_1$.

5.1. Symplectic Diffeomorphism Induced by Symplectic 2-form

Let (P, ω) be a symplectic manifold. The 2-form ω defines a mapping $\beta_{(P,\omega)} : TP \to T^*P$ characterized by the equality

$$\left\langle \beta_{(P,\omega)} \circ u_P, v_P \right\rangle = \langle \omega; u_P, v_P \rangle \tag{5.2}$$

for the vector fields $u_P, v_P : P \to TP$ with $\tau_P \circ u_P = \tau_P \circ v_P$. In particular, for $P = T^*Q$

$$\begin{tikzcd}
TT^*Q \arrow[rr, "\beta_{(T^*Q,\, \omega_Q)}"] \arrow[dr, "\tau_{T^*Q}"'] & & T^*T^*Q \arrow[dl, "\pi_{T^*Q}"] \\
& T^*Q &
\end{tikzcd}$$

Figure 5.1. The mapping beta

we recall coordinates $(q,p;v,V)\in T_{\alpha(x)}T_x^*Q$ and let $u_{T^*Q}(q,p)=(q,p;v,V)$, $v_{T^*Q}(q,p)=\left(q,p;\bar{v},\bar{V}\right)$. Then, from Eq (5.2) we have, for the right hand side

$$\begin{aligned}
\left\langle \beta_{(T^*Q,\omega_Q)}\left(u_{T^*Q}(q,p)\right),\left(v_{T^*Q}(q,p)\right)\right\rangle &= \left\langle dq^i\wedge dp_i; v\frac{\partial}{\partial q}+V\frac{\partial}{\partial p},\bar{v}\frac{\partial}{\partial q}+\bar{V}\frac{\partial}{\partial p}\right\rangle \\
&= \left\langle vdp-Vdq,\bar{v}\frac{\partial}{\partial q}+\bar{V}\frac{\partial}{\partial p}\right\rangle \\
&= v\bar{V}-V\bar{v} \qquad (5.3)
\end{aligned}$$

To evaluate the left hand side, take $\beta_{(T^*Q,\omega_Q)}(q,p;v,V)=Adq+Bdp\in T^*_{\alpha(x)}T^*_xQ$

$$\left\langle \beta_{(T^*Q,\omega_Q)}(q,p;v,V),\bar{v}\frac{\partial}{\partial q}+\bar{V}\frac{\partial}{\partial p}\right\rangle = A\bar{v}+B\bar{V} \qquad (5.4)$$

which implies by Eq (5.3) $A=-V$ and $B=v$. Thus the action of $\beta_{(T^*Q,\omega_Q)}$ on a vector field in TT^*Q is a 1-form in T^*T^*Q given locally by

$$\beta_{(T^*Q,\omega_Q)}(q,p;v,V)=-Vdq+vdp=(q,p,-V,v).$$

Proposition 5.1. $\beta_{(T^*Q,\omega_Q)}$ *is a symplectic diffeomorphism from* $(TT^*Q,d_T\omega_Q)$ *to* (T^*T^*Q,ω_{T^*Q}).

Proof. We shall show that the pull-back of canonical 1-form θ_{T^*Q} on T^*T^*Q is the 1-form $i_T\omega_Q$ on TT^*Q whose derivative is the symplectic 2-form $d_T\omega_Q$. For each $v_{T^*T^*Q}:T^*T^*Q\to TT^*T^*Q$ we have

$$\begin{aligned}
&\left\langle \beta^*_{(T^*Q,\omega_Q)}\theta_{T^*Q},v_{T^*T^*Q}\right\rangle \\
&= \left\langle \theta_{T^*Q},T\beta_{(T^*Q,\omega_Q)}\circ v_{T^*T^*Q}\right\rangle \qquad (5.5)\\
&= \left\langle \tau_{T^*T^*Q}\circ T\beta_{(T^*Q,\omega_Q)}\circ v_{T^*T^*Q},T\pi_{T^*Q}\circ T\beta_{(T^*Q,\omega_Q)}\circ v_{T^*T^*Q}\right\rangle \\
&= \left\langle \beta_{(T^*Q,\omega_Q)}\circ\tau_{TT^*Q}\circ v_{T^*T^*Q},T\tau_{T^*Q}\circ v_{T^*T^*Q}\right\rangle \\
&= \langle\omega_Q;\tau_{TT^*Q}\circ v_{T^*T^*Q},T\tau_{T^*Q}\circ v_{T^*T^*Q}\rangle \\
&= \langle i_T\omega_Q,v_{T^*T^*Q}\rangle \qquad (5.6)
\end{aligned}$$

where we used the definition of canonical 1-form and the relations

$$\tau_{T^*T^*Q} \circ T\beta_{\left(T^*Q,\omega_Q\right)} = \beta_{\left(T^*Q,\omega_Q\right)} \circ \tau_{TT^*Q} \tag{5.7}$$

$$T\pi_{T^*Q} \circ T\beta_{\left(T^*Q,\omega_Q\right)} = T\tau_{T^*Q} \tag{5.8}$$

$$\pi_{T^*Q} \circ \beta_{\left(T^*Q,\omega_Q\right)} = \tau_{T^*Q} \tag{5.9}$$

which can easily be obtained with the help of the diagram.

$$\begin{tikzcd}
TTT^*Q \arrow[r, "T\beta_{(T^*Q,\omega_Q)}"] \arrow[d, "\tau_{TT^*Q}"'] & TT^*T^*Q \arrow[d, "\tau_{T^*T^*Q}"] \arrow[dl, "T\pi_{T^*Q}"'] \\
TT^*Q \arrow[r, "\beta_{(T^*Q,\omega_Q)}"] \arrow[d, "\tau_{T^*Q}"'] & T^*T^*Q \arrow[dl, "\pi_{T^*Q}"] \\
T^*Q &
\end{tikzcd}$$

Figure 5.2. Beta mapping and its tangent

We thus showed that $\beta^*_{\left(T^*Q,\omega_Q\right)}\theta_{T^*Q} = i_T\omega_Q$ and this completes the proof. ■

5.2. The Symplectic Diffeomorphism $\alpha_Q : TT^*Q \to T^*TQ$

For $\alpha(x) = (q,p) \in T^*_xQ$ and $v(x) = (q,v) \in T_xQ$, let $v_{T^*Q}(q,p) = t\zeta(0)$ and $u_{TQ} = t\chi(0)$ be equivalence classes of curves in $T_{\alpha(x)}T^*_xQ$ and $T_{v(x)}T_xQ$, respectively. Assume that they project onto the same curve on Q, that is, $\pi_Q \circ \zeta = \tau_Q \circ \chi$. In coordinates $(q,p;v,V) \in T_{\alpha(x)}T^*_xQ$ and $(q,v;\xi,\eta) \in T_{v(x)}T_xQ$ we can take the maps

$$\zeta(s) : (q,p) \to (q+sv, p+sV) \tag{5.10}$$

$$\chi(s) : (q,v) \to (q+s\xi, v+s\eta) \tag{5.11}$$

as representative of these curves to first order in a parameter s. Note that $\zeta(s)(q,p)$ and $\chi(s)(q,v)$ are curves in T^*Q and TQ, respectively. Using natural pairings of these spaces we define the pairing of the spaces TT^*Q and TTQ as bundles over TQ with projections $T\pi_Q$ and $T\tau_Q$, respectively. For vectors $v_{T^*Q}(q,p) \in T_{\alpha(x)}T_x^*Q$ and $u_{TQ}(q,v) \in T_{v(x)}T_xQ$ we let

$$\begin{aligned}
\langle v_{T^*Q}(q,p), u_{TQ}(q,v)\rangle^{\sim} &= \langle (q,p;v,V), (q,v;\xi,\eta)\rangle^{\sim} \\
&= \frac{d}{ds}\langle \zeta(s)(q,p), \chi(s)(q,v)\rangle\rfloor_{s=0} \\
&= \frac{d}{ds}\langle (q+sv, p+sV), (q+s\xi, v+s\eta)\rangle\rfloor_{s=0} \\
&= \frac{d}{ds}(p+sV)(v+s\eta)\rfloor_{s=0} \\
&= Vv + p\eta.
\end{aligned}$$

On the other hand, TTQ has natural pairing with T^*TQ over the space TQ, that is, as bundles with projections τ_{TQ} and π_{TQ}, respectively. From the construction of dual tangent rhombic we recall the isomorphism κ_Q of bundles $T\tau_Q : TTQ \to TQ$ and $\tau_{TQ} : TTQ \to TQ$. We define the dual isomorphism α_Q of the bundles $T\pi_Q : TT^*Q \to TQ$ and $\pi_{TQ} : T^*TQ \to TQ$ by requiring the above pairing to be equal to the natural pairing of T^*TQ and TTQ. From the dual diagrams

$$\begin{tikzcd}
TTQ \arrow[rr, "\kappa_Q"] \arrow[dr, "T\tau_Q"'] & & TTQ \arrow[dl, "\tau_{TQ}"] \\
& TQ &
\end{tikzcd}
\qquad
\begin{tikzcd}
TT^*Q \arrow[rr, "\alpha_Q"] \arrow[dr, "T\pi_Q"'] & & T^*TQ \arrow[dl, "\pi_{TQ}"] \\
& TQ &
\end{tikzcd}$$

Figure 5.3. κ_Q and α_Q mappings

we have

$$\langle \alpha_Q \circ v_{T^*Q}, u_{TQ}\rangle = \langle v_{T^*Q}, \kappa_Q \circ u_{TQ}\rangle^{\sim}. \tag{5.12}$$

To find the coordinate expression for α_Q we compute

$$\begin{aligned}
\langle \alpha_Q (q,p;v,V), (q,v;\xi,\eta)\rangle &= \langle (q,p;v,V), \kappa_Q (q,v;\xi,\eta)\rangle^{\tilde{}} \\
&= \langle (q,p;v,V), (q,\xi;v,\eta)\rangle^{\tilde{}} \\
&= V\xi + p\eta \qquad (5.13)
\end{aligned}$$

for the right hand side. If $\alpha_Q (q,p;v,V) = (q,v;A,B)$ is a 1-form on TQ, we get

$$A\xi + B\eta = V\xi + p\eta \qquad (5.14)$$

which implies $A = V$ and $B = p$. That is

$$\alpha_Q (q,p;v,V) = (q,v;V,p) \qquad (5.15)$$

is the local expression for the isomorphism α_Q.

Proposition 5.2. *α_Q is a symplectomorphism from $(TT^*Q, d_T\omega_Q)$ to (T^*TQ, ω_{TQ}).*

Proof. Recall that

$$\theta_Q (q,v;\alpha,\beta) = \alpha_i dq^i + \beta_i dv^i = (q,v;\alpha,\beta) \qquad (5.16)$$

is the canonical 1-form on T^*TQ. If this is the image of (q,p,v,V) under α_Q we get $\alpha_i = V_i$ and $\beta_i = p_i$. Thus, the pull-back of canonical 1-form θ_{TQ} to TT^*Q is

$$\left(\alpha_Q^* \theta_{TQ}\right)(q,p;v,V) = V_i dq^i + p_i dv^i = d_T\theta_Q. \qquad (5.17)$$

■

6. SPECIAL SYMPLECTIC STRUCTURES

Let (P, ω) be a symplectic manifold. A **special symplectic structure** for (P, ω) is a collection $(P, Q, \pi, \theta, \alpha)$, where $\pi : P \to Q$ is a vector fibration and, $\theta = \alpha^* (\theta_Q)$ is a one-form on P with $-d\theta = \omega$, θ_Q is the canonical 1-form on T^*Q, and $\alpha : P \to T^*Q$ is a diffeomorphism uniquely characterized by $\langle \alpha (p), v_Q \rangle = \langle \theta, v_P \rangle$ for vector fields $v_Q : Q \to TQ$ and $v_P : P \to TP$ satisfying $T\pi \circ v_P = v_Q$. To see this characterization, we have, based on the following diagram

$$\begin{tikzcd}
TP \arrow[r, "T\alpha"] \arrow[d, "\tau_P"'] \arrow[rr, bend left, "T\pi", start anchor=north, end anchor=north east] & TT^*Q \arrow[d, "\tau_{T^*Q}"] \arrow[dr, "T\pi_Q"] & \\
P \arrow[r, "\alpha"] \arrow[d, "\pi"'] & T^*Q \arrow[d, "\pi_Q"] & TQ \arrow[dl, "\tau_Q"] \\
Q \arrow[r, equal] & Q &
\end{tikzcd}$$

Figure 6.1. Fibrations on special symplectic structure

the following computation. For each $v_P : P \to TP$ we have, using definition of canonical 1-form θ_Q on the second line

$$\begin{aligned}
\langle \theta, v_P \rangle &= \langle \alpha^* \theta_Q, v_P \rangle = \langle \theta_Q, T\alpha \circ v_P \rangle \\
&= \langle \tau_{T^*Q} \circ T\alpha \circ v_P, T\pi_Q \circ T\alpha \circ v_P \rangle \\
&= \langle \alpha \circ \tau_P \circ v_P, T(\pi_Q \circ \alpha) \circ v_P \rangle \\
&= \langle \alpha \circ \tau_P \circ v_P, T\pi \circ v_P \rangle . \qquad (6.1)
\end{aligned}$$

With this observation we state the following definition. Let (P, Q, π) be a fibration and θ be a 1-form on P. The quadruple (P, Q, π, θ) is called a **special symplectic manifold** if there is a diffeomorphism $\alpha : P \to T^*Q$ such that $\pi = \pi_Q \circ \alpha$ and $\theta = \alpha^* \theta_Q$.

If (P, Q, π, θ) is a special symplectic structure, so is $(TP, TQ, T\pi, d_T\theta)$ with $\alpha_Q \circ T\alpha : TP \to T^*TQ$, $T\pi = \pi_{TQ} \circ \alpha_Q \circ T\alpha$ and $d_T\theta = (\alpha_Q \circ T\alpha)^* (\theta_{TQ})$. In other words, the special symplectic structure for $(P, -d\theta)$ represented by the diagram

$$\begin{tikzcd}
(P,\theta) \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & (T^*Q,\theta_Q) \arrow[dl, "\pi_Q"] \\
& Q &
\end{tikzcd}$$

Figure 6.2. Special symplectic structure

induces a special symplectic structure

$$\begin{tikzcd}
(TP, d_T\theta) \arrow[rr, "\alpha_Q \circ T\alpha"] \arrow[dr, "T\pi"'] & & (T^*TQ,\theta_{TQ}) \arrow[dl, "\pi_{TQ}"] \\
& TQ &
\end{tikzcd}$$

Figure 6.3. Tangent of a special symplectic structure

for the exact symplectic manifold $(TP, d_T\omega)$.

6.1. Tulczyjew's Triplet

In particular, if $P = T^*Q$ and $\alpha = id_{T^*Q}$, we have the special symplectic structure

$$\begin{tikzcd}
(TT^*Q, d_T\theta_Q) \arrow[rr, "\alpha_Q"] \arrow[dr, "T\pi_Q"'] & & (T^*TQ, \theta_{TQ}) \arrow[dl, "\pi_{TQ}"] \\
& TQ &
\end{tikzcd}$$

Figure 6.4. Special symplectic structure for $TP = TT^*Q$

for the symplectic manifold $(TT^*Q, d_T\omega_Q)$.

On the other hand, using the map $\beta_{(P,\omega)} : TP \to T^*P$, we construct, from the symplectic manifold (P, ω), the diagram

$$\begin{tikzcd}
(TP, i_T\omega) \arrow[rr, "\beta_{(P,\omega)}"] \arrow[dr, "\tau_P"'] & & (T^*P, \theta_P) \arrow[dl, "\pi_P"] \\
& P &
\end{tikzcd}$$

Figure 6.5. Special symplectic structure due to beta mapping

which represents a special symplectic structure for the symplectic manifold $(TP, i_T\omega)$. As a special case, if we take $P = T^*Q$, then

$$\begin{tikzcd}
(TT^*Q, i_T\omega_Q) \arrow[rr, "\beta_{(T^*Q,\, \omega_Q)}"] \arrow[dr, "\tau_{T^*Q}"'] & & (T^*T^*Q, \theta_{T^*Q}) \arrow[dl, "\pi_{T^*Q}"] \\
& T^*Q &
\end{tikzcd}$$

Figure 6.6. Special symplectic structure of $(TT^*Q, i_T\omega_Q)$

represents a special symplectic structure for the symplectic manifold $(TT^{*}Q, di_T\omega_Q)$. Thus we have a triplet of structures that indicates two special symplectic structure for the symplectic manifold $(TT^{*}Q, di_T\omega_Q = dd_T\theta_Q)$

$$
\begin{tikzcd}
T^{*}TQ \arrow[dr, "\pi_{TQ}"'] & & TT^{*}Q \arrow[ll, "\alpha_Q"'] \arrow[rr, "\beta_{(T^{*}Q,\, \omega_Q)}"] \arrow[dl, "T\pi_Q"] \arrow[dr, "\tau_{T^{*}Q}"'] & & T^{*}T^{*}Q \arrow[dl, "\pi_{T^{*}Q}"] \\
& TQ & & T^{*}Q &
\end{tikzcd}
$$

Figure 6.7. Tulczyjew's triplet

which is called *Tulczyjew's triplet*.

7. LAGRANGIAN SUBMANIFOLDS

7.1. Lagrangian Submanifolds and Generating Functions

A **Lagrangian submanifold** of a symplectic manifold (P,ω) is a submanifold $S\subset P$ of $\dim(S)=\frac{1}{2}\dim(P)$ such that $\omega\rfloor_S=0$. Where $\rfloor_S$ represents the restriction of the two-form to the submanifold S.

If S is the image of an immersion $\sigma : M\to P$, then $\omega\rfloor_S=0$ is equivalent to $\sigma^*\omega=0$.There is a nice characterization of immersed Lagrangian submanifolds of (T^*Q,ω_Q) in terms of Lagrange brackets. We assume $S\subset T^*Q$ is the image of an immersion $\sigma : M\to T^*Q$ and let $\left(m^k\right)$ be coordinates on M. Representing the map σ by $\sigma^i(m)=q^i$ and $\sigma_j(m)=p_j$ we compute

$$\begin{aligned}
\sigma^*\omega_Q &= -\sigma^* d\theta_Q = -d\left(\sigma^*\theta_Q\right) = -d\left(\sigma_i d\sigma^i\right) = d\sigma^i\wedge d\sigma_i \\
&= \frac{\partial\sigma^i}{\partial m^k}dm^k\wedge\frac{\partial\sigma_i}{\partial m^l}dm^l \\
&= \frac{1}{2}\left(\frac{\partial\sigma^i}{\partial m^k}\frac{\partial\sigma_i}{\partial m^l}-\frac{\partial\sigma_i}{\partial m^k}\frac{\partial\sigma^i}{\partial m^l}\right)dm^k\wedge dm^l
\end{aligned}\tag{7.1}$$

which implies that the condition for S to be a Lagrangian submanifold of T^*Q is vanishing of the **Lagrange brackets**

$$\left[m^k,m^l\right] := \left(\frac{\partial\sigma^i}{\partial m^k}\frac{\partial\sigma_i}{\partial m^l}-\frac{\partial\sigma_i}{\partial m^k}\frac{\partial\sigma^i}{\partial m^l}\right). \tag{7.2}$$

We observe that if $\sigma^*\theta_Q=dU(m)$ for some function U on M, then $d\sigma^*\theta_Q=0$ identically. In this case, $U(m)$ is called a **generator** of Lagrangian submanifold S. From $dU(m)=\sigma_i(m)\,d\sigma^i(m)$ one obtains the representation of S by equations

$$\frac{\partial U(m)}{\partial m^k}=\sigma_i(m)\frac{\partial\sigma^i}{\partial m^k} \tag{7.3}$$

or, using $\sigma_i(m) = p_i$, $\sigma^i(m) = q^i$ this can be converted into

$$\frac{\partial U}{\partial q^i} = p_i \tag{7.4}$$

for proper invertibility conditions on σ. Alternatively, one can consider more general class of functions depending on some parameters, called **Lagrange multipliers**, to obtain a characterization of Lagrangian submanifolds by Eqs (7.4) together with additional equations characterizing the domain of validity, thereby replacing the invertibility conditions.

Yet another way, which we shall employ in the following, is to consider families of functions parametrized by an appropriate set $X \subset Q$ on which the dynamics takes place. The critical points of such a family turns out to be the Lagrangian submanifolds of T^*X. This can be used to generate Lagrangian submanifolds of T^*Q and special symplectic structures enable us to generalize this construction to arbitrary symplectic manifolds.

7.2. Morse Families

Let $\rho : R \to X$ be a differentiable fibration with $\dim(X) = m$ and dimension of fibers being k. A vector field $v : R \to TR$ on the total space is called a **vertical vector field** if $T\rho \circ v = 0$.

In local coordinates, $(x, r) \in R$, $x \in X$, $\rho : (x, r) \to x$, $(x, r; \dot{x}, \dot{r}) \in T_{(x,r)}R$ and the vertical space $V_{(x,r)}R$ consists of vectors of the form $(x, r; 0, \dot{r})$ or equivalently $\dot{r}\,(\partial/\partial r)$.

A function $U : R \to \mathbb{R}$ can be considered to be a **family of functions** on the fibres of the fibration $\rho : R \to X$ and parametrized by the coordinates of the base space X. We shall use the following diagram to represent a family of functions.

$$
\begin{tikzcd}
R \arrow[r, "U"] \arrow[d, "\rho"] & IR \\
X &
\end{tikzcd}
$$

Figure 7.1. Family of functions

The **critical set** for a family of functions $U : R \rightarrow \mathbb{R}$ is defined by

$$Cr\,(U,\rho) = \left\{(x,r) \in R : \langle dU\,(x,r)\,, v\,(x,r)\rangle = 0, \forall v\,(x,r) \in V_{(x,r)}R\right\} \tag{7.5}$$

and is a submanifold of R of dimension $m = \dim(X)$.

At each point $(x,r) \in Cr\,(U,\rho)$ we define a bilinear mapping

$$\begin{aligned} W\,(U,(x,r)) \;&:\; V_{(x,r)}R \times T_{(x,r)}R \rightarrow \mathbb{R} \\ &:\; (v\,(x,r)\,, w\,(x,r)) \rightarrow D^{(1,1)}\,(U \circ \chi)\,(0,0)\,, \end{aligned} \tag{7.6}$$

where $\chi : \mathbb{R}^2 \rightarrow R$ is such that $v\,(x,r) = t\chi\,(.,0)\,(0)$ and $w\,(x,r) = t\chi\,(0,.)\,(0)$. $W\,(U,(x,r))$ is in general a $k \times (m+k)$ matrix of derivatives.

A family of functions $U : R \rightarrow \mathbb{R}$ is said to be **regular** if the rank of $W\,(U,(x,r))$ is the same at each $(x,r) \in Cr\,(U,\rho)$. A family of functions $U : R \rightarrow \mathbb{R}$ is said to be a **Morse family** if the rank of $W\,(U,(x,r))$ is maximal at each $(x,r) \in Cr\,(U,\rho)$.

As an example, let X be two dimensional with coordinates (x^1,x^2) and let r be the coordinate on the one-dimensional fibers of $R \rightarrow X$. The internal energy function

$$U\left(x^1,x^2,r\right) = \frac{1}{2}\left(\left(x^1 - a\cos r\right)^2 + \left(x^2 - a\sin r\right)^2\right) \tag{7.7}$$

associated with the motion of a material point on the circle of radius a is a Morse family on R since the rank of the 1×3 matrix

$$\left(\frac{\partial^2 U}{\partial r \partial r}, \frac{\partial^2 U}{\partial r \partial x^1}, \frac{\partial^2 U}{\partial r \partial x^2}\right) = \left(a\left(x^1 \cos r + x^2 \sin r\right), a \sin r, a \cos r\right) \tag{7.8}$$

is 1, everywhere. For arbitrary vertical vectors of the form $v(x,r) = \dot{r}\frac{\partial}{\partial r}$ the condition $\langle dU, v \rangle = 0$ implies $\partial U / \partial r = 0$ which reduces to

$$\frac{x^2}{x^1} - \tan r = 0 \tag{7.9}$$

and this defines a two dimensional submanifold of R as a critical set $Cr(U, \rho)$ of the Morse family U.

7.3. Critical Set as Lagrangian Submanifold

Next result shows that critical set of a Morse family on $R \to X$ is a Lagrangian submanifold of the cotangent bundle of the base space X.

Proposition 7.1. *For points $(x,r) \in R$ with $\rho(x,r) = x = \pi_X(y)$, the set*

$$S = \left\{y \in T^*X : \langle y, T\rho \circ u_R \rangle = \langle dU, u_R \rangle, \textit{for all } u_R : R \to T_{(x,r)}R\right\} \tag{7.10}$$

is an immersed Lagrangian submanifold of the symplectic space (T^*X, ω_X).(See the figure below.)

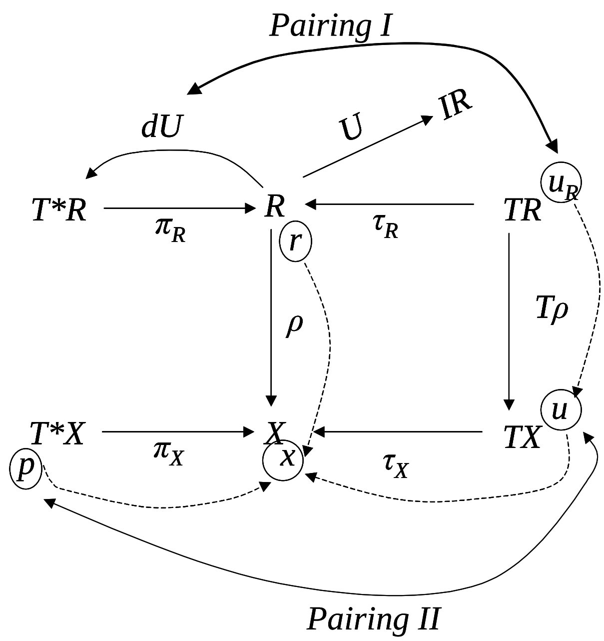

Figure 7.2. Generation of the set S.

Proof. Let us define a mapping $\kappa : Cr\,(U,\rho) \to T^*X$, such that $\pi_X\,(\kappa\,(x,r)) = \rho\,(x,r)$ and

$$\langle \kappa\,(x,r)\,, v_X\rangle = \langle dU, u_R\rangle\,, \tag{7.11}$$

where $v_X : X \to TX$ and $u_R : R \to TR$ such that $T\rho \circ u_R = v_X$. This mapping is an immersion and $S = im\,(\kappa)$. Let $u_{Cr} : Cr\,(U,\rho) \to T_r Cr\,(U,\rho)$

$$\begin{aligned} \langle \kappa^*\,(\theta_X)\,, u_{Cr}\rangle &= \langle \theta_X, T\kappa \circ u_{Cr}\rangle \\ &= \langle \tau_{T^*X} \circ T\kappa \circ u_{Cr}, T\pi_X \circ T\kappa \circ u_{Cr}\rangle \\ &= \langle \kappa, T\rho \circ u_{Cr}\rangle \\ &= \langle dU, u_{Cr}\rangle \end{aligned} \tag{7.12}$$

where we used the definition of the canonical 1-form and the definition of κ. It follows that

$$\kappa^*\omega_X = 0 \tag{7.13}$$

and $\dim(S) = \dim(Cr(U,\rho)) = m$. ■

For the example of this section we have

$$\theta_X = p_1 dx^1 + p_2 dx^2$$

and the critical set is defined by the equation $x^2 = x^1 \tan r$. κ is a 1-form on $Cr(U,\rho)$ given by

$$\kappa\left(x^1, x^2, r\right) = \frac{\partial U}{\partial x^1} dx^1 + \frac{\partial U}{\partial x^2} dx^2 = \left(x^1 - a\cos r\right) dx^1 + \left(x^2 - a\sin r\right) dx^2. \tag{7.14}$$

It follows from the property of the canonical 1-form that

$$\kappa^* \theta_X = \left(x^1 - a\cos r\right) dx^1 + \left(x^2 - a\sin r\right) dx^2 \tag{7.15}$$

and we have $d\kappa^*\theta_X = 0$ since fiber coordinate r is constant on X.

The Lagrangian submanifold S in T^*X is then defined by the equations

$$p_1 = x^1 - a\cos r, \quad p_2 = x^2 - a\cos r \tag{7.16}$$

where the fiber coordinate r assumed to be eliminated from the equation for critical set, so that above equations define a 2 dimensional submanifold of T^*X.

7.4. Lagrangian submanifolds of T^*Q

By embedding X into Q we shall obtain Lagrangian submanifolds of T^*Q from those of T^*X. Let $X \subset Q$ be a submanifold, take a regular family of functions U on the total space of $\rho : R \to X$. By above result, we have an immersed Lagrangian submanifold S of T^*X generated by U. Let $T_X^*Q = \pi_Q^{-1}(X)$ denote the inverse image of X in the total space of the cotangent bundle $\pi_Q : T^*Q \to Q$. Define the mapping

$$\xi : T_X^*Q \to T^*X \tag{7.17}$$

by $\langle \xi(p), v\rangle = \langle p, v\rangle$ for each $v : X \to TX$ so that ξ is identity if base point of $p \in T_X^*Q$ is in X, that is, if $p \in T^*X$. Hence, ξ is invertible on T^*X. Then, if $i : T_X^*Q \to T^*Q$ is the canonical injection, we can define

$$i \circ \xi^{-1} = T^*X \to T^*Q. \tag{7.18}$$

Proposition 7.2. *$S' = i \circ \xi^{-1}(S)$ is a Lagrangian submanifold of (T^*Q, ω_Q).*

Proof. If $(x, p) \in T^*Q$ with $\pi_X(x, p) = x \in X$ we can find $(x, r) \in R$ with $\rho(x, r) = x$. Then for every $w_R : R \to T_rR$ we have $T\rho \circ w_R = v : X \to T_xX \subset T_xQ$ and $\langle (x, p), v\rangle = \langle dU, w_R\rangle$. Alternatively, we can show that canonical two form ω_Q vanishes on S'. To this end, we define the functions

$$\bar{U} : Q \to \mathbb{R}, \quad V = \bar{U} \circ \pi_Q, \quad \tilde{U} = V\rfloor_{S'} \tag{7.19}$$

with $\bar{U}\rfloor_X = U$. Obviously V is on T^*Q and $\tilde{U}$ is on S'. $\bar{U}$ does not depend on the choice of $\bar{U}$ since it can also be defined directly as

$$\tilde{U} = (U \circ \pi_Q)\rfloor_{S'}, \tag{7.20}$$

for all $p \in S'$. If $w : S' \to TS'$, then $T\pi_Q \circ w : S \to TS$ since $\pi_Q(S') = S$ from the definition of S'. Now we compute

$$\begin{aligned} \langle \theta_Q, w\rangle &= \langle \tau_{T^*Q} \circ w, T\pi_Q \circ w\rangle \\ &= \langle dU, T\pi_Q \circ w\rangle = \langle d\bar{U}, T\pi_Q \circ w\rangle \\ &= \langle V, w\rangle = \left\langle d\tilde{U}, w\right\rangle, \end{aligned} \tag{7.21}$$

which implies $\theta_Q\rfloor_{S'} = d\tilde{U}$ and

$$\omega_Q\rfloor_{S'} = -d\theta_Q\rfloor_{S'} = -d\left(\theta_Q\rfloor_{S'}\right) = -dd\tilde{U} = 0. \tag{7.22}$$

So S' is a Lagrangian submanifold of (T^*Q, ω_Q). ■

7.5. Lagrangian Submanifolds of Special Symplectic manifolds

Above result combined with the structure of special symplectic manifolds enable us to generate Lagrangian submanifolds of arbitrary symplectic manifolds. Let (P, Q, π, θ) be a special symplectic structure for a symplectic manifold (P, ω) with its special symplectic isomorphism $\alpha : P \to T^*Q$. Combined with the diffeomorphism α of the special symplectic structure $(P, Q, \pi, \theta, \alpha)$ we obtain the following result.

Proposition 7.3. *$N = \alpha^{-1}(S') = \alpha^{-1} \circ i \circ \xi^{-1}(S)$ is a Lagrangian submanifold of (P, θ).*

Note that for every $w_R : R \to TR$ as above, there exists $w_P : P \to TP$ with $T\rho \circ w_R = T\pi \circ w_P$ and the generating function of N is defined by

$$\langle \theta, w_P \rangle = \langle dU, w_R \rangle .$$

We shall denote this construction diagrammatically as

$$\begin{tikzcd}
(P,\theta) \arrow[d, "\pi"] & R \arrow[r, "U"] \arrow[d, "\rho"] & IR \\
Q & X \arrow[l, hook', "i"] &
\end{tikzcd}$$

Figure 7.3. Generating object

and call it to be the **generating object** for the immersed Lagrangian submanifold N of P.

We shall obtain various examples of Lagrangian submanifolds and their generating objects as special cases of the above construction.

1. Take $\alpha : id_{T^*Q}$, $X = Q$ and $\rho = id_X$. Then $S = N$ and $U : Q \to \mathbb{R}$. The image S of the differential $dU : Q \to T^*Q$ is a Lagrangian submanifold of (T^*Q, ω_Q)

$$\begin{tikzcd}
T^*Q \arrow[d, "\pi_Q"] & \\
Q \arrow[r, "U"] & IR
\end{tikzcd}$$

Figure 7.4. Case I for generating object

since $\dim(S) = m$ and

$$(dU)^* \omega_Q = (dU)^* d\theta_Q = d(dU)^* \theta_Q = ddU = 0. \tag{7.23}$$

The submanifold S is said to be generated by the function U. In terms of coordinates (q^i, p_j) the set S is described by the equations

$$p_j = \frac{\partial U(q^i)}{\partial q^j}. \tag{7.24}$$

2. Let $\alpha : id_{T^*Q}$, $\rho = id_X$ and $X \subset Q$ be a submanifold of dimension k. Then $U : X \to \mathbb{R}$ and the set

$$N = S = \{p \in T^*Q : q = \pi_Q(p) \in X, \ \ \langle p, v\rangle = \langle dU, v\rangle, \forall v : X \to TX \subset TQ\} \tag{7.25}$$

is a Lagrangian submanifold.

$$\begin{tikzcd} T^*Q \arrow[d, "\pi_Q"] & & \\ Q & X \arrow[l, hook'] \arrow[r, "U"] & IR \end{tikzcd}$$

Figure 7.5. Case II for generating object

3. Let α be arbitrary, $\rho = id_X$ and $X = Q$. The Lagrangian submanifold generated by this object is

$$N = im\left(\alpha^{-1} \circ dU\right) = \alpha^{-1}(dU) = \{p \in P : \langle \theta, z\rangle = \langle dU, T\pi \circ z\rangle, \forall z : P \to TP\} \tag{7.26}$$

where $U : X = Q \to \mathbb{R}$. The diagrammatic representation of this construction is

$$\begin{tikzcd} (P,\theta) \arrow[d, "\pi"] & \\ Q \arrow[r, "U"] & IR \end{tikzcd}$$

Figure 7.6. Case III for generating object

4. Let α be arbitrary, $\rho : id_X$ and $X \subset Q$. Then the set

$$N = \{p \in P : q = \pi(p) \in X, \ \ \langle \theta, u\rangle = \langle dU, T\pi \circ z\rangle, \forall z.P \to TP\} \tag{7.27}$$

is a Lagrangian submanifold of $(P, d\theta)$ said to be generated with respect to (P, Q, π, θ) by the function $U : X \to \mathbb{R}$. Figure illustrates the construction of Lagrangian submanifold N.

$$
\begin{tikzcd}
(P,\theta) \arrow[d, "\pi"] & & \\
Q & X \arrow[l, hook', "i_X"'] \arrow[r, "U"] & IR
\end{tikzcd}
$$

Figure 7.7. Case IV for generating object

5. Let $\alpha = id_X$, ρ arbitrary and $X \subset Q$. Then we obtain the definition of the immersed Lagrangian submanifold of (T^*Q, ω_Q) described in proposition 3. The corresponding diagram is

$$
\begin{tikzcd}
 & R \arrow[r, "U"] \arrow[d, "\rho"] & IR \\
Q & X \arrow[l, hook'] &
\end{tikzcd}
$$

Figure 7.8. Case V for generating object

7.6. Lagrangian and Hamiltonian Dynamics

We shall show that the Lagrangian and Hamiltonian formulations of dynamics described at the beginning are indeed Lagrangian submanifolds of TT^*Q generated by the functions L and $-H$, respectively.

Let (P, Q, π, θ) be a special symplectic manifold. If the Lagrangian submanifold D of $(TP, d_T\omega)$ is generated by following generating object

$$\begin{tikzcd}
(TP,d_T\theta) \arrow[d, "T\pi"] & R \arrow[r, "L"] \arrow[d, "\rho"] & IR \\
TQ & X \arrow[l, hook'] &
\end{tikzcd}$$

Figure 7.9. Lagrangian generating object

with respect to special symplectic structure $(TP, TQ, T\pi, d_T\theta)$, then generating function L is called **Lagrangian** function and X is called **Lagrangian constraint**. Furthermore, the functions represented diagrammatically as

$$\begin{tikzcd}
R \arrow[r, "L"] \arrow[d, "\rho"] & IR \\
X &
\end{tikzcd}$$

Figure 7.10. Lagrangian family

are called a **Lagrangian family**. In particular, if we take $P = T^*Q$, then $(TT^*Q, TQ, T\pi_Q, d_T\theta_Q)$ with the generating object

$$\begin{tikzcd}
(TT^*Q, d_T\theta_Q) \arrow[d, "T\pi_Q"] & R \arrow[r, "L"] \arrow[d, "\rho"] & IR \\
TQ & X \arrow[l, hook'] &
\end{tikzcd}$$

Figure 7.11. Lagrangian system for $P = T^*Q$

is called a **Lagrangian system** for $P = T^{*}Q$. For this case, immersed Lagrangian submanifold is given as

$$D = \left\{ \begin{array}{c} w_{T^{*}Q} : T^{*}Q \to TT^{*}Q : \exists\,(x,r) \in R; \rho\,(x,r) = T\pi_{Q} \circ w_{T^{*}Q}, \\ T\rho \circ u_{R} = TT\pi_{Q} \circ v_{TT^{*}Q} \Longrightarrow \langle d_{T}\theta_{Q}, v_{TT^{*}Q} \rangle = \langle dL, u_{R} \rangle\,, \\ \forall v_{TT^{*}Q} : TT^{*}Q \to TTT^{*}Q, \forall u_{R} : R \to TR. \end{array} \right\}. \tag{7.28}$$

If on the other hand, the Lagrangian submanifold D of $(TP, d_{T}\omega)$ is generated by the following generating object

$$\begin{tikzcd} (TP, i_{T}\omega) \arrow[d, "\tau"] & R \arrow[r, "-H"] \arrow[d, "\rho"] & IR \\ P & X \arrow[l, hook'] & \end{tikzcd}$$

Figure 7.12. Hamiltonian generating object

with respect to special symplectic structure $(TP, P, \tau, i_{T}\omega)$, then H is called **Hamiltonian** of the system and X is called **Hamiltonian constraint**. In addition,

$$\begin{tikzcd} R \arrow[r, "-H"] \arrow[d, "\rho"] & IR \\ X & \end{tikzcd}$$

Figure 7.13. Hamiltonian family

is called the **Hamiltonian family** of functions. In particular, if $P = T^{*}Q$ then the special symplectic structure $(TP, P, \tau, i_T\omega)$ becomes $(TT^*Q, T^*Q, \tau_{T^*Q}, i_T\omega_Q)$, and the following generating object

$$\begin{tikzcd}
(TT^*Q, i_T\omega_Q) \arrow[d, "\tau_{T^*Q}"] & R \arrow[r, "-H"] \arrow[d, "\rho"] & IR \\
T^*Q & X \arrow[l, hook']
\end{tikzcd}$$

Figure 7.14. Hamiltonian system for $P = T^*Q$

is called **Hamiltonian generating object** for $P = T^*Q$. The dynamics is given by the Lagrangian submanifold

$$D = \left\{ \begin{array}{c} w_{T^*Q} : T^*Q \to TT^*Q : \exists\,(x, r) \in R; \rho\left((x, r)\right) = \tau_{T^*Q} \circ w_{T^*Q}, \\ T\rho \circ u_R = T\tau_P \circ v_{TT^*Q} \Longrightarrow \langle i_T\omega_Q, v_{TT^*Q} \rangle = \langle -dH, u_R \rangle, \\ \forall v_{TT^*Q} : TT^*Q \to TTT^*Q, \forall u_R : R \to TR \end{array} \right\}. \tag{7.29}$$

8. LAGRANGIAN SUBMANIFOLDS OF CARTESIAN PRODUCTS

8.1. Products of Symplectic Manifolds

Let (P_1, ω_1) and (P_2, ω_2) be symplectic manifolds and and let pr_1 and pr_2 denote the canonical projections

$$pr_1 : P_2 \times P_1 \to P_1, \quad pr_1 : P_2 \times P_1 \to P_2. \tag{8.1}$$

Proposition 8.1. $(P_2 \times P_1, \omega_2 \ominus \omega_1)$ *is a symplectic manifold*

To see this, we need to show that $\omega_2 \ominus \omega_1 = pr_2^*\omega_2 - pr_1^*\omega_1$ is closed:

$$\begin{aligned} d\left(\omega_2 \ominus \omega_1\right) &= d\left(pr_2^*\omega_2 - pr_1^*\omega_1\right) \\ &= pr_2^* d\omega_2 - pr_1^* d\omega_1 = 0 \end{aligned} \tag{8.2}$$

and it is non-degenerate by construction.

Let $\varphi : P_1 \to P_2$ be a diffeomorphism. The graph of φ is the subset of $P_2 \times P_1$ defined by

$$graph\left(\varphi\right) = \left\{\left(\varphi\left(p_1\right), p_1\right) : p_1 \in P_1\right\}. \tag{8.3}$$

Equivalently, graph(φ) can be described as the image of the map

$$\left(\varphi, id_{P_1}\right) : P_1 \to P_2 \times P_1. \tag{8.4}$$

Following result gives a connection between symplectic diffeomorphisms and Lagrangian submanifolds of $P_2 \times P_1$.

Proposition 8.2. *φ is a symplectic diffeomorphism if and only if graph(φ) is a Lagrangian submanifold of* $(P_2 \times P_1, \omega_2 \ominus \omega_1)$.

Proof. Since $(\varphi, id_{P_1})^* (\omega_2 \ominus \omega_1) = \varphi^* \omega_2 - \omega_1$, φ is a symplectic diffeomorphism if and only if $\varphi^* \omega_2 - \omega_1 = 0$ which implies $\omega_2 \ominus \omega_1 \rfloor_{graph(\varphi)} = 0$. ■

In local coordinates, if $(q, p) \in P_1$ and $(\bar{q}, \bar{p}) \in P_2$ then

$$\omega_2 \ominus \omega_1 = d\bar{q}^i \wedge d\bar{p}_i - dq^i \wedge dp_i \tag{8.5}$$

and a diffeomorphism $\varphi : P_1 \to P_2$ can be represented as

$$\bar{q} = \psi(q, p), \qquad \bar{p} = \chi(q, p). \tag{8.6}$$

Restricting $\omega_2 \ominus \omega_1$ to graph of φ we have

$$\begin{aligned}
\omega_2 \ominus \omega_1 \rfloor_{graph} \varphi &= d\bar{q}^i \wedge d\bar{p}_i - dq^i \wedge dp_i \rfloor_{graph} \varphi \\
&= d\psi^i \left(q^k, p_l\right) \wedge d\chi_i \left(q^k, p_l\right) - dq^i \wedge dp_i \\
&= \left(\frac{\partial \psi^i}{\partial q^k} dq^k + \frac{\partial \psi^i}{\partial p_l} dp_l\right) \wedge \left(\frac{\partial \chi_i}{\partial q^k} dq^k + \frac{\partial \chi_i}{\partial p_l} dp_l\right) - dq^i \wedge dp_i \\
&= \left[q^k, q^l\right] dq^k \wedge dq^l + \left[p_l, q^k\right] dq^k \wedge dp_l - \left[q^k, p_l\right] dq^k \wedge dp_l \\
&\quad + \left[p_k, p_l\right] dp_k \wedge dp_l - dq^i \wedge dp^i = 0
\end{aligned} \tag{8.7}$$

where $\left[q^k, q^l\right]$, $\left[q^k, p_l\right]$ and $[p_k, p_l]$ are Lagrange brackets. So condition is equivalent to

$$\left[q^k, q^l\right] = 0, \qquad \left[q^k, p_l\right] = \delta_l^k, \qquad [p_k, p_l] = 0. \tag{8.8}$$

8.2. Products of Special Symplectic Manifolds

Assume further that $(P_i, Q_i, \pi_i, \theta_i, \alpha_i)$, $i = 1, 2$ are special symplectic manifolds. Define the mapping

$$\alpha_{21} : P_2 \times P_1 \to T^* (Q_2 \times Q_1) = T^*Q_2 \times T^*Q_1 \tag{8.9}$$

by $\alpha_{21} (p_2, p_1) = \alpha_2 (p_2) - \alpha_1 (p_1)$. Introducing

$$\Pi_1 = \pi_1 \circ \alpha_1, \quad \Pi_2 = \pi_2 \circ \alpha_2 \tag{8.10}$$

we have $\Pi_2 \times \Pi_1 = (\pi_1 \times \pi_2) \circ \alpha_{21}$. Since $\theta_{Q_2} \oplus \theta_{Q_1} = pr_2^* \theta_{Q_2} + pr_1^* \theta_{Q_1}$ is the canonical 1-form on $T^* (Q_2 \times Q_1) = T^*Q_2 \times T^*Q_1$, the 1-form

$$\theta_2 \ominus \theta_1 = \alpha_{21}^* (\theta_{Q_2} \oplus \theta_{Q_1}) \tag{8.11}$$

is associated with the special symplectic manifold

$$(P_2 \times P_1, Q_2 \times Q_1, \Pi_2 \times \Pi_1, \theta_2 \ominus \theta_1) \tag{8.12}$$

represented by the diagram

$$\begin{tikzcd}
(P_2 x P_1, \theta_2 - \theta_1) \arrow[rr, "\alpha_{21}"] \arrow[dr, "\Pi_2 x \Pi_1"'] & & (T^*Q_2 x T^*Q_1, \theta_{Q_2} - \theta_{Q_1}) \arrow[dl, "\pi_2 x \pi_1"] \\
& Q_2 x Q_1 &
\end{tikzcd}$$

Figure 8.1. Cartesian of special symplectic structures

As a result we conclude that if (P_i, ω_i) , $i = 1, 2$ are underlying symplectic manifolds of $(P_i, Q_i, \pi_i, \theta_i)$ then $(P_2 \times P_1, \omega_2 \ominus \omega_1)$ is the underlying symplectic manifold of the special symplectic structure in Eq (8.12).

Let $M \subset Q_2 \times Q_1$ and consider the fibration $\rho_{21} : R_{21} \to M$. The Lagrangian submanifold of $(P_2 \times P_1, \omega_2 \ominus \omega_1)$ represented by the graph of symplectomorphism $\varphi : P_1 \to P_2$ can be generated according to the diagram

$$\begin{tikzcd}
(P_2 x P_1, \theta_2 - \theta_1) \arrow[d, "\Pi_2 x \Pi_1"] & R_{21} \arrow[r, "U_{21}"] \arrow[d, "\rho_{21}"] & IR \\
Q_2 x Q_1 & M \arrow[l, hook']
\end{tikzcd}$$

Figure 8.2. Generating object for $\varphi : P_1 \to P_2$

by a function $U_{21} : R_{21} \to \mathbb{R}$. Locally, the submanifold M can be described by l equation

$$W^a \left(q^i, \bar{q}^i\right) = 0, \quad a = 1, ..., l. \tag{8.13}$$

Then, if $\bar{U}_{21} : Q_2 \times Q_1 \to \mathbb{R}$ is an arbitrary continuation of $U_{21} : M \to \mathbb{R}$, an implicit description of Lagrangian submanifold generated by U_{21} or equivalently of the corresponding diffeomorphism φ can be given by

$$\theta_2 \ominus \theta_1 = \bar{p}_i d\bar{q}^i - p_i dq^i = d\left(\bar{U}_{21}\left(q, \bar{q}\right) - \nu_a W^a\left(q, \bar{q}\right)\right) \tag{8.14}$$

which implies the equations

$$\begin{aligned}
\bar{p}_i &= \frac{\partial \bar{U}_{21}}{\partial \bar{q}^i} - \nu_a \frac{\partial W^a}{\partial \bar{q}^i} \\
p_i &= -\frac{\partial \bar{U}_{21}}{\partial q^i} + \nu_a \frac{\partial W^a}{\partial q^i} \\
W^a\left(q^i\right) &= 0, \quad a = 1, .., l.
\end{aligned} \tag{8.15}$$

For $M = Q_2 \times Q_1$ these equations reduce to

$$\bar{p}_i = \frac{\partial \bar{U}_{21}}{\partial \bar{q}^i}, \quad p_i = -\frac{\partial \bar{U}_{21}}{\partial q^i}. \tag{8.16}$$

To describe the relation between generating functions of the Lagrangian submanifolds of P_1, P_2 and $(P_2 \times P_1, \omega_2 \ominus \omega_1)$ we have from Eq (8.14)

$$\bar{p}_i d\bar{q}^i = d\left(\bar{U}_{21}\left(q^i, \bar{q}^i\right) + \nu_a W^a\left(q^i, \bar{q}^i\right)\right) + p_i dq^i \tag{8.17}$$

and if the Lagrangian submanifold of P_1 is described by

$$p_i dq^i = d\left(\bar{U}_1\left(q^i\right) + \lambda_b F_1^b\left(q^i\right)\right), \tag{8.18}$$

we obtain

$$\bar{p}_i d\bar{q}^i = d\left(\bar{U}_{21}\left(q^i, \bar{q}^i\right) + \nu_a W^a\left(q^i, \bar{q}^i\right) + \bar{U}_1\left(q^i\right) + \lambda_b F_1^b\left(q^i\right)\right). \tag{8.19}$$

Hence the local expression of a generating function U_2 of N_2 is given by

$$U_2\left(\bar{q}^i\right) = \bar{U}_{21}\left(q^i, \bar{q}^i\right) + \bar{U}_1\left(q^i\right) + \nu_a W^a\left(q^i, q'^i\right) + \lambda_b F_1^b\left(q^i\right) \tag{8.20}$$

In particular if $X_1 = Q_1$ and $M = Q_2 \times Q_1$ then

$$U_2\left(\bar{q}^i\right) = \bar{U}_{21}\left(q^i, \bar{q}^i\right) + \bar{U}_1\left(q^i\right). \tag{8.21}$$

8.3. Products in Tulczyjew's Triplet

We let, further, $P_1 = P_2 = P = TT^*Q$ with its two different special symplectic structures

$$\left(TT^*Q, Q_1 = T^*Q, \pi_1 = \tau_{T^*Q}, \theta_1 = d_T\theta_Q, \alpha_1 = \alpha_Q\right) \tag{8.22}$$

$$\left(TT^*Q, Q_2 = TQ, \pi_2 = T\pi_Q, \theta_2 = i_T d\theta_Q, \alpha_2 = \beta_{\left(T^*Q,\omega_Q\right)}\right). \tag{8.23}$$

Following previous section, we define the mapping

$$\alpha_{21} : TT^*Q \times TT^*Q \to T^*\left(TQ \times T^*Q\right) = T^*TQ \times T^*T^*Q \tag{8.24}$$

which reads, in local coordinates,

$$\begin{aligned} \alpha_{21}\left(\left(q,p,v,V\right),\left(q,p,v,V\right)\right) &= \beta_{\left(T^*Q,\omega_Q\right)}\left(q,p,v,V\right) - \alpha_Q\left(q,p,v,V\right) \\ &= \left(q,p,-V,v\right) \ominus \left(q,v,V,p\right). \end{aligned} \tag{8.25}$$

The symplectic form in the product $T^*\left(TQ \times T^*Q\right)$ is

$$\theta_{TQ} \oplus \theta_{T^*Q} = pr_2^*\left(\theta_{TQ}\right) + pr_1^*\left(\theta_{T^*Q}\right), \tag{8.26}$$

hence on $TT^*Q \times TT^*Q$ we have

$$\begin{aligned} \theta_{TQ} \ominus \theta_{T^*Q} &= \alpha_{21}^*\left(\theta_{TQ} \oplus \theta_{T^*Q}\right) \\ &= \left(V_i dq^i + p_i dv^i\right) - \left(v^i dp_i - V_i dq^i\right) \end{aligned} \tag{8.27}$$

and the derivative is

$$\omega_{TQ} \ominus \omega_{T^*Q} = 2\left(dV_i \wedge dq^i + dv^i \wedge dp_i\right). \tag{8.28}$$

The projection onto the base $TQ \times T^*Q$ is given by

$$\begin{aligned}
(\Pi_2 \times \Pi_1)(q, p, v, V) &= ((\pi_1 \times \pi_2) \circ \alpha_{21})(q, p, v, V) \\
&= (\pi_1 \times \pi_2)((q, p, -V, v) \ominus (q, v, V, p)) \\
&= ((q, v), (q, p)) \in T_xQ \times T_x^*Q. \qquad (8.29)
\end{aligned}$$

We, thus, have the special symplectic manifold

$$(TT^*Q \times TT^*Q, TQ \times T^*Q, (\Pi_2 \times \Pi_1), \theta_{TQ} \ominus \theta_{T^*Q}) \qquad (8.30)$$

associated with the diagram

$$\begin{tikzcd}
TT^*Q \times TT^*Q \arrow[rr, "\beta_{(T^*Q,\, \omega_Q)} \times \alpha_Q"] \arrow[dr, "\Pi_2 \times \Pi_1"'] & & T^*TQ \times T^*T^*Q \arrow[dl, "\pi_{TQ} \times \pi_{T^*Q}"] \\
& TQ_2 \times T^*Q &
\end{tikzcd}$$

Figure 8.3. Special symplectic structure for cartesian of TT^*Q

9. LEGENDRE TRANSFORMATIONS

Let (P, ω) be the underlying symplectic manifold of two special symplectic manifolds $(P, Q_1, \pi_1, \theta_1)$ and $(P, Q_2, \pi_2, \theta_2)$. Lagrangian submanifolds of (P, ω) may be generated by generating functions with respect to both special structures. The transition from the representation of Lagrangian submanifolds of (P, ω) by generating functions with respect to $(P, Q_1, \pi_1, \theta_1)$ to the representation by generating functions with respect to $(P, Q_2, \pi_2, \theta_2)$ is called the **Legendre transformation** from $(P, Q_1, \pi_1, \theta_1)$ to $(P, Q_2, \pi_2, \theta_2)$. In this case, the symplectomorphism $P \to P$ whose graph is the Lagrangian submanifold of $P \times P$ is chosen to be the identity mapping id_P so that one ensures the change in representation of Lagrangian submanifold only. The generator of id_P will be a function E_{21} on a submanifold I_{21} of $Q_2 \times Q_1$ which is in fact the required generator of the Legendre transformation.

The graph of id_P can, alternatively, be represented as the diagonal map $\Delta : P \to P \times P$ defined as

$$\Delta(p) = (p, p) \in P \times P. \tag{9.1}$$

An important property of this map which will lead us to find the generating function E_{21} is that, the pull-back of bundles over P is their Whitney sum. This means that, each fiber of the pull-back bundle is the direct sum of the corresponding fibers. For example, the pull-back of canonical forms on P is now a direct sum.

Given a Lagrangian submanifold N of P its image, which is called a **Legendre relation**, under the mapping $(\pi_{Q_2} \times \pi_{Q_1}) \circ \Delta : P \to Q_2 \times Q_1$ can be realized as the graph of a map $\Lambda_{21} : Q_1 \to Q_2$. This map is called the **Legendre transformation** of Q_1 to Q_2 corresponding to the Lagrangian submanifold N of P.

9.1. Legendre Transformations Between TQ and T^*Q

Let Q be the configuration manifold of a mechanical system. The phase space of the system is the symplectic manifold $(T^*Q, d\theta_Q)$. There are two canonical special symplectic structures for the symplectic manifold $(TT^*Q, dd_T\theta_Q = di_T\omega_Q)$, Hamiltonian and Lagrangian special symplectic structures.

Proposition 9.1. *The Legendre transformation of a Lagrangian system is generated by the function E_{21} defined on $I_{21} = T^*Q \times_Q TQ$ and is given by*

$$E_{21} \circ (\alpha, v) = -\langle \alpha, v \rangle \tag{9.2}$$

Proof. Let Φ be the mapping defined by the commutative diagram

$$\begin{tikzcd} TT^*Q \arrow[r, "\Delta"] \arrow[d, "\Phi"'] & TT^*Q \times TT^*Q \arrow[d, "\tau_{T^*Q} \times T\pi_Q"] \\ T^*Q \times_Q TQ \arrow[r, hook] & T^*Q \times TQ \end{tikzcd}$$

Figure 9.1. Definition of the mapping phi

where Δ is the diagonal map

$$\Delta(q, p, v, V) = ((q, p, v, V), (q, p, v, V)) = graph\,(id_{TT^*Q}) \,. \tag{9.3}$$

Then for $w : T^*Q \to TT^*Q$

$$(E_{21} \circ \Phi)(w) = E_{21} \circ (\tau_{T^*Q} \circ w, T\pi_Q \circ w) = -\langle \tau_{T^*Q} \circ w, T\pi_Q \circ w \rangle \tag{9.4}$$

hence by definition of the canonical 1-form θ_Q we have

$$E_{21} \circ \Phi = -i_T \theta_Q \tag{9.5}$$

and $\Phi\left(w\left(q,p,v,V\right)\right) = \left(q,p,v\right)$, in coordinates, we recall from section (4) that $E_{21}\left(q,p,v\right) = -pv$. To see that this generates the Lagrangian submanifold we compute

$$\Delta^*\left(i_T d\theta_Q \ominus d_T\theta_Q\right) = i_T d\theta_Q - d_T\theta_Q = -di_T\theta_Q = d\left(E_{21}\circ\rho\right). \tag{9.6}$$

■

The generating objects for the Legendre transformation and its inverse can be represented by the following diagrams

$$\begin{tikzcd}
(TT^*Q x\, TT^*Q, i_T d\theta_Q - d_T\theta_Q) \arrow[d, "\tau_{T^*Q} x\, T\pi_Q"] & & \\
T^*QxTQ & T^*Qx_Q TQ \arrow[l, hook'] \arrow[r, "-\langle\, ,\, \rangle"] & IR
\end{tikzcd}$$

Figure 9.2. Legendre transformation from TQ to T^*Q

and for the inverse case we have

$$\begin{tikzcd}
(TT^*Q x\, TT^*Q, d_T\theta_Q - i_T d\theta_Q) \arrow[d, "T\pi_Q x\, \tau_{T^*Q}"] & & \\
TQxT^*Q & TQx_Q T^*Q \arrow[l, hook'] \arrow[r, "\langle\, ,\, \rangle"] & IR
\end{tikzcd}$$

Figure 9.3. Inverse Legendre transformation from T^*Q to TQ

To pass to the Hamiltonian formalism from Lagrangian formalism, let us start with a Lagrangian system $(TT^*Q, TQ, d_T\theta_Q, T\pi_Q)$ with generating object

$$\begin{tikzcd}
(TT^*Q, d_T\theta_Q) \arrow[d, "T\pi_Q"] & R_1 \arrow[r, "L"] \arrow[d, "\rho_1"] & IR \\
TQ & X_1 \arrow[l, hook'] &
\end{tikzcd}$$

Figure 9.4. Lagrangian system

We define the Hamiltonian family with

$$\begin{aligned}
X_2 &= \pi_Q^{-1}\left(\tau_Q\left(X_1\right)\right) = \left\{(q,p) \in T^*Q : \exists\,(q,v) \in X_1; \pi_Q\left(q,p\right) = \tau_Q\left(q,v\right)\right\} && (9.7)\\
R_2 &= \left\{(q,p,v,r) \in T^*Q \times TQ \times R_1 : (q,v) \in X_1, \rho_1\left(q,p,r\right) = (q,v)\right\} && (9.8)
\end{aligned}$$

and the corresponding Hamiltonian system is represented by the generating object

$$\begin{tikzcd}
(TT^*Q, i_T\omega_Q) \arrow[d, "\tau_{T^*Q}"] & R_2 \arrow[r, "-H"] \arrow[d, "\rho_2"] & IR \\
T^*Q & X_2 \arrow[l, hook'] &
\end{tikzcd}$$

Figure 9.5. Hamiltonian system

where $\rho_2 : R_2 \to X_2 : (q,p,v,r) \to (q,p)$ and

$$H : R_2 \to \mathbb{R} : (q,p,v,r) \to \langle p, v\rangle - L\left(q,v,r\right). \qquad (9.9)$$

These two systems generate the same dynamics since the generating object generates the identity relation. Conversely, given a Hamiltonian system, we can also construct a Lagrangian system with

$$X_1 = \tau_Q^{-1}\left(\pi_Q\left(X_2\right)\right) = \left\{(q,v) \in TQ : \exists\,(q,p) \in T^*Q; \pi_Q\left(q,p\right) = \tau_Q\left(q,v\right)\right\} \tag{9.10}$$

$$R_1 = \left\{(q,p,v,r) \in T^*Q \times_Q TQ \times R_2 : (q,p) \in X_2, \rho_2\left(q,p,r\right) = (q,p)\right\} \tag{9.11}$$

and the Lagrangian function

$$L : R_1 \to \mathbb{R} : (q,p,v,r) \to \langle p,v\rangle - H\left(q,p,r\right). \tag{9.12}$$

which is called the **inverse Legendre transformation**.

9.2. Examples

A Non-relativistic Particle:

Let Q be the configuration manifold of a non-relativistic particle of mass m, and let $U\left(q\right)$ be the local expression of the potential energy of the particle. Let us recall our notation higher order bundles $(q,p,v,V) \in TT^*Q$. Dynamics of the particle represented by a Lagrangian submanifold D of $(TT^*Q, d_T\omega)$ defined locally by

$$p_i = mv^i, \quad V_i = -\frac{\partial U}{\partial q^i}. \tag{9.13}$$

The submanifold D can be described by the equations

$$\begin{aligned} V_i dq^i + p_i dv^i &= -\frac{\partial U}{\partial q^i}dq^i + mv^i dv^i = d\left(-U\left(q^i\right) + \frac{1}{2}m\left(v^i\right)^2\right) && (9.14)\\ V_i dq^i - v^i dp_i &= -\frac{\partial U}{\partial q^i}dq^i - \frac{1}{m}p_i dp_i = -d\left(V\left(q^i\right) + \frac{1}{2m}\left(p_i\right)^2\right), && (9.15)\end{aligned}$$

hence the Lagrangian and Hamiltonian functions are

$$\begin{aligned} L\left(q^{i}, v^{j}\right) &= \frac{1}{2} m\left(v^{j}\right)^{2}-U\left(q^{i}\right) & (9.16) \\ H\left(q^{i}, p_{j}\right) &= U\left(q^{i}\right)+\frac{1}{2m}\left(p_{j}\right)^{2} & (9.17) \end{aligned}$$

and the relations

$$\begin{aligned} H\left(q^{i}, p_{j}\right) &= p_{j} v^{j}-L\left(q^{i}, v^{j}\right) & (9.18) \\ L\left(q^{i}, \dot{q}^{j}\right) &= p_{j} v^{j}-H\left(q^{i}, p_{j}\right) & (9.19) \end{aligned}$$

between them are local expressions of Legendre and inverse Legendre transformations.

Thermostatic:

Let P be a manifold with elements $(V, S, p, T) \in P$ interpreted as volume, metrical entropy, the pressure and the absolute temperature, respectively, of one mole of ideal gas. Symplectic 2-form on P is given as

$$\omega = dV \wedge dp + dT \wedge dS. \tag{9.20}$$

The equations determining the behavior of the ideal gas are

$$pV = RT, \quad pV^{\gamma} = K \exp \frac{S}{C_V} \tag{9.21}$$

where R, K and γ are constants and $C_V = \dfrac{R}{\gamma - 1}$.Let $(V, S) \in Q_1$, $(V, T) \in Q_2$, $(p, T) \in Q_3$ and $(S, p) \in Q_4$ be 2-dimensional manifolds. With the projections

$$\pi_i : P \to Q_i \tag{9.22}$$

and the 1-forms

$$\begin{aligned}
\theta_1 &= -pdV + TdS && (9.23)\\
\theta_2 &= -pdV - SdT && (9.24)\\
\theta_3 &= Vdp - SdT && (9.25)\\
\theta_4 &= Vdp + TdS && (9.26)
\end{aligned}$$

we have the special symplectic manifolds $(P, Q_1, \pi_1, \theta_1)$, $(P, Q_2, \pi_2, \theta_2)$, $(P, Q_3, \pi_3, \theta_3)$, and $(P, Q_4, \pi_4, \theta_4)$, respectively. The underlying symplectic manifold is

$$(P, \omega = dV \wedge dp + dT \wedge dS).$$

The relations in Eq (9.21) defines a Lagrangian submanifold of this structure. To find the generating functions with respect to above special symplectic structures we compute for the first one

$$\begin{aligned}
\theta_1 &= -pdV + TdS \\
&= -K \exp\left(\frac{S}{C_V}\right)\frac{1}{V^{\gamma}}dV + \frac{K}{R}V^{(1-\gamma)}\exp\left(\frac{S}{C_V}\right)dS. \\
&= d\left(\frac{K}{\gamma-1}V^{(1-\gamma)}\exp\left(\frac{S}{C_V}\right)\right) && (9.27)
\end{aligned}$$

which gives

$$U_1(V, S) = \frac{K}{\gamma-1}V^{(1-\gamma)}\exp\left(\frac{S}{X_V}\right). \qquad (9.28)$$

Similarly, we find

$$
\begin{aligned}
\theta_2 &= -pdV - SdT \\
&= -\frac{RT}{V}dV - C_V \ln\left(\frac{RTV^{\gamma-1}}{K}\right)dT \\
&= d\left(C_V T\left(1 - \ln T + \ln K - \ln R\right) - RT\ln V\right) \qquad (9.29) \\
\theta_3 &= Vdp - SdT \\
&= \frac{RT}{p}dp - C_V \ln\left(\frac{R^\gamma T^\gamma}{p^{\gamma-1}K}\right) \\
&= d\left(C_V T\left(1 - \ln T - \ln R\right) + C_V T\ln K + RT\ln p\right) \qquad (9.30) \\
\theta_4 &= Vdp + TdS \\
&= \left(\frac{K\exp\frac{S}{C_V}}{p}\right)dp + \frac{p}{R}\left(\frac{K\exp\frac{S}{C_V}}{p}\right)dS \\
&= d\left(\frac{\gamma}{\gamma-1}K^{\frac{1}{\gamma}}p^{\frac{\gamma-1}{\gamma}}\exp\frac{S}{C_V}\right) \qquad (9.31)
\end{aligned}
$$

so that generating functions are

$$
\begin{aligned}
U_2(V,T) &= C_V T\left(1 - \ln T + \ln K - \ln R\right) - RT\ln V \qquad (9.32) \\
U_3(p,T) &= C_V T\left(1 - \ln T - \ln R\right) + C_V T\ln K + RT\ln p \qquad (9.33) \\
U_4(S,p) &= \frac{\gamma}{\gamma-1}K^{\frac{1}{\gamma}}p^{\frac{\gamma-1}{\gamma}}\exp\frac{S}{C_V}. \qquad (9.34)
\end{aligned}
$$

The generating functions U_1, U_2, U_3, U_4 are known as thermodynamic potentials and are called the *internal energy*, the *Helmholtz function*, the *Gibbs function* and the *enthalpy,* respectively.

There are twelve Legendre transformations relating the four special symplectic structures or four thermodynamic potentials. The mapping $\pi_2 \times \pi_1$ maps the diagonal of $P \times P$ onto a submanifold I_{21} of $Q_2 \times Q_1$ its coordinates (V, S, T) related to the coordinates (V, S, p, T). The Legendre transformation from $(P, Q_1, \pi_1, \theta_1)$ to $(P, Q_2, \pi_2, \theta_2)$ is generated by the function E_{21} defined on I_{21} by

$$E_{21}(V, S, T) = -TS. \qquad (9.35)$$

Similarly, the Legendre transformation from $(P, Q_1, \pi_1, \theta_1)$ to $(P, Q_3, \pi_3, \theta_3)$ is generated by the function E_{31} defined on $I_{31} \subset Q_3 \times Q_1$ by

$$E_{31}(V, S, p, T) = pV - TS. \tag{9.36}$$

and the others can be found to be

$$E_{41} = pV, \quad E_{32} = pV, \quad E_{42} = pV + TS, \quad E_{43} = TS \tag{9.37}$$

from which one can also verifies various relations between generating functions such as

$$U_2(V, T) = U_1(V, S) - TS. \tag{9.38}$$

APPENDIX A

Submanifolds and Product Manifolds

A differentiable mapping $\alpha : S \rightarrow Q$ is called an **immersion** if at each point $m \in S$ the linear mapping

$$T_m\alpha : T_m S \rightarrow T_{\alpha(m)} Q \tag{A.1}$$

obtained by restricting the mapping $T\alpha$ to the fibre $T_m S = \tau_M^{-1}(m)$ is injective. If $(m^i) : S \rightarrow \mathbb{R}^k$ are coordinates in M and $\alpha^k = q^k \circ \alpha$, then α is an immersion if the matrix $\partial_i \alpha^k$ is of maximal rank k. The image $S = im(\alpha) \subset Q$ is called an **immersed submanifold** of Q of dimension k.

If the immersion $\alpha : M \rightarrow Q$ is injective, then it is called an **embedded submanifold**. The level sets of functions

$$S = \{q \in Q : F_A(q) = 0, \forall F_A : Q \rightarrow \mathbb{R}\} \tag{A.2}$$

is an example of an embedded submanifold. Here F_A are $m-k$ functions on Q such that the matrix $\partial_\kappa F_A$ is of maximal rank $m-k$ at points of S. (See [8])

As an example of an immersed but not embedded submanifold in $\mathbb{R}^2$ consider the graph of the function $\alpha(t) = \left(2\cos\left(t - \frac{1}{2}\pi\right), \sin 2\left(t - \frac{1}{2}\pi\right)\right)$, which is in form of figure eight. The graph is periodic but not injective. On the other hand, although the image of the function $\alpha(t) = \left(2\cos\left(2\arctan\left(t + \frac{\pi}{2}\right)\right), \sin 2\left(2\arctan\left(t + \frac{\pi}{2}\right)\right)\right)$ is similar to the above one it is an embedded submanifold for it does not intersect itself. (See [4])

The product manifold $Q \times P$ has coordinates $(q^i, p^j) : Q \times P \rightarrow \mathbb{R}^{m+n}$. The tangent bundle $(T(Q \times P), \tau_{Q\times P}, Q \times P)$ is isomorphic to the product of tangent bundles $(TQ \times TP, \tau_Q \times \tau_P, Q \times P)$ with the isomorphism $\psi : T(Q \times P) \to TQ \times TP$ given by

$$\psi(\xi) = (Tpr_1(\xi), Tpr_2(\xi)) \in T_qQ \times T_pP, \tag{A.3}$$

where $\xi \in T_{(q,p)}(Q \times P)$, $pr_1 : Q \times P \to Q$ and $pr_2 : Q \times P \to P$ are projections with tangent mappings $Tpr_1 : T(Q \times P) \to TQ$ and $Tpr_2 : T(Q \times P) \to TP$. In local coordinates,

$$\xi = \xi_q^i \frac{\partial}{\partial q^i} + \xi_p^j \frac{\partial}{\partial p^j}, \quad \psi(\xi) = \left(\xi_q^i \frac{\partial}{\partial q^i}, \xi_p^j \frac{\partial}{\partial p^j}, \right). \tag{A.4}$$

Pull-back

Let P and Q be differentiable manifolds and $\psi : Q \to P$ be a diffeomorphism. If $f : P \to \mathbb{R}$ is a function, its **pull-back** by ψ is a function on Q defined by

$$\psi^* f = f_Q = f \circ \psi = Q \to \mathbb{R}. \tag{A.5}$$

The action of ψ on vectors on Q is by push-forward and is defined before. To interchange push-forward with pull-back one replaces ψ with ψ^{-1}.

If α is a p-form on P, its pull-back by ψ is a p-form on Q defined by

$$\langle (\psi^* \alpha)(x); v_1(x), v_2(x), ...v_p(x) \rangle = \langle (\alpha)(\psi(x)); T_x\psi \circ v_1(x), ..., T_x\psi \circ v_p(x) \rangle \tag{A.6}$$

for vectors $v_i(x) \in T_xQ$. Pull-back of forms satisfies $\psi^*(\alpha \wedge \beta) = \psi^*\alpha \wedge \psi^*\beta$.

For a one-form $\alpha(y) = \alpha_i(y)\, dy^i$ and a 2-form $\omega = \frac{1}{2}\omega_{ij}(y)\, dy^i \wedge dy^j$ on P at

$y = \psi(x)$, $x \in Q$, the coordinate expressions for pull-backs are

$$(\psi^*\alpha)(x) = \alpha_i(\psi(x))\frac{\partial\psi^i(x)}{\partial q^j}dq^j \tag{A.7}$$

$$(\psi^*\omega)(x) = \frac{1}{2}\omega_{ij}(\psi(x))\frac{\partial\psi^i(x)}{\partial q^k}\frac{\partial\psi^j(x)}{\partial q^l}dq^k \wedge dq^l. \tag{A.8}$$

In particular, for one forms $\alpha(x) = \alpha_i(q)\,dq^i$ on Q pull-back by the tangent projection $\tau_Q : TQ \to Q$ defined by

$$\left(\tau_Q^*\alpha\right)(v(x)) = \alpha_i\left(\left(\varphi \circ \tau_Q\right)(v(x))\right) d\left(\varphi \circ \tau_Q\right)^i \tag{A.9}$$

is a one-form on T_xQ although both α on Q and $\left(\tau_Q^*\alpha\right)$ on TQ have the same coordinate expressions. The properties of pull-back of forms are

$$\psi^*\left(\alpha^1 + \alpha^2\right) = \psi^*\alpha^1 + \psi^*\alpha^2 \tag{A.10}$$

$$\psi^*\left(\alpha^1 \wedge \alpha^2\right) = \psi^*\alpha^1 \wedge \psi^*\alpha^2 \tag{A.11}$$

and that it commutes with d.An important characterization of the canonical 1-form is the following

Proposition A.1. *Let $\alpha(x)$ be a 1-form on Q, and θ_Q is canonical 1-form then*

$$\alpha(x)^*\theta_Q = \alpha(x). \tag{A.12}$$

To see this, we can take $\alpha(x)$ as a section $\alpha(x) : Q \to T^*Q$, then

$$\begin{aligned}
\langle \alpha(x)^*\theta_Q, v(x)\rangle &= \langle \theta_Q, T\alpha \circ v(x)\rangle \\
&= \langle \tau_{T^*Q} \circ T\alpha \circ v(x), T\pi_Q \circ T\alpha \circ v(x)\rangle \\
&= \langle \alpha \circ \tau_Q \circ v(x), T(\pi_Q \circ \alpha) \circ v(x)\rangle \\
&= \langle \alpha, v\rangle(x),
\end{aligned} \tag{A.13}$$

where $v(x) : Q \to TQ$ arbitrary vector field. The first step is coming from definition of pullback, the second is obtained from definition of Liouville form, the third is obtained from the following figure and the last step is obtained since $\pi_Q \circ \alpha(x) = x$.

Derivations

Let $\Omega(Q)$ be the exterior algebra of differential forms on Q. A linear operator

$$a : \Omega(Q) \to \Omega(Q) : \mu \to a\mu \tag{A.14}$$

is called a **derivation** for a q-form μ of $\Omega(Q)$ of degree p if $a\mu$ is a form of degree $q+p$ and

$$a(\mu \wedge \nu) = a\mu \wedge \nu + (-1)^{pq} \mu \wedge a\nu \tag{A.15}$$

for all $\nu \in \Omega(Q)$. By definition, the exterior derivative $d : \Omega(Q) \to \Omega(Q)$ is a derivation of degree 1.

Definition A.1. *Let $\Omega(Q)$ be the exterior algebra of differential forms on a differential manifold Q and let $\Omega(TQ)$ be the exterior algebra of forms on tangent bundle TQ of Q. A linear mapping*

$$\begin{aligned} a &: \Omega(Q) \to \Omega(TQ) \\ &: \mu \to a\mu \end{aligned} \tag{A.16}$$

is called a derivation of degree p of $\Omega(Q)$ into $\Omega(TQ)$ relative to τ_Q if $\deg(a\mu) = \deg(\mu) + p$ *and*

$$a(\mu \wedge \nu) = a\mu \wedge \tau_Q^* \nu + (-1)^{pq} \tau_Q^* \mu \wedge a\nu \tag{A.17}$$

where degree of μ is q and ν is any form on Q.

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