ARF SYMBOLS AND THE ABSOLUTE GALOIS GROUP OF A LOCAL FIELD

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Submitted to Graduate School of Natural and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Mathematics

Yeditepe University
2016

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DATE OF APPROVAL: ..../..../2016

ACKNOWLEDGEMENTS

First, I would like to express my sincere gratitude to my supervisors Prof. Dr. K. İlhan İkeda and Assist. Prof. Dr. Erol Serbest for their brilliant comments, guidance and help throughout my thesis.

Besides, I would like to thank Assoc. Prof. Dr. Ender Abadoğlu and Assoc. Prof. Dr. Alp Bassa for their valuable comments and encouragement at all stages of this process.

I also appreciate constant support and friendship of my friends and all faculty members of the Mathematics Department of Yeditepe University.

Finally, I would like to acknowledge the support of Yeditepe University during my graduate study.

Last but not least, special thanks go to my mother Zübeyde Fırat and my father Musa Fırat for believing in me and encouraging me at every step of my academic life. Hereby, I devote this thesis to my parents.

ABSTRACT

ARF SYMBOLS AND THE ABSOLUTE GALOIS GROUP OF A LOCAL FIELD

Arf gives a description of the separable closure of the field of formal Laurent series over the finite field with p elements and of the absolute Galois group of the maximal tamely ramified closure of this Laurent series field in terms of certain symbols. In our thesis, combining Arf's approach with Fontaine-Wintenberger theory of fields of norms and non-abelian local class field theory, we obtain a description of the absolute Galois group of *any* local field in terms of certain hybrid symbols determined from Arf-Steinberg symbols and non-abelian class formations via non-abelian topological group extensions.

ÖZET

ARF SEMBOLLERİ VE YEREL CİSMİN MUTLAK GALOIS GRUBU

Arf, p elemanlı sonlu cisim üzerine tanımlı biçimsel Laurent seriler cisminin ayrılabilir kapanışını ve bu cismin maksimal sakin dallanmış genişlemesinin mutlak Galois grubunu, belirli semboller aracılığıyla tasvir etmiştir. Tezimizde, abelyen olmayan topolojik grup genişlemeleri yardımıyla, Arf'ın yaklaşımını Fontaine-Wintenberger'in geliştirmiş olduğu norm cisimleri kuramı ve abelyen-olmayan yerel sınıf cisim kuramı ile birleştirerek *herhangi* bir yerel cismin mutlak Galois grubunun bir betimlemesi, Arf-Steinberg sembolleri tarafından belirlenen hibrit semboller ve abelyen-olmayan sınıf yapılanmaları cinsinden elde edilmiştir.

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1. INTRODUCTION

"...It is the task of young Turkish mathematicians working in these fields first to learn by heart what Cahit Arf did, and then to continue further study along the lines indicated by him."

M. Gündüz İkeda

Arguably, the most important algebraic object in number theory is the absolute Galois group G_L of a global field L. Therefore, one of the most central problems in number theory is to describe the topological group G_L and its arithmetic [1] in terms of algebraic and analytic objects related only to the base field L, which is the content of non-abelian global class field theory. This is indeed an extremely difficult problem which is still wide open. Following Chevalley's philosophy of idèles, we study a global field L via its completions L_ν at primes ν of L. Thus following this philosophy, as a first step to understand G_L, we should study[2] the topological groups G_{L_ν} at each ν and this is achieved by non-abelian local class field theory of L_ν, which describes G_{L_ν} and its arithmetic [3] in terms of certain algebraic and analytic objects related to L_ν for each prime ν of L.

Recall that abelian class field theory of the global field L gives a description of the abelianization $G_L^{ab} = \mathrm{Gal}(L^{ab}/L)$ of the topological group G_L (namely, the description of the maximal abelian part L^{ab}/L inside L^{sep}/L) via a *unique* topological isomorphism called the *"Artin-Takagi-Hasse reciprocity map"*

$$\mathsf{rec}_L : G_L^{ab} \xrightarrow{\sim} \widehat{C}_L$$

of L. Here, C_L denotes a certain topological group whose definition depends solely on the ground field L and $\widehat{C}_L$ denotes the profinite completion of C_L. Moreover, the reciprocity map of L is "natural", describes all abelian extensions of L and their arithmetic in terms of $\widehat{C}_L$, and is compatible with the abelian class field theory of L_ν (for details about abelian local class field theory, see Section 2.3). Here, the main references are Artin-Tate [2] and Fesenko-Vostokov [7].

So the next natural step then is to extend and generalize class field theory of L so that this

[1] decomposition behaviour of prime ideals $\mathfrak{p}$ of L in finite Galois extensions E of L

[2] The group theoretic description of G_{L_ν} for any prime ν of L is well known by the work of Koch (see [20]).

[3] study of higher ramification subgroups $G_{L_\nu}^{v}$ of G_{L_ν}

"extended theory" of L would describe *all* (that is, including all non-abelian) Galois extensions of L and their arithmetic in terms of certain "objects" whose definition depends only on the ground field L via the *"non-abelian reciprocity map"* of L, so that when restricted to abelian extensions over L this "extended theory" should reduce to the class field theory of L and the "non-abelian reciprocity map" of L to the Artin-Takagi-Hasse map rec_L of L. Moreover, there should be a local version of this extended theory of L. That is, there should be an extended theory of L_ν describing all Galois extensions of L_ν and their arithmetic in terms of objects whose definition depends only on L_ν via non-abelian reciprocity map of L_ν which should be compatible with the global theory.

In fact, in his revolutionary work in 1967 [24], R. P. Langlands introduced his *reciprocity* and more generally his *functoriality principles* (see [24, 25, 26]). Although conjectural in nature, these principles settled the construction of the non-abelian class field theory of L. The basic idea of Langlands is to replace the rôle played by the characters $C_L \to \mathbb{C}^\times$ of the group C_L appearing in class field theory of L with the higher-dimensional, possibly infinite-dimensional, representations of C_L, where the characters that we are interested in are the 1-dimensional automorphic representations of $C_L = \mathrm{GL}(n, \mathbb{A}_L)$. Here, $\mathbb{A}_L$ denotes the adèle ring of L which is defined by the restricted direct product of L_ν with respect to $\mathcal{O}_\nu$ as ν ranges over all Henselian and Archimedean primes of L. In the local setting, for each prime ν of L, we have $C_{L_\nu} = \mathrm{GL}(n, L_\nu)$ and we are interested in the admissible characters of C_{L_ν}. However, global reciprocity principle of Langlands depends on a conjectural group, called the automorphic Langlands group $\mathcal{L}_L$ of L, which should be the global version of Weil-Deligne groups WD_{L_ν} of L_ν for each prime ν of L, whose absence is a major obstacle to formulate global reciprocity principle.

Another, but not unrelated, approach to a non-abelian generalization of local class field theory initiated by Koch is to use Fontaine-Wintenberger theory of arithmetically profinite extensions (in short, APF-extensions) of local fields and the fields of norms attached to such extensions (see [8, 9]). In fact, generalizing Koch and de Shalit's and Fesenko's works [19], [5, 6], the non-abelian local class field theory (in the sense of Koch) has been developed by Laubie [27], Bedikyan [3], and İkeda and Serbest [12, 13, 14, 15]. There is also the global version of this theory constructed by İkeda [11], which is closely related with the hypothetical group $\mathcal{L}_L$.

Finally, there is the approach of Arf (see [1] and [17]). With the aim to develop the non-abelian local class field theory over $\mathbb{F}_p((t))$, Arf gives a description of $\mathbb{F}_p((t))^{sep}$ and of G_E where $E = \overline{\mathbb{F}}_p((t))(t^{1/n} \mid n \in \mathbb{Z}_{>0}, p \nmid n)$ is the maximal tamely ramified extension of $\mathbb{F}_p((t))$ using certain symbols. The idea of Arf is to introduce certain formal expressions satisfying Artin-Schreier type identities that "symbolizes" the process of taking the successive extensions of Artin-Schreier type over $\mathbb{F}_p((t))$. For an explanation, see [17]. However, [1] has a gap, namely Arf's work just gives a description of the group G_E not of the absolute Galois group $G_{\mathbb{F}_p((t))}$ of $\mathbb{F}_p((t))$.

1.1. THESIS PROBLEM

As described above, in his paper [1], Arf explicitly constructed the separable closure $\mathbb{F}_p((t))^{sep}$ of the field of formal Laurent series $\mathbb{F}_p((t))$ over $\mathbb{F}_p$ with one indeterminate t and the absolute Galois group G_E of E where $E = \overline{\mathbb{F}}_p((t))(t^{1/n} \mid n \in \mathbb{Z}_{>0}, p \nmid n)$ in terms of certain symbols which will be called as *Arf symbols*[4] in our thesis. Our first aim is to fill the gap of [1]; that is, to construct the absolute Galois group $G_{\mathbb{F}_p((t))}$ of $\mathbb{F}_p((t))$ using the approach of Arf with some class field theoretic modification, if necessary. Then, our second aim is to construct the absolute Galois group G_K of *any* local field K à la Arf; that is, the aim is to extend Arf's construction of G_E to G_K.

1.2. METHOD

In this section, we describe the method that we use in our thesis:

Let L/K be an APF and Galois extension. Then the field of norms $\mathbb{X}(L/K)$ corresponding to the extension L/K is a local field of characteristic $p > 0$, where $p = \text{char}(\kappa_K)$ (for details, see [7] and [8, 9]). Moreover, there exists an isomorphism

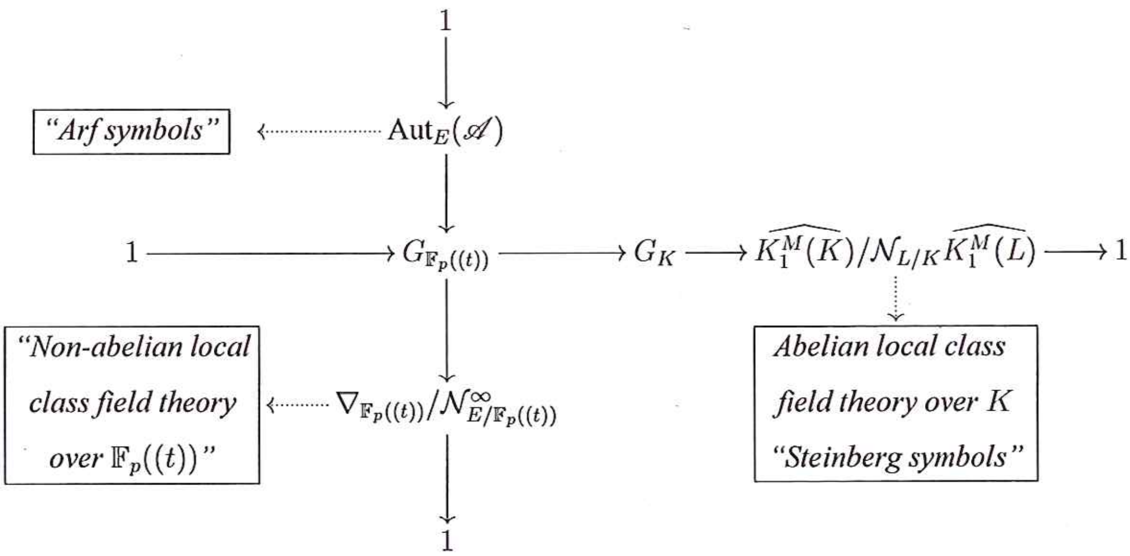
$$G_{\mathbb{X}(L/K)} \simeq \text{Gal}(K^{sep}/L),$$

under the Fontaine-Wintenberger functor. Now, $\mathbb{X}(L/K)^{sep}$ and its Galois group $G_{\mathbb{X}(L/K)}$ can

[4] called *Arf vectors* by Whaples

be described in terms of Arf symbols as $\mathbb{X}(L/K) \simeq \mathbb{F}_p((t))$. Actually, Arf symbols describe G_E, but with a group extension argument together with non-abelian local class field theory we get a description of $G_{\mathbb{F}_p((t))}$ as well. Thus, there exists a description of $\mathrm{Gal}(K^{sep}/L)$ in terms of Arf symbols and non-abelian local class formations. On the other hand, the group $\mathrm{Gal}(L/K)$, has a description in terms of the non-abelian local class field theory developed in [14, 15]. In fact, even considering a totally ramified $\mathbb{Z}_p$-extension L/K suffices for our discussion. Now, "glueing" the description of $G_{\mathbb{F}_p}((t))$ in terms of Arf symbols and non-abelian local class formations and the description of $\mathrm{Gal}(L/K)$ in terms of abelian local class field theory (for which we prefer to use the K-theoretic formulation) using the theory of group extensions, we finally get the description of G_K in terms of Arf symbols, Steinberg symbols and the non-abelian local class formations. In fact, the main theorem of this thesis can be stated as follows:

Main Theorem . *The following pair of short exact sequences*



uniquely describe G_K in terms of Arf symbols, Milnor K_1-group and non-abelian local class formation modulo the choice of continuous normalized section and ψ defined by (6.2) for the horizontal short exact sequence, and the choice of continuous normalized section for the vertical short exact sequence, which is a homomorphism, as the vertical sequence splits which yields an explicit description of $G_{\mathbb{F}_p((t))}$ (see Theorem 5.5.2).

1.3. THE LAYOUT OF THE THESIS

The layout of the thesis is as follows:

In the first part of Chapter 2, we review the basic definitions and properties related with local fields. Then, in the remainder of this chapter, we summarize the ramification theory and give the statements of abelian local class field theory.

Chapter 3 of the thesis involves APF-extensions of local fields, the fields of norms construction of Fontaine-Wintenberger and a very brief summary of non-abelian local class field theory.

In Chapter 4, we briefly discuss the construction of the separable closure of a local field of positive characteristic.

In Chapter 5, following Arf's paper [1], we introduce Arf symbols and some of the basic properties of these symbols. Thereby, employing such symbols we explain Arf's construction of the separable closure $\mathbb{F}_p((t))^{sep}$ of $\mathbb{F}_p((t))$ and the absolute Galois group G_E of E. In the last section, we describe $G_{\mathbb{F}_p((t))}$ in terms of Arf symbols and non-abelian local class formations.

Finally, in Chapter 6 which is the main part of the thesis, we construct the absolute Galois group G_K of any local field K using the theory of non-abelian topological group extensions, Fontaine-Wintenberger theory of fields of norms and the theory of Arf symbols combained with Milnor K_1-theory of K and non-abelian local class field theory of $\mathbb{F}_p((t))$.

2. PRELIMINARIES ON LOCAL FIELDS

In this chapter, we present some necessary definitions and properties concerning local fields. Moreover, we give the statements of the abelian local class field theory after reviewing the higher ramification subgroups in the upper numbering of the absolute Galois group G_K of the local field K which has a significant role in the theory of APF-extensions over K. The main refences for this chapter are [2], [7] and [12].

2.1. LOCAL FIELDS AND THEIR EXTENSIONS

In this section, we recall some basic definitions and properties of the local fields which are one of the fundamental objects of local class field theory.

Let K be a local field, that is, a complete discrete valuation field with finite residue class field. Throughout the thesis, we shall use the following notations.

- p: a fixed prime number;
- $\nu_K : K^\times \to \mathbb{Z}$: a discrete valuation normalized by $\nu_K(K^\times) = \mathbb{Z}$ with $\nu_K(0) = \infty$;
- $\mathcal{O}_K = \{\alpha \in K : \nu_K(\alpha) \geq 0\}$: the ring of integers of K with respect to ν_K;
- $\mathfrak{p}_K = \{\alpha \in K : \nu_K(\alpha) > 0\}$: the unique maximal ideal of $\mathcal{O}_K$;
- $\kappa_K = \mathcal{O}_K/\mathfrak{p}_K$: the residue class field of K of order $q = p^f$ for some $f \in \mathbb{N}$;
- $U_K = \{\alpha \in \mathcal{O}_K : \nu_K(\alpha) = 0\}$: the group of units of K;
- $\pi_K \in \mathcal{O}_K$: a prime element of K, that is, $\nu_K(\pi_K) = 1$;
- $U_K^1 = 1 + \pi_K\mathcal{O}_K$: the group of principal units of K;
- $U_K^i = 1 + \pi_K^i\mathcal{O}_K$ ($i \in \mathbb{Z}_{\geq 1}$): higher groups of units;
- K^{sep}: a fixed separable closure of K;
- $G_K = \text{Gal}(K^{sep}/K)$: the absolute Galois group of K.

Let L be a finite extension over K. Then, L is also a local field. In this case, we have the following identity

$$e(L/K)f(L/K) = [L : K].$$

Here, $e(L/K)$ is the "ramification index" defined by

$$e(L/K) = (\nu_L(L^\times) : \nu_K(K^\times)),$$

and $f(L/K)$ is the "residue degree" defined by

$$f(L/K) = [\kappa_L : \kappa_K].$$

Note that, in this case $\nu_L = \frac{1}{f(L/K)}\nu_K \circ \mathrm{N}_{L/K}$ where $\mathrm{N}_{L/K} : L^\times \to K^\times$ is the norm map.

Local fields are classified as follows:

- if $\mathrm{char}(K) = 0$, then K is a finite extension of $\mathbb{Q}_p$;
- if $\mathrm{char}(K) = p > 0$, then K is a finite extension of $\mathbb{F}_q((t))$ where $q = p^s$ for some $s \in \mathbb{N}$.

Definition 2.1.1. *Let L/K be a finite extension of local fields. Then the extension L/K is called*

(i) "unramified" if $[\kappa_L : \kappa_K] = [L : K]$ that is $e(L/K) = 1$;

(ii) "totally ramified" if $\kappa_L = \kappa_K$ that is $f(L/K) = 1$;

(iii) "tamely ramified" if κ_L/κ_K is a separable extension and $p \nmid e(L/K)$ where $\mathrm{char}(\kappa_K) = p > 0$;

(iv) "wildly totally ramified" if $\kappa_L = \kappa_K$ and the degree of L/K is a power of $p = \mathrm{char}(K)$.

The compositum of all finite unramified extensions of K in a fixed separable closure K^{sep} is called the "maximal unramified extension " of K and is denoted by K^{ur}. In general, K^{ur} is not a complete field. Its maximality implies $\sigma K^{ur} = K^{ur}$ for any automorphism σ of the separable closure K^{sep} over K. Thus, K^{ur}/K is a Galois extension. Moreover, K^{ur}/K is procyclic and its topological generator φ_K of $\mathrm{Gal}(K^{ur}/K)$ which is mapped on the topological generator Frob_q of $\text{Gal}(\mathbb{F}_q^{sep}/\mathbb{F}_q)$ is called a "Frobenius automorphism" of K. If L/K is any separable extension, then its "maximal unramified subextension" L_0/K is defined by $L_0 := L \cap K^{ur}$.

2.2. RAMIFICATION THEORY

Let K be a local field. Let K^{sep} be a fixed separable closure of K and G_K be the absolute Galois group of K. In this section, we shall summarize the higher ramification subgroups in the upper numbering of G_K which will be needed in the theory of arithmetically profinite extensions over K which will be discussed in Chapter 3. The main references for this section are [7] and [12].

Let L/K be a finite separable extension and $\sigma \in \text{Hom}_K(L, K^{sep})$. We define

$$i_{L/K}(\sigma) := \min_{x \in \mathcal{O}_L}\{\nu_L(\sigma(x) - x)\},$$

and for $t \in \mathbb{R}_{\geq -1}$

$$\gamma_t := \#\{\sigma \in \text{Hom}_K(L, K^{sep}) : i_{L/K}(\sigma) \geq t + 1\}.$$

Then, the "Hasse-Herbrand transition function" of L/K,

$$\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$$

is defined by the rule

$$\varphi_{L/K}(u) := \begin{cases} \int_0^u \frac{\gamma_t}{\gamma_0} dt & \textit{if} \quad u \in \mathbb{R}_{\geq 0}; \\ u & \textit{if} \quad -1 \leq u < 0. \end{cases}$$

The function $\varphi_{L/K}$ is monotone increasing, continuous and piecewise linear. Moreover, $\varphi_{L/K}$ induces a homeomorphism $\mathbb{R}_{\geq -1} \xrightarrow{\approx} \mathbb{R}_{\geq -1}$. So, we may define its inverse $\psi_{L/K} = \varphi_{L/K}^{-1}$.

In the remainder of this section, we further suppose that L/K is a finite *Galois* extension with Galois group $G := \text{Gal}(L/K)$. For $u \in \mathbb{R}_{\geq -1}$, the "$u$-th ramification group" G_u of G

in the lower numbering is a normal subgroup of G defined as

$$G_u = \{\sigma \in G : i_{L/K}(\sigma) \geq u + 1\}.$$

Observe that for every pair $u, u' \in \mathbb{R}_{\geq -1}$, we have the following properties:

- $G_{-1} = G$.
- G_u has order γ_u.
- If $u' \geq u$, then $G_{u'} \subseteq G_u$.

Therefore, the family $\{G_u\}_{u \in \mathbb{R}_{\geq -1}}$ induces a natural filtration on G, which we call the "lower ramification filtration" on G. Any number $u \in \mathbb{R}_{\geq -1}$ which satisfies $G_u \neq G_{u+\epsilon}$ for every $\epsilon \in \mathbb{R}_{>0}$ is called a "break in the lower ramification filtration".

Using Hasse-Herbrand function, the "v-th ramification group" G^v of G in the upper numbering is defined as

$$G^v := G_{\psi_{L/K}(v)} \tag{2.1}$$

for $v \in \mathbb{R}_{\geq -1}$. Note that for every $v', v \in \mathbb{R}_{\geq -1}$, if $v' \geq v$ then $G^{v'} \subseteq G^v$. Hence, the family $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ induces a filtration on G which we call "upper ramification filtration" on G. Similarly, any number $v \in \mathbb{R}_{\geq -1}$ which satisfies $G^v \neq G^{v+\epsilon}$ for every $\epsilon \in \mathbb{R}_{>0}$ is called a "break in the upper ramification filtration".

Remark 2.2.1. *Let F/K be a subextension of L/K and $H = \mathrm{Gal}(L/F)$. Now, we have the following properties of lower and upper ramification filtrations on G:*

(i) For $u \in \mathbb{R}_{\geq -1}$, we have

$$H_u = G_u \cap H.$$

(ii) (Herbrand's Theorem) Suppose further that H is a normal subgroup of G. Then, for $v \in \mathbb{R}_{\geq -1}$, we have

$$(G/H)^v = G^v H/H. \tag{2.2}$$

This enables us to introduce the upper ramification filtration G^v on G for an infinite Galois extensions.

(iii) The Hasse-Herbrand functions φ and ψ satisfy the following transitivity laws

$$\varphi_{L/K} = \varphi_{F/K} \circ \varphi_{L/F}$$

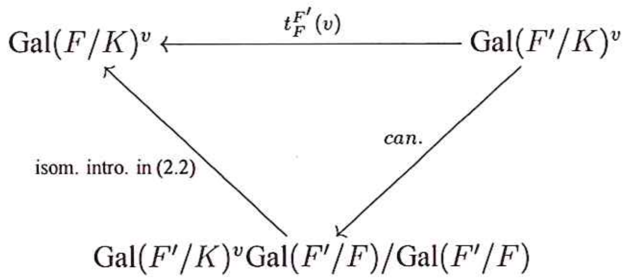
and

$$\psi_{L/K} = \psi_{L/F} \circ \psi_{F/K}.$$

Now, let L/K be an *infinite* Galois extension with Galois group G. Let $K \subseteq F \subseteq F' \subseteq L$ be a chain of finite Galois subextensions F/K and F'/K of L/K. Define the morphism

$$t_F^{F'}(v) : \mathrm{Gal}(F'/K)^v \longrightarrow \mathrm{Gal}(F/K)^v,$$

for $v \in \mathbb{R}_{\geq -1}$ which is the restriction morphisms from F' to F, by the following commutative diagram



induced by Equation (2.2). Here, "can." denotes the canonical group homomorphism. So, we get the following inverse system:

$$\{\mathrm{Gal}(F/K)^v; t_F^{F'}(v) : \mathrm{Gal}(F'/K)^v \longrightarrow \mathrm{Gal}(F/K)^v\}. \tag{2.3}$$

Then, the inverse limit of this system is denoted by

$$G^v := \varprojlim_{K \subseteq F \subset L} \mathrm{Gal}(F/K)^v$$

and called the "v-th ramification group" of G in the upper numbering. Note that for every pair $v, v' \in \mathbb{R}_{\geq -1}$, if $v \leq v'$, then $G^{v'} \subseteq G^{v}$ via the commutativity of the following diagram

$$\begin{array}{ccc} \mathrm{Gal}(F/K)^{v} & \xleftarrow{t_F^{F'}(v)} & \mathrm{Gal}(F'/K)^{v} \\ {\scriptstyle inc.}\uparrow & & \uparrow{\scriptstyle inc.} \\ \mathrm{Gal}(F/K)^{v'} & \xleftarrow{t_F^{F'}(v')} & \mathrm{Gal}(F'/K)^{v'} \end{array}$$

for every chain $K \subset F \subseteq F' \subseteq L$. Here, "inc." denotes the inclusion homomorphism of groups. The family $\{G^{v}\}_{v\in\mathbb{R}_{\geq -1}}$ induces a filtration on G called the "upper ramification filtration" on G. Moreover,

(i) For $v \in \mathbb{R}_{\geq -1}$, G^{v} is a closed subgroup of G.

(ii) $\bigcap_{v\in\mathbb{R}_{\geq -1}} G^{v} =< 1_G >$.

(iii) $G^{-1} = G$.

(iv) $G^{0} = \{\sigma \in G : \sigma(x) \equiv x \pmod{\mathfrak{p}_L}, \forall x \in \mathcal{O}_L\}$.

Any number $v \in \mathbb{R}_{\geq -1}$ is called a "break in the upper filtration" $\{G^{v}\}_{v\in\mathbb{R}_{\geq -1}}$of G if v is a break in the upper filtration of some finite quotient G/H for some normal subgroup H of G.

(v) (Hasse-Arf Theorem) The upper ramification breaks in abelian extensions over K occur at integers.

(vi) The upper ramification breaks in separable extensions over K occur at rational numbers.

2.3. ABELIAN LOCAL CLASS FIELD THEORY

Here K denotes again a local field. Let K^{sep} be a fixed separable closure of K and G_K denote the absolute Galois group $\mathrm{Gal}(K^{sep}/K)$ of K. Let G'_K be the closure of the first commutator subgroup $[G_K, G_K]$ of G_K and G_K^{ab} be the maximal abelian Hausdorff quotient group G_K/G'_K of G_K. Let $\widehat{K^{\times}}$ be the profinite completion of the multiplicative group $K^{\times}$. The main reference for this section is [12].

Theorem 2.3.1 (The local Artin reciprocity map over K). *There exists a natural algebraic* *and topological isomorphism*

$$\alpha_K : \widehat{K^\times} \xrightarrow{\sim} G_K^{ab}$$

called, "the local Artin reciprocity map of K", which is "unique" satisfying the following conditions:

(i) We have

$$\alpha_K(K^\times) = W_K^{ab},$$

where W_K is the Weil group of K.

(ii) (Isomorphism theorem) Let L/K be an abelian extension. Then the homomorphism

$$\alpha_{L/K} : \widehat{K^\times} \xrightarrow{\alpha_K} G_K^{ab} \xrightarrow{\mathrm{res}_L} \mathrm{Gal}(L/K)$$

satisfies

$$\mathcal{N}_L := \mathrm{Ker}(\alpha_{L/K}) = \bigcap_{K \subseteq F \subseteq L} \mathrm{N}_{F/K}(\widehat{F^\times})$$

where F/K runs through the finite subextension of L/K.

(iii) (Existence theorem) Let L/K be an abelian extension. The map

$$L \mapsto \mathcal{N}_L$$

defines a one-to-one correspondence

$$\{L/K : abelian\} \longleftrightarrow \{\mathcal{N} : \mathcal{N} \leq_{closed} \widehat{K^\times}\}.$$

Moreover, if L, L_1 and L_2 over K are three abelian extensions over K, we have the following properties:

- $[L : K] < \infty \Longleftrightarrow \mathcal{N}_L$ *is an open subgroup of* $\widehat{K^\times}$ *(equivalently,* $(\widehat{K^\times} : \mathcal{N}_L) < \infty$*);*
- $L_1 \subseteq L_2 \Leftrightarrow \mathcal{N}_{L_1} \supseteq \mathcal{N}_{L_2}$*;*
- $\mathcal{N}_{L_1 \cap L_2} = \mathcal{N}_{L_1}\mathcal{N}_{L_2}$*;*

- $\mathcal{N}_{L_1 L_2} = \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2}$.

(iv) (Ramification theory) Let L/K be an abelian extension. For every $i \in \mathbb{Z}_{\geq 0}$, $v \in (i-1, i] \in \mathbb{R}$ and $x \in \widehat{K^\times}$, we have

$$\alpha_{L/K}(x) \in \mathrm{Gal}(L/K)^v \Longleftrightarrow x \in U_K^i \mathcal{N}_L.$$

(v) (Functoriality) For an abelian extension L/K, we have the following functoriality properties:

- *For $\eta \in \mathrm{Aut}(K)$ and $x \in \widehat{K^\times}$,*

$$\alpha_K(\eta(x)) = \widetilde{\eta} \alpha_K(x) \widetilde{\eta}^{-1}$$

where $\widetilde{\eta}$ is any automorphism of the field K^{ab} which satisfies $\widetilde{\eta}|_K = \eta$;

- *If furthermore L/K is finite, then for every $x \in \widehat{L^\times}$,*

$$\alpha_L(x)|_{K^{ab}} = \alpha_K(N_{L/K}(x)).$$

- *If L/K is finite, for every $x \in \widehat{K^\times}$, we have*

$$\alpha_L(x) = V_{K \to L}(\alpha_K(x))$$

where $V_{K \to L} : G_K^{ab} \longrightarrow G_L^{ab}$ is the group-theoretic transfer homomorphism (Verlagerung).

3. FONTAINE-WINTENBERGER THEORY OF FIELDS OF NORMS

In this chapter, we shall briefly review the theory of APF-extensions over local fields and Fontaine-Wintenberger theory of fields of norms associated to such extensions. The main references that we follow for this chapter are [7], [8, 9] and [36].

Throughout this chapter, K denotes a local field. Fix a separable closure K^{sep} of K and let $G_K = \text{Gal}(K^{sep}/K)$ be the absolute Galois group of K.

3.1. APF-EXTENSIONS OVER LOCAL FIELDS

In this section, we summarize the theory of APF-extensions over local fields. As usual, we denote by $\{G_K^v\}_{v\in\mathbb{R}_{\geq 1}}$ the upper ramification filtration of G_K. Set

$$R^v := (K^{sep})^{G_K^v} = \{x \in K^{sep} : \sigma(x) = x, \forall \sigma \in G_K^v\}.$$

Now, we introduce an "arithmetically profinite extension" which will be frequently used in this chapter.

Definition 3.1.1. *An extension L/K is said to be an "arithmetically profinite (in short APF) extension" if one of the following equivalent conditions holds:*

(i) $G_K^v G_L \underbrace{\subseteq}_{open} G_K$, $\forall v \in \mathbb{R}_{\geq -1}$;

(ii) $(G_K : G_K^v G_L) < \infty$, $\forall v \in \mathbb{R}_{\geq -1}$;

(iii) $[L \cap R^v : K] < \infty$, $\forall v \in \mathbb{R}_{\geq -1}$.

Remark 3.1.2. *It is important to note that if L/K is APF, one can define also the lower ramification filtration. Moreover notice that if L/K is an APF-extension, then $[\kappa_L : \kappa_K] < \infty$.*

Example 3.1.3. *We have the following examples of APF-extensions.*

(i) If L/K is an abelian extension with $[\kappa_L : \kappa_K] < \infty$, then L is an APF-extension over

K. In particular, any abelian totally ramified $\mathbb{Z}_p$-extension is an APF-extension (see [7]).

Proof. For an abelian extension L/K, we know that Gal$(L/K)^v$ is the image of G^v in Gal(L/K) by the definition of upper ramification filtration (2.1) in Chapter 2. Since, every G^v has finite index in G^0 by the Hasse-Arf theorem, we conclude that every Gal$(L/K)^v$ has finite index in Gal(L/K). Hence, L/K is an APF-extension by Definition 3.1.1. □

(ii) *Every p-adic Lie extension L/K is also an APF-extension (for the positive characteristic case see [35]; for the characteristic zero case see [31]).*

Let L/K be an APF-extension. Put $G_L^0 = G_L \cap G_K^0$, and define

$$\psi_{L/K}(v) = \begin{cases} \int_0^v (G_K^0 : G_L^0 G_K^x)dx & \text{if} \quad v \in \mathbb{R}_{\geq 0}; \\ v & \text{if} \quad -1 \leq v < 0. \end{cases}$$

Note that, $\psi_{L/K}$ is a well defined map $v \mapsto \psi_{L/K}(v)$ for $v \in \mathbb{R}_{\geq -1}$ and it establishes a continuous, strictly monotone increasing and piecewise-linear bijection $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$. We denote the inverse of $\psi_{L/K}$ by $\varphi_{L/K} = \psi_{L/K}^{-1}$.
Let L/K be a *Galois* APF-extension. In this case, we can define the "u-th ramification subgroup" Gal$(L/K)_u$ of Gal(L/K) in the lower numbering by setting

$$\mathrm{Gal}(L/K)_u := \mathrm{Gal}(L/K)^{\varphi_{L/K}(u)}$$

for $u \in \mathbb{R}_{\geq -1}$.

Remark 3.1.4. *Let L/K be a finite separable extension and L'/L an APF-extension. Then, L'/K is also an APF-extension, and we have the following transitivity rules for the maps $\psi_{L'/K}$ and $\varphi_{L'/K}$*

$$\psi_{L'/K} = \psi_{L/K} \circ \psi_{L'/L}$$

and

$$\varphi_{L'/K} = \varphi_{L'/L} \circ \varphi_{L/K}.$$

The following theorem (see Proposition 1.2.3 in [36]) is important in what follows:

Theorem 3.1.5. *Given a tower of field extensions $K \subseteq F \subseteq L \subseteq K^{sep}$. Then:*

(i) if F/K is finite, then L/K is an APF-extension $\iff$ L/F is an APF-extension;

(ii) if L/F is finite, then L/K is an APF-extension $\iff$ F/K is an APF-extension;

(iii) if L/K is an APF-extension, then so is F/K.

3.2. FONTAINE-WINTENBERGER THEORY OF FIELDS OF NORMS

Let L/K be an *infinite* APF-extension and $\{L_i\}_{i\in\mathbb{Z}_{\geq 0}}$ be an increasing directed family of subextensions L_i/K in L/K such that:

(i) $[L_i : K] < \infty, \forall i \in \mathbb{Z}_{\geq 0}$;

(ii) $\bigcup_{i\in\mathbb{Z}_{\geq 0}} L_i = L$.

Then,

$$\{L_i^{\times}; \mathrm{N}_{L_{i'}/L_i} : L_{i'}^{\times} \to L_i^{\times}\}_{\substack{i,i'\in\mathbb{Z}_{\geq 0}\\ i\leq i'}} \tag{3.1}$$

is an inverse system where the transition morphisms are the norm maps

$$\mathrm{N}_{L_{i'}/L_i} : {L_{i'}}^{\times} \to {L_i}^{\times},$$

for every $i, i' \in \mathbb{Z}_{\geq 0}$ with $i \leq i'$. Let

$$\mathbb{X}(L/K)^{\times} := \varprojlim_{i} {L_i}^{\times}$$

be the inverse limit of this system (3.1).

Remark 3.2.1. *The group $\mathbb{X}(L/K)^{\times}$ is independent of the choice of the family $\{L_i\}_{i\in\mathbb{Z}_{\geq 0}}$ defined above. Let $S_{L/K}$ be the partially ordered family of all finite subextensions in L/K. Hence,*

$$\mathbb{X}(L/K)^{\times} = \varprojlim_{M\in S_{L/K}} M^{\times}$$

where as before the inverse limit is taken with respect to the norm maps

$$\mathrm{N}_{M_2/M_1} : M_2^\times \to M_1^\times,$$

for every $M_1, M_2 \in S_{L/K}$ *with* $M_1 \subseteq M_2$.

We set

$$\mathbb{X}(L/K) := \mathbb{X}(L/K)^\times \cup \{0\}$$

where 0 is a fixed formal symbol. The addition on $\mathbb{X}(L/K)$ is defined by the rule

$$\begin{aligned} + : \mathbb{X}(L/K) \times \mathbb{X}(L/K) &\to \mathbb{X}(L/K) \\ ((\alpha_M), (\beta_M)) &\mapsto (\alpha_M) + (\beta_M) = (\gamma_M) \end{aligned} \tag{3.2}$$

where $\gamma_M \in M$ is defined by the limit

$$\gamma_M = \lim_{\substack{M \subset M' \in S_{L/K} \\ [M':M] \to \infty}} \mathrm{N}_{M'/M}(\alpha_{M'} + \beta_{M'}) \in M \tag{3.3}$$

whose existence is guaranteed by the fact that L/K is an APF-extension. Now, we have the following theorem.

Theorem 3.2.2 (Fontaine-Wintenberger). *Let* L/K *be an APF-extension. Then* $\mathbb{X}(L/K)$ *is a field under the addition defined by* (3.2) *and under the componentwise multiplication on* $\mathbb{X}(L/K)^\times$. *We call the field* $\mathbb{X}(L/K)$ *"field of norms" attached to the APF-extension* L/K.

From now on, consider the following specific increasing directed family $\{L_i\}_{i \in \mathbb{Z}_{\geq 0}}$ of subextensions L_i/K in L/K such that:

(i) L_0/K is the maximal unramified subextension of L/K;

(ii) L_1/K is the maximal tamely ramified subextension of L/K;

(iii) for $i \geq 2$, let L_i/L_1 be defined inductively as finite subextensions of L/L_1 such that

- $L_i \subseteq L_{i+1}$,
- $\bigcup_{i \in \mathbb{Z}_{\geq 0}} L_i = L$.

Note that $[L_0 : K] < \infty$, and according to the part (iii) of Definition 2.1.1, it follows that $L_0 \subseteq L_1$, with $[L_1 : K] < \infty$. Therefore, we have

$$\nu_{L_i}(\alpha_{L_i}) = \nu_{L_0}(\alpha_{L_0}) \tag{3.4}$$

for any $(\alpha_{L_i})_{i\in\mathbb{Z}_{\geq 0}} \in \mathbb{X}(L/K)$. Hence, the map

$$\nu_{\mathbb{X}(L/K)} : \mathbb{X}(L/K) \to \mathbb{Z} \cup \{\infty\}$$

defined by the rule

$$\nu_{\mathbb{X}(L/K)}((\alpha_{L_i})_{i\in\mathbb{Z}_{\geq 0}}) = \nu_{L_0}(\alpha_{L_0}) \tag{3.5}$$

is a discrete valuation on $\mathbb{X}(L/K)$. Then, we have the following main theorem of Fontaine-Wintenberger theory (see Theorem 2.1.3 in [36]) :

Theorem 3.2.3 (Fontaine-Wintenberger). *Let L/K is an APF-extension and $\mathbb{X}(L/K)$ the field of norms attached to L/K. Then:*

(i) the field $\mathbb{X}(L/K)$ is complete with respect to $\nu_{\mathbb{X}(L/K)}$ defined by (3.5)*;*

(ii) $\kappa_{\mathbb{X}(L/K)} \xrightarrow{\sim} \kappa_L$*;*

(iii) $\text{char}(\mathbb{X}(L/K)) = \text{char}(\kappa_K) = p$.

3.3. THE ABSOLUTE GALOIS GROUP $G_{\mathbb{X}(L/K)}$ OF $\mathbb{X}(L/K)$ ATTACHED TO AN APF-EXTENSION L/K

Let L be an *infinite* APF-extension over K. Let F/K be a finite subextension of L/K and E/L be a finite subextension of K^{sep}/K that is we have the tower

$$K \subseteq F \subseteq L \subseteq E \subseteq K^{sep}$$

of field extensions. It follows from (i) and (ii) of Theorem 3.1.5 that L/F is an infinite APF-extension. Moreover, we have

$$\mathbb{X}(L/K) = \mathbb{X}(L/F).$$

Again by Theorem 3.1.5, E/K is also an infinite APF-extension such that

$$\mathbb{X}(L/K) \hookrightarrow \mathbb{X}(E/K)$$

under the natural topological embedding

$$\epsilon_{L,E}^{(M)} : \mathbb{X}(L/K) \to \mathbb{X}(E/K),$$

depending on a finite extension M over K with $LM = E$. Let us briefly recall the definition of the embedding $\epsilon_{L,E}^{(M)}$. Let $\{L_i\}_{i\in\mathbb{Z}_{\geq 0}}$ be an increasing directed family of subextensions L_i/K in L/K such that:

(i) $[L_i : K] < \infty, \forall i \in \mathbb{Z}_{\geq 0}$;

(ii) $\cup_{i\in\mathbb{Z}_{\geq 0}} L_i = L$.

Then, obviously, $\{L_iM\}_{i\in\mathbb{Z}_{\geq 0}}$ is an increasing directed family of subextensions in E/K such that

(i) $[L_iM : K] < \infty, \forall i \in \mathbb{Z}_{\geq 0}$;

(ii) $\cup_{i\in\mathbb{Z}_{\geq 0}} L_iM = E$.

For the above directed families, there exists sufficiently large $m = m(M) \in \mathbb{Z}_{\geq 0}$, satisfying

$$\mathrm{N}_{L_jM/L_iM}(x) = \mathrm{N}_{L_j/L_i}(x) \tag{3.6}$$

for $j \geq i \geq m$ and for each $x \in L_j$. Then for every $(\alpha_{L_i})_{i\in\mathbb{Z}_{\geq 0}} \in \mathbb{X}(L/K)^\times$,

$$\begin{aligned} \epsilon_{L,E}^{(M)} : \mathbb{X}(L/K) &\hookrightarrow \mathbb{X}(E/K) \\ (\alpha_{L_i})_{i\in\mathbb{Z}_{\geq 0}} &\mapsto (\alpha'_{L_iM})_{i\in\mathbb{Z}_{\geq 0}} \end{aligned}$$

where $\alpha'_{L_iM} \in L_iM$ for every $i \in \mathbb{Z}_{\geq 0}$,

$$\alpha'_{L_iM} = \begin{cases} \alpha_{L_i} & \text{if } i \geq m, \\ \mathrm{N}_{L_mM/L_iM}(\alpha_{L_m}) & \text{if } i < m. \end{cases}$$

Therefore, $\mathbb{X}(E/K)$ is an extension of the complete discrete valuation field $\mathbb{X}(L/K)$ under the embedding $\epsilon_{L,E}^{(M)}$.

Remark 3.3.1. *Suppose that M and M' are two finite extensions over K satisfying $LM = LM' = E$, then the corresponding topological embeddings $\epsilon_{L,E}^{(M)}, \epsilon_{L,E}^{(M')}$ are the same. So, we shall set $\epsilon_{L,E}^{(M)} = \epsilon_{L,E}$.*

Now, let E/L be *any* separable extension. Let us denote by $S_{E/L}^{sep}$ the partially ordered family of all finite separable subextensions in E/L. Then we have the following result.

Proposition 3.3.2. *The system*

$$\{\mathbb{X}(E'/K); \epsilon_{E',E''} : \mathbb{X}(E'/K) \hookrightarrow \mathbb{X}(E''/K)\}_{\substack{E',E'' \in S_{E/L}^{sep} \\ E' \subseteq E''}}$$

is inductive whose limit is given by

$$\mathbb{X}(E, L/K) := \varinjlim_{E' \in S_{E/L}^{sep}} \mathbb{X}(E'/K).$$

Theorem 3.3.3 (Fontaine-Wintenberger). *Let L/K be an APF-extension. Then, the fields of norms functor*

$$\mathbb{X}_{L/K}^{FW} : \{\textit{seperable ext of } L\} \xrightarrow{\sim} \{\textit{separable ext of } \mathbb{X}(L/K)\}$$

defines an equivalence between the category of separable extensions of L and the category of separable extensions of $\mathbb{X}(L/K)$. More precisely, if E is a separable extension of L, then the functor $\mathbb{X}_{L/K}^{FW}$ associates to E the field $\mathbb{X}(E, L/K)$. If moreover, E/L is a Galois extension, then the extension $\mathbb{X}(E, L/K)/\mathbb{X}(L/K)$ is also Galois, and we have a canonical isomorphism

$$\mathbb{X}_{L/K}^{FW} : \mathrm{Gal}(\mathbb{X}(E, L/K)/\mathbb{X}(L/K)) \xrightarrow{\sim} \mathrm{Gal}(E/L).$$

In particular, if $E = L^{sep}$, we get the following corollary.

Corollary 3.3.4. *Let L/K be an APF-extension. Then, there is a canonical isomorphism*

$$\mathrm{Gal}(\mathbb{X}(L^{sep}, L/K)/\mathbb{X}(L/K)) \simeq \mathrm{Gal}(L^{sep}/L).$$

Note that $\mathbb{X}(L^{sep}, L/K) = \mathbb{X}(L/K)^{sep}$. Therefore, Corollary 3.3.4 can be reformulated as

$$G_{\mathbb{X}(L/K)} \simeq G_L.$$

3.4. A BRIEF SUMMARY OF NON-ABELIAN LOCAL CLASS FIELD THEORY

Using Fontaine-Wintenberger theory of fields of norms attached to APF-extensions over K, it is possible to develop non-abelian local class field theory of K. That is, there exists a topological group ∇_K which depends only on the local field K and a topological group isomorphism

$$\nabla_K \xrightarrow{\sim} G_K,$$

satisfying basic functorial properties, the "existence" theorem and the "ramification" theorem. For details, see [3], [14, 15] and [27].

4. CONSTRUCTION OF THE SEPARABLE CLOSURE OF A FIELD OF POSITIVE CHARACTERISTIC

In the first part of this chapter, we give a brief discussion of cyclic extensions of fields. Then in the second part, we review the well known construction of the seperable closure of a local field K of characteristic p which will be used in the next chapter. For all details, we refer the reader to [7], [23].

4.1. CYCLIC EXTENSIONS OF FIELDS

Let K be any field.

Theorem 4.1.1. *Let K be a field containing a primitive n-th root of unity where $n \in \mathbb{Z}_{>0}$ such that $\gcd(n, \operatorname{char}(K)) = 1$. Then for a cyclic extension L/K of degree n, there is an element $\alpha \in L$ such that $L = K(\alpha)$, and α satisfies an equation $x^n - a = 0$ for some $a \in K$. Conversely, let $a \in K$ and α be a root of $x^n - a$. Then $K(\alpha)/K$ is a cyclic extension of degree d, $d \mid n$, and $\alpha^d \in K$.*

Theorem 4.1.2 (Artin-Schreier). *Let K be a field with $\operatorname{char}(K) = p > 0$. For a cyclic extension L/K with $[L : K] = p$, there is an element $\alpha \in L$ such that $L = K(\alpha)$ and α satisfies an equation $x^p - x - a = 0$ with some $a \in K$. Conversely, let $a \in K$ and consider the polynomial $p_a(x) = x^p - x - a$. Then, either*

(i) $p_a(x)$ is irreducible. In this case, if α is a root of $p_a(x)$ then the extension $K(\alpha)/K$ is cyclic with $[K(\alpha) : K] = p$;

or

(ii) $p_a(x)$ has one root in K. In this case, all roots of $p_a(x)$ belong to K.

In order to mention the notion of solvability of a finite group G, we first introduce the commutator (also known as derived) subgroup of G. The "derived subgroup" G' of G is defined as

$$G' = \langle aba^{-1}b^{-1} | a, b \in G \rangle.$$

For any $i \in \mathbb{Z}_{\geq 0}$, "$i$-th derived subgroup" $G^{(i)}$ of G is defined as follows:

(i) $G^{(0)} = G$;

(ii) $G^{(1)} = G'$;

(iii) $G^{(i)} = (G^{(i-1)})'$.

This gives a sequence of normal subgroups

$$G \supset G^{(1)} \supset G^{(2)} \supset \dots$$

of G. Now, recall that a finite group G is called a "solvable " group if $G^{(n)} = \langle e \rangle$ for some $n \geq 0$ where e is the identity element of G.

Let K be any field and E/K be a finite separable extension. Let L be the smallest Galois extension of K containing E. Then E/K is called a "solvable extension" if Gal(L/K) is a solvable group.

A finite extension F/K is called "solvable by radicals" if this extension is separable and there is a finite extension E/K which contains F, having a tower decomposition

$$K = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$$

such that each extension E_{i+1}/E_i can be obtained by adjoining one of the following elements:

(i) a root of unity,

(ii) a root of a polynomial $x^n - a \in E_i[x]$ with $gcd(n, \text{char}(E_i)) = 1$,

(iii) a root of a polynomial $p_a(x) = x^p - x - a \in E_i[x]$ if $\text{char}(E_i) = p > 0$.

Theorem 4.1.3. *Assume that E/K is a separable extension. Then*

$$E \text{ is solvable by radicals} \iff E \text{ is a solvable extension of } K. \tag{4.1}$$

4.2. SEPARABLE CLOSURE OF A LOCAL FIELD OF CHARACTERISTIC p

Now, we assume that K is a local field. The next well known theorem is a powerful tool for describing the separable closure K^{sep} of K.

Theorem 4.2.1. *Every finite separable extension of K is solvable.*

Proof. See Chapter 4, Lemma 1.2 of [7]. □

The particular case we are concerned with in this section is the case $K = \mathbb{F}_p((t))$, the field of formal Laurent series over the finite field $\mathbb{F}_p$. By Theorem 4.1.3, K is solvable by radicals. Then, immediately the following corollary holds true.

Corollary 4.2.2. *Let $K = \mathbb{F}_p((t))$. Then, the separable closure K^{sep} of K is obtained by adjoining to K:*

(i) the elements of the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$;

(ii) all n-th roots of the indeterminate t of K such that $\gcd(n, p) = 1$;

(iii) all roots of Artin-Schreier polynomial $p_a(x) = x^p - x - a \in K[x]$.

This is indeed the starting point of Arf's construction of $\mathbb{F}_p((t))^{sep}$ in terms of certain symbols which is the content of next chapter.

5. AN OVERVIEW OF ARF SYMBOLS

"...O sembolizmi hala benimsetemedim sanıyorum. Ama belki öldükten sonra benimserler. Ben devam edeceğim. Daha iş bitmedi..."

Cahit Arf

In his important paper [1], Arf explicitly constructed the separable closure $\mathbb{F}_p((t))^{sep}$ of the field of formal Laurent series $\mathbb{F}_p((t))$ with one indeterminate t over the finite field $\mathbb{F}_p$ with p elements and the absolute Galois group G_E of the maximal tamely ramified extension E of $\mathbb{F}_p((t))$ in terms of certain symbols, which will be called *Arf symbols* (also known as *Arf vectors* following Whaples) in this work. Here, $E = \overline{\mathbb{F}}_p((t))(t^{1/n} \mid n \in \mathbb{Z}_{>0}, p \nmid n)$. In our thesis, we shall construct the absolute Galois group G_K of *any* local field K à la Arf. More precisely, the aim is to extend Arf's construction of G_E to G_K using Arf symbols, Milnor K-theory and non-abelian local class field theory.

For a fixed prime number p, let $\overline{\mathbb{F}}_p((t))$ denote the field of formal Laurent series with one indeterminate t over the finite extensions of $\mathbb{F}_p$. The field $\overline{\mathbb{F}}_p((t))$ is a valued field, equipped with the valuation

$$\nu : \overline{\mathbb{F}}_p((t)) \to \mathbb{Z} \cup \{\infty\}$$

defined by $\nu(0) = \infty$ and

$$\nu : \sum_i a_i t^i \mapsto \min\{i \mid a_i \neq 0\},$$

for every $0 \neq \sum_i a_i t^i \in \overline{\mathbb{F}}_p((t))$, which is complete with respect to this valuation. From the previous chapter, it is known that the separable closure $\overline{\mathbb{F}}_p((t))^{sep}$ of $\overline{\mathbb{F}}_p((t))$ is obtained by adjoining to $\overline{\mathbb{F}}_p((t))$:

(i) the n-th roots of the indeterminate t of $\overline{\mathbb{F}}_p((t))$ where $p \nmid n$;

(ii) the roots of Artin-Schreier polynomial $p_a(x) = x^p - x - a$ with $a \in \overline{\mathbb{F}}_p((t))$.

Note that $\overline{\mathbb{F}}_p((t))^{sep} = \mathbb{F}_p((t))^{sep}$.

Let $\mathbb{Z}_{(p)}$ denote the local ring obtained by localization of $\mathbb{Z}$ at the prime p. So, any element

$\upsilon \in \mathbb{Z}_{(p)}$ has the form $\upsilon = \frac{m}{n} \in \mathbb{Q}$ where $m, n \in \mathbb{Z}$ and $p \nmid n$. If υ is furthermore a unit element in $\mathbb{Z}_{(p)}$, then p does not divide m as well.

5.1. ARF SYMBOLS

In this section, we shall briefly review the paper [1] of Arf, which is one of the main theories that we shall use in our work. Here, we first introduce the defining relations of the Arf symbols.

Definition 5.1.1. *Introduce a "symbol" of length $n \in \mathbb{N}$ by*

$$\begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{pmatrix},$$

where $\xi_j \in \overline{\mathbb{F}}_p$ and $\nu_j \in \mathbb{Z}_{(p)}$ positive with $j = 1, \dots, n$.

Let A denote the set of all possible formal E-linear combinations of the symbols defined above and a fixed formal symbol denoted by 1. Then, A has a natural E-vector space structure. Consider the natural E-linear embedding

$$E \hookrightarrow A$$

defined by

$$\alpha \mapsto \alpha.1,$$

for all $\alpha \in E$. From now on, we shall identify $\alpha.1$ with α for every $\alpha \in E$. Now, consider the subgroup N of A generated by the elements of A given by:

$$\begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi'_h + \xi''_h & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi'_h & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi''_h & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{pmatrix}, \quad (5.1)$$

$$\begin{pmatrix} \xi^p \\ \nu p \end{pmatrix} - \begin{pmatrix} \xi \\ \nu \end{pmatrix} - \frac{\xi}{t^\nu}, \tag{5.2}$$

for $n \geq 2$,

$$\begin{pmatrix} \xi_1^p & \xi_2 & \dots & \xi_n \\ \nu_1 p & \nu_2 & \dots & \nu_n \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{pmatrix} - \begin{pmatrix} \xi_1\xi_2 & \xi_3 & \dots & \xi_n \\ \nu_1+\nu_2 & \nu_3 & \dots & \nu_n \end{pmatrix}, \tag{5.3}$$

$$\begin{pmatrix} \xi_1 & \dots & \xi_{j-1} & \xi_j^p & \xi_{j+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{j-1} & \nu_j p & \nu_{j+1} & \dots & \nu_n \end{pmatrix} - \begin{pmatrix} \xi_1 & \dots & \xi_{j-1} & \xi_j & \xi_{j+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{j-1} & \nu_j & \nu_{j+1} & \dots & \nu_n \end{pmatrix}$$
$$- \begin{pmatrix} \xi_1 & \dots & \xi_{j-1} & \xi_j\xi_{j+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{j-1} & \nu_j+\nu_{j+1} & \dots & \nu_n \end{pmatrix} + \begin{pmatrix} \xi_1 & \dots & \xi_{j-1}\xi_j^p & \xi_{j+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{j-1}+\nu_j p & \nu_{j+1} & \dots & \nu_n \end{pmatrix}, \tag{5.4}$$

$$\begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} & \xi_n^p \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} & \nu_n p \end{pmatrix} - \begin{pmatrix} \xi_1 & \dots & \xi_{n-1} & \xi_n \\ \nu_1 & \dots & \nu_{n-1} & \nu_n \end{pmatrix} - \frac{\xi_n}{t^{\nu_n}} \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} \end{pmatrix}$$
$$+ \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1}\xi_n^p \\ \nu_1 & \nu_2 & \dots & \nu_{n-1}+\nu_n p \end{pmatrix}. \tag{5.5}$$

Note that, Relation (5.1) and Relations (5.2)-(5.5) encode the additivity and the p-powering operation of symbols, respectively. Denote the factor group A/N by $\mathscr{A}$, and for any symbol $\begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{pmatrix}$ in A, set

$$\begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} := \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{pmatrix} + N,$$

where $\xi_j \in \overline{\mathbb{F}}_p$ and $\nu_j \in \mathbb{Z}_{(p)}$ positive with $j = 1, \dots, n$.

Definition 5.1.2. *An element*

$$\begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix}$$

in $\mathscr{A}$, where $\xi_j \in \overline{\mathbb{F}}_p$ and positive $\nu_j \in \mathbb{Z}_{(p)}$ (respectively, $\nu_j \in \mathbb{Z}_{(p)}^\times$) with $j = 1, \dots, n$ is called a "general (respectively, fundamental) Arf symbol of length n".

Remark 5.1.3. *The factor group $\mathscr{A}$ consists of all possible formal E-linear combinations of* 1 *and of the general Arf symbols.*

Moreover, observe that the following equalities for general Arf symbols hold:

$$\begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_h' + \xi_h'' & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{Bmatrix} = \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_h' & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{Bmatrix} + \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_h'' & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{Bmatrix}, \tag{5.6}$$

$$\begin{Bmatrix} \xi^p \\ \nu p \end{Bmatrix} = \begin{Bmatrix} \xi \\ \nu \end{Bmatrix} + \frac{\xi}{t^\nu}, \tag{5.7}$$

for $n \geq 2$,

$$\begin{Bmatrix} \xi_1^p & \xi_2 & \dots & \xi_n \\ \nu_1 p & \nu_2 & \dots & \nu_n \end{Bmatrix} = \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} + \begin{Bmatrix} \xi_1\xi_2 & \xi_3 & \dots & \xi_n \\ \nu_1 + \nu_2 & \nu_3 & \dots & \nu_n \end{Bmatrix}, \tag{5.8}$$

$$\begin{aligned} \begin{Bmatrix} \xi_1 & \dots & \xi_{j-1} & \xi_j^p & \xi_{j+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{j-1} & \nu_j p & \nu_{j+1} & \dots & \nu_n \end{Bmatrix} &= \begin{Bmatrix} \xi_1 & \dots & \xi_{j-1} & \xi_j & \xi_{j+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{j-1} & \nu_j & \nu_{j+1} & \dots & \nu_n \end{Bmatrix} \\ + \begin{Bmatrix} \xi_1 & \dots & \xi_{j-1} & \xi_j\xi_{j+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{j-1} & \nu_j + \nu_{j+1} & \dots & \nu_n \end{Bmatrix} &- \begin{Bmatrix} \xi_1 & \dots & \xi_{j-1}\xi_j^p & \xi_{j+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{j-1} + \nu_j p & \nu_{j+1} & \dots & \nu_n \end{Bmatrix}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} & \xi_n^p \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} & \nu_n p \end{Bmatrix} &= \begin{Bmatrix} \xi_1 & \dots & \xi_{n-1} & \xi_n \\ \nu_1 & \dots & \nu_{n-1} & \nu_n \end{Bmatrix} + \frac{\xi_n}{t^{\nu_n}} \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} \end{Bmatrix} \\ &- \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1}\xi_n^p \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} + \nu_n p \end{Bmatrix}. \end{aligned} \tag{5.10}$$

Definition 5.1.4. *Let $\mathscr{B}$ be any $\overline{\mathbb{F}}_p$-basis over $\mathbb{F}_p$. Define a "fundamental Arf symbol of length n" with respect to the basis $\mathscr{B}$ by*

$$\begin{Bmatrix} \beta_1 & \beta_2 & \dots & \beta_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix}$$

where $\beta_j \in \mathscr{B}$, $\nu_j \in \mathbb{Z}_{(p)}^{\times}$ *with* $j = 1, \ldots n$.

Proposition 5.1.5. *Let $\mathscr{B}$ be any $\overline{\mathbb{F}}_p$-basis over $\mathbb{F}_p$. Then, 1 and the fundamental Arf symbols with respect to $\mathscr{B}$ span the vector space $\mathscr{A}$ over E.*

Proof. As $\mathscr{B}$ is an $\mathbb{F}_p$-basis of $\overline{\mathbb{F}}_p$, any $\xi \in \overline{\mathbb{F}}_p$ has a unique expression $\xi = c_1\beta_1 + \cdots + c_j\beta_j$ where $\beta_1, \ldots, \beta_j \in \mathscr{B}$ and $c_1, \ldots, c_j \in \mathbb{F}_p$. Therefore,

$$\begin{aligned}
\begin{Bmatrix} \xi^p \\ \nu p \end{Bmatrix} &= \begin{Bmatrix} c_1\beta_1 + \cdots + c_j\beta_j \\ \nu \end{Bmatrix} + \frac{c_1\beta_1 + \cdots + c_j\beta_j}{t^\nu} \\
&= c_1 \begin{Bmatrix} \beta_1 \\ \nu \end{Bmatrix} + c_2 \begin{Bmatrix} \beta_2 \\ \nu \end{Bmatrix} + \cdots + c_j \begin{Bmatrix} \beta_j \\ \nu \end{Bmatrix} + \frac{c_1\beta_1}{t^\nu} + \cdots + \frac{c_j\beta_j}{t^\nu} \\
&= c_1 \begin{Bmatrix} \beta_1^p \\ \nu \end{Bmatrix} + c_2 \begin{Bmatrix} \beta_2^p \\ \nu \end{Bmatrix} + \cdots + c_j \begin{Bmatrix} \beta_j^p \\ \nu \end{Bmatrix}.
\end{aligned}$$

That is, it is true for $n = 1$. For $n = 2$, we consider $\begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix}$ where $\xi_1, \xi_2 \in \overline{\mathbb{F}}_p$. Then, the following 4 different cases occur:

- $\nu_1, \nu_2 \in \mathbb{Z}_{(p)}^{\times}$ positive;
- $\nu_1 \in \mathbb{Z}_{(p)} - \mathbb{Z}_{(p)}^{\times}$, $\nu_2 \in \mathbb{Z}_{(p)}^{\times}$ positive;
- $\nu_1 \in \mathbb{Z}_{(p)}^{\times}$, $\nu_2 \in \mathbb{Z}_{(p)} - \mathbb{Z}_{(p)}^{\times}$ positive;
- $\nu_1, \nu_2 \in \mathbb{Z}_{(p)} - \mathbb{Z}_{(p)}^{\times}$ positive.

In Case 1, we assume that $\nu_1, \nu_2 \in \mathbb{Z}_{(p)}^{\times}$ positive. Then, we have

$$\begin{aligned}
\begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix} &= \begin{Bmatrix} c_{11}\beta_{11} + \cdots + c_{1s}\beta_{1s} & c_{21}\beta_{21} + \cdots + c_{2t}\beta_{2t} \\ \nu_1 & \nu_2 \end{Bmatrix} \\
&= c_{11} \begin{Bmatrix} \beta_{11} & c_{21}\beta_{21} + \cdots + c_{2t}\beta_{2t} \\ \nu_1 & \nu_2 \end{Bmatrix} + \cdots + c_{1s} \begin{Bmatrix} \beta_{1s} & c_{21}\beta_{21} + \cdots + c_{2t}\beta_{2t} \\ \nu_1 & \nu_2 \end{Bmatrix} \\
&= c_{11}c_{21} \begin{Bmatrix} \beta_{11} & \beta_{21} \\ \nu_1 & \nu_2 \end{Bmatrix} + c_{11}c_{22} \begin{Bmatrix} \beta_{11} & \beta_{22} \\ \nu_1 & \nu_2 \end{Bmatrix} + \cdots + c_{1s}c_{2t} \begin{Bmatrix} \beta_{11} & \beta_{21} \\ \nu_1 & \nu_2 \end{Bmatrix} \\
&= \sum_{j=1}^{t} \sum_{i=1}^{s} c_{1i}c_{2j} \begin{Bmatrix} \beta_{1i} & \beta_{2j} \\ \nu_1 & \nu_2 \end{Bmatrix}.
\end{aligned}$$

That is, 1 and the fundamental Arf symbols with respect to $\mathscr{B}$ span the vector space $\mathscr{A}$. In Case 2, $\nu_1 \in \mathbb{Z}_{(p)} - \mathbb{Z}_{(p)}^{\times}$, $\nu_2 \in \mathbb{Z}_{(p)}^{\times}$ positive. That is, $\nu_1 = p\nu'$ where $\nu' \in \mathbb{Z}_{(p)}^{\times}$. Then by using Equation (5.8), we have

$$\begin{aligned}
\begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix} &= \begin{Bmatrix} c_{11}\beta_{11} + \cdots + c_{1s}\beta_{1s} & c_{21}\beta_{21} + \cdots + c_{2t}\beta_{2t} \\ \nu_1 & \nu_2 \end{Bmatrix} \\
&= \begin{Bmatrix} \alpha^p & c_{21}\beta_{21} + \cdots + c_{2t}\beta_{2t} \\ p\nu' & \nu_2 \end{Bmatrix} \\
&= \begin{Bmatrix} \alpha & \alpha_2 \\ \nu' & \nu_2 \end{Bmatrix} + \begin{Bmatrix} \alpha\alpha_2 \\ \nu' + \nu_2 \end{Bmatrix}
\end{aligned}$$

when $c_{11}\beta_{11} + \cdots + c_{1s}\beta_{1s} = \alpha^p$ and $c_{21}\beta_{21} + \cdots + c_{2t}\beta_{2t} = \alpha_2$. Clearly, Case 3 is similar to the Case 2. Now, let us consider the Case 4. For Case 4, $\nu_1, \nu_2 \in \mathbb{Z}_{(p)} - \mathbb{Z}_{(p)}^{\times}$, then $\nu_1 = p\nu'$ and $\nu_2 = p\nu''$ where $\nu', \nu'' \in \mathbb{Z}_{(p)}^{\times}$.

Then by using Equations (5.8) and (5.9), we obtain

$$\begin{aligned}
\begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix} &= \begin{Bmatrix} c_{11}\beta_{11} + \cdots + c_{1s}\beta_{1s} & c_{21}\beta_{21} + \cdots + c_{2t}\beta_{2t} \\ \nu_1 & \nu_2 \end{Bmatrix} \\
&= \begin{Bmatrix} \alpha^p & \gamma^p \\ p\nu' & p\nu'' \end{Bmatrix} \\
&= \begin{Bmatrix} \alpha & \gamma^p \\ \nu' & p\nu'' \end{Bmatrix} + \begin{Bmatrix} \alpha\gamma^p \\ \nu' + p\nu'' \end{Bmatrix} \\
&= \begin{Bmatrix} \alpha & \gamma \\ \nu' & \nu'' \end{Bmatrix} + \frac{\gamma}{t^{\nu''}} \begin{Bmatrix} \alpha & \gamma \\ \nu' & \nu'' \end{Bmatrix} - \begin{Bmatrix} \alpha\gamma^p \\ \nu' + p\nu'' \end{Bmatrix} + \begin{Bmatrix} \alpha\gamma^p \\ \nu' + p\nu'' \end{Bmatrix} \\
&= \begin{Bmatrix} \alpha & \gamma \\ \nu' & \nu'' \end{Bmatrix} + \frac{\gamma}{t^{\nu''}} \begin{Bmatrix} \alpha & \gamma \\ \nu' & \nu'' \end{Bmatrix}.
\end{aligned}$$

Here, $\alpha^p = \xi_1 \in \overline{\mathbb{F}}_p$ and $\gamma^p = \xi_2 \in \overline{\mathbb{F}}_p$. This shows that all other cases are turned into the first case. Therefore, the desired result can evidently be obtained by applying the similar argument for an Arf symbol of arbitrary length. □

In [1], Arf proved that $\mathscr{A}$ is a ring under the natural addition and a certain multiplication on $\mathscr{A}$, which will be defined in the remainder of this section. In order to do so, we introduce the following expansion mapping

$$\begin{aligned}
E_{\substack{\xi \\ \nu}} : \mathscr{A} &\to \mathscr{A} \\
A &\mapsto \begin{Bmatrix} A; \begin{matrix} \xi \\ \nu \end{matrix} \end{Bmatrix}.
\end{aligned}$$

For any $A \in \mathscr{A}$, setting

$$A = \sum_i \frac{\eta^{(i)}}{t^{\nu(i)}} \begin{Bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \ldots & \xi_{n(i)}^{(i)} \\ \nu_1^{(i)} & \nu_2^{(i)} & \ldots & \nu_{n(i)}^{(i)} \end{Bmatrix}$$

where $\nu(i) \in \mathbb{Z}_{(p)}$ positive, we define an element $\left\{A; \begin{matrix}\xi \\ \nu\end{matrix}\right\}$ of $\mathscr{A}$ by:

$$E_{\substack{\xi \\ \nu}}(A) = \left\{A; \begin{matrix}\xi \\ \nu\end{matrix}\right\} = \sum_i \left\{\begin{matrix}\xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} & \eta^{(i)}\xi \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} & \nu(i)+\nu\end{matrix}\right\}.$$

Here, we use the convention that $\left\{1; \begin{matrix}\xi \\ \nu\end{matrix}\right\} := \left\{\begin{matrix}\xi \\ \nu\end{matrix}\right\}$.

Proposition 5.1.6. *For two elements $A, A' \in \mathscr{A}$, the following relation*

$$E_{\substack{\xi \\ \nu}}(A + A') = E_{\substack{\xi \\ \nu}}(A) + E_{\substack{\xi \\ \nu}}(A')$$

holds. This relation shows that the expansion mapping is additive on $\mathscr{A}$.

That is, by Proposition 5.1.6 for two elements $\left\{A; \begin{matrix}\xi \\ \nu\end{matrix}\right\}, \left\{A'; \begin{matrix}\xi \\ \nu\end{matrix}\right\} \in \mathscr{A}$, the following relation

$$\left\{A + A'; \begin{matrix}\xi \\ \nu\end{matrix}\right\} = \left\{A; \begin{matrix}\xi \\ \nu\end{matrix}\right\} + \left\{A'; \begin{matrix}\xi \\ \nu\end{matrix}\right\} \tag{5.11}$$

holds.

Proposition 5.1.7. *The image $E_{\substack{\xi \\ \nu}}(A) \in \mathscr{A}$ of $A \in \mathscr{A}$ under the expansion mapping*

$$E_{\substack{\xi \\ \nu}} : \mathscr{A} \to \mathscr{A}$$

does not depend on the choice of particular representation of A as a linear combination of symbols $\left\{\begin{matrix}\xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n\end{matrix}\right\}$.

Proof. For $\nu(i), \mu(j) \in \mathbb{Z}_{(p)}$ positive, suppose that any two representation of A is equal that is

$$A = \sum_i \frac{\eta^{(i)}}{t^{\nu(i)}} \left\{\begin{matrix}\xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)}\end{matrix}\right\} = \sum_j \frac{\gamma^{(j)}}{t^{\mu(j)}} \left\{\begin{matrix}\xi_1^{(j)} & \xi_2^{(i)} & \dots & \xi_{n(j)}^{(j)} \\ \nu_1^{(j)} & \nu_2^{(j)} & \dots & \nu_{n(j)}^{(j)}\end{matrix}\right\}.$$

Then,

$$0 = \sum_i \frac{\eta^{(i)}}{t^{\nu(i)}} \begin{Bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} \end{Bmatrix} - \sum_j \frac{\gamma^{(j)}}{t^{\mu(j)}} \begin{Bmatrix} \xi_1^{(j)} & \xi_2^{(i)} & \dots & \xi_{n(j)}^{(j)} \\ \nu_1^{(j)} & \nu_2^{(j)} & \dots & \nu_{n(j)}^{(j)} \end{Bmatrix}.$$

Apply the expansion mapping to both sides, then we obtain

$$E_{\substack{\xi \\ \nu}}(0) = \sum_i \begin{Bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} & \eta^{(i)}\xi \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} & \nu(i)+\nu \end{Bmatrix} - \sum_j \begin{Bmatrix} \xi_1^{(j)} & \xi_2^{(i)} & \dots & \xi_{n(j)}^{(j)} & \gamma^{(j)}\xi \\ \nu_1^{(j)} & \nu_2^{(i)} & \dots & \nu_{n(j)}^{(j)} & \mu(j)+\nu \end{Bmatrix}.$$

In addition, by using Equation (5.11), we have

$$\left\{0+0; \begin{matrix} \xi \\ \nu \end{matrix}\right\} = \left\{0; \begin{matrix} \xi \\ \nu \end{matrix}\right\} + \left\{0; \begin{matrix} \xi \\ \nu \end{matrix}\right\}.$$

Therefore,

$$E_{\substack{\xi \\ \nu}}(0) = \left\{0; \begin{matrix} \xi \\ \nu \end{matrix}\right\} = 0$$

that is,

$$\sum_i \begin{Bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} & \eta^{(i)}\xi \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} & \nu(i)+\nu \end{Bmatrix} = \sum_j \begin{Bmatrix} \xi_1^{(j)} & \xi_2^{(i)} & \dots & \xi_{n(j)}^{(j)} & \gamma^{(j)}\xi \\ \nu_1^{(j)} & \nu_2^{(i)} & \dots & \nu_{n(j)}^{(j)} & \mu(j)+\nu \end{Bmatrix}.$$

$\square$

Now, for $A = \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} \in \mathscr{A}$ introduce the following truncation mapping

$$T : \mathscr{A} \to \mathscr{A}$$
$$A \mapsto T(A) = \overline{A}$$

where $\overline{A}$ is defined by

$$\overline{A} = \begin{cases} \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} \end{Bmatrix} & \text{if } n > 1, \\ 1 & \text{if } n = 1. \end{cases}$$

By using truncation mapping, for $A = \begin{Bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \\ \nu_1 & \nu_2 & \cdots & \nu_n \end{Bmatrix}$ and $B = \begin{Bmatrix} \eta_1 & \eta_2 & \cdots & \eta_r \\ \mu_1 & \mu_2 & \cdots & \mu_r \end{Bmatrix}$ one can define their product in $\mathscr{A}$ by

$$AB = \begin{Bmatrix} A\overline{B}; & \eta_r \\ & \mu_r \end{Bmatrix} + \begin{Bmatrix} \overline{A}B; & \xi_n \\ & \nu_n \end{Bmatrix} + \begin{Bmatrix} \overline{A}\,\overline{B}; & \xi_n\eta_r \\ & \nu_n + \mu_r \end{Bmatrix}. \tag{5.12}$$

Note that the multiplication on $\mathscr{A}$ is commutative and associative. Moreover, product of two non-zero symbols is a non-zero symbol. Therefore, the ring $\mathscr{A}$ becomes an integral domain with respect to the addition and the multiplication defined by Equation (5.12).

5.2. IDENTITIES SATISFIED BY ARF SYMBOLS

One of the fundamental goal of this chapter is to give a detailed presentation of Arf's description of $\mathbb{F}_p((t))^{sep}$. We start with introducing the preparatory identities satisfied by Arf symbols.

Lemma 5.2.1. *The following identities hold:*

$$\begin{Bmatrix} \xi_1^p & \xi_2^p & \cdots & \xi_h^p & \xi_{h+1} & \cdots & \xi_n \\ \nu_1 p & \nu_2 p & \cdots & \nu_h p & \nu_{h+1} & \cdots & \nu_n \end{Bmatrix} = \begin{Bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \\ \nu_1 & \nu_2 & \cdots & \nu_n \end{Bmatrix} + \begin{Bmatrix} \xi_1 & \xi_2 & \cdots & \xi_{h-1} & \xi_h\xi_{h+1} & \cdots & \xi_n \\ \nu_1 & \nu_2 & \cdots & \nu_{h-1} & \nu_h + \nu_{h+1} & \cdots & \nu_n \end{Bmatrix}, \tag{5.13}$$

$$\begin{Bmatrix} \xi_1^p & \xi_2^p & \cdots & \xi_n^p \\ \nu_1 p & \nu_2 p & \cdots & \nu_n p \end{Bmatrix} = \begin{Bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \\ \nu_1 & \nu_2 & \cdots & \nu_n \end{Bmatrix} + \begin{Bmatrix} \xi_1 & \xi_2 & \cdots & \xi_{n-1} \\ \nu_1 & \nu_2 & \cdots & \nu_{n-1} \end{Bmatrix} \frac{\xi_n}{t^{\nu_n}}. \tag{5.14}$$

Proof. For $h = 1$, Identity (5.13) corresponds to Equation (5.8). Now, assume that the identity holds for h. Then by Equation (5.9), we have

$$\begin{Bmatrix} \xi_1^p & \cdots & \xi_h^p & \xi_{h+1}^p & \xi_{h+2} & \cdots & \xi_n \\ \nu_1 p & \cdots & \nu_h^p & \nu_{h+1} p & \nu_{h+2} & \cdots & \nu_n \end{Bmatrix} = \begin{Bmatrix} \xi_1^p & \cdots & \xi_h^p & \xi_{h+1} & \cdots & \xi_n \\ \nu_1 p & \cdots & \nu_h p & \nu_{h+1} & \cdots & \nu_n \end{Bmatrix}$$
$$+ \begin{Bmatrix} \xi_1^p & \cdots & \xi_h^p & \xi_{h+1}\xi_{h+2} & \cdots & \xi_n \\ \nu_1 p & \cdots & \nu_h p & \nu_{h+1} + \nu_{h+2} & \cdots & \nu_n \end{Bmatrix} - \begin{Bmatrix} \xi_1^p & \cdots & \xi_h^p\xi_{h+1}^p & \xi_{h+2} & \cdots & \xi_n \\ \nu_1 p & \cdots & (\nu_h + \nu_{h+1})p & \nu_{h+2} & \cdots & \nu_n \end{Bmatrix}.$$

From this equality and the validity of Identity (5.13) for h, it follows that the identity is true for $h+1$. So, this identity is true for all $h \in \mathbb{N}$ by induction.

Moreover, Identity (5.14) can be viewed as a special case of Identity (5.13) for $h = n$. According to Equation (5.10), we obtain

$$\begin{Bmatrix} \xi_1^p & \xi_2^p & \dots & \xi_n^p \\ \nu_1 p & \nu_2 p & \dots & \nu_n p \end{Bmatrix} = \begin{Bmatrix} \xi_1^p & \xi_2^p & \dots & \xi_{n-1}^p & \xi_n \\ \nu_1 p & \nu_2 p & \dots & \nu_{n-1} p & \nu_n \end{Bmatrix} + \frac{\xi_n}{t^{\nu_n}} \begin{Bmatrix} \xi_1^p & \xi_2^p & \dots & \xi_{n-1}^p \\ \nu_1 p & \nu_2 p & \dots & \nu_{n-1} p \end{Bmatrix}$$
$$- \begin{Bmatrix} \xi_1^p & \xi_2^p & \dots & \xi_n^p \xi_{n+1}^p \\ \nu_1 p & \nu_2 p & \dots & (\nu_n + \nu_{n-1}) p \end{Bmatrix}$$
$$= \begin{Bmatrix} \xi_1^p & \xi_2^p & \dots & \xi_{n-1}^p & \xi_n \\ \nu_1 p & \nu_2 p & \dots & \nu_{n-1} p & \nu_n \end{Bmatrix} + \frac{\xi_n}{t^{\nu_n}} \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} \end{Bmatrix} + \frac{\xi_{n-1}\xi_n}{t^{\nu_{n-1}+\nu_n}} \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-2} \\ \nu_1 & \nu_2 & \dots & \nu_{n-2} \end{Bmatrix}$$
$$- \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1}\xi_n \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} + \nu_n \end{Bmatrix} - \frac{\xi_{n-1}\xi_n}{t^{\nu_{n-1}+\nu_n}} \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-2} \\ \nu_1 & \nu_2 & \dots & \nu_{n-2} \end{Bmatrix}$$
$$= \begin{Bmatrix} \xi_1^p & \xi_2^p & \dots & \xi_{n-1}^p & \xi_n \\ \nu_1 p & \nu_2 p & \dots & \nu_{n-1} p & \nu_n \end{Bmatrix} - \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1}\xi_n \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} + \nu_n \end{Bmatrix} + \frac{\xi_n}{t^{\nu_n}} \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} \end{Bmatrix}$$

if we assume the validity of Identity (5.14) for $n-1$. Then, from this equality and Identity (5.13) we obtain that Identity (5.14) is true for n. Also, for $n = 1$, it corresponds to Equation (5.8). Thus, Identity (5.14) is true for all $n \in \mathbb{N}$ by induction. □

Lemma 5.2.2. *The following identity holds:*

$$\begin{Bmatrix} \xi \\ \nu \end{Bmatrix}^n = \sum_{n_1+n_2+\dots n_l = n} \frac{n!}{n_1! n_2! \dots n_l!} \begin{Bmatrix} \xi^{n_1} & \xi^{n_2} & \dots & \xi^{n_l} \\ n_1\nu & n_2\nu & \dots & n_l\nu \end{Bmatrix}. \tag{5.15}$$

Proof. From Equation (5.12) which is the definition of the multiplication, we first deduce the following by induction on l

$$\begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_l \\ \nu_1 & \nu_2 & \dots & \nu_l \end{Bmatrix} \begin{Bmatrix} \xi \\ \nu \end{Bmatrix} = \sum_{i=1}^{l} \begin{Bmatrix} \xi_1 & \dots & \xi_{i-1} & \xi & \xi_i & \dots & \xi_l \\ \nu_1 & \dots & \nu_{i-1} & \nu & \nu_i & \dots & \nu_l \end{Bmatrix}$$
$$+ \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi\xi_i & \dots & \xi_l \\ \nu_1 & \nu_2 & \dots & \nu + \nu_i & \dots & \nu_l \end{Bmatrix} + \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_l & \xi \\ \nu_1 & \nu_2 & \dots & \nu_l & \nu \end{Bmatrix}.$$

Then, Identity (5.15) is obtained by induction on n. □

Let

$$A_i = \sum_j \frac{\eta^{(i,j)}}{t^{\nu(i,j)}} \begin{Bmatrix} \xi_1^{(i,j)} & \xi_2^{(i,j)} & \dots & \xi_{n(i,j)}^{(i,j)} \\ \nu_1^{(i,j)} & \nu_2^{(i,j)} & \dots & \nu_{n(i,j)}^{(i,j)} \end{Bmatrix} \in \mathscr{A}$$

where $\nu(i,j) \in \mathbb{Z}_{(p)}$ positive. Now, we introduce the symbol

$$\left\{ A_l \quad A_{l-1} \quad \dots \quad A_1; \begin{matrix} \xi_1 & \dots & \xi_l \\ \nu_1 & \dots & \nu_l \end{matrix} \right\}$$

by the reduction formula

$$\left\{ A_l \quad A_{l-1} \quad \dots \quad A_1; \begin{matrix} \xi_1 & \dots & \xi_l \\ \nu_1 & \dots & \nu_l \end{matrix} \right\} = \left\{ A_l \left\{ A_{l-1} \quad \dots \quad A_1; \begin{matrix} \xi_1 & \dots & \xi_{l-1} \\ \nu_1 & \dots & \nu_{l-1} \end{matrix} \right\}; \begin{matrix} \xi_l \\ \nu_l \end{matrix} \right\}.$$

By using the last symbol, we obtain the following identity which indicates the form of m-th power of any Arf symbol with length n.

Lemma 5.2.3.

$$\begin{aligned} &\begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix}^m \\ &= \sum_{m_1+m_2+\dots+m_l=m} \frac{m!}{m_1!m_2!\dots m_l!} \left\{ \overline{A}^{m_l} \quad \overline{A}^{m_{l-1}} \dots \quad \overline{A}^{m_1}; \begin{matrix} \xi_n^{m_1} & \dots & \xi_n^{m_l} \\ m_1\nu_n & \dots & m_l\nu_n \end{matrix} \right\} \end{aligned} \tag{5.16}$$

holds where

$$\overline{A} = \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} \end{Bmatrix}.$$

Proof. The proof is done by using induction on m by first showing that:

$$\begin{aligned} &\left\{ \overline{A}; \begin{matrix} \xi_n \\ \nu_n \end{matrix} \right\} \left\{ \overline{A}^{m_l} \quad \overline{A}^{m_{l-1}} \dots \quad \overline{A}^{m_1}; \begin{matrix} \xi_n^{m_1} & \dots & \xi_n^{m_l} \\ m_1\nu_n & \dots & m_l\nu_n \end{matrix} \right\} \\ &= \left\{ \overline{A} \quad \overline{A}^{m_l} \dots \quad \overline{A}^{m_1}; \begin{matrix} \xi_n^{m_1} & \dots & \xi_n^{m_l} & \xi_n \\ m_1\nu_n & \dots & m_l\nu_n & \nu_n \end{matrix} \right\} \\ &+ \sum_i^l \left\{ \overline{A}^{m_l} \dots \quad \overline{A}^{m_i} \quad \overline{A} \quad \overline{A}^{m_{i-1}} \dots \quad \overline{A}^{m_1}; \begin{matrix} \xi_n^{m_1} & \xi_n^{m_{i-1}} & \xi_n & \xi_n^{m_i} \dots & \xi_n^{m_l} \\ m_1\nu_n & m_{i-1}\nu_n & \nu_n & m_i\nu_n \dots & m_l\nu_n \end{matrix} \right\} \\ &+ \sum_i^l \left\{ \overline{A}^{m_l} \dots \quad \overline{A}^{m_i+1} \quad \overline{A}^{m_{i-1}} \dots \quad \overline{A}^{m_1}; \begin{matrix} \xi_n^{m_1} \dots & \xi_n^{m_{i-1}} & \xi_n^{m_i+1} \dots & \xi_n^{m_l} \\ m_1\nu_n \dots & m_{i-1}\nu_n & (m_i+1)\nu_n \dots & m_l\nu_n \end{matrix} \right\}. \end{aligned}$$

Then, the right hand side of Identity (5.16) can be obtained by induction on l. □

Using Identities (5.15) and (5.16) with $n = p$ and $m = p$, respectively we get the following identities:

$$\begin{Bmatrix} \xi \\ \nu \end{Bmatrix}^p = \begin{Bmatrix} \xi^p \\ \nu p \end{Bmatrix} \quad \textit{and} \quad \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix}^p = \left\{ \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} \end{Bmatrix}^p ; \begin{matrix} \xi_n^p \\ \nu_n p \end{matrix} \right\}.$$

Also by induction on n, we obtain the following identity

$$\begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix}^p = \begin{Bmatrix} \xi_1^p & \xi_2^p & \dots & \xi_n^p \\ \nu_1 p & \nu_2 p & \dots & \nu_n p \end{Bmatrix}.$$

Then, from Equation (5.7) and Identity (5.14), we get the following identities which will play an important role in the proof of Theorem 5.3.5:

$$\begin{Bmatrix} \xi \\ \nu \end{Bmatrix}^p - \begin{Bmatrix} \xi \\ \nu \end{Bmatrix} = \frac{\xi}{t^\nu} \tag{5.17}$$

$$\begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix}^p - \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} = \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} \end{Bmatrix} \frac{\xi_n}{t^{\nu_n}}. \tag{5.18}$$

Moreover, Equations (5.17) and (5.18) yield the following theorem:

Theorem 5.2.4. *The integral domain $\mathscr{A}$ is a field and it is algebraic over $\mathbb{F}_p((t))$.*

Proof. Now, consider the tower

$$\mathbb{F}_p((t)) \subset E = \mathscr{A}_0 \subset \mathscr{A}_1 \subset \cdots \subset \mathscr{A}_n \subset \cdots \subseteq \mathscr{A},$$

where $\mathscr{A}_n$ is the subring of $\mathscr{A}$ obtained by adjoining length n Arf symbols to $\mathscr{A}_{n-1}$. Then, Equation (5.17) implies that $\mathscr{A}_1$ is algebraic over E and Equation (5.18) implies that $\mathscr{A}_n$ is algebraic over $\mathscr{A}_{n-1}$. So, $\mathscr{A}_n$ is algebraic over $\mathbb{F}_p((t))$ as E is algebraic over $\mathbb{F}_p((t))$ for all $n = 1, 2, \dots$. Then, $\mathscr{A} = \cup \mathscr{A}_n$ is algebraic over $\mathbb{F}_p((t))$. Moreover, $\mathscr{A}$ is an integral domain and algebraic over $\mathbb{F}_p((t))$. Therefore, $\mathscr{A}$ is a field which is algebraic over $\mathbb{F}_p((t))$. □

5.3. DESCRIPTION OF $\mathbb{F}_p((t))^{sep}$ À LA ARF

Arf symbols and the essential identities satisfied by them that are discussed previously in Section 5.1 and Section 5.2 allow us to describe the separable closure $\mathbb{F}_p((t))^{sep}$ of $\mathbb{F}_p((t))$. More precisely, we prove that there exists an isomorphism $\mathscr{A} \simeq \mathbb{F}_p((t))^{sep}$. In order to show this, we introduce p-extensions of fields following the first section of Chapter 6 of [30].

Definition 5.3.1. *An extension L/K of fields is called a p-extension if L/K is Galois and $[L : K] = p^s$ for some $s \in \mathbb{N}$.*

Lemma 5.3.2. *Any finite Galois extension M of $\mathbb{F}_p((t))$ corresponds to a p-extension $ME_{n_1,\dots,n_i}$ of $E_{n_1,\dots,n_i} := \overline{\mathbb{F}}_p((t))(t^{1/n_1},\dots,t^{1/n_i})$ for some n_i where $p \nmid n_i$, and $i \in \mathbb{N}$.*

Proof. Let $M/\mathbb{F}_p((t))$ be a finite Galois extension. Then, there exists $\alpha_1, \alpha_2, \dots, \alpha_j \in M$, $\zeta_{n_1}, \dots, \zeta_{n_i} \in \overline{\mathbb{F}}_p$ such that $M = \mathbb{F}_p((t))(\alpha_1, \alpha_2, \dots, \alpha_j; \zeta_{n_1}, \dots, \zeta_{n_i}; t^{1/n_1}, \dots, t^{1/n_i})$ where α_j is the root of an Artin-Schreier polynomial $x^p - x - a_i \in \mathbb{F}_p((t))[x]$ and ζ_{n_i} is a root of unity. Then, $[M : \mathbb{F}_p((t))] = p^j = [ME_{n_1,\dots,n_i} : E_{n_1,\dots,n_i}]$. That is, $ME_{n_1,\dots,n_i}$ is a p-extension of $E_{n_1,\dots,n_i}$. □

Definition 5.3.3. *A field L is p-closed if it has no Galois extension of degree p.*

Assume that the characteristic char(K) of K is $p > 0$. Consider the homomorphism

$$\wp : L \to L$$

defined by

$$\wp(x) = x^p - x, \forall x \in L.$$

Clearly, Ker$(\wp) = \mathbb{F}_p$. Moreover, for $a \in L$, the polynomial $f_a(t) = t^p - t - a$ is separable over L. If $\alpha \in L^{sep}$ is any root of $f_a(t)$, then $\alpha + 1, \dots, \alpha + p - 1$ are the other roots of this polynomial. Note that, we have two cases:

(i) if a is in the image of $\wp$, then there exists $\alpha_0 \in L$ such that $\wp(\alpha_0) = \alpha_0^p - \alpha_0 = a$ which implies $f_a(\alpha_0) = 0$. Therefore, all roots of $f_a(t)$ belong to L.

(ii) if a is not in the image of $\wp$, then all roots of $f_a(t)$ are in $L^{sep} - L$. Thus, the splitting field of $f_a(t)$ becomes cyclic extension of L of degree p.

Therefore, if L is assumed to be p-closed, there are no Galois extensions of degree p over L. Thus, the lemma below follows immediately.

Lemma 5.3.4. *Let L/K be an extension of fields. If the field L is p-closed with* char(K) *of K is $p > 0$, then the homomorphism*

$$\psi : L \to L$$

defined by

$$\psi(x) = x^p - x, \; \forall x \in L.$$

is surjective. Moreover, we obtain the following exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to L \xrightarrow{\psi} L \to 0$$

with $\text{Ker}(\psi) = \mathbb{F}_p$.

The important theorem obtained by Arf can be stated as follows.

Theorem 5.3.5 (Arf). *The field $\mathscr{A}$ is the separable closure of $\mathbb{F}_p((t))$.*

Proof. From the previous section, by Theorem 5.2.4, $\mathscr{A}$ is an algebraic extension of $\mathbb{F}_p((t))$. By Lemma 5.3.2, it is sufficient to prove that $\mathscr{A}$ is a p-closed. Thus, it is enough to show that each element of $\mathscr{A}$ is of the form $x^p - x$ with $x \in \mathscr{A}$. Here, each element of $\mathscr{A}$ can be written as a sum of elements of the form

$$y = a \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix},$$

where the coefficients a belong to E and can always be written as a sum

$$a = \sum_i \frac{\eta_i}{t^{\mu_i}} + b^p - b$$

where $\mu_i \in \mathbb{Z}_{(p)}$ positive. Therefore,

$$y = x_1^p - x_1 - b^p \frac{\xi_n}{t^{\nu_n}} \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} \end{Bmatrix}$$

with

$$x_1 = b \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} + \sum_i \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n & \eta_i \\ \nu_1 & \nu_2 & \dots & \nu_n & \mu_i \end{Bmatrix}$$

From this we deduce the assertion by induction on n. □

Therefore, we conclude this section with an important theorem which describes the separable closure $\mathbb{F}_p((t))^{sep}$.

5.4. CONSTRUCTION OF THE ABSOLUTE GALOIS GROUP G_E VIA ARF SYMBOLS

In this section, we shall first describe the automorphisms of $\mathscr{A}$ that leave the elements of E invariant in terms of Arf symbols. Let $\sigma \in \mathrm{Aut}_E(\mathscr{A})$. Clearly $\sigma : \mathscr{A} \to \mathscr{A}$ satisfies:

$$\begin{aligned} \sigma\left(\begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} + \begin{Bmatrix} \eta_1 & \eta_2 & \dots & \eta_m \\ \mu_1 & \mu_2 & \dots & \mu_m \end{Bmatrix} \right) & \\ = \sigma \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} + \sigma \begin{Bmatrix} \eta_1 & \eta_2 & \dots & \eta_m \\ \mu_1 & \mu_2 & \dots & \mu_m \end{Bmatrix}, & \quad (5.19) \end{aligned}$$

and

$$\begin{aligned} \sigma\left(\begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} \begin{Bmatrix} \eta_1 & \eta_2 & \dots & \eta_m \\ \mu_1 & \mu_2 & \dots & \mu_m \end{Bmatrix} \right) & \\ = \sigma \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} \sigma \begin{Bmatrix} \eta_1 & \eta_2 & \dots & \eta_m \\ \mu_1 & \mu_2 & \dots & \mu_m \end{Bmatrix}. & \quad (5.20) \end{aligned}$$

Theorem 5.4.1. *There exist symbols* $\begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_i \\ \nu_1 & \nu_2 & \cdots & \nu_i \end{bmatrix}_\sigma \in \mathbb{F}_p$ *such that*

$$\sigma \begin{Bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \\ \nu_1 & \nu_2 & \cdots & \nu_n \end{Bmatrix} = \sum_{i=0}^{n} \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_i \\ \nu_1 & \nu_2 & \cdots & \nu_i \end{bmatrix}_\sigma \begin{Bmatrix} \xi_{i+1} & \xi_{i+2} & \cdots & \xi_n \\ \nu_{i+1} & \nu_{i+2} & \cdots & \nu_n \end{Bmatrix}. \tag{5.21}$$

Here, for $i = 0$,

$$\begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_i \\ \nu_1 & \nu_2 & \cdots & \nu_i \end{bmatrix}_\sigma = 1$$

and when $i = n$,

$$\begin{Bmatrix} \xi_{i+1} & \xi_{i+2} & \cdots & \xi_n \\ \nu_{i+1} & \nu_{i+2} & \cdots & \nu_n \end{Bmatrix} = 1.$$

Proof. Start with verifying the case that $n = 1$. From Equation (5.17), we have the following

$$\begin{Bmatrix} \xi \\ \nu \end{Bmatrix}^p - \begin{Bmatrix} \xi \\ \nu \end{Bmatrix} = \frac{\xi}{t^\nu},$$

where $\frac{\xi}{t^\nu} \in E$. Hence, $\begin{Bmatrix} \xi \\ \nu \end{Bmatrix}$ is a root of Artin-Schreier equation $p_a(x) = x^p - x - a = 0$ where $a = \frac{\xi}{t^\nu} \in E$. Since $\sigma \in \mathrm{Aut}_E(\mathscr{A})$, it is obvious that $\sigma \begin{Bmatrix} \xi \\ \nu \end{Bmatrix}$ is another root of $p_a(x)$. Therefore, there exists a unique $\begin{bmatrix} \xi \\ \nu \end{bmatrix}_\sigma \in \mathbb{F}_p$ satisfying

$$\sigma \begin{Bmatrix} \xi \\ \nu \end{Bmatrix} = \begin{Bmatrix} \xi \\ \nu \end{Bmatrix} + \begin{bmatrix} \xi \\ \nu \end{bmatrix}_\sigma.$$

Now, we suppose that $n = 2$. Then, it is known that the assertion holds:

$$\begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix}^p - \begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix} = \begin{Bmatrix} \xi_1 \\ \nu_1 \end{Bmatrix} \frac{\xi_2}{t^{\nu_2}}.$$

That is, $\begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix}$ is a root of Artin-Schreier equation $p_a(x) = x^p - x - a = 0$ where this time $a = \begin{Bmatrix} \xi_1 \\ \nu_1 \end{Bmatrix} \frac{\xi_2}{t^{\nu_2}} \in \mathscr{A}_1$ which is the algebraic extension of $\mathscr{A}_0 = E$ obtained by adjoining length 1 Arf symbols. Moreover, $\sigma \begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix}$ is a root of Artin-Schreier equation $p_{\sigma(a)}(x) = x^p - x - \sigma(a) = 0$ with $\sigma(a) = \left(\begin{Bmatrix} \xi_1 \\ \nu_1 \end{Bmatrix} + \begin{bmatrix} \xi_1 \\ \nu_1 \end{bmatrix}_\sigma \right) \frac{\xi_2}{t^{\nu_2}}$, which has p distinct roots. Now, introduce p distinct linear combinations

$$\begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix} + \begin{bmatrix} \xi_1 \\ \nu_1 \end{bmatrix}_\sigma \begin{Bmatrix} \xi_2 \\ \nu_2 \end{Bmatrix} + \lambda$$

as λ runs over $\mathbb{F}_p$. Then, we obtain

$$\left(\begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix} + \begin{bmatrix} \xi_1 \\ \nu_1 \end{bmatrix}_\sigma \begin{Bmatrix} \xi_2 \\ \nu_2 \end{Bmatrix} + \lambda \right)^p - \left(\begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix} + \begin{bmatrix} \xi_1 \\ \nu_1 \end{bmatrix}_\sigma \begin{Bmatrix} \xi_2 \\ \nu_2 \end{Bmatrix} + \lambda \right) - \sigma(a) = 0.$$

Therefore, there exists $\lambda \in \mathbb{F}_p$, which we shall denote by $\lambda = \begin{bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{bmatrix}_\sigma$, so that

$$\sigma \begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix} = \begin{Bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{Bmatrix} + \begin{bmatrix} \xi_1 \\ \nu_1 \end{bmatrix}_\sigma \begin{Bmatrix} \xi_2 \\ \nu_2 \end{Bmatrix} + \begin{bmatrix} \xi_1 & \xi_2 \\ \nu_1 & \nu_2 \end{bmatrix}_\sigma .$$

Using induction, we can show that

$$\sigma \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} = \sum_{i=0}^n \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_i \\ \nu_1 & \nu_2 & \dots & \nu_i \end{bmatrix}_\sigma \begin{Bmatrix} \xi_{i+1} & \xi_{i+2} & \dots & \xi_n \\ \nu_{i+1} & \nu_{i+2} & \dots & \nu_n \end{Bmatrix}.$$

Here we make the convention that for $i = 0$, $\begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_i \\ \nu_1 & \nu_2 & \dots & \nu_i \end{bmatrix}_\sigma = 1$; and for $i = n$, $\begin{Bmatrix} \xi_{i+1} & \xi_{i+2} & \dots & \xi_n \\ \nu_{i+1} & \nu_{i+2} & \dots & \nu_n \end{Bmatrix} = 1.$

□

Furthermore, by applying σ to Equations (5.6)-(5.10), we obtain the following type of symbols which satisfy certain, similar equations (see [1]).

$$\begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_h' + \xi_h'' & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{bmatrix}_\sigma = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_h' & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{bmatrix}_\sigma + \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_h'' & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{bmatrix}_\sigma, \tag{5.22}$$

$$\begin{bmatrix} \xi^p \\ \nu p \end{bmatrix}_\sigma = \begin{bmatrix} \xi \\ \nu \end{bmatrix}_\sigma, \tag{5.23}$$

for $n \geq 2$,

$$\begin{bmatrix} \xi_1^p & \xi_2 & \dots & \xi_n \\ \nu_1 p & \nu_2 & \dots & \nu_n \end{bmatrix}_\sigma = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{bmatrix}_\sigma + \begin{bmatrix} \xi_1\xi_2 & \xi_3 & \dots & \xi_n \\ \nu_1 + \nu_2 & \nu_3 & \dots & \nu_n \end{bmatrix}_\sigma, \tag{5.24}$$

$$\begin{bmatrix} \xi_1 & \dots & \xi_{h-1} & \xi_h^p & \xi_{h+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{h-1} & \nu_j p & \nu_{h+1} & \dots & \nu_n \end{bmatrix}_\sigma = \begin{bmatrix} \xi_1 & \dots & \xi_{h-1} & \xi_h & \xi_{h+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{h-1} & \nu_h & \nu_{h+1} & \dots & \nu_n \end{bmatrix}_\sigma$$
$$+ \begin{bmatrix} \xi_1 & \dots & \xi_{h-1} & \xi_h\xi_{h+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{h-1} & \nu_h + \nu_{h+1} & \dots & \nu_n \end{bmatrix}_\sigma - \begin{bmatrix} \xi_1 & \dots & \xi_{h-1}\xi_h^p & \xi_{h+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{h-1} + \nu_h p & \nu_{h+1} & \dots & \nu_n \end{bmatrix}_\sigma, \tag{5.25}$$

$$\begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} & \xi_n^p \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} & \nu_n p \end{bmatrix}_\sigma = \begin{bmatrix} \xi_1 & \dots & \xi_{n-1} & \xi_n \\ \nu_1 & \dots & \nu_{n-1} & \nu_n \end{bmatrix}_\sigma - \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1}\xi_n^p \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} + p\nu_n \end{bmatrix}_\sigma. \tag{5.26}$$

Introduce the mapping

$$e^{(\sigma)}_{\substack{\xi \\ \nu}} : \mathscr{A} \to \mathbb{F}_p$$

given by

$$S \mapsto \left[S; \begin{matrix} \xi \\ \nu \end{matrix}\right]_\sigma,$$

for

$$S = \sum_i \frac{\eta^{(i)}}{t^{\nu(i)}} \begin{Bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} \end{Bmatrix} \in \mathscr{A}$$

where the element $\left[S; \begin{matrix}\xi \\ \nu\end{matrix}\right]_\sigma$ in $\mathbb{F}_p$ is defined by

$$\left[S; \begin{matrix}\xi \\ \nu\end{matrix}\right]_\sigma = \sum_i \begin{bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} & \eta^{(i)}\xi \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} & \nu(i)+\nu \end{bmatrix}_\sigma .$$

From Equations (5.17) and (5.18), we get

$$\sigma\left\{S; \begin{matrix}\xi \\ \nu\end{matrix}\right\} = \left\{\sigma S; \begin{matrix}\xi \\ \nu\end{matrix}\right\} + \left[S; \begin{matrix}\xi \\ \nu\end{matrix}\right]_\sigma . \tag{5.27}$$

Moreover, from Equations (5.12), (5.20) and (5.27), we obtain

$$\left[\overline{A}; \begin{matrix}\xi_n \\ \nu_n\end{matrix}\right]_\sigma \left[\overline{B}; \begin{matrix}\eta_m \\ \mu_m\end{matrix}\right]_\sigma = \left[\overline{A}\left\{\overline{B}; \begin{matrix}\eta_m \\ \mu_m\end{matrix}\right\}; \begin{matrix}\xi_n \\ \nu_n\end{matrix}\right]_\sigma + \left[\left\{\overline{A}; \begin{matrix}\xi_n \\ \nu_n\end{matrix}\right\}\overline{B}; \begin{matrix}\eta_m \\ \mu_m\end{matrix}\right]_\sigma + \left[\overline{A}\,\overline{B}; \begin{matrix}\xi_n\eta_m \\ \nu_n+\mu_m\end{matrix}\right]_\sigma . \tag{5.28}$$

Conversely, if the above Equations (5.22)-(5.26) and Equation (5.28) are satisfied, then Equation (5.21) is an automorphism of $\mathscr{A}$ that fixes the elements of E.

Now, consider the elements of the form

$$A = \sum_i \frac{\eta^{(i)}}{t^{\mu(i)}} \begin{Bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} \end{Bmatrix}$$

where $\mu(i) \in \mathbb{Z}_{(p)}$ non-negative. Then, such elements $A \in \mathscr{A}$ form an integral domain denoted by $\mathscr{A}'$. Moreover, the set of all elements of the form

$$B = \sum_i \frac{\eta^{(i)}}{t^{\mu(i)}} \begin{Bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} \end{Bmatrix}$$

where $\mu(i) \in \mathbb{Z}_{(p)}$ positive form an ideal $\mathcal{I}$ of $\mathscr{A}'$.

The residue class ring $\mathscr{A}'/\mathcal{I} = \overline{\mathscr{A}}$ is clearly an algebra over $\overline{\mathbb{F}}_p$. The residue classes

$$1 := 1 \mod \mathcal{I}$$

and

$$\begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{bmatrix} := \begin{Bmatrix} \beta_1 & \beta_2 & \dots & \beta_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} \mod \mathcal{I}$$

form an $\overline{\mathbb{F}}_p$-basis of $\overline{\mathscr{A}}$. Generally, we denote the residue class with

$$\begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{bmatrix} := \begin{Bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{Bmatrix} \mod \mathcal{I}.$$

Furthermore, for $A = \sum_i \eta^{(i)} \begin{bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} \end{bmatrix}$, by reducing $E_{\substack{\xi \\ \nu}}(A)$ modulo $\mathcal{I}$, we have

$$\begin{bmatrix} A; \begin{matrix} \xi \\ \nu \end{matrix} \end{bmatrix} = \sum_i \begin{bmatrix} \xi_1^{(i)} & \xi_2^{(i)} & \dots & \xi_{n(i)}^{(i)} & \eta^{(i)}\xi \\ \nu_1^{(i)} & \nu_2^{(i)} & \dots & \nu_{n(i)}^{(i)} & \nu \end{bmatrix}.$$

Then, the following equations are satisfied:

$$\begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_h' + \xi_h'' & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_h' & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{bmatrix} + \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_h'' & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_h & \dots & \nu_n \end{bmatrix}, \quad (5.29)$$

$$\begin{bmatrix} \xi^p \\ \nu p \end{bmatrix} = \begin{bmatrix} \xi \\ \nu \end{bmatrix}, \quad (5.30)$$

for $n \geq 2$,

$$\begin{bmatrix} \xi_1^p & \xi_2 & \dots & \xi_n \\ \nu_1 p & \nu_2 & \dots & \nu_n \end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{bmatrix} + \begin{bmatrix} \xi_1\xi_2 & \xi_3 & \dots & \xi_n \\ \nu_1 + \nu_2 & \nu_3 & \dots & \nu_n \end{bmatrix}, \quad (5.31)$$

$$\begin{bmatrix} \xi_1 & \dots & \xi_{h-1} & \xi_h^p & \xi_{h+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{h-1} & \nu_h p & \nu_{h+1} & \dots & \nu_n \end{bmatrix} = \begin{bmatrix} \xi_1 & \dots & \xi_{h-1} & \xi_h & \xi_{h+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{h-1} & \nu_h & \nu_{h+1} & \dots & \nu_n \end{bmatrix}$$
$$+ \begin{bmatrix} \xi_1 & \dots & \xi_{h-1} & \xi_h\xi_{h+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{h-1} & \nu_j + \nu_{h+1} & \dots & \nu_n \end{bmatrix} - \begin{bmatrix} \xi_1 & \dots & \xi_{h-1}\xi_h^p & \xi_{h+1} & \dots & \xi_n \\ \nu_1 & \dots & \nu_{h-1} + \nu_h p & \nu_{h+1} & \dots & \nu_n \end{bmatrix}, \quad (5.32)$$

$$\begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1} & \xi_n^p \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} & \nu_n p \end{bmatrix} = \begin{bmatrix} \xi_1 & \dots & \xi_{n-1} & \xi_n \\ \nu_1 & \dots & \nu_{n-1} & \nu_n \end{bmatrix} - \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_{n-1}\xi_n^p \\ \nu_1 & \nu_2 & \dots & \nu_{n-1} + \nu_n p \end{bmatrix}, \tag{5.33}$$

$$\left[A; \begin{matrix} \xi \\ \nu \end{matrix}\right] \left[B; \begin{matrix} \eta \\ \mu \end{matrix}\right] = \left[\left[A; \begin{matrix} \xi \\ \nu \end{matrix}\right] B; \begin{matrix} \eta \\ \mu \end{matrix}\right] + \left[A \left[B; \begin{matrix} \eta \\ \mu \end{matrix}\right]; \begin{matrix} \xi \\ \nu \end{matrix}\right] + \left[AB; \begin{matrix} \xi\eta \\ \nu + \mu \end{matrix}\right], \tag{5.34}$$

$$\begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{bmatrix}^p = \begin{bmatrix} \xi_1^p & \xi_2^p & \dots & \xi_n^p \\ \nu_1 p & \nu_2 p & \dots & \nu_n p \end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{bmatrix}. \tag{5.35}$$

Equations (5.29)-(5.34) when compared with the preceding Equations (5.22)-(5.26) and Equation (5.28) show that the mapping

$$\sigma^* : \sum \eta \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{bmatrix} \mapsto \sum \eta \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \nu_1 & \nu_2 & \dots & \nu_n \end{bmatrix}_\sigma \quad \textit{where } \eta \in \overline{\mathbb{F}}_p \tag{5.36}$$

from $\overline{\mathscr{A}}$ to $\overline{\mathbb{F}}_p$ is a homomorphism. Moreover, each homomorphisms from $\overline{\mathscr{A}}$ to $\overline{\mathbb{F}}_p$ which leaves the element of $\overline{\mathbb{F}}_p$ fixed correspond to σ. That is, we obtain the following theorem:

Theorem 5.4.2. *There exists an isomorphism*

$$\mathrm{Aut}_E(\mathscr{A}) \xrightarrow{\sim} \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\overline{\mathscr{A}}, \overline{\mathbb{F}}_p)$$

defined by

$$\sigma \mapsto \sigma^*,$$

where σ^* is given by (5.36), *for all $\sigma \in \mathrm{Aut}_E(\mathscr{A})$.*

Note that, $\mathrm{Aut}_E(\mathscr{A})$ is nothing but the absolute Galois group G_E of E. Therefore, G_E has a description in terms of Arf symbols. Now, we can describe the absolute Galois group $G_{\mathbb{F}_p((t))}$ in terms of Arf symbols and non-abelian local class field theory which is the topic of the next section.

5.5. CONSTRUCTION OF THE ABSOLUTE GALOIS GROUP $G_{\mathbb{F}_p((t))}$

In this section, we shall construct the absolute Galois group $G_{\mathbb{F}_p((t))}$ of $\mathbb{F}_p((t))$ using Arf symbols and the non-abelian local class field theory of $\mathbb{F}_p((t))$. Recall that we have the following identifications

$$G_E = \text{Aut}_E(E^{sep}) = \text{Aut}_E(\mathbb{F}_p((t))^{sep}) \simeq \text{Aut}_E(\mathscr{A}) \simeq \text{Hom}_{\overline{\mathbb{F}}_p}(\overline{\mathscr{A}}, \overline{\mathbb{F}}_p),$$

where E is the maximal tamely ramified extension of $\mathbb{F}_p((t))$; that is,

$$E = \overline{\mathbb{F}}_p((t))(t^{1/n} \mid n \in \mathbb{Z}_{>0}, p \nmid n).$$

Theorem 5.5.1. *The following natural short exact sequence*

$$1 \to G_E \hookrightarrow G_{\mathbb{F}_p((t))} \xrightarrow{r} \text{Gal}(E/\mathbb{F}_p((t))) \to 1 \tag{5.37}$$

splits.

Proof. For a proof, see Theorem 2.2 in [21]. □

Note that, there exists group theoretical description of $\text{Gal}(E/\mathbb{F}_p((t)))$ given by Iwasawa (see [18] and 7.5.3 of [30]), which states that $\text{Gal}(E/\mathbb{F}_p((t)))$ is a 2-generated profinite group with genarators α, β satisfying the unique "tame" relation $\alpha\beta\alpha^{-1} = \beta^p$. However, Iwasawa's description of $\text{Gal}(E/\mathbb{F}_p((t)))$ does not encode the arithmetic information of the non-abelian extension $E/\mathbb{F}_p((t)))$. Thus, we shall use the non-abelian local class field theory of $\mathbb{F}_p((t))$ introduced in [14].

By the non-abelian local class field theory of $\mathbb{F}_p((t))$, there exists a topological group isomorphism

$$\text{Gal}(E/\mathbb{F}_p((t))) \simeq \nabla_{\mathbb{F}_p((t))}/\mathcal{N}^{\infty}_{E/\mathbb{F}_p((t))}, \tag{5.38}$$

which in return defines the following split short exact sequence

$$1 \to G_E \hookrightarrow G_{\mathbb{F}_p((t))} \twoheadrightarrow \nabla_{\mathbb{F}_p((t))}/\mathcal{N}^{\infty}_{E/\mathbb{F}_p((t))} \to 1.$$

Therefore, there exists a section $s : \mathrm{Gal}(E/\mathbb{F}_p((t))) \to G_{\mathbb{F}_p((t))}$ of $r : G_{\mathbb{F}_p((t))} \to \mathrm{Gal}(E/\mathbb{F}_p((t)))$ which is furthermore a continuous homomorphism, inducing an action α_s of $\mathrm{Gal}(E/\mathbb{F}_p((t)))$ on G_E by conjugation in $G_{\mathbb{F}_p((t))}$ as follows.

For $\kappa \in \mathrm{Gal}(E/\mathbb{F}_p((t)))$ and $\gamma \in G_E$, we have

$$s(\kappa)\gamma s(\kappa)^{-1} = s(\kappa)\gamma s(\kappa^{-1})$$

and

$$r(s(\kappa)\gamma s(\kappa^{-1})) = r(s(\kappa))r(\gamma)r(s(\kappa^{-1})) = \kappa r(\gamma)\kappa^{-1} = \kappa\kappa^{-1} = 1$$

as $\gamma \in G_E = \mathrm{Ker}(r)$. Therefore, there exists a unique $\gamma' \in G_E$ such that $s(\kappa)\gamma s(\kappa^{-1}) = \gamma'$ which defines an action of $\mathrm{Gal}(E/\mathbb{F}_p((t)))$ on G_E. So, there exists a homomorphism

$$\theta : \mathrm{Gal}(E/\mathbb{F}_p((t))) \to \mathrm{Aut}(G_E)$$

defined by

$$\theta : \kappa \mapsto \theta_\kappa$$

for every $\kappa \in \mathrm{Gal}(E/\mathbb{F}_p((t)))$, where

$$\theta_\kappa : G_E \to G_E$$

is the homomorphism given by

$$\theta_\kappa(\gamma) = \gamma',$$

for all $\gamma \in G_E$.

From the theory of group extensions (see [4]), $G_{\mathbb{F}_p((t))}$ is topologically isomorphic to the semi-direct product $G_E \rtimes_{\theta^*} \mathrm{Gal}(E/\mathbb{F}_p((t)))$(see [20] and 7.5.13 of [30]). Thus, we have proven the following theorem:

Theorem 5.5.2. *There exists a topological group isomorphism*

$$G_{\mathbb{F}_p((t))} \xrightarrow{\sim} \mathrm{Aut}_E(\mathscr{A}) \rtimes_{\theta^*} \nabla_{\mathbb{F}_p((t))}/\mathcal{N}^{\infty}_{E/\mathbb{F}_p((t))},$$

where $\theta^* : \nabla_{\mathbb{F}_p((t))}/\mathcal{N}^{\infty}_{E/\mathbb{F}_p((t))} \to \mathrm{Aut}(\mathrm{Aut}_E(\mathscr{A}))$ *sits in the following commutative diagram*

$$\begin{array}{ccc} \mathrm{Gal}(E/\mathbb{F}_p((t))) & \xrightarrow{\theta} & \mathrm{Aut}(G_E) \\ \uparrow & & \uparrow \\ \nabla_{\mathbb{F}_p((t))}/\mathcal{N}^{\infty}_{E/\mathbb{F}_p((t))} & \xrightarrow{\theta^*} & \mathrm{Aut}(\mathrm{Aut}_E(\mathscr{A})) \end{array}$$

of continuous group homomorphisms.

6. CONSTRUCTION OF THE ABSOLUTE GALOIS GROUP G_K OF A LOCAL FIELD K

In this chapter, K denotes any local field with no restriction on $\text{char}(K)$. Our aim is to describe the absolute Galois group G_K of K by using certain hybrid symbols constructed out of Arf symbols and Steinberg symbols.

6.1. STATEMENT OF THE MAIN THEOREM

In this section, we shall state the main theorem of our thesis. However, we shall postpone its proof to Section 6.2.

Main Theorem . *The following pair of short exact sequences*

$$\begin{array}{ccccccccc} & & 1 & & & & & & \\ & & \downarrow & & & & & & \\ & & \text{Aut}_E(\mathscr{A}) & & & & & & \\ & & \downarrow & & & & & & \\ 1 & \longrightarrow & G_{\mathbb{F}_p((t))} & \longrightarrow & G_K & \longrightarrow & \widehat{K_1^M(K)}/\mathcal{N}_{L/K}\widehat{K_1^M(L)} & \longrightarrow & 1 \\ & & \downarrow & & & & & & \\ & & \nabla_{\mathbb{F}_p((t))}/\mathcal{N}^{\infty}_{E/\mathbb{F}_p((t))} & & & & & & \\ & & \downarrow & & & & & & \\ & & 1 & & & & & & \end{array}$$

uniquely describe G_K in terms of Arf symbols, Milnor K_1-group and non-abelian local class formation modulo the choice of continuous normalized section and ψ defined by (6.2) *for the horizontal short exact sequence, and the choice of continuous normalized section for the vertical short exact sequence, which is a homomorphism, as the vertical sequence splits which yields an explicit description of $G_{\mathbb{F}_p((t))}$ (see Theorem 5.5.2).*

6.2. PROOF OF THE MAIN THEOREM

Our proof consists of the following steps:

(i) Let L be an infinite Galois APF-extension of K and $\mathbb{X}(L/K)$ be the field of norms corresponding to L/K. Recall that, $\mathbb{X}(L/K)$ is a local field of $\text{char}(\mathbb{X}(L/K)) = \text{char}(\kappa_K) = p > 0$. Moreover, there is an isomorphism (see Chapter 3):

$$G_{\mathbb{X}(L/K)} \simeq \text{Gal}(K^{sep}/L) = G_L.$$

(ii) By Cohen-Gabber Structure Theorem of local fields in equal characteristic case (see [22]), the field of norms $\mathbb{X}(L/K)$ of the APF-extension L/K can be identified with a finite extension of $\mathbb{F}_p((t))$.

(iii) By [10], choose L/K in such a way that $\mathbb{X}(L/K) \simeq \mathbb{F}_p((t))$ and $\text{Gal}(L/K)$ is a closed subgroup of a p-adic Lie group such that it is isomorphic to $\mathbb{Z}_p$. Furthermore, $\text{Gal}(K^{sep}/L) \simeq G_{\mathbb{F}_p((t))}$ which has description in terms of Arf symbols and the non-abelian local class field theory as described in Section 5.5. Therefore, the following isomorphism

$$G_{\mathbb{F}_p((t))} \simeq G_{\mathbb{X}(L/K)} \simeq \text{Gal}(L^{sep}/L) = \text{Gal}(K^{sep}/L),$$

clearly holds under the mapping

$$i^{FW} : G_{\mathbb{F}_p((t))} \xrightarrow{\sim} \text{Gal}(K^{sep}/L) \hookrightarrow G_K.$$

(iv) Note that, G_K sits in the short exact sequence

$$1 \to \text{Gal}(K^{sep}/L) \hookrightarrow G_K \twoheadrightarrow \text{Gal}(L/K) \to 1 \tag{6.1}$$

of profinite groups. Observe that G_K acts on G_L via conjugation, which induces a group homomorphism

$$\psi_0 : G_K/G_L \to \text{Out}(G_L),$$

defined in the natural way. Therefore, there exists a group homomorphism

$$\psi : \mathrm{Gal}(L/K) \xrightarrow{\overline{res_L}^{-1}} G_K/G_L \xrightarrow{\psi_0} \mathrm{Out}(G_L) = \mathrm{Aut}(G_L)/\mathrm{Inn}(G_L) \tag{6.2}$$

where $\mathrm{Inn}(G_L)$ consists of all inner automorphisms of G_L and $\mathrm{Out}(G_L)$ is the outer automorphism group of G_L which is the cokernel of

$$\alpha : G_L \to \mathrm{Aut}(G_L).$$

(v) To sum up, there exists the following short exact sequence of profinite groups:

$$1 \to G_{\mathbb{X}(L/K)} \hookrightarrow G_K \twoheadrightarrow \mathrm{Gal}(L/K) \to 1. \tag{6.3}$$

Here:

- $G_{\mathbb{X}(L/K)}$ can be described by Arf symbols and the non-abelian local class field theory, as $\mathbb{X}(L/K) \simeq \mathbb{F}_p((t))$ by (iii);
- $\mathrm{Gal}(L/K)$ can be described by the non-abelian local class field theory of K (see [5, 6], [12, 13, 14]). We shall, however, choose a totally ramified $\mathbb{Z}_p$-extension, which suffices for our purposes;
- G_K is a topological group extension of $\mathrm{Gal}(L/K)$ by $G_{\mathbb{X}(L/K)}$ (see Chapter 6 of [34]).

(vi) Let $s : \mathrm{Gal}(L/K) \to G_K$ be the continuous, normalized section of (6.3) which exists in the profinite category (see Proposition 1.3.2 of [34]). The fixed section $s : \mathrm{Gal}(L/K) \to G_K$ identifies a function

$$f : \mathrm{Gal}(L/K) \times \mathrm{Gal}(L/K) \to G_{\mathbb{F}_p((t))}$$

satisfying the equation

$$s(g)s(g') = f(g, g')s(gg'),$$

for every $g, g' \in \mathrm{Gal}(L/K)$. Additionally, by using $\psi^* : \mathrm{Gal}(L/K) \to \mathrm{Aut}(G_{\mathbb{F}_p((t))})$, let us identify the lifting

$$\psi : \mathrm{Gal}(L/K) \to \mathrm{Out}(G_{\mathbb{F}_p((t))}) = \mathrm{Aut}(G_{\mathbb{F}_p((t))})/\mathrm{Inn}(G_{\mathbb{F}_p((t))})$$

where $\mathrm{Inn}(G_{\mathbb{F}_p((t))})$ consists of all inner automorphisms of $G_{\mathbb{F}_p((t))}$. Also, $\mathrm{Out}(G_{\mathbb{F}_p((t))})$ is the outer automorphism group of $G_{\mathbb{F}_p((t))}$ which is the cokernel of

$$\alpha : G_{\mathbb{F}_p((t))} \to \mathrm{Aut}(G_{\mathbb{F}_p((t))}).$$

For $g, g', g'' \in \mathrm{Gal}(L/K)$, the following gives a relation between f and ψ^* by

$$\psi^*(g)\psi^*(g') = \alpha(f(g,g'))\psi^*(gg')$$

where $\alpha : G_{\mathbb{F}_p((t))} \to \mathrm{Aut}(G_{\mathbb{F}_p((t))})$. Moreover, for every $g, g', g'' \in \mathrm{Gal}(L/K)$, the following identity, which is named cocycle condition,

$$f(g,g')f(gg',g'') = \psi^*(f(g',g''))f(g,g'g'')$$

holds. Now, introduce a composition law on

$$E_{f,\psi^*} = G_{\mathbb{F}_p((t))} \times \mathrm{Gal}(L/K)$$

by the rule

$$(m,g)(m',g') = (m\psi^*(g)(m')f(g,g'), gg')$$

for all $(m,g),(m',g') \in E_{f,\psi^*}$. Here, the center $Z(G_{\mathbb{F}_p((t))})$ is trivial by Mochizuki [29]. Therefore, if we fix $\psi : \mathrm{Gal}(L/K) \to \mathrm{Out}(G_{\mathbb{F}_p((t))})$, then there exists a unique topological extension of $\mathrm{Gal}(L/K)$ by $G_{\mathbb{F}_p((t))}$ up to equivalence (for details, see [4]). Thus, by Section 8 of the paper [14], there exists a topological group isomorphism

$$\rho : G_K \xrightarrow{\sim} E_{f,\psi^*}$$

which sits in the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{\mathbb{F}_p((t))} & \xrightarrow{i^{FW}} & G_K & \xrightarrow{\mathrm{res}_L} & \mathrm{Gal}(L/K) & \longrightarrow & 1 \\ & & \Vert & & \rho \downarrow \wr & & \Vert & & \\ 1 & \longrightarrow & G_{\mathbb{F}_p((t))} & \xrightarrow{i} & E_{f,\psi^*} & \xrightarrow{p} & \mathrm{Gal}(L/K) & \longrightarrow & 1 \end{array}$$

of short exact sequence.

From the discussion above, there exists the following short exact sequence

$$1 \to G_{\mathbb{F}_p((t))} \hookrightarrow G_K \twoheadrightarrow \mathrm{Gal}(L/K) \to 1. \tag{6.4}$$

By the abelian local class field theory of K, which we reviewed in the third part of Chapter 2 in our thesis, the short exact sequence (6.4) yields

$$1 \to G_{\mathbb{F}_p((t))} \to G_K \to \widehat{K_1^M(K)}/\mathcal{N}_{L/K}\widehat{K_1^M(L)} \to 1 \tag{6.5}$$

where $K_n^M(K) = (K^\times)^{\otimes n}/I$ denotes the n-th Milnor K-group of the local field K (actually, Milnor K-theory can be defined for an arbitrary field). Here,

$$(K^\times)^{\otimes n} = \underbrace{K^\times \otimes \cdots \otimes K^\times}_{n-copy}$$

is the n-fold tensor product of $K^\times$ with itself and I denotes the subgroup of $(K^\times)^{\otimes n}$ generated by the set of Steinberg relations:

$$\{a_1 \otimes \cdots \otimes a_m : a_1, \ldots, a_m \in K^\times \textit{ such that } a_i + a_j = 1 \textit{ for some } i \neq j\}.$$

For the abelian extension L/K, there is the K-theoretic universal norm homomorphism

$$\mathcal{N}_{L/K} : \widehat{K_n^M(L)} \to \widehat{K_n^M(K)}.$$

In particular, when $n = 1$, we have $K_1^M(K) = K^\times$ and in this case, the norm map coincides with the usual universal norm map $\mathcal{N}_{L/K} : \widehat{L^\times} \to \widehat{K^\times}$ where

$$\mathcal{N}_{L/K}(\widehat{L^\times}) = \bigcap_{\substack{K \subseteq F \subseteq L \\ \text{finite}}} \mathrm{N}_{F/K}(\widehat{F^\times}).$$

For details, see [28].

Remark 6.2.1. *Observe that, we have used the K-theoretic formulation of abelian local class field theory over K with an aim to generalize our discussion to the symbol theoretic descriptions of n-dimensional local fields and their absolute Galois groups, which is closely related to the ongoing work of İkeda and Serbest [16].*

6.3. FURTHER WORK

We finish our thesis by a list of possible future research directions that are based on our construction. It would be very interesting to answer the following natural questions:

(i) Complete the unfinished project of Arf started in [1]. Namely, it would be very interesting to construct a non-abelian local class field theory over $\mathbb{F}_p((t))$ and then over K using hybrid "Arf-Steinberg" symbols, and study its relationship with the non-abelian local class field theory developed in [3], [14] and [27].

(ii) Study the relationship between the symbol theoretic construction of G_K in our thesis and the local Langlands reciprocity principle.

(iii) Give a similar symbol theoretic description of G_K as well as a non-abelian local class field theory of K where this time K is an n-dimensional local field (see Remark 6.2.1).

(iv) Quite recently, Fontaine-Wintenberger theory has been generalized by Scholze (see [32, 33]), where he introduced certain valued fields called perfectoid fields. It seems to be possible to extend the results of our thesis without much difficulty to perfectoid fields.

7. CONCLUSION

In this thesis, we made an overview of Arf symbols and reviewed Arf's construction of G_E in terms of his symbols. We then completed a gap in Arf's paper. Namely, using Koch's group theoretic description of the absolute Galois group $G_{\mathbb{F}_p((t))}$ of $\mathbb{F}_p((t))$, we managed to construct $G_{\mathbb{F}_p((t))}$ using a modification of Arf symbols via non-abelian local class field theory. Next, combining our modification of Arf's construction with Fontaine- Wintenberger theory of fields of norms attached to APF-extensions, we obtained a description of the absolute Galois group G_K of any local field K in terms of "hybrid symbols" constructed out of Arf symbols, Steinberg symbols of Milnor's K-theory and non-abelian local class field theory. Thus, the symbols that we introduce in our thesis should be considered as the non-abelian version of $K_1^M(K)$ where K is a local field.

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