GROUPS WITH THE BERGMAN PROPERTY

by

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ABSTRACT

GROUPS WITH THE BERGMAN PROPERTY

The aim of the thesis is the study of groups with the Bergman property. This property has been named after George Bergman who proved in 2003 that infinite symmetric groups have finite width relative to all their generating sets (the property which is now called the Bergman property). We shall expand the proofs of a number of results on the Bergman property, including the results from Bergman's original paper, in order to make the material understandable by senior undergraduate students.

ÖZET

BERGMAN ÖZELLİĞİNE SAHİP GRUPLAR

Tezin amacı Bergman özelliğine sahip gruplar hakkında bir çalışma yapmaktır. Bu özellik 2003 de George Bergman tarafından ispatlandı ve bu özelliğe göre sonsuz simetri grupları üretici kümelerine göre sonlu bir genişliğe sahiptirler(bu özellik Bergman özelliği olarak adlandırılıyor). Bergman özelliği hakkında birtakım sonuçların ispatını genişleteceğiz ve bu sonuçların bazıları Bergmanın kendi makalesinden alınmıştır. Amacımız bu tezi son sınıf lisans ögrencileri için anlaşılır hale getirmektir.

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LIST OF SYMBOLS/ABBREVIATIONS

Alt(X)	The alternating group of a set X
depth(t)	The depth of a term t
dom$(\mathcal{M})$	The domain of a first-order structure $\mathcal{M}$
fix(σ)	The fixed-point set of a permutation σ
FS(X)	The finite symmetric group of a set X
$G_{(Y)}$	The pointwise stabizer of a set Y
$G_{\{Y\}}$	The setwise stabizer of a set Y
GL(V)	The general linear group of a vector space V
NC(g)	The normal closure of a element g of a given group G
orb$_\sigma(x)$	The orbit of a given element x under a permutation σ
$R(\sigma)$	The residual space of a linear operator σ
res(σ)	The residue of a linear operator σ
supp(σ)	The support of a permutation σ
Sym(X)	The symmetric group of a set X
Sym(X, λ)	A bounded subgroup of the symmetric group Sym(X)
wid(G, S)	The width of a given group G with regard to its generating set S
$\Delta(\mathcal{M})$	The diagonal subalgebra of the Cartesian power $\mathcal{M}^{\aleph_0}$ of a structure $\mathcal{M}$
UF-cofinality	Uncountable, or Finite cofinality

1. PRELIMINARIES

1.1. SYMMETRIC GROUPS

In this section we reproduce some facts on infinite symmetric groups we shall need in the subsequent sections. Occasionally we shall recall some basic facts on finite symmetric groups; such facts, though, are assumed to be known.

1.1.1. Permutations

Let X be an arbitrary nonempty set; a bijection (a one-to-one, onto mapping) of X onto itself is called a *permutation* of X. The set of all permutations of X forms a group, under composition of mappings, called the *symmetric group* on X. We shall denote this by Sym(X),and write S_n to denote the special group Sym(X) when $X = \{1, 2, \ldots, n\}$. A *permutation group* is just a subgroup of a symmetric group.

Proposition 1.1.1. *For an infinite set* X*, set* $\lambda = |X|$*. Then*

$$|\operatorname{Sym}(X)| = 2^{\lambda}.$$

Proof. Let $P(X)$ denote the power set of X. Then $|P(X)| = 2^{\lambda}$. Also

$$\begin{aligned} |\operatorname{Sym}(X)| &\leqslant |\{g : X \to X\}| \\ &\leqslant |\{g : X \to P(X)\}| \\ &= (2^{\lambda})^{\lambda} \\ &= 2^{\lambda} \end{aligned}$$

as λ is an infinite cardinal. Therefore $|\operatorname{Sym}(X)| \leqslant 2^{\lambda}$.

To prove the reverse inequality, let us write X as a disjoint union of 2-element subsets of X. That is, let $X = \bigsqcup_{i \in I} \{x_i, y_i\}$. For every subset J of I, define a permutation $\pi_J := \prod_{i \in J}(x_i, y_i)$. Then different subsets J, J' of I will produce different permutations $\pi_J, \pi_{J'}$.

Since $|I| = \lambda$, there are 2^λ such subsets of I. So there are at least that many permutations in Sym(X). □

A permutation $\sigma \in$ Sym(X) is called an *r-cycle* $(r = 1, 2, ...)$ if for r distinct points $x_1, x_2, \ldots, x_r$ of X, σ maps x_i onto x_{i+1} $(i = 1, ..., r-1)$, maps x_r onto x_1, and leaves all other points fixed; and σ is called an *infinite cycle* if for some doubly infinite sequence x_i $(i \in \mathbb{Z})$, σ maps x_i onto x_{i+1} for each i and leaves all other points fixed.

A 1-cycle (x) is the *identity permutation* and a 2-cycle is called a *transposition*. Also, the *inverse* of the cycle $(i_1 i_2 \ldots i_r)$ is the cycle $(i_r i_{r-1} \ldots i_1)$.

As we know, one of the common ways is to specify a permutation in S_n is to write it as a product of disjoint cycles, by *disjoint* we mean that no two cycles move a common point. In infinite permutation groups, such a product is only a *formal* product.

Definition (Supports, fixed-point sets, orbits of permutations)**.** Let $\sigma \in$ Sym(X) then the *support* supp(σ) of σ is the set

$$\text{supp}(\sigma) = \{x \in X : \sigma x \neq x\}$$

and the *fixed-point set* fix(σ) of σ is the set

$$\text{fix}(\sigma) = \{x \in X : \sigma x = x\}.$$

Evidently,

$$X = \text{supp}(\sigma) \sqcup \text{fix}(\sigma)$$

where $\sqcup$ denotes the disjoint union of sets.

The *orbit* orb$_\sigma(x)$ of an $x \in X$ under σ is the set

$$\text{orb}_\sigma(x) = \{\sigma^m(x) : m \in \mathbb{Z}\}.$$

Any set of the form orb$_\sigma(x)$ where $x \in X$ is called an orbit of σ. An orbit of σ is called *nontrivial* if it has at least two elements.

Claim 1.1.2. *The relation "to be in the same orbit of* σ*" is an equivalence relation on* X*.*

Proof. We know that $\text{orb}_\sigma(x) = \{\sigma^m(x) : m \in \mathbb{Z}\}$. Now let $\sigma \in \text{Sym}(X)$ and $a, b \in X$, then

$$a \equiv_\sigma b \Leftrightarrow \exists m \in \mathbb{Z} \text{ such that } b = \sigma^m(a).$$

Since $a \equiv_\sigma a \Rightarrow \sigma^0(a) = a$ (reflexive), $a \equiv_\sigma b \Rightarrow \sigma^m(a) = b \Rightarrow a = \sigma^{-m}(b) \Rightarrow b \equiv_\sigma a$ (symmetric), and $a \equiv_\sigma b, b \equiv_\sigma c \Rightarrow \sigma^m(a) = b, \sigma^n(b) = c \Rightarrow \sigma^{m+n}(a) = \sigma^n(\sigma^m(a)) = \sigma^n(b) = c \Rightarrow a \equiv_\sigma c$ (transitive). □

Claim 1.1.3. *Let* $\sigma \in \text{Sym}(X)$ *be a permutation. Then*

(i) $\text{supp}(\sigma) = \text{supp}(\sigma^{-1})$.

(ii) $\text{supp}(\sigma_1\sigma_2) \subseteq \text{supp}(\sigma_1) \cup \text{supp}(\sigma_2)$;

(iii) $\text{supp}(\pi\sigma\pi^{-1}) = \pi\,\text{supp}(\sigma)$.

Proof. (i) Clearly,

$$\text{supp}(\sigma) = \{x \in X : \sigma x \neq x\} = \{x \in X : x \neq \sigma^{-1}x\} = \text{supp}(\sigma^{-1}).$$

(ii) Now, let $x \in \text{supp}(\sigma_1\sigma_2) \Rightarrow \overbrace{\sigma_1\sigma_2(x) \neq x} \Rightarrow \overbrace{\sigma_1(x) \neq x \text{ or } \sigma_2(x) \neq x}$. Switching to the contrapositive statement, we get that

$$\overbrace{\sigma_1(x) = x \text{ and } \sigma_2(x) = x} \Rightarrow \overbrace{\sigma_1\sigma_2(x) = \sigma_1(x) = x}$$

Therefore,

$$\text{supp}(\sigma_1\sigma_2) \subseteq \text{supp}(\sigma_1) \cup \text{supp}(\sigma_2).$$

(iii) Easy. □

Claim 1.1.4. *For every $\sigma \in \mathrm{Sym}(X)$, the support $\mathrm{supp}(\sigma)$ of σ is the (disjoint) union of all nontrivial orbits of σ.*

$$\mathrm{supp}(\sigma) = \bigsqcup_{i \in I} \mathrm{orb}_\sigma(y_i)$$

where $\{y_i : i \in I\}$ is a complete set of representatives of nontrivial orbits of σ.

Proof. By Claim 1.1.2. □

For a $\sigma \in \mathrm{Sym}(X)$, let $\chi_n(\sigma)$ $(1 \leqslant n \leqslant \aleph_0)$ denote the cardinality of the set of all n-element orbits of σ.

Lemma 1.1.5. *Permutations $\sigma, \sigma' \in \mathrm{Sym}(X)$ are conjugate if and only if*

$$\chi_n(\sigma) = \chi_n(\sigma')$$

for all cardinals $1 \leqslant n \leqslant \aleph_0$.

Proof. $(\Rightarrow)$. Let $\sigma \in \mathrm{Sym}(X)$ and let

$$\mathrm{supp}(\sigma) = \bigsqcup_{i \in I} O_i$$

is the disjoint union of nontrivial orbits of σ. Then for every $\pi \in \mathrm{Sym}(X)$, for the conjugate $\pi\sigma\pi^{-1}$ of σ by π we have that

$$\mathrm{supp}(\pi\sigma\pi^{-1}) = \bigsqcup_{i \in I} \pi O_i$$

and hence the set $\{\pi O_i : i \in I\}$ is the set of all nontrivial orbits of $\pi\sigma\pi^{-1}$. Now since π is a bijection,

$$\chi_n(\sigma) = \chi_n(\pi\sigma\pi^{-1})$$

Since, further,

$$\mathrm{fix}(\pi\sigma\pi^{-1}) = \pi(\mathrm{fix}(\sigma))$$

we have that,

$$\chi_1(\pi\sigma\pi^{-1}) = |\mathrm{fix}(\pi\sigma\pi^{-1})| = |\pi\mathrm{fix}(\sigma)| = |\mathrm{fix}(\sigma)| = \chi_1(\sigma)$$

($\Leftarrow$). Take a $\sigma \in \mathrm{Sym}(X)$. For each n with $1 \leqslant n \leqslant \aleph_0$ choose an index set I_n so that sets I_n are pairwise disjoint and the cardinality of the family of all n-element orbits of σ and the cardinality of I_n are equal. Let then

$$X_n(\sigma) = \bigsqcup_{i \in I_n} O_{i,n},$$

where $O_{i,n}$ is an n-element orbit of σ, be the disjoint union of all n-element orbits of σ. Suppose now that σ, σ' have the same cardinality of n-element orbits for all n with $1 \leqslant n \leqslant \aleph_0$. Then for every n with $1 \leqslant n \leqslant \aleph_0$ we can write

$$X_n(\sigma) = \bigsqcup_{i \in I_n} O_{i,n}, X_n(\sigma') = \bigsqcup_{i \in I_n} O'_{i,n}.$$

where $O'_{i,n}$ is an n-element orbit of σ' . Set

$$\sigma_{i,n} = \sigma|_{O_{i,n}}, \sigma'_{i,n} = \sigma|_{O'_{i,n}} \qquad (1 \leqslant n \leqslant \aleph_0) \qquad (1,1)$$

Now since

$$X = \bigsqcup_{1 \leqslant n \leqslant \aleph_0} X_n(\sigma) = \bigsqcup_{1 \leqslant n \leqslant \aleph_0} \bigsqcup_{i, I_n} O_{i,n},$$

one can define the permutation π of X as follows:

$$\pi(x) = \pi_{i,n}$$

if $x \in O_{i,n}$ for suitable n and $i \in I_n$. Then it follows from (1.1) that

$$\pi\sigma\pi^{-1} = \sigma'$$

as we wished to prove. $\square$

1.1.2. Normal Subgroups of Symmetric Groups

Since the alternating group A_n is simple whenever $n \neq 4$ (that is, A_n has no nonidentity proper normal subgroups whenever $n \neq 4$) [8], it follows quite easily that the only normal subgroups of the symmetric group S_n are

$$\{\text{id}\}, A_n, \text{ and } S_n.$$

Now let X be an infinite set; we will fix X till the end of this subsection. According to the famous theorem by Baer--Schreier--Ulam, any normal subgroup of Sym(X) is (exactly) one of the following:

$$\{\text{id}\}, \text{Alt}(X), \text{Sym}(X, \lambda), \text{ and } \text{Sym}(X).$$

Let $\lambda \leqslant |X|$ be a cardinal. The subgroup Sym(X, λ), which is called a bounded symmetric group is defined as follows:

$$\text{Sym}(X, \lambda) = \{\sigma \in \text{Sym}(X) : |\text{supp}(\sigma)| < \lambda\}.$$

In the case when $\lambda = \aleph_0$,

$$\text{Sym}(X, \aleph_0) = \{\sigma \in \text{Sym}(X) : |\text{supp}(\sigma)| < \aleph_0\}$$

is the subgroup of Sym(X) which consists of all permutations of X with finite support. The group Sym$(X, \aleph_0)$ is called then the *finitary symmetric group* of X, and it also is denoted by FS(X).

Also, the subgroup Alt(X) of FS(X), consisting of all even permutations, that is, permutations that can be written as a product of even number of transpositions is called the *alternating group* of X and is denoted by Alt(X).

In what follows we shall give a brief outline of the proof of Baer--Schreier--Ulam theorem.

Lemma 1.1.6. *For every infinite cardinal* $\lambda \leqslant |X|$, Sym(X, λ) *is a normal subgroup of* Sym(X).

Proof. By Claim 1.1.3. □

Definition (Involutions)**.** Let G be a group. An element x of G is called an *involution* if x is of order two.

Clearly, an element of the finite symmetric group S_n is an involution if and only if it is a product of disjoint transpositions (2-cycles).

In the infinite symmetric group Sym(X) an element is an involution if and only if all its nontrivial orbits are of length two.

Definition (Normal closures)**.** Let G be a group and let S be a subset of G. The *normal closure* NC(S) of S in G is the intersection of all normal subgroups of G containing S:

$$\text{NC}(S) = \bigcap_{S \subseteq H \trianglelefteq G} H = \langle gsg^{-1} : g \in G, s \in S \rangle.$$

Clearly, NC(S) is a normal subgroup of G. In particular, if $S = \{s\}$ is a singleton set,

$$\text{NC}(s) = \langle gsg^{-1} : g \in G \rangle,$$

that is, NC(s) is generated by all conjugates of s.

Let $\lambda \leqslant |X|$ be an infinite cardinal. The main idea of one of the classical proofs of Baer--Schreier--Ulam theorem (see e.g. [4, Chapter 8]) is to write Sym(X, λ) as the union of a chain of normal closures of suitable involutions π_μ, where $\mu < \lambda$ is a cardinal, for every $\lambda \leqslant |X|$:

$$\text{Sym}(X, \lambda) = \bigcup_{\mu<\lambda} \text{NC}(\pi_\mu).$$

Now we explain how to define the involutions π_μ. Let μ be a cardinal with $\mu < |X|$. We define an involution π_μ as follows:

- we choose two disjoint infinite subsets Y_μ, Z_μ of X of cardinality μ;
- (ii) we use an index set I of cardinality μ to write

  $$Y_\mu = \{y_i : i \in I\}$$

  and

  $$Z_\mu = \{z_i : i \in I\}$$

- we then construct π_μ as a unique permutation of X whose nontrivial orbits are exactly

  $$\{y_i, z_i\} \qquad (i \in I);$$

- it follows that

  $$\pi_\mu(y_i) = z_i \text{ and } \pi_\mu(z_i) = y_i$$

  for every $i \in I$ and

  $$\pi_\mu(t) = t$$

for every $t \in X \setminus (Y_\mu \cup Z_\mu)$;

- we observe that

$$\begin{aligned}\text{supp}(\pi_\mu) &= \text{the union of all nontrivial orbits of } \pi_\mu \\ &= \bigcup_{i\in I}\{y_i, z_i\} \\ &= Y_\mu \cup Z_\mu\end{aligned}$$

and hence the support of π_μ is of cardinality

$$|Y_\mu \cup Z_\mu| = |Y_\mu \sqcup Z_\mu| = |Y_\mu| + |Z_\mu| = \mu + \mu = 2\mu;$$

it follows that if μ is an infinite cardinal, the support of π_μ is of cardinality μ, since in this case $\mu + \mu = \mu$.

- finally, we observe that π_μ has

$$|X \setminus (Y_\mu \cup Z_\mu)| = |X|$$

fixed points.

We also construct one more involution $\pi^* = \pi_{|X|}$ corresponding to the cardinal $|X|$ itself. To construct π^* we choose subsets Y, Z of cardinality $|X|$ of X as before, but we shall require in addition the complement $Y \cup Z$ of X also be of cardinality $|X|$:

$$|X| = |Y| = |Z| = |X \setminus (Y \cup Z)|.$$

Then the construction of π^* goes forth as above: if

$$Y = \{y_i : i \in I\} \text{ and } Z = \{z_i : i \in I\}$$

then nontrivial orbits of π^* must be exactly two-element sets

$$y_i, z_i \quad (i \in I).$$

The special feature of $\pi^* = \pi_{|X|}$ which distinguishes it from other involutions π_μ with $\mu < |X|$ is that as we shall see that π^* *normally generates* $\mathrm{Sym}(X)$, that is,

$$\mathrm{Sym}(X) = \mathrm{NC}(\pi^*).$$

Note that in the case when $\mu = 1$, π_μ is just a transposition; hence the normal closure of π_μ is the subgroup $\mathrm{FS}(X)$ of all finitary permutations of X. If, further, $\mu = 2$, then π_μ is a product of two disjoint transpositions, an *even* permutation, and hence $\mathrm{NC}(\pi_\mu) = \mathrm{Alt}(X)$. More generally, if μ is a finite cardinal, then

$$\mathrm{NC}(\pi_\mu) = \begin{cases} \mathrm{FS}(X), & \text{if } \mu < \aleph_0 \text{ is odd,} \\ \mathrm{Alt}(X), & \text{if } \mu < \aleph_0 \text{ is even.} \end{cases}$$

Proposition 1.1.7. *Let $\mu \leqslant |X|$ be an infinite cardinal and let $\sigma \in \mathrm{Sym}(X)$ be such that*

$$|\mathrm{supp}(\sigma)| \leqslant \mu.$$

Then $\sigma \in \mathrm{NC}(\pi_\mu)$, and, consequently,

$$\mathrm{NC}(\sigma) \leqslant \mathrm{NC}(\pi_\mu).$$

The following three results are rather easy corollaries of Proposition 1.1.7.

Proposition 1.1.8. (i) *Let $\lambda < |X|$ be an infinite cardinal. Then*

$$\mathrm{Sym}(X, \lambda) = \bigcup_{\mu < \lambda} NC(\pi_\mu)$$

where μ is a cardinal.

(ii) $\mathrm{Sym}(X) = \mathrm{NC}(\pi^*)$.

Corollary 1.1.9. *The symmetric group* $\mathrm{Sym}(X)$ *is generated by involutions.*

Proposition 1.1.10. *Let $\sigma \in \mathrm{Sym}(X)$ be a permutation whose support is of infinite cardinality λ. Then the normal closure of σ contains the involution π_λ, and, in effect,*

$$\mathrm{NC}(\sigma) = NC(\pi_\lambda).$$

Theorem 1.1.11 (Schreier-Ulam, R. Baer)**.** *Let X be an infinite set. A proper nonidentity normal subgroup N of* $\mathrm{Sym}(X)$ *either equals to* $\mathrm{Alt}(X)$*, or there is an infinite cardinal $\lambda \leqslant |X|$ such that $N = \mathrm{Sym}(X, \lambda)$. It follows that the normal subgroups of* $\mathrm{Sym}(X)$ *form a chain:*

$$\{\mathrm{id}\} \lhd \mathrm{Alt}(X) \lhd \mathrm{Sym}(X, \aleph_0) \lhd \ldots \lhd \mathrm{Sym}(X, |X|) \lhd \mathrm{Sym}(X)$$

in which the subgroup $\mathrm{Sym}(X, |X|)$ *is the largest proper normal subgroup and hence the quotient group*

$$\mathrm{Sym}(X)/\,\mathrm{Sym}(X, |X|)$$

is simple.

1.1.3. Writing Every Element of $\mathrm{Sym}(X)$ As a Commutator, X an Infinite Set

It is well-know that in an infinite symmetric group every element is a commutator [12]. In this subsection we reproduce one of the most recent proofs of this result due to George Bergman [1].

Definition (Replete permutations)**.** Let X be an infinite set, we call an element $f \in \mathrm{Sym}(X)$ a *replete permutation* if it has $|X|$ orbits of each positive cardinality$\leqslant \aleph_0$ (including orbits of cardinality 1). For a subset $Y \subseteq X$ of cardinality $|X|$, we call that "σ is replete on Y" if $\sigma(Y) = (Y)$ and $\sigma|_Y$ is a replete permutation of Y.

A permutation of X which is replete on a subset $Y \subseteq X$ of cardinality $|X|$ is necessarily replete on X and the replete permutations of X form a conjugacy class.

Lemma 1.1.12. *Let X be an infinite set, every $\sigma \in$ Sym(X) is the product of two replete permutations.*

Proof. Given $\sigma \in$ Sym(X), choose a moiety Y_0 of X such that σ moves only finitely many elements from Y_0 to $X - Y_0$ or from $X - Y_0$ to Y_0.

Now if X is uncountable then the existence of such a Y_0 is immediate, since X is uncountable we can break X into two families and these two families have $|X|$ orbits in each, then take $Y_0$the union of one of these families.

Now let's check when X is countable, we can apply the same method if σ has infinitely many orbits, and can get the same conclusion in an obvious way if f has more than one infinite orbit. If σ has exactly one infinite orbit, $\langle\sigma\rangle(x_0)$, finitely many finite orbits, we can take $Y_0 = \{\sigma^n(x_0) : n \leqslant 0\}$; clearly σ moves exactly one element out of Y_0, and none into it.

After choosing Y_0, let us split $X - Y_0$ into two disjoint moities Y_1 and Y_2, $\big((X - Y_0) = Y_1 \cup Y_2$ $Y_1 \cap Y_2 = \varnothing\big)$ so that Y_1 contains the finitely many elements of $(\sigma(Y_0) \cup \sigma^{-1}(Y_0)) - Y_0$.

If $\pi_0 \in$ Sym(Y_0) and $\pi_2 \in$ Sym(Y_2), then there exist a permutation ρ of X such that $(\sigma\rho = \pi_0)|_{Y_0}$ and $(\rho = \pi_2)|_{Y_2}$. This pair of conditions specifies the values of ρ on the two disjoint sets $\sigma(Y_0)$ and Y_2 in a one to one fashion, and both the set on which it leaves ρ unspecified and the set of elements that it does specify as values for ρ are of cardinality $|X|$. Hence the former set can be mapped bijectively to the latter, and the resulting bijection will complete the definition of ρ.

Now if we take $\pi_0 \in$ Sym(Y_0) and $\pi_2 \in$ Sym(Y_2) as replete permutations, then $\sigma\rho$ will be replete on Y_0 hence replete, and ρ will be replete on Y_2 hence replete.

Therefore, $\sigma = (\sigma\rho)\rho^{-1}$ is a product of two replete permutations. □

Corollary 1.1.13. *Every element of* Sym(X) *is a commutator.*

Proof. The inverse of a replete permutation is, as it easy to see, also a replete permutation. So the conjugacy class of all replete permutations in Sym(X) is closed under taking inverses.

Now if $\sigma_1\sigma_2$ is a product of two replete permutations we have for a suitable $\pi \in \mathrm{Sym}(X)$

$$\sigma_1\sigma_2 = \pi\sigma_2^{-1}\pi^{-1}\sigma_2,$$

and the result follows. □

1.2. GENERAL LINEAR GROUPS

1.2.1. Modules

Let A be a ring with identity. A *left module* M over A is an abelian group, usually written additively, together with the scalar multiplication by elements of A on M (viewing A as a multiplicative monoid) such that for all $a, b \in A$ and $x, y \in M$ we have

$$(a+b)x = ax + bx \text{ and } a(x+y) = ax + ay.$$

So $a(-x) = -(ax)$ because $a(0+(-x)) = a0 + a(-x) = -(ax)$ and $0x = 0$ because $(0+0)x = 0x + 0x = 0x \Rightarrow 0x = 0$. By definition of an operation, we have $1x = x$. In a similar way, one defines a *right A-module.*

Let M be an A-module. By a *submodule* N of M we mean an additive subgroup such that $AN \subset N$. Then N is a module(with the operation induced by that of A on M).

1.2.2. Vector Spaces

A module over a field is called a *vector space.*

Theorem 1.2.1. *Let V be a vector space over a field K, and assume that $V \neq \{0\}$. Let Γ be a set of generators of V over K and let S be a subset of Γ which is linearly independent. Then there exists a basis $\mathfrak{B}$ of V such that $S \subset \mathfrak{B} \subset \Gamma$.*

Proof. Let $\mathfrak{I}$ be the set whose elements are subsets T of Γ which contain S and are linearly independent. Then $\mathfrak{I}$ is not empty (it contains S), and we contend that $\mathfrak{I}$ is inductively ordered. Indeed, if $\{T_i\}$ is a totally ordered subset of $\mathfrak{I}$ (by ascending inclusion), then $\bigcup T_i$ is again linearly independent and contains S. By Zorn's lemma, let $\mathfrak{B}$ be a maximal element of $\mathfrak{I}$. Then $\mathfrak{B}$ is linearly independent. Let W be the subspace of V generated by $\mathfrak{B}$. If $W \neq V$, there exist some element $x \in \Gamma$ such that $x \notin W$. Then $\mathfrak{B} \cup \{x\}$ is linearly independent, for given a linear combination

$$\Sigma_{y \in \mathfrak{B}} a_y y + bx = 0, \ a_y, b \in K,$$

we must have $b = 0$, otherwise we get

$$x = -\Sigma_{y \in \mathfrak{B}} b^{-1} a_y y \in W.$$

By construction, we now see that $a_y = 0$ for all $y \in \mathfrak{B}$, thereby proving that $\mathfrak{B} \cup \{x\}$ is linearly independent, and contradicting the maximality of $\mathfrak{B}$. It follows that $W = V$, and furthermore that $\mathfrak{B}$ is not empty since $V \neq \{0\}$. This proves our theorem. □

If a vector space $\neq \{0\}$, then in particular, we see that every set of linearly independent elements of V can be extended to a basis, and that a basis may be selected from a given set of generators.

Proposition 1.2.2. *Let V be a vector space over a field K. Then two bases of V over K have the same cardinality.*

Proof. Let us first assume that there exists a basis of V with a finite number of elements, say $\{v_1, v_2, \dots, v_m\}$, $m \geqslant 1$. We shall prove that any other basis must also have m elements. For this it will suffice to prove: If $w_1, w_2, \dots, w_n$ are elements of V which are linearly independent over K, then $n \leqslant m$ (for we can then use symmetry). We proceed by induction. There exists elements, $c_1, c_2, \dots, c_m$ of K such that

$$w_1 = c_1 v_1 + \cdots + c_m v_m, \tag{1.2.1}$$

and some c_i, say c_1, is not equal to 0. Then v_1 lies in the space generated by $w_1, v_2, \dots, v_m$ over K, and this space must therefore be equal to V itself. Furthermore, $w_1, v_2, \dots, v_m$ are linearly independent, for suppose $b_1, b_2, \ldots, b_m$ are elements of K such that

$$b_1 w_1 + b_2 v_2 + \cdots + b_m v_m = 0.$$

If $b_1 \neq 0$, divide by b_1 and express w_1 as a linear combination of $v_2, v_3, \ldots, v_m$. Subtracting from (1.2.1) would yield a relation of linear dependence among the v_i, which is impossible. Hence $b_1 = 0$, and again we must have all $b_i = 0$ because the v_i are linearly independent.

Suppose inductively that after a suitable renumbering of the v_i, we have found $w_1, \ldots, w_r (r < n)$ such that

$$\{w_1, \ldots, w_r, v_{r+1}, \ldots, v_m\}$$

is a basis of V. We express w_{r+1} as a linear combination

$$w_{r+1} = c_1 w_1 + \cdots + c_r w_r + c_{r+1} v_{r+1} + \cdots + c_m v_m, \tag{1.2.2}$$

with $c_i \in K$. The coefficients of the v_i in this relation cannot all be 0; otherwise there would be a linear dependence among the w_j. Say $c_{r+1} \neq 0$. Using an argument similar to that used above, we can replace v_{r+1} by w_{r+1} and still have a basis of V. This means that we can repeat the procedure until $r = n$, and therefore that $n \leqslant m$, thereby proving our theorem. □

If a vector space V admits one basis with a finite number of elements, say m, then we shall say that V is *finite dimensional* and that m is its *dimension*. In view of Proposition 1.2.2, we see that m is the number of elements in any basis of V. If $V = \{0\}$, then we define its dimension to be 0, and say that V is 0-dimensional. We abbreviate "dimension" by "dim" or "$\dim_K$" if the reference to K is needed for clarity.

When dealing with vector spaces over a field, we use the words *subspace* and *factor space* instead of *submodule* and *factor module*.

Proposition 1.2.3. *Let V be a vector space over a field K, and let W be a subspace of V.*

Then

$$\dim_K V = \dim_K W + \dim_K V/W.$$

If $f : V \to U$ *is a homomorphism of vector spaces over* K*, then*

$$\dim V = \dim \mathrm{Ker} f + \dim \mathrm{Im} f.$$

(*the* rank-nullity *theorem*)

Proof. The first statement is a special case of the second, taking for f the canonical map. Let $\{v_i\}_{i \in I}$ be a basis of Imf, and let $\{w_j\}_{j \in J}$ be a basis of Ker$f$. Let $\{v_i\}_{i \in I}$ be a family of elements of V such that $f(v_i) = u_i$ for each $i \in I$. We contend that

$$\{v_i, w_j\}_{i \in I, j \in J}$$

is a basis for V. This will obviously prove our assertion.

Let x be an element of V. Then there exist elements $\{a_i\}_{i \in I}$ of K almost all of which are 0 such that

$$f(x) = \Sigma_{i \in I} a_i u_i.$$

Hence $f(x - \Sigma a_i v_i) = f(x) - \Sigma a_i f(v_i) = 0$. Thus

$$x - \Sigma a_i v_i$$

is in the kernel of f, and there exist elements $\{b_j\}_{j \in J}$ of K almost all of which are 0 such that

$$x - \Sigma a_i v_i = \Sigma b_j w_j.$$

From this we see that $x = \Sigma a_i v_i + \Sigma b_j w_j$, and that $\{v_i, w_j\}$ generates V.It remains to be shown that the family $\{v_i, w_j\}$ is linearly independent. Suppose that there exist elements c_i, d_j such that

$$0 = \Sigma c_i v_i + \Sigma d_j w_j.$$

Applying f yields

$$0 = \Sigma c_i f(v_i) = \Sigma c_i u_i,$$

whence all $c_i = 0$. From this we conclude at once that all $d_j = 0$, and hence that our family $\{v_i, w_j\}$ is a basis for V over K, as was to be shown. □

Corollary 1.2.4. *Let V be a vector space and W a subspace. Then*

$$\dim W \leqslant \dim V.$$

If V is finite dimensional and $\dim W = \dim V$ *then* $W = V$.

1.2.3. General Linear Groups

If V is a vector space, then the family $\mathrm{GL}(V)$ of all invertible (bijective) linear operators of V is called the *general linear group* of V.

For any element $\sigma \in \mathrm{GL}(V)$ we can define the *residual space* $R(\sigma)$ of σ as

$$R(\sigma) = \mathrm{Im}(\sigma - \mathrm{id}) = \{\sigma x - x : x \in V\}$$

and the *fixed-point subspace* $P(\sigma)$ as

$$P(\sigma) = \mathrm{Ker}(\sigma - \mathrm{id}) = \{x \in V : \sigma x = x\}$$

(notice the similarity---and more to follow---with the notions of the support and the fixed-point set of a permutation). Due to the rank-nullity theorem (Proposition 1.2.3),

$$\dim R(\sigma) + \dim P(\sigma) = \dim V.$$

The dimension $\dim R(\sigma)$ of the residual space of σ is called the *residue* and is denoted by $\operatorname{res}(\sigma)$.

Proposition 1.2.5. *Let* $\sigma, \pi, \sigma_1, \sigma_2 \in \mathrm{GL}(V)$. *Then*

(i) $\operatorname{res}(\sigma^{-1}) = \operatorname{res}(\sigma)$;

(ii) $\operatorname{res}(\pi\sigma\pi^{-1}) = \operatorname{res}(\sigma)$;

(iii) $\operatorname{res}(\sigma_1\sigma_2) \leqslant \operatorname{res}(\sigma_1) + \operatorname{res}(\sigma_2)$.

Proof. (i). We have that $R(\sigma^{-1}) = R(\sigma)$, since

$$\begin{aligned} R(\sigma^{-1}) &= \{\sigma^{-1}x - x : x \in V\} = \{\sigma^{-1}(\sigma x) - \sigma x : \sigma x \in V\} \\ &= \{x - \sigma x : x \in V\} = R(\sigma). \end{aligned}$$

(ii). It is easy to see that $R(\pi\sigma\pi^{-1}) = \pi R(\sigma)$.

(iii). We have that $R(\sigma_1\sigma_2) \subseteq R(\sigma_1) + R(\sigma_2)$, since

$$\sigma_1\sigma_2 x - x = \underbrace{\sigma_1(\sigma_2 x) - \sigma_2 x}_{\text{in } R(\sigma_1)} + \underbrace{\sigma_2 x - x}_{\text{in } R(\sigma_2)}$$

for all $x \in V$. □

Next, we shall reproduce a result by A. Rosenberg (1958) on the structure of normal subgroups of the general linear group of an infinite-dimensional vector space [13]. Fix an infinite-dimensional vector space V over a field K.

Let $\lambda \leqslant \dim V$ be an infinite cardinal. Set

$$\Gamma(\lambda) = \{\sigma \in \mathrm{GL}(V) : \operatorname{res}(\sigma) < \lambda\}.$$

Proposition 1.2.6. *For every infinite cardinal $\lambda \leqslant \dim V$, $\Gamma(\lambda)$ is a normal subgroup of* $\mathrm{GL}(V)$.

Proof. By Proposition 1.2.5. □

Theorem 1.2.7 (A. Rosenberg, [13])**.** *If N is a normal subgroup of $\Gamma = \mathrm{GL}(V)$, then either N is contained in the center of Γ, that is in the subgroup $\mathrm{RL}(V)$ of all radiations of V, or there exist an infinite cardinal $\lambda \leqslant \dim V$ and a subgroup D of the multiplicative group K^* of K such that $N = D\Gamma(\lambda)$. Consequently, the normal subgroup $K^*\Gamma(\varkappa)$ where $\varkappa = \dim V$ contains all proper normal subgroups of Γ and the quotient group $\Gamma/K^*\Gamma(\varkappa)$ is simple.*

1.3. FIRST-ORDER STRUCTURES

1.3.1. Predicates, Operations, and Constants

Recall that if A is a some set, then by $A \times A$ we denote the set of all ordered pairs (a_1, a_2), where a_1, a_2 are elements of A, by $A \times A \times A$ we denote the set of all ordered triples of elements of A, etc. In general, the set

$$A \times \ldots \times A = A^n = \{(a_1, \ldots, a_n) : a_i \in A\}$$

is called the *n-th Cartesian power of A*.

Definition (Predicates)**.** Let A be a non-empty set. Then an *n-placed predicate* on A is an arbitrary map $P : A^n \to \{$**T**,**F**$\}$. So that we assign to every n-tuple, $(a_1, \ldots, a_n)$, consisting of elements of A, the value **T** (true) or **F** (false). The set $\{(a_1, \ldots, a_n) \in A^n : P(a_1, \ldots, a_n) =$ **T**$\}$ is said to be the *set of realizations* of P.

We can consider a predicate as a *condition* which either holds (**T**), or does not hold (**F**) at a given n-tuple of elements A.

Note that one-placed predicates are also called *unary*, two-placed predicates---*binary*, three-placed ones---*ternary*. In the general case we say in such a manner about n-ary predicates.

Definition (Operations)**.** Let A be a set. Then a *n-placed operation* on A is any mapping of the form $f : A^n \to A$. Therefore, an n-ary operation assigns to each n-*tuple* of elements of A an *element* A.

Some elements of mathematical objects have special properties that differentiates (distinguish) them from other elements. As examples we can consider 0, the neutral element of the addition operation on the set of reals, and 1, the neutral element of the multiplication on the set of reals. The special symbol $\varnothing$ denotes the empty set, etc. Such distinguished elements are called *constants*.

Definition (Structures)**.** A non-empty set A equipped with a set $\{P_i : i \in I\}$ of predicates, a set $\{f_j : j \in J\}$ of operations on A, and some set $\{c_k : k \in K\}$ of constants,

$$\mathcal{A} = \langle A; \{P_i : i \in I\}, \{f_j : j \in J\}, \{c_k : k \in K\}\rangle$$

is called a *structure.* The set A is said to be the *domain* (or the *universe*) of the structure $\mathcal{A}$, symbolically,

$$\mathrm{dom}(\mathcal{A}) = A.$$

The relations in

$$\{P_i : i \in I\} \cup \{f_j : j \in J\} \cup \{c_k : k \in K\}$$

are called the *basic* or the *primitive* relations of $\mathcal{A}$.

In the case when A is equipped with no predicates, that is, in the case when $\{P_i : i \in I\} = \varnothing$, the structure $\mathcal{A}$ is also called an *algebra*.

The key word in the definition of a structure is the word `together': the domain of a structure, predicates and operations defined on the domain and constants have to be *all* considered as an *organic whole.*

Examples

(i) $\mathcal{A} = \langle \mathbf{N}; +, \cdot, \mathrm{Sc}, 0 \rangle$ (Peano's structure). The domain of this structure is the set of naturals **N**, Sc (the successor function) is the unary function

$$\mathrm{Sc}(n) = n + 1,$$

and the constant is 0.

(ii) $\langle \mathbf{R}; +, \cdot, 0, 1, < \rangle$ (the ordered field of real numbers). This structure has two operations, one binary predicate and two constants.

(iii) Groups and rings are algebras in the sense of the above definition. Indeed, a group G may be considered as: either as an algebra with one basic binary operation $\langle G; \cdot \rangle$ where $\cdot$ is the multiplication on G, or as an algebra $\langle G; \cdot, ^{-1}, e \rangle$ with two basic operations and a constant; here we add the unary operation $^{-1}$ for taking inverses, and a constant e for the identity element of G.

Similarly, one sees that any ring can be viewed an algebra.

(iv) Let V be a vector space over a division ring D. For each $\alpha \in D$, consider a unary operation f_α on V defined as follows:

$$f_\alpha(x) = \alpha \cdot x \qquad [x \in V]$$

where $\cdot$ is the scalar multiplication. Then V, viewed as an algebra, is the structure

$$\mathcal{V} = \langle V; +, \{ f_\alpha : \alpha \in D \} \rangle$$

where $+$ is the vector addition on V.

1.3.2. Structures in a First-Order Language

A *first-order language* $\mathcal{L}$ is a collection of distinct symbols partitioned into three parts:

$$\mathcal{L} = \mathcal{P} \cup \mathcal{F} \cup \mathcal{C},$$

where the set $\mathcal{P}$ is the set of *predicate symbols*, $\mathcal{F}$ is the set of *symbols of operations* (*function symbols*) and $\mathcal{C}$ is the set of *constant symbols.*

As it is almost universally done in the textbooks on model-theory, we shall assume $\mathcal{P}$ contains the two-placed predicate symbol $=^{(2)}$ (to be always interpreted by the equality relation).

Each predicate and function symbol from $\mathcal{L}$ comes in the following *specified* form:

$$P_i^{(n_i)} \text{ and } f_j^{(m_j)}$$

where superscripts (n_i) and (m_j) indicate the *arity* (number of arguments, number of places) of the associated symbol.

Definition (Interpretations of first-order languages)**.** Let $\mathcal{L} = \mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ be a first-order language. Suppose that

$$\mathcal{P} = \{P_i^{(n_i)} : i \in I\},$$
$$\mathcal{F} = \{f_j^{(m_j)} : j \in J\},$$
$$\mathcal{C} = \{c_k : k \in K\}.$$

Then the process of an *interpretation of the language* $\mathcal{L}$ is realized as follows:

- we fix some non-empty set A;

- for each predicate symbol $P_i^{(n_i)}$ from the set $\mathcal{P}$ of predicate symbols we define on A an n_i-placed predicate $P_i^{\mathcal{A}}$ ($i \in I$);

  By the agreement, the predicate symbol $=^{(2)}$ is a member of $\mathcal{P}$, and we always interpret it by the equality relation on A;

- for each function symbol $f_j^{(m_j)}$ from the set $\mathcal{F}$ of all function symbols of $\mathcal{L}$ we define on the set A an (m_j)-placed operation $f_j^{\mathcal{A}}$;

- we link each constant symbol c_k from $\mathcal{C}$ with some element $c_k^{\mathcal{A}}$ from A.

The structure $\mathcal{A}$

$$\mathcal{A} = \langle A; \{P_i^{\mathcal{A}} : i \in I\}, \{f_j^{\mathcal{A}} : j \in J\}, \{c_k^{\mathcal{A}} : k \in K\}\rangle$$

obtained in the way described above is called a *structure in the language* $\mathcal{L}$.

1.3.3. Alphabets and Words. Terms of First-Order Languages

Let X be any set which we shall call an *alphabet*, meaning that we are going to construct *words* over X.

A *word* of the alphabet X (or, simply, over X) is an ordered sequence

$$x_0 x_1 \ldots x_{n-1}$$

of elements of X. It is convenient to consider *empty* sequence of elements of X, the so-called of *empty* word, which is denoted by $\langle\rangle$. By the definition, the length of the empty word is 0, the length of a word $x_0 x_1 \ldots x_{n-1} \neq \langle\rangle$ is n.

The family of all words of length n ($n \in \mathbf{N}$) is denoted by X^n.

The family of all words over X is denoted by $W(X)$. Thus

$$W(X) = \bigcup_{n \in \mathbf{N}} X^n.$$

Lemma 1.3.1. *Let X be a nonempty alphabet. Then the cardinality $|W(X)|$ of the set of all words over X is equal to $|X| + \aleph_0$:*

$$|W(X)| = |X| + \aleph_0.$$

Accordingly, if X is infinite then, $|W(X)| = |X|$.

Proof. If X is finite, then of course there are exactly $\aleph_0$ words over X:

$$|W(X)| = \aleph_0.$$

On the other hands, as X is finite, then

$$\aleph_0 = |X| + \aleph_0.$$

Let then X be infinite. Then for all $n \geqslant 1$

$$|X^n| = |X|^n = |X|.$$

Now

$$\begin{aligned} |X| + \aleph_0 = |X| \leqslant |W(X)| = |\bigcup_{n\in\mathbf{N}} X^n| \leqslant \bigcup_{n\geqslant 1} |X^n| \\ = \bigcup_{n\geqslant 1} |X| \leqslant \aleph_0 \cdot |X| \\ = |X| = |X| + \aleph_0. \end{aligned}$$

□

We define the operation of the *concatenation* on $W(X)$:

$$u, v \in W(X) \mapsto uv.$$

If $w \in W(X)$ and w can be written as

$$w = u_1 u u_2$$

then u is called a *subword* of w.

One of the reasons to have the empty word over X is to have a workable definition of a subword of a given word.

Now let $\mathcal{L}$ be a first-order language. Our goal is to give the formal definition of a *term* in $\mathcal{L}$. First, we recall the definition of the *alphabet of the first-order logic in* $\mathcal{L}$.

Definition (The alphabet of first-order logic in $\mathcal{L}$)**.** The *alphabet* of first-order logic in the language $\mathcal{L}$ is the set that contains the following elements:

- predicate, function and constant symbols from $\mathcal{L}$;
- logical connectives $\wedge, \vee, \rightarrow$ and $\neg$;
- quantifiers $\forall$ (the universal quantifier) and $\exists$ (the existential quantifier);
- parentheses (,) ;
- the infinite list of so-called *free variables* $x_0, x_1 \ldots, x_{n-1}, \ldots$
- the infinite list of *bound variables* $y_0, y_1 \ldots, y_{n-1}, \ldots$

Definition (Terms)**.** A *term* of the language $\mathcal{L}$ (sometimes an $\mathcal{L}$-term, for convenience's sake) is a word in the alphabet of the first-order logic in $\mathcal{L}$ which can be obtained by subsequent applications of the following rules:

(T1). each free variable x_k and each constant symbol c in $\mathcal{L}$ is a term in the language $\mathcal{L}$;

(T2). if $f = f^{(n)}$ is a function symbol from $\mathcal{L}$ and words $t_1, \ldots, t_n$ are terms, then the word

$$f(t_1, \ldots, t_n)$$

is a term in the language $\mathcal{L}$.

We also state that

(T3). there are no $\mathcal{L}$-terms other than those that are obtained by application of rules (T1) and (T2).

It is helpful to observe that $\mathcal{L}$-terms are *words of the alphabet*

$$\mathcal{F} \cup \mathcal{C} \cup \{x_k : k \in \mathbf{N}\} \cup \{(,)\} \cup \{,\}.$$

This is an infinite language, and then there are at most $|\mathcal{F}| + |\mathcal{C}| + \aleph_0$ $\mathcal{L}$-terms. On the other hand, it is clear from the definition that there at least $|\mathcal{C}| + \aleph_0 + |\mathcal{F}|$ $\mathcal{L}$-terms. Thus the following result is true.

Proposition 1.3.2. *There are exactly* $|\mathcal{F}| + |\mathcal{C}| + \aleph_0$ $\mathcal{L}$*-terms.*

Suppose that $\mathcal{L} = \{=^{(2)}, f^{(1)}, g^{(2)}, c\}$ (one unary, one binary function symbol and a constant). Then the following words are $\mathcal{L}$-terms:

$$x_0, c, f(x_1), f(c), g(c, c), f(f(x_2)), g(f(x_1), f(f(x_2))).$$

1.3.4. Cartesian Products of First-Order Structures

Fix a first-order language $\mathcal{L} = \mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ and a family $\{\mathcal{M}_i : i \in I\}$ of first-order structures in $\mathcal{L}$. We shall explain how to introduce a $\mathcal{L}$-structure on the Cartesian product $\prod_{i \in I} \mathcal{M}_i$ of the said family of structures.

Let M_i denote dom$(\mathcal{M}_i)$ for every $i \in I$. Recall that the *Cartesian product* $\prod_{i \in I} M_i$ is the family $\{f\}$ of all functions of the form

$$f : I \to \bigcup_{i \in I} M_i$$

such that $f(i) \in M_i$ for all $i \in I$. Given an element $x \in \prod_{i \in I} M_i$ such that

$$x(i) = x_i \qquad [i \in I]$$

it is convenient to write x as $(x_i)_{i \in I}$.

If $P \in \mathcal{L}$ is a predicate symbol we shall denote by P_i its interpretation on $\mathcal{M}_i$, that is, the predicate $P^{\mathcal{M}_i}$ $(i \in I)$.

Similarly, if $f \in \mathcal{L}$ is a function symbol we shall denote by f_i its interpretation on $\mathcal{M}_i$, that is, the operation $f^{\mathcal{M}_i}$ on dom$\mathcal{M}_i$ $(i \in I)$.

Finally, if $c \in \mathcal{L}$ is a constant symbol we denote by c_i the element $c^{\mathcal{M}_i}$ of M_i $(i \in I)$.

We denote the structure we are going to construct, the *Cartesian product* $\prod_{i \in I} \mathcal{M}_i$ of the family $\{\mathcal{M}_i : i \in I\}$ by $\mathcal{M}$.

The domain: $\mathrm{dom}(\mathcal{M}) = \prod_{i \in I} M_i$.

Interpretation of the constants: if $c \in \mathcal{L}$ we set

$$c^{\mathcal{M}} = (c_i)_{i \in I} = (c^{\mathcal{M}_i})_{i \in I}.$$

Intepretations of the function symbols: let $f \in \mathcal{L}$ be a function symbol of arity n and let

$$(a_{1,i})_{i \in I}, \dots, (a_{n,i})_{i \in I}$$

be an n-tuple of elements of $\prod_{i \in I} M_i$. Then, by the definition,

$$f^{\mathcal{M}}((a_{1,i})_{i \in I}, \dots, (a_{n,i})_{i \in I}) = (f_i(a_{1,i}, \dots, a_{n,i}))_{i \in I}.$$

Intepretations of the predicate symbols: let $P \in \mathcal{L}$ be a predicate symbol of arity n and let

$$(a_{1,i})_{i \in I}, \dots, (a_{n,i})_{i \in I}$$

be an n-tuple of elements of $\prod_{i \in I} M_i$. Then

$$P^{\mathcal{M}}((a_{1,i})_{i \in I}, \dots, (a_{n,i})_{i \in I}) = \mathbf{T} \iff P_i(a_{1,i}, \dots, a_{n,i}) = \mathbf{T} \text{ for all } i \in I.$$

2. GROUPS WITH THE BERGMAN PROPERTY

2.1. INTRODUCTION

Let G be a group and let S be a generating set of G. Then the *width* $\mathrm{wid}(G, S)$ of G relative to S is the least natural number k such that any element of G is expressible as a product of at most k elements of $S \cup S^{-1}$, or ∞, otherwise. In the case when the width of G with regard to S is a finite number k, it is also customary to say that G is generated by S in k steps.

A group is said to have *the Bergman property* [6] (or to be a group of *finite width*) if its width relative to any generating set is finite. The property is named after George M. Bergman who found that it is satisfied by all infinite symmetric groups [1]. The first example of an infinite group with the Bergman property had been found by Shelah in the 1980s [14].

A preprint version of [1] attracted a considerable attention and very soon other examples of groups of uniformly finite width have been found: the automorphism groups of doubly transitive chains [7], the automorphism group of **R** as a Borel space [6], infinite-dimensional general linear groups over fields [15], the automorphism groups of ω-stable and ω-categorical structures and the automorphism group of the random graph [9] etc.

In the first section we give the proof of finiteness of width of infinite symmetric groups [1]. It is based on the ideas developed in papers [5, 10, 15].

In the next section we shall consider some basic properties of the class **B** of all groups with the Bergman property.

The third section will be devoted to the proof of the fact that infinite-dimensional linear groups over fields have the Bergman property [15].

In the final, fourth section we shall discuss sufficient conditions on algebras (in particular, on groups) in terms of their Cartesian powers to satisfy an analogue of the Bergman property [2].

2.2. INFINITE SYMMETRIC GROUPS

The primary of goal of this section is to show all infinite symmetric groups have the Bergman property [1]. The proof we are going to give is different from the original proof found by Bergman in [1]: it is rather an adaptation for the (easier) case of the infinite symmetric groups of the proof that infinite-dimensional linear groups have the Bergman property given in [15].

We shall use the standard notation of the theory of permutation groups. If a group G acts on a set X and if Y is a subset of X, then by $G_{(Y)}$ we shall the pointwise stabilizer of Y in G,

$$G_{(Y)} = \{g \in G : g \cdot y = y \text{ for all } y \in Y\}$$

and by $G_{\{Y\}}$ we shall denote the setwise stabilizer of Y in G:

$$G_{\{Y\}} = \{g \in G : g \cdot Y = Y\}.$$

Any notation like $G_{*_1,*_2}$ means the intersection $G_{*_1} \cap G_{*_2}$ of subgroups G_{*_1} and G_{*_2} of G.

Let us fix an infinite set X till the end of this section

2.2.1. Writing $\mathrm{Sym}(X)$ as a Power of a Conjugacy Class

Along with fixing X, fix also a partition of X

$$X = X_1 \sqcup X_2 \sqcup Y$$

into moieties. Consider an involution $\pi^* \in \mathrm{Sym}(X)$ which interchanges X_1 and X_2 and fixes Y pointwise. Thus both the support and the fixed-point sets of π^* are moieties, and as we have seen in the previous chapter, π^* normally generates $\mathrm{Sym}(X)$ (Proposition I.1.1.8). Now we claim that $C(\pi^*)$ generates $\mathrm{Sym}(X)$ in finitely many steps.

Proposition 2.2.1. *Let $C(\pi^*)$ be the conjugacy class of π^* in the group* $\mathrm{Sym}(X)$. *Then*

$$\mathrm{Sym}(X) = C(\pi^*)^{12},$$

that is, every permutation of X can be written as a product of at most 12 conjugates of π^*.

We start with a classical result on generation of Sym(X) due to J. Dixon, P. M. Neumann and S. Thomas [5].

Theorem 2.2.2. *The symmetric group $\Gamma = \text{Sym}(X)$ is generated in three steps by the union of the stabilizers*

$$\Sigma_1 = \Gamma_{\{X_1 \cup Y\},(X_2)} = \Gamma_{(X_2)}, \quad \Sigma_2 = \Gamma_{\{X_2 \cup Y\},(X_1)} = \Gamma_{(X_1)},$$

or, equivalently,

$$\text{wid}(\text{Sym}(X), \Sigma_1 \cup \Sigma_2) \leqslant 3.$$

Proof. Let $\sigma \in \text{Sym}(X)$. As Y is of cardinality $|X|$ and as

$$\sigma Y = (\sigma Y \cap (X_1 \sqcup Y)) \sqcup (\sigma Y \cap X_2),$$
$$\sigma Y = (\sigma Y \cap (X_2 \sqcup Y)) \sqcup (\sigma Y \cap X_1)$$

one of the sets

$$\sigma Y \cap (X_1 \sqcup Y), \text{ or } \sigma Y \cap (X_2 \sqcup Y)$$

is of cardinality $|X|$.

Suppose then that

$$|\sigma Y \cap (X_1 \sqcup Y)| = |X| \tag{2.2.1}$$

Let $Z \subseteq Y$ be a subset of Y such that

σZ is a moiety of $\sigma Y \cap (X_1 \sqcup Y)$

It means that there is a subset T of Y such that

$$\sigma Z \sqcup \sigma T = \sigma Y \cap (X_1 \sqcup Y).$$

Now since σ is a bijection and since σZ is a moiety of $\sigma Y \cap (X_1 \sqcup Y)$, then T is also of cardinality $|X|$ and is disjoint to Z. Consequently,

- Z is a moiety of Y (of $X_1 \sqcup Y$) and
- σZ is moiety of $X_1 \cap Y$.

It follows that there is a permutation $\rho \in \Sigma_1$ which maps σZ onto Z and, moreover, such that

$$\rho(\sigma z) = (\rho\sigma)z = z \tag{2.2.2}$$

for all $z \in Z$.

Since, further, X_1 and Z are both moieties of $X_1 \sqcup Y$, there is a $\pi \in \Sigma_1$ which takes X_1 onto Z:

$$\pi(X_1) = Z.$$

Consider the product $\pi^{-1}(\rho\sigma)\pi$. We have that for all $x_1 \in X_1$, $\pi(x_1)$ is in Z, and hence by (2.2.2),

$$(\rho\sigma)(\pi x_1) = \pi x_1,$$

whence

$$\pi^{-1}\rho\sigma\pi(x_1) = \pi^{-1}(\pi x_1) = x_1$$

for all $x_1 \in X_1$. It follows that

$$\pi^{-1}\rho\sigma\pi \in \Sigma_2,$$

or

$$\sigma \in \rho^{-1}\pi\Sigma_2\pi^{-1} = \Sigma_1\Sigma_2\Sigma_1.$$

Summing up, we see that if (2.2.1) is true, then

$$\text{Sym}(X) = \Sigma_1\Sigma_2\Sigma_1.$$

By symmetry, if

$$|\sigma Y \cap (X_2 \sqcup Y)| = |X|,$$

we get that

$$\text{Sym}(X) = \Sigma_2\Sigma_1\Sigma_2,$$

and the result follows. □

Proof of Proposition 2.2.1. Let

$$X = A \sqcup C \sqcup B$$

be a partition of X into moieties. Take an index set I of cardinality $|X|$. Suppose that

$$A = \{a_i : i \in I\} \text{ and } B = \{b_i : i \in I\}.$$

Take a bijection $f : I \to I$. Write α for the permutation

$$\alpha(a_i) = a_{f(i)} \qquad [i \in I]$$

of A induced by f. Consider involutions π_1, π_2 which both fix C pointwise and act on A as follows

$$\pi_1 a_i = b_i, \qquad [i \in I]$$
$$\pi_2 a_{f(i)} = b_i.$$

Clearly, π_1, π_2 are conjugates of π^*, since their supports $(= A \sqcup B)$ and fixed-point sets $(= C)$ are moieties. We have that

$$\pi_2\pi_1 a_i = \pi_2(b_i) = a_{f(i)} \qquad [i \in I].$$

On the other hand,

$$\pi_2\pi_1 b_i = \pi_2(a_i) = \pi_2(a_{f(f^{-1}(i))}) = b_{f^{-1}(i)}$$

for all $i \in I$. We see that

the action of $\pi_2\pi_1$ on $B = \{b_i : i \in I\}$ is isomorphic to the action of α^{-1} on $A = \{a_i : i \in I\}$,

or, informally, one can write that

$$\pi_2\pi_1 = \alpha \sqcup \mathrm{id} \sqcup \alpha^{-1}.$$

Extending the principle of the construction of $\pi_2\pi_1$, one can represent as a product of two conjugates of π^* any permutation of X of the form

$$\bigsqcup_{n \in \mathbf{N}} \alpha \sqcup \mathrm{id} \sqcup \bigsqcup_{n \in \mathbf{N}} \alpha^{-1}, \tag{2.2.3}$$

where the latter disjoint union of *maps* corresponds to a disjoint union moieties and α is the isomorphism type of a permutation of one of these moieties.

Let us consider permutations σ_1, σ_2 of X of the form (2.2.3):

$$\sigma_1 = \mathrm{id} \sqcup (\alpha \quad \sqcup \alpha^{-1} \sqcup \alpha \quad \sqcup \alpha^{-1} \sqcup \ldots) \sqcup \mathrm{id},$$
$$\sigma_2 = \alpha \sqcup (\alpha^{-1} \sqcup \alpha \quad \sqcup \alpha^{-1} \sqcup \alpha \quad \sqcup \ldots) \sqcup \mathrm{id},$$

both constructed over the *same* partition of X into a countable disjoint union of moieties. It is clear that

$$\sigma_1\sigma_2 = \alpha \sqcup \bigsqcup_{n \in \mathbf{N}} \mathrm{id} \sqcup \mathrm{id} \tag{2.2.4}$$

is a product of four conjugates of π^*.

The equation (2.2.4) demonstrates that each element of the subgroup $\Gamma_{\{X_2 \cup Y\},(X_1)}$ (resp. the subgroup $\Gamma_{\{X_1 \cup Y\},(X_2)}$) is a product of at most four conjugates of π^*. Then by Theorem 2.2.2 any element of $\mathrm{Sym}(X)$ is a product of at most $3 \cdot 4 = 12$ conjugates of π^*. □

Theorem 2.2.3. *Let U be a generating set of the group* $\mathrm{Sym}(X)$. *Then the width of* $\mathrm{Sym}(X)$ *with respect to U is finite. Consequently,* $\mathrm{Sym}(X)$ *has the Bergman property.*

Proof. We start with the diagonal argument which has been used in a number of papers on the automorphism groups of classical structures (see e.g. [10]).

Lemma 2.2.4. *Let $V = U \cup U^{-1}$. There exist a power V^m of V and a partition $X = Y \sqcup Z$ of X into moieties such that the set*

$$(V^m)_{\{Y\},\{Z\}}$$

induces the group $\mathrm{Sym}(Y)$ *on Y.*

Proof. Let

$$X = \bigsqcup_{k \geqslant 1} X_k$$

be a partition of X into moieties countably many moieties indexed by natural numbers $\geqslant 1$.

Write

$$X_k^* = \bigsqcup_{i \neq k} X_i$$

for all $k \geqslant 1$.

If for some pair (V^k, X_j) we have that

$$(V^k)_{\{X_j\},\{X_j^*\}} \text{ induces } \mathrm{Sym}(X_j) \text{ on } X_j$$

then the conclusion of the lemma is true. Suppose otherwise. Then, in particular, for all k

$$(V^k)_{\{X_k\},\{X_k^*\}} \text{ does not induce } \mathrm{Sym}(X_k) \text{ on } X_k.$$

Hence for each $k \in$ **N** we can find $\sigma_k \in \mathrm{Sym}(L_k)$ such that

σ_k is not equal to the restriction on X_k of any element from $(V^k)_{\{X_k\},\{X_k^*\}}$.

Set

$$\sigma = \bigsqcup_{k \geqslant 1} \sigma_k.$$

Since $\mathrm{Sym}(X) = \bigcup_k V^k$, we have $\sigma \in V^j$ for some $j \in$ **N**. It is clear, however, that

$$\sigma \in (V^j)_{\{X_j\},\{X_j^*\}}.$$

But then the restriction of σ to X_j is σ_j, a contradiction. □

Let V^m, Y, Z satisfy the conclusion of the lemma. Take an involution $\pi \in \mathrm{Sym}(X)$ which is conjugate to π^*, fixes Y setwise and fixes Z pointwise. Then by Proposition 2.2.1 the set of permutations

$$\Sigma = \{\sigma_1 \pi \sigma_1^{-1} \dots \sigma_{12} \pi \sigma_{12}^{-1} : \sigma_1, \dots, \sigma_{12} \in (V^m)_{\{Y\},\{Z\}}\} \tag{2.2.5}$$

is the group $\Gamma_{\{Y\},(Z)}$, since any permutation $\sigma_k \pi \sigma_k^{-1}$ acts trivially on Z due to triviality of the action of π on Z. Clearly, Σ is contained in some power of V. Now, by Theorem 2.2.2, for the conjugate set $\rho\Sigma\rho^{-1}$ by a suitable permutation $\rho \in \mathrm{Sym}(X)$ we have that

$$\mathrm{wid}(\mathrm{Sym}(X), \Sigma \cup \rho\Sigma\rho^{-1}) \leqslant 3.$$

Let l be a natural number such that $\Sigma \cup \rho\Sigma\rho^{-1} \subseteq V^l$. Then evidently

$$\mathrm{Sym}(X) = Y^{3l}.$$

□

2.3. BASIC PROPERTIES OF THE CLASS OF ALL GROUPS WITH THE BERGMAN PROPERTY

It is quite clear that the class **B** of all groups that have the Bergman property is closed under homomorphic images: whenever G is a group, any homomorphic image of G also has the Bergman property.

Next, we are going to show that the class **B** is closed under group extensions.

Definition (Group extensions)**.** We say a given group G is an *extension* of a group A by a group B, if G has a normal subgroup N such that

$$N \cong A \text{ and } G/N \cong B.$$

A class **K** of groups is said to be *closed under group extensions* if given any groups $A, B \in$ **K**, any extension of A by B is again in **K**.

An example: a direct product $A \times B$ of groups A, B is an extension of A by B.

Lemma 2.3.1. *[1] Let $H < G$ be groups and U a generating set for G. For some $n \geqslant 0$, suppose every right coset of H in G contains a group word of length $\leqslant n$ in the elements* *of U. Then the set of elements of H that can be written as words of length $\leqslant 2n+1$ in the elements of U generates H.*

Proof. Let V be a set of right coset representatives of H in G consisting of words of length $\leqslant n$ in the elements of U, with the coset H represented by the element 1, and let $r : G \to V$ be the retraction collapsing each coset to its representative. Let W denote the set of elements of H that can be written as words of length $\leqslant 2n+1$ in the elements of U.

For any $v \in V$ and $u \in U \cup U^{-1}$, note that $vu = (vur(vu)^{-1})r(vu)$. Since $r(vu)$ by definition lies in the same right coset as vu, then $vur(vu)^{-1}$ lies in H, and since v and $r(vu)$ are members of V, each have length $\leqslant n$, then the factor $vur(vu)^{-1}$ has length $\leqslant 2n+1$, then $vur(vu)^{-1} \in W$.

It follows that $vu = (vur(vu)^{-1})r(vu) \in WV$ and then

$$V(U \cup U^{-1}) \subseteq WV \subseteq \bigcup_{i \in I} W^i V \ldots (6).$$

It follows that $\bigcup_{i \in I} W^i V$ is closed under right multiplication by $U \cup U^{-1}$, hence $\bigcup_{i \in I} W^i V = G$.

If we intersect both sides by H,

$$H = G \cap H = (\bigcup_{\{i \in I\}} W^i V) \cap H = \bigcup_{i \in I} (W^i V \cap H) = \bigcup_{i \in I} W^i.$$

The intersection $(W^i V \cap H)$ has the effect of discarding elements having right factors from V other than 1, therefore $\bigcup_{i \in I} W^i = H$, and completing the proof. □

Corollary 2.3.2. *Let G be a group. Suppose that there is a normal subgroup N of G such that both N and the quotient group G/N have the Bergman property. Then G also has the Bergman property.*

Accordingly, the class of all groups with the Bergman property is closed under group extensions.

Proof. By Lemma 2.3.1. □

Definition (Groups of uncountable cofinality)**.** Let G be a group which is not finitely generated. Then G is said to have an *uncountable cofinality*, if whenever G is written as the union of a chain of subgroups

$$G_0 \leqslant G_1 \leqslant G_2 \leqslant \ldots$$

indexed by **N** (or, for short, as the union of an *exhaustive* countable chain of subgroups of G), then for some n, $G_n = G$. Equivalently, any countable exhaustive chain of subgroups of G terminates at G after finitely many steps.

Observe that in a finitely generated group any countable exhaustive chain terminates after finitely many steps. Thus the notion we introduced in the last definition is more interesting for groups that are not finitely generated.

Recall that given a group G and a metric space $\langle M, d\rangle$, an action of G on M for which

$$d(ga, gb) = d(a, b)$$

for all $g \in G$ and for all $a, b \in M$ is called an *action by isometries*.

Several authors studied the conjunction of the Bergman property and uncountable cofinality. Following [6], we call a group G having *both* these properties a group of the *strong uncountable cofinality.* The following proposition that appeared in a number of papers on the Bergman property provides a number of criteria of the strong uncountable cofinality [1, 3, 6].

Proposition 2.3.3. *Let G be a group. Then the following are equivalent:*

(i). *G has the strong uncountable cofinality;*

(ii). *every exhaustive chain* (U_k)

$$U_0 \subseteq U_1 \subseteq \ldots \subseteq U_k \subseteq \ldots \subseteq G$$

of subsets of G such that for every $i \in$ **N**

- U_i *closed under taking inverses;*

- *the product U_iU_i is contained in a suitable U_k*

terminates after finitely many steps;

(iii). *orbits of every action of G by isometries on a metric space $\langle M, d\rangle$ have bounded diameters;*

(iv). *every function $L : G \to \mathbf{R}$ such that*

- *$L(g) = 0$ if and only if $g = 1$;*
- *$L(g^{-1}) = L(g)$ and $L(gh) \leqslant L(g) + L(h)$ for all $g, h \in G$*

is bounded from above.

Proof. (i) ⇒ (ii). Clearly, the chain of subgroups of G generated by sets U_i,

$$\langle U_0\rangle \leqslant \langle U_1\rangle \leqslant \ldots \leqslant \langle U_k\rangle \leqslant \ldots \leqslant G,$$

is an exhaustive chain of subgroups of G. Then $G = \langle U_j\rangle$ for a suitable natural number j, because G is a group of uncountable cofinality. It follows that $U_j = U_j^{-1}$ is a symmetric generating set of G. As G has the Bergman property, $G = U_j^s$ for some natural number s. By the conditions on the chain (U_k), the power U_j^s is contained in some U_m for an appropriate $m \in \mathbf{N}$, whence $U_m = G$.

(ii) ⇒ (iii). Let a be an arbitrary element of a metric space M satisfying (iii). Set

$$U_n = \{g \in G : d(a, ga) \leqslant n\} \qquad (n \in \mathbf{N}).$$

Let $g, h \in U_n$. Then we have that

$$d(a, gha) \leqslant d(a, ga) + d(ga, gha) = d(a, ga) + d(a, ha) \leqslant n + n = 2n.$$

Consequently, $U_nU_n \subseteq U_{2n}$. As the chain (U_n) terminates, we get that $G = U_m$ for some $m \in \mathbf{N}$. Hence

$$d(a, ga) \leqslant m$$

for all $g \in G$. Thus the diameter of the orbit $\{ga : g \in G\}$ of $a \in M$ is at most $2m$.

(iii) $\Rightarrow$ (iv). Let $a, b \in G$. Set

$$d(a, b) = L(ab^{-1}).$$

It is easy to see that d is a metric on G satisfying the conditions in (iii) for the left action G on itself. Indeed, we have that

$$\begin{aligned}
&d(a,b) = 0 \iff L(ab^{-1}) = 0 \iff ab^{-1} = 1 \iff a = b,\\
&d(a,b) = L(ab^{-1}) = L(ba^{-1}) = d(b,a),\\
&d(a,b) = L(ab^{-1}) = L(ac^{-1} \cdot cb^{-1}) \leqslant L(ac^{-1}) + L(cb^{-1}) = d(a,c) + d(c,b),\\
&d(ga, gb) = L(gab^{-1}g^{-1}) = L(ab^{-1}) = d(a,b),
\end{aligned}$$

for all $a, b, g, h \in G$. Then the orbit of $1 \in G$ under the left action of G on itself has a bounded diameter $m \in \mathbf{N}$, or

$$L(g) = d(g1, 1) \leqslant m \qquad (g \in G).$$

(iv) $\Rightarrow$ (i). Let $S = S^{-1}$ be a symmetric generating set of G. Then the function

$$L_1(g) = |g|_S \qquad (g \in G)$$

that is, the length function with regard to S, which meets all conditions mentioned in (iv), must be bounded from above by some natural number m. Accordingly, $G = S^m$.

Let further (N_k) be an exhaustive countable chain of subgroups of G. For every $g \in G$ set

$$L_2(g) = \min\{k \in \mathbf{N} : g \in N_k\} \qquad (g \in G).$$

It is readily seen that L_2 satisfies all conditions in (iv). For example,

$$L_2(ab) \leqslant \max(L_2(a), L_2(b)) \leqslant L_2(a) + L_2(b)$$

for all $a, b \in G$. One again concludes that L_2 is bounded from above by a certain natural number m, whence $G = N_m$. □

We shall discuss the property of the strong uncountable cofinality in a more general settings, for arbitrary algebras, in Section 2.5.

2.4. INFINITE-DIMENSIONAL LINEAR GROUPS

Throughout the section we shall denote by V a left infinite-dimensional vector space over a field. Γ stands for the general linear group $\mathrm{GL}(V)$ of V, the group of all invertible linear transformations from V into itself.

Following [10], we call a subspace U of V *moietous* (clearly, it is an analogue of the notion of a moiety of an infinite set), if

$$\dim U = \operatorname{codim} U = \dim V.$$

Let U_1, U_2, W be moietous subspaces of V with

$$V = U_1 \oplus U_2 \oplus W.$$

We also fix throughout an involution π^* of V which interchanges U_1, U_2, that is,

$$\pi^* U_1 = U_2 \text{ and } \pi^* U_2 = U_1$$

and preserves all elements of W.

Similarly to the main result of Section 2.2, the main result stating that $\mathrm{GL}(V)$ has the Bergman property will be obtained as a consequence of the following statement. According to Rosenberg's theorem (Theorem I.1.2.7), π^* normally generates $\mathrm{GL}(V)$, since the residual space of π^* is of dimension $\dim V$.

Theorem 2.4.1. *The width of* $\mathrm{GL}(V)$ *relative to the conjugacy class* $C(\pi^*)$ *of* π^* *is at most* 28.

Theorem 2.4.1, in turn, is based on obtaining an estimate of the width of $\mathrm{GL}(V)$ relative to the union of a pair of naturally defined subgroups introduced by Macpherson in [10]. The reader will notice similarities with the proof of the Bergman property for the infinite symmetric groups we give in Section 2.2.

Lemma 2.4.2. *Suppose that* U_1, U_2, W *are moietous subspaces of* V *such that*

$$V = U_1 \oplus U_2 \oplus W.$$

Let $\Gamma = \mathrm{GL}(V)$ *and let*

$$\Sigma_1 = \Gamma_{\{U_1+W\},(U_2)} \text{ and } \Sigma_2 = \Gamma_{\{U_2+W\},(U_1)}.$$

Then the width of $\mathrm{GL}(V)$ *relative to the set* $\Sigma_1 \cup \Sigma_2$ *is at most* 7.

Proof. Our proof is based on Macpherson's proof of the fact that $\mathrm{GL}(V)$ is generated by $\Sigma_1 \cup \Sigma_2$ (see the proof of Proposition 2.2 in [10]).

We prove first that

$$\Gamma_{(U_1+W)} \subseteq \Sigma_2\Sigma_1\Sigma_2\Sigma_1, \tag{2.4.1}$$

that is, the width of $\Gamma_{(U_1+W)}$ with respect to $\Sigma_1 \cup \Sigma_2$ is at most 4. Take $\theta \in \Gamma_{(U_1+W)}$ and assume that $(x_i : i \in I)$ is a basis for U_2. Then we have that

$$\theta x_i = x_i' + a_i + b_i, \quad [i \in I]$$

where a_i is an element of W, b_i an element of U_1, and the system $(x_i' : i \in I)$ is a basis for

U_2. Consider the automorphism τ_1 of Σ_2 which acts trivially on $U_1 + W$ and such that

$$\tau_1 x_i' = x_i - a_i, \quad [i \in I].$$

Now let τ_2 be an element of Σ_1 which takes U_1 onto W. Therefore we have that

$$\tau_2 \tau_1 \theta \tau_2^{-1} x_i = x_i + a_i', \quad [i \in I].$$

where $a_i' = \tau_2 b_i$ is an element of W. Then, as above, for a suitable τ_3 in Σ_2 we have that

$$\tau_3 \tau_2 \tau_1 \theta \tau_2^{-1} = \mathrm{id}_V,$$

which proves (2.4.1).

Consider now an arbitrary $\sigma \in \mathrm{GL}(V)$. Let $\mathcal{B}_1 = (y_i : i \in I)$ be a basis for U_1. We have that

$$\sigma y_i = z_i + t_i, \quad [i \in I],$$

where $z_i \in U_1$ and $t_i \in U_2 + W$. There exists a moiety J of I such that the set $(t_j : j \in J)$ is contained in a moietous subspace, say L of $U_2 + W$. Then an appropriate element $\gamma_1 \in \Sigma_2$ takes L to a moietous subspace of W. It follows that the subspace $\gamma_1 \langle z_j + t_j : j \in J \rangle$ is a moietous subspace of $U_1 + W$. Let y_j' denote $\gamma_1(z_j + t_j)$, where $j \in J$.

Suppose that $\mathcal{B}_W$ is a basis for W and $\mathcal{B}_W^0$ is a moiety of $\mathcal{B}_W$. We follow $\gamma_1 \sigma$ by a transformation γ_2 of Σ_1 which takes $(y_j' : j \in J)$ onto $\mathcal{B}_1 \cup \mathcal{B}_W^0$ and then apply some $\gamma_3 \in \Sigma_2$ which maps $\mathcal{B}_W^0$ onto $\mathcal{B}_W$. We obtain therefore that

$$\gamma_3 \gamma_2 \gamma_1 \sigma y_j = \gamma_3 \gamma_2 y_j' = v_j, \quad [j \in J],$$

where $(v_j : j \in J) = \mathcal{B}_1 \cup \mathcal{B}_W$. In a similar way, the system $(y_j : j \in J)$ is taken by the product of some $\delta_2 \in \Sigma_1$ and $\delta_3 \in \Sigma_2$ onto $(v_j : j \in J)$:

$$\delta_2 \delta_3 y_j = v_j, \quad [j \in J].$$

Hence

$$\gamma_3\gamma_2\gamma_1\sigma\delta_3^{-1}\delta_2^{-1}v_j = v_j, \quad [j \in J].$$

Thus the automorphism $\gamma_3\gamma_2\gamma_1\sigma\delta_3^{-1}\delta_2^{-1}$ is in the subgroup $\Gamma_{(U_1+W)}$. Since $\gamma_3 \in \Sigma_2$ and $\delta_2 \in \Sigma_1$, the equation (2.4.1) implies that

$$\sigma \in \Sigma_2\Sigma_1(\Sigma_2\Sigma_1\Sigma_2\Sigma_1)\Sigma_2$$

and then $\mathrm{wid}(\mathrm{GL}(V), \Sigma_1 \cup \Sigma_2) \leqslant 7$, as desired. □

Proof of Theorem 2.4.1. Let L_1, L_2, M be moietous subspaces such that V is their direct sum:

$$V = L_1 \oplus M \oplus L_2.$$

Let I be an index set of cardinality $\dim V$ and

$$(a_i : i \in I), \quad (a_i^* : i \in I)$$

be bases of L_1 and

$$(b_i : i \in I)$$

a basis for L_2. Involutions π_1, π_2 which both fix M pointwise and act on the bases (a_i) and (a_i^*) as follows

$$\pi_1 a_i = b_i, \qquad [i \in I]$$
$$\pi_2 a_i^* = b_i$$

are conjugates of π^*. We have that

$$\pi_2\pi_1 a_i = a_i^*, \quad [i \in I].$$

Now let α denote the automorphism of the vector space L_1 that takes the basis (a_i) onto the basis (a_i^*). Suppose that for all $i \in I$

$$\alpha^{-1} a_i^* = \sum_j \beta_{ij} a_j^*.$$

We then have

$$\pi_2\pi_1 b_i = \pi_2 a_i = \pi_2(\alpha^{-1} a_i^*) = \pi_2(\sum_j \beta_{ij} a_j^*) = \sum_j \beta_{ij} b_j$$

for all $i \in I$. We see that the action of $\pi_2\pi_1$ on $L_2 = \langle b_i : i \in I\rangle$ is isomorphic to the action of α^{-1} on $L_1 = \langle a_i^* : i \in I\rangle$, or, again, using convenient informal notation as we did in the proof of Proposition 2.2.1, one can write that

$$\pi_2\pi_1 = \alpha \oplus \mathrm{id} \oplus \alpha^{-1}.$$

Similarly, any automorphisms of V of the form

$$\bigoplus_{n\in\mathbf{N}} \alpha \oplus \mathrm{id} \oplus \bigoplus_{n\in\mathbf{N}} \alpha^{-1}, \tag{2.4.2}$$

where the latter direct sum corresponds to a direct sum of *moietous* subspaces and α is the isomorphism type of an automorphism of one of these subspaces, can be obtained as a product of two conjugates of π^*.

Let us consider two automorphisms σ_1, σ_2 of V of the form (2.4.2):

$$\begin{aligned}\sigma_1 &= \mathrm{id} \oplus (\alpha \quad \oplus \alpha^{-1} \oplus \alpha \quad \oplus \alpha^{-1} \oplus \ldots) \oplus \mathrm{id},\\ \sigma_2 &= \alpha \oplus (\alpha^{-1} \oplus \alpha \quad \oplus \alpha^{-1} \oplus \alpha \quad \oplus \ldots) \oplus \mathrm{id}\end{aligned}$$

(both constructed over the *same* decomposition of V into a countable infinite direct sum of moietous subspaces.) Then

$$\sigma_1\sigma_2 = \alpha \oplus \bigoplus_{n\in\mathbf{N}} \mathrm{id} \oplus \mathrm{id} \tag{2.4.3}$$

is a product of four conjugates of π^*.

Let U_1, U_2, W be subspaces of V with

$$V = U_1 \oplus W \oplus U_2.$$

Now each element of the subgroup $\Gamma_{(U_1),\{U_2+W\}}$, or of the subgroup $\Gamma_{(U_2),\{U_1+W\}}$) is a product of at most four conjugates of π^*. Then by Lemma 2.4.2 any element of $\mathrm{GL}(V)$ is a product of at most $7 \cdot 4 = 28$ conjugates of π^*. □

Remark 2.4.3. It is easy to see that π^* is a commutator. Indeed, let

$$(a_i, a_i^*, b_i : i \in I)$$

be a basis for V and let I_0 be a moiety of I. We define two involutions π_1 and π_2, conjugates of π^*:

$$\begin{aligned}
&\pi_1 a_i = \pi_2 a_i = a_i^*, && [i \in I_0],\\
&\pi_1 a_i = a_i^*, && [i \notin I_0],\\
&\pi_2 a_i = a_i, && [i \notin I_0],\\
&\pi_1 b_i = \pi_2 b_i = b_i, && [i \in I].
\end{aligned}$$

As π_2 is a conjugate of π_1, the product $\pi_1\pi_2$ is a commutator which is, moreover, a conjugate of π^*. Hence π^* is a commutator. By Theorem 2.4.1 this provides an elementary proof of the fact that $\Gamma = \mathrm{GL}(V)$ is perfect, that is, Γ coincides with the commutator subgroup $[\Gamma, \Gamma]$ (this had been proved by Rosenberg in [13] as a corollary of his main result of [13], we have quoted in Theorem I.1.2.7 above)

Theorem 2.4.4. *Let X be any generating set of* $\mathrm{GL}(V)$*. Then the width of* $\mathrm{GL}(V)$ *with respect to X is finite.*

Proof. We are repeating the diagonal argument we have used in the proof of Lemma 2.2.4, making necessary changes.

Lemma 2.4.5. *Let $Y = X^{\pm 1}$. There exist a power Y^m of Y and a decomposition $V = U \oplus W$ of V into a direct sum of moietous subspaces such that the set*

$$(Y^m)_{\{U\},\{W\}}$$

induces the group $\mathrm{GL}(U)$ *on U.*

Proof. Let

$$V = \bigoplus_{k \in N} L_k$$

be decomposition of V into a direct sum of moietous subspaces. Write

$$L_k^* = \bigoplus_{i \neq k} L_i$$

for all $k \in \mathbf{N}$.

If for some pair (Y^k, L_j) we have that

$$(Y^k)_{\{L_j\},\{L_j^*\}} \text{ induces } \mathrm{GL}(L_j) \text{ on } L_j$$

then the conclusion of the lemma is true. Suppose otherwise. Then, in particular, for all k

$$(Y^k)_{\{L_k\},\{L_k^*\}} \text{ does not induce } \mathrm{GL}(L_k) \text{ on } L_k.$$

Hence for each $k \in \mathbf{N}$ we can find $\sigma_k \in \mathrm{GL}(L_k)$ such that

σ_k is not equal to the restriction on L_k of any element from $(Y^k)_{\{L_k\},\{L_k^*\}}$.

Set

$$\sigma = \bigoplus_{k \in \mathbf{N}} \sigma_k.$$

Since $\mathrm{GL}(V) = \bigcup_k Y^k$, we have $\sigma \in Y^j$ for some $j \in \mathbf{N}$. It is clear, however, that

$$\sigma \in (Y^j)_{\{L_j\},\{L_j^*\}}.$$

But then the restriction of σ on L_j is σ_j, a contradiction. □

Let Y^m, U, W be as in the Lemma 2.4.5.

Consider a conjugate $\pi \in \mathrm{GL}(V)$ of π^*, which fixes U setwise and which fixes W pointwise. By Theorem 2.4.1, if

$$\Sigma = \{\sigma_1 \pi \sigma_1^{-1} \ldots \sigma_{28} \pi \sigma_{28}^{-1} : \sigma_1, \ldots, \sigma_{28} \in (Y^m)_{\{U\},\{W\}}\} \tag{2.4.4}$$

then Σ coincides with $\Gamma_{\{U\},(W)}$, because any automorphism $\sigma_k \pi \sigma_k^{-1}$ acts trivially on W. Now Σ can be found in an appropriate power of Y. By Lemma 2.4.2, we have that

$$\mathrm{wid}(\mathrm{GL}(V), \Sigma \cup \rho\Sigma\rho) \leqslant 7.$$

for a suitable involution ρ. Finally, suppose that l is a natural number with $\Sigma \cup \rho\Sigma\rho \subseteq Y^l$. Then $\mathrm{GL}(V) = Y^{7l}$, and we are done. □

2.5. CARTESIAN POWERS OF ALGEBRAS AND THE BERGMAN PROPERTY

In this section we shall discuss some generalizations of the results from [1] George Bergman came up with in his 2006 paper [2].

Let $\mathcal{L}$ be a first-order language which contains no predicate symbols. Recall that first-order $\mathcal{L}$-structures are called *algebras*. Apart from the notion of the length of a term t of $\mathcal{L}$ we have considered in the introductory chapter, there is also a convenient notion of the *depth* $\mathrm{depth}(t)$

of a term t of $\mathcal{L}$. The definition is by induction on the length of t. Variables and constants are declared to be terms of the depth zero. Now if

$$t = f(t_1, \ldots, t_n)$$

is a term of $\mathcal{L}$ where $f \in \mathcal{L}$ is a function symbol and t_k are terms, then

$$\text{depth}(t) = \max(\text{depth}(t_1), \ldots, \text{depth}(t_n)) + 1.$$

If $\mathcal{M}$ is an $\mathcal{L}$-structure, $a \in \text{dom}(\mathcal{M})$, $B \subseteq \text{dom}(\mathcal{M})$, we say that a is a term (more formally, but less conveniently, the value of a term) of elements of B if there is an $\mathcal{L}$-term $t(v_1, \ldots, v_n)$ such that $a = t(b_1, \ldots, b_n)$ for some elements $b_1, \ldots, b_n$ of B.

Let $\mathcal{M}$ be an $\mathcal{L}$-algebra and let X be a generating set of $\mathcal{M}$: $\mathcal{M} = \langle X \rangle$. Then given $m \in \mathbf{N}$, the *m-th power* X^m of X is the set of all terms of elements of X of depth $\leqslant m$. Clearly,

$$\text{dom}(\mathcal{M}) = \bigcup_{m \geqslant 0} X^m.$$

In [2] Bergman introduced, in development the ideas of the notions of the uncountable cofinality and of the strong uncountable cofinality for groups (see discussion of these properties in Section 2 above), the following definitions.

Definition (Algebras that have a UF- , or the strong UF-cofinality, the Bergman property for algebras)**.** Let $\mathcal{M}$ be an algebra and M denote the domain of $\mathcal{M}$.

(i) $\mathcal{M}$ is said to have a *UF-cofinality* (`UF' stands for `Uncountable, or Finite') if any countable exhaustive chain of subalgebras of $\mathcal{M}$ terminates after finitely many step;

(ii) $\mathcal{M}$ is said to satisfy the *Bergman property* if any generating set X of $\mathcal{M}$ generates it in finitely many steps, or more precisely, if for every generating set X of $\mathcal{M}$ there is a natural number m such that $M = X^m$;

(ii) $\mathcal{M}$ is said to have the *strong UF-cofinality* if given any exhaustive chain

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$$

of subsets of the domain M of $\mathcal{M}$, that is, a chain of subsets of M whose union is M such that for all $k \in \mathbf{N}$ and for all basic operations f of $\mathcal{M}$

$$f(X_k, \ldots, X_k) \subseteq X_{k+1} \tag{2.5.1}$$

where f is a basic operation of $\mathcal{M}$, we have that the chain stabilizes after finitely many steps at M, that is, there is a natural number s with $X_s = M$.

Proposition 2.5.1. *An algebra $\mathcal{M}$ has the strong UF-cofinality if and only if $\mathcal{M}$ has a UF-cofinality and the Bergman property.*

Proof. The necessity part is easy, since both (exhaustive) chains of powers of generating sets and exhaustive chains of subalgebras satisfy the condition (2.5.1). Conversely, write M for dom$(\mathcal{M})$ and let (X_n) be an exhaustive chain with (2.5.1). Consider then the, evidently exhaustive, chain of subalgebras

$$\langle X_0 \rangle \subseteq \langle X_1 \rangle \subseteq \ldots \subseteq \langle X_n \rangle \subseteq \ldots$$

As $\mathcal{M}$ has a UF-cofinality, then $M = \langle X_k \rangle$ for a suitable natural number k. Then X_k is a generating set of $\mathcal{M}$. As $\mathcal{M}$ has the Bergman property, there is a natural number m such that $M = X_k^m$, which means that every element of $M =$ dom$(\mathcal{M})$ is written as a term of depth $\leqslant m$ of elements of X_k. Clearly, by definition, terms of depth one of elements of X_k are contained in X_{k+1}, terms of depth two in X_{k+2}, and so on. So $M = X_{k+m}$, as required. $\square$

Let $\mathcal{M}$ be an algebra and let $\varkappa$ be a cardinal. We write $\mathcal{M}^\varkappa$ for the Cartesian power

$$\prod_{i \in I} \mathcal{M}$$

of $\mathcal{M}$ where I is an index set of cardinality $\varkappa$. $\Delta(\mathcal{M})$ will denote the *diagonal* of $\mathcal{M}^\varkappa$, that is, the set of elements

$$\{\Delta(x) : x \in \text{dom}(\mathcal{M})\} = \{(x)_{i\in I} : x \in \text{dom}(\mathcal{M})\};$$

(we use the notation from Subsection I.1.3; recall that given a function $f \in \mathcal{M}^{\varkappa}$ we write f as $(f(i))_{i\in I}$. Thus a diagonal element of $\mathcal{M}^{\varkappa}$ is a function in $\mathcal{M}^{\varkappa}$ whose values are the same.)

In [2] Bergman found the following sufficient condition for the strong UF-cofinality.

Theorem 2.5.2. *Let $\mathcal{M}$ be an algebra. Suppose that the countable power $\mathcal{M}^{\aleph_0}$ of $\mathcal{M}$ is finitely generated over its diagonal $\Delta(\mathcal{M})$. Then $\mathcal{M}$ has the strong UF-cofinality.*

Before considering the proof of the Theorem, let us consider some examples of algebras which satisfy its condition.

Theorem 2.5.3. *Let X be an infinite set and let $\varkappa \leqslant |X|$ be a cardinal.*

(i) *Let M be the semigroup of all maps $X \to X$. Then $M^{\varkappa}$ is generated by two elements over $\Delta(M)$;*

(ii) *Let S be the symmetric group of X. Then $S^{\varkappa}$ is generated by one element over $\Delta(S)$.*

Proof. (i). Fix an index set I of cardinality $\varkappa$ and let

$$X = \bigsqcup_{i\in I} X_i$$

be a partition of X into moieties.

Let $i \in I$. Take any map $\pi_i : X \to X$ which takes X to X_i:

$$\pi_i(X) = X_i.$$

Now as X_i is a moiety of X, there is a left inverse ρ_i of π_i, that is, a map $\rho_i : X \to X$ with

$$\rho_i\pi_i = \text{id}\,.$$

Form then elements

$$\pi = (\pi_i)_{i\in I} \text{ and } \rho = (\rho_i)_{i\in I}$$

of $M^\varkappa$. We claim that $M^\varkappa$ is generated over $\Delta(M)$ by two elements, namely, by π and ρ.

Indeed, let $\sigma = (\sigma_i)_{i\in I}$ be an element of $M^\varkappa$ where $\sigma_i \in M$ ($i \in I$). We are going to construct a map $\sigma' \in M$ such that

$$\rho_i\sigma'\pi_i = \sigma_i \qquad [i \in I]. \tag{2.5.2}$$

It will follow that

$$(\rho_i)_{i\in I} \cdot (\sigma')_{i\in I} \cdot (\pi_i)_{i\in I} = (\rho_i\sigma'\pi_i)_{i\in I} = (\sigma_i)_{i\in I},$$

or

$$\rho\Delta(\sigma')\pi = \sigma,$$

which means that

$$M^\varkappa = \langle \Delta(M), \rho, \sigma\rangle,$$

as desired.

The construction of σ' is as follows: we require σ' to be equal to $\pi_i\sigma_i\rho_i$ on X_i for each $i \in I$:

$$\sigma'|_{X_i} = (\pi_i\sigma_i\rho_i)|_{X_i} \qquad [i \in I]. \tag{2.5.3}$$

Let us check (2.5.2). Fix $i \in I$ and take $x \in X$. By the definition of π_i, $\pi_i(x) \in X_i$. Then by (2.5.3),

$$\sigma'(\pi_i x) = \pi\sigma_i\rho_i(\pi_i x),$$

whence

$$\begin{aligned}\rho_i\sigma'\pi_i(x) &= \rho_i\sigma'(\pi_i x) = \rho_i\pi\sigma_i\rho_i(\pi_i x) = \rho_i\pi_i\sigma_i\rho_i\pi_i(x)\\ &= (\rho_i\pi_i)\sigma_i(\rho_i\pi_i)(x) = \mathrm{id}\,\sigma_i\,\mathrm{id}(x)\\ &= \sigma_i(x),\end{aligned}$$

which completes the proof of (i).

(ii). Write X as a disjoint union of two moieties:

$$X = A \sqcup B$$

and then partition B into $2\varkappa = \varkappa$ moieties

$$B = \bigsqcup_{i\in I}(C_i \sqcup D_i).$$

We claim that $S^{\varkappa}$ is generated over the diagonal $\Delta(S)$ by just one element

$$\pi = (\pi_i)_{i\in I}.$$

Take $i \in I$. Let us explain how π_i is to be defined:

- π_i acts on $\{A, C_i, D_i\}$ as a 3-cycle, that is,

$$\pi_i(A) = C_i, \pi_i(C_i) = D_i, \pi_i(D_i) = A;$$

- for each $j \in I \setminus \{i\}$, π_i interchanges sets C_j and D_j:

$$\pi_i(C_j) = D_j, \pi_i(D_j) = C_j.$$

Write S_1 for $S_{(B)}$ and let $\sigma \in S_1^{\varkappa}$, that is,

$$\sigma = (\sigma_i)_{i \in I}$$

where $\sigma_i \in S_1$ for all $i \in I$.

As in any infinite symmetric group every element is a commutator (Corollary I.1.1.13), for all $i \in I$ there are $\alpha_i, \beta_i \in \mathrm{Sym}(A)$ such that

$$\sigma_i^0 = [\alpha_i, \beta_i] = \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$$

where σ_i^0 is the restriction of σ_i to A.

Now we are going to construct two permutations $\lambda, \mu \in S$ for which

$$\sigma = [\pi \Delta(\lambda) \pi, \pi^2 \Delta(\nu) \pi^{-2}]. \tag{2.5.4}$$

The definition of λ, μ is as follows:

$$\lambda = \mathrm{id} \sqcup \bigsqcup_{i \in I} (\mathrm{id} \sqcup \alpha_i),$$
$$\mu = \mathrm{id} \sqcup \bigsqcup_{i \in I} (\beta_i \sqcup \mathrm{id}),$$

where disjoint union of maps in the right-hand sides both correspond to the partition

$$X = A \sqcup \bigsqcup_{i \in I} (C_i \sqcup D_i)$$

of X.

Let us check (2.5.4). Take an $i \in I$. Then

$$\pi_i \lambda \pi_i^{-1} = (\alpha_i \sqcup \mathrm{id} \sqcup \mathrm{id}) \sqcup \bigsqcup_{j \in I \setminus \{i\}} (\alpha_j \sqcup \mathrm{id}),$$
$$\pi_i^2 \mu \pi_i^{-2} = (\beta_i \sqcup \mathrm{id} \sqcup \mathrm{id}) \sqcup \bigsqcup_{j \in I \setminus \{i\}} (\mathrm{id} \sqcup \beta_j),$$

where disjoint unions of maps in the right-hand sides correspond to the partition

$$X = (A \sqcup C_i \sqcup D_i) \sqcup \bigsqcup_{j \in I \setminus \{i\}} (C_j \sqcup D_j).$$

It follows that

$$\pi_i \lambda^{-1} \pi_i^{-1} = (\alpha_i^{-1} \sqcup \mathrm{id} \sqcup \mathrm{id}) \sqcup \bigsqcup_{j \in I \setminus \{i\}} (\alpha_j^{-1} \sqcup \mathrm{id}),$$
$$\pi_i^2 \mu^{-1} \pi_i^{-2} = (\beta_i^{-1} \sqcup \mathrm{id} \sqcup \mathrm{id}) \sqcup \bigsqcup_{j \in I \setminus \{i\}} (\mathrm{id} \sqcup \beta_j^{-1}).$$

and that

$$[\pi_i \lambda \pi_i^{-1}, \pi_i^2 \mu^{-1} \pi_i^{-2}] = ([\alpha_i, \beta_i] \sqcup \mathrm{id} \sqcup \mathrm{id}) \sqcup \bigsqcup_{j \in I \setminus \{i\}} (\mathrm{id} \sqcup \mathrm{id})$$
$$= (\sigma_i^0 \sqcup \mathrm{id} \sqcup \mathrm{id}) \sqcup \bigsqcup_{j \in I \setminus \{i\}} (\mathrm{id} \sqcup \mathrm{id})$$
$$= \sigma_i$$

As i is arbitrary, we get that

$$[\pi \Delta(\lambda) \pi^{-1}, \pi^2 \Delta(\mu) \pi^{-2}] = \sigma,$$

as required.

This implies that

$$S_1^{\varkappa} \leqslant \langle \Delta(S), \pi \rangle. \tag{2.5.5}$$

Recall that S_1 together with a suitable conjugate $S_2 = \rho S_1 \rho^{-1}$ generates S (Theorem 2.2.2) in three steps:

$$S = S_1 S_2 S_1 \cup S_2 S_1 S_2 = S_1 S_2 S_1 S_2,$$

which means that

$$S^{\varkappa} = S_1^{\varkappa} S_2^{\varkappa} S_1^{\varkappa} S_2^{\varkappa}.$$

Finally, Eq. (2.5.5) implies that

$$S_2^{\varkappa} = \Delta(\rho) S_1^{\varkappa} \Delta(\rho^{-1}) \leqslant \Delta(\rho)\langle \Delta(S), \pi\rangle \Delta(\rho)^{-1} \leqslant \langle \Delta(S), \pi\rangle$$

and therefore

$$S^{\varkappa} \leqslant \langle \Delta(S), \pi\rangle.$$

□

Proof of Theorem 2.5.2. Write M for the domain of $\mathcal{M}$ and let (X_n) be an exhaustive chain of subsets of M satisfying the condition (2.5.1).

Suppose, towards a contraction, that the chain (X_n) never stabilizes. For an element $x \in M$, define the *rank* of x, symbolically rank(x), as the minimal r such that $x \in X_r$. Clearly, the failure of (X_n) to stabilize implies that the function rank : $M \to \mathbf{N}$ is unbounded.

Let Y be a finite subset of $M^{\aleph_0}$ which generates $\mathcal{M}^{\aleph_0}$ together with the diagonal $\Delta(\mathcal{M})$.

Take an $i \in \mathbf{N}$. As the function rank is unbounded, there is a $x_i \in M$ such that

$$\text{rank}(x_i) > i + \max_{y \in Y} \text{rank}(y_i) \tag{2.5.6}$$

where y_i is the i-th component of $y \in Y$. Form then the element x of $M^{\aleph_0}$ as

$$x = (x_i)_{i \in \mathbf{N}}.$$

We have that

$$x \in \langle \Delta(M), Y \rangle.$$

Hence there is a finite subset Z of M with

$$x \in \langle \Delta(Z), Y \rangle.$$

It follows that x is a term of some depth d of elements of $\Delta(Z) \cup Y$. In turn, for every $i \in \mathbf{N}$, the element x_i is also a term of depth d of elements of $Z \cup \{y_i : y \in Y\}$.

Now we get a contradiction on taking any i with

$$i > d + \max_{z \in Z} \text{rank}(z).$$

Indeed, by (2.5.6) we have that

$$\begin{aligned} \text{rank}(x_i) &> i + \max_{y \in Y} \text{rank}(y_i) \\ &> d + \max_{z \in Z} \text{rank}(z) + \max_{y \in Y} \text{rank}(y_i). \end{aligned}$$

But this is impossible: if

$$m = \max_{z \in Z} \text{rank}(z) + \max_{y \in Y} \text{rank}(y_i),$$

then

$$Z \cup \{y_i : y \in I\} \subseteq X_m$$

and since x_i is a term of depth d of elements of $Z \cup \{y_i : y \in I\}$,

$$x_i \in X_{m+d}$$

and, by the condition (2.5.1)

$$\text{rank}(x_i) \leqslant m + d = d + \max_{z \in Z} \text{rank}(z) + \max_{y \in Y} \text{rank}(y_i).$$

$\square$

Now, as a corollary of Theorem 2.5.2 and Theorem 2.5.3, we obtain a new proof of Bergman's theorem on finiteness of width of infinite symmetric groups.

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