GLOBAL BI-HAMILTONIAN STRUCTURE OF THREE DIMENSIONAL DYNAMICAL SYSTEMS

by

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ABSTRACT

GLOBAL BI-HAMILTONIAN STRUCTURE OF THREE DIMENSIONAL DYNAMICAL SYSTEMS

In this thesis the conditions for the local existence of the bi-Hamiltonian structure corresponding to a non-vanishing vector field on three dimensional manifolds are obtained by using the given vector field alone. It is shown that any non-vanishing vector field on a three dimensional manifold is locally bi-Hamiltonian. Then by working on the equations related with the local existence, the obstructions to extension of local bi-Hamiltonian structure are obtained. After that, these obstructions are expressed in terms of the characteristic classes related with the normal bundle of the given vector field. It is shown that any non-vanishing vector field on a three dimensional manifold is globally bi-Hamiltonian if and only if the Chern class of the normal bundle of the vector field and Bott class of the transversally holomorphic complex codimension one foliation defined by the vector field vanishes.

ÖZET

ÜÇ BOYUTLU DİNAMİK SİSTEMLERİN GLOBAL Bİ-HAMİLTONYEN YAPISI

Bu tezde üç boyutlu manifoldlar üzerinde hiç sıfır olmayan bir vektör alanına karşılık gelen bi-Hamiltonyen yapının lokal varlığı için koşullar sadece verilen vektör alanı kullanılarak elde edilmiştir. Üç boyutlu bir manifold üzerinde hiç sıfır olmayan her vektör alanının lokal olarak bi-Hamiltonyen olduğu gösterilmiştir. Daha sonra, lokal varlıkla ilişkili denklemler üzerinde çalışarak, lokal bi-Hamiltonyen yapıların genişletilebilmesinin engelleri elde edilmiştir. Bundan sonra, bu engeller verilen vektör alanının normal demetiyle ilişkili karakteristik sınıflar cinsinden ifade edilmiştir. Üç boyutlu bir manifold üzerinde hiç sıfır olmayan bir vektör alanının global olarak bi-Hamiltonyen olması için gerek ve yeter şartın vektör alanının normal demetinin Chern sınıfının ve vektör alanı ile tanımlanan transversal holomorfik kompleks ek-boyutu bir yapraklanmanın Bott sınıfının sıfır olması olduğu gösterilmiştir.

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1. INTRODUCTION

Three dimensional dynamical systems are widely used in mathematics, as well as physics and engineering sciences for their wide range of applications [1]. Basically it consists of three ordinary differential equations in three variables describing the local rate of change of these variables. Then, one could analyze the qualitative behavior such as stability, or the existence and uniqueness of solutions around certain initial values or boundary conditions. Here, the dimension three has a special importance from the point of view of modeling real world applications. Among these systems an important subclass is autonomous dynamical systems, that is the dynamical systems that do not explicitly depend on the independent variable. Such systems are especially useful in formulating natural systems since the laws of nature are assumed to be independent of time.

In this thesis, autonomous three dimensional dynamical systems are investigated from a geometric point of view. Namely an autonomous three dimensional dynamical system is identified with a vector field or more specifically as the local section of the tangent bundle of a three dimensional manifold. Identifying a dynamical system with geometric objects on a manifold would enable to construct relations between differential geometric structures and the dynamical system defined on a three dimensional manifold. Then, by using the topological properties of three dimensional manifolds, it would be possible to investigate some of the global properties of three dimensional dynamical systems.

An important property is the local bi-Hamiltonian structure of an autonomous three dimensional dynamical system [2]. In the analysis of dynamical systems, the invariants of the system, i.e. the quantities or properties that are invariant under the flow describing the dynamical system, are important. Many properties of the system, such as Liouville integrability and stability are usually defined or related with these invariants. A Hamiltonian function is an invariant of the dynamical system on a Poisson manifold, that is, a manifold with a Poisson structure.

The main problem that is investigated in this thesis is the following: What are the conditions to have a global bi-Hamiltonian structure? In other words, what are the obstructions on a vector field $\vec{v}\,(x)$ on a three dimensional manifold M to be globally decomposed into two factors $\vec{J}_1\,(x)$ and $\vec{J}_2\,(x)$ such that

$$\begin{aligned} \vec{v}\,(x) &= \vec{J}_1\,(x) \times \vec{J}_2\,(x) \\ \left(\nabla \times \vec{J}_i\,(x)\right) \cdot \vec{J}_i\,(x) &= 0 \ , \ \ i = 1, 2 \\ \vec{J}_1 \cdot (\nabla \times \vec{J}_2) + \vec{J}_2 \cdot (\nabla \times \vec{J}_1) &= 0 \end{aligned} \tag{1.1}$$

In three dimensions there are many examples of dynamical systems having a global bi-Hamiltonian structure, which means that the solution of this problem is not empty. On the other hand, as shown in [3] there is at least one counterexample, which means that the solution of this problem is not trivial.

In the local existence part of the problem, since we work in three dimensions we search for Poisson structures defined by the given vector field and we show that it is always possible to find a compatible pair of Poisson structures with two Hamiltonian functions. In the global existence part, we prove that it is possible to find two global compatible Poisson structures if and only if the first Chern class of the normal bundle of the vector field and Bott class of the transversally holomorphic complex codimension one foliation defined by the vector field vanishes.

2. PRELIMINARIES

For the definitions in this chapter and for more information one may consult [2], [3], [4].

Definition 2.0.1. *A Poisson structure on a manifold M is the bilinear map $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying*

i. skew-symmetry condition: $\{f, g\} = -\{g, f\}$

ii. Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$

iii. Leibniz rule: $\{fg, h\} = f\{g, h\} + g\{f, h\}$

2.1. BI-HAMILTIONIAN STRUCTURE OF AN AUTONOMOUS DYNAMICAL SYSTEM

Denote the sections of $\Lambda^p(TM)$ (that is, the space of p-multivector fields) by $A^p(M)$. Corresponding to a Poisson structure $\{\cdot,\cdot\}$, one can define a *Poisson bivector field* $\Lambda \in A^2(M)$ by $\{f, g\} =< \Lambda, df \wedge dg >$. Here $< \cdot,\cdot >$ is the pairing between multivector fields and differential forms: $< X_1 \wedge X_2 \wedge \cdots \wedge X_p, \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p >= det(< X_i, \alpha_j >)$

As a natural extension of Lie derivative, *Schouten-Nijenhuis bracket*

$[\cdot,\cdot] : A^p(M) \times A^q(M) \to A^{p+q-1}(M)$ is defined by the following proposition (for a proof see [4]).

Proposition 2.1.1. *Let M be a smooth n-dimensional manifold and let $A(M)$ be the exterior algebra of multivector fields on M. There exists a unique $\mathbb{R}$-bilinear map $A(M) \times A(M) \to A(M)$, $(P, Q) \mapsto [P, Q]$, called the Schouten-Nijenhuis bracket, which satisfies the following properties:*

i. For $f, g \in A^0(M) = C^\infty(M)$, $[f, g] = 0$

ii. For $X \in A^1(M) = \mathfrak{X}(M)$ and a multivector $Q \in A(M)$, $[X, Q] = \mathcal{L}_X Q$

iii. For $P \in A^p(M), Q \in A^q(M)$, $[P, Q] = (-1)^{pq}[Q, P]$

iv. For $P \in A^p(M), Q \in A^q(M), R \in A(M)$,

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]$$

Remark 2.1.2. *Jacobi identity for* $\{\cdot, \cdot\}$ *is equivalent to* $[\Lambda, \Lambda] = 0$.

$(< [\Lambda, \Lambda], df \wedge dg \wedge dh >= 0)$

Associated with a Poisson bivector field Λ, there is a bundle map $\mathbf{J} : T^*M \to TM$. (denote the induced map on the sections by the same letter $\mathbf{J} : \Omega^1(M) \to \mathfrak{X}(M)$) defined by $< \mathbf{J}(\alpha), \beta >= \Lambda(\alpha \wedge \beta)$.

To each $H \in C^\infty(M)$ there is associated vector field $v_{_H} = \mathbf{J}(dH)$. H is called the *Hamiltonian function* and $v_{_H}$ is called the *Hamiltonian vector field*. The equation $\dot{x} = \mathbf{J}(dH)$ is called *Hamilton's equations of motion*.

Definition 2.1.3. *A Poisson pair on a manifold* M *is a pair* (Λ_1, Λ_2) *of Poisson bivector fields such that* $[\Lambda_1, \Lambda_2] = 0$. *A bi-Hamiltonain system is prescribed by two Hamiltonian functions* H_1, H_2 *satisfying:*

$$\begin{cases} v &= \mathbf{J}_1(dH_2) \\ v &= \mathbf{J}_2(dH_1) \end{cases}$$

where $\mathbf{J}_1, \mathbf{J}_2$ *are bundle maps determined by* Λ_1, Λ_2 *respectively. The vector field* v *is called a bi-Hamiltonian vector field.*

If M is a 3 manifold with volume form Ω, associated with a Poisson bivector field Λ, there is a one form $J = \Lambda \lrcorner \Omega$ called Poisson one-form. The equation $v = \mathbf{J}(dH)$ can be written as $\iota_v \Omega = J \wedge dH$ and the Jacobi identity is given by $J \wedge dJ = 0$. (Note that this equation is invariant under the multiplication of J by a differentiable function f. That is, $(fJ) \wedge d(fJ) = f^2 J \wedge dJ$.) Then, a bi-Hamiltonian system is two linearly independent Poisson one-forms such that $\iota_v \Omega = J_1 \wedge dH_2 = J_2 \wedge dH_1$. It can be shown that locally J_1 and J_2 can be chosen to be proportional to dH_1 and dH_2, respectively. (Hence $\iota_v \Omega = \lambda dH_1 \wedge dH_2$.)

2.2. IDENTIFICATION OF VECTOR FIELDS WITH 2-FORMS AND 1-FORMS IN $\mathbb{R}^3$

Let $M = \mathbb{R}^3$. By the definition of the bundle map **J**,

$$< \mathbf{J}(\alpha), \beta >= \Lambda(\alpha \wedge \beta) = -\Lambda(\beta \wedge \alpha) = - < \mathbf{J}(\beta), \alpha > \tag{2.1}$$

Using the metric $< \cdot, \cdot >_g$, if one identifies $T_pM \cong T_p^*M$, $v \mapsto < v, \cdot >_g$, one can consider **J** as a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{aligned} < \mathbf{J}u, v >_g = - < \mathbf{J}v, u >_g = - < v, \mathbf{J}^T u >_g, \quad \forall u, v \in \mathbb{R}^3 \\ \Rightarrow \mathbf{J}^T u = -\mathbf{J}u, \quad \forall u \in \mathbb{R}^3 \end{aligned} \tag{2.2}$$

This means, one can associate a matrix in $so(3)$ to $\mathbf{J} : \mathbb{R}^3 \to \mathbb{R}^3$.

Using the isomorphism $(so(3), [,]) \cong (\mathbb{R}^3, \times)$, $\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, one can assign a *Poisson vector field* $\vec{J} \in \mathbb{R}^3$ to **J** at each point. Multiplying a column vector $u \in \mathbb{R}^3$ by the matrix corresponding to **J** is equal to $\vec{J} \times u$ so there is the following relation:
$\mathbf{J}(dH) = \vec{J} \times \nabla H$. From this one can write the bi-Hamiltonian system in $\mathbb{R}^3$ as two different compatible Hamiltonian structures such that:

$$\begin{cases} v &= \vec{J_1} \times \nabla H_2 \\ v &= \vec{J_2} \times \nabla H_1 \end{cases}$$

Jacobi identity is $\vec{J_i} \cdot (\nabla \times \vec{J_i}) = 0$, $i = 1, 2$ and the compatibility condition is
$\vec{J_1} \cdot (\nabla \times \vec{J_2}) + \vec{J_2} \cdot (\nabla \times \vec{J_1}) = 0$. Locally $\vec{J_1}$ and $\vec{J_2}$ can be chosen to be proportional to ∇H_1 and ∇H_2, respectively. (Hence $v = \lambda \nabla H_1 \times \nabla H_2$.)

Local existence of bi-Hamiltonian structure in $\mathbb{R}^3$ has been shown in [5].

For the volume form Ω in $\mathbb{R}^3$ one has the following isomorphism:

$$\begin{array}{rcl} \mathfrak{X}(\mathbb{R}^3) & \to & \Omega^1(\mathbb{R}^3) \\ u & \mapsto & *\iota_u\Omega =< u, \cdot >_g := \alpha_u \end{array} \tag{2.3}$$

Note that (Using the identity: $\alpha_u \wedge *\beta = (-1)^{p-1} * \iota_u\beta$ for any p-form β)

$$*\iota_v(\iota_u\Omega) = -\alpha_v \wedge *\iota_u\Omega = -\alpha_v \wedge \alpha_u = \alpha_u \wedge \alpha_v \tag{2.4}$$

$$\left.\begin{array}{rcl} \iota_u(\iota_v\iota_u\Omega) & = & 0 \\ \iota_v(\iota_v\iota_u\Omega) & = & 0 \end{array}\right\} \Rightarrow \iota_v\iota_u\Omega = c\alpha_{u\times v} \tag{2.5}$$

$$\begin{array}{rcl} 0 & = & \iota_v(\iota_u\Omega \wedge \alpha_u \wedge \alpha_v) \\ & = & (\iota_v\iota_u\Omega) \wedge \alpha_u \wedge \alpha_v + \iota_u\Omega \wedge \iota_v(\alpha_u \wedge \alpha_v) \\ & = & (\iota_v\iota_u\Omega) \wedge \alpha_u \wedge \alpha_v + \iota_u\Omega \wedge \iota_v\alpha_u \wedge \alpha_v - \iota_u\Omega \wedge \alpha_u \wedge \iota_v\alpha_v \\ & = & \iota_v\iota_u\Omega \wedge *\iota_v\iota_u\Omega + *\alpha_u \wedge \iota_v\alpha_u \wedge \alpha_v - *\alpha_u \wedge \alpha_u \wedge \iota_v\alpha_v \\ & = & (|\iota_v\iota_u\Omega|_g^2 + < u, v >_g^2 - |u|_g^2|v|_g^2)\Omega \end{array} \tag{2.6}$$

By Lagrange's identity:

$$|\iota_v\iota_u\Omega|_g^2 = |u|_g^2|v|_g^2 - < u, v >_g^2 = |u \times v|_g^2 \tag{2.7}$$

So $\iota_v\iota_u\Omega = \alpha_{u\times v}$ (up to sign). Hence $*\alpha_{u\times v} = \alpha_u \wedge \alpha_v$ (up to sign)

3. LOCAL EXISTENCE THEOREM FOR BI-HAMILTONIAN STRUCTURES IN THREE DIMENSIONS

3.1. THE FLOW COORDINATES

Given a manifold M and a non-vanishing vector field v on M, we have

$$\mathbf{T}(p) = \frac{v(p)}{\|v(p)\|_{T_pM}} \tag{3.1}$$

to be the unit vector field of $v(p)$ at $p \in M$. The flow Φ_s defined by $\mathbf{T}$ is

$$\frac{d\Phi_s}{ds} = \mathbf{T} \circ \Phi_s \text{ and } \Phi_0 = id \tag{3.2}$$

Let (U_p, x_p) be an orthogonal coordinate system around $p \in M$. Let $B_\delta(p)$ be the geodesic ball around $p \in M$, that is for any point $z \in B_\delta(p)$ there is a unique geodesic $\gamma_p(z)(t)$ joining p and z. Let $\widetilde{U}_p = U_p \cap B_\delta(p)$ and $\left(\widetilde{U}_p, \widetilde{x}_p\right)$ be the geodesic coordinate chart around $p \in M$. Namely, if $z \in \widetilde{U}_p$ and $\gamma_p(z)(t)$ is the geodesic joining p and z such that $\gamma_p(z)(0) = z$, denote

$$\frac{d\gamma_p(z)}{dt}(0) = \dot{\gamma}_p(z) \tag{3.3}$$

Then

$$\widetilde{x}_p(z) = (x_p)_* \dot{\gamma}_p(z) = \widetilde{z} \tag{3.4}$$

Assuming the geodesic joining p with itself to be the constant map $\gamma_p(p) = p$, we have $\dot{\gamma}_p(p) = 0$ and hence $\widetilde{x}_p(p) = 0$. Let

$$(\widetilde{x}_p)_*(\mathbf{T}) = \widetilde{\mathbf{T}} \tag{3.5}$$

be the coordinates of the vector field $\mathbf{T}$.

Now we can define

$$V_p = \left\{ z \in \widetilde{U}_p \mid \left(\dot{\gamma}_p(z), \mathbf{T}(z)\right)_{T_zM} = 0 \right\} \tag{3.6}$$

or equivalently

$$V_p = \left\{ z \in \widetilde{U}_p \mid \widetilde{\mathbf{z}} \cdot \widetilde{\mathbf{T}}\left(\widetilde{z}\right) = 0 \right\} \tag{3.7}$$

where $\widetilde{\mathbf{z}}$ is the position vector of $z \in \widetilde{U}_p$ in geodesic coordinates around p. Since $\widetilde{\mathbf{p}} = 0$ then $p \in V_p$ and the set is not empty. The following lemma proves that the set V_p can be considered as a special set of initial points for each integral curve in $\widetilde{U}_p$.

Lemma 3.1.1. *Each integral curve of* $\mathbf{T}$ *in* $\widetilde{U}_p$ *intersects* V_p *at least at one point.*

Proof. Consider the part of the integral curve C_z passing through an arbitrary $z \notin \Phi_s(p)$ in $\widetilde{U}_p$, that is $C_z = \Phi_s(z) \cap \widetilde{U}_p$. Define the function

$$\begin{array}{cccc} f & : & C_z & \longrightarrow & \mathbb{R}^+ \cup \{0\} \\ & & \Phi_s(z) & \mapsto & \left|\dot{\gamma}_p\left(\Phi_s(z)\right)\right|^2 \end{array} \tag{3.8}$$

Since $\Phi_s(z)$ with the induced metric from $\mathbb{R}^3$ is closed, it is complete, and $f\left(\Phi_s(z)\right) \geq 0$ is bounded below. Then, if $\left\{ x_n = \widetilde{\Phi}_{s_n}\left(\widetilde{z}\right) \right\}$ is any sequence in C_z for which $\left|f\left(x_n\right)\right|$ is bounded and for which $\|df_{x_n}\| \to 0$, then we are going to prove that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \to z_p$. [6]. Let

$$\widetilde{\Phi}_s\left(\widetilde{z}\right) = \left(\widetilde{x}_p\right)_* \left(\dot{\gamma}_p\left(\Phi_s(z)\right)\right) \tag{3.9}$$

be the position vector of $\Phi_s(z)$ in geodesic coordinates. By assumption

$$\|df_{x_n}\| = \widetilde{\Phi}_{s_n}\left(\widetilde{z}\right) \cdot \frac{d\widetilde{\Phi}_{s_n}}{ds}\left(\widetilde{z}\right) = \widetilde{\Phi}_{s_n}\left(\widetilde{z}\right) \cdot \widetilde{\mathbf{T}}\left(\widetilde{\Phi}_{s_n}\left(\widetilde{z}\right)\right) \to 0 \tag{3.10}$$

which implies that

$$\widetilde{\Phi}_{s_n}\left(\widetilde{z}\right) \cdot \widetilde{\mathbf{T}}\left(\widetilde{\Phi}_{s_n}\left(\widetilde{z}\right)\right) = \left\|\widetilde{\Phi}_{s_n}\left(\widetilde{z}\right)\right\| \left\|\widetilde{\mathbf{T}}\left(\widetilde{\Phi}_{s_n}\left(\widetilde{z}\right)\right)\right\| \cos\theta_{\widetilde{z}}\left(s_n\right) \to 0 \tag{3.11}$$

where $\theta_{\widetilde{z}}\left(s_n\right)$ is the angle between $\dot{\gamma}_p\left(\Phi_s(z)\right)$ and $\widetilde{\mathbf{T}}\left(\widetilde{\Phi}_{s_n}\left(\widetilde{z}\right)\right)$. Since $\left\|\widetilde{\Phi}_{s_n}\left(\widetilde{z}\right)\right\| \neq 0$ and $\left\|\widetilde{\mathbf{T}}\left(\widetilde{\Phi}_{s_n}\left(\widetilde{z}\right)\right)\right\| = 1$ we get

$$\cos\theta_{\widetilde{z}}\left(s_n\right) \to 0 \tag{3.12}$$

Now define $Q_{\widetilde{z}} = \{s \in \mathbb{R} \mid \cos\theta_{\widetilde{z}}(s)\}$ which is a homeomorphism for $-\frac{\pi}{2} < \theta_{\widetilde{z}}(s) < \frac{\pi}{2}$, and the subsequence $\{s_{n_k}\} = \{s_n\} \cap Q_{\widetilde{z}}$. Now as $\cos\theta_{\widetilde{z}}\left(s_n\right) \to 0$ then $s_{n_k} \to \left(\cos\theta_{\widetilde{z}}\right)^{-1}(0) =$

$\widetilde{z}_p$. Then, by Theorem 9.1.9 in [6] $\widetilde{z}_p$ is a critical point and $f\left(\widetilde{z}_p\right)$ is the minimum value of f. In other words, defining $\widetilde{z}_p = \widetilde{\Phi}_{s_z}\left(\widetilde{z}\right)$ implies $\widetilde{x}_p^{-1}\left(\widetilde{z}_p\right) \in V_p$. Namely $\widetilde{z}_p$ is the point on $\widetilde{\Phi}_s\left(\widetilde{z}_p\right)$ which is closest to $\widetilde{p}$. □

Now, it is possible to define another neighborhood of $p \in M$,

$$\widehat{U}_p = \Phi_s\left(V_p\right) \cap \widetilde{U}_p \tag{3.13}$$

Using the chart $\left(\widetilde{U}_p, \widetilde{x}_p\right)$ we define new coordinate functions as follows: Let $z \in \widehat{U}_p$ and $\Phi_s\left(z\right)$ be the integral curve through z. Since $z \in \widehat{U}_p$ implies $z \in \Phi_s\left(V_p\right)$ and since Φ_s is invertible, there exists a $z_p \in V_p$ and $s_z \in \mathbb{R}$ such that

$$z = \Phi_{s_z}\left(z_p\right) \tag{3.14}$$

Using (3.14) we can define s_z to be the first coordinate of the point z.

For the remaining two coordinates, consider the vector $\widetilde{\mathbf{z}}_p$ which is the tangent vector of the geodesic joining p and z_p at z_p in geodesic coordinates around p. Now, choose an orthonormal frame at $\left\{\widetilde{\mathbf{T}}\left(\widetilde{z}_p\right), \widetilde{\mathbf{N}}\left(\widetilde{z}_p\right), \widetilde{\mathbf{B}}\left(\widetilde{z}_p\right)\right\}$ at $\widetilde{z}_p$. Since

$$\widetilde{\mathbf{z}}_p \cdot \widetilde{\mathbf{T}}\left(\widetilde{z}_p\right) = 0 \tag{3.15}$$

we have

$$\widetilde{\mathbf{z}}_p = n\left(z_p\right)\widetilde{\mathbf{N}}\left(\widetilde{z}_p\right) + b\left(z_p\right)\widetilde{\mathbf{B}}\left(\widetilde{z}_p\right) \tag{3.16}$$

Now, we could not use the components $n\left(z_p\right)$ and $b\left(z_p\right)$ of $\widetilde{\mathbf{z}}_p$ as coordinates of the point z since they are defined at point z_p rather than z. Therefore, one should first map the vector $\widetilde{\mathbf{z}}_p$ to the point z then by defining an appropriate frame at z, we can take the components of this new vector at z to be the coordinates of the point z. Obviously, one may want to use the pushforward map $\left(\Phi_{s_z}\right)_*$ for this purpose but since this map may not be orthogonal, it may preserve neither the orthogonality of $\widetilde{\mathbf{z}}_p$ and $\widetilde{\mathbf{T}}\left(\widetilde{z}_p\right)$ nor the orthogonality of $\widetilde{\mathbf{N}}\left(\widetilde{z}_p\right)$ and $\widetilde{\mathbf{B}}\left(\widetilde{z}_p\right)$. To solve the former problem we introduce the following map

$$\left(\Phi_{s_z}\right)_*^{\perp}\left(\widetilde{\mathbf{u}}\right) = \left(\Phi_{s_z}\right)_*\left(\widetilde{\mathbf{u}}\right) - \left(\widetilde{\mathbf{T}}\left(\widetilde{z}\right) \cdot \left(\Phi_{s_z}\right)_*\left(\widetilde{\mathbf{u}}\right)\right)\widetilde{\mathbf{T}}\left(\widetilde{z}\right) \tag{3.17}$$

Defining the subspaces $Q_{\widetilde{z}_p}$ and $Q_{\widetilde{z}}$ of $T_{z_p}M$ and T_zM respectively as the subspaces orthogonal to $\widetilde{\mathbf{T}}(\widetilde{z}_p)$ and $\widetilde{\mathbf{T}}(\widetilde{z})$ respectively, or simply

$$Q_{\widetilde{u}} = \left\{ \widetilde{\mathbf{u}} \in T_uM \mid \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{T}}(\widetilde{u}) = 0 \right\} \tag{3.18}$$

Proposition 3.1.2. $(\Phi_{s_z})_*^{\perp}$ *is an isomorphism between* $Q_{\widetilde{z}_p}$ *and* $Q_{\widetilde{z}}$.

Proof. To prove this we only need to check the ker $\left((\Phi_{s_z})_*^{\perp}\right)$. Indeed if we let $\widetilde{\mathbf{u}} \in$ ker $\left((\Phi_{s_z})_*^{\perp}\right)$

$$(\Phi_{s_z})_*^{\perp}(\widetilde{\mathbf{u}}) = (\Phi_{s_z})_*(\widetilde{\mathbf{u}}) - \left(\widetilde{\mathbf{T}}(\widetilde{z}) \cdot (\Phi_{s_z})_*(\widetilde{\mathbf{u}})\right) \widetilde{\mathbf{T}}(\widetilde{z}) = \mathbf{0} \tag{3.19}$$

from which we get

$$(\Phi_{s_z})_*(\widetilde{\mathbf{u}}) = \left(\widetilde{\mathbf{T}}(\widetilde{z}) \cdot (\Phi_{s_z})_*(\widetilde{\mathbf{u}})\right) \widetilde{\mathbf{T}}(\widetilde{z}) \tag{3.20}$$

By (3.2) we have

$$(\Phi_{s_z})_* \widetilde{\mathbf{T}}(\widetilde{z}_p) = \widetilde{\mathbf{T}}(\widetilde{z}) \tag{3.21}$$

Therefore applying $(\Phi_{s_z})_*^{-1}$ to both sides of (3.20) gives

$$\widetilde{\mathbf{u}} = \left(\widetilde{\mathbf{T}}(\widetilde{z}) \cdot (\Phi_{s_z})_*(\widetilde{\mathbf{u}})\right) \widetilde{\mathbf{T}}(\widetilde{z}_p) \tag{3.22}$$

which projects onto **0** in $Q_{\widetilde{z}_p}$. □

Hence we have the commutative diagram

$$\begin{array}{ccc} T_{z_p}M & \xrightarrow{(\Phi_{s_z})_*} & T_zM \\ pr \downarrow & & \downarrow pr \\ Q_{\widetilde{z}_p} & \xrightarrow{(\Phi_{s_z})_*^{\perp}} & Q_{\widetilde{z}} \end{array} \tag{3.23}$$

where pr denotes the projection onto corresponding subspaces.

Although $(\Phi_{s_z})_*^{\perp}$ solves the problem of preserving the orthogonality of $\widetilde{\mathbf{z}}_p$ and $\widetilde{\mathbf{T}}(\widetilde{z}_p)$ along the integral curve, the latter problem, namely the invariance of the orthogonality of $\widetilde{\mathbf{N}}(\widetilde{z}_p)$ and $\widetilde{\mathbf{B}}(\widetilde{z}_p)$ along the curve requires $(\Phi_{s_z})_*^{\perp}$ to be orthogonal while it may not be. Now, we will use the following lemma (page 26 of [7]) to solve this problem:

Lemma 3.1.3. *Let V and W be vector spaces, provided with inner products* $\boldsymbol{k}$ *and* $\boldsymbol{l}$*. Let $L : V \to W$ be a vector space isomorphism. Then there exists a unique positive definite self-adjoint linear mapping $H : W \to W$ such that $H \circ L$ preserves the inner products.*

This Lemma follows from the polar decomposition of the linear transformation L. Defining an ambient isotopy H_t for the self-adjoint transformation H, one could find an isotopy $L_t = H_t \circ L$ such that $L_0 = L$ and $L_1 = L^O$ which is an orthogonal transformation. Applying this construction to $Q_{\widetilde{z}_p} \overset{(\Phi_{s_z})_*^{\perp}}{\longrightarrow} Q_{\widetilde{z}}$ defines the orthogonal transformation $(\Phi_{s_z})_*^{\perp O}$. Now, we may define the two-frame field on M as

$$\begin{aligned} \widetilde{\mathbf{N}}(\widetilde{z}) &= (\Phi_{s_z})_*^{\perp O}\, \widetilde{\mathbf{N}}(\widetilde{z}_p) \\ \widetilde{\mathbf{B}}(\widetilde{z}) &= (\Phi_{s_z})_*^{\perp O}\, \widetilde{\mathbf{B}}(\widetilde{z}_p) \end{aligned} \tag{3.24}$$

Then, defining $\widetilde{\mathbf{z}} \in Q_{\widetilde{z}_p}$ to be

$$\widetilde{\mathbf{z}} = (\Phi_{s_z})_*^{\perp O} \widetilde{\mathbf{z}}_p \tag{3.25}$$

and applying $(\Phi_{s_z})_*^{\perp O}$ to (3.16)

$$\widetilde{\mathbf{z}} = n(z_p)\, \widetilde{\mathbf{N}}(\widetilde{z}) + b(z_p)\, \widetilde{\mathbf{B}}(\widetilde{z}) \tag{3.26}$$

and we define the components of $\widetilde{\mathbf{z}}$ to be the second and third coordinates of the point $z \in \widehat{U}_p$, and we get the coordinates $(s_z, n(z_p), b(z_p))$ defined above.

Note that the choice of the frame at z_p is arbitrary. One may define the following specific frame that will help us later on to define the two Hamiltonian functions defined by the vector field. Let S_ε be the sphere of radius ε centered at the point with position vector $\widetilde{\mathbf{z}}_p + \varepsilon \widetilde{\mathbf{T}}(\widetilde{z}_p)$ then the unit normal of S_ε at $\widetilde{z}_p$ is $\widetilde{\mathbf{T}}(\widetilde{z}_p)$. Choose ε sufficiently small such that $y_p^{-1}(S_\varepsilon) \subset B_r(p)$. We have $\widetilde{\mathbf{z}}_p \in T_{\widetilde{z}_p} S_\varepsilon$. Let $\widetilde{\sigma}_n(\widetilde{z}_p)$ be the great circle passing through $\widetilde{z}_p$ in the direction of $\widetilde{z}_p$. Then rotate this great circle by $\pi/2$ around $\widetilde{z}_p$ to obtain another great circle, in other words, $\left\{\widetilde{\mathbf{T}}(\widetilde{z}_p), \widetilde{\mathbf{N}}(\widetilde{z}_p), \widetilde{\mathbf{B}}(\widetilde{z}_p)\right\}$ is the Darboux frame of $\widetilde{\sigma}_n(\widetilde{z}_p)$ on S_ε where

$$\widetilde{\mathbf{N}}(\widetilde{z}_p) = \frac{\widetilde{\mathbf{z}}_p}{|\widetilde{\mathbf{z}}_p|} \tag{3.27}$$

and

$$\widetilde{\mathbf{B}}(\widetilde{z}_p) = \widetilde{\mathbf{T}}(\widetilde{z}_p) \times \widetilde{\mathbf{N}}(\widetilde{z}_p) \tag{3.28}$$

The coordinates of a point z in the coordinate chart $\left(\widehat{U}_p, y_p\right)$ of a point $p \in M$, is given by

$$y_p(z) = \left(y_p^1(z), y_p^2(z), y_p^3(z)\right) = \left(s_z, n(z_p), b(z_p)\right) \tag{3.29}$$

where s_z is the arclength of the integral curve segment joining z and z_p given in (3.14). The point z_p is on the integral curve passing through z which is closest to p, and $(n(z_p), b(z_p))$ are given by (3.26). Now, consider another coordinate chart $\left(\widehat{U}_q, y_q\right)$ of a point $q \in M$. If $z \in \widehat{U}_p \cap \widehat{U}_q$, then we have a unique integral curve joining points z, z_p and z_q defined above.

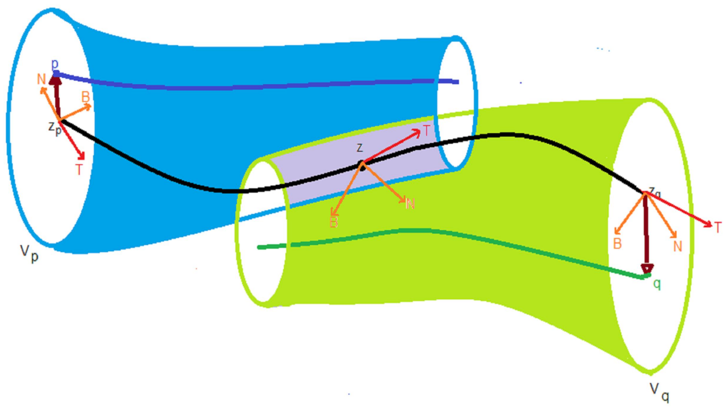

Figure 3.1. Coordinate transformation

Now we have the coordinates $y_p(z) = (s_z^p, n(z_p), b(z_p))$ and $y_q(z) = (s_z^q, n(z_q), b(z_q))$. To find the coordinate transformation first we start with the fact that

$$s_z^p - s_z^q = s_{z_p}^{z_q} \tag{3.30}$$

where $s_{z_p}^{z_q}$ is the arclength of the integral curve between z_p and z_q. In order to find the coordinate transformations first note that taking the derivative with respect to s_z^p (or equivalently s_z^q) is nothing but the derivative along the integral curve passing through z. Now, since the points z_p and z_q are the same for all points on the integral curve segment passing through z in $\widehat{U}_p \cap \widehat{U}_q$, therefore the distance between z_p and z_q is constant for all points on the segment of the integral and hence its derivative along the the integral curve vanishes.

$$\frac{\partial s_{z_p}^{z_q}}{\partial s_z^p} = 0 \tag{3.31}$$

Differentiating (3.30) with respect to s_z^p and using (3.31) lead to

$$\frac{\partial s_z^q}{\partial s_z^p} = 1 \tag{3.32}$$

Similarly, since the points p and z_p, and q and z_q are the same for all points on the integral curve segment passing through z in $\widehat{U}_p \cap \widehat{U}_q$, the tangent vectors of the geodesics joining these pairs of points remain invariant as we move along the integral curve segment in $\widehat{U}_p \cap \widehat{U}_q$. Since the choice of the frame depends only on the point z_p or z_q, the components $(n(z_p), b(z_p))$ and $(n(z_q), b(z_q))$ are constant for all points on the integral curve segment passing through z in $\widehat{U}_p \cap \widehat{U}_q$. Therefore their derivatives along the integral curve also vanish i.e.

$$\frac{\partial n(z_q)}{\partial s_z^p} = \frac{\partial b(z_q)}{\partial s_z^p} = 0 \tag{3.33}$$

Next we need to answer the following question: What happens to s_z^q when we change $(n(z_p), b(z_p))$? Note that, given a point $z \in \widehat{U}_p \cap \widehat{U}_q$ since the points z_p and z_q are determined by the points q and p which are the same for all points in the intersection of coordinate neighborhoods, the only way of varying $(n(z_p), b(z_p))$ is to change the frame at z_p. However, changing the frame at z_p does not change s_z^p and s_p^q, and by (3.30) does not change s_z^q. Therefore we have

$$\frac{\partial s_z^q}{\partial n(z_p)} = \frac{\partial s_z^q}{\partial b(z_p)} = 0 \tag{3.34}$$

Finally, for the relation between $(n(z_p), b(z_p))$ and $(n(z_q), b(z_q))$ we will use the fact that they are components of $(\Phi_{s_z^p})_*^{\perp O}\widetilde{\mathbf{z}}_p$ and $(\Phi_{s_z^q})_*^{\perp O}\widetilde{\mathbf{z}}_q$ where $\widetilde{\mathbf{z}}_p$ and $\widetilde{\mathbf{z}}_q$ are the tangent vectors of the geodesics joining pairs of points (p, z_p) and (q, z_q). Since $\widetilde{\mathbf{z}}_p$ and $\widetilde{\mathbf{z}}_q$ are unit vectors by definition and $(\Phi_{s_z^p})_*^{\perp O}$ and $(\Phi_{s_z^q})_*^{\perp O}$ are orthogonal, $(\Phi_{s_z^p})_*^{\perp O}\widetilde{\mathbf{z}}_p$ and $(\Phi_{s_z^q})_*^{\perp O}\widetilde{\mathbf{z}}_q$ are unit vectors perpendicular to $\mathbf{T}(z)$ at z. Therefore, we have

$$(\Phi_{s_z^q})_*^{\perp O}\widetilde{\mathbf{z}}_q = A_{pq}(z)\,(\Phi_{s_z^p})_*^{\perp O}\widetilde{\mathbf{z}}_p \tag{3.35}$$

where $A_{pq}(z) \in SO(2)$ defined by the rotations around $\mathbf{T}(z)$. Therefore we have the Jacobian matrix of coordinate transformations

$$\frac{\partial y_q^i(z)}{\partial y_p^j(z)} = \begin{pmatrix} 1 & 0 \\ 0 & A_{pq}(z) \end{pmatrix} \tag{3.36}$$

This coordinate transformation suggest that we have the local trivialization

$$\begin{array}{cccccc} (z) & \widehat{U}_p \subset M & \xrightarrow{y_p} & \mathbb{R}\times\mathbb{R}^2 & (y^1, y^2, y^3) \\ \downarrow & pr\downarrow & & \downarrow & \downarrow \\ (s) & \widehat{U}_p \cap C_p & \longrightarrow & \mathbb{R} & (y^1) \end{array} \tag{3.37}$$

In fact, the flow coordinate chart is nothing but the identification of the intersection of $\widehat{U}_p$ and a sufficiently small tubular neighborhood of C_p, which we identify with $\widehat{U}_p$, with the normal bundle of **T** in TM over M, which we denote by Q. The local frame field $\{\widehat{e}_1(z), \widehat{e}_2(z), \widehat{e}_3(z)\}$ is the adapted orthonormal frame.

Let (U_p, x_p) be an orthogonal chart for M and $\left(\widehat{U}_p, y_p\right)$ be the corresponding flow coordinate chart defined above, which are related by

$$y^i = y^i\left(x^1, x^2, x^3\right) \tag{3.38}$$

Let $\left\{\widehat{i}_k(x)\right\}$ be an orthonormal coordinate frame field in the former coordinate system and $\{\widehat{e}_k(y)\}$ be the orthonormal frame field $\{\mathbf{T}(y), \mathbf{N}(y), \mathbf{B}(y)\}$ defined by (3.1) and (3.24) in the latter coordinate system. We assume both to be right handed. Without restriction of generality we may assume that $\left\{\widehat{i}_k(x)\right\}$ is the standart Euclidean frame with the gradient operator

$$\nabla = \widehat{i}_k \partial_{x^k} \tag{3.39}$$

Then, one can define the new local non-coordinate basis of vector fields as the directional derivatives along the frame field $\{\widehat{e}_k(y)\}$

$$\widehat{e}_i = e_i^j(\mathbf{x})\,\partial_{x^j} = e_i^j(\mathbf{y})\,\partial_{y^j} \tag{3.40}$$

Taking the definition of the latter coordinate system we have

$$\widehat{e}_1 = \frac{\partial}{\partial y^1} \tag{3.41}$$

These basis vectors are involutive but not commutative. Now let us define the structure functions $\left(C_{ij}^{k}\left(\mathbf{y}\right)\right)$ via the relation

$$\left[\widehat{e}_i, \widehat{e}_j\right] = C_{ij}^{k}\left(\mathbf{y}\right)\widehat{e}_k \tag{3.42}$$

and subject to the integrability condition (Jacobi identity)

$$\bigodot_{i,j,k}\left[\left[\widehat{e}_i, \widehat{e}_j\right], \widehat{e}_k\right] = \left(-\widehat{e}_{[k}\left(C_{ij]}^{l}\right) + C_{m[k}^{l}C_{ij]}^{m}\right)\widehat{e}_l = 0 \tag{3.43}$$

(Here $\bigodot$ denotes a cyclic sum over the indices.) Finally, the curl and divergence of the basis vectors are given by

$$\begin{aligned} \nabla \times \widehat{e}_i\left(\mathbf{y}\right) &= -\tfrac{1}{2}\varepsilon^{jkl}C_{kl}^{i}\widehat{e}_j\left(\mathbf{y}\right) \\ \nabla \cdot \widehat{e}_i\left(\mathbf{y}\right) &= C_{ji}^{j}\left(\mathbf{y}\right) \end{aligned} \tag{3.44}$$

3.2. POISSON STRUCTURES IN THREE DIMENSIONS

Now, we will show that every three dimensional dynamical system is locally bi-Hamiltonian, namely there are two Poisson vector fields $\vec{J}_1$ and $\vec{J}_2$ and two Hamiltonian functions H_1 and H_2 such that

$$\dot{\vec{x}}\left(t\right) = \vec{v}\left(\vec{x}\left(t\right)\right) = \vec{J}_1 \times \nabla H_2 = \vec{J}_2 \times \nabla H_1 \tag{3.45}$$

And the Jacobi identity becomes

$$\left(\nabla \times \vec{J}\right) \cdot \vec{J} = 0 \tag{3.46}$$

Note that, Jacobi identity for Poisson vector fields has a dilatation symmetry in the sense that if $\vec{J}$ is a Poisson vector field, i.e. satisfies the Jacobi identity then $f\vec{J}$ is also a Poisson vector field. Since

$$\begin{aligned} \left(\nabla \times f\vec{J}\right) \cdot f\vec{J} &= \left(\nabla f \times \vec{J} + f\nabla \times \vec{J}\right) \cdot f\vec{J} \\ &= f^2\left(\nabla \times \vec{J}\right) \cdot \vec{J} = 0 \end{aligned} \tag{3.47}$$

Proposition 3.2.1. *A non-vanishing vector field v on M defines two Poisson structures on M.*

Proof. Adopting the coordinate system and frames defined before, and keeping (3.45) in mind, we will start with assuming that

$$\widehat{e}_1 \cdot \vec{J} = 0 \tag{3.48}$$

and hence we have the Poisson vector field

$$\vec{J} = \alpha \widehat{e}_2 + \beta \widehat{e}_3 \tag{3.49}$$

and its curl is subject to the Jacobi identity (3.46) which leads to

$$\vec{\nabla} \times \vec{J} = \nabla\alpha \times \widehat{e}_2 + \alpha\nabla \times \widehat{e}_2 + \nabla\beta \times \widehat{e}_3 + \beta\nabla \times \widehat{e}_3 \tag{3.50}$$

Then, applying the Jacobi identity (3.46) obtained by taking the dot product of (3.49) with (3.50), and using triple vector product identities we get

$$\beta\partial_{y^1}\alpha - \alpha\partial_{y^1}\beta - \alpha^2 C_{31}^2 - \alpha\beta\left(C_{31}^3 + C_{12}^2\right) - \beta^2 C_{12}^3 = 0 \tag{3.51}$$

If $\vec{J} = \vec{0}$ then $\|\vec{v}\| = 0$ and hence $\vec{v} = \vec{0}$ which contradicts with our assumption that the vector field is non-vanishing. Therefore we assume

$$\vec{J} \neq \vec{0} \tag{3.52}$$

which means that $\alpha \neq 0$ or $\beta \neq 0$. Now we assume $\alpha \neq 0$, while the case $\beta \neq 0$ is similar and amounts to rotation of the frame. Defining

$$\mu = \frac{\beta}{\alpha} \tag{3.53}$$

and dividing (3.51) by α^2 we get

$$\partial_{y^1}\mu = -C_{31}^2 - \mu\left(C_{31}^3 + C_{12}^2\right) - \mu^2 C_{12}^3 \tag{3.54}$$

whose characteristic curve is the integral curve of (3.45) in arclenght parametrization. Therefore on the solution curve we have

$$\frac{d\mu}{ds} = -C_{31}^2 - \mu\left(C_{31}^3 + C_{12}^2\right) - \mu^2 C_{12}^3 \tag{3.55}$$

in the arclength variable $s = y^1$. The Riccati equation (3.55) is equivalent to a linear second order equation and hence possesses two linearly independent solutions leading to two Poisson vector fields for dynamical system under consideration. Since the vector field $\mathbf{v}(\mathbf{x})$ is assumed to be non-vanishing, then for each $\mathbf{x}_0 \in \mathbb{R}$ it is possible to find a neighborhood foliated by the integral curves of $\mathbf{v}(\mathbf{x})$ which are nothing but characteristic curves of (3.54). Therefore the solutions of (3.55) can be extended to the whole neighborhood. It is a Riccati equation and has two independent solutions which we call μ_i for $i = 1, 2$. Hence we have two Poisson vector fields

$$\vec{J}_i = \alpha_i \left(\widehat{e}_2 + \mu_i \widehat{e}_3\right) \tag{3.56}$$

where the coefficients α_i are arbitrary. □

Taking the advantage of the freedom of choosing arbitrary scaling factors we may restrict the coefficients by imposing further assumptions on our Poisson vector fields. Our next condition will be compatibility of Poisson structures.

Definition 3.2.2. *Two Poisson structures $\vec{J}_1$ and $\vec{J}_2$ are compatible if $\vec{J}_1 + \vec{J}_2$ is also a Poisson structure.*

Proposition 3.2.3. *Poisson structures obtained in* (3.56) *are compatible if*

$$\partial_{y^1} \ln \frac{\alpha_i}{\alpha_j} = C^3_{12} \left(\mu_i - \mu_j\right) \tag{3.57}$$

Proof. Let

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \tag{3.58}$$

Using (3.46) for $\vec{J}_1$, $\vec{J}_2$ and $\vec{J}$

$$\begin{aligned} \left(\nabla \times \vec{J}\right) \cdot \vec{J} &= \nabla \times \left(\vec{J}_1 + \vec{J}_2\right) \cdot \left(\vec{J}_1 + \vec{J}_2\right) \\ &= \left(\nabla \times \vec{J}_2\right) \cdot \vec{J}_1 + \left(\nabla \times \vec{J}_1\right) \cdot \vec{J}_2 \end{aligned} \tag{3.59}$$

Therefore we need to show that

$$\left(\nabla \times \vec{J}_i\right) \cdot \vec{J}_j = -\left(\nabla \times \vec{J}_j\right) \cdot \vec{J}_i \tag{3.60}$$

For the Poisson vector fields defined in (3.56),

$$\nabla \times \vec{J}_i = \nabla \alpha_i \times (\widehat{e}_2 + \mu_i \widehat{e}_3) + \alpha_i (\nabla \times \widehat{e}_2 + \nabla \mu_i \times \widehat{e}_3 + \mu_i \nabla \times \widehat{e}_3) \tag{3.61}$$

Then taking the dot product of both sides by $\vec{J}_j$ and using (3.54) gives

$$\begin{aligned} \left(\nabla \times \vec{J}_i\right) \cdot \vec{J}_j &= \alpha_i \alpha_j (\nabla \times \widehat{e}_2 + \nabla \mu_i \times \widehat{e}_3 + \mu_i \nabla \times \widehat{e}_3) \cdot (\widehat{e}_2 + \mu_j \widehat{e}_3) \\ &+ \alpha_j (\nabla \alpha_i \times (\widehat{e}_2 + \mu_i \widehat{e}_3)) \cdot (\widehat{e}_2 + \mu_j \widehat{e}_3) \\ &= \alpha_i \alpha_j \left(-C^2_{31} - \partial_{y^1} \mu_i - C^3_{31} \mu_i + \left(-C^2_{12} - \mu_i C^3_{12}\right) \mu_j\right) \\ &+ \alpha_i \alpha_j \left((\widehat{e}_2 + \mu_i \widehat{e}_3) \times (\widehat{e}_2 + \mu_j \widehat{e}_3)\right) \cdot \nabla \ln \alpha_i \\ &= \alpha_i \alpha_j (\mu_i - \mu_j) \left(C^2_{12} + C^3_{12} \mu_i - \partial_{y^1} \ln \alpha_i\right) \end{aligned} \tag{3.62}$$

Therefore the compatibility condition (3.60) implies that

$$C^2_{12} + C^3_{12} \mu_i - \partial_{y^1} \ln \alpha_i = C^2_{12} + C^3_{12} \mu_j - \partial_{y^1} \ln \alpha_j \tag{3.63}$$

and hence we get

$$\partial_{y^1} \ln \frac{\alpha_i}{\alpha_j} = C^3_{12} (\mu_i - \mu_j) \tag{3.64}$$

whose characteristic curve is the solution curve of (3.45) in arclenght parametrization

$$\frac{d}{ds} \ln \frac{\alpha_i}{\alpha_j} = C^3_{12} (\mu_i - \mu_j) \tag{3.65}$$

then, by a similar line of reasoning as above the solutions of (3.65) can also be extended to the whole neighborhood. Therefore, the Poisson vector fields obtained from solutions of Riccati equation are always compatible and the proposition follows. □

3.3. BI-HAMILTONIAN SYSTEMS IN THREE DIMENSIONS

A dynamical system

$$\dot{\vec{x}}(t) = \vec{v}(\vec{x}(t)) \tag{3.66}$$

is called bi-Hamiltonian if there is a pair of Poisson structures $\vec{J}_1$ and $\vec{J}_2$, and a pair of functions H_1 and H_2 such that

$$\vec{v}\left(\vec{x}\right) = \vec{J}_1 \times \nabla H_2 = \vec{J}_2 \times \nabla H_1 \tag{3.67}$$

Having a pair of Poisson structures obtained in (3.56) and even a compatible pair obtained in (3.57) do not guarantee the existence of Hamiltonian functions even locally.

Proposition 3.3.1. *The dynamical system* (3.66) *is locally bi-Hamiltonian with a pair of Poisson structures obtained in* (3.56) *if and only if*

$$\partial_{y^1} \ln \alpha_i = \partial_{y^1} \ln \|\vec{v}\| + C^3_{31} + \mu_i C^3_{12} \tag{3.68}$$

Proof. For this purpose we first need to write down the equations for the Hamiltonian functions. The equation (3.67) implies that

$$\widehat{e}_1 \cdot \nabla H_i = \partial_{y^1} H_i = 0 \tag{3.69}$$

so the Hamiltonian functions depend only on variables y^2 and y^3. This is the invariance condition of Hamiltonian functions under the flow generated by $\vec{v}\left(\mathbf{x}\right)$. Therefore the first equation for Hamiltonian functions can be written as

$$\widehat{e}_1\left(H_i\right) = 0 \tag{3.70}$$

The gradient of the Hamiltonian functions reduce to

$$\nabla H_j = \left(\widehat{e}_2\left(H_j\right)\right)\widehat{e}_2 + \left(\widehat{e}_3\left(H_j\right)\right)\widehat{e}_3 \tag{3.71}$$

Inserting (3.56) and (3.71) into (3.67)

$$\vec{v} = \|\vec{v}\|\,\widehat{e}_1 = \alpha_i\left(\widehat{e}_3\left(H_j\right) - \mu_i \widehat{e}_2\left(H_j\right)\right)\widehat{e}_1 \tag{3.72}$$

Hence

$$\widehat{e}_3\left(H_j\right) - \mu_i \widehat{e}_2\left(H_j\right) = \frac{\|\vec{v}\|}{\alpha_i} \tag{3.73}$$

Now, defining

$$\vec{u}_i = -\mu_i \widehat{e}_2 + \widehat{e}_3 \tag{3.74}$$

the (3.73) can be written as

$$\vec{u}_i\left(H_j\right) = \frac{\|\vec{v}\|}{\alpha_i} \tag{3.75}$$

These two equations (3.70) and (3.75) for Hamiltonian functions are subject to the integrability condition

$$\widehat{e}_1\left(\vec{u}_i\left(H_j\right)\right) - \vec{u}_i\left(\widehat{e}_1\left(H_j\right)\right) = \left[\widehat{e}_1, \vec{u}_i\right]\left(H_j\right) \tag{3.76}$$

Using the commutation relation given in (3.42)

$$\begin{aligned} \left[\widehat{e}_1, \vec{u}_i\right] &= \left[\widehat{e}_1, \widehat{e}_3 - \mu_i \widehat{e}_2\right] \\ &= \left[\widehat{e}_1, \widehat{e}_3\right] - \mu_i\left[\widehat{e}_1, \widehat{e}_2\right] - \left(\partial_{y^1}\mu_i\right)\widehat{e}_2 \\ &= -C^k_{31}\widehat{e}_k - \mu_i C^k_{12}\widehat{e}_k - \left(\partial_{y^1}\mu_i\right)\widehat{e}_2 \\ &= -\left(C^1_{31} + \mu_i C^1_{12}\right)\widehat{e}_1 - \left(C^3_{31} + \mu_i C^3_{12}\right)\widehat{e}_3 \\ &+ \left(-C^2_{31} - \mu_i C^2_{12} - \partial_{y^1}\mu_i\right)\widehat{e}_2 \end{aligned} \tag{3.77}$$

Now, using the Riccati equation (3.54) defining μ_i we obtain

$$\left[\widehat{e}_1, \vec{u}_i\right] = -\left(C^1_{31} + \mu_i C^1_{12}\right)\widehat{e}_1 - \left(C^3_{31} + \mu_i C^3_{12}\right)\vec{u}_i \tag{3.78}$$

and applying H_j to both sides of (3.78), and using two equations (3.70) and (3.75) for Hamiltonian functions we get

$$\left[\widehat{e}_1, \vec{u}_i\right]\left(H_j\right) = -\left(C^3_{31} + \mu_i C^3_{12}\right)\frac{\|\vec{v}\|}{\alpha_i} \tag{3.79}$$

Therefore our integrability condition for Hamiltonian functions becomes,

$$\partial_{y^1}\left(\frac{\|\vec{v}\|}{\alpha_i}\right) = -\left(C^3_{31} + \mu_i C^3_{12}\right)\frac{\|\vec{v}\|}{\alpha_i} \tag{3.80}$$

hence

$$\partial_{y^1}\ln\left(\frac{\alpha_i}{\|\vec{v}\|}\right) = \mu_i C^3_{12} + C^3_{31} \tag{3.81}$$

and the proposition follows. □

Corollary 3.3.2. *The pair of Poisson structures* $\vec{J}_i = \alpha_i\left(\widehat{e}_2 + \mu_i \widehat{e}_3\right)$ *where* α_i *is defined by*

(3.68) *and* μ_i *is defined by* (3.54) *is compatible.*

Proof. What we need is to show that (3.57) is satisfied. Indeed, writing (3.68) for α_i and α_j

$$\begin{aligned} \partial_{y^1} \ln \alpha_i &= \partial_{y^1} \ln \|\vec{v}\| + C^3_{31} + \mu_i C^3_{12} \\ \partial_{y^1} \ln \alpha_j &= \partial_{y^1} \ln \|\vec{v}\| + C^3_{31} + \mu_j C^3_{12} \end{aligned} \tag{3.82}$$

and subtracting the second from the first proves the corollary. □

Remark 3.3.3. *Note that, for a pair of compatible Poisson structures,* $\vec{J}_1$ *and* $\vec{J}_2$*, the dilatation symmetry* $\vec{J} \to f\vec{J}$ *and the additive symmetry* $\vec{J}_1 + \vec{J}_2$ *do not imply that* $\vec{J}_1 + f\vec{J}_2$ *is a Poisson structure. Indeed if we apply the Jacobi identity condition and using triple vector identity*

$$\left(\vec{J}_1 + f\vec{J}_2\right) \cdot \nabla \times \left(\vec{J}_1 + f\vec{J}_2\right) = -\nabla f \cdot \left(\vec{J}_1 \times \vec{J}_2\right) = 0 \tag{3.83}$$

which implies that

$$\partial_{y^1} f = 0 \tag{3.84}$$

3.4. THE RELATION BETWEEN POISSON VECTOR FIELDS AND HAMILTONIANS

Now we try to describe the relation between the pair of compatible Poisson structures and Hamiltonian functions. But first, we need the following lemma to describe this relation.

Lemma 3.4.1. *For the bi-Hamiltonian system with a pair of compatible Poisson structures defined above,*

$$\nabla \cdot \widehat{e}_1 = \partial_{y^1} \ln \frac{\alpha_1 \alpha_2 (\mu_2 - \mu_1)}{\|\vec{v}\|^2} \tag{3.85}$$

Proof. Adding the equations for integrability conditions of Hamiltonian functions (3.68) for $i = 1, 2$ we get

$$\partial_{y^1} \ln (\alpha_1 \alpha_2) = \partial_{y^1} \ln \left(\|\vec{v}\|^2\right) + 2C^3_{31} + (\mu_1 + \mu_2) C^3_{12} \tag{3.86}$$

On the other hand, subtracting the Riccati equations satisfied by μ_1 and μ_2, and the dividing by $(\mu_2 - \mu_1)$ gives

$$\partial_{y^1} \ln(\mu_2 - \mu_1) = -\left(C^3_{31} + C^2_{12}\right) - (\mu_1 + \mu_2) C^3_{12} \tag{3.87}$$

Adding (3.86) to (3.87) and using (3.44) we get

$$\partial_{y^1} \ln(\alpha_1 \alpha_2 (\mu_2 - \mu_1)) = \partial_{y^1} \ln\left(\|\vec{v}\|^2\right) + \nabla \cdot \widehat{e}_1 \tag{3.88}$$

and the lemma follows. □

Proposition 3.4.2. *Given a bi-Hamiltonian system with a pair of compatible Poisson structures, then there exists a canonical pair of compatible Poisson structures $\vec{J}_1, \vec{J}_2$ with the same Hamiltonian functions H_1, H_2 such that*

$$\vec{J}_i = (-1)^{i+1} \phi \nabla H_i \tag{3.89}$$

where

$$\phi = \frac{\alpha_1 \alpha_2 (\mu_2 - \mu_1)}{\|\vec{v}\|} \tag{3.90}$$

Proof. Since Poisson vectors are linearly independent one could write Hamiltonians in terms of Poisson vectors

$$\begin{aligned} \nabla H_1 &= \sigma^1_1 \vec{J}_1 + \sigma^2_1 \vec{J}_2 \\ \nabla H_2 &= \sigma^1_2 \vec{J}_1 + \sigma^2_2 \vec{J}_2 \end{aligned} \tag{3.91}$$

Since

$$\begin{aligned} \|\vec{v}\| \widehat{e}_1 &= \vec{J}_1 \times \nabla H_2 = \sigma^2_2 \vec{J}_1 \times \vec{J}_2 \\ &= \vec{J}_2 \times \nabla H_1 = -\sigma^1_1 \vec{J}_1 \times \vec{J}_2 \end{aligned} \tag{3.92}$$

and

$$\vec{J}_1 \times \vec{J}_2 = \alpha_1 \alpha_2 (\mu_2 - \mu_1) \widehat{e}_1 \tag{3.93}$$

we get

$$\sigma^2_2 = -\sigma^1_1 = \frac{\|\vec{v}\|}{\alpha_1 \alpha_2 (\mu_2 - \mu_1)} \tag{3.94}$$

Since we have

$$\begin{aligned} \nabla \times \nabla H_1 &= \nabla \sigma^1_1 \times \vec{J}_1 + \nabla \sigma^2_1 \times \vec{J}_2 + \sigma^1_1 \vec{\nabla} \times \vec{J}_1 + \sigma^2_1 \nabla \times \vec{J}_2 = \vec{0} \\ \nabla \times \nabla H_2 &= \nabla \sigma^1_2 \times \vec{J}_1 + \nabla \sigma^2_2 \times \vec{J}_2 + \sigma^1_2 \vec{\nabla} \times \vec{J}_1 + \sigma^2_2 \nabla \times \vec{J}_2 = \vec{0} \end{aligned} \tag{3.95}$$

Taking the dot product of both sides with $\vec{J}_1$ we get

$$\begin{aligned} \partial_{y^1} \ln \sigma_1^2 &= \frac{\vec{J}_1 \cdot \left(\nabla \times \vec{J}_2\right)}{\alpha_1 \alpha_2 (\mu_2 - \mu_1)} \\ \partial_{y^1} \ln \sigma_2^2 &= \frac{\vec{J}_1 \cdot \left(\nabla \times \vec{J}_2\right)}{\alpha_1 \alpha_2 (\mu_2 - \mu_1)} \end{aligned} \tag{3.96}$$

Similarly taking the dot product of both sides with $\vec{J}_2$ we get

$$\begin{aligned} \partial_{y^1} \ln \sigma_1^1 &= -\frac{\vec{J}_2 \cdot \left(\nabla \times \vec{J}_1\right)}{\alpha_1 \alpha_2 (\mu_2 - \mu_1)} \\ \partial_{y^1} \ln \sigma_2^1 &= -\frac{\vec{J}_2 \cdot \left(\nabla \times \vec{J}_1\right)}{\alpha_1 \alpha_2 (\mu_2 - \mu_1)} \end{aligned} \tag{3.97}$$

Now, compatibility implies that

$$\partial_{y^1} \ln \sigma_1^1 = \partial_{y^1} \ln \sigma_2^1 = \partial_{y^1} \ln \sigma_1^2 = \partial_{y^1} \ln \sigma_2^2 = \frac{\vec{J}_1 \cdot \left(\nabla \times \vec{J}_2\right)}{\alpha_1 \alpha_2 \left(\mu_2 - \mu_1\right)} \tag{3.98}$$

Inserting (3.68) into (3.62),

$$\frac{\vec{J}_1 \cdot \left(\nabla \times \vec{J}_2\right)}{\alpha_1 \alpha_2 \left(\mu_2 - \mu_1\right)} = -\left(\partial_{y^1} \ln \left(\|\vec{v}\|\right) + \nabla \cdot \widehat{e}_1\right) \tag{3.99}$$

and using (3.88)

$$\partial_{y^1} \ln \sigma_j^i = -\partial_{y^1} \ln \phi \tag{3.100}$$

Therefore

$$\sigma_j^i = \frac{\Psi_j^i \left(y^2, y^3\right)}{\phi} \tag{3.101}$$

and

$$\begin{aligned} \nabla H_1 &= \tfrac{1}{\phi} \left(\Psi_1^1 \left(y^2, y^3\right) \vec{J}_1 + \Psi_1^2 \left(y^2, y^3\right) \vec{J}_2\right) \\ \nabla H_2 &= \tfrac{1}{\phi} \left(\Psi_2^1 \left(y^2, y^3\right) \vec{J}_1 - \Psi_1^1 \left(y^2, y^3\right) \vec{J}_2\right) \end{aligned} \tag{3.102}$$

Inserting (3.102) and (3.93) into (3.67)

$$\|\vec{v}\| \, \widehat{e}_1 = -\frac{\Psi_1^1 \left(y^2, y^3\right)}{\phi} \alpha_1 \alpha_2 \left(\mu_2 - \mu_1\right) \widehat{e}_1 \tag{3.103}$$

we get

$$\Psi_1^1 \left(y^2, y^3\right) = -1 \tag{3.104}$$

and finally

$$\begin{aligned} \nabla H_1 &= -\tfrac{\|\vec{v}\|}{\alpha_1\alpha_2(\mu_2-\mu_1)}\left(\vec{J}_1 - \Psi_1^2\left(y^2,y^3\right)\vec{J}_2\right) \\ \nabla H_2 &= \tfrac{\|\vec{v}\|}{\alpha_1\alpha_2(\mu_2-\mu_1)}\left(\Psi_2^1\left(y^2,y^3\right)\vec{J}_1 + \vec{J}_2\right) \end{aligned} \tag{3.105}$$

Note that

$$\nabla H_1 \times \nabla H_2 = -\left(1+\Psi_2^1\Psi_1^2\right)\frac{\|\vec{v}\|^2}{\alpha_1\alpha_2\left(\mu_2-\mu_1\right)}\widehat{e}_1 \tag{3.106}$$

For the Hamiltonians to be functionally independent RHS of (3.106) must not vanish, i.e.

$$1+\Psi_2^1\Psi_1^2 \neq 0 \tag{3.107}$$

Now let us define

$$\begin{aligned} \vec{K}_1 &= \tfrac{\vec{J}_1-\Psi_1^2\vec{J}_2}{1+\Psi_2^1\Psi_1^2} &= -\tfrac{\alpha_1\alpha_2(\mu_2-\mu_1)}{\left(1+\Psi_2^1\Psi_1^2\right)\|\vec{v}\|}\nabla H_1 \\ \vec{K}_2 &= \tfrac{\vec{J}_2+\Psi_2^1\vec{J}_1}{1+\Psi_2^1\Psi_1^2} &= \tfrac{\alpha_1\alpha_2(\mu_2-\mu_1)}{\left(1+\Psi_2^1\Psi_1^2\right)\|\vec{v}\|}\nabla H_2 \end{aligned} \tag{3.108}$$

By (3.67) and (3.108) we get

$$\begin{aligned} \vec{K}_1 \times \nabla H_1 &= \vec{0} & \vec{K}_2 \times \nabla H_2 &= \vec{0} \\ \vec{K}_2 \times \nabla H_1 &= \|\vec{v}\|\,\widehat{e}_1 & \vec{K}_1 \times \nabla H_2 &= \|\vec{v}\|\,\widehat{e}_1 \end{aligned} \tag{3.109}$$

Choosing $\vec{K}_i$ to be our new Poisson structure the proposition follows. □

Therefore, we can write the local existence theorem of bi-Hamiltonian systems in three dimensions which generalizes the result of [5].

Theorem 3.4.3. *Any three dimensional dynamical system*

$$\dot{\vec{x}}\left(t\right) = \vec{v}\left(\vec{x}\left(t\right)\right) \tag{3.110}$$

has a pair of compatible Poisson structures

$$\vec{J}_i = \alpha_i\left(\widehat{e}_2 + \mu_i\widehat{e}_3\right) \tag{3.111}$$

in which μ_i's are determined by the equation

$$\partial_{y^1}\mu_i = -C_{31}^2 - \mu_i\left(C_{31}^3 + C_{12}^2\right) - \mu_i^2 C_{12}^3 \tag{3.112}$$

and α_i's are determined by the equation

$$\partial_{y^1} \ln \alpha_i = \partial_{y^1} \ln \|\vec{v}\| + C^3_{31} + \mu_i C^3_{12} \tag{3.113}$$

Furthermore (3.110) *is a locally bi-Hamiltonian system with a pair of local Hamiltonian functions determined by*

$$\vec{J}_i = (-1)^{i+1} \phi \nabla H_i \tag{3.114}$$

where

$$\phi = \frac{\alpha_1 \alpha_2 (\mu_2 - \mu_1)}{\|\vec{v}\|} \tag{3.115}$$

3.5. BI-HAMILTONIAN SYSTEMS ON THREE MANIFOLDS BY DIFFERENTIAL FORMS

In order to obtain and express the obstructions to the global existence of bi-Hamiltonian structures on three manifolds by certain cohomology groups and characteristic classes, we will reformulate the problem by using differential forms. For this purpose, if M is a 3 manifold with volume form Ω, associated with a Poisson bivector Λ, there is a one form

$$J = \Lambda \lrcorner \Omega \tag{3.116}$$

called Poisson one-form. The equation

$$v = \mathbf{J}(dH) \tag{3.117}$$

can be written as

$$\iota_v \Omega = J \wedge dH \tag{3.118}$$

and the Jacobi identity is given by

$$J \wedge dJ = 0 \tag{3.119}$$

Then, a bi-Hamiltonian system is two linearly independent Poisson one-forms such that

$$\iota_v \Omega = J_1 \wedge dH_2 = J_2 \wedge dH_1 \tag{3.120}$$

It can be shown that locally J_1 and J_2 can be chosen to be proportional to dH_1 and dH_2, respectively, and hence

$$\iota_v \Omega = \Phi dH_1 \wedge dH_2. \tag{3.121}$$

For the rest of our work, we will use the following commutative diagrams for the identification of vector fields and differential forms on $\mathbb{R}^3$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^0(\mathbb{R}^3) & \xrightarrow{d} & \Lambda^1(\mathbb{R}^3) & \xrightarrow{d} & \Lambda^2(\mathbb{R}^3) & \xrightarrow{d} & \Lambda^3(\mathbb{R}^3) & \longrightarrow & 0 \\ & & \| & & \uparrow *\iota_v\Omega & & \uparrow \iota_v\Omega & & \uparrow * & & \\ 0 & \longrightarrow & C^\infty(\mathbb{R}^3) & \xrightarrow{\nabla} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\nabla\times} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\nabla\cdot} & C^\infty(\mathbb{R}^3) & \longrightarrow & 0 \end{array} \tag{3.122}$$

and

$$\begin{aligned} *\iota_u\iota_v\Omega &= \iota_{u\times v}\Omega \\ *\left(*\iota_u\Omega \wedge \iota_v\Omega\right) &= u\cdot v \end{aligned} \tag{3.123}$$

The Jacobi identity for Poisson vector fields (3.119) implies that there are 1-forms β_i such that

$$dJ_i = \beta_i \wedge J_i \tag{3.124}$$

for each $i = 1, 2$.

Proposition 3.5.1. *There is a 1-form β such that*

$$dJ_i = \beta \wedge J_i \tag{3.125}$$

for each $i = 1, 2$.

Proof. Applying (3.124) to the compatibility condition

$$J_1 \wedge dJ_2 + J_2 \wedge dJ_1 = 0 \tag{3.126}$$

we get

$$(\beta_1 - \beta_2) \wedge J_1 \wedge J_2 = 0 \tag{3.127}$$

which implies that

$$\beta_1 - \beta_2 = b_1 J_1 + b_2 J_2 \tag{3.128}$$

and therefore we can define

$$\beta = \beta_1 - b_1 J_1 = \beta_2 + b_2 J_2 \tag{3.129}$$

Hence

$$\beta \wedge J_i = \beta_i \wedge J_i = dJ_i \tag{3.130}$$

and the proposition follows. □

Note that β is a TM valued 1-form. namely,

$$\iota_{\widehat{e}_1} \beta \neq 0 \tag{3.131}$$

in general. Now we are going to show that by an appropriate change of Poisson vectors, we may reduce it to a Q valued 1-form.

Lemma 3.5.2.

$$\iota_{\widehat{e}_1} \beta = \iota_{\widehat{e}_1} \left(d \ln \phi \right) \tag{3.132}$$

where ϕ is the function defined in (3.90)

Proof. For the ease of computation we carry out the computation for Poisson vector fields, then map them to differential forms The Jacobi identity (3.46) implies that $\nabla \times J_i$ is orthogonal to J_i and therefore we get

$$\nabla \times J_i = a_{i1} \widehat{e}_1 + a_{i2} \widehat{e}_1 \times J_i \tag{3.133}$$

By the definition of Poisson vector fields we have

$$J_1 \times J_2 = \phi \left\| \vec{v} \right\| \widehat{e}_1 \tag{3.134}$$

we can rewrite (3.133) in the form

$$\begin{aligned} \nabla \times J_1 &= \tfrac{a_{11}}{\phi \|\vec{v}\|} J_1 \times J_2 + a_{12} \widehat{e}_1 \times J_1 \\ \nabla \times J_2 &= \tfrac{a_{21}}{\phi \|\vec{v}\|} J_1 \times J_2 + a_{22} \widehat{e}_1 \times J_2 \end{aligned} \tag{3.135}$$

or

$$\begin{aligned} \nabla \times J_1 &= \left(-\tfrac{a_{11}}{\phi\|\vec{v}\|} J_2 + a_{12}\widehat{e}_1\right) \times J_1 \\ \nabla \times J_2 &= \left(\tfrac{a_{21}}{\phi\|\vec{v}\|} J_1 + a_{22}\widehat{e}_1\right) \times J_2 \end{aligned} \tag{3.136}$$

Now, the compatibility condition (3.60) implies

$$(a_{22}\widehat{e}_1 \times J_2) \cdot J_1 = -\,(a_{12}\widehat{e}_1 \times J_1) \cdot J_2 \tag{3.137}$$

and therefore we have

$$(J_2 \times J_1) \cdot (a_{22}\widehat{e}_1) = -\,(J_1 \times J_2) \cdot (a_{12}\widehat{e}_1) \tag{3.138}$$

using (3.134) we get

$$-\phi\,\|\vec{v}\|\,\widehat{e}_1 \cdot a_{22}\widehat{e}_1 = -\phi\,\|\vec{v}\|\,\widehat{e}_1 \cdot a_{12}\widehat{e}_1 \tag{3.139}$$

and obtain

$$a_{22} = a_{12} = \frac{a}{\phi\,\|\vec{v}\|} \tag{3.140}$$

Now, we can rewrite (3.136)

$$\begin{aligned} \nabla \times J_1 &= \left(-\tfrac{a_{11}}{\phi\|\vec{v}\|} J_2 + \tfrac{a}{\phi\|\vec{v}\|}\widehat{e}_1\right) \times J_1 \\ \nabla \times J_2 &= \left(\tfrac{a_{21}}{\phi\|\vec{v}\|} J_1 + \tfrac{a}{\phi\|\vec{v}\|}\widehat{e}_1\right) \times J_2 \end{aligned} \tag{3.141}$$

Let

$$\xi = \frac{a_{21}}{\phi\,\|\vec{v}\|} J_1 - \frac{a_{11}}{\phi\,\|\vec{v}\|} J_2 + \frac{a}{\phi\,\|\vec{v}\|}\widehat{e}_1 \tag{3.142}$$

where the coefficients a_{21}, a_{11} and a are given by

$$\begin{aligned} a_{11} &= (\nabla \times J_1) \cdot \widehat{e}_1 \\ a_{21} &= (\nabla \times J_2) \cdot \widehat{e}_1 \\ a &= (\nabla \times J_1) \cdot J_2 \end{aligned} \tag{3.143}$$

Therefore ξ can be written as

$$\xi = \frac{((\nabla \times J_2) \cdot \widehat{e}_1)\, J_1 - ((\nabla \times J_1) \cdot \widehat{e}_1)\, J_2 + ((\nabla \times J_1) \cdot J_2)\, \widehat{e}_1}{\phi\,\|\vec{v}\|} \tag{3.144}$$

Now, using (3.99), (3.85) and (3.90) we get

$$(\nabla \times J_1) \cdot J_2 = \phi \, \|\vec{v}\| \, (\partial_{y^1} \ln \phi) \tag{3.145}$$

and hence

$$\xi = \frac{((\nabla \times J_2) \cdot \widehat{e}_1) \, J_1 - ((\nabla \times J_1) \cdot \widehat{e}_1) \, J_2}{\phi \, \|\vec{v}\|} + (\partial_{y^1} \ln \phi) \, \widehat{e}_1 \tag{3.146}$$

Now using the vector identities

$$(\nabla \times J_i) \cdot \widehat{e}_1 = \nabla \cdot (J_i \times \widehat{e}_1) + (\nabla \times \widehat{e}_1) \cdot J_i \tag{3.147}$$

and

$$[u, v] = -(\nabla \cdot u) \, v + (\nabla \cdot v) \, u - \nabla \times (u \times v) \tag{3.148}$$

after some computation we obtain

$$\xi = \nabla \ln \phi + \widehat{e}_1 \times \left(\frac{[\widehat{e}_1 \times J_1, \widehat{e}_1 \times J_2]}{\phi \, \|\vec{v}\|} - \widehat{e}_1 \times \nabla \ln \|\vec{v}\| \right) \tag{3.149}$$

hence

$$\xi \cdot \widehat{e}_1 = \widehat{e}_1 \cdot \nabla \ln \phi \tag{3.150}$$

defining

$$\beta = * \iota_\xi \Omega \tag{3.151}$$

and using (3.123) the lemma follows. □

Now we define new Poisson one forms K_i

$$J_i = \phi K_i \tag{3.152}$$

Taking the exterior derivatives of both sides

$$dJ_i = d\phi \wedge K_i + \phi dK_i = \beta \wedge \phi K_i \tag{3.153}$$

and dividing both sides by ϕ

$$dK_i = (\beta - d \ln \phi) \wedge K_i \tag{3.154}$$

Let

$$\gamma = \beta - d\ln\phi \tag{3.155}$$

Now, by the lemma above

$$\iota_{\widehat{e}_1}\gamma = \iota_{\widehat{e}_1}\beta - \iota_{\widehat{e}_1}\left(d\ln\phi\right) = 0 \tag{3.156}$$

and

$$dK_i = \gamma \wedge K_i \tag{3.157}$$

4. GLOBAL EXISTENCE THEOREM FOR BI-HAMILTONIAN STRUCTURES IN THREE DIMENSIONS

In this section, we investigate the conditions for which the local existence theorem holds globally. To study the global properties of the vector field v by topological means, we will relate the vector field with its normal bundle.

4.1. THE COMPLEX LINE BUNDLE $\mathbf{Q} \rightarrow \mathbf{M}$

Let E be the 1-dimensional subbundle of TM generated by the non-vanishing vector field $v\left(y\right)$. Let $Q_z = T_zM/E_z$ and $Q = \underset{z\in M}{\cup} Q_z$. Q is isomorphic to the orthogonal complement bundle to E in TM. Then we have the short exact sequence

$$\begin{array}{ccccccccc} 0 & \rightarrow & E & \overset{i}{\longrightarrow} & TM & \overset{\pi}{\longrightarrow} & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \pi_{TM} & & \downarrow \pi_Q & & \\ & & M & & M & & M & & \end{array} \tag{4.1}$$

4.1.1. The Complex Structure on Fibers of $\mathbf{Q} \rightarrow \mathbf{M}$

By using the cross product with $\mathbf{T}\left(z\right)$ we can define a complex structure $\mathbf{\Lambda}$ on the fibers of $Q \rightarrow M$

$$\begin{array}{ccc} Q & \overset{\mathbf{\Lambda}}{\longrightarrow} & Q \\ (z, \mathbf{v}) & \longmapsto & (z, \mathbf{T} \times \mathbf{v}) \end{array} \tag{4.2}$$

For a normal vector $\widehat{e}_2\left(z\right)$ fixed by the choice of coordinate system, the complex structure allows us to identify fibers of $Q \rightarrow M$ with $\mathbb{C}$ as follows. Let

$$Q^{(1,0)} = \left\{\mathbf{v} - \sqrt{-1}\mathbf{\Lambda}\left(\mathbf{v}\right) \mid \mathbf{v} \in Q\right\} \tag{4.3}$$

and

$$Q^{(0,1)} = \left\{\mathbf{v} + \sqrt{-1}\mathbf{\Lambda}\left(\mathbf{v}\right) \mid \mathbf{v} \in Q\right\} \tag{4.4}$$

Therefore we have the isomorphism $\pi^{(1,0)}$ defined by

$$\begin{array}{ccc} Q & \xrightarrow{\pi^{(1,0)}} & Q^{(1,0)} \\ (z, \mathbf{v}) & \longmapsto & \left(z, \mathbf{v} - \sqrt{-1}\mathbf{\Lambda}(\mathbf{v})\right) \end{array} \tag{4.5}$$

which satisfies

$$\pi^{(1,0)}(\mathbf{\Lambda}(\mathbf{v})) = \sqrt{-1}\pi^{(1,0)}(\mathbf{v}) \tag{4.6}$$

Hence we have

$$\begin{aligned} \pi^{(1,0)}(y^2\widehat{e}_2 + y^3\widehat{e}_3) &= y^2\pi^{(1,0)}(\widehat{e}_2) + y^3\pi^{(1,0)}(\mathbf{\Lambda}(\widehat{e}_2)) \\ &= y^2\pi^{(1,0)}(\widehat{e}_2) + \sqrt{-1}y^3\pi^{(1,0)}(\widehat{e}_2) \\ &= \left(y^2 + \sqrt{-1}y^3\right)\pi^{(1,0)}(\widehat{e}_2) \end{aligned} \tag{4.7}$$

for all $(y, y^2\widehat{e}_2 + y^3\widehat{e}_3) \in Q$.

With the help of this complex structure the coordinate transformations given by (3.36) suggests that in any flow coordinate neighborhood around $p \in M$ has the structure of a complex line bundle over the integral curve passing through $p \in M$. Since we now have $A_{pq}(z) \in U(1)$ and therefore

$$A_{pq}(z) = e^{\sqrt{-1}\theta_{pq}(z)} \tag{4.8}$$

the coordinate transformations has the form

$$\frac{\partial y_q^i(z)}{\partial y_p^j(z)} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\sqrt{-1}\theta_{pq}(z)} \end{pmatrix} \tag{4.9}$$

and gives the local coordinates in the form

$$\begin{array}{ccc} \widehat{U}_p \subset M & \xrightarrow{y_p} & \mathbb{R} \times \mathbb{C} \\ (z) & \mapsto & \left(y^1, y^2 + \sqrt{-1}y^3\right) \end{array} \tag{4.10}$$

4.1.2. The Basic Connection on $\mathbf{Q} \to \mathbf{M}$

By definition of Q we have the commutative diagram for tangent bundles

$$\begin{array}{ccc} TQ & \overset{\pi_{TQ}}{\longrightarrow} & Q \\ \downarrow (\pi_Q)_* & & \downarrow \pi_Q \\ TM & \overset{\pi_{TM}}{\longrightarrow} & M \end{array} \tag{4.11}$$

Using the pullback bundle

$$\begin{array}{ccc} \pi_Q^*(TM) & \overset{\pi_{TQ}}{\longrightarrow} & Q \\ (y,u;v,0) & & (y;u) \\ \downarrow (\pi_Q)_* & & \downarrow \pi_Q \\ TM & \overset{\pi_{TM}}{\longrightarrow} & M \\ (y;v) & & (y) \end{array} \tag{4.12}$$

we obtain the short exact sequence of bundles over Q

$$\begin{array}{ccccccccc} 0 & \to & \pi_Q^*(TM) & \overset{i_{TM}}{\longrightarrow} & TQ & \overset{\pi_{VQ}}{\longrightarrow} & VQ & \to & 0 \\ & & (y,u;v,0) & & (y,u;v,w) & & (y,u;0,w) & & \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \pi_Q^*(E) & \overset{i_E}{\longrightarrow} & \pi_Q^*(TM) & \overset{\pi_Q}{\longrightarrow} & \pi_Q^*(Q) & \to & 0 \\ & & (y,u;e_1) & & (y,u;v) & & (y,u;w) & & \end{array} \tag{4.13}$$

Definition 4.1.1. *A connection on Q is an $\mathbb{R}-$bilinear map*

$$\begin{array}{cccc} \nabla & : & \Gamma(TM) \times \Gamma(Q) & \to & \Gamma(Q) \\ & & (v,s) & \mapsto & \nabla_v s \end{array} \tag{4.14}$$

satisfying the properties

i. $\nabla_v (fs) = v(f)\, s + f \nabla_v s$

ii. $\nabla_{fv} s = f \nabla_v s$

Since E is generated by a non-vanishing vector field, it is an integrable subbundle of TM.

Given a section s of Q, it is possible to lift it to a section $\widetilde{s}$ of TM such that

$$s = pr\left(\widetilde{s}\right) \tag{4.15}$$

where pr is the projection from TM to Q. Then it is possible to define a connection according to which the covariant derivative of a section s of Q in the direction of a section e of E is given by

$$\nabla_e s = pr\left(\mathcal{L}_e \widetilde{s}\right) = pr[e, \widetilde{s}] \tag{4.16}$$

where $\mathcal{L}_e$ is the Lie derivative in the direction of e. Such a connection is called a basic connection or a Bott connection [8].

Now, it is possible to define a similar connection for the complexified bundle $\pi^{(1,0)}Q$, and this connection is called the complex Bott connection. Note that, one may also repeat the same calculation for the dual bundle Q^* and also of its complexification $\pi^{(1,0)}Q^*$. We will keep using the same name and notation for the connections on the dual bundle.

4.2. THE FIRST OBSTRUCTION: CHERN CLASS OF Q

Now, we try to find conditions for which a non-vanishing vector field $\vec{v}$ decomposes

$$\vec{v} = \phi \nabla H_1 \times \nabla H_2 \tag{4.17}$$

for some globally defined function ϕ, H_1 and H_2. with the Poisson vector fields

$$J_i = (-1)^{i+1} \phi \nabla H_i \tag{4.18}$$

This implies that we have

$$\phi \vec{v} = J_1 \times J_2 \tag{4.19}$$

For a vector field to be decomposed into the form (4.19), first of all the vector field must be written as a product of two globally defined, linearly independent non-vanishing vector fields J_1 and J_2. For this purpose let

$$\iota_{\vec{v}} \Omega = w \tag{4.20}$$

Our question is to decompose the w into a product of two globally defined one forms ρ_1 and ρ_2.

$$w = \rho_1 \wedge \rho_2 \tag{4.21}$$

Since v is a non-vanishing vector field then w is a 2-form of constant rank 2 for $\iota_u \iota_v \Omega = 0$ for $u \in \mathfrak{X}(M)$ implies $u_p = \lambda_p v_p \in T_p M$ at each $p \in M$. If we let S_w to be the subbundle of TM on which w is of maximal rank, then we have $S_w \cong Q$ defined in (4.1) .

Theorem 4.2.1. *[9] Let Σ be an $\mathbb{R}^n$-bundle over a connected base space M. Let w be a 2-form on Σ of constant rank $2s$. Let S_w be the subbundle of Σ on which w is of maximal rank. w decomposes if and only if*

i. S_w is a trivial bundle.

ii. The representation of its normalization as a map $w_1 : M \to SO(2s)/U(s)$ arising from any trivialization of S_w lifts to $SO(2s)$.

In our case, when $s = 1$, since $U(1) \cong SO(2)$, $SO(2)/U(1)$ is a point so the second condition in the theorem is satisfied. Hence, w decomposes if and only if S_w is trivial.

Note that, a complex line bundle Q is trivial if and only if $c_1(Q) = 0$, or equivalently if and only if it has a global section.

Since the decomposition of the 2-form w into a globally defined 1-forms ρ_1 and ρ_2 is a necessary condition for the existence a global bi-Hamiltonian structure. However, the decomposition does not imply that the factors ρ_i satisfy the condition

$$\rho_i \wedge d\rho_i = 0 \tag{4.22}$$

In order to determine the effect of a vanishing Chern class condition on the constructions made so far, we are going to investigate the Riccati equation defining the Poisson vector fields.

Since our Poisson vectors and related integrability conditions are determined by the local solutions of the Riccati equation (3.112), they are defined locally on each chart. Let $\{J_i^p\}$ and $\{J_i^q\}$ be the Poisson vector fields in flow coordinate charts (U_p, y_p) and (U_q, y_q) around points $p \in M$ and $q \in M$, respectively. Around point $p \in M$, the Poisson vector fields $\{J_i^p\}$

are determined by μ_i^p, α_i^p and the local frame $\{\widehat{e}_2^p, \widehat{e}_3^p\}$. Given the local frame, we can write the Riccati equation (3.112) whose solutions are μ_i^p's, and using μ_i^p's we can determine α_i^p's by the equation (3.113). Now, if $c_1(Q) = 0$, which is a necessary condition for the existence of global bi-Hamiltonian structure, then we have a global section of Q, i.e. a global vector field normal to $\vec{v}$. Using the metric on M, we can normalize the global section of Q and take it as $\widehat{e}_2$ and define $\widehat{e}_3 = \widehat{e}_1 \times \widehat{e}_2$. Then we have the global frame field

$$\widehat{e}_i^p = \widehat{e}_i^q \tag{4.23}$$

In order to understand the relation between local Poisson vector fields obtained in two different coordinate neighborhoods, we first need the following lemma

Lemma 4.2.2. *If two solutions $\mu_1(t)$ and $\mu_2(t)$ of the Riccati equation*

$$\frac{d\mu_i}{dt} = -C_{31}^2 - \mu_i\left(C_{31}^3 + C_{12}^2\right) - \mu_i^2 C_{12}^3 \tag{4.24}$$

are known, then the general solution $\mu(t)$ is given by

$$\mu - \mu_1 = K(\mu - \mu_2)\, e^{\int C_{12}^3(\mu_2 - \mu_1)dt} \tag{4.25}$$

where K is an arbitrary constant.

Proof.

$$\frac{d}{dt}(\mu - \mu_1) = -(\mu - \mu_1)\left(C_{31}^3 + C_{12}^2\right) - (\mu^2 - \mu_1^2)C_{12}^3 \tag{4.26}$$

Dividing by $\mu - \mu_1$ gives

$$\frac{d}{dt}\ln(\mu - \mu_1) = -\left(C_{31}^3 + C_{12}^2\right) - (\mu + \mu_1)C_{12}^3 \tag{4.27}$$

Similarly

$$\frac{d}{dt}\ln(\mu - \mu_2) = -\left(C_{31}^3 + C_{12}^2\right) - (\mu + \mu_2)C_{12}^3 \tag{4.28}$$

Hence

$$\frac{d}{dt}\ln\left(\frac{\mu - \mu_1}{\mu - \mu_2}\right) = -(\mu_2 - \mu_1)C_{12}^3 \tag{4.29}$$

And the result follows. □

Theorem 4.2.3. *If $c_1(Q) = 0$, then two pairs of compatible Poisson vector fields $\{J_i^p\}$ and $\{J_i^q\}$ obtained on two intersecting coordinate neighborhoods U_p and U_q are related on $U_p \cap U_q$ by*

$$\frac{J_i^q}{\|J_i^q\|} = \frac{J_i^p}{\|J_i^p\|} \tag{4.30}$$

Proof. Given the global frame field $\{\widehat{e}_2, \widehat{e}_3\}$ defined both on coordinate neighborhoods U_p and U_q, Riccati equations for μ_i's can be written as

$$\partial_{y^1}\mu_i^r = (\nabla \times \widehat{e}_2) \cdot \widehat{e}_2 + \mu_i^r \left((\nabla \times \widehat{e}_2) \cdot \widehat{e}_3 + (\nabla \times \widehat{e}_3) \cdot \widehat{e}_2\right) + (\mu_i^r)^2 (\nabla \times \widehat{e}_3) \cdot \widehat{e}_3 \tag{4.31}$$

for $r = p, q$ Therefore, on $U_p \cap U_q$, μ_i^p and μ_i^q are four solutions of the same Riccati equation for $i = 1, 2$. By the lemma above we have

$$\mu_i^q - \mu_1^p = K_i^{pq} (\mu_i^q - \mu_2^p)\, e^{\int C_{12}^3 \left(\mu_2^p - \mu_1^p\right) dy^1} \tag{4.32}$$

Now, using the compatibility condition (3.57)

$$C_{12}^3 (\mu_2^p - \mu_1^p) = \partial_{y^1} \ln \frac{\alpha_2^p}{\alpha_1^p} \tag{4.33}$$

(4.32) becomes

$$\mu_i^q - \mu_1^p = K_i^{pq} (\mu_i^q - \mu_2^p) \frac{\alpha_2^p}{\alpha_1^p} \tag{4.34}$$

where

$$K_i^{pq} = K_i^{pq} \left(y_p^2, y_p^3\right) \tag{4.35}$$

is constant w.r.t. y_p^1. Multiplying both sides by $\alpha_1^p \alpha_i^q$ in (4.34) gives

$$\alpha_1^p \alpha_i^q (\mu_i^q - \mu_1^p) = K_i^{pq} \alpha_2^p \alpha_i^q (\mu_i^q - \mu_2^p) \tag{4.36}$$

which leads to

$$J_i^q \times J_1^p = K_i^{pq} J_i^q \times J_2^p \tag{4.37}$$

Rearranging (4.37) we obtain

$$J_i^q \times (J_1^p - K_i^{pq} J_2^p) = 0 \tag{4.38}$$

Since

$$\partial_{y^1} K_i^{pq} = 0 \tag{4.39}$$

by compatibility we can take

$$\widetilde{J}_i^p = J_1^p - K_i^{pq} J_2^p \tag{4.40}$$

to be our new Poisson vector fields on the neighborhood U_p, and obtain

$$J_i^q \times \widetilde{J}_i^p = 0 \tag{4.41}$$

By compatibility this new Poisson vector fields $\widetilde{J}_i^p$ produce functionally dependent Hamiltonians and therefore, for the simplicity of notation, we will assume without restriction of generality that

$$\widetilde{J}_i^p = J_i^p \tag{4.42}$$

and get the result

$$J_i^q = \lambda^{qp} J_i^p \tag{4.43}$$

and the theorem follows. □

This theorem states clearly the reason one may fail to extend local Poisson vector fields into global vector fields even if $c_1(Q) = 0$. In order to extend the local Poisson vector fields into a global one, one should have

$$J_i^q = J_i^p \tag{4.44}$$

on $U_p \cap U_q$. However not the Poisson vectors but their unit vectors are global. Then, we have the following result.

Corollary 4.2.4. *If $c_1(Q) = 0$, then we have two global sections $\widehat{j}_i$ of Q satisfying*

$$\widehat{j}_i \cdot \left(\nabla \times \widehat{j}_i\right) = 0 \tag{4.45}$$

and

$$J_i^p \times \widehat{j}_i = 0 \tag{4.46}$$

Obviously, $\widehat{j}_i$'s provide the global Poisson vector fields but since

$$\partial_{y^1} \frac{\|J_2^p\|}{\|J_1^p\|} \neq 0 \tag{4.47}$$

in general, they may not lead to a pair of compatible Poisson structures. Now, we take $\widehat{j}_1$ as our first global Poisson vector field, and then going to check if we can find another global Poisson vector field compatible with this one by rescaling $\widehat{j}_2$.

4.3. THE SECOND OBSTRUCTION: BOTT CLASS OF THE COMPLEX CODIMENSION ONE FOLIATION

Since v is a nonvanishing vector field on M, it defines a real codimension two foliation on M. Since $Q = TM/E$ is a complex line bundle on M, this foliation has complex codimension one. Now, by assuming our primary obstruction, which is the vanishing of the Chern class, we will define the Bott class of the complex codimension 1 foliation and then show that the system admits two globally defined Hamiltonian functions if and only if the Bott Class is trivial. First, we will give some definitions.

4.3.1. Poisson Vector Fields of Trivial Normal Bundle

For the rest of our work, we will assume that Q and its dual Q^* are trivial bundles. By (4.45) it has two global sections $\widehat{j}_i$ satisfying

$$\widehat{j}_i \wedge d\widehat{j}_i = 0 \tag{4.48}$$

and

$$\left\|\widehat{j}_i\right\| = 1 \tag{4.49}$$

satisfying

$$d\widehat{j}_i = \Gamma_i \wedge \widehat{j}_i \tag{4.50}$$

for globally defined Γ_i's. These $\widehat{j}_i$'s are related with the local Poisson vector fields J_i^P by

$$J_i^p = \|J_i^p\| \widehat{j}_i \tag{4.51}$$

and

$$K_i^p = \psi_i^p \widehat{j}_i \tag{4.52}$$

where

$$\psi_i^p = \frac{\|J_i^p\|}{\phi^p} \tag{4.53}$$

By (3.157) we have

$$dK_i^p = \gamma^p \wedge K_i^p \tag{4.54}$$

Using (4.52) into (4.54) leads to

$$d\widehat{j}_i = (\gamma^p - d\ln\psi_i^p) \wedge \widehat{j}_i \tag{4.55}$$

Redefining Γ_i's if necessary, comparing (4.50) with (4.55) we get

$$\Gamma_i = \gamma^p - d\ln\psi_i^p \tag{4.56}$$

Since, $\widehat{j}_1$ and $\widehat{j}_2$ are not compatible we introduce a local Poisson form j^p defined on the coordinate neighborhood U_p of $p \in M$, which is compatible with $\widehat{j}_1$ and parallel to $\widehat{j}_2$ i.e.

$$j^p = f^p \widehat{j}_2 \tag{4.57}$$

and

$$\widehat{j}_1 \wedge dj^p + j^p \wedge d\widehat{j}_1 = 0 \tag{4.58}$$

Now, (4.57) implies that

$$dj^p = (\Gamma_2 + d\ln f^p) \wedge j^p \tag{4.59}$$

Putting (4.50) and (4.59) into (4.58) we get

$$\widehat{j}_1 \wedge (\Gamma_2 + d\ln f^p) \wedge j^p + j^p \wedge \Gamma_1 \wedge \widehat{j}_1 = 0 \tag{4.60}$$

or equivalently

$$(\Gamma_1 - \Gamma_2 - d\ln f^p) \wedge \widehat{j}_1 \wedge j^p = 0 \tag{4.61}$$

and using (4.57) we find

$$(\Gamma_1 - \Gamma_2 - d\ln f^p) \wedge \widehat{j}_1 \wedge \widehat{j}_2 = 0 \tag{4.62}$$

which amounts to

$$(\Gamma_1 - \Gamma_2) \wedge \widehat{j}_1 \wedge \widehat{j}_2 = d\ln f^p \wedge \widehat{j}_1 \wedge \widehat{j}_2 \tag{4.63}$$

Our aim here is to find the obstruction for the extending f^p to M, or for (4.63) to hold globally. For this purpose we consider the connections on Q defined by Γ_i's. By (4.56) we define the curvature of these connections to be

$$\kappa = d\Gamma_i = d\gamma^p \tag{4.64}$$

Taking the exterior derivative of (4.54) we get

$$d\gamma^p \wedge K_i^p = 0 \tag{4.65}$$

and hence

$$d\gamma^p \wedge \widehat{j}_i = 0 \tag{4.66}$$

which leads to

$$\kappa = d\gamma^p = \varphi\widehat{j}_1 \wedge \widehat{j}_2 \tag{4.67}$$

Now multiplying both sides of (4.63) with φ

$$(\Gamma_1 - \Gamma_2) \wedge \kappa = d\ln f^p \wedge \kappa = d\left((\ln f^p)\,\kappa\right) \tag{4.68}$$

Let

$$\Xi = (\Gamma_1 - \Gamma_2) \wedge \kappa \tag{4.69}$$

We have a compatible pair of global Poisson structures, i.e. f^p is globally defined, if and only if Ξ is exact. Now we are going to show that the cohomology class of Ξ vanishes if and only if the Bott class of the complex codimension 1 foliation vanishes. Since Q is a complex line bundle we have

$$c_1(Q) = [\kappa] \tag{4.70}$$

Since the vanishing of $c_1(Q)$ is a necessary condition let

$$c_1 = dh_1 \tag{4.71}$$

therefore we have

$$dh_1 = \kappa = d\gamma^p \tag{4.72}$$

which implies that on U_p

$$h_1 = \gamma^p + d\ln h^p \tag{4.73}$$

Then the Bott class [8]

$$h_1 \wedge c_1 = (\gamma^p + d\ln h^p) \wedge d\gamma^p = d\ln h^p \wedge \kappa + \gamma^p \wedge d\gamma^p \tag{4.74}$$

Now by (3.156) and (4.67) we have

$$\gamma^p \wedge d\gamma^p = 0 \tag{4.75}$$

hence we get

$$h_1 \wedge c_1 = d\left((\ln h^p)\,\kappa\right) \tag{4.76}$$

Our last step is to compute the function h^p. Since h_1 is globally defined, on $U_p \cap U_q$ we have

$$h_1 = \gamma^p + d\ln h^p = \gamma^q + d\ln h^q \tag{4.77}$$

Hence

$$\gamma^p - \gamma^q = d\ln\frac{h^q}{h^p} \tag{4.78}$$

On the other hand, since Γ_i's are globally defined, by (4.56) we have

$$\gamma^p - \gamma^q = d\ln\frac{\psi_i^p}{\psi_i^q} \tag{4.79}$$

and hence

$$d\ln\frac{h^p\psi_i^p}{h^q\psi_i^q} = 0 \tag{4.80}$$

Now, we have the following theorem.

Theorem 4.3.1. *The cohomology class of Ξ vanishes if and only if the Bott class vanishes.*

Proof. If the Bott class vanishes, then we have a globally defined function h such that

$$d\left(\left(\ln h\right)\kappa\right)=0 \tag{4.81}$$

Then choosing $f=h$ leads to a compatible pair of global Poisson structures. Conversely if there is a pair of global Poisson structures then γ becomes a global form and by (4.78) we have

$$d\ln\frac{h^q}{h^p}=0 \tag{4.82}$$

on $U_p\cap U_q$. Therefore

$$\ln h^q-\ln h^p=c^{qp} \tag{4.83}$$

where c^{qp} is a constant on $U_p\cap U_q$. Now fixing a point $x_0\in U_p\cap U_q$

$$c^{qp}=\ln h^q\left(x_0\right)-\ln h^p\left(x_0\right)=\ln c^q-\ln c^p \tag{4.84}$$

we obtain

$$\frac{h^p}{c^p}=\frac{h^q}{c^q}=h \tag{4.85}$$

where h is a globally defined function. Therefore

$$d\ln h=d\ln h^p \tag{4.86}$$

and

$$\left[h_1\wedge c_1\right]=\left[d\left(\left(\ln h\right)\kappa\right)\right]=0 \tag{4.87}$$

and the theorem follows. □

5. CONCLUSION

In this work, the bi-Hamiltonian structure defined by a nonvanishing vector field on a three dimensional manifold is investigated. Bi-Hamiltonian structures are defined by symplectic or Poisson structures on a manifold. Since our manifold is odd dimensional, we begin with searching for Poisson structures defined by the given vector field. As Poisson structures defined by Jacobi identities we compute the Jacobi identity in the flow coordinate system defined by the vector field and found that the Jacobi identity amounts to a Riccati equation. By the local existence theorem, the Riccati equation has at least two independent solutions and these solutions define two families of Poisson structures. Then, it is shown that in these families of solutions, it is possible to find families of compatible pairs of Poisson structures. Next, we investigate the condition for the existence of two functionally independent Hamiltonian functions and show that these conditions satisfy the conditions obtained for the compatible Poisson structures. Therefore, we formulated the local existence theorem for bi-Hamiltonian structures stating that given a non-vanishing vector field on a three dimensional manifold, locally it is always possible to find a compatible pair of Poisson structures with two Hamiltonian functions.

The second part of our work is devoted to the investigation of global existence of bi-Hamiltonian structures. For this purpose, first we study the space of all bi-Hamiltonian structures which is a subbundle of the normal bundle of the nonvanishing vector field. Therefore, the existence of a global section of the subbundle, which is the global bi-Hamiltonian structure, implies the existence of a global nontrivial section of the normal bundle. Since, it is possible to define a complex structure on the fibers of the normal bundle given by taking the cross product with the given vector field, the normal bundle has the structure of a complex line bundle. Therefore, the existence of a global section of a complex line bundle implies that it is trivial and its first Chern class vanishes. Hence, the first Chern class of the normal bundle becomes the primary obstruction to the existence of bi-Hamiltonian structure. Obviously this condition is necessary but not sufficient since the existence of a global section of the subbundle implies a global section of the bundle itself but the converse may not be true. Therefore with the assumption of the vanishing first Chern class we investigate the relation among local

Poisson structures defined on different neighborhoods on the intersection of these neighborhoods. This problem is called the extension problem, which can be stated as follows: Given locally defined objects on coordinate neighborhoods, is it possible to find an extension which is well-defined on the union of neighborhoods? The answer of this question lies on the relation of quantities on the intersection of neighborhoods. Since our local object is essentially defined by a Riccati equation, with the help of the relations between solutions of Riccati equation we are managed to obtain the relation between local bi-Hamiltonian structures on the intersection of neighborhoods and obstructions to such an extension.Then we showed that it is possible to find two global Poisson structures if and only if the first Chern class vanishes. However, this pair of global Poisson structures may fail to be compatible.

In order to study the global existence of a compatible pair of global Poisson structures, we investigate the complex codimension one foliation defined by the nonvanishing vector field. Then we formulate a globally defined 3-form on the manifold and showed that compatibility is equivalent to the vanishing of the cohomology class of this 3-form. Furthermore we computed the Bott class of the complex codimension one foliation and showed that above defined 3-form vanishes if and only if the Bott class vanishes. Therefore, vanishing of the Bott class of the complex codimension one foliation defined by the nonvanishing vector field is the secondary obstruction which is sufficient to imply the global existence of bi-Hamiltonian structures.

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