NONLINEAR SECOND ORDER PARABOLIC AND HYPERBOLIC EQUATIONS: BLOW UP AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS

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ABSTRACT

NONLINEAR SECOND ORDER PARABOLIC AND HYPERBOLIC EQUATIONS: BLOW UP AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS

In this thesis blow up in a finite time and asymptotic behavior of solutions of initial boundary value problems for second order nonlinear parabolic and hyperbolic equations are studied. Sufficient conditions for blow up of solutions of initial boundary value problems for nonlinear non-autonomous parabolic and damped hyperbolic equations under Robin boundary conditions, and solutions with arbitrary positive initial energy of initial boundary value problems, under the Robin and Dirichlet boundary conditions, for nonlinear parabolic and damped wave equations are obtained. Besides, sufficient condition for decay of solutions of initial boundary value problems for non-autonomous parabolic and damped wave equations with time dependent coefficients are investigated.

ÖZET

DOĞRUSAL OLMAYAN PARABOLİK VE HİPERBOLİK DENKLEMLER: ÇÖZÜMÜN PATLAMASI VE ASİMTOTİK DAVRANIŞI

Tezde ikinci mertebeden doğrusal olmayan parabolik ve hiperbolik denklemler icin, başlangıç sınırdeğer problemlerinin çozümlerinin sonlu zamanda patlaması ve asimptotik davranışı problemleri incelenmiştir. Ikinci mertbeden otonom olmayan ve doğrusal olmayan parabolik denklemler için, Robin sınırdeğer koşulu altında ve yeterince büyük başlangıç enerjisi olan, doğrusal olmayan sönümlü hiperbolik denklemler icin, Robin ve Dirichlet sınırdeğer koşullari altında, baslagıç sınırdeğer problemlerinin çözümlerinin sonlu zamanda patlaması ispat edilmistir. Ayrıca, ikinci mertebeden otonom olmayan ve katsayları zamana bağlı olan parabolik ve sönümlü dalga denklemleri icin başlangıç sınırdeğer problemlerinin çözümlerinin sıfıra yaklaşması icin yeterli koşullar elde edilmiştir.

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1. INTRODUCTION

The thesis is devoted to the study of initial boundary value problems for second order nonlinear parabolic and hyperbolic equations under various boundary conditions.
The main problems discussed here are:

- The blow up of solutions of initial boundary value problems for nonlinear parabolic and hyperbolic equations under various boundary conditions.
- The decay and growth of solutions of initial boundary value problems for second order parabolic and damped hyperbolic equations.

One of the most interesting features that distinguish nonlinear parabolic and hyperbolic equations from the corresponding linear equations is that solutions of nonlinear equations starting from smooth initial data may blow up in a finite time, i.e. some norm of a solution of a problem may tend to infinity as $t \to t_0^-$ for some $t_0 < \infty$.
The interest to problems of blow up of solutions of initial and initial boundary value problems for nonlinear partial differential equations is inspired by two main reasons. First is to describe precise as possible the classes of nonlinear partial differential equations for which the initial or initial-boundary value problems have unique global in time solution. The second is to give a rigorous mathematical justification and analysis of real processes where the blow up effects are observed.
The theory of blow up of solutions of PDEs is an important area of qualitative theory of PDEs. It worth mentioning that during last decades several books on blow up of solutions of nonlinear PDEs are published: The books of Samarskii, Galaktionov, Kurdyumov and Mikhailov [1] and Hu [2] are devoted to the study of blow up of solutions of nonlinear parabolic equations and systems , the book of Pokhozhaev and Mitidieri [3] is devoted to problems of blow up of solutions of nonlinear parabolic and hyperbolic equations and inequalities, [4] is devoted to problems of blow up of solutions of various nonlinear evolution equations of continuum mechanics. We would like to mention also the book of Al'shin, Korpusov and Sveshnikov [5] which is completely devoted to the problem of blow up of solutions of initial boundary value problems for various nonlinear pseudoparabolic equations.

1.1. BACKGROUND OF PROBLEM

There are many papers devoted to the problem of blow up of solutions to the Cauchy problem and initial boundary value problems for nonlinear evolution equations (see e.g. [6], [7], [8], [9], [10], [11], [12] and references therein).

There are also many publications devoted to the study of asymptotic behavior of solutions of initial boundary value problems for second order nonlinear parabolic and damped nonlinear hyperbolic equations (see [13], [14], [15], [16] and references therein).

1.1.1. Previous Results on Blow up of Solutions

Simple and effective examples of nonlinear parabolic equations whose solutions may blow up in a finite time are demonstrated in [17]. One can find also examples of nonlinear second order parabolic equations constructed by Friedman [18] and [19] whose solutions blow up in a finite time for some classes of initial functions. The following example constructed in [17]: Suppose that the problem

$$\begin{cases} u_t - u_{xx} = u^2, \;\; x \in [0,1], \; t \in [0,T], \\ u(0,t) = g_0(t), \;\; u(1,t) = g_1(t), \;\; t \in [0,T], \\ u(x,0) = u_0(x), \;\; x \in [0,1] \end{cases}$$

has a classical solution $u(x,t)$, corresponding to smooth initial and boundary functions u_0, g_0, g_1, which is bounded by some constant $c = \frac{c_1}{c_2}, \; c_1 > 0, \; c_2 > 0$ in $\overline{Q}_T := \{0 \leq x \leq 1, 0 \leq t \leq T\}$. It is easy to see that the function

$$z(x,t) = \frac{c_1}{c_2 - tx(1-x)} \;\; \text{for} \;\; t < 4c_2$$

satisfies the inequality $z_t - z_{xx} \leq z^2$ and the condition $z = 0$ on the parabolic boundary Γ_T of the domain Q_T for $c_1 \geq \frac{1}{4} + 8c_2$. The function $v(x,t) = (z(x,t) - u(x,t))e^{-\lambda t}$ is non-positive on Γ_T and satisfies the inequality

$$v_t - v_{xx} + (\lambda - z - u)v \leq 0.$$

It is clear that for $\lambda > 0$ large enough the function $v(x,t)$ can not attain a positive maximum value on $Q_{4c_2} \setminus \Gamma_{4c_2}$. Therefore the function $v(x,t)$ is non-positive, i.e. $u \geq z$. But $z(\frac{1}{2}, t) \to \infty$ as $t \to 4c_2$. Hence $u(\frac{1}{2}, t)$ tends to infinity in a finite time.
The conditions for the blow up of solutions of equations of the form

$$u_t = Lu + f(u),\ x \in \Omega \subset \mathbb{R}^n, \tag{1.1.1}$$

$$u_{tt} = Lu + f(u),\ x \in \Omega \subset \mathbb{R}^n, \tag{1.1.2}$$

where L is a second order self-adjoint uniformly elliptic operator with smooth coefficients depending on $x \in \Omega$, are obtained by using the comparison theorems which are valid for second order parabolic equations. In the papers of [20], [21], [22], [23], [24], [25], [26] , [27], [28], [29] and in [30] the conditions on the data and the nonlinear term $f(\cdot)$ for an equation of the form (1.1.1) and for the equation (1.1.2) are obtained by employing the fact that the Green's function of the main linear part for these equations is positive or positiveness of the first eigenvalue of the first eigenfunction their linear stationary parts.
In [22] it is proved that for each initial function $u_0(x) \neq 0$ the solutions of the Cauchy problem for the equation (1.1.1) ($\Omega = \mathbb{R}^n$) with

$$L = \Delta \ \text{ and } \ f(u) = u^{1+\alpha}$$

blows up in a finite time whenever $\alpha \in (0, \frac{2}{n})$.
The positiveness of the Green's function of the main linear part, is used by many authors to construct the lower solutions of problems they study. This lower solutions are solutions of nonlinear ordinary differential equations. Analyzing solutions of these ODE's the authors find conditions of blow up of solutions of corresponding nonlinear PDE's.
The proof of the blow up theorems for equations of type (1.1.1) and (1.1.2) by eigenfunctions method (employing positivity of the first eigenfunction of the stationary problem) when the

nonlinear term $f(u)$ is convex and satisfies the condition

$$\int_1^\infty \frac{dv}{f(v)} < \infty \tag{1.1.3}$$

usually follow the scheme:

- The equations (1.1.1) and (A.0.17) are multiplied in $L^2(\Omega)$ by a normalized first eigenfunction $\psi_1(x)$ of the operator generated by the differential expression $-L$ with the zero Dirichlet boundary condition: and obtained the equations

$$\frac{d}{dt}\int\limits_\Omega u\psi_1 dx + \lambda_1 \int\limits_\Omega u\psi_1 dx = \int\limits_\Omega f(u)u\psi_1 dx, \tag{1.1.4}$$

$$\frac{d^2}{dt^2}\int\limits_\Omega u\psi_1 dx + \lambda_1 \int\limits_\Omega u\psi_1 dx = \int\limits_\Omega f(u)u\psi_1 dx. \tag{1.1.5}$$

- The Jensen inequality

$$\int_\Omega f(u(x,t))\psi_1(x)dx \geq f\left(\int_\Omega u(x,t)\psi_1(x)dx\right), \tag{1.1.6}$$

  is used in (1.1.4) and (1.1.5) and the following ordinary differential inequality for the function $\Phi(t) = \int_\Omega u(x,t)\psi_1(x)dx$ are obtained

$$\frac{d}{dt}\Phi(t) + \lambda_1\Phi(t) \geq f\left(\Phi(t)\right), \tag{1.1.7}$$

$$\frac{d^2}{dt^2}\Phi(t) + \lambda_1\Phi(t) \geq f\left(\Phi(t)\right). \tag{1.1.8}$$

- The conditions of blow up of solutions of the initial boundary value problem for (1.1.1) and (1.1.2) are obtained by studying the Cauchy problem for the obtained ordinary differential inequalities (1.1.7) and (1.1.8).

The method of eigenfunction is used also in the study of equations of the form (1.1.1) when the operator L is also a nonlinear one.
By using the eigenfunction method the equation in [31]

$$u_t - \phi(u)u_{xx} = \psi(u), x \in (0, l),$$

and in [32] the initial boundary value problem for the equation of the form

$$u_t - \Delta\phi(u) = \psi(u), x \in \Omega \subset \mathbb{R}^n, \tag{1.1.9}$$

are studied.
It is shown in [32] that if non-negative functions $\phi(\cdot)$ and $\psi(\cdot)$ involved in (A.0.18) satisfy the conditions

$$\phi''(s) \geq 0,\ \psi''(s) \geq 0,\ \psi''(s)\phi'(s) - \psi'(s)\phi''(s) \geq 0,\ \forall s \in \mathbb{R},$$
$$\psi'(s)\phi(s) - \psi(s)\phi'(s) \geq 0,\ \forall s \in \mathbb{R},\ \int_1^\infty \frac{d\eta}{\psi(\eta)} d\eta < \infty,$$

then for a certain class of initial data there exists $t_1 < \infty$ such that

$$\limsup_{t \to t_1} \left(\sup_{x \in \Omega} |u(x,t)| \right) = \infty. \tag{1.1.10}$$

In [33] sufficient conditions on data that guarantee blow up of solutions of a class of equations of the form

$$a(u)u_t = (K(u)u_{x_i})_{x_i} + g(u) \tag{1.1.11}$$

are found. Blow up of solutions of an equation of the form 1.1.11 with $a(\cdot) = const$ is established in [34] by employing a method based on criticality of the initial function.
In [20], [35], [36], [37] sufficient conditions of blow up of solutions of initial and initial boundary value problems for equations of the form (1.1.2) are found by the method of comparison of solutions of nonlinear PDE's with the solutions of nonlinear ODE's. This method

is based on the Huygens's Principle.

In [38], [27], [39], [40], sufficient conditions of blow up of solutions of the Cauchy problem and initial boundary value problems for nonlinear hyperbolic equations of the form

$$u_{tt} = (a(u_x))_x\,, \tag{1.1.12}$$

where $a(\cdot) \in C^1$ is an increasing function, and essentially nonlinear hyperbolic systems of the form

$$\vec{u}_t + A(x,t,\vec{u})\vec{u}_x = \vec{f}(x,t,\vec{u}) \tag{1.1.13}$$

are found.

In [38] blow up of the function $u_x(x,t)$ (the gradient catastrophe) , where $u(x,t)$ is a solutions of the equation (1.1.12) is established for

$$a(s) = c^2(1+\epsilon s^2), \epsilon > 0$$

(which is a continuum analog of the famous system of nonlinear ODEs - the so called Fermi-Pasta-Ulam chain). The gradient catastrophe of solutions to non isentropic flow of an ideal gas is established in [39]. In [40] it is shown that second derivatives of all solutions of the equation (1.1.12) with

$$a(s) = s\tau(s)(1+s^2)^{-1/2}, \tag{1.1.14}$$

where $\tau(\cdot)$ is an odd and smooth function, blow up in a finite time if the initial functions are twice differentiable functions with a small amplitude.

The papers of [41], and [42] are the problem of blow up of solutions of the Cauchy problem for the nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + f(|\psi|^2)\psi = 0,\ x \in \mathbb{R}^n. \tag{1.1.15}$$

is investigated. The effect of blow up in a finite time of solutions of the Cauchy problem for the cubic nonlinear Schrödinger equation in two dimensional case (i.e. when $f(s) = s$ and $n = 2$) was first observed in the paper of Talanov.
In [42] it is proved that for some class of initial functions the gradient of solutions of the Cauchy problem for (1.1.15) blows up in a finite time provided the following conditions are satisfied

$$sf(s) - c_n F(s) \geq 0, c_n > 1 + \frac{2}{n}, \ \forall s \in \mathbb{R}^+, \tag{1.1.16}$$

$$F(s) = \int_0^s f(\tau) d\tau. \tag{1.1.17}$$

A result on blow up of solutions of initial boundary value problem for the nonlinear Schrödinger equation in a bounded domain $\Omega \subset \mathbb{R}^n$ under the conditions (1.1.16) is obtained in [43].
Let us note that the above mentioned papers are devoted to nonlinear second order and first order equations and systems of equations. The methods employed in these works are not applicable in the study of higher order equations.
The energy method of finding conditions of blow up of solutions to initial boundary value problems for equations of the form (1.1.1) and (1.1.2), that can be used in the study of higher order equations. These methods was first suggested in [10]. and later used in [44], [45].
In [44] it is shown that if $f(\cdot)$ satisfies the condition

$$sf(s) - \mu F(s) \geq c_0 |u|^{2+\epsilon}, \tag{1.1.18}$$

$$F(s) = \int_0^s f(\tau) d\tau, \ \epsilon > 0, c_0 > 0, \mu > 2, \tag{1.1.19}$$

then for a certain class of initial functions the solutions of initial boundary value problems for the equations (1.1.1) and (1.1.2) blow up in a finite time.
In [25], [46] a powerful method of finding sufficient conditions of blow up of solution to the Cauchy problem for differential operator equations of the form

$$Pu_t = -Au + F(u), \tag{1.1.20}$$

$$Pu_{tt} = -Au + F(u), \tag{1.1.21}$$

in a Hilbert space H is suggested.
Here P and A are linear symmetric operators satisfying the conditions $P > 0, A \geq 0$ and $F(\cdot)$ is a nonlinear gradient operator that satisfies the condition

$$(F(u), u)_H \geq \beta G(u),$$

where $\beta > 2$ is a given number, $(\cdot,\cdot)_H$ is the dot product in H and $G(u)$ is a functional whose gradient is the operator $F(u)$. The results obtained for the equations (1.1.20) and (1.1.21) allow to get sufficient conditions of blow up of solutions of initial boundary value problems for a wide class of parabolic, hyperbolic, pseudo-parabolic equations and systems of equations, including higher order equations.
In [32], [47], [48], [25]- [49], [50] the concavity method and its modifications were used for finding sufficient conditions of blow up of solutions to the Cauchy problem for non-linear differential operator equations, differential operator equations with dissipative term, initial boundary value problems for linear parabolic and hyperbolic equations with nonlinear boundary conditions and various equations and systems of continuum mechanics.
This approach allowed the authors of above mentioned works to cover not only the problems considered in preceding papers [21], [20], [22], [24], [19] and other works, but also a wide class of new nonlinear problems for which the mentioned methods were not applicable.
By using the concavity method Levine and Payne obtained also interesting results on global nonexistence of solutions to the initial boundary value problems for linear parabolic equations under nonlinear boundary conditions of the form

$$\begin{cases} u_t - \Delta u = 0, \;\; x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} = f(u), \;\; x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \;\; x \in \Omega, \end{cases} \tag{1.1.22}$$

where Ω is a bounded omain of $\mathbb{R}^N$ with suffisiently smooth boundary $\partial\Omega, f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a given nonlinear term. The concavity method was used also in the study of initial boundary value problems for higher order parabolic equations under nonlinear boundary conditions and in the study of the linear wave equation under nonlinear boundary condition of the form:

$$\begin{cases} u_{tt} - \Delta u = 0, \ \ x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} = f(u), \ \ x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ \ x \in \Omega. \end{cases}$$

The nonlinear term here and in (1.1.22) satisfy the condition

$$f(s)s \geq (2+\epsilon)F(s), \ \ \forall s \in \mathbb{R}. \tag{1.1.23}$$

However, as it was noted in [47], [25], [46], in the frames of the concavity method the conditions of non-negativity and symmetricity of the linear operator A in (1.1.20) and (1.1.21) are essential.

The integral method (generalized concavity method) suggested in [51] allowed to get rid of this restriction.

This method, generalizing the concavity method, is based on a construction of some positive functional $\Psi(t) = \psi(u(t))$, which is defined in terms of the local solution of the problem (the local solvability of the problem is therefore required) and proving that the function $\Psi(t)$ satisfies the inequality

$$\Psi''(t)\Psi(t) - \beta\left[\Psi'(t)\right]^2 \geq -C_1\Psi^2(t) - C_2\Psi(t)\Psi'(t), \ t > 0,$$

where $\beta > 2$, $C_1 \geq 0$ and $C_2 \geq 0$ are given numbers.

The last inequality, thanks to the Lemma A.0.8 of the Chapter 1 allows to see that for some class of initial data, a solution of a problem under consideration blows up in a finite time.

The results obtained for differential-operator equations are used in [51] for finding conditions of blow up of solutions for a wide class of parabolic and hyperbolic equations with non-symmetric main parts of the form under the homogeneous Dirichlet boundary conditions.

The concavity method and its generalizations were used in the study of many nonlinear partial differential equations and systems By using the generalized concavity method Qin and Rivera [11] found sufficient conditions of blow up in a finite time of solutions of the

Cauchy problem for the system of thermoelasticity of the form

$$u_{tt} = au_{xx} + b\theta_x + du_x - mu_t + f(t,u),$$
$$\theta_t = k\theta_{xx} + g * \theta_{xx} + bu_{xt} + pu_x + q\theta_x.$$

In ([52], [53]) the concavity method is employed to find sufficient conditions of blow of solutions to Cauchy problem for nonlocal nonlinear equations of elasticity of the form

$$u_{tt} - (\beta * (u_1 + g(u)))_{xx} \tag{1.1.24}$$

and to the Cauchy problem for the system of equations of one-dimensional elasticity

$$\begin{cases} u_{1tt} - (\beta_1 * (u_1 + g_1(u_1, u_2)))_{xx}\,, \\ u_{2tt} - (\beta_2 * (u_2 + g_2(u_1, u_2)))_{xx}\,. \end{cases} \tag{1.1.25}$$

An interesting method of finding sufficient conditions of blow up of solutions of the Cauchy problem for nonlinear hyperbolic equations with nonlinear damping term of the form

$$u_{tt} - \Delta u + |u_t|^m u_t = |u|^p u,$$

where $p, q > 0$ are given numbers, was introduced in [54]. It was shown in [54] that if $p > m$, then there are initial data for which solution of the Cauchy problem for this equation blows up in a finite time. The method introduced in [54] based on the construction of a perturbed functional energy $\Psi(t)$. The result on blow up is obtained by showing that $\Psi(t)$ satisfies an ordinary differential inequality of the form

$$\Psi'(t) \geq \beta \left[\Psi(t)\right]^{1+\nu}, \quad \beta > 0,\ \nu > 0.$$

1.1.2. Previous Results on Asymptotic Behavior of Solutions

Initial boundary value problems for many mathematical models described by nonlinear parabolic and hyperbolic equations have global in time solutions. For this kind of problems an interesting and important problem is the problem of investigation of asymptotic behavior of solutions of corresponding initial boundary value problems as $t \to \infty$.

We would like to note that most of techniques employed in the the study of problems of blow up of solutions of initial boundary value problems for nonlinear parabolic and hyperbolic equations are based on the idea of transfer the study of initial boundary value problems for nonlinear PDEs to the study of Cauchy problems for appropriate nonlinear ordinary differential inequalities. In this way results on blow up of solutions of nonlinear PDEs are established by analysis of qualitative properties of solutions of nonlinear ordinary differential inequalities.

Most of results on asymptotic behavior of solutions of initial boundary value problems for second order nonlinear parabolic and hyperbolic equations are devoted to equations with constant coefficients or equations with coefficients depending only on spatial variables.

In [55] the authors got a result on exponential decay of global solutions of the problem

$$\begin{cases} u_t = \Delta u + f(u), \ \ x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} + \alpha u = 0, \ \ x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \ x \in \Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary, $u_0(x) \geq 0, \ \forall x \in \Omega$ is a smooth initial function, and the nonlinear term satisfies the conditions $f(0) = 0, \ \ f(s) > 0$, and $f(s) \leq p(s), \ \forall s > 0$, for a positive nondecreasing function $p(s)$.

One of the recent results of this type is result obtained in [56]. Here the author studied the decay rate to 0, as $t \to \infty$ of the solution of the initial boundary value problem for the equation

$$\psi_t - \Delta\psi - \lambda_1\psi + |\psi|^{p-1}\psi = 0, \ \ p > 1,$$

under homogeneous Dirichlet boundary conditions in a bounded smooth open connected domain of R^n. It is shown that either $\psi(\cdot,t)$ converges to 0 faster than any negative power of t, or $\psi(\cdot,t)$ decreases like $t^{-\frac{1}{p-1}}$.

Less is studied the problem of asymptotic behavior of solutions of nonlinear non-autonomous parabolic and hyperbolic equations with time dependent coefficients. In [57] the Cauchy problem for the following first order differential-operator equation in a Hilbert space H is considered

$$\frac{du}{dt} = A(t)u + F(t,u) + b(t),\ t \geq 0; \quad u(0) = u_0.$$

Here $A(t)$ is a linear dissipative operator, i.e.

$$Re(A(t)u,u) \leq \gamma(t)(u,u),\ \gamma(t) \geq 0,$$

$F(t,u)$ is a nonlinear operator, which satisfies the condition

$$\|F(t,u)\| \leq c_0\|u\|^p,\ p > 1,\ \|b(t)\| \leq \beta(t),$$

where $\beta(t) \geq 0$ is a continuous function. It is shown that under appropriate conditions on $\gamma(t)$ and $\beta(t)$

$$\|u(t)\| \to 0 \ \text{as} \ \ t \to \infty.$$

A number of papers devoted to the decay of solutions of the Cauchy problem for nonlinear wave equations with time dependent damping coefficient appeared last years (see e.g. [58], [59]). In these papers the decay estimates of solutions to the Cauchy problem for second order nonlinear wave equations of the form

$$u_{tt} - \Delta u + b(t)u_t = f(u),\ x \in \mathbb{R}^n,\ t > 0,$$

are considered. For special type of damping terms and nonlinearities it is estalished that the solutions of problems under consideration tend to zero as $t \to \infty$.

1.2. STATEMENT OF PROBLEMS

1.2.1. Second Order Nonlinear Parabolic Equations

Chapter 2 of the thesis is devoted to the study of initial boundary value problems for second order nonlinear parabolic equations under various boundary conditions.
First we considered the problem of blow up of solutions in a finite time

$$\begin{cases} u_t - \Delta u = f(u) + h(x,t), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} + \gamma u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2.1}$$

Next, by using the energy method, we studied the problem on blow up of solutions of the problem (4.0.1) when $h = h(x)$ and in addition to the condition (4.0.2) the the following condition holds

$$F(u) \geq D_0|u|^p - D_1 \quad \forall u \in \mathbb{R}$$

for some $p > 2, D_0 > 0, D_1 \geq 0$.
Finally the decay of solutions of initial boundary value problems for non-autonomous nonlinear parabolic equations with time dependent coefficients are investigated.

1.2.2. Second Order Nonlinear Hyperbolic Equations

Chapter 3 is devoted to study of initial boundary value problems for second order nonlinear hyperbolic equations. We obtained here sufficient conditions of blow up of solutions of initial boundary value problems for nonlinear wave equations in a finite time. For a wide class of second order nonlinear non-autonomous wave equations with time dependent damping terms conditions under natural conditions on nonlinear terms asimptotic behavior of solutions is studied. It is shown that all solutions of the problem under consideration tend to zero as $t \to \infty$.

The first problem on this chapter is the result on blow up of solutions of the problem

$$\begin{cases} u_{tt} + bu_t = \Delta u + f(u) + h(x,t), \quad x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} + \gamma u = 0, \qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1, \quad x \in \Omega, \end{cases} \tag{1.2.2}$$

where $b > 0, \gamma \in \mathbb{R}$ are given number, h is a given source term, u_0, u_1 are given initial functions, and $f(\cdot)$ is a nonlinear term.

Next we studied the decay of solutions:

$$\begin{cases} u_{tt} + b(t)u_t - \Delta u + f(u) = 0, \quad x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad x \in \Omega, \end{cases} \tag{1.2.3}$$

where $b(t)$ is a positive differentiable function defined on $[0,\infty)$ that satisfies the conditions

$$0 \leq b(t) \leq b_0, \quad |b'(t)| \leq \alpha b(t), \;\; 0 < \alpha \leq 2, \;\; \forall t \geq 0, \tag{1.2.4}$$

$$\int_0^t b(s)ds \to \infty \quad \text{as} \quad t \to \infty \tag{1.2.5}$$

and the function $f(u)$ satisfies the condition (4.0.7).

The proofs of auxiliary propositions which we have used to get main results are given as an Appendix.

1.3. NOTATIONS AND AUXILIARY PROPOSITINS

Throughout the thesis we are using the following notations:

- $\mathbb{R} := (-\infty, \infty)$.
- $\mathbb{R}^+ := (0, \infty)$.
- $L^2(\Omega)$ is a usual Lebesgue space with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$.
- $L^\infty(\Omega)$ is a usual Lebesgue space equipped with the norm

$$\|u\|_{L^\infty(\Omega)} := \sup_{x\in\Omega} |u(x)|. \tag{1.3.1}$$

- $C(\mathbb{R}^+)$ is the class of all functions that belong to $C[0, T]$ for each $T > 0$,
- $H^1(\Omega)$ is a Sobolev space of functions $v \in L^2(\Omega)$ whose weak derivatives also belong to $L^2(\Omega)$. This space is a Hilbert space with the inner product

$$(u, v)_{H^1(\Omega)} = \int_\Omega (u(x)v(x) + \nabla u(x) \cdot \nabla v(x))\, dx \tag{1.3.2}$$

  and the norm

$$\|v\|_{H^1(\Omega)} = \left(\|v\|^2 + \|\nabla v\|^2\right)^{1/2}. \tag{1.3.3}$$

- $H_0^1(\Omega)$ is the Sobolev space obtained by completion of $C_0^\infty(\Omega)$ with respect to the norm of $H^1(\Omega)$. The inner product and the norm in this space are defined as follows

$$(u, v)_{H_0^1(\Omega)} = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx \tag{1.3.4}$$

  and

$$\|v\|_{H_0^1(\Omega)} = \|\nabla v\|. \tag{1.3.5}$$

- $L^p(0,T;B),\ p \in [1,\infty)$ denotes a Banach space of all vector-functions with values in a Banach space B equipped with the norm

$$\|v\|_{L^p(0,T;B)} := \left(\int_0^T \|v(t)\|_B^p dt\right)^{1/p}.$$

- $L^\infty(0,T;B)$ denotes a Banach space of all vector-functions with values in a Banach space B equipped with the norm

$$\|v\|_{L^\infty(0,T;B)} := \sup_{t\in(0,T)} \|v(t)\|_B.$$

We will need the following inequalities:

- **Cauchy inequality "with ε"**

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \tag{1.3.6}$$

which is valid for each $a, b \ge 0$ and $\varepsilon > 0$.

- **Holder inequality** is the inequality

$$\int_\Omega |f(x)g(x)|\,dx \le \|f\|_{L^p(\Omega)} \cdot \|g\|_{L^{p'}(\Omega)}. \tag{1.3.7}$$

which holds for each $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ the inequality

- **Jensen inequality for integrals** is the inequality

$$\frac{\int_\Omega f(u(x))\psi(x)dx}{\int_\Omega \psi(x)dx} \ge f\left(\frac{\int_\Omega u(x)\psi(x)dx}{\int_\Omega \psi(x)dx}\right), \tag{1.3.8}$$

where f is a convex function on $\mathbb{R}$, $u \in C(\overline{\Omega}), \psi \in L^1(G)$ and ψ is positive on the domain Ω.

- **Poincare-Friedrichs Inequality** is the inequality

$$\|w\| \leq \lambda_1^{1/2} \|\nabla w\| \tag{1.3.9}$$

which holds for each $w \in H_0^1(\Omega)$. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, λ_1 is the first eigenvalue of the problem

$$\begin{cases} -\Delta\phi = \lambda\phi, \ x \in \Omega, \\ \phi = 0, \quad x \in \partial\Omega. \end{cases}$$

If $w \in H^2(\Omega) \cap H_0^1(\Omega)$, then (1.3.9) inequality implies that

$$\|\nabla w\| \leq \lambda_1^{-\frac{1}{2}} \|\Delta w\|. \tag{1.3.10}$$

- **Poincare Inequality** is the inequality

$$\|w\|^2 \leq a_0 \left[\int\limits_{\partial\Omega} w^2(x)dx + \|\nabla w\|^2 \right], \quad a_0 > 0. \tag{1.3.11}$$

which holds for each $w \in H^1(\Omega)$. Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, a_0 is a positive number which depends on $|\Omega|$. We will use also the following version of the Poincaré inequality which is valid for each function u from the Sobolev space $H^1(\Omega)$ (see e.g. [60] Ch. I):

$$\int_{\partial\Omega} v^2 d\sigma \leq \epsilon \int_{\Omega} |\nabla v|^2 dx + C_\epsilon \int_{\Omega} v^2 dx, \tag{1.3.12}$$

where ϵ is a positive parameter, and C_ϵ is a positive parameter which depends on ϵ.

In the study of asymptotic behavior of solutions to initial boundary value problems for nonlinear non-autonomous parabolic and hyperbolic equations we will use the following Lemma:

Lemma 1.3.1. *Suppose that* $a, q \in C[0, \infty),\ a(t) > 0, q(t) \geq 0,\ \ \forall t \geq 0,$

$$\int_0^t a(s)ds \to \infty,\ \ q(t) \to 0 \ \ \textit{as} \ \ t \to \infty. \tag{1.3.13}$$

Then all nonnegative solutions of the differential inequality

$$z'(t) + p(t)z(t) \leq q(t) \tag{1.3.14}$$

tend to zero as $t \to \infty$.

The proofs of results on blow up of solutions of problems we considered are based on the following propositions:

Lemma 1.3.2. *(see [24]) Suppose that a function* $a(t)$ *is twice continuously differentiable on some interval* $[0, T)$,
a function $H(r)$ *is continuous on* $[a_0, \infty)$ *and the condition*

$$H(r) \geq 0,\ \ \forall r \geq a_0 \tag{1.3.15}$$

holds. Assume also that

$$a''(t) \geq H(a(t)),\ t \geq 0, \tag{1.3.16}$$

$$a(0) = a_0 > 0,\ a'(0) = a_1 > 0. \tag{1.3.17}$$

Then

$$(1)\ \ a(t)\ \ \textit{is continuous and}\ \ \ a'(t) > 0,\ \ \forall t \in [0, T)$$

$$(2) \qquad\qquad t \leq 2 \int\limits_{a_0}^{a(t)} \left[a_1^2 + 2 \int\limits_{a_0}^{s} H(r)dr \right]^{-1/2} ds. \tag{1.3.18}$$

Lemma 1.3.3. *(see [25]) Let $\Psi(t)$ be a positive, twice differentiable function, which satisfies, for $t > t_0 \geq 0$, the inequality*

$$\Psi^{''}(t)\Psi(t) - (1+\alpha)\left[\Psi^{'}(t)\right]^2 \geq 0 \tag{1.3.19}$$

with some $\alpha > 0$.
If $\Psi(t_0) > 0$ and $\Psi'(t_0) > 0$, then there exists a time

$$T_0 \in (t_0, T_1), \ \ T_1 = \frac{\Psi(t_0)}{\alpha \Psi'(t_0)} + t_0$$

such that

$$\Psi(t) \to +\infty \ \ as \ \ t \to T_0^-. \tag{1.3.20}$$

Lemma 1.3.4. *(see [51]) Let twice continuously differentiable function $\Psi(t)$ satisfies for each $t \geq 0$ the inequality*

$$\Psi''(t)\Psi(t) - (1+\alpha)\left[\Psi(t)\right]^2 \geq 2C_1\Psi(t)\Psi'(t) - C_2\Psi^2(t) \tag{1.3.21}$$

and

$$\Psi(0) > 0, \Psi'(0) > -\gamma_2\alpha^{-1}\Psi(0), \tag{1.3.22}$$

where $\alpha > 0, C_1, C_2 \geq 0, C_1 + C_2 > 0$ and $\gamma_2 = -C_1 - \sqrt{C_1^2 + \alpha C_2}$. Then there exists

$$t_1 \leq T_1 = \left(2\sqrt{C_1^2 + \alpha C_2}\right)^{-1} \ln \frac{\gamma_1\Psi(0) + \alpha\Psi(0)}{\gamma_2\Psi(0) + \alpha\Psi'(0)},$$

with $\gamma_1 = -C_1 + \sqrt{C_1^2 + \alpha C_2}$ such that

$$\Psi(t) \to \infty \ \ as \ \ t \to t_1^-.$$

If $\Psi(0) > 0, \Psi'(0) > 0$ *and* $C_1 = C_2 = 0$*, then there exists*

$$t_2 \le T_2 = \frac{\Psi(0)}{\alpha \Psi'(0)}$$

such that

$$\Psi(t) \to \infty \ \textit{ as } \ t \to t_2^-.$$

Lemma 1.3.5. *(see [61]) Suppose* $\Psi(t) \in \mathbb{C}^{(2)}([0,T])$*, satisfies inequality*

$$\Psi\Psi'' - \alpha(\Psi')^2 + \kappa\Psi'\Psi \ge -\beta\Psi, \quad \alpha > 1, \ \beta \ge 0, \ \kappa \ge 0, \tag{1.3.23}$$

and

$$\Psi'(0) > \frac{\gamma}{\alpha - 1}\Psi(0), \tag{1.3.24}$$

$$\left[\Psi'(0) - \frac{\gamma}{\alpha - 1}\Psi(0)\right]^2 > \frac{2\beta}{2\alpha - 1}\Psi(0), \tag{1.3.25}$$

where $\Psi(t) \ge 0$*,* $\Psi(0) > 0$*. Then there exists*

$$T_0 \le \Psi^{1-\alpha}(0)(\alpha - 1)^{-1}\Psi^{\alpha}(0)\left[\left(\Psi'(0) - \frac{\gamma}{\alpha - 1}\Psi(0)\right)^2 - \frac{2\beta}{2\alpha - 1}\Psi(0)\right]^{-\frac{1}{2}}$$

such that

$$\limsup_{t \to T_0^-} \Psi(t) = +\infty.$$

For the convenience we give the proofs of these propositions in the Appendix.

2. SECOND ORDER NONLINEAR NONAUTONOMOUS PARABOLIC EQUATIONS

This chapter is devoted to the study of initial boundary value problems for second order nonlinear parabolic equations. Employing the energy methods, we find the sufficient conditions of blow up in a finite time of solutions to initial boundary value problems for second order nonlinear non-autonomous parabolic equations under the Robin boundary conditions. We study also the asymptotic behavior of solutions of nonlinear non-autonomous equations (whose solutions exist globally) as $t \to \infty$: Results on decay and growth of solutions of the considered problems are obtained.

2.1. BLOW UP OF SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS

In this section we study the initial boundary value problem for the second order nonlinear non-autonomous equation of the following form:

$$u_t - \Delta u = f(u) + h(x,t), \quad x \in \Omega, \quad t > 0, \tag{2.1.1}$$

$$\frac{\partial u}{\partial \nu} + \gamma u = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{2.1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{2.1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, γ is a given scalar, h is a given source term and $f(\cdot)$ is a given nonlinear term.
We show that if the nonlinear term satisfies the following conditions

$$f(u)u \geq 2(1+\alpha)F(u), \quad F(u) = \int_0^u f(s)\,ds, \ \text{ for all } u \in \mathbb{R} \tag{2.1.4}$$

with some positive α,

$$h \in L^2(\mathbb{R}^+; L^2(\Omega)) \cap L^\infty(\mathbb{R}^+; L^2(\Omega)) \tag{2.1.5}$$

then solutions of the problem (2.1.1)-(2.1.3) corresponding to a wide class of initial conditions blow-up in a finite time. Let us note that our study of the problem of blow up of solutions of the problem (2.1.1)-(2.1.3) is inspired by the work of Payne-Schaefer [62] . In the this paper, using the energy method, the authors established blow-up of solutions and obtained a lower bound of blow up time for the solutions of the problem (2.1.1)-(2.1.3) with $h \equiv 0$, essentially using positiveness of the coefficient γ and the initial function u_0. Later on in [63] blow up theorem and estimate of blow up time for nonlinear heat equation with time dependent coefficient is also obtained.

In this section, by using the concavity method of Levine [25] we will derive sufficient conditions for the finite-time blow-up of solutions of the problem (2.1.1)-(2.1.3) regardless of the sign of γ and the initial functions u_0 under the Robin boundary conditions.

For the blow-up of solutions of nonlinear parabolic partial differential equations there is a wide literature, we refer to [2], [25], [29], [64], [65] and references therein. The blow-up theorem will be established by using the Lemma A.0.7 In Section 2 the sufficient conditions of the blow up of solutions are obtained. In addition to that some remarks on blow up solutions are given.

2.2. BLOW UP OF SOLUTIONS

In this section, by using the Lemma A.0.7 we obtain sufficient conditions of blow up in a finite time of solutions of the initial boundary value problem (2.1.1)-(2.1.3).

Main result of this section is the following theorem:

Theorem 2.2.1. *Suppose that u is alocal soluton of the problem* (2.1.1)-(2.1.3)*, the initial function u_0 satisfies the condition*

$$-\|\nabla u_0\|^2 - \gamma \int_{\partial\Omega} u_0^2(x)d\sigma + 2\int_\Omega F(u_0(x))dx \geq (4+\frac{4}{\alpha})H_1 + \frac{H_2}{4\alpha|\gamma|C_\gamma(\alpha+1)} + \left(\frac{\alpha+2}{\alpha+1} + |\gamma|C_\gamma\right)\|u_0\|^2, \quad (2.2.1)$$

where C_γ is a positive t constant of the Poincaré inequality (1.3.12) *with $\epsilon = \frac{1}{\gamma}$, and $H_1 := \int_0^\infty \|h(t)\|^2 dt$ and $H_2 := \sup_{t\in\mathbb{R}^+} \|h(t)\|^2$. And suppose that the conditions* (2.7.7) *and* (2.1.5) *are also satisfied. Then the solution of the problem* (2.1.1)-(2.1.3) *blows up in a* *finite time, i.e. there exists $t_1 \le t_2 := \frac{1}{2\alpha}$ such that*

$$\lim_{t \to t_1^-} \int_0^t \|u(s)\|^2 ds = \infty.$$

Suppose that $u(x,t)$ is a local solution of the problem (2.1.1)-(2.1.3). It is clear that the function $v(x,t) = e^{-mt}u(x,t)$, $m > 0$ satisfies the equation

$$mv + v_t = \Delta v + e^{-mt} f(e^{mt} v) + e^{-mt} h(x,t), \tag{2.2.2}$$

the boundary condition

$$\frac{\partial v}{\partial \nu} + \gamma v = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{2.2.3}$$

and the initial condition

$$v(x,0) = u_0(x), \; x \in \; \Omega. \tag{2.2.4}$$

So our aim now is to find sufficient conditions of blow up in a finite time of solutions of the problem (3.2.17)-(2.2.4). First we prove the following Lemma:

Lemma 2.2.2. *Let v be a local solution og the problem* (3.2.17)-(2.2.4). *The the function*

$$E(t) = -\frac{m}{2}\|v\|^2 - \frac{1}{2}\|\nabla v\|^2 - \frac{\gamma}{2}\int_{\partial\Omega} v^2 d\sigma + e^{-2mt}\int_{\Omega} F(e^{mt}v)dx \tag{2.2.5}$$

satisfies the differential inequality

$$\frac{d}{dt}E(t) \ge 2m\alpha E(t) + m\alpha\left[m\|v\|^2 + \|\nabla v\|^2 + \gamma\int_{\partial\Omega} v^2 d\sigma\right] + (1-\varepsilon_1)\|v_t\|^2 - \frac{1}{4\varepsilon_1}\|h\|^2 e^{-2mt}, \tag{2.2.6}$$

and the following estimate from below with a positive parameter $\epsilon_1 \in (0,1)$ for the function

$E(t)$ *holds true*

$$E(t) \geq e^{2m\alpha t}E(0) + (1-\varepsilon_1)e^{2m\alpha t}\int_0^t \|v_s(s)\|^2 e^{-2ms}ds - \frac{1}{4\varepsilon_1}e^{2m\alpha t}\int_0^t \|h(s)\|^2 e^{-2m(\alpha+1)s}ds. \quad (2.2.7)$$

Proof. Multiplying the equation (3.2.17) by v_t and integrating over Ω and using (2.2.3) we obtain

$$\frac{d}{dt}\left[\frac{m}{2}\|v\|^2 + \frac{1}{2}\|\nabla v\|^2 + \frac{\gamma}{2}\int_{\partial\Omega} v^2 d\sigma\right] + \|v_t\|^2 = e^{-mt}\int_\Omega f(e^{mt}v)v_t dx + e^{-mt}\int_\Omega h v_t dx. \quad (2.2.8)$$

It is easy to see that

$$\frac{d}{dt}F(e^{mt}v) = f(e^{mt}v)(e^{mt}v_t + me^{mt}v).$$

Plugging the expression

$$e^{-mt}f(e^{mt})v_t = e^{-2mt}\frac{d}{dt}F(e^{mt}v) - me^{-mt}f(e^{mt}v)v$$

into (3.2.18)we obtain

$$\frac{d}{dt}\left[\frac{m}{2}\|v\|^2 + \frac{1}{2}\|\nabla v\|^2 + \frac{\gamma}{2}\int_{\partial\Omega} v^2 d\sigma\right] + \|v_t\|^2 - e^{-2mt}\frac{d}{dt}\int_\Omega F(e^{mt}v)dx + me^{-mt}\int_\Omega f(e^{mt}v)v dx = e^{-mt}\int_\Omega h v_t dx. \quad (2.2.9)$$

Since

$$e^{-2mt}\frac{d}{dt}\int_\Omega F(e^{mt}v)dx = \frac{d}{dt}\left[e^{-2mt}\int_\Omega F(e^{mt}v)dx\right] + 2me^{-2mt}\int_\Omega F(e^{mt}v)dx \quad (2.2.10)$$

we have

$$\frac{d}{dt}\left[\frac{m}{2}\|v\|^2+\frac{1}{2}\|\nabla v\|^2+\frac{\gamma}{2}\int_{\partial\Omega}v^2d\sigma-e^{-2mt}\int_\Omega F(e^{mt}v)dx\right]+\|v_t\|^2$$
$$+me^{-mt}\int_\Omega f(e^{mt}v)vdx-2me^{-2mt}\int_\Omega F(e^{mt}v)dx=e^{-mt}\int_\Omega hv_tdx. \quad (2.2.11)$$

By using the condition (2.7.7) we see that

$$e^{-mt}f(e^{mt}v)v=e^{-2mt}f(e^{mt}v)e^{mt}v\geq 2(\alpha+1)e^{-2mt}F(e^{mt}v). \quad (2.2.12)$$

Employing this inequality and the Cauchy inequality with ε we deduce from (2.2.11) the following inequality

$$\frac{d}{dt}\left[\frac{m}{2}\|v\|^2+\frac{1}{2}\|\nabla v\|^2+\frac{\gamma}{2}\int_{\partial\Omega}v^2d\sigma-e^{-2mt}\int_\Omega F(e^{mt}v)dx\right]+\|v_t\|^2$$
$$+2m\alpha e^{-2mt}\int_\Omega F(e^{mt}v)dx\leq\varepsilon_1\|v_t\|^2+\frac{1}{4\varepsilon_1}\|h\|^2e^{-2mt}.$$

From this inequality we obtain

$$\frac{d}{dt}\left[-\frac{m}{2}\|v\|^2-\frac{1}{2}\|\nabla v\|^2-\frac{\gamma}{2}\int_{\partial\Omega}v^2d\sigma+e^{-2mt}\int_\Omega F(e^{mt}v)dx\right]\geq$$
$$2m\alpha\left[-\frac{m}{2}\|v\|^2-\frac{1}{2}\|\nabla v\|^2-\frac{\gamma}{2}\int_{\partial\Omega}v^2d\sigma+e^{-2mt}\int_\Omega F(e^{mt}v)dx\right] \quad (2.2.13)$$
$$+m\alpha\left[m\|v\|^2+\|\nabla v\|^2+\gamma\int_{\partial\Omega}v^2d\sigma\right]+(1-\varepsilon_1)\|v_t\|^2-\frac{1}{4\varepsilon_1}\|h\|^2e^{-2mt}.$$

So we get (2.2.6). By using (1.3.12) in (2.2.6) we get

$$\frac{d}{dt}E(t)\geq 2m\alpha E(t)+(1-\varepsilon_1)\|v_t\|^2+m\alpha(m-|\gamma|C_\gamma)\|v\|^2-\frac{1}{4\varepsilon_1}\|h\|^2e^{-2mt}.$$

Choosing in the last inequality $m=|\gamma|C_\gamma$, and then solving the obtained differential inequality we obtain the estimate (2.2.7). □

Lemma 2.2.3. *Let* $v(x,t)$ *be the solution of the problem* (3.2.17)-(2.2.4) *and define*

$$\Psi(t) := \int_0^t \|v(s)\|^2 ds + c_0,$$

where c_0 *is some positive parameter to be specified. Then we have:*

$$\Psi'(t) = \|v(t)\|^2 = 2\int_0^t (v(s), v_s(s))ds + \|u_0\|^2 \tag{2.2.14}$$

and

$$\Psi''(t) \geq 4(\frac{\alpha}{2}+1)\left[\int_0^t \|v_s(s)\|^2 ds + c_0\right], \tag{2.2.15}$$

where α *is a positive number in* (2.7.7).

Proof. Proof of (2.2.14) is trivial, and it is obvious that $\Psi''(t) = 2(v(t), v_t(t))$. By using the equation (3.2.17) and the inequality (2.2.12) we obtain the following estimate from below for the function $\Psi''(t)$:

$$\begin{aligned}\Psi''(t) = 2\int_\Omega v\Big[-mv + \Delta v + e^{-mt}f(e^{mt}v) + e^{-mt}h(x,t)\Big]dx \geq -2m\|v\|^2 \\ -2\|\nabla v\|^2 - 2\gamma\int_{\partial\Omega} v^2 d\sigma + 4(\alpha+1)e^{-2mt}\int_\Omega F(e^{mt}v)dx + 2e^{-mt}(h,v) \\ = 4(\alpha+1)\Big[-\frac{m}{2}\|v\|^2 - \frac{1}{2}\|\nabla v\|^2 - \frac{\gamma}{2}\int_{\partial\Omega} v^2 d\sigma + e^{-2mt}\int_\Omega F(e^{mt}v)dx\Big] + \\ 2m\alpha\|v\|^2 + 2\alpha\|\nabla v\|^2 + 2\alpha\gamma\int_{\partial\Omega} v^2 d\sigma + 2e^{-mt}(h,v).\end{aligned} \tag{2.2.16}$$

Since

$$2e^{-mt}(h,v) \geq -2m\alpha\|v\|^2 - e^{-2mt}\frac{1}{2\alpha m}\|h\|^2$$

we deduce from (2.2.16) that

$$\Psi''(t) \geq 4(\alpha+1)E(t) - e^{-2mt}\frac{1}{2\alpha m}\|h(t)\|^2.$$

Thus employing the lower estimate (2.2.7) for $E(t)$ we obtain the estimate

$$\Psi''(t) \geq 4(\alpha+1)(1-\epsilon_1)\Big[\int_0^t \|v_s(s)\|^2\,ds + c_0\Big] + 4(\alpha+1)\Big[E(0) - \frac{1}{\epsilon_1}\int_0^t \|h(s)\|^2\,ds\Big] - \frac{1}{2\alpha m}e^{-2mt}\|h(t)\|^2 - 4(\alpha+1)(1-\epsilon_1)c_0. \quad (2.2.17)$$

Now, by assuming

$$E(0) \geq (2+\frac{2}{\alpha})\int_0^\infty \|h(t)\|^2 + \frac{1}{8\alpha m(\alpha+1)}\sup_{t\in\mathbb{R}^+}\|h\|^2 + \frac{\alpha+2}{2(\alpha+1)}c_0$$

and choosing $\epsilon_1 = \frac{\alpha}{2(\alpha+1)}$ we get $4(\alpha+1)(1-\frac{\varepsilon_1}{2}) = 4(\frac{\alpha}{2}+1)$ we see that (2.2.17) implies (2.2.15). □

Proof. Using (2.2.14) and (2.2.15) we get

$$\Psi''(t)\Psi(t) - (\alpha_1+1)(\Psi'(t))^2 \geq 4(\frac{\alpha}{2}+1)\Big[\int_0^t \|v_s(s)\|^2\,ds + c_0\Big]\Big[\int_0^t \|v(s)\|^2\,ds + c_0\Big] - 4(\frac{\alpha}{2}+1)\Big[\int_0^t (v(s),v_s(s))ds + \frac{1}{2}\|u_0\|^2\Big]^2. \quad (2.2.18)$$

Finally we choose $c_0 = \frac{1}{2}\|u_0\|^2$. Then due to the Cauchy-Schwarz inequality we deduce from (2.2.18) the desired inequality $\Psi''(t)\Psi(t) - (\frac{\alpha}{2}+1)(\Psi'(t))^2 \geq 0$. The proof of the theorem follows from the Lemma A.0.7. □

2.3. SOME REMARKS ON BLOW UP

We proved the following propositions on blow-up under the Robin boundary conditions:

Remark 2.3.1. *If* $h(x,t) \equiv h(x) \in L^2$ *and in addition to* (2.7.7)

$$F(u) \geq D_0|u|^p - D_1, \quad \forall u \in \mathbb{R}, \quad (2.3.1)$$

for some $D_0 > 0$, $D_1 \geq 0$, then by an easier argument a blow up result can be obtained employing the energy equalities for solutions of initial boundary value problem (2.1.1)-(2.1.3)*:*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = -\|\nabla u\|^2 - \gamma \int_{\partial\Omega} u^2 d\sigma + (f(u), u) + (h, u) \tag{2.3.2}$$

$$\frac{d}{dt}E(t) = \|u_t(t)\|^2, \tag{2.3.3}$$

where

$$E(t) \equiv -\frac{1}{2}\|\nabla u\|^2 - \frac{\gamma}{2}\int_{\partial\Omega} u^2 d\sigma + (F(u), 1) + (h, u).$$

In fact the following proposition holds.

Proposition 2.3.2. *If the nonlinear term $f(.)$ satisfies the conditions* (2.7.7), (2.3.1) *and the initial function satisfies the condition*

$$E(0) - \alpha D_1 \geq 0, \tag{2.3.4}$$

then the solution of the problem (2.1.1)-(2.1.3) *blows up in a finite time.*

Proof. Utilizing the conditions (2.7.7) and (2.3.1) we obtain from (2.3.2) the inequality

$$\frac{d}{dt}\|u(t)\|^2 \geq -2\|\nabla u(t)\|^2 - 2\gamma \int_{\partial\Omega} u^2 d\sigma + 4(\alpha+1)(F(u), 1) + 2(h, u)$$
$$= 4E(t) + 4\alpha(F(u), 1) \geq 4E(t) + 4\alpha D_0 \int_\Omega |u(x,t)|^p dx - 4\alpha D_1. \tag{2.3.5}$$

Due to the inequality $E(t) \geq E(0)$ which can be obtained by integration of the energy equality (2.3.3), and the condition (2.3.4) we obtain from (2.3.5) the following first order ordinary differential inequality for the function $\Psi(t) \equiv \|u(t)\|^2$:

$$\Psi'(t) \geq K_0[\Psi(t)]^{\frac{p}{2}}, \tag{2.3.6}$$

where $K_0 \equiv |\Omega|^{-\frac{p-2}{2}}(4\alpha D_0)^{-\frac{p}{2}}$.

Integrating (2.3) we see that

$$\Psi(t) \to \infty \quad \text{as} \quad t \to (p-2)[2K_0]^{-1}[\Psi(0)]^{\frac{2-p}{2}}.$$

□

We would like also to note that a result on blow up of solutions to a class of nonlinear parabolic equations under the Robin boundary condition can be obtained by using the so called method of eigenfunctions. In fact the following proposition holds true:

Proposition 2.3.3. *Suppose that* $u_0(x) \geq 0$, $\forall x \in \Omega$, $\gamma > 0$, *the source term* $h(x,t) \equiv h(x) \in L^2(\Omega)$, *depends only on* $x \in \Omega$, *the nonlinear term is a convex, continuous function that satisfies also the conditions;*

$$f(u) - \lambda_1 u - h_0 > 0, \quad \forall u \geq \alpha_0 > 0,$$

with

$$\int_{\alpha_0}^{\infty} \frac{d\eta}{f(\eta) - \lambda_1 \eta - h_0} < \infty, \tag{2.3.7}$$

where $h_0 = \int_\Omega h(x)\psi_1(x)dx$, $\alpha_0 = \int_\Omega u_0(x)\psi_1(x)$, $\lambda_1 > 0$ *is the eigenvalue corresponding to the normalized principal eigenfunction* $\psi_1(x)$ *of the problem*

$$\begin{cases} -\Delta\psi = \lambda\psi, & x \in \Omega; \\ \frac{\partial\psi}{\partial\nu} + \gamma\psi = 0, & x \in \partial\Omega. \end{cases} \tag{2.3.8}$$

Then the solution of the problem (2.1.1)-(2.1.3) *blows up in a finite time.*

Proof. In fact multiplying the equation (2.1.1) by the positive function ψ_1, then integrating the obtined relation over Ω and using the boundary condition (2.1.2) we obtain the followong integral equality

$$\int_\Omega u_t\psi_1 dx + \lambda_1 \int_\Omega u\psi_1 dx = \int_\Omega f(u)\psi_1 dx + \int_\Omega hudx. \tag{2.3.9}$$

Due to the Jensen inequality for integrals(1.3.8) we have

$$\int_\Omega f(u)\psi_1 dx \geq f\left(\int_\Omega u\psi_1 dx\right).$$

Thus from (3.3.5) we get the following differential inequality for the function $E(t) = \int_\Omega u(x,t)\psi_1(x)dx$:

$$E'(t) \geq f(E(t)) - \lambda_1 E(t) - h_0.$$

Integrating this inequality and using the condition (4.0.3) we obtain the desired result. □

2.4. BLOW UP WHEN THE ENERGY IS POSITIVE

In this section we will prove that there is a wide class of initial functions for which solutions of the problem Consider the initial and boundary value problem

$$u_t - \Delta u = f(u), \quad x \in \Omega \quad t > 0, \tag{2.4.1}$$

$$\frac{\partial u}{\partial \eta} + \gamma u = 0, \quad x \in \partial\Omega, \tag{2.4.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{2.4.3}$$

with arbitrary positive initial energy blow up in a finite time. More precisely we prove the following theorem.

Theorem 2.4.1. *Assume that the nonlinear term $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a contnuous function that satisfies the following condition*

$$f(s)s - 2(1+\alpha)F(s) \geq -D_0, \quad \forall s \in \mathbb{R}, \tag{2.4.4}$$

where $\alpha > 0$, $\gamma \geq 0$, and $D_0 \geq 0$ are given numbers, and $F(s) = \int_0^s f(\tau)d\tau$. Suppose also that

$$\nu(\alpha,\gamma)\|u_0\|^2 > 2|\Omega|D_0 + 4(1+\alpha)E(0), \tag{2.4.5}$$

where

$$E(t) := \frac{1}{2}\|\nabla u(t)\|^2 + \frac{\gamma}{2}\int_{\partial\Omega} u^2(x,t)d\sigma - (F(u(t)), 1), \tag{2.4.6}$$

and $\nu(\alpha,\gamma)$ *is a positive parameter depending on positive scalars* α *and* γ. *Then the corresponding local solution of the problem* (2.4.1)- (2.4.3) *blows up in a finite time.*

Proof. Multiplication of the equation (2.4.1) by the function u_t and integration of the obtained relation over the domain Ω gives us the following equality

$$\|u_t\|^2 + \frac{d}{dt}\left[\frac{1}{2}\|\nabla u\|^2 + \frac{\gamma}{2}\int_{\partial\Omega} u^2 d\sigma - (F(u), 1)\right] = 0. \tag{2.4.7}$$

Integrating (2.4.7) over the interval $(0, t)$ we obtain

$$E(t) = E(0) - \int_0^t \|u_\tau(\tau)\|^2 d\tau \tag{2.4.8}$$

which implies that

$$E(t) \leq E(0), \ \ \forall t > 0.$$

We consider now the function

$$\Psi(t) := \int_0^t \|u(\tau)\|^2 d\tau,$$

where u is a solution of the initial boundary valuemproblem (2.4.1)- (2.4.3) . By using the equation (2.4.1), the boundary condition (2.4.2) and the condition (4.0.4) on the nonlinear term, we obtain the following estimate:

$$\Psi''(t) = 2(u, u_t) = 2(u, \Delta u + f(u))$$

$$\geq -2\|\nabla u\|^2 - 2\gamma \int\limits_{\partial\Omega} u^2 d\sigma + 4(1+\alpha)(F(u),1) - 2|\Omega|D_0.$$

By using the energy equality (2.4.8) in the last inequality we obtain the following estimate from below for the function $\Psi''(t)$:

$$\Psi''(t) \geq 4(1+\alpha)\left[-\frac{1}{2}\|\nabla u\|^2 - \frac{\gamma}{2}\int_{\partial\Omega} u^2 d\sigma + (F(u),1)\right]$$
$$- 2|\Omega|D_0 + 2\alpha\|\nabla u\|^2 + 2\alpha\gamma\int_{\partial\Omega} u^2 d\sigma = 2\alpha\|\nabla u\|^2 + 2\alpha\gamma\int_{\partial\Omega} u^2 d\sigma$$
$$- 4(1+\alpha)E(0) + 4(1+\alpha)\int_0^t \|u_\tau(\tau)\|^2 d\tau - 2|\Omega|D_0. \quad (2.4.9)$$

By using (2.4.5) we get

$$\Psi''(t) \geq \left[\nu(\alpha,\gamma)\|u(t)\|^2 - D_1\right] + 4(1+\alpha)\int_0^t \|u_\tau(\tau)\|^2 d\tau, \quad (2.4.10)$$

where $D_1 = 2|\Omega|D_0 + 4(1+\alpha)E(0)$.
Since (2.4.5) holds we deduce from (2.4.10) that

$$\Psi''(t) \geq 4(1+\alpha)\int_0^t \|u_\tau(\tau)\|^2 d\tau. \quad (2.4.11)$$

By using the equality

$$\Psi'(t) = \|u(t)\|^2 = 2\int_0^t\int_\Omega u u_t dx d\tau + \|u_0\|^2,$$

the estimate (2.4.11) and the Cauchy-Schwarz inequality we obtain then the following inequality

$$\Psi''(t)\Psi(t) - (1+\alpha)\big(\Psi'(t) - \|u_0\|^2\big)^2 \geq$$
$$4(1+\alpha)\left[\int_0^t \|u(\tau)\|^2 d\tau \int_0^t \|u(\tau)\|^2 d\tau - \left(\int_0^t (u,u_\tau) d\tau\right)^2\right] \geq 0 \quad (2.4.12)$$

Thanks to the Cauchy -Schwarz inequality the expression in square brackets on the right hand side of the last inequality is positive. Therefore we have

$$0 \leq \Psi''(t)\Psi(t) - (1+\alpha)\left(\Psi'(t)\right)^2 = \Psi''(t)\Psi(t) - (1+\frac{\alpha}{2})\left(\Psi'(t)\right)^2 - M(t), \quad (2.4.13)$$

where

$$M(t) := \frac{\alpha}{2}\left(\Psi'(t)\right)^2 - 2(1+\alpha)\Psi'(t)\|u_0\|^2 + (1+\alpha)\|u_0\|^4.$$

It follows from (2.4.10) that

$$\frac{d}{dt}\left(\Psi'(t) - M_1\right) \geq \nu(\alpha,\gamma)\left(\Psi'(t) - M_1\right),$$

where $M_1 = \frac{D_1}{\nu(\alpha,\gamma)}$.
From the last inequality we deduce that

$$\Psi'(t) \geq M_1 + e^{\nu(\alpha,\gamma)t}\left(\Psi'(0) - M_1\right).$$

Hence

$$\Psi'(t) \to \infty \quad \text{as} \quad t \to \infty,$$

and therefore there exists some $t_0 > 0$ such that

$$M(t) \geq 0, \quad \forall t \geq t_0.$$

Therefore (2.4.13) implies that

$$\Psi''(t)\Psi(t) - (1+\frac{\alpha}{2})\left(\Psi'(t)\right)^2, \ \ \forall t \geq t_0. \quad (2.4.14)$$

Finally thanks to the inequality (2.4.14) we can use the Lemma A.0.7 and get the desired result. □

2.5. NONLINEAR PARABOLIC EQUATIONS WITH CUBIC NONLINEARITY. BLOW UP FOR POSITIVE INITIAL ENERGY

In this section we consider the initial boundary value problem for the heat equation with cubic nonlinearity:

$$u_t - \Delta u = u^3, \quad x \in \Omega, t > 0, \tag{2.5.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{2.5.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.5.3}$$

Our aim is to show that there are inital data with positive initial energy for which the corresponding solutions of the problem (2.5.1)-(2.5.3) blow up in a finite time.
To prove this result we will show that the the function

$$\Psi(t) = \int_0^t \|u(\tau)\|^2 d\tau.$$

satisfies the conditions of the Lemma A.0.7.
Employing the equation (2.5.1) we can easily get

$$\Psi''(t) = 2(u(t), u_t(t)) = -2\|\nabla u(t)\|^2 + 2\int_\Omega u^4(x,t)dx. \tag{2.5.4}$$

Multiplying the equation (2.5.1) by u_t and integrating over the domain Ω we obtain the energy equality

$$\|u_t(t)\|^2 + \frac{d}{dt}\left[\frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{4}\int_\Omega u^4(x,t)\right] = 0.$$

After integration of the last equality over the interval $(0,t)$ we arrive at the following energy equality

$$E(t) := \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{4}\int_\Omega u^4(x,t)dx = E(0) - \int_0^t \|u(\tau)\|^2 d\tau, \tag{2.5.5}$$

where

$$E(0) = \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{4}\int_\Omega u_0(x)dx.$$

By using the energy equality (2.5.5) and the inequality

$$\int_\Omega u^4(x,t)dx \geq |\Omega|^{-1}\left(\|u(t)\|^2\right)^2$$

we obtain from the relation (2.5.4) the following inequality

$$\begin{aligned}\Psi''(t) &= 5\left[-\frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{4}\int_\Omega u^4(x,t)\right] + \frac{1}{4}\int_\Omega u^4(x,t)dx + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{3}{4}\int_\Omega u^4(x,t)dx \\ &= -5E(0) + 5\int_0^t \|u(\tau)\|^2 d\tau + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{3}{4}\int_\Omega u^4 dx \\ &\geq \frac{3}{4|\Omega|}\left(\int_\Omega u^2(x,t)dx\right)^2 - 5E(0) = m_0\left[\left(\|u(t)\|^2\right)^2 - m_1^2\right], \quad (2.5.6)\end{aligned}$$

where $m_0 = \frac{3}{4|\Omega|}$ and $m_1^2 = \frac{20|\Omega|E(0)}{3}$.
So we have

$$\frac{d}{dt}\|u(t)\|^2 \geq m_0(\|u(t)\|^2 - m_1)(\|u(t)\|^2 + m_1). \quad (2.5.7)$$

From the last inequality we deduce that the function $\|u(t)\|^2 - m_1$ satisfies the following first order differential inequality

$$\frac{d}{dt}(\|u(t)\|^2 - m_1) \geq m_0 m_1(\|u(t)\|^2 - m_1).$$

Integrating this inequality over the interval $(0,t)$, we obtain the following estimate from below for the function $\Psi'(t)$

$$\|u(t)\|^2 \geq m_1 + e^{m_0 m_1 t}A_0,$$

where

$$A_0 = \|u_0\|^2 - \left[\frac{20|\Omega|}{3}E(0)\right]^{\frac{1}{2}} > 0$$

and

$$E(0) := \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{4}\int_\Omega u_0^4(x)dx \geq 0.$$

Thus

$$\Psi'(t) = \|u(t)\|^2 \to \infty \quad \text{as} \quad t \to \infty, \tag{2.5.8}$$

On the other side from (2.5.6) we get

$$\Psi''(t) \geq -5E(0) + 5\int_0^t \|u(\tau)\|^2 d\tau + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{3}{4}\int_\Omega u^4(x,t)dx. \tag{2.5.9}$$

Thanks to (2.5.8) the functions $\|\nabla u(t)\|^2$ and $\int_\Omega u^4(x,t)dx$ tend to infinity as $t \to \infty$. Therefore there exists some $t_1 > 0$ such that

$$-5E(0) + 5\int_0^t \|u(\tau)\|^2 d\tau + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{3}{4}\int_\Omega u^4(x,t)dx \geq 0, \;\; \forall t \geq t_1.$$

Hence (2.5.9) implies that

$$\Psi''(t) \geq 4(1+\alpha_0)\int_0^t \|u(\tau)\|^2 d\tau$$

with $\alpha_0 = \frac{1}{4}$.

Employing the last inequality and the Cauchy-Schwarz inequality we arrive at the desired inequality

$$\Psi''(t)\Psi(t) - (1+\alpha_0)\left[\Psi'(t)\right]^2 \geq 0, \;\; \forall t \geq t_1.$$

Thus thanks to the Lemma A.0.7 there exists $T_1 > t_1$ such that

$$\Psi(t) \to \infty, \quad \text{as} \quad t \to T_1^-.$$

So we have proved the following Theorem.

Theorem 2.5.1. *Suppose that*

$$\|u_0\| > 0, \;\; E_0 := \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{4}\int_\Omega u_0^4(x)dx \geq 0,$$

and

$$\|u_0\|^2 > \left[\frac{20|\Omega|}{3}E(0)\right]^{\frac{1}{2}}.$$

Then the solution of the problem (2.5.1)-(2.5.3) *blows up in a finite time.*

2.6. ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF IBVP FOR NON-AUTONOMOUS PARABOLIC EQUATIONS

In this section we consider semilinear second order parabolic equations under the Dirichlet boundary condition whose energy integrals are sign preserved (in contrary to problems we considered in the previous section).

First we study the following problem:

$$u_t - \Delta u + c(t)f(u) = h(x,t), \quad x \in \Omega \quad t > 0, \tag{2.6.1}$$

$$u = 0, \quad x \in \partial\Omega, \tag{2.6.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2.6.3}$$

where Ω is a bouded domain with sufficiently smooth boudary $\partial\Omega$, $c(t)$ is a given damping coefficient, $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a given nonlinear term, u_0 is a given initial function and $h(x,t)$ is a given source term. We obtained the following result about behavior of solutions to the initial boundary value problem (2.6.1)-(2.6.3) as $t \to +\infty$:

Theorem 2.6.1. *Suppose that*

$$c \in C^1(\mathbb{R}^+), \;\; \textit{and} \;\; c(t) \geq c'(t) \;\; \forall t > 0, \tag{2.6.4}$$

$$h \in L^2(0,T;L^2(\Omega)), \textit{ for each } T > 0 \textit{ and } \|h(t)\| \to 0, \; \textit{as } t \to \infty \tag{2.6.5}$$

Suppose also that $f(\cdot) : \mathbb{R} \to \mathbb{R}$ *is a continuous function that satisfies the following conditions*

$$f(u)u - F(u) \geq 0, \quad F(u) := \int_0^u f(s)ds \geq 0, \quad \forall u \in \mathbb{R}. \tag{2.6.6}$$

Then all solutions of the initial boundary value problem (2.6.1)-(2.6.3) *tend to zero as* $t \to \infty$*, i.e.*

$$\lim_{t\to\infty} \left[\frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 + c(t)(F(u),1)\right] = 0.$$

Proof. Multiplying (2.6.1) by u_t and by u, then integrating over Ω we get the following equalities:

$$\|u_t\|^2 + \frac{d}{dt}\left[\frac{1}{2}\|\nabla u\|^2 + c(t)(F(u),1)\right] = c'(t)(F(u),1) + (h,u_t), \tag{2.6.7}$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + c(t)(f(u),u) = (h,u). \tag{2.6.8}$$

Adding (2.6.7) and (2.6.8) we obtain

$$\|u_t\|^2 + \frac{d}{dt}\left[\frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 + c(t)(F(u),1)\right] + \|\nabla u\|^2 + c(t)(f(u),u) = (h,u) + (h,u_t) + c'(t)(F(u),1). \tag{2.6.9}$$

Now applying Hölder, Cauchy and Poincaré-Friedrichs inequalities we estimate the first two terms on the right hand side of the equality (2.6.9):

$$|(h, u_t)| \leq \frac{1}{4}\|h\|^2 + \|u_t\|^2$$

and

$$|(h, u)| \leq \|h\|\|u\|^2 \leq \|h\|\lambda_1^{\frac{-1}{2}}\|\nabla u\| \leq \lambda_1^{-1}\|h\|^2 + \frac{1}{4}\|\nabla u\|^2.$$

Employing these inequalities, and the condition (2.6.6) on the nonlinear term $f(\cdot)$ we obtain from the relation (2.6.9) the following inequality

$$\frac{d}{dt}\left[\frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 + c(t)(F(u), 1)\right] + \frac{3}{4}\|\nabla u\|^2 + 2c(t)(F(u), 1) \leq (\frac{1}{4} + \lambda_1^{-1})\|h\|^2 + c'(t)(F(u), 1).$$

Using the condition (2.6.4) we deduce from the last inequality that the following inequality holds true:

$$\frac{d}{dt}\left[\frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 + c(t)(F(u), 1)\right] + \frac{3}{4}\|\nabla u\|^2 + c(t)(F(u), 1) \leq (\frac{1}{4} + \lambda_1^{-1})\|h\|^2. \quad (2.6.10)$$

Due to the Poincar$\acute{e}$- Friedrich's inequality (1.3.9) we have

$$\frac{3}{4}\|\nabla u\|^2 = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{4}\|\nabla u\|^2 \geq \frac{1}{2}\|\nabla u\|^2 + \frac{\lambda_1}{4}\|u\|^2$$

By using the last inequality in (2.6.10) we get

$$\frac{d}{dt}\left[\frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 + c(t)(F(u), 1)\right] + \frac{1}{2}\|\nabla u\|^2 + c(t)(F(u), 1) + \frac{\lambda_1}{4}\|u\|^2 \leq (\frac{1}{4} + \lambda_1^{-1})\|h\|^2.$$

Since

$$\frac{1}{2}\|\nabla u\|^2 + c(t)(F(u),1) + \frac{\lambda_1}{4}\|u\|^2 \geq \min\left\{1, \frac{\lambda_1}{2}\right\}\left[\frac{1}{2}\|u\|^2 + \frac{1}{2}\|\nabla u\|^2 + c(t)(F(u),1)\right],$$

we have

$$\frac{d}{dt}E_1(t) + d_0 E_1(t) \leq (\frac{1}{4} + \lambda_1^{-1})\|h(t)\|^2,$$

where $d_0 = \min\{1, \frac{\lambda_1}{2}\}$. Finally we use Lemma A.0.5 and deduce that $E_1(t) \to 0$ as $t \to 0$ and get the desired result. □

2.7. NONLINEAR PARABOLIC EQUATION WITH TIME DEPENDENT COEFFICIENTS: DECAY OF SOLUTIONS

In this section we consider the problem

$$u_t - a(t)\Delta u + f(u) = h(x,t), \quad x \in \Omega, \quad t > 0, \tag{2.7.1}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{2.7.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{2.7.3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ and $a(t), h(x,t), f(\cdot)$ are given functions.

We prove that under some restriction on the functions $a(t)$ and $h(x,t)$ all solutions of the problem (2.7.1)-(2.7.3) tend to zero as $t \to \infty$. More precisely we prove the following theorem:

Theorem 2.7.1. *Assume that* $h \in L^2(0,T;L^2(\Omega))$, $\forall T > 0$, $a(t) > 0, \forall t \geq 0$ *is a continuous function on* $[0,\infty)$, *such that*

$$\int_0^t a(s)ds \to \infty \quad as \quad t \to \infty, \tag{2.7.4}$$

and

$$\lim_{t\to\infty} a^{-1}(t)\|h(t)\|^2 = 0. \tag{2.7.5}$$

Suppose also that $f(\cdot) : \mathbb{R} \to \mathbb{R}$ *is a continuous function that satisfies the condition* (2.6.6). *Then*

$$\lim_{t\to\infty} \|u(t)\| = 0. \tag{2.7.6}$$

Proof. We multiply the equation (2.7.1) in $L^2(\Omega)$ by u:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + a(t)\|\nabla u(t)\|^2 + (f(u), u) = (h, u(t)) \leq \|h(t)\|\|u(t)\|.$$

By using the Cauchy-Schwarz inequality and the Poincare inequality we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \lambda_1 a(t)\|u(t)\|^2 \leq \frac{\lambda_1 a(t)}{2}\|u(t)\|^2 + \frac{1}{2\lambda_1 a(t)}\|h(t)\|^2$$

or

$$\frac{d}{dt}\|u(t)\|^2 + \lambda_1 a(t)\|u(t)\|^2 \leq \lambda_1^{-1}a^{-1}(t)\|h(t)\|^2. \tag{2.7.7}$$

We can apply the Lemma A.0.5 with

$$z(t) = \|u(t)\|^2, p(t) = \lambda_1 a(t), q(t) = \lambda_1^{-1}a^{-1}(t)\|h(t)\|^2$$

and deduce that (2.7.6) holds true, i.e. $\|u(t)\|^2 \to 0$ as $t \to \infty$. □

Theorem 2.7.2. *If the function* $a(t)$ *satisfies the conditions of the Theorem 2.7.1 and* $f(\cdot)$ *is a differentiable nondecreasing function, then*

$$\|\nabla u(t)\| \to 0 \quad \textit{as} \quad t \to \infty. \tag{2.7.8}$$

Proof. Multiplication of the equation (2.7.1) by $-\Delta u$ now gives us the following equality

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2 + a(t)\|\Delta u(t)\|^2 + (f'(u), |\nabla u(t)|^2) = -(h(t), \Delta u(t)). \tag{2.7.9}$$

Since the function $a(t)$ is a positive function we can estimate the right hand side of (3.3.2) as follows

$$|(h(t), \Delta u(t))| \leq \frac{1}{2}a(t)\|\Delta u(t)\|^2 + \frac{1}{2}\|h(t)\|^2 a^{-1}(t).$$

Since $f(\cdot)$ is nondecreasing the expression $(f'(u), |\nabla u(t)|^2)$ is non-negative. Therefore using the last inequality we obtain from (3.3.2) the following inequality

$$\frac{d}{dt}\|\nabla u(t)\|^2 + a(t)\|\Delta u(t)\|^2 \leq \|h(t)\|^2 a^{-1}(t).$$

Finally by using the inequality (1.3.10) we obtain

$$\frac{d}{dt}\|\nabla u(t)\|^2 + a(t)\lambda_1\|\nabla u(t)\|^2 \leq \|h(t)\|^2 a^{-1}(t).$$

From the last inequality thanks to the Lemma A.0.5 we deduce the desired result. □

Remark 2.7.3. *It follows from Theorem 2.7.1 that solutions of the problem* (2.7.1)-(2.7.3) *tend to zero even when heat conductivity coefficient may tend zero as* $t \to \infty$*, and* $\|h(t)\|$ *may tend to* $+\infty$ *as* $t \to \infty$*. For instance solutions of* (2.7.1)-(2.7.3) *tend to zero as* $t \to \infty$ *when* $a(t) = \frac{1}{1+t}$, $h(x,t) = h(x)\sqrt{t}$*, where* $h \in L^2(\Omega)$ *is a given function.*

3. SECOND ORDER NONLINEAR WAVE EQUATIONS

In this chapter we study the problems of blow up and decay of solutions of initial boundary value problems for second order nonlinear wave equations under various boundary conditions.

3.1. BLOW UP OF SOLUTIONS TO DAMPED NONLINEAR WAVE EQUATIONS

Here we study the problem of blow up of solutions to the following initial boundary value problem.

$$u_{tt} + bu_t = \Delta u + f(u) + h(x,t), \quad x \in \Omega, \quad t > 0, \tag{3.1.1}$$

$$\frac{\partial u}{\partial \nu} + \gamma u = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{3.1.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1, \quad x \in \Omega, \tag{3.1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $b > 0$ is a given damping coefficient, $\gamma \in \mathbb{R}$ is a given number, h is a given source term, u_0, u_1 are given initial functions, and $f(\cdot)$ is a nonlinear term.
Our aim is to find sufficient conditions of finite time blow up of local solutions to non-autonomous semilinear damped wave equations with damping term and source term under the Robin boundary conditions.
There are many papers devoted to the blow up of solutions to initial boundary value problems for nonlinear wave equations (see, e.g., [25], [46], [61]). In majority of these papers initial boundary value problems for various nonlinear wave equations under the homogeneous Dirichlet or Neumann boundary conditions are considered. The main novelty compared to preceding results is studying the blow up of solutions of nonlinear wave equations under the Robin boundary conditions, and we obtained results on blow up of solutions for more wide class of non-autonomous equations with arbitrary large initial energy. The main tool we used in the proof of our results is Levine' s concavity method and its modifications

(see [25], [46], [61]).

3.2. BLOW UP OF SOLUTIONS TO DAMPED SEMILINEAR WAVE EQUATION UNDER THE ROBIN BOUNDARY EQUATION

In this section we will find sufficient conditions for blow up of solutions to the problem (2.1.1)-(2.1.3) under some restrictions on initial functions and the source term, when the nonlinear term satisfies the condition

$$f(s)s - 2(2\alpha+1)F(s) \geq 0, \ \ \forall s \in \mathbb{R}, \tag{3.2.1}$$

with some $\alpha > 0$. Here $F(s) = \int_0^s f(\tau)d\tau$.
First we consider the case when the number γ is nonnegative. Multiplication of (3.1.1) in $L^2(\Omega)$ by u_t gives us the energy equality:

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - (F(u),1) + \frac{\gamma}{2}\int_{\partial\Omega} u^2 d\sigma\right] + b\|u_t\|^2 = (u_t, h), \tag{3.2.2}$$

$$\frac{d}{dt}E_\gamma(t) + b\|u_t(t)\|^2 = (u_t(t), h(t)), \tag{3.2.3}$$

where

$$E_\gamma(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 - (F(u(t)),1) + \frac{\gamma}{2}\int_{\partial\Omega} u^2(t) d\sigma.$$

Following [46] we introduce the function

$$\Psi(t) = \|u(t)\|^2 + b\int_0^t \|u(s)\|^2 ds + c_0, \tag{3.2.4}$$

with a positive parameter c_0 to be chosen below.
By using the equation (3.1.1) the boundary condition (3.1.2) and the condition (3.2.1) on the

nonlinear term, we get:

$$
\begin{aligned}
\frac{d^2}{dt^2}\Psi(t) &= 2(u,u_{tt}) + 2\|u_t\|^2 + 2b(u,u_t) \\
&= 2(u,u_{tt}+bu_t) + 2\|u_t\|^2 = 2(u,\Delta u + f(u) + h) + 2\|u_t\|^2 \\
&= -2\|\nabla u\|^2 + 2(u,f(u) + 2(u,h) + 2\|u_t\|^2 - 2\gamma\int_{\partial\Omega} u^2 d\sigma \\
&\geq -2\|\nabla u\|^2 + 4(2\alpha+1)(F(u),1) + 2\|u_t\|^2 + 2(u,h) - 2\gamma\int_{\partial\Omega} u^2 d\sigma \\
&= -4(2\alpha+1)E_\gamma(t) + 4\alpha\|\nabla u\|^2 + 4(\alpha+1)\|u_t\|^2 + 4\alpha\gamma\int_{\partial\Omega} u^2 d\sigma + 2(u,h). \quad (3.2.5)
\end{aligned}
$$

It follows from (3.2.3) that

$$
E_\gamma(t) = E_\gamma(0) - b\int_0^t \|u_s(s)\|^2 ds + \int_0^t (u_s(s),h(s))ds.
$$

Thus we obtain from (3.2.5):

$$
\begin{aligned}
\Psi''(t) \geq -4(2\alpha+1)E_\gamma(0) &+ 4(2\alpha+1)b\int_0^t \|u_s\|^2 ds - 4(2\alpha+1)\int_0^t (u_s,h)ds \\
&+ 4\alpha\|\nabla u\|^2 + 4(\alpha+1)\|u_t\|^2 + 4\alpha\gamma\int_{\partial\Omega} u^2 d\sigma + 2(u,h). \quad (3.2.6)
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality and the Cauchy inequality with ϵ (1.3.6) we obtain

$$
\begin{aligned}
\left|4(2\alpha+1)\int_0^t (u_s(s),h(s))ds\right| &\leq 4(2\alpha+1)\int_0^t \|u_s(s)\|\|h(s)\|ds \\
&\leq \delta\int_0^t \|u_s(s)\|^2 ds + \frac{4(2\alpha+1)^2}{\delta}\int_0^t \|h(s)\|^2 ds \quad (3.2.7)
\end{aligned}
$$

$$
2|(u,h)| \leq \delta\|u\|^2 + \frac{1}{\delta}\|h\|^2. \quad (3.2.8)
$$

By using the inequality (3.2.7) with $\delta = 4\alpha b$, and the inequality (3.2.8) with $\delta = \nu_0 := 4\alpha a_0 \min\{1,\gamma\}$, where a_0 is a constant in the Poincare inequality (1.3.11), we get from (3.2.6) the estimate:

$$\Psi''(t) \geq -4(2\alpha+1)E_\gamma(0) + 4(\alpha+1)\left[\|u_t(t)\|^2 + b\int_0^t \|u_s(s)\|^2 ds\right] - \frac{(2\alpha+1)^2}{\alpha b}\int_0^\infty \|h(t)\|^2 dt - \frac{1}{\nu_0}\|h(t)\|_\infty^2 \quad (3.2.9)$$

or by choosing $c_0 = \frac{b}{2}\|u_0\|^2$ we get

$$\Psi''(t) \geq 4(\alpha+1)\left[\|u_t\|^2 + b\int_0^t \|u_s\|^2 ds + \frac{b}{2}\|u_0\|^2\right] - d_0, \quad (3.2.10)$$

where

$$d_0 := 4(2\alpha+1)E_\gamma(0) + \frac{(2\alpha+1)^2}{\alpha b}\int_0^\infty \|h(t)\|^2 dt + 2(\alpha+1)b\|u_0\|^2 + \frac{1}{\nu_0}\|h(t)\|_\infty^2. \quad (3.2.11)$$

Hence thanks to Cauchy-Schwarz inequality we obtain:

$$\Psi''(t)\Psi(t) - (\alpha+1)\left[\Psi'(t)\right]^2 \geq -d_0\Psi(t).$$

Therefore due to the Lemma 1.3.5 we have following result:

Theorem 3.2.1. *Suppose that*

i) $\gamma \geq 0,\ (u_0, u_1) > 0,\ h \in L^\infty(\mathbb{R}^+; L^2(\Omega))$,

and

ii) $\left[2(u_0,u_1) + b\|u_0\|^2\right]^2 > \frac{2d_0(b+2)}{2\alpha+1}\|u_0\|^2$*, where* d_0 *is defined in* (3.2.11).

Then there exist $t_0 < \infty$ *such that*

$$\lim_{t\to t_0^-} \|u(t)\| = \infty,$$

i.e. solution of the problem (3.1.1)-(3.1.3) *blows up in a finite time.*

Now we consider the case when $\gamma < 0$. First we obtain sufficient conditions of blow up of solutions of the problem (3.1.1)-(3.1.3) by using Lemma [51]. To get the desired result we again use the function $\Psi(t)$ defined in (3.2.4). Arguing as in the proof of the previous theorem we arrive at the inequality (3.2.6). Employing the inequality (3.2.7) in (3.2.6) we get

$$\Psi''(t) \geq -4(2\alpha+1)E_\gamma(0) + 4(\alpha+1)b\int_0^t \|u_s\|^2 ds - \frac{4(2\alpha+1)^2}{\alpha b}\int_0^t \|h(t)\|^2 ds$$
$$+ 4\alpha\|\nabla u\|^2 + 4(\alpha+1)\|u_t\|^2 + 4\alpha\gamma\int_{\partial\Omega} u^2 d\sigma + 2(u,h). \quad (3.2.12)$$

Utilizing Cauchy Schwarz inequality and the inequality (1.3.12) with $\epsilon = \frac{1}{|\gamma|}$ we obtain

$$\left|4\alpha\gamma\int_{\partial\Omega} u^2 d\sigma\right| \leq 4\alpha\|\nabla u\|^2 + 4\alpha C_{\frac{1}{|\gamma|}}\|u\|^2, \ |2(u,h)| \leq \|u\|^2 + \|h\|^2.$$

Employing last two inequalities we obtain from (3.2.12) the following inequality:

$$\Psi''(t) \geq -4(2\alpha+1)E_\gamma(0) + 4(\alpha+1)\left[\|u_t\|^2 + b\int_0^t \|u_s\|^2 ds\right]$$
$$- \|h\|^2 - \frac{4(2\alpha+1)^2}{\alpha b}\int_0^t \|h(s)\|^2 ds - \left(1 + 4\alpha C_{\frac{1}{|\gamma|}}\right)\|u\|^2. \quad (3.2.13)$$

Suppose that

$$-4(2\alpha+1)E_\gamma(0) - \|h\|_\infty^2 - \frac{4(2\alpha+1)^2}{\alpha b}\int_0^\infty \|h(t)\|^2 dt \geq 0. \quad (3.2.14)$$

Then we obtain from (3.2.13) the estimate

$$\Psi''(t) \geq 4(\alpha+1)\left[\|u_t\|^2 + b\int_0^t \|u_s\|^2 ds + \frac{b}{2}\|u_0\|^2\right] - d_1\|u\|^2 - 4(\alpha+1)\frac{b}{2}\|u_0\|^2, \quad (3.2.15)$$

where $d_1 = \left(1 + 4\alpha C_{\frac{1}{|\gamma|}}\right)$. Thus employing Cauchy-Schwarz inequality we arrive at the inequality

$$\Psi''(t)\Psi(t) - (1+\alpha)\left[\Psi'(t)\right]^2 \geq -C_0\Psi^2(t), \ \ C_0 = d_1 + 2(\alpha+1)b.$$

Therefore it follows from Lemma [51] that the following theorem holds.

Theorem 3.2.2. *Suppose that* $E_\gamma(0) < 0$ *and the condition* (3.2.14) *is satisfied. Suppose also that*

$$2(u,u_0) > \left[\sqrt{\alpha C_0}(1+\frac{b}{2}) - b\right]\|u_0\|^2.$$

Then the solution of the problem (3.1.1)-(3.1.3) *blows up in a finite time.*

Finally we will prove blow up of solutions to the problem by employing Lemma A.0.7. It is convenient to make the following change :

$$u(x,t) = e^{mt}v(x,t), \tag{3.2.16}$$

where m is some positive number. It is easy to see that

$$(mb+m^2)e^{mt}v + (b+2m)e^{mt}v_t + e^{mt}v_{tt} = e^{mt}\Delta v + f(e^{mt}v) + h(x,t).$$

Thus the function $v(x,t)$ defined by (3.2.16) is a solution of the problem

$$(mb+m^2)v + (b+2m)v_t + v_{tt} = \Delta v + e^{-mt}f(e^{mt}v) + e^{-mt}h(x,t), \tag{3.2.17}$$

$$\frac{\partial v}{\partial \nu} + \gamma v = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{3.2.18}$$

$$v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x) + mu_0(x). \tag{3.2.19}$$

By using the Levine's Lemma first we prove the following:

Theorem 3.2.3. *Suppose that the condition* (3.2.1) *holds, and*

$$4(\alpha+1)E_1(0) - \frac{1}{2m\alpha}\int_0^\infty \|h(s)\|^2 ds - \frac{1}{2(mb+m^2)\alpha}\|h\|^2_{L^\infty(R^+)} - 4(\alpha+1)c_0 \geq 0, \tag{3.2.20}$$

where $E_1(0)$ *is defined in* (3.2.25)*, and* m *is a positive solution of the equation*

$$m^2 + mb - |\gamma|C(|\gamma|^{-1}) = 0, \; c_0 = (b+2m)\|v_0\|^2. \tag{3.2.21}$$

Then the corresponding solution of the problem (3.2.17)-(3.2.19) *blows up in a finite time.*

Proof. Multiplying (3.2.17) by v_t and integrating over Ω we obtain

$$(mb+m^2)\int_\Omega vv_t dx + (b+2m)\int_\Omega {v_t}^2 dx + \int_\Omega v_{tt}v_t = \int_\Omega \Delta v v_t dx + \int_\Omega e^{-mt} f(e^{mt}v)v_t dx + \int_\Omega e^{-mt}h(x,t)v_t dx.$$

$$\frac{1}{2}(mb+m^2)\frac{d}{dt}\|v\|^2 + (b+2m)\|v_t\|^2 dx + \frac{1}{2}\frac{d}{dt}\|v_t\|^2 = -\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 - \frac{\gamma}{2}\frac{d}{dt}\int_{\partial\Omega} v^2 d\sigma + \frac{d}{dt}\left[e^{-2mt}\int_\Omega F(e^{mt}v)dx\right] + 2me^{-2mt}\int_\Omega F(e^{mt}v)dx - me^{-mt}\int_\Omega f(e^{mt}v)vdx + \int_\Omega e^{-mt}h(x,t)v_t dx.$$

From the last inequality we get

$$\frac{d}{dt}\left[\frac{mb+m^2}{2}\|v\|^2 + \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\nabla v\|^2 + \frac{\gamma}{2}\int_{\partial\Omega} v^2 d\sigma - e^{-2mt}\int_\Omega F(e^{mt}v)dx\right] + (b+2m)\|v_t\|^2 - 2me^{-2mt}\int_\Omega F(e^{mt}v)dx + me^{-mt}\int_\Omega f(e^{-mt}v)vdx \le \varepsilon_1\|v_t\|^2 + \frac{1}{4\varepsilon_1}\|h\|^2 e^{-2mt} \quad (3.2.22)$$

Thanks to (3.2.1) we have:

$$e^{-mt}f(e^{mt}v)v = e^{-2mt}f(e^{mt}v)e^{-mt}v \ge 2(2\alpha+1)e^{-2mt}F(e^{mt}v).$$

By using the last inequality on the left hand side of the onequality (3.2.22) we obtain the following estimate

$$\frac{d}{dt}\left[\frac{mb+m^2}{2}\|v\|^2 + \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\nabla v\|^2 + \frac{\gamma}{2}\int_{\partial\Omega} v^2 d\sigma - e^{-2mt}\int_\Omega F(e^{mt}v)dx\right] + (b+2m)\|v_t\|^2 + 4\alpha me^{-2mt}\int_\Omega F(e^{mt}v)dx \le \varepsilon_1\|v_t\|^2 + \frac{1}{4\varepsilon_1}\|h\|^2 e^{-2mt}.$$

We can rewrite the last inequality in the following form

$$\frac{d}{dt}\left[-\frac{mb+m^2}{2}\|v\|^2 - \frac{1}{2}\|v_t\|^2 - \frac{1}{2}\|\nabla v\|^2 - \frac{\gamma}{2}\int_{\partial\Omega} v^2 d\sigma + e^{-2mt}\int_\Omega F(e^{mt}v)dx\right] \geq$$
$$4m\alpha\left[-\frac{mb+m^2}{2}\|v\|^2 - \frac{1}{2}\|v_t\|^2 - \frac{1}{2}\|\nabla v\|^2 - \frac{\gamma}{2}\int_{\partial\Omega} v^2 d\sigma + e^{-2mt}\int_\Omega F(e^{mt}v)dx\right]$$
$$+ 2m\alpha\left[(mb+m^2)\|v\|^2 + \|\nabla v\|^2 + \gamma\int_{\partial\Omega} v^2 d\sigma\right] +$$
$$(-\varepsilon_1 + (b+2m) + 2m\alpha)\|v_t\|^2 - \frac{1}{4\varepsilon_1}\|h\|^2 e^{-2mt}. \quad (3.2.23)$$

So we have

$$\frac{d}{dt}E_1(t) \geq 4m\alpha E_1(t) + 2m\alpha\left[(mb+m^2)\|v\|^2 + \|\nabla v\|^2 + \gamma\int_{\partial\Omega} v^2 d\sigma\right] +$$
$$(2m\alpha + b + 2m - \varepsilon_1)\|v_t\|^2 - \frac{1}{4\varepsilon_1}\|h\|^2 e^{-2mt}, \quad (3.2.24)$$

where

$$E_1(t) := -\frac{mb+m^2}{2}\|v\|^2 - \frac{1}{2}\|v_t\|^2 - \frac{1}{2}\|\nabla v\|^2 - \frac{\gamma}{2}\int_{\partial\Omega} v^2 d\sigma$$
$$+ e^{-2mt}\int_\Omega F(e^{mt}v)dx. \quad (3.2.25)$$

Employing the inequality (1.3.12) with $\epsilon = |\gamma^{-1}|$ we get from (3.2.24) the estimate

$$\frac{d}{dt}E_1(t) \geq 4m\alpha E_1(t) +$$
$$(2m\alpha + b + 2m - \varepsilon_1)\|v_t\|^2 + m\alpha\left[(mb+m^2) - |\gamma| C(|\gamma|^{-1})\right]\|v\|^2 - \frac{1}{4\varepsilon_1}\|h\|^2 e^{-2mt}.$$

Taking in the last inequality $\varepsilon_1 = 2m\alpha$, and integrating it we obtain the following estimate from below for $E_1(t)$

$$E_1(t) \geq e^{4m\alpha t}E_1(0) + (b+2m)e^{4m\alpha t}\int_0^t \|v_s(s)\|^2 e^{-4ms} ds$$
$$- \frac{1}{4\varepsilon_1}e^{4m\alpha t}\int_0^t \|h(s)\|^2 e^{-m(4\alpha+2)s} ds. \quad (3.2.26)$$

Let us consider the following function

$$\Psi(t) = \|v(t)\|^2 + (b+2m)\int_0^t \|v(\tau)\|^2 d\tau + c_0,$$

where v is the solution of the problem and c_0 is a positive parameter to be chosen later. It is easy to see that

$$\Psi'(t) = 2(v(t), v_t(t)) + (b+2m)\|v(t)\|^2$$
$$= 2(v(t), v_t(t)) + 2(b+2m)\int_0^t (v(\tau), v_\tau(\tau))d\tau + (b+2m)\|v_0\|^2. \quad (3.2.27)$$

and

$$\Psi''(t) = 2\|v_t(t)\|^2 + 2(v(t), v_{tt}(t)) + 2(b+2m)(v(t), v_t(t))$$
$$= 2\|v_t(t)\|^2 + 2(v_{tt}(t) + (b+2m)v_t(t), v(t)).$$

Employing here the equation (3.2.17) and the condition (3.2.1) we obtain

$$\Psi''(t) = 2\|v_t(t)\|^2 + 2(\Delta v(t) + e^{-mt}f(e^{mt}v(t)) + e^{-mt}h - (mb+m^2)v(t), v(t))$$
$$\geq -2(mb+m^2)\|v\|^2 - 2\|\nabla v\|^2$$
$$-2\gamma\int_{\partial\Omega} v^2 d\sigma + 4(2\alpha+1)e^{-2mt}\int_\Omega F(e^{mt}v)dx + 2e^{-mt}(h,v) + 2\|v_t\|^2$$
$$= 4(2\alpha+1)\left[-\frac{(mb+m^2)}{2}\|v\|^2 - \frac{1}{2}\|v_t\|^2 - \frac{1}{2}\|\nabla v\|^2 - \frac{\gamma}{2}v^2 d\sigma + e^{-2mt}\int_\Omega F(e^{mt}v)dx\right]$$
$$+ 4(mb+m^2)\alpha\|v\|^2 + 4\alpha\|\nabla v\|^2 - 4\alpha\gamma\int_{\partial\Omega} v^2 d\sigma$$
$$+ 2e^{-mt}(h,v) + 4(\alpha+1)\|v_t\|^2. \quad (3.2.28)$$

Thanks to the Cauchy- Schwarz inequality we have:

$$2e^{-mt}(h,v) \geq -2(mb+m^2)\alpha\|v\|^2 - e^{-2mt}\frac{1}{2(mb+m^2)\alpha}\|h\|^2.$$

Employing the last inequality and the notation (3.2.25) we obtain from the equality (3.2.28) the following estimate

$$\Psi''(t) \geq 4(2\alpha+1)E_1(t) - e^{-2mt}\frac{1}{2(mb+m^2)\alpha}\|h\|^2 + 4(\alpha+1)\|v_t\|^2.$$

From the last inequality due to (3.2.26) we have

$$\begin{aligned}\Psi''(t) \geq 4(\alpha+1)\left[(b+2m)\int_0^t \|v_s\|^2 ds + \|v_t\|^2 + c_0\right] - 4(\alpha+1)c_0 \\ + 4(\alpha+1)e^{4m\alpha t}\left[E_1(0) - \frac{1}{2m\alpha}\int_0^t e^{-m(4\alpha+2)s}\|h\|^2 ds\right] \\ - e^{-2mt}\frac{1}{2(mb+m^2)\alpha}\|h\|^2\end{aligned}$$

By using the condition (3.2.20) we infer from the last inequality that

$$\Psi''(t) \geq 4(\alpha+1)\left[(b+2m)\int_0^t \|v_s\|^2 ds + \|v_t\|^2 + c_0\right].$$

Thus employing the Schwarz inequality we get

$$\Psi''(t)\Psi(t) - (\alpha+1)\left[\Psi'(t)\right]^2 \geq 0.$$

So the statement of the theorem follows from the Lemma A.0.7. □

3.3. BLOW UP OF SOLUTIONS OF NONLINEAR WAVE EQUATION. METHOD OF EIGENFUNCTION.

In this section we consider the initial boundary value problem for second order nonlinear wave equation of the form

$$u_{tt} - \Delta u = f(u) + h(x), \quad x \in \Omega, \quad t > 0, \tag{3.3.1}$$

$$\frac{\partial u}{\partial \nu} + \gamma u = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{3.3.2}$$

$$u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{3.3.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, γ is a given positive number, h is a given source term that depends on space variables, u_0, u_1 are given initial functions, and $f(\cdot)$ is a given nonlinear term. Our aim is to use the so called method of eigenfunctions to find sufficient conditions of blow up of solutions to the problem (3.3.1)-(3.3.3).

Proposition 3.3.1. *Suppose that $u_0(x), u_1(x), h(x)$ are given smooth functions on the domain Ω, $\gamma > 0$, the nonlinear term $f(\cdot)$ is a convex function that satisfies also the conditions;*

$$f(u) - \lambda_1 u - h_0 > 0, \quad \forall u \geq \alpha_0 > 0,$$

with

$$\int_{\alpha_0}^{\infty} \left[a_1^2 + 2\int_{a_0}^{s} (f(\eta) - \lambda_1\eta - h_0)\, d\eta\right]^{-1/2} ds < \infty, \tag{3.3.4}$$

where $\lambda_1 > 0$ is the eigenvalue corresponding to the normalized principal eigenfunction $\psi_1(x)$ of the problem (2.3.8)*, $h_0 = \int_\Omega h(x)\psi_1(x)dx$, and*

$$\alpha_0 = \int_\Omega u_0(x)\psi_1(x) > 0, \;\; \alpha_1 = \int_\Omega u_1(x)\psi_1(x) > 0.$$

Then the solution of the initial boundary value problem (3.3.1)-(3.3.3) *blows up in a finite time.*

Proof. Multiplying the equation (3.3.1) by ψ_1, and then integrating the obtained relation over Ω and using the boundary condition (2.1.2) we get

$$\int_\Omega u_{tt}\psi_1 dx + \lambda_1 \int_\Omega u\psi_1 dx = \int_\Omega f(u)\psi_1 dx + \int_\Omega hudx. \tag{3.3.5}$$

By using the the Jensen inequality for integrals(1.3.8) we obtain

$$\int_\Omega f(u)\psi_1 dx \geq f\left(\int_\Omega u\psi_1 dx\right).$$

Thus from (3.3.5) we get the following second order differential inequality for the function

$V(t) = \int_\Omega u(x,t)\psi_1(x)dx$:

$$V''(t) \geq f(V(t)) - \lambda_1 V(t) - h_0. \tag{3.3.6}$$

Since $V(0) = \alpha_0 > 0,\ V'(0) = \alpha_1 > 0$ and the condition (3.3.4) holds, we can use the Lemma 1.3.2 with

$$H(V) = f(V) - \lambda_1 V - h_0$$

and deduce that solution of the problem (3.3.1)-(3.3.3) blows up in a finite time. □

3.4. DECAY OF SOLUTIONS TO DAMPED NONAUTONOMOUS NONLINEAR WAVE EQUATION

In this section we study the initial boundary value problem for nonlinear wave equation with time dependent damping coefficient:

$$u_{tt} + b(t)u_t - \Delta u + f(u) = 0, \quad x \in \Omega, \quad t > 0, \tag{3.4.1}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{3.4.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad x \in \Omega, \tag{3.4.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, b is time dependent damping coefficient, $\gamma \in \mathbb{R}$ is a given number, h is a given source term, u_0, u_1 are given initial functions, and $f(\cdot)$ is a nonlinear term.

Theorem 3.4.1. *Suppose that* $b(t)$ *is a positive differentiable function defined on* $[0,\infty)$ *that satisfies the conditions*

$$0 \leq b(t) \leq b_0, \quad |b'(t)| \leq \alpha b(t), \quad 0 < \alpha \leq 2, \quad \forall t \geq 0, \tag{3.4.4}$$

$$\int_0^t b(s)ds \to \infty \quad as \quad t \to \infty \tag{3.4.5}$$

and the function $f(u)$ satisfies the condition (2.6.6)*. Then all solutions of the problem* (3.4.1)-(3.4.3) *tend to zero as $t \to \infty$.*

Proof. Multiplying (3.4.1) by $u_t + \varepsilon b(t)u$, where ε is a positive number to be determined below:

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}b(t)\varepsilon b(t)\|u(t)\|^2 + (F(u),1) + \varepsilon b(t)(u,u_t)\right]$$
$$+ [b(t) - \varepsilon b(t)]\,\|u_t(t)\|^2 - \varepsilon b'(t)(u,u_t) + \varepsilon b(t)\|\nabla u\|^2 + \varepsilon b(t)(f(u),u)$$
$$- \frac{1}{2}\varepsilon(b^2(t))'\|u(t)\|^2 = (h(t), u_t + \varepsilon b(t)u(t)). \quad (3.4.6)$$

By using the notation

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \varepsilon b(t)(u,u_t) + (F(u),1) + \frac{1}{2}b(t)\varepsilon(t)\|u(t)\|^2 \quad (3.4.7)$$

we can rewrite the equality (3.4.6) in the following form:

$$\frac{d}{dt}E(t) + \delta b(t)E(t) + b(t)(1-\varepsilon)\|u_t\|^2 + \varepsilon b(t)\|\nabla u\|^2 - \varepsilon b'(t)(u,u_t) + \varepsilon b(t)(f(u),u)$$
$$- \frac{\delta}{2}\|u_t\|^2 - \frac{\delta}{2}\|\nabla u\|^2 - \delta(F(u),1) - \delta\varepsilon b(t)(u_t,u) - \frac{\delta}{2}\varepsilon b^2(t)\|u\|^2 = 0.$$

Here δ is a positive parameter to be chosen below.

$$\frac{d}{dt}E(t) + \delta b(t)E(t) + b(t)(1-\varepsilon-\frac{\delta}{2})\|u_t\|^2 + b(t)(\varepsilon - \frac{\delta}{2}\varepsilon b_0^3)\|u\|^2$$
$$+ b(t)(\varepsilon-\delta)[-(F(u),1) + (f(u),u)] - \varepsilon b'(t)(u_t,u) - \delta\varepsilon b^2(t)(u_t,u) \le 0. \quad (3.4.8)$$

Due to the Cauchy-Schwarz inequality we have:

$$\delta\varepsilon b^2(t)|(u_t,u)| \le \frac{\delta\varepsilon b_0^2}{2} - \frac{\delta\varepsilon b_0^2}{2}\|u_t\|^2 + \frac{\delta\varepsilon b_0^2}{2} - \frac{\delta\varepsilon b_0^2}{2}\|u\|^2,$$
$$\varepsilon b'(t)|(u_t,u)| \le \frac{\varepsilon\alpha b(t)}{2}\|u(t)\|^2 + \frac{\varepsilon\alpha b(t)}{2}\|u(t)\|^2\|u_t\|^2$$

By using the last two inequalities we obtain from the inequality (3.4.8) the following differential inequality:

$$\frac{d}{dt}E(t) + \delta b(t)E(t) + b(t)\left[1 - \varepsilon - \frac{\delta}{2} - \frac{\varepsilon\alpha}{2} - \frac{\delta\varepsilon b_0^2}{2}\right]\|u_t(t)\|^2 + b(t)\left[\varepsilon - \frac{\delta\varepsilon b_0^2}{2} - \frac{\delta\varepsilon b_0^2}{2} - \frac{\varepsilon\alpha}{2}\right]\|u(t)\|^2 \leq 0.$$

The last inequality implies that

$$\frac{d}{dt}E(t) + \delta b(t)E(t) \leq 0 \tag{3.4.9}$$

if $\varepsilon = \frac{1}{3}$ and δ is small enough. Integrating (3.4.9) we get

$$E(t) \leq E(0)e^{-\delta\int_0^t b(\tau)d\tau}. \tag{3.4.10}$$

Due to the condition (3.4.5) it follows from (3.4.10) that $E(t)$ tends to zero as $t \to \infty$. On the other hand we have

$$E(t) \geq \frac{1}{6}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + (F(u(t)), 1) + \frac{b^2(t)}{16}\|u(t)\|^2.$$

Thus

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \to 0 \quad \text{as} \quad t \to \infty.$$

□

Arguing as in the proof of Theorem 3.4.1 and using the Lemma we can show that solutions of the initial boundary value problem for second order nonlinear non-autonomous wave equation with time dependent damping term under the Dirichlet boundary condition tends to zero as $t \to \infty$ under some restrictions on the damping coefficient and the source term.

In fact the following Proposition holds true:

Proposition 3.4.2. *Suppose that all conditions of the Theorem 3.4.1 are satisfied,* $h \in L^2(\mathbb{R}^+; L^2(\Omega)$ *and suppose that the damping term and the source term satisfy the follow-*

ing conditions

$$b(t) > 0, \ \ \forall t \geq 0, \ \ \|h(t)\| \, (b(t))^{-1} \to 0, \quad \textit{as} \ \ t \to \infty.$$

Then all solutions of the problem

$$u_{tt} + b(t)u_t - \Delta u + f(u) = h(x,t), \quad x \in \Omega, \quad t > 0, \tag{3.4.11}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{3.4.12}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad x \in \Omega, \tag{3.4.13}$$

all solutions of the problem (3.4.11)-(3.4.13) *tend to zero as* $t \to \infty$, *i.e.*

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \to 0 \ \ \textit{as} \ \ t \to \infty.$$

4. CONCLUSION

This thesis is devoted to the study of initial boundary value problems for second order nonlinear parabolic and hyperbolic equations.
The main results of the thesis are given in Chapter 2 and Chapter 3.
The second chapter of the thesis is devoted to the study of initial boundary value problems for second order nonlinear parabolic equations under various boundary conditions.
First we considered the initial boundary value problem under the Robin boundary conditions for second order nonlinear parabolic equation:

$$\begin{cases} u_t - \Delta u = f(u) + h(x,t), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} + \gamma u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, γ is a given number, h is a given source term and $f(\cdot)$ is a given nonlinear term.
We proved that if the nonlinear term satisfies the conditions

$$f(u)u \geq 2(1+\alpha)F(u), \quad F(u) = \int_0^u f(s)ds, \text{ for all } u \in \mathbb{R} \tag{4.0.2}$$

with some positive $\alpha, \;\; h \in L^2(\mathbb{R}^+; L^2(\Omega)) \cap L^\infty(\mathbb{R}^+; L^2(\Omega))$, then solutions of the problem (4.0.1) corresponding to a wide class of initial data blow-up in a finite time. This result can be considered as development of the result obtained in [62], where the authors, using the energy method, established blow-up of solutions of the problem (4.0.1) with $h \equiv 0$, essentially using positiveness of the coefficient γ in the boundary condition and the initial function u_0. We found sufficient conditions for the finite-time blow-up of solutions of the problem (4.0.1) regardless of the sign of γ and the sign of the initial function u_0.
Next, by using the energy method, we obtained result on blow up of solutions of the initial boundary value problem (4.0.1) when the source term depends only on the space variables, i.e. $h = h(x)$ and in addition to the condition (4.0.2) we assume that the following condition

on the nonlinear term holds true

$$F(u) \geq D_0|u|^p - D_1 \quad \forall u \in \mathbb{R}$$

for some $p > 2, D_0 > 0$, $D_1 \geq 0$.
Employing Kaplan's eigenfunction method we proved also the following proposition.
Proposition 4.0.3. *Suppose that $u_0(x) \geq 0$, $\forall x \in \Omega$, $\gamma > 0$, the source term $h = h(x)$, the nonlinear term $f(\cdot)$ is a convex function that satisfies also the conditions;*

$$f(u) - \lambda_1 u - h_0 > 0, \quad \forall u \geq \alpha_0 > 0,$$

with

$$\int_{\alpha_0}^{\infty} \frac{d\eta}{f(\eta) - \lambda_1 \eta - h_0} < \infty, \tag{4.0.3}$$

where $h_0 = \int_\Omega h(x)\psi_1(x)dx$, $\alpha_0 = \int_\Omega u_0(x)\psi_1(x)$, $\lambda_1 > 0$ is the eigenvalue corresponding to the normalized principal eigenfunction $\psi_1(x)$ of the Laplace operator $-\Delta$ under the Robin boundary condition Then the solution of the problem (4.0.1) *follows up in a finite time.*

We obtained in this chapter also result on blow up of solutions with positive initial energy of initial boundary value problems for nonlinear parabolic equations under Robin boundary conditions.
We proved that if $h \equiv 0$ and the nonlinear term satisfies the condition

$$f(s)s - 2(1+\alpha)F(s) \geq -D_0, \quad \forall s \in \mathbb{R}, \tag{4.0.4}$$

where $\alpha > 0$, $\gamma \geq 0$, and $D_0 \geq 0$ are given numbers, then there exist a wide class of initial data with arbitrary positive initial energy for which solutions of the problem (4.0.1) blow up in a finite time.
As far as we know it is the first result on blow up of solutions with arbitrary positive initial energy of nonlinear parabolic equations under the Robin condition. Blow up of solutions with arbitrary positive initial energy of the initial boundary value problem for parabolic equation

with cubic nonlinearity of the form

$$u_t - \Delta u = u^3,$$

is also obtained.

Finally in Chapter 2, two results on decay of solutions of initial boundary value problems for non-autonomous nonlinear parabolic equations with time dependent coefficients are obtained. It is shown that if

$$c \in C^1(\mathbb{R}^+), \text{ and } c(t) \geq c'(t) \ \forall t > 0, \tag{4.0.5}$$

$$h \in L^2(0, T; L^2(\Omega)), \text{ for each } T > 0 \text{ and } \|h(t)\| \to 0, \text{ as } t \to \infty \tag{4.0.6}$$

and $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the condition

$$f(u)u - F(u) \geq 0, \quad F(u) := \int_0^u f(s)ds \geq 0, \quad \forall u \in \mathbb{R}, \tag{4.0.7}$$

then all solutions of the initial boundary value problem for the equation

$$u_t - \Delta u + c(t)f(u) = h(x,t), \quad x \in \Omega \quad t > 0, \tag{4.0.8}$$

under the homogeneous Dirichlet boundary condition tend to zero as $t \to \infty$. We considered here also the problem

$$\begin{cases} u_t - a(t)\Delta u + f(u) = h(x,t), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{4.0.9}$$

We proved that if the nonlinear term $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the condition (4.0.7) , $h \in L^2(0, T; L^2(\Omega)),$ for each $T > 0, \ a(t) > 0, \ \forall t \geq 0$ is a continuous function on $[0, \infty)$, such that $\int_0^t a(s)ds \to \infty \quad as \quad t \to \infty$, and also the following condition holds true $\lim_{t\to\infty} a^{-1}(t)\|h(t)\|^2 = 0$, then the solution of the initial boundary value problem

$$\lim_{t\to\infty} \|u(t)\| = 0.$$

Under an additional assumption that $f(\cdot)$ is a differentiable nondecreasing function, we proved that

$$\lim_{t\to\infty} \|\nabla u(t)\| = 0.$$

Chapter 3 is devoted to study of initial boundary value problems for second order nonlinear hyperbolic equations.

The first result in this chapter is the result on blow up of solutions of the problem

$$\begin{cases} u_{tt} + bu_t = \Delta u + f(u) + h(x,t), \quad x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} + \gamma u = 0, \qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1, \quad x \in \Omega, \end{cases} \tag{4.0.10}$$

where $b > 0, \gamma \in \mathbb{R}$ are given number, h is a given source term, u_0, u_1 are given initial functions, and $f(\cdot)$ is a nonlinear term.

Employing the Lemma 1.3.5 we proved that if

$$f(s)s - 2(2\alpha+1)F(s) \geq 0, \ \ \forall s \in \mathbb{R}, \tag{4.0.11}$$

with some $\alpha > 0$,

$$\gamma \geq 0, \ (u_0, u_1) > 0, \ h \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \tag{4.0.12}$$

and

$$\left[2(u_0,u_1) + b\|u_0\|^2\right]^2 > \frac{2d_0(b+2)}{2\alpha+1}\|u_0\|^2, \tag{4.0.13}$$

then the solution of the problem (4.0.10) blows up in a finite time.

Let us note that from this result it follows that there are solutions of the problem (4.0.10) with arbitrary positive energy that blow up in a finite time.

Under a different restrictions on data a result on blow up of solutions for the case when nonlinear term satisfies the condition (4.0.11) and $\gamma < 0$ is obtained by using the Levine's Lemma.

Finally in Chapter 3 we considered the problem:

$$\begin{cases} u_{tt} + b(t)u_t - \Delta u + f(u) = 0, \quad x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad x \in \Omega, \end{cases} \tag{4.0.14}$$

where $b(t)$ is a positive differentiable function defined on $[0, \infty)$ that satisfies the conditions

$$0 \leq b(t) \leq b_0, \ \ |b'(t)| \leq \alpha b(t), \ \ 0 < \alpha \leq 2, \ \ \forall t \geq 0, \tag{4.0.15}$$

$$\int_0^t b(s)ds \to \infty \quad \text{as} \quad t \to \infty \tag{4.0.16}$$

and the function $f(u)$ satisfies the condition (4.0.7).

We proved that under these restrictions all solutions of the problem (4.0.14) tend to zero as $t \to \infty$.

In the last Chapter 4 (Appendix) we gave the proofs of auxiliary propositions which we have used to get main results.

Finally we would like to note that throughout the thesis we deal with strong solutions of problems considered, i.e. solutions for which all terms involved in the corresponding equations belong to $L^2(0,T); L^2(\Omega))$. For local solution $T < \infty$, and for global solutions $T = \infty$. For results on existence and uniqueness of strong and classical solutions of initial boundary value problems for nonlinear parabolic and hyperbolic equations under the Dirichlet and Robin boundary conditions and even more general nonlinear boundary conditions, we refer to [17], Pao [29], [66] , [67] and references therein.

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APPENDIX A: AUXILARY INEQUALITIES

In this chapter we give proofs of auxiliary propositions we used in the proofs of main results in Chapter 2 and Chapter 3.

Lemma A.0.4. *If* $w \in H^2(\Omega) \cap H_0^1(\Omega)$*, then*

$$\|\nabla w\| \le \lambda_1^{-1/2} \|\Delta w\|. \tag{A.0.1}$$

Proof. Since $C_0^\infty(\Omega)$ is dense in $H^2(\Omega) \cap H_0^1(\Omega)$, it suffices to prove the inequality (A.0.1) for $w \in C_0^\infty(\Omega)$. In fact integrating the equality :

$$\nabla \cdot (w(x)\nabla w(x)) = |\nabla w(x)|^2 + w(x)\Delta w(x),$$

over the domain Ω we get

$$\|\nabla w\|^2 = -(w, \Delta w).$$

Thanks to the Cauchy-Schwarz inequality and the Poincare - Friedrich inequality (1.3.9) we deduce form the last equality that

$$\|\nabla w\|^2 = \|w\|\|\Delta w\| \le \lambda_1^{-1/2} \|\nabla w\|\|\Delta w\|.$$

Hence (A.0.1) follows. □

Lemma A.0.5. *Suppose that* $p, q \in C[0,\infty),\ p(t) > 0, q(t) \ge 0,\ \ \forall t \ge 0,$ *and*

$$\int_0^t p(s)ds \to \infty,\ \ q(t)\,(p(t))^{-1} \to 0 \ \ \textit{as} \ \ t \to \infty. \tag{A.0.2}$$

Then all nonnegative solutions of the differential inequality

$$z'(t) + p(t)z(t) \le q(t) \tag{A.0.3}$$

tend to zero as $t \to \infty$.

Proof. Multiplication of the inequality (A.0.3) by $e^{\int_0^t p(s)ds}$ gives

$$\frac{d}{dt}\left(e^{\int_0^t p(s)ds} z(t)\right) \le q(t)e^{\int_0^t p(s)ds}.$$

Integrating this inequality over the interval $(0, t)$:

$$e^{\int_0^t p(s)ds} z(t) \le z(0) + \int_0^t q(s)e^{\int_0^s p(\tau)d\tau} ds.$$

This inequality implies

$$z(t) \le z(0)e^{-\int_0^t p(s)ds} + e^{-\int_0^t p(s)ds} \int_0^t q(s)e^{\int_0^s p(\tau)d\tau} ds. \tag{A.0.4}$$

Thanks to the condition (A.0.2) the first term on the right hand side of (A.0.4) tends to zero as $t \to \infty$. Employing L'Hospital's rule and the condition (A.0.2) we obtain:

$$\lim_{t\to\infty} \int_0^t q(s)e^{\int_0^s p(\tau)d\tau} ds \left(e^{\int_0^t p(s)ds}\right)^{-1}$$
$$= \lim_{t\to\infty} q(t)e^{\int_0^s p(s)ds} \left(p(t)e^{\int_0^t p(s)ds}\right)^{-1} = \lim_{t\to\infty} q(t)\,(p(t))^{-1} = 0.$$

Hence the second term on the right hand side of (A.0.4) also tends to zero as $t \to \infty$. □

Lemma A.0.6. *(see [24]) Suppose that a function* $a(t)$ *is twice continuously differentiable on some interval* $[0, T)$,
a function $H(r)$ *is continuous on* $[a_0, \infty)$ *and the condition*

$$H(r) \ge 0, \ \ \forall r \ge a_0 \tag{A.0.5}$$

holds. Assume also that

$$a''(t) \geq H(a(t)),\ t \geq 0, \tag{A.0.6}$$

$$a(0) = a_0 > 0,\ a'(0) = a_1 > 0. \tag{A.0.7}$$

Then

$$(1)\ \ a(t)\ \ \textit{is continuous and}\ \ \ a'(t) > 0,\ \ \forall t \in [0, T)$$

$$(2) \qquad\qquad t \leq \int\limits_{a_0}^{a(t)} \left[a_1^2 + 2 \int\limits_{a_0}^{s} H(r)dr \right]^{-1/2} ds. \tag{A.0.8}$$

Proof. Let us prove (1). Assume that this property is not satisfied. Then there exists a $t_1 > 0$ such that $a'(t) > 0, t \in [0, t_1)$ and $a'(t_1) = 0$.

Integrating the inequality (A.0.6) over the interval $(0, t_1)$:

$$0 = a'(t_1) = a_0 + \int_0^{t_1} H(a(t))dt. \tag{A.0.9}$$

Since $a(t) > a_0$ and a is increasing on the interval $[0, t_1)$, $a(t) > a_0,\ \forall t \in [0, t_1$. Since H satisfies the condition (A.0.5) the right hand side of (A.0.9) is not lesser than $a_1 > 0$.

To prove (A.0.8) multiply both sides of (A.0.6) by $a'(t)$:

$$a''(t)a'(t) - h(a(t)a'(t) \geq 0.$$

This inequality, rewrite in the form:

$$\frac{d}{dt} \left[\frac{1}{2}[a'(t)]^2 - \int\limits_{a_0}^{a(t)} H(r)dr \right] \geq 1.$$

Integrating the last inequality

$$\frac{1}{2}[a'(t)]^2 \geq a_1^2 + 2\int\limits_{a_0}^{a(t)} H(r)dr.$$

From the last inequality:

$$a'(t) \geq \left(a_1^2 + 2\int\limits_{a_0}^{a(t)} H(r)dr\right)^{1/2}. \tag{A.0.10}$$

Writing (A.0.10) in the form

$$\frac{d}{dt}\int\limits_{a_0}^{a(t)}\left[a_1^2 + 2\int\limits_{a_0}^{s} H(r)dr\right]^{-1/2} ds \geq 1.$$

Finally integrating the last inequality and see that the inequality (A.0.8) holds true. □

Lemma A.0.7. *(see [25]) Let $\Psi(t)$ be a positive, twice differentiable function,which satisfies, for $t > t_0 \geq 0$, the inequality*

$$\Psi''(t)\Psi(t) - (1+\alpha)\left[\Psi'(t)\right]^2 \geq 0 \tag{A.0.11}$$

with some $\alpha > 0$.

If $\Psi(t_0) > 0$ and $\Psi'(t_0) > 0$, then there exists a time

$$T_0 \in (t_0, T_1), \ \ T_1 = \frac{\Psi(t_0)}{\alpha\Psi'(t_0)} + t_0$$

such that

$$\Psi(t) \to +\infty \ \ as \ \ t \to T_0^-. \tag{A.0.12}$$

Proof. Consider the function

$$\Phi(t) := \Psi^{-\alpha}(t).$$

It is clear that

$$\Phi'(t) = -\alpha\Psi^{(-\alpha-1)}(t)\Psi'(t) \tag{A.0.13}$$

and

$$\begin{aligned}\Phi''(t) &= \alpha(\alpha+1)\Psi^{(-\alpha-2)}(t)(\Psi'(t))^2 - \alpha\Psi^{(-\alpha-1)}\Psi''(t)\\ &= -\alpha\Phi^{(-\alpha-2)}(t)\left[\Phi''(t)\Phi(t) - (1+\alpha)(\Phi'(t))\right].\end{aligned} \tag{A.0.14}$$

Thus thanks to the condition (A.0.11)

$$\Phi^{''}(t) = -\alpha\left[\Psi^{''}(t)\Psi(t) - (1+\alpha)\left[\Psi^{'}(t)\right]^2\right] \le 0.$$

Therefore the function $\Phi(t)$ is a concave function.

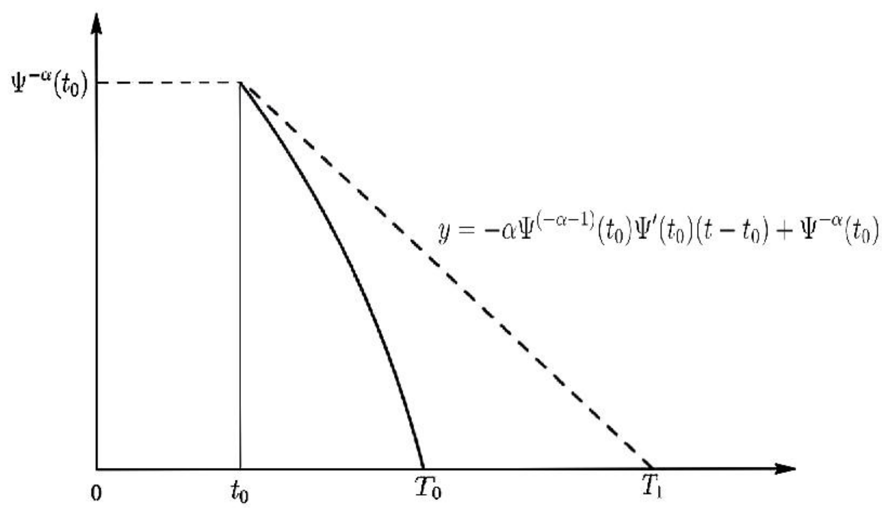

Figure A.1.

Since

$$\Phi(t_0) = \Psi^{-\alpha}(t_0) > 0 \tag{A.0.15}$$

and

$$\Phi'(t_0) = -\alpha\Psi^{(-\alpha-1)}(t_0)\Psi'(t_0) < 0$$

the function $\Phi(t)$ must tend to zero as $t \to T_0^-$ for some $T_0 > t_0$ (see Fig. 1).
Hence $\Psi(t)$ must tend to ∞ as $t \to T_0^-$. □

Lemma A.0.8. *(see e.g. [51])Let twice continuously differentiable function $\Psi(t)$ satisfies for each $t \geq 0$ the inequality*

$$\Psi''(t)\Psi(t) - (1+\alpha)\left[\Psi(t)\right]^2 \geq 2C_1\Psi(t)\Psi'(t) - C_2\Psi^2(t) \tag{A.0.16}$$

and

$$\Psi(0) > 0, \Psi'(0) > -\gamma_2\alpha^{-1}\Psi(0), \tag{A.0.17}$$

where $\alpha > 0, C_1, C_2 \geq 0, C_1 + C_2 > 0$ and $\gamma_2 = -C_1 - \sqrt{C_1^2 + \alpha C_2}$. Then there exists

$$t_1 \leq T_1 = \left(2\sqrt{C_1^2 + \alpha C_2}\right)^{-1} \ln \frac{\gamma_1\Psi(0) + \alpha\Psi(0)}{\gamma_2\Psi(0) + \alpha\Psi'(0)},$$

with $\gamma_1 = -C_1 + \sqrt{C_1^2 + \alpha C_2}$ such that

$$\Psi(t) \to \infty \;\; as \;\; t \to t_1^-.$$

If $\Psi(0) > 0, \Psi'(0) > 0$ and $C_1 = C_2 = 0$, then there exists

$$t_2 \leq T_2 = \frac{\Psi(0)}{\alpha\Psi'(0)}$$

such that

$$\Psi(t) \to \infty \ \ as \ \ t \to t_2^-.$$

Proof. Make the notation

$$\Phi(t) = \Psi^{-\alpha}(t).$$

Then

$$\Phi'(t) = -\frac{\alpha\Psi'(t)}{\Psi^{1+\alpha}(t)}, \ \Phi''(t) = -\alpha\frac{\Psi''(t)\Psi(t) - (1+\alpha)\left[\Psi'(t)\right]^2}{\Psi^{2+\alpha}(t)}.$$

Thus it follows from (A.0.16) that the function $\Phi(t)$ satisfies the differential inequality

$$\Phi''(t) + 2C_1\Phi'(t) - \alpha C_2\Phi(t) \equiv f(t) \leq 0. \tag{A.0.18}$$

Integrating this equation for $C_1 + C_2 > 0$:

$$\Phi(t) = \beta_1 e^{\gamma_1 t} + \beta_2 e^{\gamma_2 t} + (\gamma_1 - \gamma_2)^{-1}\int_0^t \left[e^{\gamma_1(t-\tau)} - e^{\gamma_2(t-\tau)}\right] d\tau \leq \beta_1 e^{\gamma_1 t} + \beta_2 e^{\gamma_2 t}.$$

The numbers β_1 and β_2 are solutions of the system

$$\begin{cases} \beta_1 + \beta_2 = \Phi(0), \\ \beta_1\gamma_1 + \beta_2\gamma_2 = \Phi'(0), \end{cases}$$

i.e.

$$\beta_1 = (\gamma_1 - \gamma_2)^{-1}\left[\Phi'(0) - \gamma_2\Phi(0)\right] = -(\gamma_1 - \gamma_2)^{-1}\left[\alpha\Phi'(0) + \gamma_2\Psi(0)\right]\Psi^{-1-\alpha}(0) > 0,$$
$$\beta_2 = (\gamma_1 - \gamma_2)^{-1}\left[\alpha\Phi'(0) - \gamma_1\Phi(0)\right]\Psi^{-1-\alpha}(0) > 0.$$

Thus from the assumptions of the Lemma A.0.8 it follows that $\Phi(t)$ tends to zero as t tends to some $t_1 \leq t_2 = (\gamma_1 - \gamma_2)^{-1} \ln(-\beta_2/\beta_1)$. Hence

$$\Psi(t) \to \infty \text{ as } t \to t_1.$$

□

Lemma A.0.9. *(see e.g. [5]) Suppose that a non-negative function $\Psi(t) \in \mathbb{C}^2[0,T]$ satisfies the inequality,*

$$\Psi''(t)\Psi(t) - (1+\alpha)(\Psi'(t))^2 + \gamma\Psi'(t)\Psi(t) + \beta\Psi(t) \geq 0, \quad \alpha > 0, \beta \geq 0, \gamma \geq 0, \quad \text{(A.0.19)}$$

and $\Psi(0) > 0$. Suppose also that the conditions

$$\Psi'(0) > \frac{\gamma}{\alpha - 1}\Psi(0), \quad \text{(A.0.20)}$$

$$A_0 := \left(\Psi'(0) - \frac{\gamma}{\alpha}\Psi(0)\right)^2 - \frac{2\beta}{2\alpha}\Psi(0) \quad \text{(A.0.21)}$$

are satisfied. Then the time $T > 0$ cannot be arbitrarily large:
The inequality $T \leq T_\infty \leq \Psi^{-\alpha}(0)A^{-1}$ holds, where A is given by the equality

$$A^2 = (\alpha)^2\Psi^{-2(1+\alpha)}(0)A_0 > 0.$$

Moreover in this case,

$$\limsup_{t \to T^-} \Psi(t) = +\infty.$$

Proof. Dividing both sides of the inequality (A.0.19) by $\Psi^{2+\alpha}$ and using the equality

$$\frac{\Psi''(t)\Psi(t) - (1+\alpha)(\Psi'(t))^2}{\Psi^{2+\alpha}(t)} = \left(\frac{\Psi'(t)}{\Psi^{1+\alpha}(t)}\right)',$$

we obtain

$$\left(\frac{\Psi'(t)}{\Psi^{1+\alpha}(t)}\right)' + \gamma\frac{\Psi'(t)}{\Psi^{1+\alpha}(t)} + \beta\frac{1}{\Psi^{1+\alpha}(t)} \geq 0.$$

The last inequality is equivalent to the inequality:

$$-\frac{1}{\alpha}(\Psi^{-\alpha}(t))'' - \frac{\gamma}{\alpha}(\Psi^{-\alpha})' + \beta\Psi^{-1-\alpha}(t) \geq 0. \tag{A.0.22}$$

It follows from (A.0.22) that the function $Z(t) = \Psi^{-\alpha}(t)$ satisfies the inequality

$$Z''(t) + \gamma Z'(t) - \beta(\alpha)Z^{\alpha_1}(t) \leq 0, \tag{A.0.23}$$

where $\alpha_1 = \frac{1+\alpha}{\alpha}$. Introducing now a new function $Y(t) = e^{\gamma t}Z(t)$, and by using (A.0.23) we can write

$$Y''(t) - \gamma Y'(t) - \beta\alpha e^{-\delta t}Y^{\alpha_1}(t) \leq 0, \quad \delta = \frac{\gamma}{\alpha}. \tag{A.0.24}$$

We now note that

$$Y'(t) = \left(\Psi^{-\alpha}(t)e^{\gamma t}\right)' = \alpha\Psi^{-1-\alpha}(t)e^{\gamma t}\left(-\Psi'(t) + \frac{\gamma}{\alpha}\Psi(t)\right). \tag{A.0.25}$$

Thanks to the condition (A.0.20), $\Psi'(0) > \frac{\gamma}{\alpha}\Psi(0)$ there exits $t_0 > 0$ such that the inequality

$$\Psi'(t) > \frac{\gamma}{\alpha}\Psi(t)$$

is satisfied for all $t \in [0, t_0)$. Hence, taking relations (A.0.25) into account, $Y'(t) < 0$ for $t \in [0, t_0)$. Because $-\gamma Y'(t) \geq 0$ for $t \in [0, t_0)$, the inequality

$$Y''(t) - \beta\alpha e^{-\delta t}Y^{\alpha_1}(t) \leq 0, \quad \delta = \frac{\gamma}{\alpha} \tag{A.0.26}$$

follows from inequality (A.0.24) for $t \in [0, t_0)$. Multiply both sides of (A.0.26) by $Y'(t)$ and obtain the inequality

$$Y'(t)Y''(t) - \beta\alpha e^{-\delta t}Y^{\alpha_1}(t)Y'(t) \geq 0 \tag{A.0.27}$$

for $t \in [0, t_0)$. Note that

$$e^{-\delta t}Y^{\alpha_1}(t)Y'(t) = \frac{d}{dt}(e^{-\delta t}Y^{1+\alpha_1}(t)) + \delta e^{-\delta t}Y^{1+\alpha_1}(t) - \alpha_1 e^{-\delta t}Y^{\alpha_1}(t)Y'(t).$$

The last equality writing in the form:

$$e^{-\delta t}Y^{\alpha_1}(t)Y'(t) = \frac{1}{1+\alpha_1}\frac{d}{dt}\left(e^{-\delta t}Y^{1+\alpha_1}(t)\right) + \frac{1}{1+\alpha_1}\delta e^{-\delta t}Y^{1+\alpha_1}(t).$$

Utilizing this relation in (A.0.27) and obtain the inequality

$$Y'(t)Y'(t) - \frac{\beta\alpha}{1+\alpha_1}\frac{d}{dt}(e^{-\delta t}Y^{1+\alpha_1}(t)) - \frac{\beta(\alpha-1)}{1+\alpha_1}\delta e^{-\delta t}Y^{1+\alpha_1}(t) \geq 0$$

for $t \in [0, t_0)$. The last inequality implies:

$$\frac{1}{2}\frac{d}{dt}\left(Y'(t)\right)^2 - \frac{\beta\alpha}{1+\alpha_1}\frac{d}{dt}\left(e^{-\delta t}Y^{1+\alpha_1}(t)\right) \geq 0,\ \forall t \in (0, t_0).$$

Integrating this inequality:

$$(Y'(t))^2 \geq A^2 + \frac{2\beta\alpha^2}{2\alpha}e^{-\delta t}Y^{1+\alpha_1}(t) \geq A^2, \tag{A.0.28}$$

where

$$A^2 = \left(Y'(0)\right)^2 - \frac{2\beta\alpha^2}{2\alpha}e^{-\delta t}Y^{1+\alpha_1}(0) = \alpha^2\Psi^{-2-2\alpha}(0)A_0 > 0. \tag{A.0.29}$$

Using inequalities (A.0.28) and (A.0.29), concludes that $Y'(t) \leq -A < 0,\ \ \forall t \in [0, t_0)$. Hence $Y'(t_0) < 0$. Clearly $Y'(t) < 0, \forall t \in [0, T]$. Consequently,

$$\Psi^{-\alpha}(t) \leq e^{-\gamma t}(\Psi^{-\alpha}(0) - At),\ \ \forall t \in [0, T].$$

Therefore $\Psi(t) \geq \frac{e^{\frac{\gamma t}{\alpha}}}{(\Psi^{-\alpha}(0)-At)^{-\frac{1}{\alpha}}}$. So the function $\Psi(t)$ must tend to $+\infty$ as $t \to t_0 \leq \Psi^{-\alpha}(0)A^{-1}$. □