FIBER BUNDLES, DIFFEOMORPHISM GROUPS AND PLASMA DYNAMICS

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ABSTRACT

FIBER BUNDLES, DIFFEOMORPHISM GROUPS AND PLASMA DYNAMICS

In the thesis, we investigate the geometrical framework of the Lie-Poisson description of the Poisson-Vlasov equations which govern the motion of the collisionless plasma. To this aim, we review symplectic and Poisson manifolds, connections on smooth bundles and symplectic reduction theory. An element of the first order jet bundle can be considered as a connection and hence, decomposes vector fields into vertical representative and holonomic parts, which are generalized vector fields of order one. The complete and the vertical lifts of vector fields and one-forms are presented and, in the existence of a connection, the decompositions of iterated bundles TT, T^*T, TT^* and T^*T^* into the direct sums of the first order bundles T and T^* are given.

Poisson equation is obtained as the preimage of a regular value of a momentum mapping coming from the gauge invariance of the Hamiltonian dynamics. We take the configuration space of the collisionless plasma as the space of canonical diffeomorphisms and attach Green's function solution of Poisson equation as a constraint in the calculations. Lie algebra of the group of canonical diffeomorphisms is the space of Hamiltonian vector fields. For the dual of the Lie algebra, there are two possibilities, namely density and momentum representations. From the Lie-Poisson formulation of Vlasov equation on the momentum representation, we obtain an intermediate system, which is called momentum-Vlasov equations. It is shown that, momentum-Vlasov equations are generated by the vertical representative of the complete cotangent lift of the Hamiltonian vector field whose associated Hamiltonian function is the energy of a charged particle in momentum phase space.

The algebra of vector fields on a symplectic manifold is decomposed into semi-direct product algebra of Hamiltonian vector fields and its complement which is isomorphic to its dual, is presented. A similar discussion on the algebraic properties of the decomposition of the one-form section into closed and non-closed one-forms is made.

ÖZET

LİF DEMETLERİ, DİFEOMORFİZMALAR GRUBU VE PLAZMA DİNAMİĞİ

Bu tezde, çarpışmasız plazma için hareket denklemleri olan Poisson-Vlasov denklem takımının Lie-Poisson formu ve bu formun geometrik altyapısı çalışılmıştır. Bu amaç doğrultusunda, ilk olarak simplektik ve Poisson katmanları, düzgün demetler üzerinde bağlantı ve simplektik indirgeme teoremi gözden geçirilmiştir. Bir düzgün demetin birinci jet uzayının elemanları, demet üzerinde bağlantı olarak kullanılmış ve vektör alanları dik temsiller ve holonomik parçaların direk toplamı olarak ifade edilmiştir. Vektör alanların ve bir-formların tam ve dik kaldırılışları tanımlanmış, ikinci derece demetler TT, T*T, TT* ve T*T*, birinci dereceden demetlerin T ve T* direk toplamları cinsinden ifade edilmiştir.

Poisson denklemi Hamilton dinamiğinin ayar simetrisinden kaynaklanan momentum dönüşümünün düzenli bir noktadaki öngörüntüsü olarak elde edilmiştir. Kanonik dönüşümler grubu plazma hareketi için konfigurasyon uzayı olarak alınmış ve Poisson denkleminin Green fonksiyonu ile çözümü hesaplar için bir kısıt olarak kullanılmıştır. Kanonik dönüşüm grubunun Lie cebirini Hamiltonyen vektör alanları oluşturmaktadır. Bu cebirin dual uzayında standart Lie-Poisson denklemleri momentum-Vlasov denklem takımını vermektedir. Yüklü parçacığın faz uzayındaki hareketini yöneten Hamiltonyen vektör alanının tam kotanjant kaldırılışının dik temsilinin momentum-Vlasov denklemlerini ürettiği gösterilmiştir.

Son olarakta, bir simplektik katman üzerindeki vektör alanları uzayı Hamiltonyen olan ve olmayan vector alanların oluşturduğu iki alt uzayın yarı-direk toplamı olarak ifade edilmiştir. Benzer olarak bir-form kesitlerin oluşturduğu uzay, kapalı olan ve olmayan bir form kesitlerin olusturduğu altuzayların direk toplamı olarak ifade edilmiş ve bu altuzayların üzerindeki cebirsel yapılar incelenmiştir.

TABLE OF CONTENTS

LIST OF SYMBOLS / ABBREVIATIONS

d	Exterior derivative
$Diff(Q)$	Group of diffeomorphisms on Q
(E,π,M)	Smooth bundle with projection π
$F(Q)$	The space of smooth functions on the manifold Q
G, H	Lie groups
i_X	Interior product
$J^1(\pi)$	First jet manifold of π
L_X	Lie derivative
$LieG, \mathfrak{g}$	Lie algebra
$Lie^*G, \mathfrak{g}^*$	The dual of a Lie algebra
Q, M	Manifolds
TQ	Tangent bundle of the manifold Q
T^*Q	Cotangent bundle of the manifold Q
$T\rho$	Tangent (lifting) mapping of ρ
$T^*\rho$	Cotangent (lifting) mapping of ρ
$\mathcal{X}(Q)$	Space of vector fields on Q
θ_{T^*Q}	Liouville 1-form on T^*Q
$\Lambda(Q)$	Graded algebra of the differential forms on Q
Ω_{T^*Q}	Canonical 2-form on T^*Q
ρ^*	Pull-back by a mapping ρ
ρ_*	Push-forward by a mapping ρ
π_Q	Cotangent fibration
τ_Q	Tangent fibration
$[.,.]_{JL}$	Jacobi-Lie bracket of vector fields
$\{\ ,\ \}_{T^*Q}$	Canonical Poisson bracket on T^*Q

1. PROLOGUE

1.1. GEOMETRIC MECHANICS AND REDUCTION

Mechanics has two main points of view, Lagrangian mechanics and Hamiltonian mechanics. Lagrangian approach is based on the observation that there are variational principles behind Newton's second law. In this approach, the dynamics of a system is formulated by Lagrangian function L on velocity phase space of the configuration space, formally on the tangent bundle $T\mathcal{M}$ of the configuration manifold $\mathcal{M}$. The main orientation of the Lagrangian approach is to extremize an action integral

$$\mathcal{A}(L) = \int L dt \tag{1.1}$$

with a variational principle called Hamilton's variational principle [1,2]. In local coordinates $\left(x^a, \dot{x}^b\right)$ on $T\mathcal{M}$, one obtains the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0. \tag{1.2}$$

On the other hand, the Hamiltonian view of mechanics is based on symplectic geometry. In this approach, the dynamics is represented by a Hamiltonian function H on the momentum phase space, formally on the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$. In the framework of symplectic geometry, the Hamiltonian dynamics is governed by Hamilton's equations

$$i_{X_H}\left(\Omega_{T^*\mathcal{M}}\right) = dH, \tag{1.3}$$

where X_H is the Hamiltonian vector field for the Hamiltonian function H, i_{X_H} is the interior product and $\Omega_{T^*\mathcal{M}}$ is the canonical symplectic form on the cotangent bundle $T^*\mathcal{M}$ [3,4]. In canonical coordinates (x^a, y_b) on $T^*\mathcal{M}$, the equations of the motion are given by

$$\dot{x}^a = \frac{\partial H}{\partial y_a} \quad \text{and} \quad \dot{y}_a = -\frac{\partial H}{\partial x^a}. \tag{1.4}$$

Having a correspondence between dynamics X_H and functions H on $\mathcal{M}$, an algebra of functions on $\mathcal{M}$ can be defined by the Poisson bracket

$$\{H, F\} = X_H(F) = -X_F(H)$$

which is nondegenerate. The dynamics of the system for a given Hamiltonian function H is governed by the equations

$$\dot{x}^a = \{x^a, H\} \quad \text{and} \quad \dot{y}_a = \{y_a, H\}. \tag{1.5}$$

This generalization of the symplectic geometry is the Poisson geometry [5]. A Poisson structure is a bilinear skew-symmetric binary operation $\{\ ,\ \}$ on the space of smooth functions that satisfies Jacobi and Leibniz identities. For the general and more rigorous constructions of Hamiltonian and Lagrangian mechanics, some references are [6, 7].

Transformations between Lagrangian mechanics and Hamiltonian mechanics are called Legendre transformations. If a nondegeneracy condition, called Hessian condition, is satisfied then the transformation is immediate. Although, a general construction is presented in [8] without the nondegeneracy condition, one faces with serious complications in transforming different descriptions of particular mechanical systems [9, 10].

Many physical systems such as rigid bodies, fluids and plasmas can not be expressed neither in the framework of canonical Hamiltonian formalism nor the variational principles of usual form due to existence of symmetries and constraints. One of the interests of the reduction theory is to obtain a noncanonical system from a canonical one by dividing out the symmetries of the physical system [11]. In Lagrangian reduction theory, one emphasizes how the variational principles pass to a quotient space whereas Hamiltonian reduction is interested in how to reduce the symplectic structure and Hamiltonian function.

Although the origins of Hamiltonian reduction theory can be found in the works of Euler, Lagrange, Hamilton, Jacobi and Poincaré, in the literature, start of the modern history of the theory has been considered as the pioneering papers of Arnold [12] and Smale [13], where one can find the reduction procedure for the systems whose configuration spaces are Lie groups. The papers of Marsden and Weinstein [14] and Meyer [15] developed the Hamiltonian reduction theory in the general context of symplectic manifolds. The Lagrangian version of the reduction theory is developed much more later then the Hamiltonian one. One expects that, the two methodologies must be in relation by the Legendre transform, but most of the cases are not straightforward way, such as thermodynamics and plasma. A detailed and comprehensive history of the reduction theory can be found in [11] and [16].

The particular case in which the configuration space of a dynamical system is a Lie group, say G, attracts deep interests since the configuration space of the systems such as Euler's top, incompressible fluid and collisionless plasma, are Lie groups [17,18]. A Lie group G acts, say on right, on its tangent TG and cotangent T^*G bundles by the lifts of group multiplication. A Hamiltonian system on T^*G under the symmetry of the lifted action can be reduced to a Hamiltonian system on the quotient space $T^*G/G = \mathfrak{g}^*$ which is the linear algebraic dual of the Lie algebra $\mathfrak{g} = T_eG$ of G. $\mathfrak{g}^*$ is a Poisson manifold with the Lie-Poisson bracket

$$\{F, H\}(\alpha) = \left\langle \alpha, \left[\frac{\delta F}{\delta\alpha}, \frac{\delta H}{\delta\alpha}\right]\right\rangle, \tag{1.6}$$

where $\alpha \in \mathfrak{g}^*$, $[\ ,\ ]$ is the Lie bracket on $\mathfrak{g}$, $\langle\ ,\ \rangle$ is the pairing between Lie algebra and its dual and we assume that the Fréchet derivatives $\delta F/\delta\alpha, \delta H/\delta\alpha \in \mathfrak{g}^{**} \simeq \mathfrak{g}$. For a Hamiltonian functional H, the equation of motion, namely the Lie-Poisson equation, is

$$\dot{\alpha} = -ad^*_{\frac{\delta H}{\delta\alpha}}\alpha, \tag{1.7}$$

where ad^* denotes the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ [11,19]. Similarly, a right invariant Lagrangian L on TG uniquely defines a real valued function l on $\mathfrak{g}$ by reduction. The Euler-Lagrange equations for L reduces to the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta l}{\delta\xi} = ad^*_{\xi}\frac{\delta l}{\delta\xi} \tag{1.8}$$

for reduced Lagrangian l, where $\xi \in \mathfrak{g}$ and $\delta l/\delta\xi \in \mathfrak{g}^*$ [20–22].

1.2. COLLISIONLESS PLASMA

Plasma consists of free positive and negative charge carriers and looks neutral from the outside and it is estimated to constitute more than 99 percent of the visible universe [23,24]. To describe the motion of plasma in $\mathcal{Q} \subset \mathbb{R}^3$, one may start to write down the whole microscopic data and Newton formulas and interactions for whole particles, which is very difficult. The kinetic plasma theory uses statistical and probabilistic concepts to handle the practical problems of the microscopic theory. The basic element in the kinetic description of a plasma is the plasma density (distribution) function $f = f(\mathbf{q}, \mathbf{p})$ that describes how particles are distributed in position-momentum phase space $T^*\mathcal{Q}$ [25–27]. We use $\mathbf{z} = (\mathbf{q}, \mathbf{p})$ as local coordinates on $T^*\mathcal{Q}$.

Charged particles in an electromagnetic field are described by the **Maxwell-Vlasov** equations consisting of the Maxwell's equations

$$\begin{array}{c} \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{E} = \rho \end{array} \quad \text{and} \quad \begin{array}{c} \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c}\mathbf{J} \\ \nabla \cdot \mathbf{B} = 0 \end{array} \tag{1.9}$$

and the Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_q f + e\left(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}\right) \cdot \nabla_p f = 0 \tag{1.10}$$

where $\mathbf{E}$ and $\mathbf{B}$ denotes the electric and magnetic fields,

$$\rho = -e \int f(\mathbf{q}, \mathbf{p})\, d^3\mathbf{p} \tag{1.11}$$

is the charge density, $\mathbf{J}$ is the current density vector, c is the speed of the light and $\mathbf{p} = m\mathbf{v}$ is the momentum of plasma particles. Some of the references involving the Hamiltonian and Lagrangian descriptions of Maxwell-Vlasov equations are [28–33].

In case of an unmagnetized plasma $\mathbf{B} = \mathbf{0}$ and absence of any current $\mathbf{J} = \mathbf{0}$, the electric field $\mathbf{E}$ is the purely potential, that is, $\mathbf{E} = \nabla_q \phi_f(\mathbf{q})$. The potential $\phi_f(\mathbf{q})$ is determined through the **Poisson equation**

$$\nabla_q^2 \phi_f = -e \int f(\mathbf{q}, \mathbf{p})\, d^3p, \tag{1.12}$$

and Vlasov equation reduces to

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_q f - e \nabla_q \phi_f \cdot \nabla_p f = 0. \tag{1.13}$$

The set of equations Eq.(1.12) and Eq.(1.13) are called the **Poisson-Vlasov equations**. One may alternatively regard the Poisson-Vlasov equations as an approximation of the Maxwell-Vlasov equations in the nonrelativistic zero-magnetic field limit, that is the limit $c \to \infty$ [39]. Some of the references for the first attempts to the investigation of the geometric foundations of the Hamiltonian and Lagrangian descriptions of Poisson-Vlasov equations are [34–38].

The main orientation of this thesis is to study geometric framework for Lie-Poisson formulations of the Poisson-Vlasov equations. We present some purely geometrical concepts in second, third and fourth sections. The rest of the thesis, fifth and sixth sections, is devoted to the application of these geometric constructions to particular case of the group of canonical diffeomorphisms, which is the configuration space of the collisionless nonrelativistic plasma. Foundations of this study can be found in some unpublished works of H. Gümral, J.E. Marsden, P.J. Morrison and T.S. Ratiu at the beginnings of 90s.

1.3. CONTENTS OF THE THESIS

In the following section, we start with the definition of a smooth bundle $(\mathcal{E}, \pi, \mathcal{M})$ and present tangent $(T\mathcal{M}, \tau_{\mathcal{M}}, \mathcal{M})$ and cotangent $(T^*\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$ bundles. Given a smooth bundle $(\mathcal{E}, \pi, \mathcal{M})$ we define the first jet manifold $J^1\pi$ and pull-back bundle $\rho^*(\pi)$ by a mapping $\rho : \mathcal{N} \to \mathcal{M}$. The first order generalized vector fields are defined as the sections of the fibration $J^1\pi \times_{\mathcal{E}} T\mathcal{E} \to J^1\pi$. It is argued that every smooth bundle $(\mathcal{E}, \pi, \mathcal{M})$ admits a short exact sequence

$$0 \;\rightarrow\; V\mathcal{E} \;\rightarrow\; T\mathcal{E} \;\rightarrow\; \pi^{*}\left(T\mathcal{M}\right) \;\rightarrow\; 0 \;, \qquad (1.14)$$

where $V\mathcal{E}$ is the bundle of vertical vectors and $\pi^{*}\left(T\mathcal{M}\right)$ is the pull-back bundle of $T\mathcal{M}$. There is no canonical way to split the sequence presented in Eq.(1.14), one needs an additional geometric structure on the sequence, so called connection, to do that. Defining a connection decomposes the cotangent bundle $T^{*}\mathcal{E}$, because the connection splits also the dual

$$0 \;\rightarrow\; \left(\pi^{*}\left(T\mathcal{M}\right)\right)^{*} \;\rightarrow\; T^{*}\mathcal{E} \;\rightarrow\; V^{*}\pi \;\rightarrow\; 0 \qquad (1.15)$$

of the sequence Eq.(1.14). We show that, an element of the jet manifold $J^{1}\pi$ can be regarded as a connection and hence decomposes the tangent bundle $T\mathcal{E}$ of $\mathcal{E}$ into direct sum of vertical and horizontal components, namely vertical representative and holonomic part. In the last subsection, we discuss the theory of exterior calculus and review some constructions on manifolds, such as volume, symplectic and Poisson structures.

In section 3, we start with a vector field X on a manifold $\mathcal{M}$ and, in a canonical way, define vector fields X^{c} and X^{c*}, called complete tangent and cotangent lifts of X, on the tangent bundle $T\mathcal{M}$ and cotangent bundle $T^{*}\mathcal{M}$ of $\mathcal{M}$, respectively. Hamiltonian structures of the complete lifts are discussed. Vertical lift X^{v} of a vector field X on $\mathcal{M}$ to the tangent bundle $T\mathcal{M}$ and vertical lift α^{v} of a one-form α on $\mathcal{M}$ to the cotangent bundle $T^{*}\mathcal{M}$ are defined. Under the existence of a connection, the iterated bundles $TT\mathcal{M}$, $TT^{*}\mathcal{M}$, $T^{*}T\mathcal{M}$ and $T^{*}T^{*}\mathcal{M}$ are expressed as the direct sums of the first order bundles $T\mathcal{M}$ and $T^{*}\mathcal{M}$.

In the fourth section, theory of symmetry and reduction is summarized. Symmetries of a mechanical system are described by invariance of the system under some Lie group action on its configuration manifold. Momentum maps, which play the fundamental role in the theory, are defined, and the link between momentum maps and Noether's theorem is established. The symplectic and the Lie-Poisson reductions are presented. By several ways, the Lie-Poisson structure on the dual space $\mathfrak{g}^{*}$ of the Lie algebra $\mathfrak{g}$ is derived.

The fifth section is devoted to the applications of geometric constructions described in previous sections to continuum theories. The momentum map realization of the Poisson equation gives that the true configuration space for the Poisson-Vlasov dynamics must be the semi-direct product structure $\mathcal{F}(\mathcal{Q})\circledS Diff_{can}(T^{*}\mathcal{Q})$ with the action of additive group $\mathcal{F}(\mathcal{Q})$ of functions given by fiber translation on $T^{*}\mathcal{Q}$ and by composition on right with canonical transformations. Following [39], the group of canonical transformations $Diff_{can}(T^{*}\mathcal{Q})$ is considered as the configuration space for collisionless plasma and Green's function solution of the Poisson equation is adapted as a constraint while performing variational derivations. We take Lie algebra of $Diff_{can}(T^{*}\mathcal{Q})$ as the space $\mathfrak{X}_{ham}(T^{*}\mathcal{Q})$ of globally Hamiltonian vector fields which is isomorphic to the space $\mathcal{F}(T^{*}\mathcal{Q})$ of smooth functions on $T^{*}\mathcal{Q}$. The dual of $\mathfrak{X}_{ham}(T^{*}\mathcal{Q})$ is the space of one-form densities whereas the dual of $\mathcal{F}(T^{*}\mathcal{Q})$ is the space $Den(T^{*}\mathcal{Q})$ of densities on $T^{*}\mathcal{Q}$, that is the space of nonvanishing top-forms. Therefore, for the dual space, two equivalent representations are possible, namely density $Den(T^{*}\mathcal{Q})$ and momentum $\mathfrak{X}^{*}_{ham}(T^{*}\mathcal{Q})$ representations. Hamiltonian functionals H_{LP} corresponding to the Lie-Poisson formulations of the Poisson-Vlasov equations for both dual spaces are given. For the density representation, the constraint variational derivative of

$$H_{LP}(f)=\int_{T^{*}\mathcal{Q}} f(\mathbf{z})h_{f}(\mathbf{z})\,\mu(\mathbf{z}), \tag{1.16}$$

is computed as $\delta H_{LP}/\delta f=h=p^{2}/2m+e\phi_{f}(\mathbf{q})$, hence Lie-Poisson equation

$$\frac{df}{dt}=-\left\{f,\frac{\delta H_{LP}}{\delta f}\right\} \tag{1.17}$$

on $Den(T^{*}\mathcal{Q})$ gives Vlasov equation in Eq.(1.13), where $h_{f}(\mathbf{z})=p^{2}/2m+e\phi_{f}(\mathbf{q})/2$. Hamiltonian functional H_{LP} is transferred to $\mathfrak{X}^{*}_{ham}(T^{*}\mathcal{Q})$ with coordinates $\Pi_{id}=(\Pi_{q},\Pi_{p})$ and obtained

$$H_{LP}(\Pi_{id})=\int_{T^{*}\mathcal{Q}}\left\langle \Pi_{id}(\mathbf{z}),X_{h_{f}}(\mathbf{z})\right\rangle\mu(\mathbf{z}) \tag{1.18}$$

up to modulo divergence. The constraint variational derivative

$$\frac{\delta H_{LP}}{\delta \Pi_{id}}=X_{h}=\frac{1}{m}\mathbf{p}\cdot\nabla_{q}-e\nabla_{q}\phi_{f}(\mathbf{q})\cdot\nabla_{p} \tag{1.19}$$

of $H_{LP}(\Pi_{id})$ is the Hamiltonian vector field for the Hamiltonian function h. Lie-Poisson equations gives the intermediate system

$$
\begin{aligned}
\dot{\Pi}_q &= -X_h\left(\Pi_q\right)+e\left(\Pi_p\cdot\nabla_q\right)\nabla_q\phi_f(\mathbf{q}) \\
\dot{\Pi}_p &= -X_h\left(\Pi_p\right)-\frac{1}{m}\Pi_q
\end{aligned}
\tag{1.20}
$$

named the **momentum-Vlasov equations**. Momentum variables (Π_q, Π_p) represent equivalence classes up to additions of the terms $\nabla_q k$ and $\nabla_p k$ for an arbitrary function $k(\mathbf{q},\mathbf{p})$, respectively. It is shown that, the momentum-Vlasov equations are generated by the vertical representative

$$
VX_h^{c*}=\big(-X_h\left(\Pi_q\right)+e\left(\Pi_p\cdot\nabla_q\right)\nabla_q\phi_f(\mathbf{q})\big)\cdot\nabla_{\Pi_q}-(\frac{1}{m}\Pi_q+X_h\left(\Pi_p\right))\cdot\nabla_{\Pi_p} \tag{1.21}
$$

of the complete cotangent lift X_h^{c*} of Hamiltonian vector field X_h thereby the precise relation between the particle motion and the Vlasov equation is established.

In section 6, algebra of vector fields $\mathfrak{X}\left(\mathcal{M}\right)$ on a symplectic manifold $\mathcal{M}$ is decomposed into a semi-direct product algebra of Hamiltonian vector fields $\mathfrak{X}_{ham}\left(\mathcal{M}\right)$ and its complement isomorphic to the dual of $\mathfrak{X}_{ham}\left(\mathcal{M}\right)$, is presented. Some other subalgebras in the space of vector fields, such as homotheties and locally Hamiltonian vector fields, are presented. A similar discussion on the decomposition of one-form sections on $\mathcal{M}$ into the spaces of closed and non-closed one-forms is made.

2. THEORY OF MANIFOLDS AND BUNDLES

2.1. SMOOTH BUNDLES

A **smooth (fiber) bundle** is a quadruple $(\mathcal{E}, \pi, \mathcal{M}, \mathcal{F})$ where

1. $\mathcal{E}, \mathcal{M}$ and $\mathcal{F}$ are manifolds, called **total**, **base** and **fiber manifolds**, respectively,
2. $\pi : \mathcal{E} \to \mathcal{M}$ is a smooth surjective map, called **projection** (or **fibration)**.

In addition, a fiber bundle admits **local trivialization property**, that is, the existence of an open cover $\{U_i\}$ of $\mathcal{M}$ with diffeomorphisms $\phi_i : U_i \times \mathcal{F} \to \pi^{-1}(U_i)$ commuting the following diagram

$$\begin{array}{ccc} U_i \times \mathcal{F} & \xrightarrow{\phi_i} & \pi^{-1}(U_i) \\ \mathrm{pr}_{U_i} \searrow & & \swarrow \pi \\ & U_i & \end{array} \tag{2.1}$$

where $\mathrm{pr}_{U_i} : U_i \times \mathcal{F} \to U_i$ is the projection operator to the first factor [**?**, 40–44]. By fixing $\mathbf{x} \in U_i$, the diffeomorphism $\phi_i : U_i \times \mathcal{F} \to \pi^{-1}(U_i)$ is reduced to

$$\phi_i|_x : \mathcal{F} \to \pi^{-1}(\mathbf{x}) = \mathcal{E}_x, \tag{2.2}$$

which identifies $\pi^{-1}(\mathbf{x}) = \mathcal{E}_x$ and $\mathcal{F}$. We will denote a smooth bundle with triple $(\mathcal{E}, \pi, \mathcal{M})$ or with projection π or with the total space $\mathcal{E}$, interchangeably, if there is no risk of confusion.

If there exists a trivialization over the entire manifold $\mathcal{M}$, then the bundle is called the **trivial bundle**. In this case the total space $\mathcal{E}$ can be identified with the product manifold $\mathcal{M} \times \mathcal{F}$.

A **global section** ν of a bundle $(\mathcal{E}, \pi, \mathcal{M})$ is a smooth map $\nu : \mathcal{M} \to \mathcal{E}$ such that $\pi \circ \nu = id_{\mathcal{M}}$, where $id_{\mathcal{M}}$ is the identity mapping on the base manifold $\mathcal{M}$. We denote the set of all sections of π by $\mathfrak{S}(\pi)$. For an open cover $\{U_i\}$ of $\mathcal{M}$, a mapping $\nu_i : U_i \to \pi^{-1}(U_i)$ satisfying $\pi \circ \nu_i = id_{U_i}$ is called a **local section**. It is important to note that, not all smooth bundles admit global sections [45].

A $k-$**dimensional real (complex) vector bundle** is a smooth bundle whose fiber $\mathcal{F}$ is a $k-$dimensional real (complex) vector space V. $k = \operatorname{rank} \pi$ is **rank of the vector bundle.** The tangent and the cotangent bundles are examples of vector bundles and they will be discussed in detail in forthcoming sections. A vector bundle $(\mathcal{E}_1, \pi_1, \mathcal{M}_1)$ is called a **subbundle** of a vector bundle $(\mathcal{E}_2, \pi_2, \mathcal{M}_2)$ if the fibers of π_1 are the linear subspaces of the fibers of π_2 at every $\mathbf{x} \in \mathcal{M}$ and $\pi_2|_{\mathcal{E}_1} = \pi_1$.

2.1.1. Bundle Map and Fiber Product

Let $(\mathcal{E}_1, \pi_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \pi_2, \mathcal{M}_2)$ be two smooth bundles. A smooth map $\hat{\varphi} : \mathcal{E}_1 \to \mathcal{E}_2$ is called a **bundle map (morphism),** if $\hat{\varphi}$ is fiber-preserving, that is, it induces a map $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\hat{\varphi}} & \mathcal{E}_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathcal{M}_1 & \xrightarrow{\varphi} & \mathcal{M}_2. \end{array} \tag{2.3}$$

If the base manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are identical, that is $\mathcal{M}_1 = \mathcal{M}_2$ then $\hat{\varphi}$ satisfies $\pi_1 = \pi_2 \circ \hat{\varphi}$.

For the case of vector bundles, we require in addition that $\hat{\varphi}$ is a linear operator from the vector space $(\mathcal{E}_1)_x = \pi_1^{-1}(\mathbf{x})$ to $(\mathcal{E}_2)_{\varphi(x)} = \pi_2^{-1}(\varphi(\mathbf{x}))$ for all $\mathbf{x} \in \mathcal{M}$. The set of all such linear mappings at $\mathbf{x} \in \mathcal{M}$ is denoted by $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2)_x$. The triple $(\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2), \pi, \mathcal{M})$ is the **homomorphism bundle,** where the total space is

$$\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2) = \bigcup_{x \in \mathcal{M}} \operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2)_x . \tag{2.4}$$

Consider the **trivial line bundle** $(\mathcal{M} \times \mathbb{R}, pr_1, \mathcal{M})$ with pr_1 is the projection to the first factor. Given a vector bundle $(\mathcal{E}, \pi, \mathcal{M})$, the **dual bundle** $(\mathcal{E}^*, \pi^*, \mathcal{M})$ is defined to be

$$(\operatorname{Hom}(\mathcal{E}, \mathcal{M} \times \mathbb{R}), \pi^*, \mathcal{M}) = (\mathcal{E}^*, \pi^*, \mathcal{M}). \tag{2.5}$$

If V is the fiber of a vector bundle $(\mathcal{E}, \pi, \mathcal{M})$ then its linear algebraic dual V^* is the fiber of the dual bundle $(\mathcal{E}^*, \pi^*, \mathcal{M})$.

Consider two bundles $(\mathcal{E}_1, \pi_1, \mathcal{M})$ and $(\mathcal{E}_2, \pi_2, \mathcal{M})$ over the same base manifold $\mathcal{M}$. The manifold

$$\mathcal{E}_1 \times_{\mathcal{M}} \mathcal{E}_2 = \{(\mathbf{e}_1, \mathbf{e}_2) \in \mathcal{E}_1 \times \mathcal{E}_2 : \pi_1(\mathbf{e}_1) = \pi_2(\mathbf{e}_2)\} \tag{2.6}$$

is called the **Whitney product**. In addition, with the projection

$$(\pi_1 \times_{\mathcal{M}} \pi_2) : \mathcal{E}_1 \times_{\mathcal{M}} \mathcal{E}_2 \to \mathcal{M} : (\mathbf{e}_1, \mathbf{e}_2) \to \pi_1(\mathbf{e}_1) = \pi_2(\mathbf{e}_2), \tag{2.7}$$

$(\mathcal{E}_1 \times_{\mathcal{M}} \mathcal{E}_2, \pi_1 \times_{\mathcal{M}} \pi_2, \mathcal{M})$ is called a **bundle product**.

2.1.2. Tangent Bundle

Let $\mathcal{M}$ be an $m-$dimensional manifold. We denote by $T_x\mathcal{M}$ the vector space of all tangent vectors at $\mathbf{x} \in \mathcal{M}$. The union of the tangent spaces

$$T\mathcal{M} = \bigcup_{x \in \mathcal{M}} T_x\mathcal{M} \tag{2.8}$$

is a $2m-$dimensional manifold called **tangent manifold** of $\mathcal{M}$. **Tangent projection (**or **fibration)** $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$ maps a tangent vector to its base and makes $(T\mathcal{M}, \tau_{\mathcal{M}}, \mathcal{M})$ a vector bundle with fibers isomorphic to $m-$dimensional Euclidean space. We will compactly denote the tangent bundle with its total space $T\mathcal{M}$ or with projection $\tau_{\mathcal{M}}$. We will use the local coordinates (x^a) and $\left(x^a, v^b\right)$ on $\mathcal{M}$ and $T\mathcal{M}$, respectively. The canonical basis for $T_x\mathcal{M}$ is given by the set $\{\partial/\partial x^a|_x\}$.

Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a smooth mapping between two manifolds $\mathcal{M}$ and $\mathcal{N}$, the **tangent lifting** (or **mapping**) $T\varphi : T\mathcal{M} \to T\mathcal{N}$ of φ is defined in a local chart U as

$$T\varphi\left(\mathbf{x}, \mathbf{v}\right) = \left(\varphi\left(\mathbf{x}\right), D\varphi\left(\mathbf{x}\right) \cdot \mathbf{v}\right), \tag{2.9}$$

that is, $T\varphi$ maps the base point $\mathbf{x}$ to the point $\varphi\left(\mathbf{x}\right) \in \mathcal{N}$ and maps the vector $\mathbf{v} \in T_x\mathcal{M}$ to a vector in $T_{\varphi(x)}\mathcal{N}$ via the Jacobian $D\varphi$ of the transformation φ. $T\varphi$ and φ make the following diagram commutative

$$\begin{array}{ccc} T\mathcal{M} & \xrightarrow{T\varphi} & T\mathcal{N} \\ \big\downarrow \tau_{\mathcal{M}} & & \big\downarrow \tau_{\mathcal{N}} \\ \mathcal{M} & \xrightarrow{\varphi} & \mathcal{N}. \end{array} \tag{2.10}$$

A section $X : \mathcal{M} \to T\mathcal{M}$ of the tangent bundle is called a **vector field on** $\mathcal{M}$. We denote the space of all vector fields by $\mathfrak{X}\left(\mathcal{M}\right)$. Let $X \in \mathfrak{X}\left(\mathcal{M}\right)$ and $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ be a differentiable mapping then the commutative diagram

$$\begin{array}{ccc} T\mathcal{M} & \xrightarrow{T\varphi} & T\mathcal{N} \\ \big\uparrow X & & \big\uparrow X_{\mathcal{N}} \\ \mathcal{M} & \xrightarrow{\varphi} & \mathcal{N} \end{array} \tag{2.11}$$

defines a vector field $X_{\mathcal{N}}$, called the **push-forward of** X **by** φ, and denoted by $X_{\mathcal{N}} = \varphi_{*}X$.

$\mathfrak{X}(\mathcal{M})$ acts on the space of smooth function $\mathcal{F}(\mathcal{M})$ on $\mathcal{M}$. Action

$$X[f] = X^{a}\frac{\partial f}{\partial x^{a}}. \tag{2.12}$$

of a vector field $X = X^{a}\partial/\partial x^{a}$ on $f \in \mathcal{F}(\mathcal{M})$ is the directional derivative of f in the direction of X. There is a bilinear, skew-symmetric binary operation on the space of vector fields $\mathfrak{X}(\mathcal{M})$ called **Jacobi-Lie bracket**, defined in terms of the actions of the vector fields as

$$[X,Y]_{JL}[f] = X[Y[f]] - Y[X[f]], \tag{2.13}$$

for $X, Y \in \mathfrak{X}(\mathcal{M})$ and $\forall f \in \mathcal{F}(\mathcal{M})$. The Jacobi-Lie bracket satisfies the **Jacobi identity**

$$[[X,Y]_{JL}, Z]_{JL} + [[Y,Z]_{JL}, X]_{JL} + [[Z,X]_{JL}, Y]_{JL} = 0. \tag{2.14}$$

With $[\ ,\ ]_{JL}$, $\mathfrak{X}(\mathcal{M})$ has the structure of a Lie algebra, that means, it is equipped with a bilinear, antisymmetric binary operation which satisfies the Jacobi identity. We will turn back to the concept of the Lie algebra in a more general setting in forthcoming sections.

2.1.3. Dual Tangent Rhombic and Canonical Involution

$TT\mathcal{M}$ is a $4m$-dimensional tangent manifold of $T\mathcal{M}$ with induced coordinates $\left(x^{a}, v^{b}; \dot{x}^{d}, \dot{v}^{e}\right)$. $TT\mathcal{M}$ has two fibrations over $T\mathcal{M}$, one is the natural tangent bundle fibration $\tau_{T\mathcal{M}}$: $TT\mathcal{M} \to T\mathcal{M}$ and the other is induced from $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$ as a tangent mapping $T\tau_{\mathcal{M}} : TT\mathcal{M} \to T\mathcal{M}$. The following commutative diagram, known as **dual tangent rhom-**

bic

$$\begin{array}{ccc} TT\mathcal{M} & \xrightarrow{\tau_{T\mathcal{M}}} & T\mathcal{M} \\ \Big\downarrow T\tau_{\mathcal{M}} & & \Big\downarrow \tau_{\mathcal{M}} \\ T\mathcal{M} & \xrightarrow{\tau_{\mathcal{M}}} & \mathcal{M}. \end{array} \tag{2.15}$$

summarizes the situation [3]. The diagram in Eq.(2.15) is an example of a double vector bundle structure [46]. Dual tangent rhombic leads to the existence of an isomorphism $\kappa_{\mathcal{M}}$ via commutativity of the following diagram

$$\begin{array}{ccc} TT\mathcal{M} & \xrightarrow{\kappa_{\mathcal{M}}} & TT\mathcal{M} \\ \Big\downarrow T\tau_{\mathcal{M}} & & \Big\downarrow \tau_{T\mathcal{M}} \\ T\mathcal{M} & \longleftrightarrow & T\mathcal{M}. \end{array} \tag{2.16}$$

$\kappa_{\mathcal{M}}$ is an involutive map, that is, $\kappa_{\mathcal{M}} \circ \kappa_{\mathcal{M}} = \mathrm{id}_{TT\mathcal{M}}$ [8]. In local coordinates, if the local representatives of $\tau_{T\mathcal{M}}$ and $T\tau_{\mathcal{M}}$ are

$$\begin{aligned} \tau_{T\mathcal{M}}\left(x^a, v^b; \dot{x}^d, \dot{v}^e\right) &= \left(x^a, v^b\right), \\ T\tau_{\mathcal{M}}\left(x^a, v^b; \dot{x}^d, \dot{v}^e\right) &= \left(x^a, \dot{x}^d\right), \end{aligned} \tag{2.17}$$

then $\kappa_{\mathcal{M}}$ is given by

$$\kappa_{\mathcal{M}}\left(x^a, v^b; \dot{x}^d, \dot{v}^e\right) = \left(x^a, \dot{x}^d; v^b, \dot{v}^e\right). \tag{2.18}$$

2.1.4. Cotangent Bundle

Given a manifold $\mathcal{M}$, the linear algebraic dual $T_x^*\mathcal{M}$ of the vector space $T_x\mathcal{M}$ is called the **cotangent (covector) space** of $\mathcal{M}$ at $\mathbf{x} \in \mathcal{M}$. The union of all cotangent spaces

$$T^*\mathcal{M} = \bigcup_{x\in\mathcal{M}} T_x^*\mathcal{M} \tag{2.19}$$

is the **cotangent manifold** of $\mathcal{M}$. $(T^*\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$ is the dual bundle of the tangent bundle and called the **cotangent bundle of** $\mathcal{M}$, where the projection $\pi_{\mathcal{M}} : T^*\mathcal{M} \to \mathcal{M}$ is the **cotangent bundle projection.** $T^*\mathcal{M}$ is a $2m-$dimensional manifold with coordinates (x^a, y_b). We choose a basis $\{ dx^a|_x\}$ for $T_x^*\mathcal{M}$ dual to the basis $\{ \partial/\partial x^a|_x\}$ of $T_x\mathcal{M}$, in the sense that,

$$\left\langle dx^a, \frac{\partial}{\partial x^b} \right\rangle_x = \delta_b^a, \tag{2.20}$$

where $\langle\ ,\ \rangle_x$ is the natural pairing at $\mathbf{x}$, and δ_b^a is the Kronecker delta. A Section $\theta : \mathcal{M} \to T^*\mathcal{M}$ of the cotangent bundle is a **differential one-form** which can locally be written as $\theta(\mathbf{x}) = \theta_a(\mathbf{x})\, dx^a$. The space of one-forms on $\mathcal{M}$ is denoted by $\Lambda^1(\mathcal{M})$. We denot the space of one-forms on $\mathcal{M}$ by $\Lambda^1(\mathcal{M})$.

Consider two differentiable manifolds $\mathcal{M}$ and $\mathcal{N}$ and a smooth map $\varphi : \mathcal{M} \to \mathcal{N}$, **cotangent lift**

$$T^*\varphi : T^*\mathcal{N} \to T^*\mathcal{M} \tag{2.21}$$

of φ is defined by

$$\langle T^*\varphi(\mathbf{z}), \mathbf{v}\rangle = \langle \mathbf{z}, T\varphi(\mathbf{v})\rangle, \tag{2.22}$$

where $T\varphi$ is the tangent mapping of φ, $\mathbf{z} \in T^*\mathcal{N}$ and $\mathbf{v} \in T\mathcal{M}$. Note that, cotangent lift switches the order of composition, that is, if $\varphi : \mathcal{M} \to \mathcal{N}$ and $\psi : \mathcal{N} \to \mathcal{Q}$, then

$$T^* (\psi \circ \varphi) = T^* \varphi \circ T^* \psi. \tag{2.23}$$

2.1.5. Pull-back Bundle

Let $(\mathcal{E}, \pi, \mathcal{M})$ be a bundle and $\rho : \mathcal{N} \to \mathcal{M}$ be a map from a manifold $\mathcal{N}$ to the base manifold $\mathcal{M}$, **pull-back bundle of** π by ρ is the triplet $(\rho^* (\mathcal{E}), \rho^* (\pi), \mathcal{N})$, where

$$\rho^* (\mathcal{E}) = \mathcal{N} \times_{\mathcal{M}} \mathcal{E} = \{(\mathbf{n}, \mathbf{e}) : \mathcal{N} \times \mathcal{E} : \rho (\mathbf{n}) = \pi (\mathbf{e})\} \tag{2.24}$$

is product of manifolds and

$$\rho^* (\pi) : \rho^* (\mathcal{E}) = \mathcal{N} \times_{\mathcal{M}} \mathcal{E} \to \mathcal{N} : (\mathbf{n}, \mathbf{e}) \to \mathbf{n}$$

is the projection pr_1 to first factor. Diagrammatically, we have

$$\begin{array}{ccc} \rho^* (\mathcal{E}) = \mathcal{N} \times_{\mathcal{M}} \mathcal{E} & \xrightarrow{\mathrm{pr}_2} & \mathcal{E} \\ \mathrm{pr}_1 \big\downarrow \rho^* (\pi) & & \big\downarrow \pi \\ \mathcal{N} & \xrightarrow{\rho} & \mathcal{M} \end{array} \tag{2.25}$$

where the projection

$$\mathrm{pr}_2 : \rho^* (\mathcal{E}) = \mathcal{N} \times_{\mathcal{M}} \mathcal{E} \to \mathcal{E} : (\mathbf{n}, \mathbf{e}) \to \mathbf{e} \tag{2.26}$$

is a bundle map.

As an example, we take $(\mathcal{E}, \pi, \mathcal{M})$ to be the tangent bundle $(T\mathcal{M}, \tau_{\mathcal{M}}, \mathcal{M})$ and $\mathcal{N} = T^*\mathcal{M}$ so that ρ is the cotangent bundle projection $\pi_{\mathcal{M}} : T^*\mathcal{M} \to \mathcal{M}$. The pull-back of

$(T\mathcal{M}, \tau_{\mathcal{M}}, \mathcal{M})$ by $\pi_{\mathcal{M}}$ is then the smooth bundle $(\pi^*_{\mathcal{M}}(T\mathcal{M}), \pi^*_{\mathcal{M}}(\tau_{\mathcal{M}}), T^*\mathcal{M})$ where

$$\pi^*_{\mathcal{M}}(T\mathcal{M}) = T^*\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} = \{(\mathbf{z}, \mathbf{v}) : T^*\mathcal{M} \times T\mathcal{M} : \pi_{\mathcal{M}}(\mathbf{z}) = \tau_{\mathcal{M}}(\mathbf{v})\} \tag{2.27}$$

is the Whitney product and $\pi^*_{\mathcal{M}}(\tau_{\mathcal{M}})$ is the projection $\mathrm{pr}_1 : (\mathbf{z}, \mathbf{v}) \to \mathbf{z}$ to the first factor. The following diagram

$$\begin{array}{ccc} \pi^*_{\mathcal{M}}(T\mathcal{M}) = T^*\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} & \xrightarrow{\mathrm{pr}_2} & T\mathcal{M} \\ \mathrm{pr}_1 \big\downarrow \pi^*_{\mathcal{M}}(\tau_{\mathcal{M}}) & & \big\downarrow \tau_{\mathcal{M}} \\ T^*\mathcal{M} & \xrightarrow{\pi_{\mathcal{M}}} & \mathcal{M} \end{array} \tag{2.28}$$

summarizes the argument. Similarly, the pull-back of $(T\mathcal{M}, \tau_{\mathcal{M}}, \mathcal{M})$ by the tangent bundle projection $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$ is the bundle $(\tau^*_{\mathcal{M}}(T\mathcal{M}), \tau^*_{\mathcal{M}}(\tau_{\mathcal{M}}) = \mathrm{pr}_1, T\mathcal{M})$ where

$$\tau^*_{\mathcal{M}}(T\mathcal{M}) = T\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} = \{(\mathbf{v}, \mathbf{u}) : T\mathcal{M} \times T\mathcal{M} : \tau_{\mathcal{M}}(\mathbf{v}) = \tau_{\mathcal{M}}(\mathbf{u})\}\,. \tag{2.29}$$

Observe the diagram

$$\begin{array}{ccc} \tau^*_{\mathcal{M}}(T\mathcal{M}) = T\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} & \xrightarrow{\mathrm{pr}_2} & T\mathcal{M} \\ \mathrm{pr}_1 \big\downarrow \tau^*_{\mathcal{M}}(\tau_{\mathcal{M}}) & & \big\downarrow \tau_{\mathcal{M}} \\ T\mathcal{M} & \xrightarrow{\tau_{\mathcal{M}}} & \mathcal{M}. \end{array} \tag{2.30}$$

As another example, we take $(\mathcal{E}, \pi, \mathcal{M})$ to be the cotangent bundle $(T^*\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$ and $\mathcal{N} = T\mathcal{M}$ so that ρ is the tangent bundle projection $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$. The pull-back of $(T^*\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$ by $\tau_{\mathcal{M}}$ is the smooth bundle $(\tau^*_{\mathcal{M}}(T^*\mathcal{M}), \tau^*_{\mathcal{M}}(\pi_{\mathcal{M}}) = \mathrm{pr}_1, T\mathcal{M})$ where the total space

$$\tau^*_{\mathcal{M}}(T^*\mathcal{M}) = T\mathcal{M} \times_{\mathcal{M}} T^*\mathcal{M} = \{(\mathbf{v}, \mathbf{z}) : T\mathcal{M} \times T^*\mathcal{M} : \tau_{\mathcal{M}}(\mathbf{v}) = \pi_{\mathcal{M}}(\mathbf{z})\}\,. \tag{2.31}$$

is again a Whitney product. Diagrammatically,

$$\begin{array}{ccc} \tau^*_{\mathcal{M}}(T^*\mathcal{M}) = T\mathcal{M} \times_{\mathcal{M}} T^*\mathcal{M} & \xrightarrow{\mathrm{pr}_2} & T^*\mathcal{M} \\ \mathrm{pr}_1 \big\downarrow \tau^*_{\mathcal{M}}(\pi_{\mathcal{M}}) & & \big\downarrow \pi_{\mathcal{M}} \\ T\mathcal{M} & \xrightarrow{\tau_{\mathcal{M}}} & \mathcal{M} \end{array} \tag{2.32}$$

from which we observe that, $\tau^*_{\mathcal{M}}(T^*\mathcal{M})$ is isomorphic to $\pi^*_{\mathcal{M}}(T\mathcal{M})$ by isomorphism

$$\pi^*_{\mathcal{M}}(T\mathcal{M}) \leftrightarrow \tau^*_{\mathcal{M}}(T^*\mathcal{M}) : (\mathbf{z}, \mathbf{v}) \leftrightarrow (\mathbf{v}, \mathbf{z}) . \tag{2.33}$$

Similarly, the pull-back of $(T^*\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$ by $\pi_{\mathcal{M}} : T^*\mathcal{M} \to \mathcal{M}$ is

$$\left(\pi^*_{\mathcal{M}}(T^*\mathcal{M}), \pi^*_{\mathcal{M}}(\pi_{\mathcal{M}}) = \mathrm{pr}_1, T^*\mathcal{M}\right), \tag{2.34}$$

where the total space is

$$\pi^*_{\mathcal{M}}(T^*\mathcal{M}) = T^*\mathcal{M} \times_{\mathcal{M}} T^*\mathcal{M} = \{(\mathbf{w}, \mathbf{z}) : T^*\mathcal{M} \times T^*\mathcal{M} : \pi_{\mathcal{M}}(\mathbf{w}) = \pi_{\mathcal{M}}(\mathbf{z})\} \tag{2.35}$$

and we have

$$\begin{array}{ccc} \pi^*_{\mathcal{M}}(T^*\mathcal{M}) = T^*\mathcal{M} \times_{\mathcal{M}} T^*\mathcal{M} & \xrightarrow{\mathrm{pr}_2} & T^*\mathcal{M} \\ \mathrm{pr}_1 \big\downarrow \pi^*_{\mathcal{M}}(\pi_{\mathcal{M}}) & & \big\downarrow \pi_{\mathcal{M}} \\ T^*\mathcal{M} & \xrightarrow{\pi_{\mathcal{M}}} & \mathcal{M}. \end{array} \tag{2.36}$$

2.1.6. First Order Jet Bundle and Generalized Vector Fields

Let $(\mathcal{E}, \pi, \mathcal{M})$ be a bundle with coordinates (x^a) and (x^a, u^α) for a local atlas on $\mathcal{M}$ and $\mathcal{E}$, respectively. Two sections $\phi, \psi \in \mathfrak{S}(\pi)$ of the bundle $(\mathcal{E}, \pi, \mathcal{M})$ at a point $\mathbf{x} \in \mathcal{M}$

are called equivalent if their tangent mappings are equal at that point, that is, $T_x\phi = T_x\psi$. Given a point $\mathbf{x}$, an equivalence class containing a section ϕ is denoted by $j^1_x\phi$ and is called **one-jet of** ϕ. The **first order jet manifold**

$$J^1\pi = \left\{ j^1_x\phi : \mathbf{x} \in \mathcal{M} \text{ and } \phi \in \mathfrak{S}(\pi) \right\} \tag{2.37}$$

associated with $(\mathcal{E}, \pi, \mathcal{M})$ is the set of equivalence classes at every point $\mathbf{x} \in \mathcal{M}$ with induced coordinates

$$\left(x^a, u^\lambda, u^\lambda_a\right) : J^1\pi \to \mathbb{R}^{m+k+mk} : j^1_x\phi \to \left(x^a, u^\lambda\left(\phi\left(\mathbf{x}\right)\right), \left.\frac{\partial \phi^\lambda}{\partial x^a}\right|_x \right), \tag{2.38}$$

where $m = \dim \mathcal{M}$ and $k = \operatorname{rank} \pi$. We have the fibrations

$$\begin{aligned} \pi_0 &: J^1\pi \to \mathcal{E} : j^1_x\phi \to \phi\left(\mathbf{x}\right) \\ \pi_1 &: J^1\pi \to \mathcal{M} : j^1_x\phi \to \mathbf{x} \end{aligned}$$

of $J^1\pi$ over $\mathcal{E}$ and $\mathcal{M}$, respectively. These form the commutative diagram

$$\begin{array}{ccc} J^1\pi & \xrightarrow{\pi_0} & \mathcal{E} \\ & \pi_1 \searrow & \downarrow \pi \\ & & \mathcal{M}. \end{array} \tag{2.39}$$

Consider the pull back bundle

$$\left(\pi_0^*\left(T\mathcal{E}\right) = J^1\pi \times_{\mathcal{E}} T\mathcal{E}, \pi_0^*\left(\tau_{\mathcal{E}}\right) = \mathrm{pr}_1, J^1\pi\right) \tag{2.40}$$

of the tangent bundle $(T\mathcal{E}, \tau_{\mathcal{E}}, \mathcal{E})$ by the projection $\pi_0 : J^1\pi \to \mathcal{E}$, that is

$$\begin{array}{ccc} \pi_0^*(T\mathcal{E}) = J^1\pi \times_{\mathcal{E}} T\mathcal{E} & \longrightarrow & T\mathcal{E} \\ \mathrm{pr}_1 \big\downarrow \pi_0^*(\tau_{\mathcal{E}}) & & \big\downarrow \tau_{\mathcal{E}} \\ J^1\pi & \xrightarrow{\pi_0} & \mathcal{E}. \end{array} \tag{2.41}$$

A section of $\pi_0^*(\tau_{\mathcal{E}})$ is called a generalized vector field of order 1 [48]- [50]. One may regard a section of $\pi_0^*(\tau_{\mathcal{E}})$ as a map from $J^1\pi$ to $T\mathcal{E}$. We additionally require that generalized vector fields are projectable. In coordinates, a generalized vector field is then given by

$$\xi\left(j_x^1\phi\right) = \xi^a(\mathbf{x}) \left.\frac{\partial}{\partial x^a}\right|_x + \xi^\lambda\left(j_x^1\phi\right) \left.\frac{\partial}{\partial u^\lambda}\right|_{\phi(x)}. \tag{2.42}$$

The **prolongation** $pr^1\xi$ of a generalized vector field ξ is defined by

$$pr^1\xi = \xi + \Phi_a^\alpha \frac{\partial}{\partial u_a^\alpha}, \quad \Phi_a^\alpha = D_{x^a}\left(\xi^\alpha - \xi^b u_b^\alpha\right) + \xi^b u_{ba}^\alpha \tag{2.43}$$

where D_{x^a} is an operator which differentiates functions on $J^1\pi$ with respect to x^a and $u_{ba}^\alpha(j_x\phi) = \partial^2\phi^\alpha/\partial x^a\partial x^b\big|_x$ is an element of the second order jet bundle [51]. **Lie bracket of two first order generalized vector fields**

$$\xi = \xi^a \frac{\partial}{\partial x^a} + \xi^\alpha \frac{\partial}{\partial u^\alpha} \quad \text{and} \quad \eta = \eta^a \frac{\partial}{\partial x^a} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$$

is the unique first order generalized vector field

$$[\xi, \eta]_{pro} = \sum_{a=1}^{m} \left(pr^1\xi(\eta^a) - pr^1\eta(\xi^a)\right) \frac{\partial}{\partial x^a} + \sum_{\alpha=1}^{k} \left(pr^1\xi(\eta^\alpha) - pr^1\eta(\xi^\alpha)\right) \frac{\partial}{\partial u^\alpha}. \tag{2.44}$$

If ξ and η are ordinary vector fields on $\mathcal{E}$, then $[\ ,\ ]_{pro}$ reduces to the Jacobi-Lie bracket of vector fields as in Eq.(2.13).

Similarly, we define the generalized one-forms as follows. Consider the pull-back

$$\left(\pi_0^*\left(T^*\mathcal{E}\right) = J^1\pi \times_{\mathcal{E}} T^*\mathcal{E}, \left(\pi_0\right)^*\left(\pi_{\mathcal{E}}\right), J^1\pi\right). \tag{2.45}$$

of the cotangent bundle $(T^*\mathcal{E}, \pi_{\mathcal{E}}, \mathcal{E})$ by $\pi_0 : J^1\pi \to \mathcal{E}$. A **generalized one-form of order 1** is a section of this pull-back bundle

$$\begin{array}{ccc} J^1\pi \times_{\mathcal{E}} T^*\mathcal{E} & \longrightarrow & T^*\mathcal{E} \\ \mathrm{pr}_1 \Big\downarrow & & \Big\downarrow \pi_E \\ J^1\pi & \xrightarrow{\pi_0} & \mathcal{E}. \end{array} \tag{2.46}$$

In coordinates, a generalized one-form λ is written as

$$\lambda\left(j_x^1\phi\right) = \lambda_a\left(j_x^1\phi\right) dx^a|_{\phi(x)} + \lambda_\alpha\left(j_x^1\phi\right) du^\alpha|_{\phi(x)}. \tag{2.47}$$

2.2. CONNECTIONS ON BUNDLES

2.2.1. Vertical Vectors

Let $(\mathcal{E}, \pi, \mathcal{M})$ be a smooth bundle with local coordinates (x^a) and (x^a, u^α). The **vertical bundle associated with** π is a vector subbundle of the tangent bundle $T\mathcal{E}$ consisting of vectors that are parallel along the fibers, that is,

$$V\mathcal{E} = \ker T\pi = \left\{\xi \in T\mathcal{E} : T\pi\left(\xi\right) = 0\right\}, \tag{2.48}$$

where $T\pi$ is the tangent mapping of the projection π. We denote the vector fields on $\mathcal{E}$ by the same notation ξ, since there is no risk of confusion. A vector field ξ on $\mathcal{E}$ is a vertical vector field if $\mathrm{Im}\left(\xi\right) \subset V\mathcal{E}$. In coordinates, a vertical vector field ξ is of the form $\xi = \xi^\alpha \partial/\partial u^\alpha$.

Consider the pull-back bundle

$$\pi^*(T\mathcal{M}) = \mathcal{E} \times_{\mathcal{M}} T\mathcal{M} = \{(\mathbf{e}, \mathbf{v}) \in \mathcal{E} \times T\mathcal{M} : \pi(\mathbf{e}) = \tau_{\mathcal{M}}(\mathbf{v})\} \tag{2.49}$$

of $(T\mathcal{M}, \tau_{\mathcal{M}}, \mathcal{M})$ by the projection $\pi : \mathcal{E} \to \mathcal{M}$. There is a homomorphism

$$\begin{aligned} \mathcal{S}_\pi \quad &: \quad T\mathcal{E} \to \pi^*(T\mathcal{M}) = \mathcal{E} \times_{\mathcal{M}} T\mathcal{M} \\ &: \quad \xi \to (\tau_{\mathcal{E}}(\xi), T\pi(\xi)), \end{aligned} \tag{2.50}$$

from $(T\mathcal{E}, \tau_{\mathcal{E}}, \mathcal{E})$ to $(\pi^*(T\mathcal{M}), \pi^*(\tau_{\mathcal{M}}), \mathcal{E})$ [52]. The kernel of $\mathcal{S}_\pi$ consists of vertical vectors on π and hence, one has the following short exact sequence of bundle morphisms

$$0 \quad \to \quad V\mathcal{E} \quad \xrightarrow{\imath} \quad T\mathcal{E} \quad \xrightarrow{\mathcal{S}_\pi} \quad \pi^*(T\mathcal{M}) \quad \to \quad 0 \tag{2.51}$$

where $\imath : V\mathcal{E} \hookrightarrow T\mathcal{E}$ is the inclusion mapping.

There is no canonical way to define a direct complement to $V\mathcal{E}$ in $T\mathcal{E}$. To establish a decomposition of $T\mathcal{E}$, one needs an additional geometric structure on π, called a **connection** [53–56]. The splitting of sequence in Eq.(2.51) is the same as finding a direct complement to $V\mathcal{E}$. To split the sequence, one needs to define an operator

$$\Gamma : \pi^*(T\mathcal{M}) = \mathcal{E} \times_{\mathcal{M}} T\mathcal{M} \to T\mathcal{E} \tag{2.52}$$

such that $\mathcal{S}_\pi \circ \Gamma = id$. We consider Γ as a fiber-preserving mapping from $T_x\mathcal{M}$ to $T_e\mathcal{E}$, where $\pi(\mathbf{e}) = \mathbf{x}$ and hence, take $\Gamma \in T^*\mathcal{M} \otimes T\mathcal{E}$. Once a connection Γ is introduced, the tangent bundle $T\mathcal{E}$ decomposes into vertical $V\mathcal{E}$ and horizontal

$$H\mathcal{E} = \Gamma(\pi^*(T\mathcal{M})) \tag{2.53}$$

subbundles and the total space can be written as the direct sum of vertical and horizontal subbundles, that is $T\mathcal{E} = V\mathcal{E} \oplus H\mathcal{E}$.

2.2.2. Semi-simple Forms

Consider the pull-back bundle $(\pi^*(T^*\mathcal{M}), \pi^*(\pi_{\mathcal{M}}), \mathcal{E})$ of $(T^*\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$ by $\pi : \mathcal{E} \to \mathcal{M}$, where $\pi^*(T^*\mathcal{M})$ is the Whitney product $\mathcal{E} \times_{\mathcal{M}} T^*\mathcal{M}$ and $\pi^*(\pi_{\mathcal{M}}) = \mathrm{pr}_1$ is the projection. We define a mapping μ_π from $\pi^*(T^*\mathcal{M})$ to $T^*\mathcal{E}$ by the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{E} \times_{\mathcal{M}} T^*\mathcal{M} & \xrightarrow{\pi^*_{\mathcal{E}}(\pi) = \mathrm{pr}_2} & T^*\mathcal{M} \\ \mathrm{pr}_1 \downarrow & \mu_\pi \searrow & \downarrow T^*\pi \\ \mathcal{E} & \xleftarrow{\pi_{\mathcal{E}}} & T^*\mathcal{E}. \end{array} \tag{2.54}$$

The image space of the mapping μ_π is called the **space of semi-simple forms** (or **horizontal covectors)** and denoted as $H^*\mathcal{E}$ [49]. In other words, if $\lambda \in H^*\mathcal{E}$, we require existence of $\alpha \in T^*\mathcal{M}$ such that $\lambda = T^*\pi(\alpha)$. For $\xi \in V\mathcal{E}$

$$\langle \lambda, \xi \rangle = \langle T^*\pi(\alpha), \xi \rangle = \langle \alpha, T\pi(\xi) \rangle = 0, \tag{2.55}$$

which means that, horizontal covectors annihilate vertical vectors.

Recall that a connection $\Gamma \in T^*\mathcal{M} \otimes T\mathcal{E}$, since μ_π identifies $T^*\mathcal{M}$ with $H^*\mathcal{E}$ we take $\Gamma \in H^*\mathcal{E} \otimes T\mathcal{E}$. If we add a normalization condition $\Gamma(\lambda) = \lambda$, for all $\lambda \in H^*\mathcal{E}$, Γ can be written, in local coordinates (x^a, u^α) on $\mathcal{E}$, as

$$\Gamma = dx^a \otimes \left(\frac{\partial}{\partial x^a} + \Gamma^\alpha_a \frac{\partial}{\partial u^\alpha} \right), \tag{2.56}$$

where a compatibility condition on the scalars $\Gamma^\alpha_a = \Gamma^\alpha_a(\mathbf{x}, \mathbf{u})$ is imposed by demanding that the local structure is preserved under coordinate transformations. In particular, if Γ^α_a's are linear with respect to fiber coordinates, namely $\Gamma^\alpha_a(\mathbf{x}, \mathbf{u}) = \Gamma^\alpha_{\beta a}(\mathbf{x})\, u^\beta$, then the compatibility condition reduces to the well-known relation

$$\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{a}} = \frac{\partial u^{\bar{\alpha}}}{\partial u^{\alpha}} \frac{\partial u^{\beta}}{\partial u^{\bar{\beta}}} \frac{\partial x^{a}}{\partial x^{\bar{a}}} \Gamma^{\alpha}_{\beta a} + \frac{\partial u^{\bar{\alpha}}}{\partial u^{\alpha}} \frac{\partial^2 u^{\alpha}}{\partial x^{\bar{a}} \partial u^{\bar{\beta}}} \tag{2.57}$$

for transformation of components $\Gamma^{\alpha}_{\beta a}$ of a connection under coordinate transformation (x^a, u^α) to $(x^{\bar{a}}, u^{\bar{\alpha}})$ [57].

We thus reduce the problem of finding a direct complement to $V\mathcal{E}$ to deciding the scalars Γ^{α}_{a}. Once these transformations are chosen, the **horizontal part** $H\xi$ and **vertical part** $V\xi$ of a vector ξ are

$$H(\xi) = \Gamma(\xi) := \xi \cdot \Gamma \quad \text{and} \quad V(\xi) = \xi - H\xi. \tag{2.58}$$

In coordinates, if $\xi = \xi^a \dfrac{\partial}{\partial x^a} + \xi^\alpha \dfrac{\partial}{\partial u^\alpha}$, then

$$H\xi = \xi^a \left(\frac{\partial}{\partial x^a} + \Gamma^{\alpha}_{a} \frac{\partial}{\partial u^\alpha} \right) \quad \text{and} \quad V\xi = \left(\xi^\alpha - \Gamma^{\alpha}_{a} \xi^a\right) \frac{\partial}{\partial u^\alpha}. \tag{2.59}$$

2.2.3. Connection as a Projection on Cotangent Bundle

The definition $V\mathcal{E}$ in Eq.(2.48) gives no clue for the dual bundle $V^*\mathcal{E}$ of the vertical bundle. One may attempt to construct a dual space by choosing $\{du^\alpha\}$ as the generators. This fails to be globalized because the set $\{du^\alpha\}$ is not invariant under coordinate transformations. Hence, one must introduce a dual basis in the form

$$\{du^\alpha - \Gamma^{\alpha}_{a} dx^a : \alpha = 1, ..., \mathrm{rank}(\pi)\} \tag{2.60}$$

where Γ^{α}_{a} are the same as scalars in Eq.(2.56) [58]. Thus, defining a connection is the same as defining the dual $V^*\mathcal{E}$ of $V\mathcal{E}$. The dual of the sequence in Eq.(2.51) is given by

$$0 \;\rightarrow\; (\pi^*(T\mathcal{M}))^* \;\xrightarrow{\;\mathcal{S}^*_\pi\;}\; T^*\mathcal{E} \;\xrightarrow{\;s\;}\; V^*\mathcal{E} \;\rightarrow\; 0 \tag{2.61}$$

where s is a surjective operator and $\mathcal{S}_\pi^*$ is the dual of $\mathcal{S}_\pi$. Γ decomposes simultaneously the dual sequence in Eq.(2.61) and $T^*\mathcal{E}$ can be written as $V^*\mathcal{E} \oplus H^*\mathcal{E}$ where $H^*\mathcal{E}$ is the space of semi-simple one-forms.

Locally, let $\lambda = \lambda_a dx^a + \lambda_\alpha du^\alpha$ be a one-form on $\mathcal{E}$. Then, the image of λ under Γ is a semi-simple one-form

$$H^*\lambda = \Gamma \cdot \lambda = \left(\lambda_a + \Gamma_a^\alpha \lambda_\alpha\right) dx^a. \tag{2.62}$$

The vertical component of λ is

$$V^*\lambda = \lambda - H^*\lambda = \lambda_\alpha \left(du^\alpha - \Gamma_a^\alpha dx^a\right). \tag{2.63}$$

We have two alternative definitions of $H^*\mathcal{E}$ given by $\mathcal{S}_\pi^* \left(\left(\pi^* \left(T\mathcal{M}\right)\right)^*\right)$ and $\mu_\pi \left(\pi^* \left(T^*\mathcal{M}\right)\right)$, where the former one is coming from the exact sequence in Eq.(2.61) and the letter one is coming from the commutative diagram in Eq.(2.54). From this identification, we have

$$\left(\pi^* \left(T\mathcal{M}\right)\right)^* \simeq \pi^* \left(T^*\mathcal{M}\right). \tag{2.64}$$

In particular, we replace $\pi : \mathcal{E} \to \mathcal{M}$ with $\tau_\mathcal{M} : T\mathcal{M} \to \mathcal{M}$ and $\pi_\mathcal{M} : T^*\mathcal{M} \to \mathcal{M}$ in Eq.(2.64) to obtain

$$\left(\tau_\mathcal{M}^* \left(T\mathcal{M}\right)\right)^* \simeq \tau_\mathcal{M}^* \left(T^*\mathcal{M}\right), \quad \left(\pi_\mathcal{M}^* \left(T\mathcal{M}\right)\right)^* \simeq \pi_\mathcal{M}{}^* \left(T^*\mathcal{M}\right), \tag{2.65}$$

respectively.

2.2.4. Holonomic Lifts and Vertical Representatives

Tangent mapping of a section $\phi : \mathcal{M} \to \mathcal{E}$ of the bundle $\pi : \mathcal{E} \to \mathcal{M}$ satisfies the requirements of being a connection. Two sections which have the same tangent mappings define the same connection. Equivalently, choosing an element of the jet bundle $J^1\pi$ splits the exact sequence in Eq.(2.51) uniquely [49, 50]. Geometrically, this hides behind the fact that, $J^1\pi$ is a subbundle of $T^*\mathcal{M} \otimes T\mathcal{E}$ [59].

Let $(\mathcal{E}, \pi, \mathcal{M})$ be a smooth bundle. Consider a vector field $X \in \mathfrak{X}(\mathcal{M})$ on the base manifold $\mathcal{M}$, and let ϕ be a section of the bundle π, then the **holonomic lift of the vector** $X(\mathbf{x}) \in T_x\mathcal{M}$ by ϕ is defined by

$$\left(j_x^1\phi, T\phi\left(X\left(\mathbf{x}\right)\right)\right) \in \pi_0^*\left(T\mathcal{E}\right) = J^1\pi \times_{\mathcal{E}} T\mathcal{E}, \tag{2.66}$$

where $J^1\pi \times_{\mathcal{E}} T\mathcal{E}$ is the Whitney product. Let $X = X^a\partial/\partial x^a$, then the **holonomic lift of the vector field** X is given by

$$X^{hol} = X^a \frac{\partial}{\partial x^a} + X^a \frac{\partial \phi^\lambda}{\partial x^a}\frac{\partial}{\partial u^\lambda} = X^a \frac{\partial}{\partial x^a} + X^a u_a^\lambda \left(j_x^1\phi\right)\frac{\partial}{\partial u^\lambda}. \tag{2.67}$$

We define the **holonomic part of a projectable vector field** $\xi \in \mathfrak{X}(\mathcal{E})$ as the holonomic lift of its push forward by π, that is

$$H\xi = \left(\pi_*\xi\right)^{hol}. \tag{2.68}$$

$H\xi$ is a generalized vector field of order 1. We define a connection $1-1$ tensor

$$\Gamma_{\mathbf{J}} = dx^a \otimes \left(\frac{\partial}{\partial x^a} + u_a^\alpha \frac{\partial}{\partial u^\alpha}\right) \tag{2.69}$$

satisfying $H\xi = \Gamma_{\mathbf{J}}\xi$. The **vertical (or evolutionary) representative**

$$V\xi = \xi - \Gamma_{\mathbf{J}}\left(\xi\right) = \left(\xi^\alpha - \xi^a u_a^\lambda\right)\frac{\partial}{\partial u^\lambda} \tag{2.70}$$

of the vector field ξ is vertical valued generalized vector field of order 1 [49–51].

Lemma 2.1. The operation in Eq.(2.68) is a Lie algebra isomorphism into.

We consider two projectable vector fields ξ and η on $\mathcal{E}$. A straight forward calculation gives

$$[\Gamma_{\mathbf{J}}(\xi), \Gamma_{\mathbf{J}}(\eta)]_{pro} = [H\xi, H\eta]_{pro} = H[\xi, \eta] = \Gamma_{\mathbf{J}}[\xi, \eta]. \tag{2.71}$$

where $[\ ,\ ]_{pro}$ is the Lie bracket for generalized vector fields in Eq.(2.44). On the other hand, the generalized bracket of vertical representatives satisfies

$$[V\xi, V\eta]_{pro} = V[\xi, \eta]_{pro} + \mathfrak{B}(\xi, \eta), \tag{2.72}$$

where $\mathfrak{B}$ is a vertical-vector valued two-form

$$\mathfrak{B}(\xi, \eta) = [H\eta, V\xi]_{pro} - [H\xi, V\eta]_{pro}. \tag{2.73}$$

The connection $\Gamma_{\mathbf{J}}$ decomposes the one-form sections into direct sum of two generalized one-forms. For $\lambda = \lambda_a dx^a + \lambda_\alpha du^\alpha$ the horizontal and the vertical parts are

$$H^*\lambda = \Gamma_{\mathbf{J}}(\lambda) = (\lambda_a + \lambda_\alpha u_a^\alpha) dx^a \quad \text{and} \quad V^*\lambda = \lambda_\alpha (du^\alpha - u_a^\alpha dx^a). \tag{2.74}$$

$V^*\lambda$ is particularly called a **contact one-form** [60].

2.3. CALCULUS ON MANIFOLDS

2.3.1. Differential Forms and Exterior Derivative

A **differential p-form** on $\mathcal{M}$ is a skew-symmetric $p-$multilinear functional on $\mathfrak{X}(\mathcal{M})$ and can be uniquely represented in local coordinates by

$$\omega(\mathbf{x}) = \frac{1}{p!}\omega_{a_1 a_2 \ldots a_p}(\mathbf{x})\, dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_p}. \tag{2.75}$$

Here, the **wedge product** $\wedge$ of two differential one-forms $dx^a \wedge dx^b$ is defined by an alternating **tensor product**

$$dx^a \otimes dx^b - dx^b \otimes dx^a \tag{2.76}$$

and extended by linearity [61, 62]. The set of differential $p-$forms are closed under addition and scalar multiplication, and they form the vector space $\Lambda^p \mathcal{M}$ over the field of reals. The direct sum

$$\Lambda(\mathcal{M}) := \Lambda^o(\mathcal{M}) \oplus \Lambda^1(\mathcal{M}) \oplus ... \oplus \Lambda^m(\mathcal{M}) \tag{2.77}$$

is a graded algebra called the **Grassmann algebra**, where the space of zero forms $\Lambda^o(\mathcal{M})$ is assumed to be the space of smooth functions $\mathcal{F}(\mathcal{M})$ on $\mathcal{M}$ and the space of one-forms $\Lambda^1(\mathcal{M})$ is the space of sections of the cotangent bundle $T^*\mathcal{M}$.

On $\Lambda(\mathcal{M})$, the exterior derivative

$$d : \Lambda^p(\mathcal{M}) \to \Lambda^{p+1}(\mathcal{M}) \tag{2.78}$$

is a linear mapping over $\mathbb{R}$ and is defined by

$$\begin{aligned} d\omega(X_0, X_1, , ..., X_p) &= \sum_{i=0}^{p} X_i \left((-1)^i \omega \left(X_0, X_1, ..., \widehat{X_i}, ..., X_p \right) \right) \\ &\quad - \sum_{i<j} \left((-1)^{i+j} \omega \left([X_i, X_j], X_0, X_1, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_p \right) \right), \end{aligned} \tag{2.79}$$

where the vector fields with hat, e.g. $\widehat{X_i}$, are omitted. Exterior derivation is a nilpotent operator, that is, $d^2 = 0$ and satisfies the generalized Leibniz rule

$$d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^p \omega \wedge d\alpha, \tag{2.80}$$

for $\omega \in \Lambda^p(\mathcal{M})$ and $\alpha \in \Lambda(\mathcal{M})$. In coordinates, the exterior derivative of ω in Eq.(2.75) is

$$d\omega = \frac{1}{p!} \frac{\partial \omega_{a_1 a_2 ... a_p}}{\partial x^a} dx^a \wedge dx^{a_1} \wedge ... \wedge dx^{a_p}, \tag{2.81}$$

and, in particular, $df = (\partial f / \partial x^a)\, dx^a$, for $f \in \mathcal{F}(\mathcal{M})$.

A p-form ω is called a **closed form,** if $d\omega = 0$. ω is called an **exact form** if there exist a $(p-1)$-form θ such that $\omega = d\theta$. Every exact form is closed whereas the inverse of this fact is not true. **Poincaré lemma** states that, a closed p-form ω on open contractible subsets of $\mathcal{M}$ is exact [47,60].

Let $\mathcal{N}$ and $\mathcal{M}$ be two smooth manifolds and $\varphi : \mathcal{N} \to \mathcal{M}$ be a differentiable mapping. The pull back operator

$$\varphi^* : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{N}) \tag{2.82}$$

maps differential forms on $\mathcal{M}$ to differential forms on $\mathcal{N}$. In particular, the pull-back of a 0-form f on $\mathcal{M}$ is $\varphi^* f = f \circ \varphi$ and, the pull-back of a p-form ω on $\mathcal{M}$ is a p-form $\varphi^*\omega$ on $\mathcal{N}$ defined by

$$\varphi^*\omega(X_1, X_1, ..., X_p) = \omega(\varphi_*(X_1), \varphi_*(X_2), ..., \varphi_*(X_p)), \tag{2.83}$$

where $X_1, X_1, ..., X_p$ are vector fields on $\mathcal{N}$ and $\varphi_*(X_i)$ denotes the push forward of X_i by φ. In coordinates, if ω is given by Eq.(2.75) and (q^i) denotes the coordinates of $\mathcal{N}$, then

$$(\varphi^*\omega)(\mathbf{q}) = \frac{1}{p!}\omega_{a_1 a_2 ... a_p}(\varphi(\mathbf{q})) \frac{\partial \varphi^{a_1}}{\partial q^{i_1}} \frac{\partial \varphi^{a_2}}{\partial q^{i_2}} ... \frac{\partial \varphi^{a_p}}{\partial q^{i_p}} dq^{i_1} \wedge dq^{i_2} \wedge ... \wedge dq^{i_p}. \tag{2.84}$$

The pull back operation respects to the exterior derivative and wedge product operation, that is,

$$\varphi^* d\omega = d\varphi^*\omega \quad \text{and} \quad \varphi^*(\omega \wedge \alpha) = \varphi^*\omega \wedge \varphi^*\alpha$$

$\forall \omega, \alpha \in \Lambda(\mathcal{M})$.

2.3.2. Interior product and Lie Derivative

The exterior derivative raises the degree of a differential form one up, whereas the interior product

$$i_X : \Lambda^p\left(\mathcal{M}\right) \rightarrow \Lambda^{p-1}\left(\mathcal{M}\right) \tag{2.85}$$

lowers the degree of a differential form one down. For $\omega \in \Lambda^p\left(\mathcal{M}\right)$ and $X, X_1, ..., X_{p-1} \in \mathfrak{X}\left(\mathcal{M}\right)$, this is defined by

$$i_X\omega\left(X_1, ..., X_{p-1}\right) = \omega\left(X, X_1, ..., X_{p-1}\right). \tag{2.86}$$

As in the case of exterior derivative, interior product satisfies the generalized Leibniz identity, that is, for $\omega \in \Lambda^p\left(\mathcal{M}\right)$ and $\alpha \in \Lambda\left(\mathcal{M}\right)$,

$$i_X\left(\omega \wedge \alpha\right) = i_X\omega \wedge \alpha + \left(-1\right)^p \omega \wedge i_X\alpha. \tag{2.87}$$

One has also that $i_{X_1} \circ i_{X_2} = -i_{X_2} \circ i_{X_1}$.

The Lie derivative $\pounds_X$ is a linear operation acting on functions, vector fields and one-forms. Lie derivative $\pounds_X f$ of a function f is the directional derivative $X(f)$ of f in the direction X. Lie derivative with respect to X of a vector field Y is the Jacobi-Lie bracket of vector fields

$$\pounds_X Y = [X, Y]_{JL}, \tag{2.88}$$

as given in Eq.(2.13). Lie derivative of a $p-$form ω is a $p-$form and is defined by the Cartan's formula

$$\pounds_X\omega = di_X\omega + i_Xd\omega. \tag{2.89}$$

We have the following commutation relations

$$d \circ \pounds_X = \pounds_X \circ d \quad \text{and} \quad [\pounds_X, i_Y] = [i_X, \pounds_Y] = i_{\pounds_X Y}, \tag{2.90}$$

where $[\pounds_X, i_Y] = \pounds_X \circ i_Y - i_Y \circ \pounds_X$ [1, 47, 60].

2.4. GEOMETRIC STRUCTURES ON MANIFOLDS

2.4.1. Volume Manifolds

A **volume form** on an m dimensional manifold $\mathcal{M}$ is a nowhere vanishing top form, i.e. $m-$form μ, with $\mu(\mathbf{x}) \neq 0$, for all $\mathbf{x} \in \mathcal{M}$. The pair $(\mathcal{M}, \mu)$ is called a **volume (or orientable) manifold**. An **orientation of** $\mathcal{M}$ is the class of $[\mu] = \{f\mu : f > 0\}$. Let $(\mathcal{M}, [\mu])$ and $(\mathcal{N}, [\eta])$ be two volume manifolds, a smooth map $\varphi : \mathcal{M} \to \mathcal{N}$ is **volume preserving** if

$$\varphi^* \eta = \mu \tag{2.91}$$

and **orientation preserving** if $\varphi^* [\eta] = [\mu]$.

Let X be a vector field on $(\mathcal{M}, \mu)$. The **divergence,** $\operatorname{div}_\mu X$, of X is defined by

$$(\operatorname{div}_\mu X)\, \mu = \pounds_X \mu. \tag{2.92}$$

A vector field X is called **divergence free** (or **solenoidal**) if $\operatorname{div}_\mu X = 0$. The equality

$$\operatorname{div}_\mu [X, Y] = X (\operatorname{div}_\mu Y) - Y (\operatorname{div}_\mu X) \tag{2.93}$$

shows that, if two vector fields X and Y are divergence free then so is $[X, Y]_{JL}$ and hence, we conclude that the space of divergence free vector fields $\mathfrak{X}_{\text{div}}(\mathcal{M})$ constitutes a Lie algebra, that is, the space of divergence free vector fields is closed under the Jacobi-Lie bracket.

2.4.2. Symplectic Manifolds

A two-form $\Omega_{\mathcal{M}}$ on $\mathcal{M}$ is called nondegenerate if $\Omega_{\mathcal{M}}(X,Y)=0$ for all $X\in\mathfrak{X}(\mathcal{M})$ implies $Y=0$. A **symplectic manifold** is a pair $(\mathcal{M},\Omega_{\mathcal{M}})$ where $\Omega_{\mathcal{M}}$ is a closed, nondegenerate two-form on $\mathcal{M}$ [63–65]. Nondegeneracy of the symplectic form $\Omega_{\mathcal{M}}$ leads to the isomorphism

$$\Omega^{\flat}_{\mathcal{M}}:\mathfrak{X}(\mathcal{M})\to\Lambda^{1}(\mathcal{M}):X\to i_{X}\Omega_{\mathcal{M}}. \tag{2.94}$$

The fiberwise inverse of $\Omega^{\flat}_{\mathcal{M}}$ is $\Omega^{\sharp}_{\mathcal{M}}:\Lambda^{1}(\mathcal{M})\to\mathfrak{X}(\mathcal{M})$. $\Omega^{\flat}_{\mathcal{M}}$ and $\Omega^{\sharp}_{\mathcal{M}}$ are called **musical isomorphisms**.

Proposition 2.2. The cotangent manifold is a symplectic manifold.

The existence of symplectic structure on a cotangent bundle $T^{*}\mathcal{M}$ follows from the double vector bundle structure of $TT^{*}\mathcal{M}$, given diagrammatically by

$$\begin{array}{ccc} TT^{*}\mathcal{M} & \xrightarrow{\tau_{T^{*}\mathcal{M}}} & T^{*}\mathcal{M} \\ \Big\downarrow T\pi_{\mathcal{M}} & & \Big\downarrow \pi_{\mathcal{M}} \\ T\mathcal{M} & \xrightarrow{\tau_{\mathcal{M}}} & \mathcal{M}. \end{array} \tag{2.95}$$

where $T\pi_{\mathcal{M}}$ denotes the tangent mapping of the fibration $\pi_{\mathcal{M}}:T^{*}\mathcal{M}\to\mathcal{M}$ and $\tau_{T^{*}\mathcal{M}}:TT^{*}\mathcal{M}\to T^{*}\mathcal{M}$ is the natural projection. We define the canonical (Liouville) one-form $\theta_{T^{*}\mathcal{M}}$ on $T^{*}\mathcal{M}$ as

$$\theta_{T^{*}\mathcal{M}}(\xi)=\left\langle \tau_{T^{*}\mathcal{M}}(\xi),T\pi_{\mathcal{M}}(\xi)\right\rangle, \tag{2.96}$$

where $\xi\in TT^{*}\mathcal{M}$ and $\langle\ ,\ \rangle$ is the natural pairing between $T\mathcal{M}$ and $T^{*}\mathcal{M}$. Locally, $\theta_{T^{*}\mathcal{M}}=y_{a}dx^{a}$. The exterior derivative

$$\Omega_{T^{*}\mathcal{M}}=-d\theta_{T^{*}\mathcal{M}}=dx^{a}\wedge dy_{a} \tag{2.97}$$

of $\theta_{T^*\mathcal{M}}$ defines a symplectic two-form on $T^*\mathcal{M}$ called the **canonical symplectic two-form.** Darboux's theorem states that all symplectic two-forms can be put into the form in Eq.(2.97) in a coordinate system called **Darboux's coordinates**.

A **(globally) Hamiltonian vector field** on a symplectic manifold $(\mathcal{M}, \Omega_{\mathcal{M}})$ is the unique vector field X_h satisfying

$$i_{X_h}\Omega = dh, \tag{2.98}$$

for a real valued function h called **Hamiltonian function**. Eq.(2.98) can be recast as $\Omega^\flat_{\mathcal{M}}(X_h) = dh$. Since $\Omega^\flat_{\mathcal{M}}$ is an isomorphism, one can always find the corresponding Hamiltonian vector field X_h for a given smooth function h. $X \in \mathfrak{X}(\mathcal{M})$ is called a **locally Hamiltonian vector field** if $i_X\Omega_{\mathcal{M}}$ is a closed form, that is $di_{X_h}\Omega_{\mathcal{M}} = 0$. Poincaré's lemma guaranties the existence of a local Hamiltonian function corresponding to a locally Hamiltonian vector field, but not necessarily a global one.

A symplectic manifold $(\mathcal{M}, \Omega_{\mathcal{M}})$ of dimension $2m$ is a volume manifold with **symplectic volume** $\mu_\Omega = (\Omega_{\mathcal{M}})^m$. A Hamiltonian vector field, even if it is local, is divergence free with respect to the symplectic volume, because

$$\mathcal{L}_X\mu_\Omega = \mathcal{L}_X(\Omega_{\mathcal{M}})^m = (\mathcal{L}_X\Omega_{\mathcal{M}})^m = (di_X\Omega_{\mathcal{M}} + i_X d\Omega_{\mathcal{M}})^m = 0. \tag{2.99}$$

In Darboux's coordinates, the Hamiltonian vector field on $T^*\mathcal{M}$ for the Hamiltonian function $h \in \mathcal{F}(\mathcal{M})$ is

$$X_h = \frac{\partial h}{\partial y_a}\frac{\partial}{\partial x^a} - \frac{\partial h}{\partial x^a}\frac{\partial}{\partial y_a} \tag{2.100}$$

and the equations

$$\dot{x}^a = \frac{\partial h}{\partial y_a}, \quad \dot{y}_a = -\frac{\partial h}{\partial x^a} \tag{2.101}$$

are called **Hamilton's equations.** In Darboux's coordinate, the symplectic volume is

$$\mu_{\Omega} = dx^1 \wedge ... \wedge dx^m \wedge dy_1 \wedge ... \wedge dy_m \tag{2.102}$$

and the divergence of $X = X^a \partial / \partial x^a + X_a \partial / \partial y_a$ with respect to the symplectic volume is

$$\operatorname{div}_{\mu_{\Omega}}(X) = \frac{\partial X^a}{\partial x^a} + \frac{\partial X_a}{\partial y_a}. \tag{2.103}$$

Proposition 2.3. Tangent manifold of a symplectic manifold is symplectic.

We use the dual tangent rhombic in Eq.(2.15) to define a one-form $\theta_{T\mathcal{M}}$ on the tangent bundle $T\mathcal{M}$ of a symplectic manifold $(\mathcal{M}, \Omega_{\mathcal{M}})$ as follows

$$\theta_{T\mathcal{M}}(\xi) = \Omega_{\mathcal{M}}(T\tau_{\mathcal{M}}(\xi), \tau_{T\mathcal{M}}(\xi)), \quad \forall \xi \in TT\mathcal{M},$$

where $T\tau_{\mathcal{M}}$ is the tangent mapping of $\tau_{\mathcal{M}}$ and $\tau_{T\mathcal{M}}$ is the natural tangent bundle projection of $TT\mathcal{M}$ to $T\mathcal{M}$. The exterior derivative $\Omega_{T\mathcal{M}} = d\theta_{T\mathcal{M}}$ of $\theta_{T\mathcal{M}}$ is the Tulczyjew's symplectic two-form on $T\mathcal{M}$ [8].

In particular, we take $\mathcal{M}$ to be the canonical symplectic manifold $\left(T^*\mathcal{Q}, \Omega_{T^*\mathcal{Q}} = dq^i \wedge dp_i\right)$ with coordinates $\left(q^i, p_j\right)$. The one-form $\theta_{TT^*\mathcal{Q}}$ and Tulczyjew's two-form $d\theta_{TT^*\mathcal{Q}}$ are, in induced coordinates $\left(q^i, p_j; \dot{q}^i, \dot{p}_j\right)$ on $TT^*\mathcal{Q}$, given by

$$\theta_{TT^*\mathcal{Q}} = \dot{q}^i dp_i - \dot{p}_i dq^i, \quad \Omega_{TT^*\mathcal{Q}} = d\theta_{TT^*\mathcal{Q}} = d\dot{q}^i \wedge dp_i - d\dot{p}_i \wedge dq^i. \tag{2.104}$$

Let $(\mathcal{M}_1, \Omega_1)$ and $(\mathcal{M}_2, \Omega_2)$ be two symplectic manifolds and $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ be a diffeomorphism. φ is called a **symplectic** (or **canonical) diffeomorphism** if $\varphi^*\Omega_2 = \Omega_1$. Compositions of two canonical diffeomorphisms is canonical.

2.4.3. Poisson Manifolds

A **Poisson structure** on a manifold $\mathcal{P}$ is a bilinear map

$$\{\ ,\ \} : \mathcal{F}(\mathcal{P}) \times \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P}), \tag{2.105}$$

which takes two smooth functions to a new one with properties:

1. skewsymmetry: $\{h, f\} = -\{f, h\}$,
2. Jacobi identity: $\{f, \{h, g\}\} + \{h, \{g, f\}\} + \{g, \{f, h\}\} = 0$,
3. Leibniz identity: $\{fh, g\} = f\{h, g\} + \{f, g\}h$.

The pair $(\mathcal{P}, \{\ ,\ \})$ is called a **Poisson manifold** [5]. For a function $h \in \mathcal{F}(\mathcal{P})$, we define the **Hamiltonian vector field** X_h on $\mathcal{P}$ by

$$X_h(f) = \{f, h\}. \tag{2.106}$$

If the Hamiltonian vector field of a non-constant function C is identically zero, then the function C is called a **distinguished** (or **Casimir) function**. Poisson brackets of Casimir functions vanishes $\{C, f\} = 0$, for all $f \in \mathcal{F}(\mathcal{P})$. If X_h is a Hamiltonian vector field for a Hamiltonian function h, then it is also Hamiltonian vector field of the function $h + C$, that is, $X_{h+C} = X_h$. We have the following lemma [3, 4].

Lemma 2.4. If X_h and X_f are Hamiltonian vector fields for h and f, respectively, then

$$[X_h, X_f]_{JL} = -X_{\{h,f\}}. \tag{2.107}$$

Every symplectic manifold $(\mathcal{M}, \Omega_{\mathcal{M}})$ is a Poisson manifold with the Poisson structure

$$\{f, h\}_{\Omega_{\mathcal{M}}} = \Omega_{\mathcal{M}}(X_f, X_h), \tag{2.108}$$

where X_f and X_h are Hamiltonian vector fields in the sense of Eq.(2.98). The closedness of $\Omega_{\mathcal{M}}$ corresponds to the Jacobi identity for $\{\ ,\ \}_{\Omega_{\mathcal{M}}}$.

Let $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ be a diffeomorphism between two Poisson manifolds $(\mathcal{P}_1, \{.,.\}_1)$ and $(\mathcal{P}_2, \{.,.\}_2)$. φ is called a **Poisson** (or **canonical**) **map,** if

$$\{f,h\}_2 \circ \varphi = \{f \circ \varphi, h \circ \varphi\}_1 , \tag{2.109}$$

for all $f, h \in \mathcal{F}(\mathcal{P}_2)$. If a Poisson structure is induced from a symplectic form, then a Poisson map is also a symplectic diffeomorphism.

We define an equivalence relation $\sim$ on the Poisson manifold $\mathcal{P}$. We call $\mathbf{z} \in \mathcal{P}$ and $\mathbf{w} \in \mathcal{P}$ equivalent, if there exists a finite sequence $\mathbf{z}_0, ..., \mathbf{z}_k$ such that $\mathbf{z}_0 = \mathbf{z}$ and $\mathbf{z}_k = \mathbf{w}$, and all $\mathbf{z}_i$ and $\mathbf{z}_{i+1}$ can be joined by a flow of a Hamiltonian vector field. The equivalence classes of this relationship, that is elements of $\mathcal{P}/\sim$, are symplectic manifolds. They are called **symplectic leaves** of the Poisson manifold [66]. If a Poisson structure is induced from a symplectic form, then there exists a unique symplectic leaf.

Proposition 2.5. Given a Poisson manifold $\mathcal{P}$, centered at any point $\mathbf{z} \in \mathcal{P}$, there are coordinates

$$\left(x^1, x^2, ..., x^n, y_1, y_2, ..., y_n, w^1, w^2, ..., w^k\right)$$

such that

$$\begin{aligned} \left\{x^a, x^b\right\} &= 0, \quad \{y_a, y_b\} = 0, \quad \left\{x^a, w^i\right\} = 0 \\ \left\{y_a, w^i\right\} &= 0, \quad \left\{w^i, w^j\right\} = 0, \quad \{x^a, y_b\} = \delta^a_b. \end{aligned}$$

In this coordinates system, Poisson bracket of two functions f and h is

$$\{f,h\} = \frac{\partial f}{\partial x^a}\frac{\partial h}{\partial y_a} - \frac{\partial f}{\partial y_a}\frac{\partial h}{\partial x^a}. \tag{2.110}$$

For a Hamiltonian function h, the Hamiltonian vector field

$$X_h = \frac{\partial h}{\partial y_a}\frac{\partial}{\partial x^a} - \frac{\partial h}{\partial x^a}\frac{\partial}{\partial y_a} \tag{2.111}$$

looks like as in symplectic case whereas the equations of motion are

$$\dot{x}^a = \{h, x^a\}, \quad \dot{y}_a = \{h, y_a\} \quad \text{and} \quad \dot{w}^i = 0. \tag{2.112}$$

A Casimir function is a function of variables $\left(w^i, i = 1, ..., k\right)$. When $k = 0$, Poisson structure can be induced from a symplectic form and the proposition is reduced to Darboux's theorem.

3. LIFTS OF VECTOR FIELDS AND FORMS

3.1. COMPLETE TANGENT AND COTANGENT LIFTS

The flow $\varphi_t : \mathcal{M} \to \mathcal{M}$ of a vector field X on $\mathcal{M}$ satisfies

$$X(\mathbf{x}) = \frac{d}{dt}\,\varphi_t(\mathbf{x})\big|_{t=0}\,, \tag{3.1}$$

for all $\mathbf{x} \in \mathcal{M}$. Induced mappings φ_t^c on the tangent bundle $T\mathcal{M}$ are defined through the following equation

$$\tau_{\mathcal{M}} \circ \varphi_t^c = \varphi_t \circ \tau_{\mathcal{M}}. \tag{3.2}$$

φ_t^c constitutes a one-parameter group of diffeomorphisms on $T\mathcal{M}$ and called **complete tangent lift of the flow** φ_t [67], [68]. Note that the tangent map $T\varphi_t$ of φ_t satisfies the conditions of being a tangent lift. The infinitesimal generator of the flow φ_t^c is denoted by X^c and is called the **complete tangent lift of** X. From differentiation of Eq.(3.2) with respect to t at $t = 0$ we obtain the equation

$$T\tau_{\mathcal{M}} \circ X^c = X \circ \tau_{\mathcal{M}}. \tag{3.3}$$

involving X and X^c. In local coordinates (x^a, v^a) of $T\mathcal{M}$, the complete tangent lift of $X = X^a\partial/\partial x^a$ is computed to be

$$X^c = X^a \frac{\partial}{\partial x^a} + v^b \frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial v^a}. \tag{3.4}$$

The tangent map $TX : T\mathcal{M} \to TT\mathcal{M}$ of a vector field X is called the **linearization** of the vector field X. The value of the linearization at a point $\mathbf{x} \in \mathcal{M}$ is given locally and explicitly as

$$TX = (x^a, X^a, v^a, \frac{\partial X^a}{\partial x^b} v^b). \tag{3.5}$$

The tangent lift X^c and linearization TX of X are connected to each other with

$$\kappa_{\mathcal{M}} \circ TX = X^c,$$

where $\kappa_{\mathcal{M}}$ is the canonical involution on $TT\mathcal{M}$ given in Eq.(2.18) [45].

Similarly, the **complete cotangent lift of a flow** φ_t is a one-parameter group of diffeomorphisms $\varphi_t^{c*} : T^*\mathcal{M} \to T^*\mathcal{M}$ satisfying

$$\pi_{\mathcal{M}} \circ \varphi_t^{c*} = \varphi_t \circ \pi_{\mathcal{M}}, \tag{3.6}$$

where $\pi_{\mathcal{M}}$ is the natural projection of $T^*\mathcal{M}$ to $\mathcal{M}$. The cotangent lift of the inverse flow $T^*\varphi_{-t}$ satisfies the argument in Eq.(3.6). The vector field X^{c*}, which has the flow φ_t^{c*}, is called the **complete cotangent lift of** X [69]. The infinitesimal version of the Eq.(3.6) is

$$T\pi_{\mathcal{M}} \circ X^{c*} = X \circ \pi_{\mathcal{M}}. \tag{3.7}$$

One may relate a vector field $X \in \mathfrak{X}(\mathcal{M})$ and its complete tangent and cotangent lifts $X^c \in \mathfrak{X}(T\mathcal{M})$ and $X^{c*} \in \mathfrak{X}(T^*\mathcal{M})$ via the mappings

$$\begin{aligned} {}^c &: \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(T\mathcal{M}) : X \to X^c \\ {}^{c*} &: \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(T^*\mathcal{M}) : X \to X^{c*}. \end{aligned} \tag{3.8}$$

We have the following proposition [11, 70].

Proposition 3.1. Maps given in Eqs.(3.8) are Lie algebra isomorphism intos, that is,

$$[X, Y]^c = [X^c, Y^c] \quad \text{and} \quad [X, Y]^{c*} = [X^{c*}, Y^{c*}], \tag{3.9}$$

for all $X, Y \in \mathfrak{X}(\mathcal{M})$.

3.1.1. Hamiltonian Structures of Complete Lifts

The complete cotangent lift X^{c*} of a vector field X on $\mathcal{M}$ is a Hamiltonian vector field on $T^*\mathcal{M}$ for the Hamiltonian function $\mathcal{P}(X) = i_{X^{c*}}\theta_{T^*\mathcal{M}}$ called the **momentum function** [11, 52, 68]. Let $X = X^a\partial/\partial x^a$. In Darboux's coordinates (x^a, y_b) on $T^*\mathcal{M}$, the momentum function is $\mathcal{P}(X)(x^a, y_b) = y_b X^b$ and the complete cotangent lift has the expression

$$X^{c*} = X_{\mathcal{P}(X)} = X^a \frac{\partial}{\partial x^a} - y_b \frac{\partial X^b}{\partial x^a} \frac{\partial}{\partial y_a}. \tag{3.10}$$

$\mathcal{P}$ can be considered as a map $\mathcal{P} : \mathfrak{X}(\mathcal{M}) \rightarrow \mathcal{F}_L(T^*\mathcal{M})$, where $\mathcal{F}_L(T^*\mathcal{M})$ is the space of functions on $T^*\mathcal{M}$ which are linear on fibers.

If $(\mathcal{M}, \Omega_{\mathcal{M}})$ is a symplectic manifold then its tangent bundle $T\mathcal{M}$ carries a symplectic structure as given in Eq.(2.104).

Proposition 3.2. The complete tangent lift of a Hamiltonian vector field X_h on a symplectic manifold $(\mathcal{M}, \Omega_{\mathcal{M}})$ is Hamiltonian.

In particular, we take $(\mathcal{M}, \Omega_{\mathcal{M}})$ to be the canonical symplectic manifold $(T^*\mathcal{Q}, \Omega_{T^*\mathcal{Q}})$ and compute the tangent lift of a generic Hamiltonian vector field on $T^*\mathcal{Q}$. In Darboux's coordinates $\left(q^i, p_j\right)$ on $T^*\mathcal{Q}$, the symplectic two-form is $\Omega_{T^*\mathcal{Q}} = dq^i \wedge dp_i$ and the Hamiltonian vector field for h becomes

$$X_h(\mathbf{q}, \mathbf{p}) = \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i} \frac{\partial}{\partial p_i} \in \mathfrak{X}(T^*\mathcal{Q}). \tag{3.11}$$

For the induced coordinates $\left(q^i, p_j; \dot{q}^i, \dot{p}_j\right)$ on $TT^*\mathcal{Q}$, the Tulczyjew symplectic two-from on $TT^*\mathcal{Q}$ is given by

$$\Omega_{TT^*\mathcal{Q}} = dq^i \wedge d\dot{p}_i + d\dot{q}^i \wedge dp_i. \tag{3.12}$$

The complete tangent lift is computed as

$$X_h^c(\mathbf{z},\dot{\mathbf{z}}) = X_h + X_{T^*\mathcal{Q}}\left(\frac{\partial h}{\partial p_i}\right)\frac{\partial}{\partial \dot{q}^i} - X_{T^*\mathcal{Q}}\left(\frac{\partial h}{\partial q^i}\right)\frac{\partial}{\partial \dot{p}_i} \in \mathfrak{X}(TT^*\mathcal{Q}), \tag{3.13}$$

where $X_{T^*\mathcal{Q}} = \tau_{TT^*\mathcal{Q}}(X_h^c) = \dot{q}^i\partial/\partial q^i + \dot{p}_i\partial/\partial p_i$. X_h^c is a Hamiltonian vector field for the Hamiltonian function

$$\tilde{H} = \Omega_{T^*Q}(X_h, X_{T^*Q}) = \frac{\partial h}{\partial p_i}\dot{p}_i + \frac{\partial h}{\partial q^i}\dot{q}^i \in \mathcal{F}(TT^*\mathcal{Q}). \tag{3.14}$$

The complete cotangent lift $X_h^{c*} \in \mathfrak{X}(T^*T^*\mathcal{Q})$ of $X_h \in \mathfrak{X}(T^*\mathcal{Q})$ with induced coordinates $\Pi = \left(q^i, p_i; \Pi_i, \Pi^i\right)$ on $T^*T^*\mathcal{Q}$ is

$$X_h^{c*}(\Pi) = X_h(z) + \Pi^\sharp\left(\frac{\partial h}{\partial q^i}\right)\frac{\partial}{\partial \Pi_i} + \Pi^\sharp\left(\frac{\partial h}{\partial p_i}\right)\frac{\partial}{\partial \Pi^i} \in \mathfrak{X}(T^*T^*\mathcal{Q}), \tag{3.15}$$

where $\Pi^\sharp$ is image of Π by the musical isomorphism $\Omega_{T^*\mathcal{Q}}^\sharp$, which, in coordinates, is given by $\Pi^\sharp = \Pi^i\partial/\partial q^i - \Pi_i\partial/\partial p_i$. The corresponding degenerate Hamiltonian function of the Hamiltonian vector field X_h^{c*} is

$$H = \langle \Pi, X_h \rangle = \frac{\partial h}{\partial p_i}\Pi_i - \frac{\partial h}{\partial q^i}\Pi^i \in \mathcal{F}(T^*T^*\mathcal{Q}) \tag{3.16}$$

where $\langle\ ,\ \rangle$ is the natural pairing between TT^*Q and T^*T^*Q. We have the relation $\tilde{H} = \left(\Omega_{T^*Q}^\flat\right)^* H$ between Hamiltonian functions $\tilde{H}$ and H.

3.1.2. Decompositions of Complete Lifts

Recall the complete tangent lift

$$X^c = X^a(\mathbf{x})\frac{\partial}{\partial x^a} + v^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial v^a} \in \mathfrak{X}(T\mathcal{M}) \tag{3.17}$$

of a vector field $X = X^a(\mathbf{x})\,\partial/\partial x^a \in \mathfrak{X}(\mathcal{M})$. The vertical representatives and the horizontal part of X^c are

$$VX^c = (v^b \frac{\partial X^a}{\partial x^b} - X^a \frac{\partial v^b}{\partial x^a})\frac{\partial}{\partial v^b}, \tag{3.18}$$

$$HX^c = X^a \frac{\partial}{\partial x^a} + X^a \frac{\partial v^b}{\partial x^a}\frac{\partial}{\partial v^b}. \tag{3.19}$$

For the mapping $\kappa_{\mathcal{M}} : TT\mathcal{M} \to TT\mathcal{M}$ defined in Eq.(2.18), we have the following commutations

$$\kappa_{\mathcal{M}} \circ HX^c = H \circ \kappa_{\mathcal{M}}(X^c) = H(TX) \quad \text{and} \quad \kappa_{\mathcal{M}} \circ VX^c = V \circ \kappa_{\mathcal{M}}(X^c). \tag{3.20}$$

The holonomic lift X^{hol} of X coincides with horizontal part of X^c, that is, $HX^c = X^{hol}$.

Similarly, the complete cotangent lift

$$X^{c*} = X^a \frac{\partial}{\partial x^a} - y_b \frac{\partial X^b}{\partial x^a}\frac{\partial}{\partial y_a} \tag{3.21}$$

of the vector field $X \in \mathfrak{X}(\mathcal{M})$ is defined in Eq.(3.10). The vertical representative and the horizontal part of X^{c*} are

$$VX^{c*} = -(y_b \frac{\partial X^b}{\partial x^a} + X^b \frac{\partial y_a}{\partial x^b})\frac{\partial}{\partial y_a}, \tag{3.22}$$

$$HX^{c*} = X^a \frac{\partial}{\partial x^a} + X^a \frac{\partial y_b}{\partial x^a}\frac{\partial}{\partial y_b}. \tag{3.23}$$

Lemma 3.3. The mappings defined by

$$\begin{aligned} V^c &: \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(T\mathcal{M}) : X \to VX^c, \\ V^{c*} &: \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(T^*\mathcal{M}) : X \to VX^{c*} \end{aligned} \tag{3.24}$$

are Lie algebra isomorphism intos.

Indeed, the vector valued two-form

$$\mathfrak{B}(\xi,\eta) = [H\eta, V\xi]_{pro} - [H\xi, V\eta]_{pro}, \tag{3.25}$$

in Eq.(2.73) vanishes for the lifts, that is, $\mathfrak{B}(X^c, Y^c) = 0$ and $\mathfrak{B}(X^{c*}, Y^{c*}) = 0$ for all $X, Y \in \mathfrak{X}(\mathcal{M})$, therefore, one has the isomorphisms

$$V[X^c, Y^c] = [VX^c, VY^c]_{pro} \quad \text{and} \quad V[X^{c*}, Y^{c*}] = [VX^{c*}, VY^{c*}]_{pro}. \tag{3.26}$$

Using Eqs.(3.9) the desired results

$$V^c[X, Y] = [VX^c, VY^c]_{pro} \quad \text{and} \quad V^{c*}[X, Y] = [VX^{c*}, VY^{c*}]_{pro} \tag{3.27}$$

are obtained.

3.2. VERTICAL LIFTS

3.2.1. Vertical Lifts of Vectors

Vertical lift operator

$$\text{ver} : T\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} \to TT\mathcal{M} : (\mathbf{v}, \mathbf{u}) \to \frac{d}{dt} (\mathbf{v} + t\mathbf{u})|_{t=0} \tag{3.28}$$

is a mapping which takes an element of the Whitney product

$$T\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} = \{(\mathbf{v}, \mathbf{u}) \in T\mathcal{M} \times T\mathcal{M} : \tau_{\mathcal{M}}(\mathbf{v}) = \tau_{\mathcal{M}}(\mathbf{u})\} \tag{3.29}$$

to the iterated tangent bundle $TT\mathcal{M}$ [3, 52]. Image space of ver consists of vertical vectors, that is the vectors in $\ker T\tau_{\mathcal{M}} = VT\mathcal{M}$. In local coordinates, if $\mathbf{v} = (x^a, v^b)$ and $\mathbf{u} = (x^a, u^b)$

then

$$\text{ver}\left(\left(x^a, v^b\right), \left(x^a, u^d\right)\right) = \left(x^a, v^b; 0, u^d\right). \tag{3.30}$$

If the first entry $\mathbf{v}$ of the Whitney product $(\mathbf{v}, \mathbf{u})$ is fixed then

$$\text{ver}_v : T\mathcal{M} \rightarrow VT\mathcal{M} : \mathbf{u} \rightarrow \text{verlift}\left(\mathbf{v}, \mathbf{u}\right) \tag{3.31}$$

is a vertical vector field. **Vertical lift X^v of a vector field** X on $\mathcal{M}$ is a vector field on $T\mathcal{M}$, that is an element of $\mathfrak{X}\left(T\mathcal{M}\right)$ defined by

$$X^v\left(\mathbf{v}\right) = \text{ver}\left(\mathbf{v}, X\left(\mathbf{x}\right)\right). \tag{3.32}$$

If $X = X^a\partial/\partial x^a$, then vertical lift of X is $X^v = X^a\partial/\partial v^a$.

The Jacobi-Lie bracket of two vertical lifts is zero, that is,

$$\left[X^v, Y^v\right] = 0 \tag{3.33}$$

hence, the space of vertical lifts is a Lie subalgebra of the space of vector fields on $T\mathcal{M}$. The Lie bracket of a vertical lift X^v and a complete lift Y^c is a vertical lift

$$\left[X^v, Y^c\right] = \left[X, Y\right]^v. \tag{3.34}$$

If $\{X_1, X_2, .., X_m\}$ is a local basis for $T\mathcal{M}$, then

$$\{X_1^v, X_2^v, .., X_m^v, X_1^c, X_2^c, .., X_m^c\} \tag{3.35}$$

is a local basis for $TT\mathcal{M}$ [71].

3.2.2. Decompositions of $TT\mathcal{M}$ and $T^*T\mathcal{M}$

Consider the pull-back bundle $(\tau^*_{\mathcal{M}}(T\mathcal{M}), \mathrm{pr}_1, T\mathcal{M})$ in Eq.(2.30), where the total space $\tau^*_{\mathcal{M}}(T\mathcal{M}) = T\mathcal{M} \times_{\mathcal{M}} T\mathcal{M}$ is the Whitney product in Eq.(2.29). The short exact sequence in Eq.(2.51) for the tangent bundle $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$ takes the particular form

$$0 \;\to\; VT\mathcal{M} \;\xrightarrow{\;\imath\;}\; TT\mathcal{M} \;\xrightarrow{\;\mathcal{S}_{\tau_{\mathcal{M}}}\;}\; \tau^*_{\mathcal{M}}(T\mathcal{M}) \;\to\; 0\;, \tag{3.36}$$

where $\mathcal{S}_{\tau_{\mathcal{M}}}(\xi) = (\tau_{T\mathcal{M}}(\xi), T\tau_{\mathcal{M}}(\xi))$. For every $\mathbf{v} \in T\mathcal{M}$, ver_v is an isomorphism of $T_x\mathcal{M}$ with the vertical subspace $V_vT_x\mathcal{M}$ of $T_vT_x\mathcal{M}$, where $\tau_{\mathcal{M}}(\mathbf{v}) = \mathbf{x}$. Thus, ver establishes an isomorphism between $T\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} = \tau^*_{\mathcal{M}}(T\mathcal{M})$ and $VT\mathcal{M}$. If a connection Γ is introduced on the bundle $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$, the iterated tangent bundle $TT\mathcal{M}$ is decomposed into the direct sum of two copies of $\tau^*_{\mathcal{M}}(T\mathcal{M})$, that is

$$\begin{aligned} TT\mathcal{M} &\simeq VT\mathcal{M} \oplus HT\mathcal{M} \simeq \tau^*_{\mathcal{M}}(T\mathcal{M}) \oplus \tau^*_{\mathcal{M}}(T\mathcal{M}) \\ &\simeq \tau^*_{\mathcal{M}}(T\mathcal{M} \oplus T\mathcal{M}), \end{aligned} \tag{3.37}$$

since $HT\mathcal{M} \simeq \Gamma(\tau^*_{\mathcal{M}}(T\mathcal{M}))$ and $\mathrm{ver}(\tau^*_{\mathcal{M}}(T\mathcal{M})) = VT\mathcal{M}$. The identification of the pull back bundle $\tau^*_{\mathcal{M}}(T\mathcal{M} \oplus T\mathcal{M})$ and the iterated bundle $TT\mathcal{M}$ is summarized in the diagram

$$\begin{array}{ccc} TT\mathcal{M} & \xrightarrow{\;\Xi_\Gamma\;} & T\mathcal{M} \oplus T\mathcal{M} \\ \downarrow \tau_{T\mathcal{M}} & & \downarrow \tau_{\mathcal{M}} \\ T\mathcal{M} & \xrightarrow{\;\tau_{\mathcal{M}}\;} & \mathcal{M} \end{array} \tag{3.38}$$

where the bundle morphism Ξ_Γ is explicitly given by

$$\Xi_\Gamma(\xi) = \left(\mathrm{pr}_2 \circ \mathrm{ver}^{-1} \circ (I - \Gamma)(\xi), \mathrm{pr}_2 \circ \mathcal{S}_{\tau_{\mathcal{M}}}(\xi)\right). \tag{3.39}$$

For the holonomic lift operator $\Gamma_{\mathbf{J}} = dx^a \otimes \left(\partial/\partial x^a + \left(\partial v^b/\partial x^a\right)\left(\partial/\partial v^b\right)\right)$ in Eq.(2.69), we have

$$\Xi_{\Gamma_{\mathbf{J}}} : \xi^a \frac{\partial}{\partial x^a} + \bar{\xi}^a \frac{\partial}{\partial v^a} \rightarrow \left(\left(\bar{\xi}^a - \frac{\partial v^a}{\partial x^b}\xi^b\right)\frac{\partial}{\partial x^a}, \xi^a \frac{\partial}{\partial x^a}\right). \tag{3.40}$$

For an alternative exposition of this decomposition we refer to [72] and the references therein.

Assume that a connection Γ on $\tau_{\mathcal{M}} : T\mathcal{M} \rightarrow \mathcal{M}$ is defined. Then, we obtain an invariant way of defining the dual space $V^*T\mathcal{M}$ of the vertical bundle $VT\mathcal{M}$. The linear algebraic dual of the map $\mathrm{ver}_v : T_x\mathcal{M} \rightarrow V_vT\mathcal{M}$ is $\mathrm{ver}_v^* : V_v^*T\mathcal{M} \rightarrow T_x^*\mathcal{M}$. We define the mapping

$$\begin{aligned} \mathrm{ver}^* \quad &: \quad V^*T\mathcal{M} \rightarrow T\mathcal{M}\times_{\mathcal{M}}T^*\mathcal{M} \\ &: \quad \lambda \rightarrow \left(\pi_{T\mathcal{M}}\lambda = \mathbf{v}, \mathrm{ver}_v^* \lambda\right), \end{aligned} \tag{3.41}$$

which may be considered as the dual of ver. The image space $T\mathcal{M}\times_{\mathcal{M}}T^*\mathcal{M}$ of ver^* is the Whitney product which is the total space of the pull-back bundle $\left(\tau_{\mathcal{M}}^*\left(T^*\mathcal{M}\right), \mathrm{pr}_1, T\mathcal{M}\right)$ in Eq.(2.32). ver^* identifies $V^*T\mathcal{M}$ with $T\mathcal{M}\times_{\mathcal{M}}T^*\mathcal{M} = \tau_{\mathcal{M}}^*\left(T^*\mathcal{M}\right)$. After the identification $\tau_{\mathcal{M}}^*\left(T^*\mathcal{M}\right) \simeq \left(\tau_{\mathcal{M}}^*\left(T\mathcal{M}\right)\right)^* = H^*T^*\mathcal{M}$ in Eq.(2.65) the dual of the sequence in Eq.(3.36) takes the form

$$0 \rightarrow \tau_{\mathcal{M}}^*\left(T^*\mathcal{M}\right) \xrightarrow{\mathcal{S}_{\tau_{\mathcal{M}}}^*} T^*T\mathcal{M} \xrightarrow{s} \tau_{\mathcal{M}}^*\left(T^*\mathcal{M}\right) \rightarrow 0 , \tag{3.42}$$

where $\mathcal{S}_{\tau_{\mathcal{M}}}^*$ is the dual of $\mathcal{S}_{\tau_{\mathcal{M}}}$ and s is a surjection. Thus, the cotangent bundle $T^*T\mathcal{M}$ is decomposed as

$$\begin{aligned} T^*T\mathcal{M} \quad &\simeq \quad V^*T^*\mathcal{M} \oplus H^*T^*\mathcal{M} \simeq \tau_{\mathcal{M}}^*\left(T^*\mathcal{M}\right) \oplus \tau_{\mathcal{M}}^*\left(T^*\mathcal{M}\right) \\ &\simeq \quad \tau_{\mathcal{M}}^*\left(T^*\mathcal{M} \oplus T^*\mathcal{M}\right) \end{aligned} \tag{3.43}$$

and we have the commutative diagram

$$\begin{array}{ccc} T^*T\mathcal{M} & \xrightarrow{\Theta_\Gamma} & T^*\mathcal{M} \oplus T^*\mathcal{M} \\ \Big\downarrow \tau_{T^*\mathcal{M}} & & \Big\downarrow \pi_{\mathcal{M}} \\ T\mathcal{M} & \xrightarrow{\tau_{\mathcal{M}}} & \mathcal{M} \end{array} \tag{3.44}$$

where Θ_Γ is a bundle morphism given explicitly by

$$\Theta_\Gamma(\lambda) = \left(\mathrm{pr}_2 \circ \mathrm{ver}^* \circ (I - \Gamma)(\lambda), \mathrm{pr}_2 \circ \mu_{\tau_{\mathcal{M}}}^{-1} \circ \Gamma(\lambda)\right), \tag{3.45}$$

with $\mu_{\tau_{\mathcal{M}}}^{-1} : T^*T\mathcal{M} \to T\mathcal{M} \times_{\mathcal{M}} T^*\mathcal{M}$ being the inverse of the mapping $\mu_{\tau_{\mathcal{M}}}$ obtained from the diagram in Eq.(2.54) by replacing $\mathcal{E}$ with $T\mathcal{M}$. In particular, for the case $\Gamma_{\mathbf{J}} = dx^a \otimes \left(\partial/\partial x^a + \left(\partial v^b/\partial x^a\right)\left(\partial/\partial v^b\right)\right)$, we have

$$\Theta_{\Gamma_{\mathbf{J}}} : \left(\lambda_a dx^a + \bar{\lambda}_a dv^a\right) \to \left(\left(\lambda_a + \frac{\partial v^b}{\partial x^a}\bar{\lambda}_b\right) dx^a, \bar{\lambda}_a dx^a\right). \tag{3.46}$$

3.2.3. Vertical Lifts of Covectors

Consider the cotangent lift $T^*\pi_{\mathcal{M}} : T^*\mathcal{M} \to T^*T^*\mathcal{M}$ of the projection $\pi_{\mathcal{M}} : T^*\mathcal{M} \to \mathcal{M}$ and recall the musical isomorphism $\Omega^\sharp_{T^*\mathcal{M}} : T^*T^*\mathcal{M} \to TT^*\mathcal{M}$ associated with the symplectic two-form $\Omega_{T^*\mathcal{M}}$ on the cotangent bundle $T^*\mathcal{M}$. We define **Euler vector field**

$$\mathcal{X}_E : T^*\mathcal{M} \to TT^*\mathcal{M} : \mathbf{z} \to \Omega^\sharp_{T^*\mathcal{M}} \circ T^*\pi_{\mathcal{M}}(\mathbf{z}), \tag{3.47}$$

which is given diagrammatically

$$\begin{array}{ccc} T^*T^*\mathcal{M} & \xrightarrow{\Omega^\sharp_{T^*\mathcal{M}}} & TT^*\mathcal{M} \\ & \nwarrow T^*\pi_{\mathcal{M}} & \uparrow \mathcal{X}_E \\ & & T^*\mathcal{M}. \end{array} \tag{3.48}$$

$\mathcal{X}_E$ is a vertical vector field, that is, $\operatorname{Im}(\mathcal{X}_E) \subset \ker(T\pi_{\mathcal{M}})$. Indeed,

$$\begin{aligned} \langle \mathbf{z}, T\pi_{\mathcal{M}} \circ \mathcal{X}_E(\mathbf{z})\rangle &= \left\langle T^*\pi_{\mathcal{M}}(\mathbf{z}), \Omega^\sharp_{T^*\mathcal{M}} \circ T^*\pi_{\mathcal{M}}(\mathbf{z})\right\rangle \\ &= \Omega_{T^*\mathcal{M}}(T^*\pi_{\mathcal{M}}(\mathbf{z}), T^*\pi_{\mathcal{M}}(\mathbf{z})) = 0, \end{aligned} \tag{3.49}$$

$\forall \mathbf{z} \in T^*\mathcal{M}$, where we use the skew-symmetry property of the symplectic form $\Omega_{T^*\mathcal{M}}$. Euler vector field is the unique field satisfying the following equalities

$$i_{\mathcal{X}_E}\Omega_{T^*\mathcal{M}} = \theta_{T^*\mathcal{M}}, \quad \mathcal{L}_{\mathcal{X}_E}\Omega_{T^*\mathcal{M}} = -\Omega_{T^*\mathcal{M}} \text{ and } \mathcal{L}_{\mathcal{X}_E}\theta_{T^*\mathcal{M}} = -\theta_{T^*\mathcal{M}}, \tag{3.50}$$

where $i_{\mathcal{X}_E}$ and $\mathcal{L}_{\mathcal{X}_E}$ are interior product and Lie derivative operators [4]. In coordinates $\mathbf{z} = (x^a, y_b)$, Euler vector field is computed as $\mathcal{X}_E = -y_a\partial/\partial y_a$. Its divergence is

$$\operatorname{div}_{\mu_\Omega} \mathcal{X}_E = -\dim(\mathcal{M}),$$

where μ_Ω is the symplectic volume on $T^*\mathcal{M}$ and $\dim(\mathcal{M})$ is the dimension of $\mathcal{M}$.

Let α be a one-form on $\mathcal{M}$. The **vertical lift**

$$\alpha^v = \mathcal{X}_E \circ \alpha \circ \pi_{\mathcal{M}} : T^*\mathcal{M} \to TT^*\mathcal{M} \tag{3.51}$$

of the one-from α is a vertical vector field on $T^*\mathcal{M}$. In coordinates, the vertical lift of the one-form $\alpha = \alpha_a dx^a$ is $\alpha^v = -\alpha_a\partial/\partial y_a$. If $\{X_1, X_2, .., X_m\}$ is a basis for the space of vector fields on $\mathcal{M}$ and $\{\alpha_1, \alpha_2, .., \alpha_m\}$ is a basis for the space of one-forms then

$$\{X_1^{c*}, X_2^{c*}, .., X_m^{c*}, \alpha_1^v, \alpha_2^v, .., \alpha_m^v\} \tag{3.52}$$

forms a basis for the vector fields on $T^*\mathcal{M}$. The Jacobi-Lie bracket

$$[X^{c*}, \alpha^v] = (\mathcal{L}_X \alpha)^v \tag{3.53}$$

of a complete cotangent lift and a vertical lift is a vertical lift [69]. The following lemma establishes the link between the vertical lifts of one-forms and vertical representatives of complete cotangent lifts of vector fields.

Lemma 3.4. Let $\alpha(\mathbf{x}) = y_a(\mathbf{x})dx^a$ be a one-form, then

$$(\mathcal{L}_X(y_a dx^a))^v = VX^{c*}(x^a, y_a)\,. \tag{3.54}$$

For any function $f \in \mathcal{F}(\mathcal{M})$, $(df)^v : T^*\mathcal{M} \to TT^*\mathcal{M}$ is a Hamiltonian vector field with respect to the canonical symplectic two-form $\Omega_{T^*\mathcal{M}}$ for the Hamiltonian function $\hat{f} = f \circ \pi_{\mathcal{M}} \in \mathcal{F}(T^*\mathcal{M})$, we actually have

$$(d\mathcal{F}(\mathcal{M}))^v \simeq VT^*\mathcal{M} \cap \mathfrak{X}_{ham}(T^*\mathcal{M})\,. \tag{3.55}$$

The Jacobi-Lie bracket of two vector fields α^v and β^v obtained from the one-forms α and β is zero, therefore we have the commutative subalgebra of the algebra $\mathfrak{X}_{ham}(T^*\mathcal{M})$ of all Hamiltonian vector fields.

3.2.4. Decompositions of $TT^*\mathcal{M}$ and $T^*T^*\mathcal{M}$

Consider the pull-back bundle $\pi^*_{\mathcal{M}}(T^*\mathcal{M}) = T^*\mathcal{M}\times_{\mathcal{M}}T^*\mathcal{M}$ in Eq.(2.35). The following commutative diagram defines a mapping χ from $\pi^*_{\mathcal{M}}(T^*\mathcal{M})$ to $VT^*\mathcal{M}$,

$$\begin{array}{ccc} \pi^*_{\mathcal{M}}(T^*\mathcal{M}) = T^*\mathcal{M}\times_{\mathcal{M}}T^*\mathcal{M} & \xrightarrow{\mathrm{pr}_2} & T^*\mathcal{M} \\ \mathrm{pr}_1 \downarrow & \chi \searrow & \downarrow -\mathcal{X}_E \\ T^*\mathcal{M} & \xleftarrow{\tau_{T^*\mathcal{M}}} & VT^*\mathcal{M} \end{array} \tag{3.56}$$

where $\mathcal{X}_E$ is the Euler vector field in Eq.(3.47). Using χ, we define

$$\chi_z : T^*\mathcal{M} \to VT^*\mathcal{M} : \mathbf{w} \to \chi(\mathbf{z}, \mathbf{w}) \tag{3.57}$$

by fixing an element $\mathbf{z} \in T^*\mathcal{M}$. In coordinates, for $\mathbf{z} = (x^a, y_b)$ and $\mathbf{w} = (x^a, w_b)$,

$$\chi(\mathbf{z}, \mathbf{w}) = \chi(x^a; y_b, w_b) = (x^a, y_b, 0, w_b) \tag{3.58}$$

The inverse of the mapping χ is

$$\chi^{-1} : TT^*\mathcal{M} \to T^*\mathcal{M}\times_{\mathcal{M}}T^*\mathcal{M} : \xi \to \left(\tau_{T^*\mathcal{M}}(\xi), -\pi_{T^*\mathcal{M}} \circ \Omega^\flat_{T^*\mathcal{M}}(\xi)\right). \tag{3.59}$$

χ identifies the Whitney product $T^*\mathcal{M}\times_{\mathcal{M}}T^*\mathcal{M}$ with the space of vertical vectors $VT^*\mathcal{M}$.

The exact sequence in Eq.(2.51), with the choice $\mathcal{E} = T^*\mathcal{M}$, takes the form

$$0 \to VT^*\mathcal{M} \xrightarrow{\chi} TT^*\mathcal{M} \xrightarrow{\mathcal{S}_{\pi_{\mathcal{M}}}} \pi^*_{\mathcal{M}}(T\mathcal{M}) \to 0 \tag{3.60}$$

where $\mathcal{S}_{\pi_{\mathcal{M}}}(\xi) = (\tau_{T^*\mathcal{M}}(\xi), T\pi_{\mathcal{M}}(\xi))$. With a connection $\Gamma : \pi^*_{\mathcal{M}}(T\mathcal{M}) \to TT^*\mathcal{M}$, we have the decomposition of $TT^*\mathcal{M}$ as

$$\begin{aligned} TT^*\mathcal{M} &\simeq VT^*\mathcal{M}\oplus HT^*\mathcal{M}\simeq \pi^*_{\mathcal{M}}\left(T^*\mathcal{M}\right)\oplus \pi^*_{\mathcal{M}}\left(T\mathcal{M}\right) \\ &\simeq \pi^*_{\mathcal{M}}\left(T^*\mathcal{M}\oplus T\mathcal{M}\right), \end{aligned} \tag{3.61}$$

where we used the identification $VT^*\mathcal{M}\simeq \pi^*_{\mathcal{M}}\left(T^*\mathcal{M}\right)$ and $HT^*\mathcal{M}\simeq \pi^*_{\mathcal{M}}\left(T\mathcal{M}\right)$. The following diagram

$$\begin{array}{ccc} TT^*\mathcal{M} & \xrightarrow{\Upsilon_\Gamma} & T^*\mathcal{M}\oplus T\mathcal{M} \\ \Big\downarrow \tau_{T^*\mathcal{M}} & & \Big\downarrow \pi_{\mathcal{M}} \\ T^*\mathcal{M} & \xrightarrow{\pi_{\mathcal{M}}} & \mathcal{M} \end{array} \tag{3.62}$$

defines the bundle morphism

$$\Upsilon_\Gamma\left(\xi\right)=\left(\mathrm{pr}_2\circ\chi^{-1}\circ\left(I-\Gamma\right)\left(\xi\right),\mathrm{pr}_2\circ S_{\pi_{\mathcal{M}}}\left(\xi\right)\right).$$

In particular, for $\Gamma_{\mathbf{J}}=dx^a\otimes\left(\partial/\partial x^a+\left(\partial y_b/\partial x^a\right)\partial/\partial y_b\right)$, we have

$$\Upsilon_{\Gamma_{\mathbf{J}}}\left(\xi^a\frac{\partial}{\partial x^a}+\xi_a\frac{\partial}{\partial y_a}\right)=\left(\left(\xi_b-\xi^a\frac{\partial y_a}{\partial x^b}\right)dx^b,\xi^a\frac{\partial}{\partial x^a}\right). \tag{3.63}$$

Using the connection $\Gamma:\pi^*_{\mathcal{M}}\left(T\mathcal{M}\right)\to TT^*\mathcal{M}$, one defines the dual bundle $V^*T^*\mathcal{M}\subset T^*T^*\mathcal{M}$ and hence the dual $\chi^*_z: V^*T^*\mathcal{M}\to T\mathcal{M}$ of the mapping χ_z. First we define the map

$$\chi^*: V^*T^*\mathcal{M}\to\pi^*_{\mathcal{M}}\left(T\mathcal{M}\right)=T^*\mathcal{M}\times_{\mathcal{M}}T\mathcal{M}:\lambda\to\left(\pi_{T^*\mathcal{M}}\left(\lambda\right)=\mathbf{z},\chi^*_z\left(\lambda\right)\right) \tag{3.64}$$

to identify the bundle of vertical one-forms $V^*T^*\mathcal{M}$ and the pull-back bundle $\pi^*_{\mathcal{M}}\left(T\mathcal{M}\right)$. As discussed in the previous section, the decomposition of $TT^*\mathcal{M}$ simultaneously decompose the cotangent bundle $T^*T^*\mathcal{M}$. Formally,

$$\begin{aligned} T^*T^*\mathcal{M} &\simeq V^*T^*\mathcal{M} \oplus H^*T^*\mathcal{M} \simeq \pi^*_{\mathcal{M}}(T\mathcal{M}) \oplus \pi^*_{\mathcal{M}}(T^*\mathcal{M}) \\ &\simeq \pi^*_{\mathcal{M}}(T\mathcal{M} \oplus T^*\mathcal{M}), \end{aligned} \tag{3.65}$$

where we used the identification induced by χ^* at the first term and the one to one mapping $\mu_{\pi_{\mathcal{M}}} : \pi^*_{\mathcal{M}}(T^*\mathcal{M}) \to H^*T^*\mathcal{M}$ in Eq.(2.54) for the second term at the right hand side. We have the following commutative diagram

$$\begin{array}{ccc} T^*T^*\mathcal{M} & \xrightarrow{\Delta_\Gamma} & T\mathcal{M} \oplus T^*\mathcal{M} \\ \downarrow \pi_{T^*\mathcal{M}} & & \downarrow \pi \\ T^*\mathcal{M} & \xrightarrow{\pi_{\mathcal{M}}} & \mathcal{M} \end{array}$$

where the bundle morphism is

$$\Delta_\Gamma(\lambda) = \left(\mathrm{pr}_2 \circ \chi^* \circ (I - \Gamma)(\lambda), \mathrm{pr}_2 \circ \mu^{-1}_{\pi_{\mathcal{M}}} \circ \Gamma(\lambda)\right)$$

and $\mu^{-1}_{\pi_{\mathcal{M}}} : H^*T^*\mathcal{M} \to \pi^*_{\mathcal{M}}(T^*\mathcal{M})$ is the inverse of $\mu_{\pi_{\mathcal{M}}}$. For $\Gamma_{\mathbf{J}}$, we have the following decomposition

$$\Delta_{\Gamma_{\mathbf{J}}} : \lambda_a dx^a + \lambda^a dy_a \to \left(\lambda^a \frac{\partial}{\partial x^a}, \left(\lambda_b + \lambda^a \frac{\partial y_a}{\partial x^b}\right) dx^b\right). \tag{3.66}$$

4. THEORY OF SYMMETRY AND REDUCTION

4.1. LIE GROUPS AND LIE ALGEBRAS

A **Lie group** G is a C^∞ manifold having a group structure compatible with its manifold structure, in the sense that, the group multiplication and the inversion

$$\varsigma : G \times G \to G : (g,h) \to gh \quad \text{and} \quad \imath : G \to G : g \to g^{-1} \tag{4.1}$$

are C^∞ maps [73]. The induced maps

$$L_g : G \to G : h \to \varsigma(g,h) \quad \text{and} \quad R_h : G \to G : g \to \varsigma(g,h) \tag{4.2}$$

from the group multiplication are called **left and right translation maps**, respectively. A differentiable map φ between two Lie groups, say (G_1, ς_1) and (G_2, ς_2), is called a **Lie group homomorphism**, if it respects the group operations, that is

$$\varphi \circ \varsigma_1 (g,h) = \varsigma_2 (\varphi(g), \varphi(h)), \quad \forall g,h \in G_1.$$

If φ is a bijection, then it is called a **Lie group isomorphism**. A **(linear) representation** of a Lie group G is a Lie group homomorphism $G \to Gl(V)$ for some **representation space** V. Here $Gl(V)$ denotes the group of all invertible linear mappings on the vector space V [41].

A **Lie algebra** $\mathfrak{g}$ is a vector space with a skew-symmetric $\mathbb{R}-$bilinear operation

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \tag{4.3}$$

called the **Lie bracket** satisfying the **Jacobi identity**

$$[\vartheta, [\eta, \zeta]] + [\eta, [\zeta, \vartheta]] + [\zeta, [\vartheta, \eta]] = 0, \quad \forall \vartheta, \eta, \zeta \in \mathfrak{g}. \tag{4.4}$$

A linear operator φ from a Lie algebra $(\mathfrak{g}_1, [\ ,\ ]_1)$ to a Lie algebra $(\mathfrak{g}_2, [\ ,\ ]_2)$ is called a **Lie algebra homomorphism** if

$$\varphi([\zeta, \eta]_1) = [\varphi(\zeta), \varphi(\eta)]_2, \quad \forall \zeta, \eta \in \mathfrak{g}_1. \tag{4.5}$$

A Lie algebra homomorphism is called a **Lie algebra isomorphism** if it is bijective.

In finite dimensions, a Lie algebra $\mathfrak{g}$ with basis $\{\eta^1, ..., \eta^m\}$ gives rise to the **structure constants** c_k^{ij} $(i, j, k = 1, ..., \dim \mathfrak{g})$ obtained through the Lie bracket of the basis elements

$$[\eta^i, \eta^j] = c_k^{ij} \eta^k. \tag{4.6}$$

We have the following properties of the structure constants

$$c_k^{ij} = -c_k^{ji} \quad \text{and} \quad c_k^{ij} c_m^{kl} + c_k^{li} c_m^{kj} + c_k^{jl} c_m^{ki} = 0 \tag{4.7}$$

as a manifestation of the skew-symmetry property of $[\ ,\ ]$ and the Jacobi identity.

Example 1 The space of all vector fields $\mathfrak{X}(\mathcal{M})$ on a manifold $\mathcal{M}$ with the Jacobi-Lie bracket is a Lie algebra. If $\mathcal{M}$ is a symplectic manifold, the space of Hamiltonian vector fields $\mathfrak{X}_{ham}(\mathcal{M})$ is a Lie algebra since

$$[X_h, X_f] = -X_{\{h,f\}}.$$

Similarly, the space of all divergence free vector fields $\mathfrak{X}_{div}(\mathcal{M})$ on an orientable manifold $\mathcal{M}$ has the structure of a Lie algebra.

We call a vector field $X : G \to TG$ on a Lie group G to be **left invariant**, if

$$T_h L_g \cdot X(h) = X(L_g h), \quad \forall g, h \in G, \tag{4.8}$$

where $T_h L_g$ is the tangent mapping of the left translation L_g at $h \in G$. The condition in Eq.(4.8) can also be written as $(L_g)_* X = X$, $\forall g \in G$. $\mathfrak{X}_L(G)$ will denote the set of all left-invariant vector fields on G. Since, the push forward operation is natural, that means, for $X, Y \in \mathfrak{X}_L(G)$

$$(L_g)_* [X, Y] = [(L_g)_* X, (L_g)_* Y] = [X, Y], \quad \forall g \in G, \tag{4.9}$$

$\mathfrak{X}_L(G)$ is a Lie algebra. There is a one to one correspondence between the space of left invariant vector fields and the tangent space T_eG at the identity element $e \in G$ given by

$$T_eG \to \mathfrak{X}_L(G) : \eta \to X_\eta, \quad X_\eta(g) = T_eL_g(\eta). \tag{4.10}$$

Identification of $\mathfrak{X}_L(G)$ and T_eG enables us to define a bracket on T_eG,

$$[\zeta, \eta] := [X_{\zeta,} X_\eta](e), \tag{4.11}$$

for $\zeta, \eta \in T_eG$ and $X_{\zeta,} X_\eta \in \mathfrak{X}_L(G)$. The **Lie algebra of a Lie group** is T_eG with the bracket defined in Eq. (4.11), and is denoted as $\mathrm{Lie}(G)$.

A **right invariant vector field** X on G satisfies $(R_g)_* X = X$, $\forall g \in G$. The space of right invariant vector fields $\mathfrak{X}_R(G)$ has the structure of a Lie algebra. The tangent mapping $T\imath : TG \to TG$ of the inversion map $\imath : g \to g^{-1}$ is a Lie algebra isomorphism between $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$ [42]. We may define a Lie algebra structure on T_eG induced from $\mathfrak{X}_R(G)$ as well. At the identity, the tangent mapping

$$T_e\imath : T_eG \to T_eG : \eta \to -\eta \tag{4.12}$$

manifests that, the Lie algebra structure on T_eG induced from the left invariant vector fields and the Lie algebra structure on T_eG induced from the right invariant vector fields are anti-isomorphic. In this thesis, Lie algebra of a Lie group is taken as the vector space T_eG with the Lie bracket structure obtained from the left invariant vector fields.

Let X_η be a left invariant vector field as defined in Eq.(4.10). There is a unique integral curve $\gamma_\eta : \mathbb{R} \to G$ of X_η passing through $e \in G$. The **exponential map** takes an element η

of a Lie algebra to an element of the underlying Lie group G and is given by

$$\exp : \mathfrak{g} \to G : \eta \to \gamma_\eta(1). \tag{4.13}$$

4.1.1. Actions of Lie Groups on Manifolds

The **left action** of a Lie group G on a manifold $\mathcal{M}$ is a smooth mapping

$$\Phi : G \times \mathcal{M} \to \mathcal{M} \tag{4.14}$$

such that $\Phi(e, \mathbf{x}) = \mathbf{x}$, and $\Phi(g, \Phi(h, \mathbf{x})) = \Phi(gh, \mathbf{x})$, $\forall g, h \in G$, $\forall \mathbf{x} \in \mathcal{M}$.

We call an action $\Phi : \mathcal{M} \times G \to \mathcal{M}$ a right action if

$$\Phi(\mathbf{x}, e) = \mathbf{x} \quad \text{and} \quad \Phi(\Phi(h, \mathbf{x}), g) = \Phi(\mathbf{x}, hg), \tag{4.15}$$

$\forall g, h \in G$ and $\forall \mathbf{x} \in \mathcal{M}$. In this section, we mean by an action to be a left action and denote it by $\Phi(g, \mathbf{x}) = g \cdot \mathbf{x}$. By fixing the first and second arguments of Φ, we define the following mappings

$$\Phi_g : \mathcal{M} \to \mathcal{M} : \mathbf{x} \to \Phi(g, \mathbf{x}) \quad \text{and} \quad \Phi_x : G \to \mathcal{M} : g \to \Phi(g, \mathbf{x}). \tag{4.16}$$

An action is called a **transitive action** if for every $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{M}$ there is a $g \in G$ such that $g \cdot \mathbf{x}_1 = \mathbf{x}_2$. If $\Phi_g = \mathrm{id}_{\mathcal{M}}$ implies $g = e$; that is, $g \to \Phi_g$ is one-to-one, then Φ is said to be a **faithful** (or **effective**) **action**. An action is **free** if it has no fixed points, that is, $\Phi_g(\mathbf{x}) = \mathbf{x}$ implies $g = e$, or alternatively, if for each $\mathbf{x} \in \mathcal{M}$, Φ_x is one-to-one.

Under the action of G, **orbit of a point** $\mathbf{x} \in \mathcal{M}$ is

$$\mathrm{Orb}(\mathbf{x}) = \{\Phi_g(\mathbf{x}) : g \in G\} \subset \mathcal{M}. \tag{4.17}$$

If the action is free and proper so that inverse images of compact sets are compact under the action Φ, then the set of orbits $\mathcal{M}/G = \{\mathrm{Orb}(\mathbf{x}) : \mathbf{x} \in \mathcal{M}\}$ has a manifold structure [3]. We define a mapping $\pi : \mathcal{M} \to \mathcal{M}/G$ which takes an element of $\mathcal{M}$ to its orbit. The quadruple $(\mathcal{M}, \pi, \mathcal{M}/G, G)$ is a **principle fiber bundle**, where $\mathcal{M}$ is the **total space**, $\mathcal{M}/G$ is the **base space**, G is the **structure group** and π is the **canonical projection** [22, 42].

The **isotropy** (or **stabilizer**) **group**

$$G_x = \{g \in G : \Phi_g(\mathbf{x}) = \mathbf{x}\} \subset G \tag{4.18}$$

of Φ at $\mathbf{x} \in \mathcal{M}$ is a Lie subgroup of G. The mapping

$$\check{\Phi}_x : G/G_x \to \mathrm{Orb}(\mathbf{x}) : gG_x \to \Phi_x\left(gG_x\right) = g \cdot \mathbf{x} \tag{4.19}$$

is a bijection between the coset space G/G_x and $\mathrm{Orb}(\mathbf{x})$, that means $G/G_x \simeq \mathrm{Orb}(\mathbf{x})$ [55].

The **inner automorphism**

$$I : G \times G \to G : (g, h) \to g^{-1}hg \tag{4.20}$$

is an action of G on itself. The mapping $I_g : G \to G : h \to I\left(g, h\right)$ is a Lie group isomorphism on G. The **adjoint action**

$$Ad : G \times \mathfrak{g} \to \mathfrak{g} : (g, \eta) \to T_e\left(R_{g^{-1}} \circ L_g\right)(\eta)\,, \tag{4.21}$$

is an action of a Lie group G on its Lie algebra $\mathfrak{g}$. The induced mapping $Ad_g : \mathfrak{g} \to \mathfrak{g} : \eta \to Ad\left(g, \eta\right)$ is a Lie algebra homomorphism, that is,

$$Ad_g\left[\zeta, \eta\right] = \left[Ad_g\left(\zeta\right), Ad_g\left(\eta\right)\right], \quad \forall \zeta, \eta \in \mathfrak{g} \quad \text{and} \quad \forall g \in G. \tag{4.22}$$

Observe that Ad_g is the differential of I_g at the identity, that is $Ad_g := T_e I_g$. Adjoint action satisfies the following identities

$$Ad_e = Id_{\mathfrak{g}}, \quad Ad_g \circ Ad_h = Ad_{gh}, \quad (Ad_g)^{-1} = Ad_{g^{-1}}, \quad \forall g, h \in G. \tag{4.23}$$

Let g^t be a curve in G passing through the identity element at $t = 0$ in the direction $\eta \in \mathfrak{g}$, that means $g^0 = e$ and $dg^t/dt\big|_{t=0} = \eta$. From the differentiation of the adjoint action, we define

$$ad_\eta \zeta = \frac{d}{dt}\, Ad_{g^t}\zeta|_{t=0}\,. \tag{4.24}$$

Observe that *ad* is the action of $\mathfrak{g}$ onto itself and it is equal to the Lie algebra bracket on $\mathrm{Lie}\,(G)$, that is

$$ad_\eta \zeta = [\eta, \zeta]\,. \tag{4.25}$$

The **coadjoint action** is the mapping

$$Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^* : (g, \alpha) \to \left(Ad_{g^{-1}}\right)^* \alpha, \tag{4.26}$$

where $\left(Ad_{g^{-1}}\right)^*$ is the linear algebraic dual of $Ad_{g^{-1}}$ [11]. The coadjoint representation ad^* of $\mathfrak{g}$ on its dual $\mathfrak{g}^*$ is defined by means of the linear algebraic dual of ad_η, that is

$$\left\langle ad^*_\eta \alpha, \zeta \right\rangle = \langle \alpha, ad_\eta \zeta\rangle = \langle \alpha, [\eta, \zeta]\rangle\,, \quad \forall \zeta, \eta \in \mathfrak{g} \quad \text{and} \quad \forall \alpha \in \mathfrak{g}^*. \tag{4.27}$$

Let Φ be an action of a group G on a manifold $\mathcal{M}$. The **left tangent lift of** Φ to $T\mathcal{M}$ is a left action defined by

$$T_L\Phi : G \times T\mathcal{M} \to T\mathcal{M} : (g, \mathbf{v}) \to T_x\Phi_g(\mathbf{v})\,, \quad \mathbf{v} \in T_x\mathcal{M} \tag{4.28}$$

The **left cotangent lift of** Φ to $T^*\mathcal{M}$ is a left action defined by

$$T_L^*\Phi : G \times T^*\mathcal{M} \to T^*\mathcal{M} : (g, \mathbf{z}) \to T^*_{\Phi(g,x)}\Phi_{g^{-1}}(\mathbf{z}), \quad \mathbf{z} \in T^*_x\mathcal{M}. \tag{4.29}$$

$T^*\Phi_{g^{-1}}$ is the cotangent lift of the diffeomorphism $\Phi_{g^{-1}}$.

4.1.2. Infinitesimal Generators

Let $\Phi : G \times \mathcal{M} \to \mathcal{M}$ be an action of a Lie group G to a manifold $\mathcal{M}$. For every element η of the Lie algebra $\mathfrak{g} = T_eG$ we construct a vector field

$$\eta_{\mathcal{M}}(\mathbf{x}) = \frac{d}{dt}\Phi(\exp(t\eta), \mathbf{x})|_{t=0} \tag{4.30}$$

on $\mathcal{M}$, called **infinitesimal generator (**or **fundamental vector field)** corresponding to η. $\eta_{\mathcal{M}}$ is the vector field generating the flow $\Phi_{\exp t\eta} : \mathcal{M} \to \mathcal{M}$ hence, tangent to the orbits in $\mathcal{M}/G$, or in other words, the tangent space of an orbit $\mathrm{Orb}(\mathbf{x})$ at $\tilde{\mathbf{x}} \in \mathrm{Orb}(\mathbf{x})$ is

$$T_{\tilde{x}}\,\mathrm{Orb}(\mathbf{x}) = \{\eta_{\mathcal{M}}(\tilde{\mathbf{x}}) : \eta \in \mathfrak{g}\}. \tag{4.31}$$

$\eta_{\mathcal{M}}$ is a vertical vector field with respect to the smooth bundle structure $(\mathcal{M}, \pi, \mathcal{M}/G)$ [41,42].

The **infinitesimal action** of a Lie algebra on $\mathcal{M}$ is defined by

$$\mathfrak{g} \times \mathcal{M} \to T\mathcal{M} : (\eta, \mathbf{x}) \to \eta_{\mathcal{M}}(\mathbf{x}). \tag{4.32}$$

The mapping $\mathfrak{g} \to \mathfrak{X}(\mathcal{M}) : \eta \to \eta_{\mathcal{M}}$ is an anti-homomorphism, that is,

$$[\zeta_{\mathcal{M}}, \eta_{\mathcal{M}}]_{JL} = -[\zeta, \eta]_{\mathfrak{g}}, \tag{4.33}$$

where $[\ ,\ ]_{JL}$ is the Jacobi Lie bracket of vector fields and the bracket at the right hand side is the Lie algebra bracket on $\mathfrak{g}$ [11].

The action Φ is said to be a **canonical action** if G acts on a symplectic manifold $(\mathcal{M}, \Omega_{\mathcal{M}})$ by canonical diffeomorphisms, that is, $\Phi_g^* \Omega_{\mathcal{M}} = \Omega_{\mathcal{M}}$, $\forall g \in G$. The action of a Lie group G on a Poisson manifold $(\mathcal{M}, \{\ ,\ \})$ is called canonical if

$$\{k, f\} \circ \Phi_g = \{k \circ \Phi_g, f \circ \Phi_g\}, \tag{4.34}$$

for all $f, k \in \mathcal{F}(\mathcal{M})$ and $g \in G$.

4.1.3. Gauge Transformations

Let $\mathcal{F}(\mathcal{Q})$ be the additive group of functions on $\mathcal{Q}$ and $\mathcal{F}(\mathcal{Q})$ acts on $T^*\mathcal{Q}$ by momentum translations

$$\Phi : \mathcal{F}(\mathcal{Q}) \times T^*\mathcal{Q} \to \left(\phi, \left(q^i, p_i\right)\right) \to \left(q^i, p_i - \frac{\partial \phi}{\partial q^i}\right). \tag{4.35}$$

This action is the gauge invariance of canonical Hamiltonian dynamics. The infinitesimal generator

$$X_\phi(\mathbf{q}, \mathbf{p}) = -\frac{\partial \phi}{\partial q^i} \frac{\partial}{\partial p_i} \tag{4.36}$$

of the action Φ is a Hamiltonian vector field on $T^*\mathcal{Q}$ for the Hamiltonian function $\hat{\phi} = \phi \circ \pi_{\mathcal{Q}} \in \mathcal{F}(T^*\mathcal{Q})$. The Jacobi-Lie bracket of two such generators is zero, that means, they constitute a commutative Lie algebra.

$\mathcal{F}(\mathcal{Q})$ acts on $TT^*\mathcal{Q}$ by the tangent lift, in coordinates,

$$\begin{aligned} \Phi^c &: \mathcal{F}(\mathcal{Q}) \times TT^*\mathcal{Q} \to TT^*\mathcal{Q} \\ &: \left(\phi, \left(q^i, p_i; \dot{q}^i, \dot{p}_i\right)\right) \to \left(q^i, p_i - \frac{\partial \phi}{\partial q^i}; \dot{q}^i, -\dot{q}^j \frac{\partial \phi}{\partial q^j \partial q^i} + \dot{p}_i\right), \end{aligned} \tag{4.37}$$

where the infinitesimal generator

$$X_{\phi}^{c}\left(\mathbf{q},\mathbf{p};\dot{\mathbf{q}},\dot{\mathbf{p}}\right)=-\frac{\partial\phi}{\partial q^{i}}\frac{\partial}{\partial p_{i}}-\dot{q}^{j}\frac{\partial\phi}{\partial q^{j}\partial q^{i}}\frac{\partial}{\partial\dot{p}_{i}},\tag{4.38}$$

is the complete tangent lift of X_{ϕ}. X_{ϕ}^{c} is a Hamiltonian vector field with respect to the symplectic structure

$$\Omega_{TT^{*}\mathcal{Q}}=d\theta_{TT^{*}\mathcal{Q}}=d\dot{q}^{i}\wedge dp_{i}+dq^{i}\wedge d\dot{p}_{i}\tag{4.39}$$

for the Hamiltonian function $\tilde{H}=\dot{q}^{j}\partial\phi/\partial q^{j}\in\mathcal{F}\left(TT^{*}\mathcal{Q}\right)$, that is

$$i_{X_{\phi}^{c}}\Omega_{TT^{*}\mathcal{Q}}=d\tilde{H}.$$

Cartan's formula $\mathfrak{L}_{X_{\phi}^{c}}=di_{X_{\phi}^{c}}+i_{X_{\phi}^{c}}d$ gives

$$\mathfrak{L}_{X_{\phi}^{c}}\Omega_{TT^{*}\mathcal{Q}}=0,\tag{4.40}$$

which means that $\mathcal{F}\left(\mathcal{Q}\right)$ is also the gauge group of the Hamiltonian dynamics on $TT^{*}\mathcal{Q}$ with respect to the symplectic two-form $\Omega_{TT^{*}\mathcal{Q}}$.

The cotangent lift of the action Φ in Eq.(4.35) is

$$\begin{aligned}\Phi^{c*}\quad&:\quad\mathcal{F}\left(\mathcal{Q}\right)\times T^{*}T^{*}\mathcal{Q}\rightarrow T^{*}T^{*}\mathcal{Q}\\&:\quad\left(\phi,\left(q^{i},p_{i};\Pi_{i},\Pi^{i}\right)\right)\rightarrow\left(q^{i},p_{i}+\frac{\partial\phi}{\partial q^{i}};\Pi_{i}+\Pi^{j}\frac{\partial\phi}{\partial q^{j}\partial q^{i}},\Pi^{i}\right),\end{aligned}\tag{4.41}$$

whose infinitesimal generator

$$X_{\phi}^{c*}=\frac{\partial\phi}{\partial q^{i}}\frac{\partial}{\partial p_{i}}+\Pi^{j}\frac{\partial\phi}{\partial q^{j}\partial q^{i}}\frac{\partial}{\partial\Pi_{i}}\tag{4.42}$$

is a Hamiltonian vector field on $T^{*}T^{*}\mathcal{Q}$ for the Hamiltonian function $H=-\Pi^{i}\partial\phi/\partial q^{i}$ with respect to the canonical symplectic two-form

$$\Omega_{T^*T^*\mathcal{Q}} = dq^i \wedge d\Pi_i + dp_i \wedge d\Pi^i \tag{4.43}$$

on $T^*T^*\mathcal{Q}$. Hence, we find that $\mathcal{F}(\mathcal{Q})$ is also a gauge group of the canonical Hamiltonian dynamics on $T^*T^*\mathcal{Q}$.

4.1.4. Momentum Maps

Let $(\mathcal{P}, \{\ ,\ \})$ be a Poisson manifold and $\Phi : G \times \mathcal{P} \to \mathcal{P}$ be a canonical action of a Lie group G on $\mathcal{P}$. In this case, infinitesimal generators are locally Hamiltonian. We will assume that they are globally Hamiltonian, that is, there exists a globally defined function $J(\eta)$ on $\mathcal{P}$ such that

$$\eta_{\mathcal{P}} = X_{J(\eta)}. \tag{4.44}$$

Eq.(4.44) implies the existence of a mapping $J : \mathfrak{g} \to \mathcal{F}(\mathcal{P})$ and determines J up to an addition of a Casimir function and, for symplectic and connected manifolds, up to addition of a constant [11,63].

Proposition 4.1. Let f be a G invariant function on the Poisson manifold $\mathcal{P}$, then the function $J(\eta)$ is a constant of the motion for the dynamics generated by f.

Indeed,

$$\begin{aligned} \{f, J(\eta)\}(\mathbf{z}) &= df(\mathbf{z}) \cdot X_{J(\eta)}(\mathbf{z}) = df(\mathbf{z}) \cdot \frac{d}{dt} \Phi(\exp t\eta, \mathbf{z})|_{t=0} \\ &= \frac{d}{dt} f\left(\Phi(\exp t\eta, \mathbf{z})|_{t=0} \right) = \frac{d}{dt} f(\mathbf{z}) = 0, \end{aligned} \tag{4.45}$$

$\forall \mathbf{z} \in \mathcal{P}$ [19,32]. The map $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$ defined by

$$\langle \mathbf{J}(\mathbf{z}), \eta \rangle = J(\eta)(\mathbf{z}) \tag{4.46}$$

$\forall\eta \in \mathfrak{g}$ and $\forall\mathbf{z} \in \mathcal{P}$ is called a **momentum mapping** of the action Φ.

Example 2 The dual of any Lie algebra homomorphism is a momentum map.

Theorem 4.2. (Noether's Theorem) Let $\mathbf{J}$ be a momentum mapping for the canonical action Φ of G on $(\mathcal{M}, \Omega_{\mathcal{M}})$. Then $\mathbf{J}$ is a constant of the motion for any G invariant Hamiltonian function f, that is

$$\mathbf{J} \circ \phi_t = \mathbf{J} \tag{4.47}$$

where ϕ_t is the flow of f.

From $\{f, J(\eta)\}(\mathbf{x}) = 0$ we compute

$$\begin{aligned} 0 &= dJ(\eta)(\mathbf{x}) \cdot X_f(\mathbf{x}) = dJ(\eta)(\mathbf{x}) \cdot \frac{d}{dt} \phi_t(\mathbf{x})\big|_{t=0} \\ &= \frac{d}{dt} J(\eta) \circ \phi_t(\mathbf{x})\big|_{t=0} = \frac{d}{dt} \langle \mathbf{J} \circ \phi_t(\mathbf{x}), \xi \rangle\big|_{t=0} \end{aligned} \tag{4.48}$$

$\forall\mathbf{x} \in \mathcal{M}$, $\eta \in \mathfrak{g}$, and the result follows [11, 19]. We say that a momentum mapping $\mathbf{J}$ of an action Φ is an **equivariant momentum mapping**, if

$$\mathbf{J} \circ \Phi_g = Ad_g^* \circ \mathbf{J}, \quad \forall g \in G. \tag{4.49}$$

A canonical action is called a **Hamiltonian action** if $\mathbf{J}$ is equivariant.

Example 3 Let a Lie algebra acts on a manifold $\mathcal{M}$ with $\mathfrak{g} \to \mathfrak{X}(\mathcal{M}) : \xi \to \xi_{\mathcal{M}}$. The left cotangent lift of this action is $\mathfrak{g} \to \mathfrak{X}(\mathcal{M}) : \xi \to \xi_{\mathcal{M}}^{c*}$, where $\xi_{\mathcal{M}}^{c*}$ is the complete cotangent lift of $\xi_{\mathcal{M}}$. This action is a Hamiltonian action with the momentum mapping

$$\langle \mathbf{J}(\mathbf{z}), \xi \rangle = \langle \mathbf{z}, \xi_{\mathcal{M}}(\mathbf{x}) \rangle = J(\xi)(\mathbf{z}),$$

for $\mathbf{x} \in \mathcal{M}$ and $\mathbf{z} \in T_x^*\mathcal{M}$.

4.2. SYMPLECTIC AND POISSON REDUCTION

Let Φ be a canonical action of a Lie group G on a symplectic manifold $(\mathcal{M}, \Omega_{\mathcal{M}})$ and $\mathbf{J} : \mathcal{M} \to \mathfrak{g}^*$ be an equivariant momentum map for this action. We denote G_α the **isotropy group of** $\alpha \in \mathfrak{g}^*$ under the coadjoint action Ad^* of G, that is

$$G_\alpha = \left\{ g \in G : Ad^*_{g^{-1}} \alpha = \alpha \right\}. \tag{4.50}$$

G_α is a Lie group, being a closed subgroup of G [73]. Let α be a regular value of $\mathbf{J}$, that is, $\mathbf{J}^{-1}(\alpha)$ is a submanifold of $\mathcal{M}$. If $g \in G_\alpha$ and $\mathbf{x} \in \mathbf{J}^{-1}(\alpha)$ then from the equivariance we have

$$\mathbf{J} \circ \Phi_g (\mathbf{x}) = Ad^*_g \mathbf{J}(\mathbf{x}) = \mathbf{J}(\mathbf{x}) \tag{4.51}$$

which says $g \cdot \mathbf{z} \in \mathbf{J}^{-1}(\alpha)$. Thus $\mathbf{J}^{-1}(\alpha) \subset \mathcal{M}$ is an invariant set for dynamics. Hence we restrict the action of Lie group G_α on $\mathbf{J}^{-1}(\alpha)$, that is

$$G_\alpha \times \mathbf{J}^{-1}(\alpha) \to \mathbf{J}^{-1}(\alpha). \tag{4.52}$$

Assume this action be free and proper. Then, $\mathbf{J}^{-1}(\alpha)/G_\alpha = P_\alpha$ is a manifold, which is called the **reduced phase space,** with projection $\pi_\alpha : \mathbf{J}^{-1}(\alpha) \to \mathbf{J}^{-1}(\alpha)/G_\alpha$. Let $i : \mathbf{J}^{-1}(\alpha) \to \mathcal{M}$ be a natural injection. We have the symplectic reduction theorem [14,15,19].

Theorem 4.3. There is a unique symplectic structure Ω_α on P_α satisfying

$$i^*\Omega = \pi^*_\alpha \Omega_\alpha. \tag{4.53}$$

Let $[\mathbf{x}] \in P_\alpha$ and $\mathbf{v}_{[\mathbf{x}]}, \mathbf{u}_{[\mathbf{x}]} \in T_{[z]}P_\alpha$. The value of symplectic form is

$$\Omega_\alpha([\mathbf{x}]) \left(\mathbf{v}_{[x]}, \mathbf{u}_{[x]} \right) = \Omega(\mathbf{x})|_{\mathbf{J}^{-1}(\alpha)} (\mathbf{v}_x, \mathbf{u}_x), \tag{4.54}$$

where $T_x\pi_\alpha(\mathbf{v}_x) = \mathbf{v}_{[x]}$, $T_x\pi_\alpha(\mathbf{u}_x) = \mathbf{u}_{[x]}$ and $\mathbf{x} \in \pi_\alpha^{-1}([\mathbf{x}])$. Given a $G-$invariant Hamiltonian function h, the reduced Hamiltonian function is $h_\alpha = h \circ \pi_\alpha$. Corresponding Hamiltonian vector fields X_h and X_{h_α} are π_α related and the trajectories of X_h project into those of X_{h_α}.

More generally, let us consider free, proper and canonical action of a Lie group G on a Poisson manifold $(\mathcal{P}, \{\ ,\ \}_\mathcal{P})$. We define a Poisson structure on the quotient manifold $\mathcal{P}/G$ by requiring that the projection $\pi : \mathcal{P} \to \mathcal{P}/G$ to be a Poisson map. For the G invariant functions f, h on $\mathcal{P}$, one has that

$$\{f \circ \pi, h \circ \pi\}_{\mathcal{P}/G} = \{f, h\}_\mathcal{P} \circ \pi. \tag{4.55}$$

This procedure is called Poisson reduction [74]. It is important to remark that, a canonical Lie group action on a Poisson manifold does not necessarily preserve its symplectic leaves [75].

4.2.1. Coadjoint Orbits

We will focus on the particular case when a Lie group G acts on its cotangent bundle T^*G by the cotangent lifts of left and right translations

$$G \times T^*G \to T^*G \quad \text{and} \quad T^*G \times G \to T^*G. \tag{4.56}$$

Momentum mappings of these actions are

$$\mathbf{J}_L : T_g^*G \to \mathfrak{g} : \Upsilon_g \to T_e R_g^* \Upsilon_g \quad \text{and} \quad \mathbf{J}_R : T_g^*G \to \mathfrak{g} : \Upsilon_g \to T_e L_g^* \Upsilon_g, \tag{4.57}$$

respectively. We are particularly interested in the left action. The inverse image of a regular value α is

$$\mathbf{J}_L^{-1}(\alpha) = \left\{\Upsilon_g \in T_g^*G : T_e R_g^* \Upsilon_g = \alpha, \forall g \in G\right\} \tag{4.58}$$

and can be identified with the image of a right invariant one-form Υ_α on G, given by $\Upsilon_\alpha(g) = \Upsilon_g$, that means, $\Upsilon_\alpha(g) = T_g R^*_{g^{-1}}\alpha$. The orbit $\mathrm{Orb}(\alpha)$ and isotropy subgroup G_α of the coadjoint action are given by

$$\begin{aligned} \mathrm{Orb}(\alpha) &= \left\{ L_g^* \Upsilon_\alpha(g) : g \in G \right\} \\ G_\alpha &= \left\{ g \in G : L_g^* \Upsilon_\alpha = \Upsilon_\alpha \right\}. \end{aligned} \tag{4.59}$$

We know that $\mathrm{Orb}(\alpha) \simeq G/G_\alpha$ and deduce the fact that,

$$G_\alpha \times \mathbf{J}_L^{-1}(\alpha) \to \mathbf{J}_L^{-1}(\alpha) : (g, \Upsilon_\alpha(h)) \to \Upsilon_\alpha(gh), \tag{4.60}$$

is a well defined left action of G_α on $\mathbf{J}_L^{-1}(\alpha)$ [19].

We identify the preimage $\mathbf{J}_L^{-1}(\alpha)$ with G by the mapping $\Upsilon_g \to g^{-1}$, therefore we have the reduced phase space T^*G_α as the coadjoint orbit of α, that is

$$\mathrm{Orb}(\alpha) \simeq \mathbf{J}_L^{-1}(\alpha)/G_\alpha. \tag{4.61}$$

Coadjoint orbits have unique symplectic structures, literarily called Krillov-Kostant-Souriau two-form, given explicitly by

$$\Omega_\alpha(\gamma)\left(\xi_{\mathfrak{g}^*}(\gamma), \eta_{\mathfrak{g}^*}(\gamma)\right) = -\langle \gamma, [\xi, \eta] \rangle, \tag{4.62}$$

for $\xi, \eta \in \mathfrak{g}$ and $\xi_{\mathfrak{g}^*}(\gamma) = ad^*_\xi \gamma, \eta_{\mathfrak{g}^*}(\gamma) = ad^*_\eta \gamma \in T_\gamma \mathrm{Orb}(\alpha)$ are obtained from Eq.(4.31) [76, 77].

4.2.2. Lie-Poisson Structure

The linear algebraic dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is a Poisson manifold with the **Lie-Poisson bracket**

$$\{F, G\}_{\mp}(\alpha) = \mp \left\langle \alpha, \left[\frac{\delta F}{\delta \alpha}, \frac{\delta G}{\delta \alpha}\right]_{\mathfrak{g}} \right\rangle, \tag{4.63}$$

where $[\ ,\ ]_{\mathfrak{g}}$ is the Lie algebra bracket and, $\delta F/\delta\alpha \in \mathfrak{g}^{**} \simeq \mathfrak{g}$ is the Fréchet derivative of F with respect to $\alpha \in \mathfrak{g}$ defined by

$$\left\langle \beta, \frac{\delta F}{\delta \alpha} \right\rangle = \lim_{\epsilon \to 0} \frac{F(\alpha + \epsilon\beta) - F(\alpha)}{\epsilon}, \tag{4.64}$$

$\forall \alpha, \beta \in \mathfrak{g}$. Observe that we have two Poisson structures, one is with plus sign and the other is with minus sign. Hamiltonian vector field X_H for a given Hamiltonian function $H \in \mathcal{F}(\mathfrak{g}^*)$ is obtained from

$$X_H(F) = \{F, H\} = \mp \left\langle \alpha, \left[\frac{\delta F}{\delta \alpha}, \frac{\delta H}{\delta \alpha}\right]_{\mathfrak{g}} \right\rangle = \pm \left\langle ad^*_{\frac{\delta H}{\delta \alpha}} \alpha, \frac{\delta F}{\delta \alpha} \right\rangle. \tag{4.65}$$

The equations of motion, called the **Lie-Poisson equations,** are

$$\dot{\alpha} = \pm ad^*_{\frac{\delta H}{\delta \alpha}} \alpha. \tag{4.66}$$

There are several alternative ways to define the Lie-Poisson structure on $\mathfrak{g}^*$. We may define the Lie-Poisson bracket on $\mathfrak{g}^*$ directly from Eq.(4.62) as

$$\{F, G\}(\alpha) = \left\{ F|_{\mathrm{Orb}(\alpha)}, G|_{\mathrm{Orb}(\alpha)} \right\}_{\mathrm{Orb}(\alpha)}(\alpha) \tag{4.67}$$

where $F|_{\mathrm{Orb}(\alpha)}$ is the restriction of the function on $\mathfrak{g}^*$ to the orbit $\mathrm{Orb}(\alpha)$, and $\{\ ,\ \}_{\mathrm{Orb}(\alpha)}$ is the nondegenerate Poisson structure on $\mathrm{Orb}(\alpha)$ induced from the symplectic structure in Eq.(4.62) [77]. From Eq.(4.67) we arrive at the Lie-Poisson structure with minus sign. For the Lie-Poisson bracket with plus sign one needs to start with right action instead of left action.

If $\mathfrak{g}$ is a Lie algebra of a Lie group G, then the dual space $\mathfrak{g}^*$ is the space of covectors at the identity $e \in G$, that is $\mathfrak{g}^* = T_e^* G$. Let F and G be two functions on $\mathfrak{g}^*$ and $\hat{F}, \hat{G} : T^*G \rightarrow \mathbb{R}$ be their right invariant extensions, that is, for every $g \in G$, we require that the following diagram commutes

$$\begin{array}{ccc} T_g^* G & \xrightarrow{T_e^* R_g} & \mathfrak{g}^* \\ \downarrow \hat{F} & \swarrow F & \\ \mathbb{R} & & \end{array} \tag{4.68}$$

where $T_e^* R_g$ is the cotangent lift of right translation. The Lie-Poisson structure with plus sign is the restriction of the canonical Poisson bracket $\{\ ,\ \}_{T^*G}$ on T^*G to the identity. Left invariant extension gives the Lie-Poisson structure with minus sign.

Example: The configuration space of the rigid body is the special orthogonal group $SO(3, \mathbb{R})$ whose associated Lie algebra is the algebra of skew-symmetric matrices $\mathfrak{so}(3, \mathbb{R})$ in which the matrix commutator is the Lie algebra bracket. $\mathfrak{so}(3, \mathbb{R})$ can be identified with $\mathbb{R}^3$ where the Lie algebra bracket is the cross product of vectors. The Lie-Poisson structure with minus sign on $\left(\mathbb{R}^3\right)^* \simeq \mathbb{R}^3$ is

$$\{F, H\}(\Pi) = -\Pi \cdot (\nabla F \times \nabla H), \tag{4.69}$$

where ∇F and ∇H are the gradients of functions $F, G \in \mathcal{F}\left(\mathbb{R}^3\right)$. In this case, Hamilton's equations for a Hamiltonian function $H \in \mathcal{F}\left(\mathbb{R}^3\right)$ are $\dot{\Pi} = \Pi \times \nabla H$. In particular, for

$$H(\Pi) = \frac{(\Pi_1)^2}{2I_1} + \frac{(\Pi_2)^2}{2I_2} + \frac{(\Pi_3)^2}{2I_3}, \tag{4.70}$$

where I_1, I_2 and I_3 are constants that refer to the moments of inertia of rigid body, and $\Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathbb{R}^3$, the equations of motion

$$\dot{\Pi}_1 = \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3, \quad \dot{\Pi}_2 = \frac{I_3 - I_1}{I_1 I_3}\Pi_1\Pi_3, \quad \dot{\Pi}_3 = \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 \tag{4.71}$$

are the Euler's equations for rigid body [11].

5. GROUP OF CANONICAL DIFFEOMORPHISMS AND PLASMA DYNAMICS

5.1. MAXWELL-VLASOV AND POISSON-VLASOV EQUATIONS

To describe the motion of the plasma, one may start to write down the whole microscopic data, Newton formulas and interactions for whole particles, which is very difficult. The kinetic theory of plasma uses statistical and probabilistic concepts to handle practical problems of microscopic theory. The basic element in kinetic description of plasma is the plasma density (distribution) function $f = f(\mathbf{q}, \mathbf{p})$ which describes particle distribution in momentum phase space. We consider a plasma consisting only of one species of particles with charge e and mass m.

It is known that a charged particle with mass m and charge e is subjected to the **Lorentz force law**

$$\dot{\mathbf{p}} = -e\left(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}\right), \tag{5.1}$$

where $\mathbf{E}$ is the electrical field, $\mathbf{B}$ is the magnetic field and $\dot{\mathbf{p}}$ is the time derivative of the momenta. Electromagnetic field is described by the **Maxwell's equations**

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c}\mathbf{J} \\ \nabla \cdot \mathbf{E} &= \rho, \qquad \nabla \cdot \mathbf{B} = 0, \end{aligned} \tag{5.2}$$

where ρ is the charge density, $\mathbf{J}$ is the current density vector and c is the speed of the light.

We let $\mathcal{Q} \subset \mathbb{R}^3$ be the region in which plasma particles move and $f = f(\mathbf{q}, \mathbf{p})$ be the plasma density at $\mathbf{z} = (\mathbf{q}, \mathbf{p}) \in T^*\mathcal{Q}$. In this case, charge and plasma densities are connected each other with

$$\rho = -e \int f(\mathbf{q}, \mathbf{p})\, d^3\mathbf{p}. \tag{5.3}$$

The number N of particles in a volume of phase space $T^*\mathcal{Q}$ is given by

$$N = \int_{T^*\mathcal{Q}} f(\mathbf{q}, \mathbf{p})\, d^3\mathbf{z} \tag{5.4}$$

and the conservation of the number of particles requires that the total time derivative of N must vanish, that is,

$$\frac{dN}{dt} = \int_{T^*\mathcal{Q}} \frac{\partial f}{\partial t} d^3\mathbf{z} + \int_{\partial(T^*\mathcal{Q})} f\dot{\mathbf{z}} dS = 0, \tag{5.5}$$

where $\dot{\mathbf{z}} = (\dot{\mathbf{q}}, \dot{\mathbf{p}})$ is the **phase velocity** of the plasma [26]. We apply the divergence theorem to the second integral on the right hand side and obtain

$$\int_{T^*\mathcal{Q}} \left(\frac{\partial f}{\partial t} + \operatorname{div}(f\dot{\mathbf{z}}) \right) d^3\mathbf{z} = 0. \tag{5.6}$$

Thus, we have the **equation of continuity**

$$\frac{\partial f}{\partial t} + \nabla_z f \cdot \dot{\mathbf{z}} + f \operatorname{div}(\dot{\mathbf{z}}) = 0$$

for the plasma. Liouville theorem states that the phase space volumes are preserved under the motion of plasma, so that $\operatorname{div}(\dot{\mathbf{z}}) = 0$ [25]. Hence, substituting the Lorentz force law in the equation of continuity, we obtain the **Vlasov equation**

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_q f - e\left(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}\right) \cdot \nabla_p f = 0. \tag{5.7}$$

The coupled system of equations in Eqs.(5.2) and (5.7) are called the **Maxwell-Vlasov equations.**

Let us consider an unmagnetized plasma $\mathbf{B} = 0$ and the absence of the current $\mathbf{J} = 0$. Then the electrical field $\mathbf{E}$ becomes the gradient of a potential ϕ_f, that is $\mathbf{E} = \nabla_q \phi_f(\mathbf{q})$, in which ϕ_f is determined through the **Poisson equation**

$$\nabla_q^2 \phi_f = -e \int f(\mathbf{q}, \mathbf{p})\, d^3\mathbf{p} \tag{5.8}$$

and the Vlasov equation in Eq.(5.7) reduces to

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_q f - e \nabla_q \phi_f \cdot \nabla_p f = 0. \tag{5.9}$$

The system of equations in Eqs.(5.8) and (5.9) are called the **Poisson-Vlasov equations**. One may alternatively regard the Poisson-Vlasov equations as an approximation of the Maxwell-Vlasov equations in the nonrelativistic zero-magnetic field limit, that is the limit $c \to \infty$ [39].

We consider the cotangent bundle $T^*\mathcal{Q}$ of $\mathcal{Q} \subset \mathbb{R}^3$ position space in which the plasma particles move. The momentum phase space $T^*\mathcal{Q}$ is a 6−dimensional symplectic manifold with symplectic structure $\Omega_{T^*\mathcal{Q}} = dq^i \wedge dp_i$. The induced Poisson bracket $\{\ ,\ \}_{T^*\mathcal{Q}}$ on $T^*\mathcal{Q}$ from the symplectic structure $\Omega_{T^*\mathcal{Q}}$ is given, in Darboux's coordinates, as

$$\{f, h\}_{T^*\mathcal{Q}} = \frac{\partial f}{\partial q^i}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q^i}. \tag{5.10}$$

If we take the Hamiltonian function

$$h = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} + e\phi_f(\mathbf{q}), \tag{5.11}$$

for the motion of individual plasma particles, then the Vlasov equation can be written in form

$$\frac{\partial f}{\partial t} = -\{f, h\}_{T^*\mathcal{Q}}. \tag{5.12}$$

Thus, the plasma density f evolves by the canonical transformations [32]. This is a clue for us to arrive the point that, the appropriate framework for the configuration space of the plasma is the group of canonical diffeomorphism $Diff_{can}(T^*\mathcal{Q})$ on $T^*\mathcal{Q}$ [18,28].

On the contrary, the momentum map realization of the Poisson equation gives that the configuration space for the Poisson-Vlasov dynamics must be the semi-direct product structure $\mathcal{F}(\mathcal{Q})$ Ⓢ $Diff_{can}(T^*\mathcal{Q})$ with the action of the additive group $\mathcal{F}(\mathcal{Q})$ of functions given by fiber translation on $T^*\mathcal{Q}$ and by composition on right with the canonical transformations. In this thesis, we take the group of canonical transformations $Diff_{can}(T^*\mathcal{Q})$ as our configuration space for collisionless plasma and adapt a constraint variational derivative instead of dealing with the complications of the semi-direct product structure.

5.2. GROUP OF DIFFEOMORPHISMS

Let $\mathcal{M}$ be a smooth volume manifold (possibly with boundary). The group $Diff(\mathcal{M})$ of diffeomorphisms on $\mathcal{M}$ is an infinite dimensional Lie group with multiplication

$$Diff(\mathcal{M}) \times Diff(\mathcal{M}) \to Diff(\mathcal{M}) : (\varphi, \psi) \to \varphi \circ \psi \tag{5.13}$$

and inversion $\varphi \to \varphi^{-1}$. The unit element of the group is the identity automorphism $\mathrm{id}_{\mathcal{M}}$. As a manifold, $Diff(\mathcal{M})$ is locally diffeomorphic to an $\infty-$dimensional vector space, which can be a Banach, Hilbert or Fréchet space, and called respectively Banach Lie group, Hilbert Lie group or Fréchet Lie Group [78]. We will not discuss the details of the functional analytical issues and refer [79–82].

The elements of the tangent space $T_{\varphi}Diff(\mathcal{M})$ at $\varphi \in Diff(\mathcal{M})$ are **material velocity fields**

$$V_{\varphi} : \mathcal{M} \to T\mathcal{M}, \tag{5.14}$$

satisfying $\tau_{Diff(\mathcal{M})} \circ V_{\varphi} = \varphi$. In particular, the tangent space at the identity $T_{\mathrm{id}}Diff(\mathcal{M})$ is the space of smooth vector fields on $\mathcal{M}$, that is,

$$T_{\mathrm{id}}Diff(\mathcal{M}) = \mathfrak{X}(\mathcal{M}). \tag{5.15}$$

A vector field on $Diff(\mathcal{M})$ is a map $V : Diff(\mathcal{M}) \to TDiff(\mathcal{M})$, whose value at $\varphi \in Diff(\mathcal{M})$ is the material velocity field $V_{\varphi} \in T_{\varphi}Diff(\mathcal{M})$. V_{φ} can be represented as a composition of a diffeomorphism φ and a vector field X, that is $V_{\varphi} = X \circ \varphi$. This is the manifestation of the parallelizability of the tangent group $TDiff(\mathcal{M}) \simeq Diff(\mathcal{M})$ Ⓢ $\mathfrak{X}(\mathcal{M})$ [39].

We assume that a continuum rests in $\mathcal{M}$ and $Diff(\mathcal{M})$ acts on left by evaluation on the space $\mathcal{M}$

$$Diff(\mathcal{M}) \times \mathcal{M} \rightarrow \mathcal{M} : (\varphi, \mathbf{x}) \rightarrow \varphi(\mathbf{x}) \tag{5.16}$$

to produce the motion of particles. The right action of $Diff(\mathcal{M})$ commutes with the particle motion and constitute an infinite dimensional symmetry group of the kinematical description. This is the particle relabelling symmetry [17].

The first attempt to use the diffeomorphism group as the configuration space of a continuum is the one introduced in [12], which concerns the geometrical background for the dynamics of the ideal fluid. Since that time, ideal fluid has been worked by several authors such as [17,83,84,86,87]. In this section, we apply the pure geometrical constructions described in the previous sections to the case of canonical diffeomorphisms group and present the geometrization of Hamiltonian structure of the Poisson-Vlasov equations in Eqs.(5.8) and (5.9).

5.2.1. Lie Algebra of $Diff(\mathcal{M})$

The inner automorphism on the group $Diff(\mathcal{M})$ is

$$I_{\psi}(\varphi^t) = \psi \circ \varphi^t \circ \psi^{-1}, \tag{5.17}$$

and its differentiation at $t=0$ along the direction X gives adjoint operator, that is

$$\begin{aligned} Ad_{\psi}(X) &= T_e I_{\psi}(X) = T_e I_{\psi}\left(\frac{d}{dt}\, \varphi^t\big|_{t=0}\right) = \frac{d}{dt}\, I_{\psi}\varphi^t\big|_{t=0} \\ &= \frac{d}{dt}\, \psi \circ \varphi^t \circ \psi^{-1}\big|_{t=0} = T\psi \circ X \circ \psi^{-1} = \psi_* X. \end{aligned} \tag{5.18}$$

Thus, the adjoint action of $Diff(\mathcal{M})$ on its Lie algebra $\mathfrak{X}(\mathcal{M})$ is the push-forward operation

$$Ad_{\psi}(X) = \psi_* X. \tag{5.19}$$

The tangent space of $Diff(\mathcal{M})$ at the identity $\mathrm{id}_{\mathcal{M}}$ consists of vector fields on $\mathcal{M}$. The Lie algebra bracket on $T_{\mathrm{id}}Diff(\mathcal{M})$ can be calculated as the differential of the adjoint representation at the identity. We differentiate $Ad_{\psi^t}(X)$ with respect to t at $t=0$ and in the direction of Y to obtain

$$[Y,X]_{Diff(\mathcal{M})} = ad_Y X = \frac{d}{dt}\,\psi^t_* X\big|_{t=0} = -[Y,X]_{JL} = -\mathcal{L}_Y X, \tag{5.20}$$

where $[\ ,\ ]_{JL}$ is the standard Jacobi-Lie bracket of vector fields and $\mathcal{L}_Y X$ is the Lie derivative of X with respect to Y. Thus, the Lie algebra structure on $\mathrm{Lie}(Diff(\mathcal{M}))$ is minus the Jacobi-Lie bracket.

The dual space $\mathrm{Lie}^*(Diff(\mathcal{M}))$ of $\mathrm{Lie}(Diff(\mathcal{M}))$ is the space of one-forms densities on $\mathcal{M}$, that is,

$$\mathrm{Lie}^*(Diff(\mathcal{M})) = \mathfrak{X}^*(\mathcal{M}) \simeq \Lambda^1(\mathcal{M}) \otimes Den(\mathcal{M}). \tag{5.21}$$

The pairing between $\mathrm{Lie}(Diff(\mathcal{M}))$ and $\mathrm{Lie}^*(Diff(\mathcal{M}))$ is

$$\langle \alpha \otimes \mu, X\rangle = \int_{\mathcal{M}} \langle \alpha(\mathbf{x}), X(\mathbf{x})\rangle\, \mu(\mathbf{x}), \tag{5.22}$$

where $X \in \mathfrak{X}(\mathcal{M})$, $\alpha \in \Lambda^1(\mathcal{M})$ and μ is a volume form on $\mathcal{M}$. The pairing inside the integral is the natural pairing of finite dimensional spaces $T_x\mathcal{M}$ and $T^*_x\mathcal{M}$ [17, 30, 83]. The dual ad^* of the adjoint action ad is defined by

$$\langle ad^*_X(\alpha\otimes\mu), Y\rangle = \langle (\alpha\otimes\mu), ad_X Y\rangle = -\int_{\mathcal{M}} \langle \alpha(\mathbf{x}), [X,Y]_{JL}(\mathbf{x})\rangle\, \mu(\mathbf{x}), \tag{5.23}$$

and after applying integration by parts, we find the explicit expression

$$ad^*_X(\alpha\otimes\mu) = \mathcal{L}_X(\alpha\otimes\mu) = (\mathcal{L}_X\alpha + (\mathrm{div}_{d\mu} X)\,\alpha)\otimes\mu, \tag{5.24}$$

of the coadjoint action ad^*, where $\mathrm{div}_\mu X$ is the divergence of the vector field X with respect to the volume form μ. For the case of divergence free vector fields Eq.(5.24) reduces to

$$ad_X^*\alpha = \mathcal{L}_X\alpha. \tag{5.25}$$

5.2.2. Canonical Diffeomorphisms

The group of canonical diffeomorphisms $Diff_{can}(T^*\mathcal{Q})$ on the canonical symplectic manifold $(T^*\mathcal{Q}, \Omega_{T^*\mathcal{Q}})$ consists of diffeomorphisms φ on $T^*\mathcal{Q}$ preserving the symplectic form $\Omega_{T^*\mathcal{Q}}$, that is,

$$\varphi^*\Omega_{T^*\mathcal{Q}} = \Omega_{T^*\mathcal{Q}}. \tag{5.26}$$

The differential companion of Eq.(5.26) is $\mathfrak{L}_X\Omega_{T^*\mathcal{Q}} = 0$ and the Cartan's formula $\mathfrak{L}_X = di_X + i_X d$ leads to

$$di_X\Omega_{T^*\mathcal{Q}} = 0. \tag{5.27}$$

That means, Lie $Diff_{can}(T^*\mathcal{Q})$ of $Diff_{can}(T^*\mathcal{Q})$ is the space of locally Hamiltonian vector fields $\mathbf{g}_{lh} = \mathfrak{X}_{ham}^{loc}(T^*\mathcal{Q})$ [88]. We assume that Lie $Diff_{can}(T^*\mathcal{Q})$ consists of globally Hamiltonian vector fields $\mathbf{g} = \mathfrak{X}_{ham}(T^*\mathcal{Q})$ and postpone discussions on the subalgebras of the space of vector fields to the next section. The following equalities

$$[X_h, X_f]_{\mathbf{g}} = -[X_h, X_f]_{JL} = X_{\{h,f\}_{\Omega_{T^*\mathcal{Q}}}} \tag{5.28}$$

link $\mathfrak{X}_{ham}(T^*\mathcal{Q})$ with the space of smooth functions $\mathcal{F}(T^*\mathcal{Q})$. Namely, we have the Lie algebra isomorphism

$$h \to X_h : (\mathcal{F}(T^*\mathcal{Q}), \{\ ,\ \}_{\Omega_{T^*\mathcal{Q}}}) \to (\mathbf{g} = \mathfrak{X}_{ham}(T^*\mathcal{Q}), [\ ,\ ]_{\mathbf{g}}) \tag{5.29}$$

up to additions of constants to the real valued function h. Here, $\{\ ,\ \}_{\Omega_{T^*\mathcal{Q}}}$ is the (nondegenerate) canonical Poisson bracket of smooth functions in $\mathcal{F}(T^*\mathcal{Q})$.

Lemma 5.1. The dual space of the Lie algebra $\mathfrak{X}_{ham}\left(T^{*}\mathcal{Q}\right)$ is

$$\mathbf{g}^{*}=\mathfrak{X}_{ham}^{*}\left(T^{*}\mathcal{Q}\right)=\{\Pi_{\mathrm{id}}\in\Lambda^{1}(T^{*}\mathcal{Q}):\operatorname{div}_{\Omega_{T^{*}Q}}\Pi_{\mathrm{id}}^{\sharp}\neq 0\}. \tag{5.30}$$

To find the precise definition of the dual space $\mathfrak{X}_{ham}^{*}\left(T^{*}\mathcal{Q}\right)$, we require the L_2 pairing $\langle X_h,\Pi_{\mathrm{id}}\rangle$ to be nondegenerate. We take the volume $\mu=\Omega_{T^{*}\mathcal{Q}}^{3}$ and compute

$$\begin{aligned}\int_{T^{*}\mathcal{Q}}\left\langle X_h\left(\mathbf{z}\right),\Pi_{\mathrm{id}}\left(\mathbf{z}\right)\right\rangle\mu\left(\mathbf{z}\right) &= -\int_{T^{*}\mathcal{Q}}\left\langle dh,\Pi_{\mathrm{id}}^{\sharp}\right\rangle\mu=-\int_{T^{*}\mathcal{Q}}i_{\Pi_{\mathrm{id}}^{\sharp}}\left(dh\right)\mu\\ &= -\int_{T^{*}\mathcal{Q}}dh\wedge i_{\Pi_{\mathrm{id}}^{\sharp}}\mu=\int_{T^{*}\mathcal{Q}}hdi_{\Pi^{\sharp}}\mu\\ &= \int_{T^{*}\mathcal{Q}}h\operatorname{div}_{\Omega_{T^{*}Q}}\Pi_{\mathrm{id}}^{\sharp}\mu,\end{aligned} \tag{5.31}$$

where we use the musical isomorphism $\Omega_{T^{*}\mathcal{Q}}^{\sharp}:\Pi_{\mathrm{id}}\rightarrow\Pi_{\mathrm{id}}^{\sharp}$ induced from the symplectic two-form $\Omega_{T^{*}\mathcal{Q}}$ at the first step and apply integration by parts at the last step. Thus the dual of the isomorphism $h\rightarrow X_h$ is

$$\Pi_{\mathrm{id}}\left(\mathbf{z}\right)\rightarrow\operatorname{div}_{\Omega_{T^{*}Q}}\Pi_{\mathrm{id}}^{\sharp}\left(\mathbf{z}\right). \tag{5.32}$$

In Darboux's coordinates $\mathbf{z}=\left(\mathbf{q},\mathbf{p}\right)=\left(q^{i},p_{i}\right)$ on $T^{*}\mathcal{Q}$, if $\Pi_{\mathrm{id}}=\Pi_{i}\left(\mathbf{z}\right)dq^{i}+\Pi^{i}\left(\mathbf{z}\right)dp_{i}$ then its image under the momentum mapping in Eq.(5.32) is

$$f\left(\mathbf{z}\right)=\operatorname{div}_{\Omega_{T^{*}Q}}\Pi_{\mathrm{id}}^{\sharp}\left(\mathbf{z}\right)=\frac{\partial\Pi^{i}\left(\mathbf{z}\right)}{\partial q^{i}}-\frac{\partial\Pi_{i}\left(\mathbf{z}\right)}{\partial p_{i}}, \tag{5.33}$$

which is defined to be the plasma density function. Note that, if $\Pi_{id}=\delta_{ij}\frac{\partial\psi}{\partial p_i}dq^{j}-\delta^{ij}\frac{\partial\psi}{\partial q^{i}}dp_{j}$ for some function ψ, then the identification in Eq.(5.33) reduces to the following Laplace equation

$$f=\Delta\psi. \tag{5.34}$$

It is important to remark that, the action of $Diff_{can}(T^*\mathcal{Q})$ on $T^*\mathcal{Q}$ is a canonical action with the momentum mapping $\mathbf{J} : T^*\mathcal{Q} \to \mathrm{Lie}^*(Diff_{can}(T^*\mathcal{Q}))$ defined by

$$\langle \mathbf{J}(\mathbf{z}), X_h \rangle = h(\mathbf{z}), \tag{5.35}$$

where X_h is the Hamiltonian vector field for the Hamiltonian function h.

5.3. HAMILTONIAN STRUCTURE OF VLASOV EQUATION

5.3.1. Density Formulation of Vlasov Equation

The dual of the space of smooth functions $\mathcal{F}(T^*\mathcal{Q})$ is the space of densities $Den(T^*\mathcal{Q})$ through the following pairing

$$\langle f\mu, h \rangle = \int_{T^*\mathcal{Q}} h(\mathbf{z}) f(\mathbf{z}) \mu(\mathbf{z}), \tag{5.36}$$

where $h \in \mathcal{F}(T^*\mathcal{Q})$ and $f\mu \in Den(T^*\mathcal{Q})$ with $\mu = \Omega^3_{T^*\mathcal{Q}}$ being the symplectic volume. The adjoint action is the canonical Poisson bracket, that is,

$$ad_h f = \{h, f\}_{\Omega_{T^*\mathcal{Q}}}. \tag{5.37}$$

For the coadjoint action ad^*, we compute

$$\begin{aligned} \langle ad_h^* f, k \rangle &= \langle f, ad_h k \rangle = \left\langle f, \{h,k\}_{\Omega_{T^*\mathcal{Q}}} \right\rangle = \int_{T^*\mathcal{Q}} f \{h,k\}_{\Omega_{T^*\mathcal{Q}}} \mu \\ &= -\int_{T^*\mathcal{Q}} \{h,f\}_{\Omega_{T^*\mathcal{Q}}} k\mu = -\left\langle \{h,f\}_{\Omega_{T^*\mathcal{Q}}}, k \right\rangle. \end{aligned} \tag{5.38}$$

and deduce that

$$ad_h^* f = -\{h, f\}_{\Omega_{T^*\mathcal{Q}}}. \tag{5.39}$$

The Lie-Poisson bracket on the dual space $Den\,(T^*\mathcal{Q})$ is

$$\{F,H\}\,(f) = \int_{T^*\mathcal{Q}} f(\mathbf{z}) \left\{ \frac{\delta F}{\delta f}, \frac{\delta H}{\delta f} \right\}_{\Omega_{T^*\mathcal{Q}}} (\mathbf{z})\mu(\mathbf{z}) \tag{5.40}$$

where $F, H \in \mathcal{F}\,(Den\,(T^*\mathcal{Q}))$ and $\delta F/\delta f, \delta H/\delta f \in \mathcal{F}\,(T^*\mathcal{Q})$. The equation of the motion for a Hamiltonian functional H is

$$\dot{f} = -ad^*_{\frac{\delta H}{\delta f}} f = \left\{ \frac{\delta H}{\delta f}, f \right\}_{\Omega_{T^*\mathcal{Q}}}. \tag{5.41}$$

The Poisson equation

$$\nabla_q^2 \phi_f = -e \int f\,(\mathbf{q},\mathbf{p})\, d^3\mathbf{p}. \tag{5.42}$$

has a Green's function solution

$$\phi_f(\mathbf{q},t) = e \int_{T^*\mathcal{Q}} K(\mathbf{q}|\bar{\mathbf{q}}) f(\bar{\mathbf{z}})\mu(\bar{\mathbf{z}}), \tag{5.43}$$

where $K(\mathbf{q}|\bar{\mathbf{q}})$ is the symmetric Green's function, that is $K(\mathbf{q}|\bar{\mathbf{q}}) = K(\bar{\mathbf{q}}|\mathbf{q})$. We consider Eq.(5.43) as a constraint in variations. The constraint variational derivative of

$$H_{LP}(f) = \int_{T^*\mathcal{Q}} f(\mathbf{z}) h_f\,(\mathbf{z})\,\mu(\mathbf{z}), \tag{5.44}$$

where $h_f\,(\mathbf{z}) = \delta^{ij} p_i p_j / 2m + e\phi_f(\mathbf{q})/2$, is

$$\begin{aligned} \left\langle \frac{\delta H_{LP}}{\delta f}, \delta f \right\rangle &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} H_{LP}\,(f + \epsilon \delta f) \\ &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_{T^*\mathcal{Q}} \delta^{ij} \frac{p_i p_j}{2m} (f + \epsilon\delta f)\,(\mathbf{z})\mu(\mathbf{z}) \\ &\quad + \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \frac{e^2}{2} \iint_{T^*\mathcal{Q}} (f+\epsilon\delta f)\,(\mathbf{z}) K(\mathbf{q}|\bar{\mathbf{q}})\,(f+\epsilon\delta f)\,(\bar{\mathbf{z}})\mu(\bar{\mathbf{z}})\mu(\mathbf{z}) \\ &= \int_{T^*\mathcal{Q}} \left(\delta^{ij} \frac{p_i p_j}{2m} + e\phi_f(\mathbf{q}) \right) \delta f(\mathbf{z})\mu(\mathbf{z}). \end{aligned} \tag{5.45}$$

The symmetry of the Green's function is used at the second step and the Poisson constraint is used at the first and last steps [18, 31, 35]. We deduce that

$$\frac{\delta H_{LP}}{\delta f} = \delta^{ij}\frac{p_i p_j}{2m} + e\phi_f(\mathbf{q}) = h \tag{5.46}$$

is the Hamiltonian function in Eq.(5.11) and thus the Lie-Poisson equation for the Hamiltonian H_{LP} gives the Vlasov equation

$$\frac{\partial f}{\partial t} = -\left\{f, \frac{\delta H_{LP}}{\delta f}\right\}_{\Omega_{T^*\mathcal{Q}}} = -\left\{f, h\right\}_{\Omega_{T^*\mathcal{Q}}}. \tag{5.47}$$

5.3.2. Momentum Formulation of Vlasov Equation

The precise definition of the dual space is

$$\mathbf{g}^* = \mathfrak{X}^*_{ham}\left(T^*\mathcal{Q}\right) = \{\Pi_{id} \in \Lambda^1(T^*\mathcal{Q}) : \operatorname{div}_{\Omega_{T^*\mathcal{Q}}} \Pi^\sharp_{id} \neq 0\}. \tag{5.48}$$

The Lie-Poisson structure on $\mathbf{g}^*$ is

$$\{H(\Pi_{id}), K(\Pi_{id})\}_{LP} = \int_{T^*\mathcal{Q}} \Pi_{id}\left(\mathbf{z}\right) \cdot \left[\frac{\delta H}{\delta \Pi_{id}}, \frac{\delta K}{\delta \Pi_{id}}\right]\left(\mathbf{z}\right)\mu(\mathbf{z}) \tag{5.49}$$

where H and K are functionals on $\mathfrak{g}^*$ and $\delta H/\delta\Pi_{id}, \delta K/\delta\Pi_{id} \in \mathfrak{g}$ [11, 39]. For a Hamiltonian functional H, the Lie-Poisson equations are

$$\frac{d\Pi_{id}}{dt} = -ad^*_{\delta H/\delta\Pi_{id}}(\Pi_{id}) = -\mathcal{L}_{\delta H/\delta\Pi_{id}}(\Pi_{id}). \tag{5.50}$$

In momentum formulation, the Poisson equation takes the form

$$\nabla^2\phi_{\mathbf{\Pi}}\left(\mathbf{q}\right) = -e\int \frac{\partial \Pi^i}{\partial q^i} d^3\mathbf{p}, \tag{5.51}$$

and its Green's function solution becomes

$$\phi_{\mathbf{\Pi}}\left(\mathbf{q}\right) = -e\int_{T^*\mathcal{Q}} \Pi^i\left(\bar{\mathbf{z}}\right)\frac{\partial}{\partial \bar{q}^i}K(\mathbf{q}|\bar{\mathbf{q}})\mu\left(\bar{\mathbf{z}}\right). \tag{5.52}$$

In general, we consider the following functional

$$F\left(\Pi_i,\Pi^i\right)=\int_{T^*\mathcal{Q}}P^k\left(\Pi_i\left(\mathbf{z}\right),\Pi^i\left(\mathbf{z}\right)\right)\frac{\partial\phi_{\mathbf{\Pi}}}{\partial q^k}\mu\left(\mathbf{z}\right) \tag{5.53}$$

on the dual space $\mathfrak{g}^*=\mathfrak{X}^*_{ham}\left(T^*\mathcal{Q}\right)$, where

$$P^k\left(\Pi_i\left(\mathbf{z}\right),\Pi^i\left(\mathbf{z}\right)\right)=\alpha^{ki_1\ldots i_m}_{j_1\ldots j_n}\Pi^{j_1}\Pi^{j_2}\ldots\Pi^{j_n}\Pi_{i_1}\Pi_{i_2}\ldots\Pi_{i_m} \tag{5.54}$$

is a mixed monomial whose degree is $n+m$. $\alpha^{ki_1\ldots i_m}_{j_1\ldots j_n}$ are scalars and if the parentheses ( ) denotes the symmetry of the indices we have $\alpha^{k(i_1\ldots i_m)}_{(j_1\ldots j_n)}$. The constraint variational derivatives of the functional F with respect to Π_i and Π^i are

$$\begin{aligned}
\frac{\delta F}{\delta\Pi_r} &= m\alpha^{ki_1\ldots i_{m-1}r}_{j_1\ldots j_n}\Pi_{i_1}\Pi_{i_2}\ldots\Pi_{i_{m-1}}\Pi^{j_1}\Pi^{j_2}\ldots\Pi^{j_n}\frac{\partial\phi_{\mathbf{\Pi}}}{\partial q^k}\\
\frac{\delta F}{\delta\Pi^r} &= n\alpha^{ki_1\ldots i_m}_{j_1\ldots j_{n-1}r}\Pi^{j_1}\Pi^{j_2}\ldots\Pi^{j_{n-1}}\Pi_{i_1}\Pi_{i_2}\ldots\Pi_{i_m}\frac{\partial\left(\phi_f\left(\mathbf{q}\right)\right)}{\partial q^k}\\
&\quad+e\frac{\partial}{\partial q^r}\int_{T^*Q}\alpha^{ki_1\ldots i_m}_{j_1\ldots j_n}\Pi^{j_1}\Pi^{j_2}..\Pi^{j_n}\Pi_{i_1}\Pi_{i_2}\ldots\Pi_{i_m}\frac{\partial K(\mathbf{q}|\bar{\mathbf{q}})}{\partial q^k}\mu\left(\mathbf{z}\right)
\end{aligned} \tag{5.55}$$

In particular, we take $m=0$, $n=1$ and $\alpha^k_j=\delta^k_j$, that is

$$F=\int_{T^*Q}\Pi^k\frac{\partial}{\partial q^k}\left(\phi_f\left(\mathbf{q}\right)\right)\mu\left(\mathbf{z}\right). \tag{5.56}$$

Then, the constraint variational derivative of F with respect to Π_r vanishes and the constraint variational derivative of F with respect to Π^r is

$$\frac{\delta F}{\delta\Pi^r}=2\frac{\partial\phi_f\left(\mathbf{q}\right)}{\partial q^r}. \tag{5.57}$$

Consider the Hamiltonian functional

$$\begin{aligned}
H_{LP}(\Pi_{id}) &= \int_{T^*Q}\left\langle\Pi_{id}\left(\mathbf{z}\right),X_{h_f}\left(\mathbf{z}\right)\right\rangle\mu\left(\mathbf{z}\right)\\
&= \int_{T^*Q}[\frac{1}{m}\delta^{ij}p_i\Pi_j-\frac{e}{2}\Pi^i\frac{\partial\phi_{\Pi}(\mathbf{q})}{\partial q^i}]\mu(\mathbf{z}).
\end{aligned} \tag{5.58}$$

We compute the constraint variational derivatives

$$\frac{\delta H_{LP}(\Pi_{id})}{\delta \Pi_i} = \frac{1}{m}\delta^{ij} p_i, \quad \text{and} \quad \frac{\delta H_{LP}(\Pi_{id})}{\delta \Pi^i} = -e\frac{\partial \phi_{\Pi}(\mathbf{q})}{\partial q^i} \tag{5.59}$$

of $H_{LP}(\Pi_{id})$ with respect to Π_i and Π^i and obtain

$$\frac{\delta H_{LP}(\Pi_{id})}{\delta \Pi_{id}} = X_h. \tag{5.60}$$

Thus, the Lie-Poisson equations generating by the functional $H_{LP}(\Pi_{id})$ is

$$\frac{d\Pi_{id}}{dt} = -ad^*_{\delta H_{LP}/\delta \Pi_{id}}(\Pi_{id}) = -\mathcal{L}_{\delta H_{LP}/\delta \Pi_{id}}(\Pi_{id}) = -\mathcal{L}_{X_h}\Pi_{id}. \tag{5.61}$$

In coordinates, the equations of motion in Eq.(5.61) read

$$\begin{aligned} \frac{d\Pi_i(\mathbf{z})}{dt} &= -X_h(\Pi_i(\mathbf{z})) + e\frac{\partial^2 \phi_{\mathbf{\Pi}}(\mathbf{q})}{\partial q^i \partial q^j}\Pi^j(\mathbf{z}) \\ \frac{d\Pi^i(\mathbf{z})}{dt} &= -X_h(\Pi^i(\mathbf{z})) - \frac{1}{m}\delta^{ij}\Pi_j(\mathbf{z}) \end{aligned} \tag{5.62}$$

with the constraint

$$\nabla^2 \phi_{\mathbf{\Pi}}(\mathbf{q}) = e\int \frac{\partial \Pi^i}{\partial q^i} d^3\mathbf{p}. \tag{5.63}$$

Eqs.(5.62) are the **momentum-Vlasov equations** [39]. The back-substitution

$$f(\mathbf{z}) = \frac{\partial \Pi^i}{\partial q^i} - \frac{\partial \Pi_i}{\partial p_i} \tag{5.64}$$

defines the plasma density function f and the momentum-Vlasov equations give the Vlasov equation in form Eq.(5.9). By definition, the momentum variables (Π_i, Π^i) represents equivalence classes up to additions of the terms $\frac{\partial k}{\partial q^i}(\mathbf{q},\mathbf{p})$ and $\frac{\partial k}{\partial p_i}(\mathbf{q},\mathbf{p})$, respectively. Thus, the reduced dynamics on $\mathbf{g}^*$ has a further symmetry given by the action of the additive group $\mathcal{F}(T^*\mathcal{Q})$ of functions on $T^*\mathcal{Q}$. In the following proposition, we show the equivalence of Hamiltonian functionals in Eqs.(5.44) and (5.58).

Proposition 5.2. $H_{LP}(\Pi_{id}) = H_{LP}(f) \pmod{\text{div}}$

We take $H_{LP}(f)$ and replace f by $\operatorname{div}_{\Omega_{T^*\mathcal{Q}}} \Pi^{\sharp}_{\text{id}}$, then

$$\begin{aligned}
H_{LP}(f) &= \int_{T^*\mathcal{Q}} \frac{1}{2m}\delta^{kj}p_k p_j(\frac{\partial \Pi^i}{\partial q^i} - \frac{\partial \Pi_i}{\partial p_i})\mu(\mathbf{z}) + \frac{e^2}{2}\iint\limits_{T^*\mathcal{Q}} \frac{\partial \Pi^i}{\partial q^i}\frac{\partial \bar{\Pi}^j}{\partial \bar{q}^j}K(\mathbf{q}|\bar{\mathbf{q}})\mu(\mathbf{z})\mu(\mathbf{z}') \\
&\quad - e^2 \iint\limits_{T^*\mathcal{Q}} \frac{\partial \Pi^i}{\partial q^i}\frac{\partial \bar{\Pi}_j}{\partial \bar{p}_j}K(\mathbf{q}|\bar{\mathbf{q}})\mu(\mathbf{z})\mu(\mathbf{z}') + \frac{e^2}{2}\iint\limits_{T^*\mathcal{Q}} \frac{\partial \Pi_i}{\partial p_i}\frac{\partial \bar{\Pi}_j}{\partial \bar{p}_j}K(\mathbf{q}|\bar{\mathbf{q}})\mu(\mathbf{z})\mu(\mathbf{z}') \\
&= \int_{T^*\mathcal{Q}} \frac{1}{m}\delta^{kj}p_k \Pi_j \mu(\mathbf{z}) + \frac{e}{2}\int_{T^*\mathcal{Q}} \Pi^i \frac{\partial}{\partial q^i} e\left(\int_{T^*\mathcal{Q}} \bar{\Pi}^j \frac{\partial}{\partial \bar{q}^j} K(\mathbf{q}|\bar{\mathbf{q}}) d\mu(\mathbf{z})\right)\mu(\mathbf{z}') \quad (\text{mod div}) \\
&= \int_{T^*\mathcal{Q}} \left\langle \Pi_{id}, X_{h_f} \right\rangle (\mathbf{z})\, \mu(\mathbf{z}) = H_{LP}(\Pi_{id}) \qquad (5.65)
\end{aligned}$$

where we omit divergence terms at the second step. Conversely, starting from the functional $H_{LP}(\Pi_{id})$ we compute

$$\begin{aligned}
H_{LP}(\Pi_{id}) &= \int_{T^*\mathcal{Q}} \left\langle X_{h_f}, \Pi_{id} \right\rangle (\mathbf{z})\, \mu(\mathbf{z}) \\
&= \int_{T^*\mathcal{Q}} \left\{ \frac{\partial h_f(\mathbf{z})}{\partial p_i}\Pi_i(\mathbf{z}) - \frac{\partial h_f(\mathbf{z})}{\partial q^i}\Pi^i(\mathbf{z}) \right\} \mu(\mathbf{z}) \\
&= \int_{T^*\mathcal{Q}} h_f(\mathbf{z})(\frac{\partial \Pi^i}{\partial q^i} - \frac{\partial \Pi_i}{\partial p_i})\mu(\mathbf{z}) \quad (\text{mod div}) \\
&= \int_{T^*\mathcal{Q}} h_f(\mathbf{z})\, f(\mathbf{z})\, \mu(\mathbf{z}) = H_{LP}(f). \qquad (5.66)
\end{aligned}$$

5.3.3. Gauge Symmetries of Hamiltonian Dynamics and Poisson Equation

Recall that action of additive group of functions $\mathcal{F}(\mathcal{Q})$ on $T^*\mathcal{Q}$ by momentum translations

$$\Phi : \mathcal{F}(\mathcal{Q}) \times T^*\mathcal{Q} \to \left(\phi, \left(q^i, p_i\right)\right) \to \left(q^i, p_i - \frac{\partial \phi}{\partial q^i}\right), \qquad (5.67)$$

with the infinitesimal generator

$$X_\phi(\mathbf{q}, \mathbf{p}) = -\frac{\partial \phi}{\partial q^i}\frac{\partial}{\partial p_i}. \qquad (5.68)$$

Lie algebra of the additive group $\mathcal{F}(\mathcal{Q})$ can be identified by itself with the trivial bracket as the Lie algebra bracket. Thus, the mapping

$$J_{\phi} := \mathcal{X}_{E} \circ d : \mathcal{F}(\mathcal{Q}) \rightarrow \mathbf{g} = \mathfrak{X}_{ham}\left(T^{*}\mathcal{Q}\right) : \phi \rightarrow X_{\phi} \tag{5.69}$$

is a Lie algebra homomorphism, where $\mathcal{X}_{E}$ is Euler vector field in Eq.(3.47). The dual $\mathbf{J}_{\phi} : \mathbf{g}^{*} \rightarrow Den\left(\mathcal{Q}\right)$ of the Lie algebra homomorphism J_{ϕ} is a Poisson and a momentum map. In coordinates, associated momentum mapping $\mathbf{J}_{\phi}$ is computed as

$$\begin{aligned} \left\langle \mathbf{J}_{\phi}\left(\Pi_{id}\right), \phi\left(\mathbf{q}\right)\right\rangle &= \left\langle \Pi_{id}, J_{\phi}\left(\phi\right)\right\rangle = \left\langle \Pi_{id}\left(\mathbf{z}\right), \mathcal{X}_{E}\left(d\phi\left(\mathbf{q}\right)\right)\right\rangle \\ &= \int_{T^{*}\mathcal{Q}} -\frac{\partial \phi\left(\mathbf{q}\right)}{\partial q^{i}} \Pi^{i}\left(\mathbf{z}\right)\mu\left(\mathbf{z}\right) = \int_{T^{*}\mathcal{Q}} \phi\left(\mathbf{q}\right)\frac{\partial \Pi^{i}\left(\mathbf{z}\right)}{\partial q^{i}}\mu\left(\mathbf{z}\right) \end{aligned} \tag{5.70}$$

where we apply integration by parts at the last step. Thus, we have

$$\mathbf{J}_{\phi}\left(\Pi_{id}\right)\left(\mathbf{q}\right) = \left(\int \frac{\partial \Pi^{i}\left(\mathbf{z}\right)}{\partial q^{i}} d^{3}\mathbf{p}\right) d^{3}\mathbf{q}. \tag{5.71}$$

The exterior derivative $d : \mathcal{F}(\mathcal{Q}) \rightarrow d\mathcal{F}(\mathcal{Q})$ is a Lie algebra homomorphism between additive algebras $\mathcal{F}(\mathcal{Q})$ and $d\mathcal{F}(\mathcal{Q})$. The dual of the homomorphism is a momentum mapping $\mathbf{J}_{d}$ given diagrammatically

$$\begin{array}{ccc} \mathcal{F}(\mathcal{Q}) & \xrightarrow{\ d\ } & d\mathcal{F}(\mathcal{Q}) \\ \Updownarrow & & \Updownarrow \\ Den\left(\mathcal{Q}\right) & \xleftarrow{\ \mathbf{J}_{d}\ } & d\mathcal{F}(\mathcal{Q})^{*} \simeq *d\mathcal{F}(\mathcal{Q}) \end{array} \tag{5.72}$$

where we identify the two-forms with one-forms using the Hodge $*$ operator induced from the scalar product on $\mathcal{Q} \subset \mathbb{R}^{3}$ [54]. We have

$$\left\langle \psi, \mathbf{J}_{d}\left(*d\phi\right)\right\rangle = \left\langle d\psi, *d\phi\right\rangle = \int d\psi \wedge *d\phi = -\int \psi d * d\phi, \tag{5.73}$$

that is $\mathbf{J}_{d}\left(*d\phi\right) = -\left(\nabla_{q}^{2}\phi\right) d^{3}\mathbf{q}$, because $*d*d = \nabla_{q}^{2}$ on $\mathbb{R}^{3}$ [60].

We combine the Lie algebra homomorphisms $J_\phi : \mathcal{F}(\mathcal{Q}) \to \mathbf{g}$ in Eq.(5.69) and $d : \mathcal{F}(\mathcal{Q}) \to d\mathcal{F}(\mathcal{Q})$ and obtain Lie algebra homomorphism

$$(J_\phi, d) : \mathcal{F}(\mathcal{Q}) \to \mathbf{g} \times d\mathcal{F}(\mathcal{Q}) : \phi \to (X_\phi, d\phi)\,, \tag{5.74}$$

where the Lie bracket on $\mathbf{g} \times d\mathcal{F}(\mathcal{Q})$ is the trivial one. Then the dual

$$\begin{aligned}(J_\phi, d)^* \quad &: \quad \mathbf{g}^* \times (*d\mathcal{F}(\mathcal{Q})) \to Den\,(\mathcal{Q}) \\ &: \quad (\Pi_{id}, *d\phi_\Pi) \to \left(-\nabla_q^2\phi_\Pi + \int \frac{\partial \Pi^i\,(\mathbf{z})}{\partial q^i} d^3\mathbf{p}\right) d^3\mathbf{q},\end{aligned} \tag{5.75}$$

of (J_ϕ, d) is a momentum and Poisson map. The zero value of the mapping leads to

$$\nabla_q^2\phi_\Pi\,(\mathbf{q}) = \int \frac{\partial \Pi^i\,(\mathbf{z})}{\partial q^i} d^3\mathbf{p}. \tag{5.76}$$

the Poisson equation in momentum formulation.

5.4. LIFTS OF HAMILTONIAN VECTOR FIELDS AND MOMENTUM-VLASOV EQUATIONS

We consider a Hamiltonian vector field

$$X_h(\mathbf{q}, \mathbf{p}) = \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i} \in \mathfrak{X}(T^*\mathcal{Q}). \tag{5.77}$$

whose complete cotangent lift $X_h^{c*} \in \mathfrak{X}(T^*T^*\mathcal{Q})$ is

$$X_h^{c*}(\Pi_{id}) = X_h(z) + \Pi_{id}^\sharp\left(\frac{\partial h}{\partial q^i}\right)\frac{\partial}{\partial \Pi_i} + \Pi_{id}^\sharp\left(\frac{\partial h}{\partial p_i}\right)\frac{\partial}{\partial \Pi^i} \in \mathfrak{X}(T^*T^*\mathcal{Q}), \tag{5.78}$$

where $\Pi_{id} = \left(q^i, p_i; \Pi_i, \Pi^i\right)$ are the induced coordinates on $T^*T^*\mathcal{Q}$ and $\Pi_{id}^\sharp = \Omega_{T^*\mathcal{Q}}^\sharp\,(\Pi_{id}) = \Pi^i\partial/\partial q^i - \Pi_i\partial/\partial p_i$. $\Pi_{id}^\sharp\left(\partial h/\partial q^i\right)$ denotes simply the action of $\Pi_{id}^\sharp$ to $\partial h/\partial q^i$. The vertical representative VX_h^{c*} and the holonomic part HX_h^{c*} of cotangent lift X_h^{c*} of a Hamiltonian vector field X_h are computed as

$$\begin{aligned} VX_h^{c*} &= \left(\Pi_{id}^{\sharp}\left(\frac{\partial h}{\partial q^i}\right) - X_h\left(\Pi_i\right)\right)\frac{\partial}{\partial \Pi_i} + \left(\Pi_{id}^{\sharp}\left(\frac{\partial h}{\partial p_i}\right) - X_h\left(\Pi^i\right)\right)\frac{\partial}{\partial \Pi^i}, \\ HX_h^{c*} &= X_h + X_h\left(\Pi_i\right)\frac{\partial}{\partial \Pi_i} + X_h\left(\Pi^i\right)\frac{\partial}{\partial \Pi^i}. \end{aligned} \tag{5.79}$$

The complete tangent lift of X_h is

$$X_h^c\left(\mathbf{z}, \dot{\mathbf{z}}\right) = X_h + X_{T^*\mathcal{Q}}\left(\frac{\partial h}{\partial p_i}\right)\frac{\partial}{\partial \dot{q}^i} - X_{T^*\mathcal{Q}}\left(\frac{\partial h}{\partial q^i}\right)\frac{\partial}{\partial \dot{p}_i} \in \mathfrak{X}(TT^*\mathcal{Q}), \tag{5.80}$$

where $X_{T^*\mathcal{Q}} = \tau_{TT^*\mathcal{Q}}\left(X_h^c\right) = \dot{q}^i\partial/\partial q^i + \dot{p}_i\partial/\partial p_i$. The vertical representative and the holonomic part of complete tangent lift X_h^c are

$$\begin{aligned} VX_h^c &= \left(X_{T^*\mathcal{Q}}\left(\frac{\partial h}{\partial p_i}\right) - X_h(\dot{q}^i)\right)\frac{\partial}{\partial \dot{q}^i} - \left(X_{T^*\mathcal{Q}}\left(\frac{\partial h}{\partial q^i}\right) + X_h(\dot{p}_j)\right)\frac{\partial}{\partial \dot{p}_i} \\ HX_h^c &= X_h + X_h(\dot{q}^i)\frac{\partial}{\partial \dot{q}^i} + X_h(\dot{p}_j)\frac{\partial}{\partial \dot{p}_i}. \end{aligned} \tag{5.81}$$

In particular, we consider the Hamiltonian function

$$h = \frac{1}{2m}\delta^{ij}p_ip_j + e\phi_f\left(q\right). \tag{5.82}$$

The corresponding Hamiltonian vector field

$$X_h(\mathbf{z}) = \frac{1}{m}\delta^{ij}p_i\frac{\partial}{\partial q^j} - e\frac{\partial \phi_f}{\partial q^i}\frac{\partial}{\partial p_i} \tag{5.83}$$

generates the motion of a charged particle. The complete tangent and cotangent lifts of X_h are the Hamiltonian vector fields

$$X_h^c = X_h + \frac{1}{m}\delta^{ij}\dot{p}_i\frac{\partial}{\partial \dot{q}^i} - e\dot{q}^j\frac{\partial^2 \phi_f}{\partial q^j \partial q^i}\frac{\partial}{\partial \dot{p}_i}, \tag{5.84}$$

$$X_h^{c*} = X_h - \delta^{ij}\frac{1}{m}\Pi_i\frac{\partial}{\partial \Pi^j} + e\Pi^j\frac{\partial^2 \phi_f}{\partial q^j \partial q^i}\frac{\partial}{\partial \Pi_i} \tag{5.85}$$

for the Hamiltonian functions

$$\tilde{H}=\frac{1}{m}\delta^{ij}\dot{p}_{i}p_{j}+e\dot{q}^{i}\frac{\partial\phi_{f}}{\partial q^{i}}\quad\text{and}\quad H=\frac{1}{m}\delta^{ij}p_{i}\Pi_{j}-e\frac{\partial\phi_{f}}{\partial q^{i}}\Pi^{i}, \tag{5.86}$$

respectively. In this case, the vertical representatives of tangent and cotangent lifts are given by

$$VX_{h}^{c} = (\frac{1}{m}\delta^{ji}\dot{p}_{j}-X_{h}(\dot{q}^{i}))\frac{\partial}{\partial\dot{q}^{i}}-(e\dot{q}^{i}\frac{\partial^{2}\phi}{\partial q^{i}\partial q^{j}}+X_{h}(\dot{p}_{j}))\frac{\partial}{\partial\dot{p}_{j}}, \tag{5.87}$$

$$VX_{h}^{c*} = \left(e\Pi^{j}\frac{\partial^{2}\phi}{\partial q^{j}\partial q^{i}}-X_{h}(\Pi_{i})\right)\frac{\partial}{\partial\Pi_{i}}-(\frac{1}{m}\Pi_{j}\delta^{ji}+X_{h}(\Pi^{i}))\frac{\partial}{\partial\Pi^{i}}. \tag{5.88}$$

Thus, we have proved the following lemma.

Lemma 5.3. Momentum-Vlasov equations

$$\begin{aligned}\dot{\Pi}_{i} &= -X_{h}(\Pi_{i})+e\frac{\partial^{2}\phi_{f}}{\partial q^{i}\partial q^{j}}\Pi^{j},\\ \dot{\Pi}^{i} &= -X_{h}(\Pi^{i})-\frac{1}{m}\delta^{ij}\Pi_{j}.\end{aligned} \tag{5.89}$$

are obtained as the flow generated by vertical representative of complete cotangent lift of Hamiltonian vector field corresponding to the Hamiltonian function in Eq.(5.82).

The coadjoint action of $X_{h}\in\mathbf{g}=\mathfrak{X}_{ham}\left(T^{*}Q\right)$ on $\mathbf{g}^{*}=\mathfrak{X}_{ham}^{*}\left(T^{*}Q\right)$ generates the Lie-Poisson dynamics which is identical to the dynamics generated by Eq.(5.88). More precisely, we can recast the Poisson-Vlasov dynamics into the form

$$VX_{h}^{c*}\left(\Pi_{id}\right)=\left(\mathcal{L}_{X_{h}}(\Pi_{id})\right)^{v}, \tag{5.90}$$

with the choice of the particular Hamiltonian function in Eq.(5.82).

6. ALGEBRA OF DIFFERENTIAL FORMS AND VECTOR FIELDS

6.1. ALGEBRA OF THE DIFFERENTIAL FORMS

Let $(\mathcal{M},\Omega_{\mathcal{M}})$ be a symplectic manifold. Recall the musical isomorphism $\Omega^{\flat}_{\mathcal{M}}:\mathfrak{X}(\mathcal{M})\rightarrow\Lambda^{1}(\mathcal{M})$ given in Eq.(2.94) and its fiberwise inverse $\Omega^{\sharp}_{\mathcal{M}}$. Using the notation $\Omega^{\flat}_{\mathcal{M}}(X)=X^{\flat}$ and $\Omega^{\sharp}_{\mathcal{M}}(\alpha)=\alpha^{\sharp}$, we have the following identities

$$\Omega_{\mathcal{M}}\left(\alpha^{\sharp},\beta^{\sharp}\right)=\left\langle\alpha,\beta^{\sharp}\right\rangle=-\left\langle\beta,\alpha^{\sharp}\right\rangle \tag{6.1}$$

and $\Omega_{\mathcal{M}}\left(\alpha^{\sharp},X\right)=\langle\alpha,X\rangle$, where X,Y and α,β are sections of tangent and cotangent bundles of $\mathcal{M}$, respectively. $\langle\ ,\ \rangle$ denotes the pairing between the vector and covector fields.

We define a Lie algebra bracket

$$\begin{aligned}\{\ ,\ \}_{\Lambda^{1}} &: \Lambda^{1}(\mathcal{M})\times\Lambda^{1}(\mathcal{M})\rightarrow\Lambda^{1}(\mathcal{M}) &(6.2)\\ &: (\alpha,\beta)\rightarrow\Omega^{\flat}\left(\left[\alpha^{\sharp},\beta^{\sharp}\right]\right) &(6.3)\end{aligned}$$

on the space of sections of cotangent bundle $T^{*}\mathcal{M}\rightarrow\mathcal{M}$ in such a way that, the musical isomorphisms $\Omega^{\flat}_{\mathcal{M}}$ and $\Omega^{\sharp}_{\mathcal{M}}$ become Lie algebra isomorphisms, that is,

$$\{\alpha,\beta\}^{\sharp}_{\Lambda^{1}}=\left[\alpha^{\sharp},\beta^{\sharp}\right]\quad\text{and}\quad\left\{X^{\flat},Y^{\flat}\right\}_{\Lambda^{1}}=[X,Y]^{\flat}. \tag{6.4}$$

To have a more explicit definition for $\{\ ,\ \}_{\Lambda^{1}}$, we make the following calculation,

$$\begin{aligned}d\Omega_{\mathcal{M}}\left(\alpha^{\sharp},\beta^{\sharp},X\right) &= \alpha^{\sharp}\left(\Omega_{\mathcal{M}}\left(\beta^{\sharp},X\right)\right)-\beta^{\sharp}\left(\Omega_{\mathcal{M}}\left(\alpha^{\sharp},X\right)\right)+X\left(\Omega_{\mathcal{M}}\left(\alpha^{\sharp},\beta^{\sharp}\right)\right)\\ &\quad-\Omega_{\mathcal{M}}\left(\left[\alpha^{\sharp},\beta^{\sharp}\right],X\right)+\Omega_{\mathcal{M}}\left(\left[\alpha^{\sharp},X\right],\beta^{\sharp}\right)-\Omega_{\mathcal{M}}\left(\left[\beta^{\sharp},X\right],\alpha^{\sharp}\right)\\ &= \alpha^{\sharp}\left(\langle\beta,X\rangle\right)-\beta^{\sharp}\left(\langle\alpha,X\rangle\right)+\left\langle d\left(i_{\beta^{\sharp}}i_{\alpha^{\sharp}}\Omega_{\mathcal{M}}\right),X\right\rangle\\ &\quad-\left\langle\{\alpha,\beta\}_{\Lambda^{1}},X\right\rangle-\left\langle\beta,\left[\alpha^{\sharp},X\right]\right\rangle+\left\langle\alpha,\left[\beta^{\sharp},X\right]\right\rangle. \end{aligned} \tag{6.5}$$

Since

$$\begin{aligned}\left\langle \mathfrak{L}_{\alpha^{\sharp}}(\beta), X\right\rangle &= \alpha^{\sharp}(\langle \beta, X\rangle) - \left\langle \beta, \left[\alpha^{\sharp}, X\right]\right\rangle, \\ \left\langle \mathfrak{L}_{\beta^{\sharp}}(\alpha), X\right\rangle &= \beta^{\sharp}(\langle \alpha, X\rangle) - \left\langle \alpha, \left[\beta^{\sharp}, X\right]\right\rangle \end{aligned} \tag{6.6}$$

the nondegeneracy of pairing and the fact that the symplectic form $\Omega_{\mathcal{M}}$ is closed imply

$$\{\alpha, \beta\}_{\Lambda^1} = \mathfrak{L}_{\alpha^{\sharp}}(\beta) - \mathfrak{L}_{\beta^{\sharp}}(\alpha) + d\left(i_{\beta^{\sharp}} i_{\alpha^{\sharp}} \Omega_{\mathcal{M}}\right). \tag{6.7}$$

We identify the dual space $\mathbf{g}^*$ of $\mathfrak{X}_{ham}(\mathcal{M})$ with the non-closed one-forms. Let α and β be two closed forms, then we have

$$\{\alpha, \beta\}_{\Lambda^1} = d\left(i_{\alpha^{\sharp}}\beta - i_{\beta^{\sharp}}\alpha + i_{\beta^{\sharp}} i_{\alpha^{\sharp}}\Omega\right) \tag{6.8}$$

which is exact. In particular, if $\alpha = df$ and $\beta = dh$ are two exact one-forms, then

$$\begin{aligned}\{df, dh\}_{\Lambda^1} &= \mathcal{L}_{X_f}(dh) - \mathcal{L}_{X_h}(X_f) + d\left(i_{X_h} i_{X_f}\Omega\right) \\ &= d\left(X_f(h)\right) - d\left(X_h(f)\right) + d\{f, h\}_{\Omega_{\mathcal{M}}} \\ &= d\{h, f\}_{\Omega_{\mathcal{M}}} - d\{f, h\}_{\Omega_{\mathcal{M}}} + d\{f, h\}_{\Omega_{\mathcal{M}}} \\ &= d\{h, f\}_{\Omega_{\mathcal{M}}}. \end{aligned} \tag{6.9}$$

where $\{h, f\}_{\Omega_{\mathcal{M}}}$ is the Poisson bracket on the space of functions induced from the symplectic structure $\Omega_{\mathcal{M}}$. Thus the set of all exact one-forms is a Lie subalgebra of the Lie algebra of all one-forms. From the calculation in Eq.(6.9) we see that exterior derivative d is a Lie algebra homomorphism between the algebra of smooth functions on the symplectic manifold $\mathcal{M}$ and the algebra of exact one-forms on $\mathcal{M}$. If α is closed and $d\beta \neq 0$, then

$$\{\alpha, \beta\}_{\Lambda^1} = i_{\alpha^{\sharp}} d\beta + d\left(i_{\alpha^{\sharp}}\Pi_{id} - i_{\beta^{\sharp}}\alpha + i_{\beta^{\sharp}} i_{\alpha^{\sharp}}\Omega_{\mathcal{M}}\right), \tag{6.10}$$

where the condition that the first term to be closed requires the invariance relations $di_{\alpha^{\sharp}} d\beta = \mathfrak{L}_{\alpha^{\sharp}} d\beta = d\mathfrak{L}_{\alpha^{\sharp}}\beta = 0$.

Let X_h be a Hamiltonian vector field for the Hamiltonian function h on the symplectic manifold $(\mathcal{M}, \Omega_{\mathcal{M}})$, that is $X_h^{\flat} = dh$ and $(dh)^{\sharp} = X_h$. Since $X_{\{h,f\}} = -\left[X_h, X_f\right]_{JL}$, we have that

$$[X_h, X_f]^{\flat}_{Diff_{can}(\mathcal{M})} = -\left[X_h, X_f\right]^{\flat}_{JL} = X^{\flat}_{\{h,f\}} = d\left\{h, f\right\}_{\Omega_{\mathcal{M}}} = \{dh, df\}_{\Lambda^1}, \tag{6.11}$$

where $[\ ,\ ]_{Diff_{can}(\mathcal{M})}$ is the Lie algebra bracket on $\operatorname{Lie} Diff_{can}(\mathcal{M})$. In addition, if X is locally Hamiltonian, then $X^{\flat}$ is closed by definition. Thus, the set of all Hamiltonian vector fields in $\mathfrak{X}(\mathcal{M})$ is isomorphic to the set of all closed one-forms in $\Lambda^1(\mathcal{M})$. Note that, the set of locally but not globally Hamiltonian vectors in $\mathfrak{X}(\mathcal{M})$ corresponds to the first de Rham cohomology space consisting of closed and non-exact one-forms. Hence, we have the vector space isomorphism

$$\Omega^{\flat}_{\mathcal{M}} : \mathbf{g} = \mathfrak{X}_{ham}(\mathcal{M}) \rightarrow \mathbf{g}^{\flat} = \ker d \cap \Lambda^1(\mathcal{M}). \tag{6.12}$$

Proposition 6.1. If $\mathbf{g}^*$ denotes the space of non-closed one forms we have

$$\Lambda^1(\mathcal{M}) = \mathbf{g}^{\flat} \oplus \mathbf{g}^* \tag{6.13}$$

furthermore,

$$\left\{\mathbf{g}^{\flat}, \mathbf{g}^{\flat}\right\}_{\Lambda^1} \subset \mathbf{g}^{\flat}, \quad \{\mathbf{g}^*, \mathbf{g}^*\}_{\Lambda^1} \subset \mathbf{g}^*, \quad \text{and} \quad \left\{\mathbf{g}^{\flat}, \mathbf{g}^*\right\}_{\Lambda^1} \subset \mathbf{g}^*. \tag{6.14}$$

This result implies that $\left(\mathbf{g}^{\flat}, \{\ ,\ \}_{\Lambda^1}\right)$ and $(\mathbf{g}^*, \{\ ,\ \}_{\Lambda^1})$ are Lie subalgebras of $\left(\Lambda^1(\mathcal{M}), \{\ ,\ \}_{\Lambda^1}\right)$ Moreover, $\mathbf{g}^*$ is an ideal of the algebra. Observe that Eq.(6.10) implies

$$\{\mathbf{g}^{\flat}_{lh}, \mathbf{g}^{\flat}\}_{\Lambda^1} \subset \mathbf{g}^{\flat}, \tag{6.15}$$

where $\mathbf{g}_{lh}$ denotes the space of locally Hamiltonian vector fields.

6.2. DECOMPOSITION OF SPACE OF VECTOR FIELDS

Let X_h be a Hamiltonian vector field. The identity

$$i_{[X,X_h]} = \mathcal{L}_X i_{X_h} - i_{X_h}\mathcal{L}_X \tag{6.16}$$

implies that if X is locally Hamiltonian then the Lie bracket $[X, X_h]$ is globally Hamiltonian, that is

$$[\mathbf{g}_{lh}, \mathbf{g}] \subset \mathbf{g}. \tag{6.17}$$

More precisely, we compute

$$i_{[X,X_h]}\Omega = \mathcal{L}_X i_{X_h}\Omega - i_{X_h}\mathcal{L}_X\Omega = \mathcal{L}_X dh = di_X dh \tag{6.18}$$

so that the Hamiltonian function is $X(h)$. Note that for two locally Hamiltonian vector fields X and Y we get $i_{[X,Y]}\Omega = d\Omega(X,Y)$, that means $[X,Y]$ is also Hamiltonian. Hence, we have

$$[\mathbf{g}_{lh}, \mathbf{g}_{lh}] \subset \mathbf{g}. \tag{6.19}$$

One may show that Eq.(6.19) is actually an equality, see [4] and references therein. If, on the other hand, X is (locally) Hamiltonian and $\alpha^\sharp \in (\mathbf{g}^*)^\sharp$ is not Hamiltonian in any sense we get

$$i_{[X,\alpha^\sharp]}\Omega = \mathcal{L}_X i_{\alpha^\sharp}\Omega = d\Omega\left(\alpha^\sharp, X\right) + i_X di_{\alpha^\sharp}\Omega \tag{6.20}$$

which need not be closed for arbitrary choices of $X \in \mathbf{g}$ and $\alpha^\sharp \in (\mathbf{g}^*)^\sharp$ and hence not Hamiltonian,

$$\left[\mathbf{g}, (\mathbf{g}^*)^\sharp\right] \subset (\mathbf{g}^*)^\sharp. \tag{6.21}$$

We summarize the above discussion in the following proposition.

Proposition 6.2. The space of vector fields on a symplectic manifold $\mathcal{M}$ can be decomposed as the direct sum

$$\mathfrak{X}(\mathcal{M}) = \mathbf{g} \oplus (\mathbf{g}^*)^\sharp, \tag{6.22}$$

where $\mathbf{g}$ denotes the set of Hamiltonian vector fields and $(\mathbf{g}^*)^\sharp$ denotes the set of non-Hamiltonian vector fields. $\mathbf{g}$ and $(\mathbf{g}^*)^\sharp$ are Lie subalgebras of $\mathfrak{X}(\mathcal{M})$, the space of vector fields, that is

$$[\mathbf{g}, \mathbf{g}] \subset \mathbf{g} \quad \text{and} \quad \left[(\mathbf{g}^*)^\sharp, (\mathbf{g}^*)^\sharp\right] \subset (\mathbf{g}^*)^\sharp. \tag{6.23}$$

In addition, the space of non-Hamiltonian vector fields is an ideal in the algebra of vector fields

$$\left[\mathbf{g}, (\mathbf{g}^*)^\sharp\right] \subset (\mathbf{g}^*)^\sharp, \quad \text{and} \quad \left[\mathbf{g}_{lh}, (\mathbf{g}^*)^\sharp\right] \subset (\mathbf{g}^*)^\sharp, \tag{6.24}$$

and the following inclusions hold

$$[\mathbf{g}_{lh}, \mathbf{g}] \subset \mathbf{g}, \quad \text{and} \quad [\mathbf{g}_{lh}, \mathbf{g}_{lh}] . \subset \mathbf{g}. \tag{6.25}$$

6.3. HOMOTHETIES

In the formulation of dynamics with the Lagrangian representation of kinematics, the plasma density is a constant. The fixed values of density in the space $Den(\mathcal{M})$ of densities corresponds via definition in Eq.(5.48) to vector fields $\Pi^\sharp$ in $T\mathcal{M}$ with constant divergence. Such vector fields, called homotheties, generate similarity transformations (of the first kind) which are also called homotheties [89]. In this section we shall investigate the properties of homotheties in the algebra of vector fields $\mathfrak{X}(\mathcal{M}) = \mathbf{g} \oplus (\mathbf{g}^*)^\sharp$ with the Jacobi-Lie bracket.

We denote the set of all vectors in $(\mathbf{g}^*)^\sharp$ with constant divergence by $(\mathbf{g}_c^*)^\sharp$, and we let μ be the symplectic volume on $\mathcal{M}$. We also observe that if $\Pi_c^\sharp \in (\mathbf{g}_c^*)^\sharp$ is of constant divergence $\mathcal{L}_{\Pi_c^\sharp}\mu = c\mu$ with respect to symplectic volume (in n dimension) then it follows from the non-degeneracy of symplectic two-form that

$$\mathcal{L}_{\Pi_c^\sharp}\Omega_{\mathcal{M}} = \frac{c}{n}\Omega_{\mathcal{M}}. \tag{6.26}$$

We recall that X is a locally Hamiltonian vector if and only if $i_X\Omega_{\mathcal{M}}$ is closed. Then previous equation is equivalent to $di_{\Pi_c^\sharp}\Omega_{\mathcal{M}} = (c/n)\,\Omega_{\mathcal{M}}$ which is a manifestation of the fact that $\Pi_c^\sharp$ is not even locally Hamiltonian and hence is in $(\mathbf{g}^*)^\sharp$. However, it follows from the identity

$$\mathcal{L}_{[X,Y]}\mu = \mathcal{L}_X\mathcal{L}_Y\mu - \mathcal{L}_Y\mathcal{L}_X\mu$$

that the Lie bracket of two vectors with constant divergence is locally Hamiltonian. Next result summarizes the algebraic relations between (locally) Hamiltonian vector fields, non-Hamiltonian vector fields and homotheties.

Proposition 6.3. $\left[\mathbf{g}, (\mathbf{g}_c^*)^\sharp\right] \subset \mathbf{g}$, $\left[(\mathbf{g}^*)^\sharp, (\mathbf{g}_c^*)^\sharp\right] \subset (\mathbf{g}^*)^\sharp$, $\left[(\mathbf{g}_c^*)^\sharp, (\mathbf{g}_c^*)^\sharp\right] \subset \mathbf{g}$.

Indeed, for the first assertion we have

$$\begin{aligned} i_{\left[X_h,\Pi_c^\sharp\right]}\Omega_{\mathcal{M}} &= \mathcal{L}_{X_h}i_{\Pi_c^\sharp}\Omega_{\mathcal{M}} - i_{\Pi_c^\sharp}\mathcal{L}_{X_h}\Omega_{\mathcal{M}} \\ &= i_{X_h}di_{\Pi_c^\sharp}\Omega_{\mathcal{M}} + di_{X_h}i_{\Pi_c^\sharp}\Omega_{\mathcal{M}} \\ &= i_{X_h}\frac{c}{n}\Omega_{\mathcal{M}} + d\Omega_{\mathcal{M}}\left(X_h, \Pi_c^\sharp\right) \\ &= d\left(\frac{c}{n}h + \Omega_{\mathcal{M}}\left(X_h, \Pi_c^\sharp\right)\right). \end{aligned} \tag{6.27}$$

If we replace X_h with a locally Hamiltonian vector field then a similar computation implies that the bracket is locally Hamiltonian. For the second, we compute from the definition of locally Hamiltonian vector fields

$$
\begin{aligned}
di_{[\Pi_{id}^{\sharp},\Pi_{c}^{\sharp}]}\Omega_{\mathcal{M}} &= \mathcal{L}_{\Pi_{id}^{\sharp}} di_{\Pi_{c}^{\sharp}}\Omega_{\mathcal{M}} - di_{\Pi_{c}^{\sharp}} \frac{\operatorname{div}_{\mu}\Pi_{id}^{\sharp}}{n}\Omega_{\mathcal{M}} \\
&= \frac{c}{n^2}\operatorname{div}_{\mu}\Pi_{id}^{\sharp}\Omega_{\mathcal{M}} - d\left(\frac{1}{n}\operatorname{div}_{\mu}\Pi_{id}^{\sharp}\right)\wedge i_{\Pi_{c}^{\sharp}}\Omega_{\mathcal{M}} - \frac{1}{n}\left(\operatorname{div}_{\mu}\Pi_{id}^{\sharp}\right) di_{\Pi_{c}^{\sharp}}\Omega_{\mathcal{M}} \\
&= -d\left(\frac{1}{n}\operatorname{div}_{\mu}\Pi_{id}^{\sharp}\right)\wedge i_{\Pi_{c}^{\sharp}}\Omega_{\mathcal{M}}. \qquad (6.28)
\end{aligned}
$$

This can be zero only if $\Pi_c^{\sharp}$ is globally Hamiltonian with divergence of the arbitrary element $\Pi_{id}^{\sharp}$ of $(\mathbf{g}^*)^{\sharp}$, which is not possible.

We compute the action of homotheties on the space of densities using the identity in the Eq.(4.11) as

$$
\begin{aligned}
\mathcal{L}_{[\Pi_{id}^{\sharp},\Pi_{c}^{\sharp}]}\mu &= \mathcal{L}_{\Pi_{id}^{\sharp}}(c\mu) - \mathcal{L}_{\Pi_{c}^{\sharp}}(f\mu) \\
&= cf\mu - df\wedge i_{\Pi_{c}^{\sharp}}(\mu) - cf\mu \\
&= -df\wedge i_{\Pi_{c}^{\sharp}}(\mu) = -i_{\Pi_{c}^{\sharp}}(df)\,\mu = -\Pi_{c}^{\sharp}(f)\,\mu. \qquad (6.29)
\end{aligned}
$$

That means, if f is the density associated with $\Pi_{id} \in \mathbf{g}^*$, then we have

$$
\begin{aligned}
\Pi_{id} &\to f\mu \\
\left[\Pi_{c}^{\sharp},\Pi_{id}^{\sharp}\right] &\to \Pi_{c}^{\sharp}(f)\,\mu. \qquad (6.30)
\end{aligned}
$$

7. DISCUSSION AND CONCLUSIONS

Starting from a vector field X on a manifold $\mathcal{M}$, we have defined vector fields X^c and X^{c*} on $T\mathcal{M}$ and $T^*\mathcal{M}$, respectively. X^{c*} is a canonically Hamiltonian vector field whereas X^c is Hamiltonian if $\mathcal{M}$ is symplectic and if $T\mathcal{M}$ is equipped with Tulczyjew's symplectic form. A connection on a smooth bundle $(\mathcal{E},\pi,\mathcal{M})$ was considered as a mapping from $T\mathcal{M}$ to $T\mathcal{E}$. The first order generalized vector fields are sections $T\mathcal{E} \to J^1\pi$, where $J^1\pi$ is the first jet bundle. Elements of $J^1\pi$ were used as connections on the bundle π hence, $T\mathcal{E}$ was decomposed into subbundles, namely bundle of vertical representatives and bundle of holonomic lifts. From this decomposition, iterated bundles TT, T^*T, TT^* and T^*T^* were expressed as the direct sums of first order bundles T and T^*.

The momentum map realization of the Poisson equation has given that the true configuration space for the Poisson-Vlasov dynamics must be the semi-direct product structure $\mathcal{F}(\mathcal{Q})\circledS Diff_{can}(T^*\mathcal{Q})$. We have taken the group of canonical transformations $Diff_{can}(T^*\mathcal{Q})$ as our configuration space for collisionless plasma and adapted the Green's function solution of the Poisson equation as a constraint while taking variational derivatives.

Lie algebra of $Diff_{can}(T^*\mathcal{Q})$ is the space of Hamiltonian vector fields $\mathfrak{X}_{ham}(T^*\mathcal{Q})$ which is isomorphic to the space of smooth functions $\mathcal{F}(T^*\mathcal{Q})$ on $T^*\mathcal{Q}$. The dual

$$\mathfrak{X}^*_{ham}(T^*\mathcal{Q}) \rightarrow Den(T^*\mathcal{Q}) : \Pi_{id} \rightarrow \operatorname{div}_{\Omega_{T^*\mathcal{Q}}} \Pi^\sharp_{id} = f \tag{7.31}$$

of the isomorphism $\mathcal{F}(T^*\mathcal{Q}) \rightarrow \mathfrak{X}_{ham}(T^*\mathcal{Q})$ is a momentum mapping and defines the plasma density function f. For the dual space of the Lie algebra of $Diff_{can}(T^*\mathcal{Q})$, two equivalent representations are possible, namely the density and the momentum formulations on $Den(T^*\mathcal{Q})$ and $\mathfrak{X}^*_{ham}(T^*\mathcal{Q})$, respectively. In density representation, constraint variational derivative of

$$H_{LP}(f) = \int_{T^*\mathcal{Q}} f(\mathbf{z}) h_f(\mathbf{z})\, \mu(\mathbf{z}), \tag{7.32}$$

where $h_f(\mathbf{z}) = \delta^{ij} p_i p_j / 2m + e\phi_f(\mathbf{q})/2$, is

$$\frac{\delta H_{LP}}{\delta f} = h = \delta^{ij} \frac{p_i p_j}{2m} + e\phi_f(\mathbf{q}) \tag{7.33}$$

thus the Lie-Poisson equation for the Hamiltonian H_{LP} gives the Vlasov equation

$$\frac{\partial f}{\partial t} = -\left\{ f, \frac{\delta H_{LP}}{\delta f} \right\}_{T^*\mathcal{Q}} = -\{f, h\}_{T^*\mathcal{Q}}. \tag{7.34}$$

In momentum representation, we have considered the functional

$$H_{LP}(\Pi_{id}) = \int_{T^*\mathcal{Q}} \left\langle \Pi_{id}\left(\mathbf{z}\right), X_{h_f}\left(\mathbf{z}\right)\right\rangle d\mu\left(\mathbf{z}\right)$$

which is equivalent to $H_{LP}(f)$ up to some divergence factors. The constraint variational derivative

$$\frac{\delta H_{LP}(\Pi_{id})}{\delta \Pi_{id}} = X_h \tag{7.35}$$

has given the Lie-Poisson equations

$$\frac{d\Pi_{id}}{dt} = -ad^*_{\delta H_{LP}/\delta\Pi_{id}}(\Pi_{id}) = -\mathcal{L}_{\delta H_{LP}/\delta\Pi_{id}}(\Pi_{id}) = -\mathcal{L}_{X_h}\Pi_{id}. \tag{7.36}$$

In the momentum variables (Π_i, Π^i), we have arrived the intermediate system

$$\begin{aligned} \dot{\Pi}_i &= -X_h\left(\Pi_i\right) + e\frac{\partial^2\phi}{\partial q^i\partial q^j}\Pi^j \\ \dot{\Pi}^i &= -X_h\left(\Pi^i\right) - \frac{1}{m}\delta^{ij}\Pi_j \end{aligned} \tag{7.37}$$

named the momentum-Vlasov equations, where

$$X_h = \frac{1}{m}\delta^{ij}p_i\frac{\partial}{\partial q^j} - e\frac{\partial\phi_f}{\partial q^i}\frac{\partial}{\partial p_i} \tag{7.38}$$

is the Hamiltonian vector field corresponding to the Hamiltonian function h. Back-substitution of the plasma density $f = \operatorname{div}_{\Omega_{T^*\mathcal{Q}}}\Pi_{id}^{\sharp}$ in the momentum-Vlasov equations has given the Vlasov equation.

We have obtained the momentum-Vlasov equations from Hamiltonian vector field X_h for the Hamiltonian function h, which is the total energy of a single charged particle. The vertical representative

$$VX_h^{c*} = \left(e\Pi^j\frac{\partial^2\phi}{\partial q^j\partial q^i} - X_h(\Pi_i)\right)\frac{\partial}{\partial\Pi_i} - \left(\frac{1}{m}\Pi_j\delta^{ji} + X_h(\Pi^i)\right)\frac{\partial}{\partial\Pi^i} \tag{7.39}$$

of the complete cotangent lift X_h^{c*} of the Hamiltonian vector field X_h, generates the momentum-Vlasov equations.

A Lie algebra structure

$$\{\alpha,\beta\}_{\Lambda^1} = \mathfrak{L}_{\alpha^\sharp}(\beta) - \mathfrak{L}_{\beta^\sharp}(\alpha) + d\left(i_{\beta^\sharp} i_{\alpha^\sharp}\Omega_{\mathcal{M}}\right)$$

on the space of one-form section $\Lambda^1(\mathcal{M})$ on a symplectic manifold $\mathcal{M}$ was defined. It was shown that, the space of non-closed one-forms $\mathbf{g}^*$ and the space of exact one-forms $\mathbf{g}^\flat$, which is the isomorphic copy of the space of Hamiltonian vector fields $\mathbf{g}$, are two Lie subalgebras of $\left(\Lambda^1(\mathcal{M}), \{\ ,\ \}_{\Lambda^1(\mathcal{M})}\right)$. The space of vector fields on $\mathcal{M}$ was decomposed into the direct sum of the spaces of Hamiltonian $\mathbf{g}$ and nonHamiltonian $(\mathbf{g}^*)^\sharp$ vector fields, that is $\mathfrak{X}(\mathcal{M}) = \mathbf{g} \oplus (\mathbf{g}^*)^\sharp$.

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